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Uncertainty principles for locally compact quantum groups

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A B S T R A C T

In this paper, we prove the Donoho–Stark uncertainty principle for locally compact quantum groups and characterize the minimizer which are bi-shifts of group-like projections. We also prove the Hirschman–Beckner uncertainty principle for compact quantum groups and discrete quantum groups. Furthermore, we show Hardy's uncertainty principle for locally compact quantum groups in terms of bi-shifts of group-like projections.

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1. Introduction

Uncertainty principles were first studied in quantum mechanics and then widely developed in harmonic analysis, information theory, and quantum information etc. In [9], Donoho and Stark proved a support-version uncertainty principle for cyclic groups and applied it in signal recovery. Later Candes, Romberg, and Tao [7] developed this un-

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certainty principle in the theory of compressed sensing. The Donoho–Stark uncertainty principle was proved for finite abelian groups [25], locally compact abelian groups[23], compact groups [1], Kac algebras [6,20]. The minimizers of the Donoho–Stark uncertainty principle for locally compact abelian groups [9,25,23] were the translations and modulations of characteristic functions of compact open subgroups. For noncommutative case, the authors [14] showed the Donoho–Stark uncertainty principle for subfactors and introduced bi-shifts of biprojections for subfactors which generalized the modulations and translations of characteristic functions of subgroups. The authors showed that the minimizers of the uncertainty principle are bi-shifts of biprojections. For infinite case, Liu and Wu [20] characterized the minimizers of the Donoho–Stark uncertainty principle for Kac algebras with biprojections.

Hirschman uncertainty principle in terms of entropies was first introduced by Hirschman in [13]. In [2], Beckner proved the uncertainty principle with sharp constant for the real line $\mathbb{R}$. The Hirschman–Beckner uncertainty principle generalized Heisenberg's uncertainty principle in quantum mechanics. This uncertainty principle was studied for locally compact abelian groups [23], Kac algebras [6,20], and subfactors [14]. The minimizers of the Hirschman–Beckner uncertainty principle were characterized in [23] and [14].

Hardy's uncertainty principle for $\mathbb{R}$ was proved in [11]. Hardy's uncertainty principles for arbitrary locally compact group were studied rarely. In [14], the authors showed that Hardy's uncertainty principle for subfactors by using the minimizers of the Donoho–Stark and the Hirschman–Beckner uncertainty principle. In [20], Liu and Wu proved Hardy's uncertainty principle for Kac algebras with biprojections. Note that the authors [14] showed that there are eight forms of a bi-shift of a biprojection and Hardy's uncertainty principle in [14,20] implies that the uniqueness of a bi-shift of a biprojection.

Locally compact quantum groups introduced by Kustermanns and Vaes [16,17] generalized locally compact groups and their duals. Compact quantum groups introduced by Woronowicz [27–30] are locally compact quantum groups. In this paper, we prove the Donoho–Stark uncertainty principle for locally compact quantum groups and characterize the minimizers of the uncertainty principle. We introduce the notion of a bi-shift of a group-like projection and show that the minimizers are bi-shifts of group-like projections. For finite abelian groups, bi-shifts of group-like projections are wave packets [10]. Wave packets are widely used in quantum mechanics, information theory, etc.

Main Theorem 1 *(Donoho–Stark uncertainty principle, Theorem 4.2, Proposition 4.7, Proposition 6.5). Suppose $\mathbb{G}$ is a locally compact quantum group. Then for any ω in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, $1 \le t \le 2$, $2 \le s \le \infty$, we have*

$$\mathcal{S}_r(\xi_t(\omega))\mathcal{S}_r(\iota^s(\lambda(\omega))) \ge 1.$$

Moreover the following are equivalent:

1. *$\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a minimizer of the Donoho–Stark uncertainty principle.*
2. *ω is an extremal bi-partial isometry such that $|\omega|\sigma_t^\varphi = |\omega|$, $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$, $\forall t \in \mathbb{R}$.*
3. *ω is a bi-partial isometry, $|\omega|\sigma_t^\varphi = |\omega|$, $\forall t \in \mathbb{R}$, and $\lambda(\omega)$ is in $L^1(\hat{\mathbb{G}})$ such that $\|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty = \|\lambda(\omega)\hat{\varphi}\|$.*
4. *$\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ satisfies that $\mathcal{S}_r(\omega)\mathcal{S}_r(\lambda(\omega)) = 1$ and $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$.*
5. *$\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ satisfies that $\mathcal{S}_r(\omega)\mathcal{S}_r(\lambda(\omega)) = 1$ and $\mathcal{S}_r(\xi(\omega))\mathcal{S}_r(\hat{\Lambda}(\lambda(\omega))) = 1$.*
6. *ω is a bi-shift of a group-like projection $B \in L^1(\mathbb{G})$.*

Note that $\mathcal{S}_r(x)$ is the φ-value of the support projection of x, and it will be explained in Section 3.

The Donoho–Stark uncertainty principle for locally compact quantum groups is a series of inequalities which is different from the case for unimodular Kac algebras. For a locally compact quantum group $\mathbb{G}$, we define an entropy $H(\xi)$ of ξ in $L^2(\mathbb{G})$. Then we proved the Hirschman–Beckner uncertainty principle for compact quantum groups or discrete quantum groups.

Main Theorem 2 *(Hirschman–Beckner uncertainty principle, Theorem 5.15). Suppose $\mathbb{G}$ is a compact quantum group or a discrete quantum group and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Let $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ such that $\|\xi(\omega)\| = 1$. If $H(\xi(\omega))$, $H(\hat{\Lambda}(\lambda(\omega)))$, $\langle |\log d| J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle$ $\langle |\log \hat{d}| \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle$ are finite, then*

$$H(\xi(\omega)) + H(\hat{\Lambda}(\lambda(\omega))) \geq -\langle (\log d) J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle - \langle (\log \hat{d}) \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle.$$

Moreover, for any $\xi \in L^2(\mathbb{G})$ if $H(\xi)$, $H(\mathcal{F}_2(\xi))$, $\langle |\log d| J_\varphi \xi, J_\varphi \xi\rangle$, $\langle |\log \hat{d}| \hat{J}\mathcal{F}_2(\xi), \hat{J}\mathcal{F}_2(\xi)\rangle$ are finite, then

$$H(\xi) + H(\mathcal{F}_2(\xi)) \geq -\langle (\log d) J_\varphi \xi, J_\varphi \xi\rangle - \langle (\log \hat{d}) \hat{J}\mathcal{F}_2(\xi), \hat{J}\mathcal{F}_2(\xi)\rangle.$$

By using bi-shifts of group-like projections, we show Hardy's uncertainty principle for locally compact quantum groups with group-like projections.

Main Theorem 3 *(Hardy's uncertainty principle, Theorem 6.6). Suppose $\mathbb{G}$ is a locally compact quantum group with a bi-shift w of a group-like projection. Let $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ be such that*

$$|x^*| \leq C|w^*|, \quad |\lambda(x\varphi)| \leq C'|\lambda(w\varphi)|,$$

for some $C, C' > 0$. Then x is a multiple of w.

The techniques in [4,12,16] for locally compact quantum groups and noncommutative L^t spaces will be frequently used in the paper. This paper is organized as follows. In Section 2, we go over the definition and some basic properties of locally compact quantum groups and noncommutative L^t spaces for $1 \leq t \leq \infty$. In Section 3, we study support projections or range projections of elements in noncommutative L^t spaces. In Section 4, we show the Donoho–Stark uncertainty principle for locally compact quantum groups and obtain the properties of the minimizers of the Donoho–Stark uncertainty principle. In Section 5, we study the derivative of the t-norm and show the Hirschman–Beckner uncertainty principle for compact quantum groups or discrete quantum groups. In Section 6, we prove Hardy's uncertainty principle for locally compact quantum groups. In Section 7, we obtain more results on Young's inequality for locally compact quantum groups.

2. Preliminaries

In this section, we will recall some properties of noncommutative L^t spaces and the definition and properties of locally compact quantum groups.

Let $\mathcal{M}$ be a von Neumann algebra with a normal semi-finite faithful weight φ and

$$\mathfrak{N}_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) < \infty\}, \quad \mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi.$$

Denote by ∇_φ, J_φ, σ^φ the modular operator, modular conjugation and modular automorphism group associated with φ. It is known that φ is invariant under σ^φ, i.e. $\varphi\sigma_t^\varphi = \varphi$ for any $t \in \mathbb{R}$. For any $x \in \mathcal{D}(\sigma_{i/2}^\varphi)$, the domain of $\sigma_{i/2}^\varphi$, we have that $\varphi(x^*x) = \varphi(\sigma_{i/2}^\varphi(x)\sigma_{i/2}^\varphi(x)^*)$. Denote by $\mathfrak{M}_\varphi^+$ the set of all positive elements in $\mathfrak{M}_\varphi$.

Let $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$ be the Gelfand–Naimark–Segal (GNS) semi-cyclic representation, where $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathcal{H}_\varphi$ is an inclusion map. We assume that $\mathcal{M}$ act on $\mathcal{H}_\varphi$. Therefore we will omit π_φ. The modular conjugation J_φ satisfies that $J_\varphi\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_{i/2}^\varphi(x)^*)$ for any $x \in \mathfrak{N}_\varphi \cap \mathcal{D}(\sigma_{i/2}^\varphi)$. For any $x \in \mathfrak{M}_\varphi$ and $a \in \mathcal{D}(\sigma_{-i}^\varphi)$, we have $ax, xa \in \mathfrak{M}_\varphi$ and $\varphi(ax) = \varphi(x\sigma_{-i}(a))$.

Denote by $\mathcal{M}_*$ the predual of $\mathcal{M}$. Denote by $\mathcal{M}_*^+$ the set of all positive linear functionals in $\mathcal{M}_*$. For any $\omega \in \mathcal{M}_*$, the linear functional $\overline{\omega} \in \mathcal{M}_*$ is given by $\overline{\omega}(x) = \overline{\omega(x^*)}$ for any x in $\mathcal{M}$. Given x in $\mathcal{M}$ and $\omega \in \mathcal{M}_*$, the linear functionals $x\omega$ and ωx are given by $(x\omega)(y) = \omega(yx)$ and $(\omega x)(y) = \omega(xy)$ for any y in $\mathcal{M}$.

The Tomita algebra $\mathcal{T}_\varphi$ is given by

$$\mathcal{T}_\varphi = \{x \in \mathcal{M} : x \text{ is analytic w. r. t. } \sigma^\varphi \text{ and } \sigma_z^\varphi(x) \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi, \forall z \in \mathbb{C}\}.$$

It is known that $\mathcal{T}_\varphi, \mathcal{T}_\varphi^2$ are σ-strongly dense in $\mathcal{M}$.

Let $\mathcal{L}_\varphi = \{x \in \mathfrak{N}_\varphi : x\varphi \in \mathcal{M}_*\}$ and $\mathcal{R}_\varphi = \{x \in \mathfrak{N}_\varphi^* : \varphi x \in \mathcal{M}_*\}$. By the results in [3], we have that $\mathcal{T}_\varphi^2 \subset \mathcal{L}_\varphi$ and $\mathcal{R}_\varphi = \mathcal{L}_\varphi^*$.

The noncommutative L^t space $L^t(\mathcal{M})$ is the complex interpolation space $(\mathcal{M}, \mathcal{M}_*)_{[1/t]}$ of $\mathcal{M}_*$ and $\mathcal{M}$ for $1 \leq t \leq \infty$. In [3], $L^t(\mathcal{M})$ is written as $L^t(\mathcal{M})_{\text{left}}$ or $L^t_{(-1/2)}(\mathcal{M})$. Note that $L^1(\mathcal{M}) = \mathcal{M}_*$, $L^\infty(\mathcal{M}) = \mathcal{M}$, and $L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) = \mathcal{L}_\varphi$. Denote by $\iota^t : \mathcal{L}_\varphi \mapsto L^t(\mathcal{M})$ the embedding of $\mathcal{L}_\varphi$ into $L^t(\mathcal{M})$. It is known that $\iota^t(\mathcal{L}_\varphi)$ is dense in $L^t(\mathcal{M})$ for any $1 \leq t \leq \infty$. By results in [3], we identify $L^2(\mathcal{M})$ with $\mathcal{H}_\varphi$. Moreover,

$$L^1(\mathcal{M}) \cap L^2(\mathcal{M}) = \mathcal{I} = \{\omega \in \mathcal{M}_* | \Lambda_\varphi(x) \mapsto \omega(x^*), x \in \mathfrak{N}_\varphi \text{ is a bounded map.}\}$$
$$= \mathcal{M}_* \cap \mathcal{H}_\varphi,$$

and

$$L^2(\mathcal{M}) \cap L^\infty(\mathcal{M}) = \mathfrak{N}_\varphi = \mathcal{H}_\varphi \cap \mathcal{M}.$$

Denote by $\xi_t : \mathcal{I} \to L^t(\mathcal{M})$ for $1 \leq t \leq 2$ the embedding from $\mathcal{I}$ to $L^t(\mathcal{M})$. For $t = 2$, we use $\xi(\omega)$ instead of $\xi_2(\omega)$ whenever $\omega \in \mathcal{I}$. Denote by $\iota^s : \mathfrak{N}_\varphi \to L^s(\mathcal{M})$ for $2 \leq s \leq \infty$ the embedding from $\mathfrak{N}_\varphi$ to $L^s(\mathcal{M})$. For $s = 2$, we use $\Lambda_\varphi(x)$ instead of $\iota^2(x)$ whenever $x \in \mathfrak{N}_\varphi$. Note that we use the same notation as $\iota^t : \mathcal{L}_\varphi \to L^t(\mathcal{M})$ here, since there is no confusion.

Let ϕ be a fixed normal semifinite faithful weight on the commutant $\mathcal{M}'$ of $\mathcal{M}$ on $\mathcal{H}_\varphi$. A closed densely defined operator a on the Hilbert space $\mathcal{H}_\varphi$ is γ-homogeneous with $\gamma \in \mathbb{R}$ if $xa \subseteq a\sigma^\phi_{i\gamma}(x)$ for all $x \in \mathcal{M}'$ analytic with respect to the modular automorphism σ^ϕ of $\mathcal{M}'$. The Hilsum L^t space $L^t(\phi)$ is the space of all closed densely defined operators a on $\mathcal{H}_\varphi$ such that if $x = u|x|$ is the polar decomposition, then $|x|^t$ is the spatial derivative of a positive linear functional $\omega \in \mathcal{M}_*$ and $u \in \mathcal{M}$. Note that $L^\infty(\phi) = \mathcal{M}$. The distinguished spatial derivative $d = \frac{d\varphi}{d\phi}$ is a strictly positive self-adjoint operator acting on $\mathcal{H}_\varphi$ and $\sigma^\varphi_t(x) = d^{it}xd^{-it}$ for every x in $\mathcal{M}$; $\sigma^\phi_t(y) = d^{-it}yd^{it}$ for every y in $\mathcal{M}'$. Throughout the paper, we will identify strong product, strong sum as product, sum respectively. There is an isometric isomorphism $\Phi_t : L^t(\phi) \to L^t(\mathcal{M})$ for $1 \leq t \leq \infty$ such that

$$\Phi_t(xd^{1/t}) = \iota^t(x), \quad x \in \mathcal{T}^2_\varphi.$$

If $t \geq 2$, we have that $\Phi_t(xd^{1/t}) = \iota^t(x)$ for any $x \in \mathfrak{N}_\varphi$.

By Theorem 2.4 in [5], for any $x \in \mathcal{T}_\varphi$, $t \in [2, \infty]$, we have $xd^{1/t} = d^{1/t}\sigma^\varphi_{i/t}(x)$.

Now we will recall the definition of locally compact quantum groups. A locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ consists of

(1) A von Neumann algebra $\mathcal{M}$;
(2) A unital normal *-homomorphism $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$ satisfying $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, where $\iota : \mathcal{M} \to \mathcal{M}$ is the identity.
(3) Two normal semifinite faithful weight φ, ψ on $\mathcal{M}$ such that

$$\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1), \ \forall \omega \in \mathcal{M}^+_*, x \in \mathfrak{M}^+_\varphi,$$
$$\psi((\iota \otimes \omega)\Delta(x)) = \psi(x)\omega(1), \ \forall \omega \in \mathcal{M}^+_*, x \in \mathfrak{M}^+_\psi,$$

φ is the left Haar weight and ψ is the right Haar weight.

Suppose $\mathcal{H}_\varphi$ is the Hilbert space arising from the GNS representation of $\mathcal{M}$ with respect to φ and assume that $\mathcal{M}$ acts on $\mathcal{H}_\varphi$.

The multiplicative unitary operator $W \in \mathcal{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$ is defined by

$$W^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)(x \otimes 1))$$

for any $x, y \in \mathfrak{N}_\varphi$. We have $\Delta(x) = W^*(1 \otimes x)W$ for any $x \in \mathcal{M}$.

There is a dual locally compact quantum group $\hat{\mathbb{G}} = (\hat{\mathcal{M}}, \hat{\Delta}, \hat{\varphi}, \hat{\psi})$. Recall that

$$\hat{\mathcal{M}} = \overline{\{(\omega \otimes \iota)(W) | \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\sigma\text{-strong-}*},$$

$\hat{W} = \Sigma W^* \Sigma$, where Σ is the flip on $\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi$. $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$ for any $x \in \hat{\mathcal{M}}$. The Fourier representation $\lambda : \mathcal{M}_* \to \hat{\mathcal{M}}$ is given by $\lambda(\omega) = (\omega \otimes \iota)(W)$ for any ω in $\mathcal{M}_*$. The dual left Haar weight $\hat{\varphi}$ is the unique normal semifinite faithful weight on $\hat{\mathcal{M}}$ having $(\mathcal{H}_\varphi, \iota, \hat{\Lambda})$ as a GNS-construction where $\hat{\Lambda}$ is a linear map from $\mathfrak{N}_{\hat{\varphi}}$ to $\mathcal{H}_\varphi$ given by $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega)$ for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$. There are the antipode S, the unitary antipode R, and a scaling automorphism τ on $\mathcal{M}$. The left and the right Haar weights φ, ψ satisfies $\psi(\cdot) = \varphi(\delta^{1/2} \cdot \delta^{1/2})$, where δ is the modular element of the locally compact quantum group $\mathbb{G}$. Note that δ is a unique strictly positive element affiliated with $\mathcal{M}$ and

$$\Delta(\delta) = \delta \otimes \delta, \quad R(\delta) = \delta^{-1}, \quad \tau_t(\delta) = \delta, \quad \forall t \in \mathbb{R}.$$

There are norm continuous one-parameter representations ρ, δ^*, τ^* of $\mathbb{R}$ on $\mathcal{M}_*$ given by

$$\rho_t(\omega)(x) = \omega(\delta^{-it}\tau_{-t}(x)), \quad \delta_t^*(\omega)(x) = \omega(\delta^{it}x), \quad \tau_t^*(\omega)(x) = \omega(\tau_t(x))$$

respectively for all $\omega \in \mathcal{M}_*$, $x \in \mathcal{M}$ and $t \in \mathbb{R}$. For the dual locally compact quantum group $\hat{\mathbb{G}}$, there are norm continuous one-parameter groups $\hat{\sigma}$, $\hat{\tau}$ on $\hat{\mathcal{M}}$ such that

$$\hat{\sigma}_t(\lambda(\omega)) = \lambda(\rho_t(\omega)), \quad \hat{\tau}_t(\lambda(\omega)) = \lambda(\omega\tau_{-t})$$

respectively for all $t \in \mathbb{R}$ and $\omega \in \mathcal{M}_*$. There is the unitary antipode $\hat{R}$ on $\hat{\mathcal{M}}$ such that $\hat{R}(\lambda(\omega)) = \lambda(\omega R)$ for all $\omega \in \mathcal{M}_*$. There is also an antipode $\hat{S} = \hat{R}\hat{\tau}_{-i/2}$ on $\hat{\mathcal{M}}$. Denote by $\hat{J}$ the modular conjugation associated to $\hat{\varphi}$.

Recall that the antipode $S = R\tau_{-i/2}$ on $\mathcal{M}$ has the following properties:

1. $S(\psi \otimes \iota)((x^* \otimes 1)\Delta(y)) = (\psi \otimes \iota)(\Delta(x^*)(y \otimes 1)),\ \forall x, y \in \mathfrak{N}_\psi$.
2. $S((\iota \otimes \varphi)(\Delta(x^*)(1 \otimes y))) = (\iota \otimes \varphi)((1 \otimes x^*)\Delta(y)),\ \forall x, y \in \mathfrak{N}_\varphi$.
3. $S((\iota \otimes \omega)(W)) = (\iota \otimes \omega)(W^*),\ \forall \omega \in \mathcal{B}(\mathcal{H})_*$.

Denote by ν the scaling constant of the locally compact quantum group $\mathbb{G}$. Then

$$\varphi\tau_t = \nu^{-t}\varphi, \quad \varphi\sigma_t^\psi = \nu^t\varphi, \quad \forall t \in \mathbb{R}.$$

Some fundamental commutation relations for the locally compact quantum group $\mathbb{G}$ are

$$\Delta\sigma_t^{\varphi} = (\tau_t \otimes \sigma_t^{\varphi})\Delta, \ \ \Delta\tau_t = (\tau_t \otimes \tau_t)\Delta.$$

For a locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ and $t \in [1, \infty]$, we denote by $L^t(\mathbb{G})$ the complex interpolation space $L^t(\mathcal{M})$. For any $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, the L^t-Fourier transform $\mathcal{F}_t : L^t(\mathbb{G}) \to L^s(\hat{\mathbb{G}})$, $1/t + 1/t' = 1$ is given by

$$\mathcal{F}_t(\xi_t(\omega)) = \iota^{t'}(\lambda(\omega)).$$

By the Hausdorff–Young inequality [5,3], we have that $\|\mathcal{F}_t\| \leq 1$.

Suppose $\mathbb{G}$ is a locally compact quantum group. A projection B in $L^\infty(\mathbb{G})$ is a group-like projection if $\Delta(B)(1 \otimes B) = B \otimes B$ and $B \neq 0$. A projection B in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ is a biprojection if $\lambda(B\varphi)$ is a multiple of a projection. For more details on group-like projections, we refer to [18]. In [19], Liu, Wang and Wu show that a biprojection is a group-like projection in $L^1(\mathbb{G})$ if $\varphi = \psi$ or φ is tracial.

In the end of the section, we recall the resolvent convergence for unbounded self-adjoint operators, let $a_n, n = 1, \dots$ and a be (unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$. The elements a_n is said to converge to a in the strong resolvent sense if $(z - a_n)^{-1} \to (z - a)^{-1}$ in the strong operator topology for all $z \in \mathbb{C}$ with $\Im z \neq 0$. a_n is said to converge to a in the weak resolvent sense if $(z - a_n)^{-1} \to (z - a)^{-1}$ in the weak operator topology for all $z \in \mathbb{C}$ with $\Im z \neq 0$. The weak resolvent convergence implies the strong resolvent convergence. For more details, we refer the readers to [24].

3. Support projections

In this section, we define the range projections and the support projections of elements in the noncommutative L^t space $L^t(\mathcal{M})$ with the help of the Hilsum L^t spaces and investigate their properties, where $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ.

For any x in $L^t(\mathcal{M})$, $1 \leq t \leq \infty$, we define the range projection $\mathcal{R}_l(x)$ (which is on the left hand side) of x to be the range projection $\mathcal{R}(\Phi_t^{-1}(x))$ of $\Phi_t^{-1}(x)$ and the support projection $\mathcal{R}_r(x)$ (which is on the right hand side) of x to be $\mathcal{R}(\Phi_t^{-1}(x)^*)$. Note that by the definition of the Hilsum L^t space, we see that $\mathcal{R}(\Phi_t^{-1}(x)), \mathcal{R}(\Phi_t^{-1}(x)^*) \in \mathcal{M}$ for any $x \in L^t(\mathcal{M})$. We denote by $\mathcal{S}_l(x) = \varphi(\mathcal{R}_l(x))$ and $\mathcal{S}_r(x) = \varphi(\mathcal{R}_r(x))$. If $t = \infty$, then for any $x \in L^\infty(\mathcal{M})(= \mathcal{M})$, we have that

$$\mathcal{R}_l(x) = \mathcal{R}(x), \quad \mathcal{R}_r(x) = \mathcal{R}(x^*).$$

For any ω in $L^1(\mathcal{M})(= \mathcal{M}_*)$, we denote the support projection of ω by $\mathcal{R}(\omega)$ which is given by

$$\mathcal{R}(\omega) = 1 - p,$$

where p is the union of all projection p_α in $\mathcal{M}$ such that $\omega(p_\alpha x) = 0, \forall x \in \mathcal{M}$.

For any ξ in $L^2(\mathcal{M})(= \mathcal{H}_\varphi)$, we denote the support projection of ξ by $\mathcal{R}(\xi)$ given by

$$\mathcal{R}(\xi) = 1 - p,$$

where p is the union of all projection p_α in $\mathcal{M}$ such that $p_\alpha \xi = 0$. For any $x \in \mathcal{M}$ and $\xi \in L^2(\mathcal{M})$, the right action of x on ξ is given by $\xi x = J_\varphi x^* J_\varphi \xi$.

Lemma 3.1. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Then for any $x \in \mathcal{M}$, $\omega \in L^1(\mathcal{M})$, $\xi \in L^2(\mathcal{M})$, we have that

$$x\Phi_1^{-1}(\omega) = \Phi_1^{-1}(x\omega), \quad \Phi_1^{-1}(\omega)x = \Phi_1^{-1}(\omega x),$$

and

$$x\Phi_2^{-1}(\xi) = \Phi_2^{-1}(x\xi), \quad \Phi_2^{-1}(\xi)x = \Phi_2^{-1}(\xi x).$$

Proof. Suppose that $x \in \mathcal{T}_\varphi^2$ and assume that $\{x_\beta\}_\beta$ is a net in $\mathcal{T}_\varphi^2$ such that $\lim_\beta \|x_\beta\varphi - \omega\| = 0$. Then

$$\begin{aligned}
\|x\Phi_1^{-1}(\omega) - \Phi_1^{-1}(x\omega)\|_{1,\phi} &= \lim_\beta \|x\Phi_1^{-1}(\omega) - xx_\beta d + xx_\beta d - \Phi_1^{-1}(x\omega)\|_{1,\phi} \\
&\le \lim_\beta \|x\Phi_1^{-1}(\omega) - xx_\beta d\|_{1,\phi} + \lim_\beta \|xx_\beta d - \Phi_1^{-1}(x\omega)\|_{1,\phi} \\
&\le \|x\|_\infty \lim_\beta \|\omega - x_\beta\varphi\| + \lim_\beta \|xx_\beta\varphi - x\omega\| = 0,
\end{aligned}$$

i.e. $x\Phi_1^{-1}(\omega) = \Phi_1^{-1}(x\omega)$ for any x in $\mathcal{T}_\varphi^2$.

Suppose that $x \in \mathcal{M}$ and take a bounded net $\{x_\alpha\}_\alpha \subseteq \mathcal{T}_\varphi^2$ such that x_α converges to x σ-strongly. Let $\Phi_1^{-1}(\omega) = w|\Phi_1^{-1}(\omega)|$ be the polar decomposition and ω_0 the positive normal linear functional such that $|\Phi_1^{-1}(\omega)| = \frac{d\omega_0}{d\phi}$. Then

$$\begin{aligned}
\|x\Phi_1^{-1}(\omega) - \Phi_1^{-1}(x\omega)\|_{1,\phi} &= \lim_\alpha \|x\Phi_1^{-1}(\omega) - x_\alpha\Phi_1^{-1}(\omega) + \Phi_1^{-1}(x_\alpha\omega) - \Phi_1^{-1}(x\omega)\|_{1,\phi} \\
&\le \lim_\alpha \|(x - x_\alpha)\Phi_1^{-1}(\omega)\|_{1,\phi} + \lim_\alpha \|x_\alpha\omega - x\omega\| \\
&\le \lim_\alpha \|(x - x_\alpha)w|\Phi_1^{-1}(\omega)|^{1/2}\|_{2,\phi} \||\Phi_1^{-1}(\omega)|^{1/2}\|_{2,\phi} \\
&= \lim_\alpha \omega_0(w^*(x - x_\alpha)^*(x - x_\alpha)w)^{1/2}\omega_0(1)^{1/2} = 0,
\end{aligned}$$

i.e. $x\Phi_1^{-1}(\omega) = \Phi_1^{-1}(x\omega)$ for all x in $\mathcal{M}$. The rest of the Lemma can be proved similarly. □

Proposition 3.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Then for any ω in $L^1(\mathcal{M})$, we have that*

$$\mathcal{R}_l(\omega) = \mathcal{R}(\overline{\omega}), \quad \mathcal{R}_r(\omega) = \mathcal{R}(\omega);$$

for any ξ in $L^2(\mathcal{M})$, we have that

$$\mathcal{R}_l(\xi) = \mathcal{R}(\xi), \quad \mathcal{R}_r(\xi) = \mathcal{R}(J_\varphi \xi).$$

Proof. This can be obtained from Lemma 3.1. □

Proposition 3.3. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Then for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, $y \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M})$, $x \in \mathcal{M}$, we have that*

$$x\Phi_t^{-1}(\xi_t(\omega)) = \Phi_t^{-1}(\xi_t(x\omega)), 1 \leq t \leq 2, \quad x\Phi_s^{-1}(\iota^s(y)) = \Phi_s^{-1}(\iota^s(xy)), 2 \leq s \leq \infty.$$

Moreover, for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, $y \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M})$, we have

$$\mathcal{R}_l(\omega) = \mathcal{R}_l(\xi_t(\omega)), 1 \leq t \leq 2, \quad \mathcal{R}_l(\iota^s(y)) = \mathcal{R}_l(y), 2 \leq s \leq \infty.$$

Proof. By Result 8.6 in [16], we have that $x\xi(\omega) = \xi(x\omega)$ for any $x \in \mathcal{M}$, $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$. That is to say that $x\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$. By Proposition 3.4 in [3], for a given $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, there is a net $\{x_\beta\}_\beta \subset \mathcal{T}_\varphi^2$ such that $\lim_\beta \|\omega - x_\beta\varphi\| = 0$ and $\lim_\beta \|\xi(\omega) - \Lambda_\varphi(x_\beta)\| = 0$. Then for any $1 \leq t \leq 2$, $x \in \mathcal{T}_\varphi^2$, we have that

$$\begin{aligned}
&\|x\Phi_t^{-1}(\xi_t(\omega)) - \Phi_t^{-1}(\xi_t(x\omega))\|_{t,\phi} \\
&= \lim_\beta \|x\Phi_t^{-1}(\xi_t(\omega)) - xx_\beta d^{1/t} + xx_\beta d^{1/t} - \Phi_t^{-1}(\xi_t(x\omega))\|_{t,\phi} \\
&= \lim_\beta \|x\Phi_t^{-1}(\xi_t(\omega)) - xx_\beta d^{1/t}\|_{t,\phi} + \|xx\beta d^{1/t} - \Phi_t^{-1}(\xi_t(x\omega))\|_{t,\phi} \\
&\leq \|x\|_\infty \|\Phi_t^{-1}(\xi_t(\omega)) - x_\beta d^{1/t}\|_{t,\phi} + \lim_\beta \|\xi_t(xx_\beta\varphi) - \xi_t(x\omega)\|_t \\
&\leq \lim_\beta \|xx_\beta\varphi - x\omega\|^{2/t-1} \|\Lambda_\varphi(xx_\beta) - \xi(x\omega)\|^{2-2/t} \\
&= 0,
\end{aligned}$$

i.e. $x\Phi_t^{-1}(\xi_t(\omega)) = \Phi_t^{-1}(\xi_t(x\omega))$ for any x in $\mathcal{T}_\varphi^2$.

For any x in $\mathcal{M}$, there is a bounded net $\{x_\alpha\}_\alpha \subset \mathcal{T}_\varphi^2$ such that x_α converges to x σ-strongly. Then for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, let $\Phi_t^{-1}(\xi_t(\omega)) = w_t|\Phi_t^{-1}(\xi_t(\omega))|$ be the polar decomposition, and we have

$$\begin{aligned}
&\|x\Phi_t^{-1}(\xi_t(\omega)) - \Phi_t^{-1}(\xi_t(x\omega))\|_{t,\phi} \\
&= \lim_\alpha \|x\Phi_t^{-1}(\xi_t(\omega)) - x_\alpha\Phi_t^{-1}(\xi_t(\omega)) + \Phi_t^{-1}(\xi_t(x_\alpha\omega)) - \Phi_t^{-1}(\xi_t(x\omega))\|_{t,\phi} \\
&\leq \lim_\alpha \|(x - x_\alpha)w_t|\Phi_t^{-1}(\xi_t(\omega))|^{t/2}|\Phi_t^{-1}(\xi_t(\omega))|^{1-t/2}\|_{t,\phi}
\end{aligned}$$

$$+\lim_{\alpha} \|x_\alpha\omega - x\omega\|_1^{2/t-1} \|\xi(x_\alpha\omega) - \xi(x\omega)\|_2^{2-2/t}$$
$$\leq \lim_{\alpha} \|(x - x_\alpha)w_t|\Phi_t^{-1}(\xi_t(\omega))|^{t/2}\|_{2,\phi} \||\Phi_t^{-1}(\xi_t(\omega))|^{1-t/2}\|_{(2-t)/2t}$$
$$= 0$$

i.e. $x\Phi_t^{-1}(\xi_t(\omega)) = \Phi_t^{-1}(\xi_t(x\omega))$ for any x in $\mathcal{M}$.

For the second equation, we can use a similar argument as above and Proposition 3.6 in [3] to see it. The remaining two equations are obtained directly from the first two equations. □

Now we can define the left action of $\mathcal{M}$ on $L^t(\mathcal{M})$ for any $1 \leq t \leq \infty$. The left action is given as

$$xy = \Phi_t(x\Phi_t^{-1}(y)), \quad \forall x \in \mathcal{M}, y \in L^t(\mathcal{M}), 1 \leq t \leq \infty.$$

Similarly, we can define the right action on $L^t(\mathcal{M})$ for $1 \leq t \leq \mathcal{M}$ by

$$yx = \Phi_t(\Phi_t^{-1}(y)x), \quad \forall x \in \mathcal{M}, y \in L^t(\mathcal{M}), 1 \leq t \leq \infty.$$

Proposition 3.4. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Then for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, $y \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M})$, $x \in \mathcal{T}_\varphi^2$, we have*

$$\Phi_2^{-1}(\xi(\omega))x = \Phi_2^{-1}(\xi(\omega\sigma_{i/2}^\varphi(x))), \quad \Phi_2^{-1}(\Lambda_\varphi(y))x = \Phi_2^{-1}(\Lambda_\varphi(y\sigma_{-i/2}^\varphi(x))).$$

Proof. For any $y \in \mathfrak{N}_\varphi$, we have that

$$\begin{aligned}
\langle \xi(\omega)x, \Lambda_\varphi(y)\rangle &= \langle J_\varphi x^* J_\varphi \xi(\omega), \Lambda_\varphi(y)\rangle \\
&= \langle \xi(\omega), J_\varphi x J_\varphi \Lambda_\varphi(y)\rangle \\
&= \langle \xi(\omega), \Lambda_\varphi(y\sigma_{-i/2}^\varphi(x^*))\rangle \\
&= \omega(\sigma_{-i/2}^\varphi(x^*)^* y^*) \\
&= \omega(\sigma_{i/2}^\varphi(x))(y^*) \\
&= \langle \xi(\omega(\sigma_{i/2}^\varphi(x)), \Lambda_\varphi(y)\rangle,
\end{aligned}$$

i.e. $\xi(\omega)x = \xi(\omega(\sigma_{i/2}^\varphi(x)))$ for any $x \in \mathcal{T}_\varphi^2$. By Lemma 3.1, we have the first equation is true. Similarly, we can prove the second equation. □

Generalizing the results in the proposition above, we have the following proposition.

Proposition 3.5. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Then for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, $y \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M})$, $1 \leq t \leq 2$, $2 \leq s \leq \infty$, $x \in \mathcal{T}_\varphi^2$, we have*

$$\Phi_t^{-1}(\xi_t(\omega))x = \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x))), \quad \Phi_s^{-1}(\iota^s(y))x = \Phi_s^{-1}(\iota^s(y\sigma^{\varphi}_{-i/s}(x)))$$

Proof. By Proposition 3.4 in [3], there exists a net $\{x_\beta\}_\beta \subset \mathcal{T}_\varphi^2$ such that $\lim_\beta \|x_\beta\varphi - \omega\| = 0$ and $\lim_\beta \|\Lambda_\varphi(x_\beta) - \xi(\omega)\| = 0$. Then we have

$$\begin{aligned}
&\|\Phi_t^{-1}(\xi_t(\omega))x - \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x)))\|_{t,\phi} \\
&= \lim_\beta \|\Phi_t^{-1}(\xi_t(\omega))x - x_\beta d^{1/t}x + x_\beta d^{1/t}x - \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x)))\|_{t,\phi} \\
&\leq \lim_\beta \|\Phi_t^{-1}(\xi_t(\omega))x - x_\beta d^{1/t}x\|_{t,\phi} + \lim_\beta \|x_\beta d^{1/t}x - \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x)))\|_{t,\phi} \\
&\leq \lim_\beta \|\Phi_t^{-1}(\xi_t(\omega)) - x_\beta d^{1/t}\|_{t,\phi}\|x\|_\infty + \lim_\beta \|x_\beta d^{1/t}x - \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x)))\|_{t,\phi} \\
&= \lim_\beta \|x_\beta\sigma^{\varphi}_{-i/t}(x)d^{1/t} - \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x)))\|_{t,\phi} \\
&\leq \lim_\beta \max\{\|x_\beta\sigma^{\varphi}_{-i/t}(x)\varphi - \omega\sigma^{\varphi}_{i-i/t}(x)\|, \|\Lambda_\varphi(x_\beta\sigma^{\varphi}_{-i/t}(x)) - \xi(\omega\sigma^{\varphi}_{i-i/t}(x))\|\} \\
&= \lim_\beta \max\{\|x_\beta\varphi\sigma^{\varphi}_{i-i/t}(x) - \omega\sigma^{\varphi}_{i-i/t}(x)\|, \|\Lambda_\varphi(x_\beta)\sigma^{\varphi}_{i/2-i/t}(x) - \xi(\omega)\sigma^{\varphi}_{i/2-i/t}(x)\|\} \\
&= 0,
\end{aligned}$$

i.e. $\Phi_t^{-1}(\xi_t(\omega))x = \Phi_t^{-1}(\xi_t(\omega\sigma^{\varphi}_{i-i/t}(x)))$. Similarly, one can show that the second equation is true. □

For any $\omega \in L^1(\mathcal{M})$, we let $\omega = v_\omega|\omega|$ be the polar decomposition. By Theorem 4.2 in Chapter 3 of [26], we see that $v_\omega^* v_\omega = \mathcal{R}(|\omega|) = \mathcal{R}(\omega)$. If $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, then $|\omega| \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$.

Proposition 3.6. *Let $\mathbb{G}$ be a locally compact quantum group. For any $\omega \in L^1(\mathbb{G})$, $t \in \mathbb{R}$, we have*

$$\mathcal{R}(\omega \circ \sigma_t^{\varphi}) = \sigma^{\varphi}_{-t}(\mathcal{R}(\omega)), \quad \mathcal{R}(\omega \circ \tau_t) = \tau_{-t}(\mathcal{R}(\omega))$$
$$\mathcal{R}(\delta_t^*(\omega)) = \delta^{-it}\mathcal{R}(\omega)\delta^{it}, \quad \mathcal{R}(\omega \circ R) = R(\mathcal{R}(\overline{\omega}))$$

Proof. We leave the proof to the reader. □

4. Donoho–Stark uncertainty principle

In this section, we prove the Donoho–Stark uncertainty principle for locally compact quantum groups and show a series of equivalent statements for the characterizations of minimizers of the uncertainty principle. Moreover, we obtain biprojections from minimizers of the uncertainty principle.

Proposition 4.1. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Then for any $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, we have that*

$$\mathcal{S}_r(\xi_t(\omega)) \geq \frac{\|\omega\|^2}{\|\xi(\omega)\|^2}, \quad 1 \leq t \leq 2;$$

for any $y \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M})$, we have that

$$\mathcal{S}_r(\iota^s(y)) \geq \frac{\|\Lambda_\varphi(y)\|^2}{\|y\|_\infty^2}, \quad 2 \leq s \leq \infty.$$

Proof. Fix some $1 \leq t \leq 2$. Suppose that $\varphi(\mathcal{R}_r(\xi_t(\omega))) < \infty$. Then when $1 < t \leq 2$, we have

$$\begin{aligned}
\|\omega\| &= \sup_{\|x\|_\infty = 1} |\omega(x)| \\
&= \sup_{\|x\|_\infty = 1, x \in \mathcal{T}_\varphi^2} |\omega(x)| \\
&= \sup_{\|x\|_\infty = 1, x \in \mathcal{T}_\varphi^2} \langle \xi_t(\omega), x \rangle_{\mathcal{R}_\varphi^*, \mathcal{R}_\varphi} \\
&= \sup_{\|x\|_\infty = 1, x \in \mathcal{T}_\varphi^2} \int \Phi_t^{-1}(\xi_t(\omega)) d^{\frac{t-1}{t}} x d\phi \\
&= \sup_{\|x\|_\infty = 1, x \in \mathcal{T}_\varphi^2} \int \Phi_t^{-1}(\xi_t(\omega)) \mathcal{R}_r(\xi_t(\omega)) d^{\frac{t-1}{t}} x d\phi \\
&\leq \sup_{\|x\|_\infty = 1, x \in \mathcal{T}_\varphi^2} \|\xi_t(\omega)\|_t \|\mathcal{R}_r(\xi_t(\omega)) d^{\frac{t-1}{t}}\|_{\frac{t}{t-1}, \phi} \|x\|_\infty \\
&= \|\xi_t(\omega)\|_t \|\iota^{\frac{t}{t-1}}(\mathcal{R}_r(\xi_t(\omega)))\|_{\frac{t}{t-1}} \\
&\leq \|\omega\|^{\frac{2}{t}-1} \|\xi(\omega)\|^{2-\frac{2}{t}} \|\mathcal{R}_r(\xi_t(\omega))\|_2^{2-\frac{2}{t}} \|\mathcal{R}_r(\xi_t(\omega))\|_\infty^{\frac{2}{t}-1} \\
&= \|\omega\|^{\frac{2}{t}-1} \|\xi(\omega)\|^{2-\frac{2}{t}} \mathcal{S}_r(\xi_t(\omega))^{1-\frac{1}{t}}
\end{aligned}$$

i.e.

$$\mathcal{S}_r(\xi_t(\omega)) \geq \frac{\|\omega\|^2}{\|\xi(\omega)\|^2}, \quad 1 < t \leq 2.$$

When $t = 1$, suppose $\varphi(\mathcal{R}(\omega)) < \infty$, we have

$$\begin{aligned}
\|\omega\| &= \sup_{x^* \in \mathfrak{N}_\varphi, \|x\|_\infty = 1} |\omega(x)| \\
&= \sup_{x^* \in \mathfrak{N}_\varphi, \|x\|_\infty = 1} |\omega(\mathcal{R}(\omega) x)| \\
&= \sup_{x^* \in \mathfrak{N}_\varphi, \|x\|_\infty = 1} |\langle \xi(\omega), \Lambda_\varphi(x^* \mathcal{R}(\omega)) \rangle|
\end{aligned}$$

$$\leq \sup_{x^* \in \mathfrak{N}_\varphi, \|x\|_\infty = 1} |\|\xi(\omega)\|\|x^*\|\|\Lambda_\varphi(\mathcal{R}(\omega))\|$$
$$= \|\xi(\omega)\|\varphi(\mathcal{R}(\omega))^{1/2}$$
$$= \|\xi(\omega)\|\mathcal{S}_r(\omega)^{1/2}.$$

Hence

$$\mathcal{S}_r(\xi_t(\omega)) \geq \frac{\|\omega\|^2}{\|\xi(\omega)\|^2}, \quad 1 \leq t \leq 2.$$

Suppose that $\varphi(\mathcal{R}_r(\iota^s(y))) < \infty$ for $2 < s \leq \infty$. Then we have

$$\begin{aligned}
\|\Lambda_\varphi(y)\| &= \sup_{\|\xi\|=1} |\langle \Lambda_\varphi(y), \xi\rangle| \\
&= \sup_{x \in L^2(\phi), \|x\|_{2,\phi}=1} |\int y d^{1/2} x d\phi| \\
&= \sup_{x \in L^2(\phi), \|x\|_{2,\phi}=1} |\int y d^{1/s} \mathcal{R}_r(\iota^s(y)) d^{\frac{1}{2}-\frac{1}{s}} x d\phi| \\
&\leq \|y\|_s \|\iota^{\frac{s-2}{2s}}(\mathcal{R}_r(\iota^s(y)))\|_{\frac{2s}{s-2}} \\
&\leq \|\Lambda_\varphi(y)\|^{\frac{2}{s}} \|y\|_\infty^{1-\frac{2}{s}} \|\Lambda_\varphi(\mathcal{R}_r(\iota^s(y)))\|^{\frac{s-2}{s}} \|\mathcal{R}_r(\iota^s(y))\|_\infty^{\frac{s}{2}} \\
&= \|\Lambda_\varphi(y)\|^{\frac{2}{s}} \|y\|_\infty^{1-\frac{2}{s}} (\mathcal{S}_r(\iota^s(y)))^{\frac{s-2}{2s}},
\end{aligned}$$

i.e.

$$\mathcal{S}_r(\iota^s(y)) \geq \frac{\|\Lambda_\varphi(y)\|^2}{\|y\|_\infty^2}, \quad 2 < s \leq \infty.$$

When $s = 2$, suppose $\varphi(\mathcal{R}_r(\Lambda_\varphi(y))) < \infty$, we have

$$\begin{aligned}
\|\Lambda_\varphi(y)\| &= \|\Lambda_\varphi(y)\mathcal{R}_r(\Lambda_\varphi(y))\| \\
&= \|y J_\varphi \Lambda_\varphi(\mathcal{R}_r(\Lambda_\varphi(y)))\| \\
&\leq \|y\|_\infty \|\Lambda_\varphi(\mathcal{R}_r(\Lambda_\varphi(y)))\| \\
&= \|y\|_\infty \varphi(\mathcal{R}_r(\Lambda_\varphi(y))^{1/2} \\
&= \|y\|_\infty \mathcal{S}_r(\Lambda_\varphi(y))^{1/2}.
\end{aligned}$$

Hence

$$\mathcal{S}_r(\iota^s(y)) \geq \frac{\|\Lambda_\varphi(y)\|^2}{\|y\|_\infty^2}, \quad 2 \leq s \leq \infty. \qquad \square$$

Theorem 4.2 *(Donoho–Stark uncertainty principle). Suppose $\mathbb{G}$ is a locally compact quantum group. Then for any ω in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, $1 \leq t \leq 2$, $2 \leq s \leq \infty$, we have*

$$\mathcal{S}_r(\xi_t(\omega))\mathcal{S}_r(\iota^s(\lambda(\omega))) \geq 1.$$

Proof. Suppose that $\varphi(\mathcal{R}(\omega)) < \infty$ and $\hat{\varphi}(\mathcal{R}(\lambda(\omega)^*)) < \infty$. Then by Proposition 4.1, we have

$$\mathcal{S}_r(\xi_t(\omega))\mathcal{S}_r(\iota^s(\lambda(\omega))) \geq \frac{\|\omega\|^2}{\|\xi(\omega)\|^2}\frac{\|\hat{\Lambda}(\lambda(\omega))\|^2}{\|\lambda(\omega)\|_\infty^2} = \frac{\|\omega\|^2}{\|\lambda(\omega)\|_\infty^2} \geq 1.$$

Hence we have the theorem proved. □

Definition 4.3. Suppose $\mathbb{G}$ is a locally compact quantum group. An element $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is said to be a minimizer of the Donoho–Stark uncertainty principle in Theorem 4.2 if

$$\mathcal{S}_r(\xi_t(\omega))\mathcal{S}_r(\iota^s(\lambda(\omega))) = 1$$

for all $1 \leq t \leq 2$, $2 \leq s \leq \infty$.

Proposition 4.4. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Let $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$ be such that $\varphi(\mathcal{R}_r(\omega)) < \infty$. If*

$$\mathcal{S}_r(\omega) = \frac{\|\omega\|^2}{\|\xi(\omega)\|^2},$$

then there is a partial isometry $v \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$ such that $\omega = \mu_\omega v\varphi$ for some $\mu_\omega > 0$ and $\mathcal{R}_r(\xi_t(\omega)) = \mathcal{R}_r(\omega) = \sigma_s^\varphi(\mathcal{R}_r(\omega))$ for any $1 \leq t \leq 2$, $s \in \mathbb{R}$.

Proof. Let $\omega = v_\omega|\omega|$ be the polar decomposition. We have that

$$\varphi(v_\omega^* v_\omega) = \varphi(\mathcal{R}(\omega)) = \varphi(\mathcal{R}_r(\omega)) < \infty.$$

Then

$$\begin{aligned} \|\omega\| &= \omega(v_\omega^*) \\ &= \langle \xi(\omega), \Lambda_\varphi(v_\omega) \rangle \\ &\leq \|\xi(\omega)\| \varphi(\mathcal{R}(\omega))^{1/2} \\ &= \|\omega\|. \end{aligned}$$

By the Cauchy–Schwarz inequality, we have that

$$\xi(\omega) = \mu_\omega \Lambda_\varphi(v_\omega),$$

for some $\mu_\omega \in \mathbb{C}$. Therefore $v_\omega \in L^1(\mathcal{M})$ and

$$\omega = \mu_\omega v_\omega \varphi.$$

Note that

$$\|\omega\| = \langle \xi(\omega), \Lambda_\varphi(v_\omega)\rangle = \mu_\omega \|\Lambda_\varphi(v_\omega)\|^2,$$

we see that $\mu_\omega > 0$. Since $\|v_\omega\varphi\| = \varphi(v_\omega^* v_\omega) = \varphi(|v_\omega|)$, we obtain that $|v_\omega|\varphi$ is a positive linear functional. Then for any $y > 0$ in $\mathfrak{M}_\varphi$, we have that

$$\varphi(y|v_\omega|) = \overline{\varphi(y|v_\omega|)} = \varphi(|v_\omega|y).$$

Hence for any y in $\mathfrak{M}_\varphi$, we have $\varphi(y|v_\omega|) = \varphi(|v_\omega|y)$. By Theorem 2.6 in [26], we see that $\sigma_s^\varphi(|v_\omega|) = |v_\omega|$ for any $s \in \mathbb{R}$. By Proposition 3.5, we have that for any $1 \leq t \leq 2$,

$$\begin{aligned}\Phi_t^{-1}(\xi_t(\omega))\mathcal{R}_r(\omega) &= \Phi_t^{-1}(\xi_t(\omega\mathcal{R}_r(\omega))) \\ &= \Phi_t^{-1}(\xi_t(\omega)).\end{aligned}$$

Hence $\mathcal{R}_r(\xi_t(\omega)) \leq \mathcal{R}_r(\omega)$. By Proposition 4.1, we have that $\mathcal{R}_r(\xi_t(\omega)) = \mathcal{R}_r(\omega)$ for any $1 \leq t \leq 2$. □

Proposition 4.5. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Let $y \in L^2(\mathcal{M}) \cap L^\infty(\mathcal{M})$. If*

$$\mathcal{S}_r(y) = \frac{\|\Lambda_\varphi(y)\|^2}{\|y\|_\infty^2},$$

then there is a partial isometry v in $\mathcal{M}$ such that $y = \mu_y v$ for some $\mu_y > 0$. If

$$\mathcal{S}_r(y) = \mathcal{S}_r(\Lambda_\varphi(y)) = \frac{\|\Lambda_\varphi(y)\|^2}{\|y\|_\infty^2},$$

then $\sigma_t^\varphi(\mathcal{R}_r(y)) = \mathcal{R}_r(y)$ for $t \in \mathbb{R}$ and

$$\mathcal{R}_r(\iota^s(y)) = \mathcal{R}_r(y), \quad \forall 2 \leq s \leq \infty.$$

Moreover $y \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$.

Proof. Since

$$\begin{aligned}\|\Lambda_\varphi(y)\|^2 &= \|\Lambda_\varphi(y\mathcal{R}_r(y))\|^2 \\ &= \langle y^* y \Lambda_\varphi(\mathcal{R}_r(y)), \Lambda_\varphi(\mathcal{R}_r(y))\rangle\end{aligned}$$

$$\begin{aligned}&\leq \|y^*y\Lambda_\varphi(\mathcal{R}_r(y))\|\|\Lambda_\varphi(\mathcal{R}_r(y))\|\\&\leq \|y\|_\infty^2\varphi(\mathcal{R}_r(y))\\&= \|\Lambda_\varphi(y)\|^2,\end{aligned}$$

we have that

$$y^*y\Lambda_\varphi(\mathcal{R}_r(y)) = \mu\Lambda_\varphi(\mathcal{R}_r(y))$$

for some $\mu > 0$, i.e. $y^*y = \mu\mathcal{R}_r(y)$. Therefore $y = v_y\mu^{1/2}\mathcal{R}_r(y)$ by the polar decomposition. Let $\mu_y = \mu^{1/2}$, $v = v_y$, we have that $y = \mu_y v$.

Now we assume that $\mathcal{S}_r(y) = \mathcal{S}_r(\Lambda_\varphi(y))$. Let $p_2 = \mathcal{R}_r(\Lambda_\varphi(y))$ and $p_\infty = \mathcal{R}_r(y)$. Then we have

$$p_2 = \mathcal{R}(d^{1/2}y^*) = \mathcal{R}(d^{1/2}p_\infty).$$

Furthermore,

$$\begin{aligned}\varphi(p_\infty) = \|p_\infty d^{1/2}p_2\|_{2,\phi} &= \int p_2 d^{1/2}p_\infty d^{1/2}p_2 d\phi\\&= \int d^{1/2}p_2p_2d^{1/2}p_\infty d\phi\\&\leq \|p_\infty\|_\infty\int d^{1/2}p_2p_2d^{1/2}d\phi\\&= \varphi(p_2) = \varphi(p_\infty).\end{aligned}$$

By Hölder's inequality, we have that

$$\mathcal{R}(d^{1/2}p_2p_2d^{1/2}) = \mathcal{R}(d^{1/2}p_2) \leq p_\infty.$$

By Proposition 4.1, we have that

$$\varphi(\mathcal{R}(d^{1/2}p_2)) = \mathcal{S}_r(\Lambda_\varphi(p_2)) \geq \frac{\|\Lambda_\varphi(p_2)\|^2}{\|p_2\|_\infty^2} = \varphi(p_2) = \varphi(p_\infty).$$

Hence $\mathcal{R}(d^{1/2}p_2) = p_\infty$. Note that $p_2 = \mathcal{R}(d^{1/2}p_\infty)$, we then obtain that

$$p_\infty d^{1/2}p_2 = p_\infty d^{1/2} = d^{1/2}p_2.$$

Applying σ_t^φ for any $t \in \mathbb{R}$, we see that

$$\sigma_t^\varphi(p_\infty)d^{1/2} = d^{1/2}\sigma_t^\varphi(p_2).$$

Let $p_2^{(n)} = \frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-n^2t^2}\sigma_t^\varphi(p_2)dt$ and $p_\infty^{(n)} = \frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-n^2t^2}\sigma_t^\varphi(p_\infty)dt$. Then for any $n \in \mathbb{N}$,

$$p_\infty^{(n)} d^{1/2} = d^{1/2} p_2^{(n)}.$$

Now we have that

$$p_\infty^{(n)} = \sigma_{-i/2}^{\varphi}(p_2^{(n)}),$$

and

$$p_\infty^{(n)} = p_\infty^{(n)*} = \sigma_{i/2}^{\varphi}(p_2^{(n)}).$$

Hence $\sigma_i^{\varphi}(p_2^{(n)}) = p_2^{(n)}$ and $\sigma_t^{\varphi}(p_2^{(n)}) = p_2^{(n)}$ for any $t \in \mathbb{R}$. Finally, we obtain that $\sigma_t^{\varphi}(p_2) = p_2 = p_\infty$ for any $t \in \mathbb{R}$ and

$$\mathcal{R}_r(\iota^s(y)) = \mathcal{R}(d^{1/s} y^*) = p_\infty = \mathcal{R}_r(y)$$

for any $2 \leq s \leq \infty$. To see y in $L^1(\mathcal{M})$, we actually have

$$\begin{aligned}\sup_{\|x\|_\infty \leq 1, x \in \mathfrak{N}_\varphi} |\varphi(xy)| &= \sup_{\|x\|_\infty \leq 1, x \in \mathfrak{N}_\varphi} |\varphi(p_\infty xy)| \\ &\leq \sup_{\|x\|_\infty \leq 1, x \in \mathfrak{N}_\varphi} \varphi(p_\infty x x^* p_\infty)^{1/2} \varphi(y^* y)^{1/2} \\ &\leq \varphi(p_\infty)^{1/2} \|\Lambda_\varphi(y)\| < \infty. \qquad \square\end{aligned}$$

Definition 4.6. Suppose that $\mathbb{G}$ is a locally compact quantum group. An element $\omega \in L^1(\mathbb{G})$ is extremal if $\|\lambda(\omega)\|_\infty = \|\omega\|$. An element $\omega \in L^1(\mathbb{G})$ is a bi-partial isometry if $\omega = \mu_\omega v_\omega \varphi$ for some $\mu_\omega > 0$ and some partial isometry $v_\omega \in L^\infty(\mathbb{G})$, and $\lambda(\omega) = \hat{\mu}_{\lambda(\omega)} \hat{v}_{\lambda(\omega)}$ for some $\hat{\mu}_{\lambda(\omega)} > 0$ and some partial isometry $\hat{v}_{\lambda(\omega)} \in L^\infty(\hat{\mathbb{G}})$. An element $\omega \in L^1(\mathbb{G})$ is an extremal bi-partial isometry if ω is a bi-partial isometry, ω is extremal, and $\lambda(\omega) \in L^1(\hat{\mathbb{G}}) \cap L^\infty(\hat{\mathbb{G}})$ is extremal.

Proposition 4.7. *Suppose that $\mathbb{G}$ is a locally compact quantum group. Then the following are equivalent:*

1. $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ *is a minimizer of the Donoho–Stark uncertainty principle.*
2. ω *is an extremal bi-partial isometry such that* $|\omega|\sigma_t^{\varphi} = |\omega|$, $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$, $\forall t \in \mathbb{R}$.
3. ω *is a bi-partial isometry,* $|\omega|\sigma_t^{\varphi} = |\omega|$, $\forall t \in \mathbb{R}$, *and* $\lambda(\omega)$ *is in* $L^1(\hat{\mathbb{G}})$ *such that* $\|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty = \|\lambda(\omega)\hat{\varphi}\|$.
4. $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ *satisfies that* $\mathcal{S}_r(\omega)\mathcal{S}_r(\lambda(\omega)) = 1$ *and* $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$.
5. $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ *satisfies that* $\mathcal{S}_r(\omega)\mathcal{S}_r(\lambda(\omega)) = 1$ *and* $\mathcal{S}_r(\xi(\omega))\mathcal{S}_r(\hat{\Lambda}(\lambda(\omega))) = 1$.

Proof. 1. ⇒ 2.: Suppose that $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a minimizer of the Donoho–Stark uncertainty principle. By Proposition 4.4, Proposition 4.5, we have that ω is a bi-partial isometry, i.e. $\omega = \mu_\omega v_\omega \varphi$ for some $\mu_\omega > 0$, a partial isometry v_ω in $L^\infty(\mathbb{G})$, $\lambda(\omega) = \hat{\mu}_{\lambda(\omega)}\hat{v}_{\lambda(\omega)} \in L^1(\hat{\mathbb{G}}) \cap L^\infty(\hat{\mathbb{G}})$ and

$$\sigma_t^\varphi(|v_\omega|) = |v_\omega|, \quad \hat{\sigma}_t(|\hat{v}_{\lambda(\omega)}|) = |\hat{v}_{\lambda(\omega)}|, \forall t \in \mathbb{R}.$$

By Theorem 4.2, we have that $\|\lambda(\omega)\|_\infty = \|\omega\|$, i.e. ω is extremal. Now we have to show $\lambda(\omega)\hat{\varphi}$ is extremal. Note that

$$\|\omega\| = \mu_\omega \varphi(|v_\omega|), \quad \|\lambda(\omega)\|_\infty = \hat{\mu}_{\lambda(\omega)},$$
$$\|\lambda(\omega)\hat{\varphi}\| = \hat{\mu}_{\lambda(\omega)}\hat{\varphi}(|\hat{v}_{\lambda(\omega)}|), \quad \|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty = \mu_\omega.$$

We have

$$\begin{aligned}
\|\lambda(\omega)\hat{\varphi}\| &= \hat{\mu}_{\lambda(\omega)}\hat{\varphi}(|\hat{v}_{\lambda(\omega)}|) \\
&= \frac{1}{\hat{\mu}_{\lambda(\omega)}}\langle \hat{\Lambda}(\hat{\mu}_{\lambda(\omega)}\hat{v}_{\lambda(\omega)}), \hat{\Lambda}(\hat{\mu}_{\lambda(\omega)}\hat{v}_{\lambda(\omega)})\rangle \\
&= \frac{1}{\hat{\mu}_{\lambda(\omega)}}\langle \Lambda_\varphi(\mu_\omega v_\omega), \Lambda_\varphi(\mu_\omega v_\omega)\rangle \\
&= \frac{\mu_\omega^2 \varphi(|v_\omega|)}{\|\lambda(\omega)\|_\infty} \\
&= \|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty \frac{\|\omega\|}{\|\lambda(\omega)\|_\infty} \\
&= \|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty,
\end{aligned} \tag{1}$$

i.e. $\|\lambda(\omega)\hat{\varphi}\| = \|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty$. Hence $\lambda(\omega)\hat{\varphi}$ is extremal and then ω is an extremal bi-partial isometry such that $|\omega|\sigma_t^\varphi = |\omega|$, $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$.

2. $\Rightarrow$ 1.: Suppose that $\omega = \mu_\omega v_\omega \varphi$ is an extremal bi-partial isometry such that $|\omega|\sigma_t^\varphi = |\omega|$, $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$, $\forall t \in \mathbb{R}$ and $\lambda(\omega) = \hat{\mu}_{\lambda(\omega)}\hat{v}_{\lambda(\omega)}$. Then for any $1 \leq t \leq 2$, $2 \leq s \leq \infty$,

$$\mathcal{S}_r(\xi_t(\omega)) = \mathcal{S}_r(\omega) = \varphi(|v_\omega|) = \frac{\|\omega\|^2}{\|\xi(\omega)\|^2},$$
$$\mathcal{S}_r(\iota^s(\lambda(\omega))) = \mathcal{S}_r(\lambda(\omega)) = \hat{\varphi}(|\hat{v}_{\lambda(\omega)}|) = \frac{\|\hat{\Lambda}(\lambda(\omega))\|^2}{\|\lambda(\omega)\|_\infty^2}.$$

Hence $\mathcal{S}_r(\xi_t(\omega))\mathcal{S}_r(\iota^s(\lambda(\omega))) = 1$.

2. $\Rightarrow$ 3.: By the argument of "1. $\Rightarrow$ 2.", it is obvious.

3. $\Rightarrow$ 2.: Suppose that $\omega = \mu_\omega v_\omega \varphi$ is a bi-partial isometry, $\sigma_t^\varphi(|v_\omega|) = |v_\omega|$ and $\lambda(\omega) = \hat{\mu}_{\lambda(\omega)}\hat{v}_{\lambda(\omega)} \in L^1(\hat{\mathbb{G}})$, $\|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty = \|\lambda(\omega)\hat{\varphi}\|$. By the computation of the Equation (1), we obtain that

$$\begin{aligned}\|\lambda(\omega)\hat{\varphi}\| &\geq \hat{\mu}_{\lambda(\omega)}\hat{\varphi}(|\hat{v}_{\lambda(\omega)}|)\\ &= \frac{\mu_\omega^2\varphi(|v_\omega|)}{\|\lambda(\omega)\|_\infty}\\ &= \|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty\frac{\|\omega\|}{\|\lambda(\omega)\|_\infty}\\ &\geq \|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty.\end{aligned}$$

By the assumption, we have that $\|\lambda(\omega)\hat{\varphi}\| = \hat{\mu}_{\lambda(\omega)}\hat{\varphi}(|\hat{v}_{\lambda(\omega)}|)$ and $\|\omega\| = \|\lambda(\omega)\|_\infty$. By the argument of Proposition 4.4, we have that $\hat{\sigma}_t(|\hat{v}_{\lambda(\omega)}|) = |\hat{v}_{\lambda(\omega)}|$. Hence ω is an extremal bi-partial isometry such that $|\omega|\sigma_t^\varphi = |\omega|$, $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$, $\forall t \in \mathbb{R}$.

1. ⇒ 4.: It is obvious.

4. ⇒ 1.: By Theorem 4.2, we have that

$$\mathcal{S}_r(\omega) = \frac{\|\omega\|^2}{\|\xi(\omega)\|^2}, \quad \mathcal{S}_r(\lambda(\omega)) = \frac{\|\hat{\Lambda}(\lambda(\omega))\|^2}{\|\lambda(\omega)\|_\infty^2}, \quad \|\lambda(\omega)\|_\infty = \|\omega\|.$$

By Proposition 4.4, we have $\mathcal{S}_r(\xi_t(\omega)) = \mathcal{S}_r(\omega)$. By Proposition 4.5, we have that $\mathcal{S}_r(\iota^s(\lambda(\omega))) = \mathcal{S}_r(\lambda(\omega))$. Hence ω is a minimizer of the Donoho–Stark uncertainty principle.

1. ⇒ 5. It is obvious.

5. ⇒ 1.: By Theorem 4.2, we have that

$$\mathcal{S}_r(\omega) = \mathcal{S}_r(\xi(\omega)) = \frac{\|\omega\|^2}{\|\xi(\omega)\|^2}, \quad \mathcal{S}_r(\lambda(\omega)) = \mathcal{S}_r(\hat{\Lambda}(\lambda(\omega))) = \frac{\|\hat{\Lambda}(\lambda(\omega))\|^2}{\|\lambda(\omega)\|_\infty^2},$$
$$\|\lambda(\omega)\|_\infty = \|\omega\|.$$

By Proposition 4.5, we have that $\hat{\sigma}_s(\mathcal{R}_r(\lambda(\omega))) = \mathcal{R}_r(\lambda(\omega))$ for any $s \in \mathbb{R}$ and $\mathcal{S}_r(\iota^s(\lambda(\omega))) = \mathcal{S}_r(\lambda(\omega))$ for any $2 \leq s \leq \infty$. By Proposition 4.4, we have that $\mathcal{S}_r(\xi_t(\omega)) = \mathcal{S}_r(\omega)$ for any $1 \leq t \leq 2$. Hence ω is a minimizer of the Donoho–Stark uncertainty principle. □

Next, we will construct a biprojection by using a minimizer of the Donoho–Stark uncertainty principle. Throughout the paper, we say $\omega = v\varphi$ is a minimizer with assumption that v is a partial isometry. We also say v is a minimizer when $v\varphi$ is a minimizer and v is a partial isometry.

Proposition 4.8. *Suppose $\mathbb{G}$ is a locally compact quantum group and $\omega = v\varphi \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a minimizer of the Donoho–Stark uncertainty principle. Then*

$$|(\varphi v^* \otimes \iota)\Delta(v)|^2 = \varphi(|v|)(\varphi|v| \otimes \iota)\Delta(|v|)$$

and $(\varphi v^* \otimes \iota)\Delta(v)$ is a multiple of a partial isometry, $\frac{1}{\varphi(|v|)}(\varphi|v| \otimes \iota)\Delta(|v|)$ is a projection.

Proof. By Proposition 4.4, we have that $\sigma_t^\varphi(|v|) = |v|$ for any $t \in \mathbb{R}$. Thus $v \otimes 1$ and $\Delta(v)(|v| \otimes 1)$ are in $\mathfrak{N}_{\varphi\otimes\iota}$ and $(\varphi \otimes \iota)((v^* \otimes 1)\Delta(v)) = (\varphi \otimes \iota)((v^* \otimes 1)\Delta(v)(|v| \otimes 1))$. By Proposition 1.24 in [16], we have that

$$\begin{aligned} |(\varphi \otimes \iota)((v^* \otimes 1)\Delta(v)(|v| \otimes 1))|^2 &\leq \|\varphi(|v|)1\|_\infty (\varphi \otimes \iota)((|v| \otimes 1)\Delta(|v|)(|v| \otimes 1)) \\ &= \varphi(|v|)(\varphi \otimes \iota)(\Delta(|v|)(|v| \otimes 1)). \end{aligned} \tag{2}$$

Applying φ to the right hand side of Inequality (2), we obtain

$$\begin{aligned} \varphi(|v|)(\varphi \otimes \varphi)(\Delta(|v|)(|v| \otimes 1)) &= \varphi(|v|)(|v|\varphi \otimes \varphi)\Delta(|v|) \\ &= \varphi(|v|)\varphi(|v|)\varphi(|v|) = \varphi(|v|)^3. \end{aligned}$$

Note that $\lambda(\omega)^*\Lambda_\varphi(v) = (\overline{\omega} \otimes \iota)(W^*)\Lambda_\varphi(v) = \Lambda_\varphi((\varphi v^* \otimes \iota)\Delta(v))$. Applying φ to the left hand side of Inequality (2), we have that

$$\begin{aligned} \varphi(|(\varphi v^* \otimes \iota)\Delta(v)|^2) &= \|\Lambda_\varphi(\varphi v^* \otimes \iota)\Delta(v))\|^2 \\ &= \|\lambda(\omega)^*\Lambda_\varphi(v)\|^2 \\ &= \|\hat{\Lambda}(\lambda(\omega)^*\lambda(\omega))\|^2 \\ &= \|\lambda(\omega)\|^2\hat{\varphi}(\lambda(\omega)^*\lambda(\omega)) \quad \text{Proposition 4.4}, 1. \Rightarrow 2. \\ &= \||v|\|_1^2\varphi(|v|) = \varphi(|v|)^3. \end{aligned}$$

Combining the computation above, we obtain that

$$|(\varphi v^* \otimes \iota)\Delta(v)|^2 = \varphi(|v|)(\varphi|v| \otimes \iota)\Delta(|v|).$$

Repeating the argument above, we have that

$$((\varphi|v| \otimes \iota)\Delta(|v|))^2 = \varphi(|v|)(\varphi|v| \otimes \iota)\Delta(|v|).$$

Hence $(\varphi|v| \otimes \iota)\Delta(|v|)$ is a multiple of a projection and $(\varphi v^* \otimes \iota)\Delta(v)$ is a multiple of a partial isometry. □

Proposition 4.9. *Suppose that $\mathbb{G}$ is a locally compact quantum group and $\omega = v\varphi$ is a minimizer of the Donoho–Stark uncertainty principle. Then we have*

1. $\tau_t((\varphi v^* \otimes \iota)\Delta(v)) = (\varphi v^* \otimes \iota)\Delta(v)\delta^{it}$;
2. $\tau_t((\varphi|v| \otimes \iota)\Delta(|v|)) = (\varphi|v| \otimes \iota)\Delta(|v|)$;
3. $\nu = 1$;
4. $R((\varphi|v| \otimes \iota)\Delta(|v|)) = (\varphi|v| \otimes \iota)\Delta(|v|)$;
5. $\delta(\varphi|v| \otimes \iota)\Delta(|v|) = (\varphi|v| \otimes \iota)\Delta(|v|)$.

Proof. For any x in $L^\infty(\mathbb{G})$, we have that

$$\begin{aligned}\rho_t(\omega)(x) &= \varphi(\delta^{-it}\tau_{-t}(x)v) \\ &= \varphi(\tau_{-t}(x)v\sigma_{-i}(\delta^{-it})) \\ &= \nu^{-t}\varphi(\tau_{-t}(x)v\delta^{-it}) \\ &= \varphi(x\tau_t(v)\delta^{-it}),\end{aligned}$$

i.e. $\rho_t(\omega) = \tau_t(v)\delta^{-it}\varphi$.

Note that $\hat{\sigma}_t(\lambda(\omega)^*\lambda(\omega)) = \lambda(\omega)^*\lambda(\omega)$ for any $t \in \mathbb{R}$. We have

$$\begin{aligned}\Lambda_\varphi((\varphi v^* \otimes \iota)\Delta(v)) &= \hat{\Lambda}(\lambda(\omega)^*\lambda(\omega)) \\ &= \hat{\Lambda}(\hat{\sigma}_t(\lambda(\omega)^*\lambda(\omega))) \\ &= \hat{\Lambda}(\lambda(\rho_t(\omega))^*\lambda(\rho_t(\omega))) \\ &= \Lambda_\varphi((\varphi\delta^{it}\tau_t(v^*) \otimes \iota)\Delta(\tau_t(v)\delta^{-it})).\end{aligned}$$

Hence

$$\begin{aligned}(\varphi v^* \otimes \iota)\Delta(v) &= (\varphi\delta^{it}\tau_t(v^*) \otimes \iota)\Delta(\tau_t(v)\delta^{-it}) \\ &= \nu^{-t}\tau_t(\varphi\delta^{it}v^* \otimes \iota)\Delta(v\delta^{-it}) \\ &= \tau_t((\varphi v^* \otimes \iota)\Delta(v))\delta^{-it},\end{aligned}$$

and $\tau_t((\varphi|v| \otimes \iota)\Delta(|v|)) = \delta^{it}(\varphi|v| \otimes \iota)\Delta(|v|)\delta^{-it}$.

Now we define

$$e_n = \frac{n}{\sqrt{\pi}}\int \exp(-n^2t^2)\delta^{it}dt.$$

Then $v\delta^{-1/2}e_n \in \mathfrak{N}_\psi$. Since

$$(\varphi v^* \otimes \iota)\Delta(v)e_n = \frac{n}{\sqrt{\pi}}\int \exp(-n^2t^2)\tau_t(\varphi v^* \otimes \iota)\Delta(v))dt,$$

we have that $(\varphi v^* \otimes \iota)\Delta(v)e_n \in \mathcal{D}(\tau_{-i/2})$ and

$$\tau_{-i/2}((\varphi v^* \otimes \iota)\Delta(v)e_n) = (\varphi v^* \otimes \iota)\Delta(v)e_n\delta^{1/2}.$$

Then

$$S((\varphi v^* \otimes \iota)\Delta(v)e_n) = R((\varphi v^* \otimes \iota)\Delta(v)e_n\delta^{1/2}).$$

On the other hand, we have that

$$\begin{aligned}
&S(((\varphi\otimes\iota)(e_k v^*\otimes 1)\Delta(ve_m))\delta^{-1/2}e_n)\\
&=e_nS(\psi\otimes\iota)(e_k\delta^{-1/2}v^*\otimes 1)\Delta(v\delta^{-1/2}e_m)\\
&=e_n(\psi\otimes\iota)(\Delta(e_k\delta^{-1/2}v^*)(v\delta^{-1/2}e_m\otimes 1))\\
&=e_n\delta^{-1/2}(\psi\otimes\iota)((\delta^{-1/2}\otimes 1)\Delta(e_kv^*)(ve_m\delta^{-1/2}\otimes 1))\\
&=e_n\delta^{-1/2}(\varphi\otimes\iota)(\Delta(e_kv^*)(ve_m\otimes 1)).
\end{aligned}$$

Since S is σ-strong-*/σ-strong-* closed, by taking the limits as $k\to\infty$ and $m\to\infty$, we obtain that

$$R((\varphi v^*\otimes\iota)\Delta(v)e_n\delta^{1/2})=e_n\delta^{-1/2}((v\varphi\otimes\iota)\Delta(v^*)).$$

Taking the limit as $n\to\infty$, we have that

$$R((\varphi v^*\otimes\iota)\Delta(v))=(v\varphi\otimes\iota)\Delta(v^*),$$

and hence

$$R((\varphi|v|\otimes\iota)\Delta(|v|))=(|v|\varphi\otimes\iota)\Delta(|v|).$$

Let

$$a_n=\frac{n}{\sqrt{\pi}}\int\exp(-n^2t^2)\delta^{it}(\varphi|v|\otimes\iota)\Delta(|v|)\delta^{-it}dt$$

Then $a_n\in\mathcal{D}(\tau_{-i/2})$, and we see that

$$\begin{aligned}
&S(\frac{n}{\sqrt{\pi}}\int\exp(-n^2t^2)\delta^{1/2}\delta^{it}((\varphi\otimes\iota)(e_k|v|\otimes 1)\Delta(|v|e_m))\delta^{-1/2}\delta^{-it})\\
&=\frac{n}{\sqrt{\pi}}\int\exp(-n^2t^2)\delta^{1/2}\delta^{-1/2}(\varphi\otimes\iota)(\Delta(e_k|v|)(|v|e_m\otimes 1)).
\end{aligned}\tag{3}$$

By Equation (3) and taking the limits, we have that

$$(|v|\varphi\otimes\iota)\Delta(|v|)=\delta(|v|\varphi\otimes\iota)\Delta(|v|).$$

Then

$$\tau_t((\varphi|v|\otimes\iota)\Delta(|v|))=(\varphi|v|\otimes\iota)\Delta(|v|),$$

and hence $\nu=1$. $\square$

Proposition 4.10. *Suppose $\mathbb{G}$ is a locally compact quantum group and $\omega = v\varphi \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a minimizer of the Donoho–Stark uncertainty principle. Let $\hat{v} = \lambda(\omega)/\|\lambda(\omega)\|_\infty$. Then*

$$\Lambda_\varphi((\varphi|v| \otimes \iota)\Delta(|v|)) = \varphi(|v|)^3 \hat{\Lambda}((\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|)).$$

Moreover $\sigma_t^\varphi((\varphi|v| \otimes \iota)\Delta(|v|)) = (\varphi|v| \otimes \iota)\Delta(|v|) \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, $\forall t \in \mathbb{R}$.

Proof. Since $\lambda(\omega) \in L^1(\hat{\mathbb{G}}) \cap L^\infty(\hat{\mathbb{G}})$, we have $\lambda(\omega)^*\lambda(\omega) \in L^1(\hat{\mathbb{G}}) \cap L^\infty(\hat{\mathbb{G}})$ and

$$\begin{aligned}\Lambda_\varphi((v\varphi \otimes \iota)\Delta(v^*)(\varphi v^* \otimes \iota)\Delta(v)) &= ((\varphi v^* \otimes \iota)\Delta(v))^* \Lambda_\varphi((\varphi v^* \otimes \iota)\Delta(v)) \\ &= (\hat{\lambda}(\lambda(\omega)^*\lambda(\omega)\hat{\varphi})^* \hat{\Lambda}(\lambda(\omega)^*\lambda(\omega)) \\ &= \|\lambda(\omega)\|^4 \hat{\lambda}(|\hat{v}|\hat{\varphi})^* \hat{\Lambda}(|\hat{v}|) \\ &= \varphi(|v|)^4 \hat{\Lambda}((\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|)).\end{aligned}$$

By Proposition 4.8, we have

$$\Lambda_\varphi((\varphi|v| \otimes \iota)\Delta(|v|)) = \varphi(|v|)^3 \hat{\Lambda}((\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|)).$$

Recall that $\hat{P}^{it}\hat{\Lambda}(x) = \nu^{t/2}\hat{\Lambda}(\hat{\tau}_t(x))$ and $\hat{J}\hat{\delta}^{it}\hat{J}\hat{\Lambda}(x) = \nu^{-t/2}\hat{\Lambda}(x\hat{\delta}^{-it})$ for any $t \in \mathbb{R}$ and $x \in \mathfrak{N}_{\hat{\varphi}}$. By Proposition 8.14 in [16] and Proposition 4.9, we have

$$\begin{aligned}\Lambda_\varphi(\sigma_t^\varphi((\varphi|v| \otimes \iota)\Delta(|v|))) &= \hat{P}^{it}\hat{J}\hat{\delta}^{it}\hat{J}\Lambda_\varphi((\varphi|v| \otimes \iota)\Delta(|v|)) \\ &= \varphi(|v|)^3 \hat{P}^{it}\hat{J}\hat{\delta}^{it}\hat{J}\hat{\Lambda}((\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|)) \\ &= \varphi(|v|)^3 \hat{\Lambda}(\hat{\tau}_t((\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|))\hat{\delta}^{-it}) \\ &= \varphi(|v|)^3 \hat{\Lambda}((\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|)) \\ &= \Lambda_\varphi((\varphi|v| \otimes \iota)\Delta(|v|)),\end{aligned}$$

i.e. $\sigma_t^\varphi((\varphi|v| \otimes \iota)\Delta(|v|)) = (\varphi|v| \otimes \iota)\Delta(|v|)$. Therefore $(\varphi|v| \otimes \iota)\Delta(|v|) \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. $\square$

Proposition 4.11. *Suppose $\mathbb{G}$ is a locally compact quantum group and $\omega = v\varphi$ is a minimizer of the Donoho–Stark uncertainty principle. Then $\tau_t(|v|) = |v|$ for any $t \in \mathbb{R}$.*

Proof. By Proposition 4.10, we have that $\sigma_t^\varphi((\varphi|v| \otimes \iota)\Delta(|v|)) = (\varphi|v| \otimes \iota)\Delta(|v|)$. Applying the commutation relation $\Delta\sigma_t^\varphi = (\tau_t \otimes \sigma_t^\varphi)\Delta$ and Proposition 4.4, we obtain that

$$(\varphi\tau_t(|v|) \otimes \iota)\Delta(|v|) = (\varphi|v| \otimes \iota)\Delta(|v|).$$

Hence $\lambda(\tau_t(|v|)\varphi)^*\lambda(|v|\varphi) = \lambda(|v|\varphi)^*\lambda(|v|\varphi)$. Let $\tilde{v}_t$ be the polar part of the polar decomposition of $\lambda(\tau_t(|v|)\varphi)$. Then

$$|\lambda(\tau_t(|v|)\varphi)|\tilde{v}_t^*\tilde{v}_0|\lambda(|v|\varphi)| = |\lambda(|v|\varphi)|^2.$$

By Proposition 4.9, we have $\tau_t((\varphi|v| \otimes \iota)\Delta(|v|)) = (\varphi|v| \otimes \iota)\Delta(|v|)$, and then

$$|\lambda(\tau_t(|v|)\varphi)| = |\lambda(|v|\varphi)|, \quad |\tilde{v}_t| = |\tilde{v}_0|.$$

Hence

$$\tilde{v}_t^*\tilde{v}_0 = |\tilde{v}_0|.$$

Now taking the adjoint and multiplying it from the right, we obtain

$$\tilde{v}_t^*|\tilde{v}_0^*|\tilde{v}_t = |\tilde{v}_0| = |\tilde{v}_t|,$$

i.e. $\tilde{v}_t^*(1 - |\tilde{v}_0^*|)\tilde{v}_t = 0$ and then

$$\tilde{v}_t^* = \tilde{v}_t^*|\tilde{v}_0^*| = \tilde{v}_t^*\tilde{v}_0\tilde{v}_0^* = |\tilde{v}_0|\tilde{v}_0^* = \tilde{v}_0^*.$$

Finally, we have that $\lambda(\tau_t(|v|)\varphi) = \lambda(|v|\varphi)$ and then $\tau_t(|v|) = |v|$. □

Theorem 4.12. *Suppose $\mathbb{G}$ is a locally compact quantum group and $\omega = v\varphi$ is a minimizer of the Donoho–Stark uncertainty principle. Then*

1. *$(\varphi v^* \otimes \iota)\Delta(v)$ and $|v|$ are minimizers of the Donoho–Stark uncertainty principle.*
2. *$\frac{1}{\varphi(|v|)}(\varphi|v| \otimes \iota)\Delta(|v|)$ is a biprojection and a minimizer of the Donoho–Stark uncertainty principle.*
3. *$\tau_t(v)\varphi$ is a minimizer of the Donoho–Stark uncertainty principle for any t in $\mathbb{R}$.*
4. *$\delta_t^*(v\varphi)$ is a minimizer of the Donoho–Stark uncertainty principle for any t in $\mathbb{R}$.*

Proof. 1. By Proposition 4.7 and Proposition 4.8, we have that $(\varphi v^* \otimes \iota)\Delta(v)$ is a bi-partial isometry. By Proposition 4.10, we have that $(\varphi v^* \otimes \iota)\Delta(v) \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. By the argument in Proposition 4.9, we have

$$\Lambda_\varphi((\varphi v^* \otimes \iota)\Delta(v)) = \hat{\Lambda}(\lambda(\omega)^*\lambda(\omega)).$$

This is to say that $\lambda(((\varphi v^* \otimes \iota)\Delta(v))\varphi) = \lambda(\omega)^*\lambda(\omega)$. Suppose that $\lambda(\omega) = \hat{\mu}\hat{v}$ where $\hat{\mu} > 0$ and $\hat{v}$ is a partial isometry. By Proposition 4.8, we have

$$\mathcal{S}_r((\hat{\varphi}\hat{v}^* \otimes \iota)\hat{\Delta}(\hat{v})) = \hat{\varphi}\left(\frac{1}{\hat{\varphi}(|\hat{v}|)}(\hat{\varphi}|\hat{v}| \otimes \iota)\hat{\Delta}(|\hat{v}|)\right) = \hat{\varphi}(|\hat{v}|).$$

By Proposition 4.8 and Proposition 4.10, we have

$$\mathcal{S}_r((\varphi v^* \otimes \iota)\Delta(v)\varphi) = \varphi\left(\frac{1}{\varphi(|v|)}(\varphi|v| \otimes \iota)\Delta(|v|)\right) = \varphi(|v|).$$

Since $\mathcal{S}_r(v\varphi)\mathcal{S}_r(\hat{v}) = 1$, we obtain that

$$\mathcal{S}_r((\varphi v^* \otimes \iota)\Delta(v)\varphi)\mathcal{S}_r((\hat{\varphi}\hat{v}^* \otimes \iota)\hat{\Delta}(\hat{v})) = 1.$$

By Proposition 4.4 and Proposition 4.7, we see that $(\varphi v^* \otimes \iota)\Delta(v)\varphi$ is a minimizer of the Donoho–Stark uncertainty principle. Then $|\lambda(\omega)|$ is a minimizer of the uncertainty principle and so is $|v|$.

2. By Proposition 4.8 and Proposition 4.10, we have that $\frac{1}{\varphi(|v|)}(\varphi|v| \otimes \iota)\Delta(|v|)$ is a biprojection. By the argument above, we see that it is a minimizer of the Donoho–Stark uncertainty principle.

3. Since σ_t^{φ}, τ_t commute, we have that

$$\mathcal{S}_r(\tau_t(v)\varphi) = \varphi(\tau_t(|v|)) = \varphi(|v|) = \mathcal{S}_r(v\varphi).$$

By Proposition 4.9 and Proposition 4.11, we have

$$\begin{aligned}\mathcal{S}_r(\lambda(\tau_t(v)\varphi)) &= \mathcal{S}_r(\hat{\tau}_{-t}(\lambda(v\varphi)))\\ &= \mathcal{S}_r(\hat{\tau}_{-t}(\lambda(v\varphi)^*\lambda(v\varphi)))\\ &= \mathcal{S}_r(\lambda(v\varphi)^*\lambda(v\varphi))\\ &= \mathcal{S}_r(\lambda(v\varphi)).\end{aligned}$$

Hence $\tau_t(v)\varphi$ is a minimizer of the Donoho–Stark uncertainty principle by Proposition 4.7.

4. By Proposition 3.6 and the argument above, we see that $\delta_t^*(v\varphi)$ is a minimizer of the Donoho–Stark uncertainty principle. □

Remark 4.13. In general, we don' t know if v^*, $\lambda(v\varphi)^*$, $\sigma_t^{\varphi}(v)$ are minimizers of the Donoho–Stark uncertainty principle when v is a minimizer.

5. Hirschman–Beckner uncertainty principle

In this section, we will prove the Hirschman–Beckner uncertainty principle for certain locally compact quantum groups.

Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite weight φ acting on the Hilbert space $\mathcal{H}_\varphi = L^2(\mathcal{M}, \varphi)$. Let ϕ be a normal semifinite weight on the commutant $\mathcal{M}'$ of $\mathcal{M}$ on $\mathcal{H}_\varphi$. A vector ξ in $\mathcal{H}_\varphi$ is ϕ-bounded if the map $y \mapsto y\xi$ for any $y \in \mathfrak{N}_\phi \subset \mathcal{M}'$ is bounded. Denote by $R^\phi(\xi) : L^2(\mathcal{M}', \phi) \to \mathcal{H}_\varphi$ the operator given by $R^\phi(\xi)y = y\xi$ for any $y \in \mathfrak{N}_\phi$. Moreover, $\theta^\phi(\xi,\xi) = |R^\phi(\xi)^*|^2$. We denote by $\mathcal{D}(\mathcal{H}_\varphi, \phi)$ the set of ϕ-bounded vector in $\mathcal{H}_\varphi$. We have that (see [12] for more details)

$$\mathcal{D}(\mathcal{H}_\varphi, \phi) = \bigcap\{\mathcal{D}(T), T \in L^2(\phi)\}.$$

Proposition 5.1. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ and $x \in \mathcal{T}_\varphi^2$. Then $f_x(t) = \|xd^{1/t}\|_{t,\phi}$ is continuous with respect to t, $t \in [1, \infty)$.*

Proof. Following the proof of Proposition 3.4 in [8], we see that the proposition holds. □

Proposition 5.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ and $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$, $x \in \mathfrak{N}_\varphi$. Then we have $f_\omega(t) = \|\xi_t(\omega)\|_t$ is continuous on $[1, 2]$ and $f_x(t) = \|xd^{1/t}\|_{t,\phi}$ is continuous on $[2, \infty)$.*

Proof. By Proposition 3.4 in [3], for any $\epsilon > 0$, there exists $y \in \mathcal{T}_\varphi^2$ such that $\|\omega - y\varphi\| < \epsilon$ and $\|\xi(\omega) - \Lambda_\varphi(y)\| < \epsilon$. Hence $\|\xi_t(\omega) - \iota^t(y)\|_t \leq \epsilon$ for any $1 \leq t \leq 2$. Since

$$\begin{aligned}&|\|\xi_{t+\varepsilon}(\omega)\|_{t+\varepsilon} - \|\xi_t(\omega)\|_t| \\ &\leq \|\xi_t(\omega) - \iota^t(y)\|_t + \|\xi_{t+\varepsilon}(\omega) - \iota^{t+\varepsilon}(y)\|_{t+\varepsilon} + |\|\iota^t(y)\|_t - \|\iota^{t+\varepsilon}(y)\|_{t+\varepsilon}|\end{aligned}$$

and $\|\iota^t(y)\|_t$ is continuous, we obtain that $f_\omega(t)$ is continuous. By Proposition 3.6 in [3], we see the rest of the proposition is true. □

Proposition 5.3. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ and $x \in \mathcal{T}_\varphi^2$. Then*

$$\frac{d}{dt}\langle |d^{1/t}x^*|^t\xi_n, \xi_n\rangle\Big|_{t=2} = \langle |d^{1/2}x^*|^2 \log |d^{1/2}x^*|\xi_n, \xi_n\rangle - \frac{1}{2}\langle x(d\log d)x^*\xi_n, \xi_n\rangle,$$

where $\xi_n = \frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-n^2t^2}d^{it}\xi dt$ for any $n \in \mathbb{N}$ and $\xi \in \mathcal{D}(\mathcal{H}_\varphi, \phi)$.

Proof. Note that

$$\langle |d^{1/2}x^*|^2 |\log |d^{1/2}x^*||\xi_n, \xi_n\rangle < \|\xi_n\|^2 + \frac{1}{2}\||d^{1/2}x^*|^2\xi_n\|^2 < \infty,$$

we have that $\xi_n \in \mathcal{D}(|d^{1/2}x^*||\log |d^{1/2}x^*||^{1/2})$. To show that

$$\begin{aligned}&\frac{\langle |d^{\frac{1}{2+\epsilon}}x^*|^{2+\epsilon}\xi_n, \xi_n\rangle - \langle |d^{1/2}x^*|^2\xi_n, \xi_n\rangle}{\epsilon} \\ &\to \frac{1}{2}\langle (xdx^*)\log(xdx^*)\xi_n, \xi_n\rangle - \frac{1}{2}\langle x(d\log d)x^*\xi_n, \xi_n\rangle\end{aligned}$$

as $\epsilon \to 0$, we have to estimate the following three terms

$$\mathrm{I} = \frac{\langle |d^{\frac{1}{2+\epsilon}}x^*|^{2+\epsilon}\xi_n, \xi_n\rangle - \langle |d^{\frac{1}{2+\epsilon}}x^*|^2\xi_n, \xi_n\rangle}{\epsilon} - \frac{1}{2}\langle (xd^{\frac{2}{2+\epsilon}}x^*)\log(xd^{\frac{2}{2+\epsilon}}x^*)\xi_n, \xi_n\rangle,$$

$$\mathrm{II} = \frac{1}{\epsilon}(\langle xd^{\frac{2}{2+\epsilon}}x^*\xi_n, \xi_n\rangle - \langle xdx^*\xi_n, \xi_n\rangle) + \frac{1}{2}\langle x(d\log d)x^*\xi_n, \xi_n\rangle,$$

$$\mathrm{III} = \langle (xd^{\frac{2}{2+\epsilon}}x^*)\log(xd^{\frac{2}{2+\epsilon}}x^*)\xi_n, \xi_n\rangle - \langle (xdx^*)\log(xdx^*)\xi_n, \xi_n\rangle.$$

Note that for any $t > 0$, we have that

$$\frac{t^{2+\epsilon} - t^2}{\epsilon} - t^2\log t = \epsilon t^2(\log t)^2\sum_{k=0}^{\infty}\frac{(\epsilon\log t)^k}{(k+2)!}.$$

When $e^{-1/|\epsilon|^{1/4}} < t < e^{1/|\epsilon|^{1/4}}$, we see that

$$|\frac{t^{2+\epsilon} - t^2}{\epsilon} - t^2\log t| < |\epsilon|^{1/2}t^2.$$

When $t \geq e^{1/|\epsilon|^{1/4}}$, we have

$$|\frac{t^{2+\epsilon} - t^2}{\epsilon} - t^2\log t| < |\epsilon|(t^3 - t^2 - t^2\log t) < |\epsilon|t^4.$$

For the first term, we have

$$\left|\frac{\langle |d^{\frac{1}{2+\epsilon}}x^*|^{2+\epsilon}\xi_n, \xi_n\rangle - \langle |d^{\frac{1}{2+\epsilon}}x^*|^2\xi_n, \xi_n\rangle}{\epsilon} - \frac{1}{2}\langle (xd^{\frac{2}{2+\epsilon}}x^*)\log(xd^{\frac{2}{2+\epsilon}}x^*)\xi_n, \xi_n\rangle\right|$$

$$= \left|\langle \chi_{[0,1/m]}(|d^{\frac{1}{2+\epsilon}}x^*|)(\frac{|d^{\frac{1}{2+\epsilon}}x^*|^{2+\epsilon} - |d^{\frac{1}{2+\epsilon}}x^*|^2}{\epsilon} - |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|)\xi_n, \xi_n\rangle\right|$$

$$+ \left|\langle \chi_{[1/m,m]}(|d^{\frac{1}{2+\epsilon}}x^*|)(\frac{|d^{\frac{1}{2+\epsilon}}x^*|^{2+\epsilon} - |d^{\frac{1}{2+\epsilon}}x^*|^2}{\epsilon} - |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|)\xi_n, \xi_n\rangle\right|$$

$$+ \left|\langle \chi_{[m,\infty)}(|d^{\frac{1}{2+\epsilon}}x^*|)(\frac{|d^{\frac{1}{2+\epsilon}}x^*|^{2+\epsilon} - |d^{\frac{1}{2+\epsilon}}x^*|^2}{\epsilon} - |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|)\xi_n, \xi_n\rangle\right|$$

$$\leq \frac{1}{|\epsilon|}(\frac{1}{m^{2+\epsilon}} + \frac{1}{m^2} + \frac{|\epsilon|}{m^2}\log m) + |\epsilon|^{1/2}\langle |d^{\frac{1}{2+\epsilon}}x^*|^2\xi_n, \xi_n\rangle$$

$$+ |\epsilon|\langle \chi_{[m,\infty)}(|d^{\frac{1}{2+\epsilon}}x^*|)|d^{\frac{1}{2+\epsilon}}x^*|^4\xi_n, \xi_n\rangle$$

Note that

$$\langle |d^{\frac{1}{2+\epsilon}}x^*|^2\xi_n, \xi_n\rangle = \langle xd^{\frac{2}{2+\epsilon}}x^*\xi_n, \xi_n\rangle \leq \|x^*\xi_n\|^2 + \|dx^*\xi_n\|^2 < \infty,$$

and

$$\begin{aligned}\langle \chi_{[m,\infty)}(|d^{\frac{1}{2+\epsilon}}x^*|)|d^{\frac{1}{2+\epsilon}}x^*|^4\xi_n,\xi_n\rangle &\leq \langle |d^{\frac{1}{2+\epsilon}}x^*|^4\xi_n,\xi_n\rangle\\ &= \langle xd^{\frac{2}{2+\epsilon}}x^*xd^{\frac{2}{2+\epsilon}}x^*\xi_n,\xi_n\rangle\\ &\leq \|x\|_\infty^2\langle xd^{\frac{4}{2+\epsilon}}x^*\xi_n,\xi_n\rangle\\ &\leq \|x\|_\infty^2(\|x^*\xi_n\|^2+\|d^2x^*\xi_n\|^2)<\infty\end{aligned}$$

Take $m=e^{1/|\epsilon|^{1/4}}$, we see the first term can be estimated.

For the second term, we apply a similar argument to d, we see the second term can be estimated.

Now we will estimate the third term. For any $\varepsilon>0$, there exist $m\in\mathbb{N}$ such that

$$\|\exp(-|d^{1/2}x^*|^2/m)\xi_n-\xi_n\|<\varepsilon.$$

Note that $xd^{\frac{2}{2+\epsilon}}x^*\to xdx^*$ as $\epsilon\to 0$ in strong resolvent sense. Then for ϵ small enough, we have

$$\||d^{\frac{1}{2+\epsilon}}x^*|\xi_n-|d^{1/2}x^*|\xi_n\|<\varepsilon,$$
$$\|\exp(-|d^{\frac{1}{2+\epsilon}}x^*|^2/m)\xi_n-\exp(-|d^{1/2}x^*|^2/m)\xi_n\|<\varepsilon,$$

and

$$\begin{aligned}&|\langle |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|\exp(-|d^{\frac{1}{2+\epsilon}}x^*|^2/m)\xi_n,\xi_n\rangle\\ &-\langle |d^{1/2}x^*|^2\log|d^{1/2}x^*|\exp(-|d^{1/2}x^*|^2/m)\xi_n,\xi_n\rangle|<\varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}&|\langle |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|\xi_n,\xi_n\rangle-\langle |d^{1/2}x^*|^2\log|d^{1/2}x^*|\xi_n,\xi_n\rangle|\\ &\leq |\langle |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|(1-\exp(-|d^{\frac{1}{2+\epsilon}}x^*|^2/m))\xi_n,\xi_n\rangle|\\ &\quad+|\langle |d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|\exp(-|d^{\frac{1}{2+\epsilon}}x^*|^2/m)\xi_n,\xi_n\rangle\\ &\quad-\langle |d^{1/2}x^*|^2\log|d^{1/2}x^*|\exp(-|d^{1/2}x^*|^2/m)\xi_n,\xi_n\rangle|\\ &\quad+|\langle |d^{1/2}x^*|^2\log|d^{1/2}x^*|(1-\exp(-|d^{1/2}x^*|^2/m))\xi_n,\xi_n\rangle|\\ &\leq \|\xi_n-\exp(-|d^{\frac{1}{2+\epsilon}}x^*|^2/m)\xi_n\|\||d^{\frac{1}{2+\epsilon}}x^*|^2\log|d^{\frac{1}{2+\epsilon}}x^*|\xi_n\|\\ &\quad+\varepsilon+\varepsilon\||d^{1/2}x^*|^2\log|d^{1/2}x^*|\xi_n\|\\ &\leq 2\varepsilon(\|\xi_n\|^2+\||d^{\frac{1}{2+\epsilon}}x^*|^3\xi_n\|^2)^{1/2}+\varepsilon+\varepsilon\||d^{1/2}x^*|^2\log|d^{1/2}x^*|\xi_n\|\\ &\leq 2\varepsilon(\|\xi_n\|^2+(\||d^{1/2}x^*|^3\xi_n\|+\varepsilon)^2)^{1/2}+\varepsilon+\varepsilon\||d^{1/2}x^*|^2\log|d^{1/2}x^*|\xi_n\|.\end{aligned}$$

Finally, we see that $\langle |d^{1/t}x^*|^t\xi_n,\xi_n\rangle$ is differentiable at $t=2$. □

Lemma 5.4. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Let $\{x_\beta\}_\beta \subset L^2(\phi)$ be a net with the limit x in $L^2(\phi)$. Then $\lim_\beta |x_\beta| = |x|$ in strong resolvent sense.

Proof. Suppose that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$. For any $\xi \in \mathcal{D}(\mathcal{H}, \phi)$, we have that

$$\begin{aligned}\|(|x_\beta| - |x|)\xi\|^2 &\leq \||| x_\beta| - |x||^2\|_1 \|\theta^\phi(\xi,\xi)\| = \||x_\beta| - |x|\|_2^2 \|\theta^\phi(\xi,\xi)\| \\ &\leq \|x_\beta - x\|_2^2 \|\theta^\phi(\xi,\xi)\|.\end{aligned}$$

Since $\mathcal{D}(\mathcal{H}, \phi)$ is dense in $\mathcal{H}$, we see that $\lim_\beta |x_\beta| = |x|$ in strong resolvent sense. □

Proposition 5.5. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. For any $\xi \in \mathcal{D}(\mathcal{H}_\varphi, \phi)$, we denote ξ_n by $\frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2t^2} d^{it}\xi dt$. Let $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$ be such that $\xi_n \in \mathcal{D}(\Phi_2^{-1}(\xi(\omega))^*)$ and*

$$\begin{aligned}&\langle |\Phi_2^{-1}(\xi(\omega))^*|^2 |\log|\Phi_2^{-1}(\xi(\omega))^*||\xi_n, \xi_n\rangle < \infty, \\ &\langle |\log d| \Phi_2^{-1}(\xi(\omega))^*\xi_n, \Phi_2^{-1}(\xi(\omega))^*\xi_n\rangle < \infty.\end{aligned}$$

Then

$$\begin{aligned}&\frac{d}{dt}\langle |\Phi_t^{-1}(\xi_t(\omega))^*|^t \xi_n, \xi_n\rangle\Big|_{t=2^-} \\ &= \langle |\Phi_2^{-1}(\xi(\omega))^*|^2 \log|\Phi_2^{-1}(\xi(\omega))^*|\xi_n, \xi_n\rangle - \frac{1}{2}\langle \Phi_2^{-1}(\xi(\omega))(\log d)\Phi_2^{-1}(\xi(\omega))^*\xi_n, \xi_n\rangle.\end{aligned}$$

Proof. There exists a net $\{x_\beta\}_\beta \subset \mathcal{T}_\varphi^2$ such that $\lim_\beta \|\xi_t(\omega) - \xi_t(x_\beta\varphi)\|_t = 0$ for $1 \leq t \leq 2$ uniformly. Then

$$\|\Phi_t^{-1}(\xi_t(\omega))^* - d^{1/t}x_\beta^*\|_{t,\phi} < \epsilon$$

for all $1 \leq t \leq 2$. Note that $\langle |d^{1/t}x_\beta^*|\xi_n, \xi_n\rangle$ is differentiable. We have that $d^{1/2}x_\beta^* \to \Phi_2^{-1}(\xi(\omega))^*$ in strong resolvent sense. By using a similar argument in Proposition 5.3, applying $\exp(-|d^{1/2}x_\beta^*|^2/m)$ we can see that $\langle |d^{1/2}x_\beta^*|^2 \log|d^{1/2}x_\beta^*|\xi_n, \xi_n\rangle \to \langle |\Phi_2^{-1}(\xi(\omega))^*|^2 \log|\Phi_2^{-1}(\xi(\omega))^*|\xi_n, \xi_n\rangle$. Note that $d^{1/2}x_\beta^*\xi_n \in \mathcal{D}(|\log d|^{1/2})$ and $d^{1/2}x_\beta^*\xi_n \to \Phi_2^{-1}(\xi(\omega))^*\xi_n$, we have that

$$\langle x_\beta d^{1/2}(\log d)d^{1/2}x_\beta^*\xi_n, \xi_n\rangle \to \langle \Phi_2^{-1}(\xi(\omega))(\log d)\Phi_2^{-1}(\xi(\omega))^*\xi_n, \xi_n\rangle.$$

Hence $\langle |\Phi_t^{-1}(\xi_t(\omega))^*|^t\xi_n, \xi_n\rangle$ is differentiable at 2^-. □

Proposition 5.6. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. For any $\xi \in \mathcal{D}(\mathcal{H}_\varphi, \phi)$, we denote ξ_n by $\frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2t^2} d^{it}\xi dt$. Let $x \in \mathfrak{N}_\varphi$ be such that*

$$\langle |d^{1/2}x^*|^2 | \log |d^{1/2}x^*| |\xi_n, \xi_n\rangle < \infty, \quad \langle |\log d| d^{1/2}x^*\xi_n, d^{1/2}x^*\xi_n\rangle < \infty$$

Then

$$\begin{aligned}
&\frac{d}{dt}\langle |d^{1/t}x^*|^t\xi_n, \xi_n\rangle\Big|_{t=2^+} \\
&= \langle |d^{1/2}x^*|^2 \log |d^{1/2}x^*|\xi_n, \xi_n\rangle - \frac{1}{2}\langle x d^{1/2}(\log d) d^{1/2}x^*\xi_n, \xi_n\rangle.
\end{aligned}$$

Proof. Note that $\xi_n \in \mathcal{D}(d^{1/2}x^*)$ and the proof is similar to the one of Proposition 5.5 □

Remark 5.7. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful tracial weight φ and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Then $d = 1$ and for any $x \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$, $\|x\|_t$ is differentiable for any $1 \le t < \infty$.

Lemma 5.8. Suppose that $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Then $\Phi_2^{-1}(\xi)^*\Lambda_\varphi(e) = J_\varphi e^*\xi$ for any $\xi \in L^2(\mathcal{M})$ and $e \in \mathcal{T}_\varphi$.

Proof. Suppose $x \in \mathcal{T}_\varphi^2$. We will show that $d^{1/2}x^*\Lambda_\varphi(e) = J_\varphi e^*\Lambda_\varphi(x)$ for any $e \in \mathcal{T}_\varphi$. Since $\phi = \varphi(J_\varphi \cdot J_\varphi)$, we have that $d = \nabla_\varphi$ and $\theta^\phi(\Lambda_\varphi(e_1), \Lambda_\varphi(e_2)) = e_1 e_2^*$ for any $e_1, e_2 \in \mathcal{T}_\varphi$.

$$d^{1/2}x^*\Lambda_\varphi(e) = \Lambda_\varphi(\sigma^\varphi_{-i/2}(x^*e)) = \Lambda_\varphi(\sigma^\varphi_{i/2}(e^*x)^*) = J_\varphi\Lambda_\varphi(e^*x).$$

For any $\xi \in L^2(\mathcal{M})$, there exists a net $\{x_\beta\}_\beta \subset \mathcal{T}_\varphi^2$ such that $\lim_\beta \|\xi - \Lambda_\varphi(x_\beta)\| = 0$. Since

$$\begin{aligned}
\|\Phi_2^{-1}(\xi)^*\Lambda_\varphi(e) - J_\varphi e^*\xi\| &= \|(\Phi_2^{-1}(\xi)^*\Lambda_\varphi(e) - d^{1/2}x_\beta^*\Lambda_\varphi(e)) + (J_\varphi e^*\Lambda_\varphi(x_\beta) - J_\varphi e^*\xi)\| \\
&\le \|\Phi_2^{-1}(\xi)^* - d^{1/2}x_\beta^*\|_{2,\phi}\|e\| + \|e\|\|\Lambda_\varphi(x_\beta) - \xi\| \\
&= 2\|e\|\|\Lambda_\varphi(x_\beta) - \xi\|.
\end{aligned}$$

we have that $\Phi_2^{-1}(\xi)^*\Lambda_\varphi(e) = J_\varphi e^*\xi$. □

Proposition 5.9. *Suppose $\mathcal{M}$ is a von Neumann algebra with a normal faithful state φ and $x \in \mathcal{T}_\varphi^2$. If $\phi = \varphi(J_\varphi \cdot J_\varphi)$, then $\|xd^{1/t}\|_{t,\phi}^t$ is differentiable at $t = 2$ and*

$$\frac{d}{dt}\|xd^{1/t}\|_{t,\phi}^t\Big|_{t=2} = \langle \log |xd^{1/2}| J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle - \frac{1}{2}\langle (\log d) J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle.$$

Moreover,

$$\frac{d}{dt}\|xd^{1/t}\|_{t,\phi}\bigg|_{t=2} = -\frac{1}{2}\|\Lambda_\varphi(x)\|\log\|\Lambda_\varphi(x)\| + \frac{1}{2\|\Lambda_\varphi(x)\|}\langle \log|xd^{1/2}|J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle$$
$$-\frac{1}{4\|\Lambda_\varphi(x)\|}\langle(\log d)J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle.$$

Proof. Let $d^{1/2}x^* = v_x|d^{1/2}x^*|$ be the polar decomposition. Then by Lemma 5.8 we have

$$\frac{d}{dt}\|xd^{1/t}\|_{t,\phi}^t\bigg|_{t=2} = \frac{d}{dt}\langle|d^{1/t}x^*|^t\Lambda_\varphi(1), \Lambda_\varphi(1)\rangle\bigg|_{t=2}$$
$$= \frac{1}{2}\langle(xdx^*)\log(xdx^*)\Lambda_\varphi(1), \Lambda_\varphi(1)\rangle - \frac{1}{2}\langle x(d\log d)x^*\Lambda_\varphi(1), \Lambda_\varphi(1)\rangle$$
$$= \langle v_x(\log|d^{1/2}x^*|)v_x^*d^{1/2}x^*\Lambda_\varphi(1), d^{1/2}x^*\Lambda_\varphi(1)\rangle - \frac{1}{2}\langle(\log d)d^{1/2}x^*\Lambda_\varphi(1), d^{1/2}x^*\Lambda_\varphi(1)\rangle$$
$$= \langle\log|xd^{1/2}|J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle - \frac{1}{2}\langle(\log d)J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle.$$

Differentiating $\|xd^{1/t}\|_{t,\phi}$ with respect to t, we obtain that

$$\frac{d}{dt}\|xd^{1/t}\|_{t,\phi}\bigg|_{t=2} = -\frac{1}{t^2}\|xd^{1/t}\|_t\log\|xd^{1/t}\|_t^t\bigg|_{t=2} + \frac{1}{t\|xd^{1/t}\|_t^{t-1}}\frac{d}{dt}\|xd^{1/t}\|_t^t\bigg|_{t=2}$$
$$= -\frac{1}{2}\|\Lambda_\varphi(x)\|\log\|\Lambda_\varphi(x)\| + \frac{1}{2\|\Lambda_\varphi(x)\|}\langle\log|xd^{1/2}|J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle$$
$$-\frac{1}{4\|\Lambda_\varphi(x)\|}\langle(\log d)J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle. \qquad \square$$

Remark 5.10. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal faithful state φ and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Let $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$. Then $\|\xi_t(\omega)\|_t$ is differentiable at $t = 2^-$ when

$$\langle|\log d|J_\varphi\xi(\omega), J_\varphi\xi(\omega)\rangle < \infty, \quad \langle|\log|\Phi_2^{-1}(\xi(\omega))||J_\varphi\xi(\omega), J_\varphi\xi(\omega)\rangle < \infty.$$

Let $x \in \mathfrak{N}_\varphi$. Then $\|xd^{1/t}\|_{t,\phi}$ is differentiable at $t = 2^+$ when

$$\langle|\log d|J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle < \infty, \quad \langle|\log|xd^{1/2}||J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle < \infty.$$

Corollary 5.11. *Suppose that $\mathcal{M} = \oplus_{j\in J}\mathcal{M}_j$ be a von Neumann algebra with a normal semifinite faithful weight φ such that $\varphi|_{\mathcal{M}_j}$ is bounded for any $j \in J$ and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Then for any $x \in \mathcal{T}_\varphi^2$, $\|xd^{1/t}\|_{t,\phi}$ is differentiable at $t = 2$.*

Proof. Let p_j be the central projection in $\mathcal{M}$ corresponding to $\mathcal{M}_j$. Then $p_jd = dp_j$ for any $j \in J$. $\|xd^{1/t}\|_{t,\phi}^t = \sum_j\|p_jxd^{1/t}\|_{t,\phi}^t$. Note that $\sum_{j \text{ finite}}\|p_jxd^{1/t}\|_t^t$ is differentiable at $t = 2$ and $\sum_{j \text{ finite}}\|p_jxd^{1/t}\|_t^t$ converges to $\|xd^{1/t}\|_t^t$ uniformly on $1/4 \le t \le 4$. Then we see that $\|xd^{1/t}\|_t^t$ is differentiable at $t = 2$. $\square$

Definition 5.12. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Let $\xi \in L^2(\mathcal{M})$. Then the entropy $H(\xi)$ of ξ is defined to be

$$H(\xi) = -\langle \log |\Phi_2^{-1}(\xi)|^2 J_\varphi \xi, J_\varphi \xi \rangle.$$

We say the entropy $H(\xi)$ of $\xi \in L^2(\mathcal{M})$ is finite if $\langle | \log |\Phi_2^{-1}(\xi)|| J_\varphi \xi, J_\varphi \xi \rangle < \infty$.

For more entropy of operator, we refer to an interesting book [22] by M. Ohya and D. Petz.

Proposition 5.13. *Suppose that $\mathcal{M} = \oplus_{j \in J} \mathcal{M}_j$ is a von Neumann algebra with a normal semifinite faithful weight φ such that $\varphi|_{\mathcal{M}_j}$ is bounded and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Let $\xi \in L^2(\mathcal{M})$ and $x_\beta \in \mathcal{T}_\varphi^2$ such that $\lim_\beta \|\xi - \Lambda_\varphi(x_\beta)\| = 0$ and $H(\xi)$ is finite. Then*

$$H(\xi) = \lim_\beta H(\Lambda_\varphi(x_\beta)).$$

Proof. For any $x \in \mathcal{T}_\varphi^2$, we have that $J_\varphi \Lambda_\varphi(x) \in \mathcal{D}(|\log |xd^{1/2}||^{1/2})$ (as follows by the properties of the function $\log t$ and Lemma 5.8). Let $h_m(t) = \exp(-t^2/m)t^2 \log t$, $m \in \mathbb{N}$. Then $h_m(t)$ is a bounded continuous function on $\mathbb{R}_{>0}$. Since $\lim_\beta \|\xi - \Lambda_\varphi(x_\beta)\| = 0$, we have that $x_\beta d^{1/2} \to \Phi_2^{-1}(\xi)$ in the strongly resolvent sense. Then for any $m \in \mathbb{N}$, by Lemma 5.8,

$$\begin{aligned}
&\lim_\beta \langle \exp(-|x_\beta d^{1/2}|^2/m) \log |x_\beta d^{1/2}| J_\varphi \Lambda_\varphi(x_\beta), J_\varphi \Lambda_\varphi(x_\beta) \rangle \\
&= \langle \exp(-|\Phi_2^{-1}(\xi)|^2/m) \log |\Phi_2^{-1}(\xi)| J_\varphi \xi, J_\varphi \xi \rangle
\end{aligned}$$

Note that as $m \to \infty$, we have

$$\langle \exp(-|\Phi_2^{-1}(\xi)|^2/m) | \log |\Phi_2^{-1}(\xi)|| J_\varphi \xi, J_\varphi \xi \rangle \to \langle | \log |\Phi_2^{-1}(\xi)|| J_\varphi \xi, J_\varphi \xi \rangle < \infty.$$

Then for m, β large enough, we have

$$\begin{aligned}
&|\langle \exp(-|x_\beta d^{1/2}|^2/m) \log |x_\beta d^{1/2}| J_\varphi \Lambda_\varphi(x_\beta), J_\varphi \Lambda_\varphi(x_\beta) \rangle \\
&- \langle \log |x_\beta d^{1/2}| J_\varphi \Lambda_\varphi(x_\beta), J_\varphi \Lambda_\varphi(x_\beta) \rangle|
\end{aligned}$$

is small enough. Therefore, we see that

$$H(\xi) = \lim_\beta H(\Lambda_\varphi(x_\beta)). \qquad \square$$

Remark 5.14. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful tracial weight φ and $\phi = \varphi(J_\varphi \cdot J_\varphi)$. Then for any $\xi \in L^2(\mathcal{M})$, and any net $\{x_\beta\}_\beta \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$ such that $\lim_\beta x_\beta = \xi$ in 2-norm, we have that $H(\xi) = \lim_\beta H(x_\beta)$.

Theorem 5.15 *(Hirschman–Beckner uncertainty principle). Suppose* $\mathbb{G}$ *is a compact quantum group or a discrete quantum group. Let* $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ *such that* $\|\xi(\omega)\| = 1$. *If* $H(\xi(\omega))$, $H(\hat{\Lambda}(\lambda(\omega)))$, $\langle |\log d| J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle$, $\langle |\log \hat{d}| \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle$ *are finite, then*

$$H(\xi(\omega)) + H(\hat{\Lambda}(\lambda(\omega))) \geq -\langle (\log d) J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle - \langle (\log \hat{d}) \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle.$$

Moreover, for any $\xi \in L^2(\mathbb{G})$ *if* $H(\xi)$, $H(\mathcal{F}_2(\xi))$, $\langle |\log d| J_\varphi \xi, J_\varphi \xi\rangle$, $\langle |\log \hat{d}| \hat{J}\mathcal{F}_2(\xi), \hat{J}\mathcal{F}_2(\xi)\rangle$ *are finite, then*

$$H(\xi) + H(\mathcal{F}_2(\xi)) \geq -\langle (\log d) J_\varphi \xi, J_\varphi \xi\rangle - \langle (\log \hat{d}) \hat{J}\mathcal{F}_2(\xi), \hat{J}\mathcal{F}_2(\xi)\rangle.$$

Proof. Let $f_\omega(t) = \|\xi_t(\omega)\|_t - \|\iota^{\frac{t}{t-1}}(\lambda(\omega))\|_{\frac{t}{t-1}}$. Since $f(t) \geq 0$, $1 \leq t \leq 2$ and $f(2) = 0$, we have for any $f'(2^-) \leq 0$. By Proposition 5.9 and Corollary 5.11, we have

$$\begin{aligned}
f'(2^-) &= \frac{1}{4}\langle \log |\Phi_2^{-1}(\xi(\omega))|^2 J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle - \frac{1}{4}\langle (\log d) J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle \\
&\quad + \frac{1}{4}\langle \log |\lambda(\omega)\hat{d}^{1/2}|^2 \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle - \frac{1}{4}\langle (\log \hat{d}) \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle \\
&= -\frac{1}{4} H(\xi(\omega)) - \frac{1}{4} H(\hat{\Lambda}(\lambda(\omega))) \\
&\quad - \frac{1}{4}\langle (\log d) J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle - \frac{1}{4}\langle (\log \hat{d}) \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle \leq 0
\end{aligned}$$

i.e.

$$H(\xi(\omega)) + H(\hat{\Lambda}(\lambda(\omega))) \geq -\langle (\log d) J_\varphi \xi(\omega), J_\varphi \xi(\omega)\rangle - \langle (\log \hat{d}) \hat{J}\hat{\Lambda}(\lambda(\omega)), \hat{J}\hat{\Lambda}(\lambda(\omega))\rangle.$$

For any $\xi \in L^2(\mathbb{G})$, there exists a net $\{x_\beta\}_\beta \subset \mathcal{T}_\varphi^2$ such that $\lim_\beta \|\Lambda_\varphi(x_\beta) - \xi\| = 0$. Let

$$x_{\beta,m} = \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-m^2 t^2) \tau_t(x_\beta) \delta^{-it} dt.$$

Then $x_{\beta,m} \in \mathcal{T}_\varphi^2$ and $\lim_{\beta,m} \|\Lambda(x_{\beta,m}) - \xi\| = 0$. Since

$$\lambda(x_{\beta,m}\varphi) = \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-m^2 t^2) \lambda(\rho_t(x_\beta \varphi)) dt = \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-m^2 t^2) \hat{\sigma}_t(\lambda(x_\beta \varphi)) dt,$$

we see that $\lambda(x_{\beta,m}\varphi) \in \mathcal{T}_{\hat{\varphi}}$. Hence we have that

$$\begin{aligned}
&H(\Lambda_\varphi(x_{\beta,m})) + H(\hat{\Lambda}(\lambda(x_{\beta,m}\varphi))) \\
&\geq -\langle (\log d) J_\varphi \Lambda_\varphi(x_{\beta,m}), J_\varphi \Lambda_\varphi(x_{\beta,m})\rangle - \langle (\log \hat{d}) \hat{J}\hat{\Lambda}(\lambda(x_{\beta,m}\varphi)), \hat{J}\hat{\Lambda}(\lambda(x_{\beta,m}\varphi))\rangle.
\end{aligned}$$

By Proposition 5.13, we have that

$$\lim_{\beta,m} H(\Lambda_\varphi(x_{\beta,m})) = H(\xi), \quad \lim_{\beta,m} H(\hat{\Lambda}(\lambda(x_{\beta,m}\varphi))) = H(\mathcal{F}_2(\xi)).$$

Note that

$$\lim_{\beta,m}\langle(\log d)J_\varphi\Lambda_\varphi(x_{\beta,m}), J_\varphi\Lambda_\varphi(x_{\beta,m})\rangle = \langle(\log d)J_\varphi\xi, J_\varphi\xi\rangle$$
$$\lim_{\beta,m}\langle(\log \hat{d})\hat{J}\hat{\Lambda}(\lambda(x_{\beta,m}\varphi)), \hat{J}\hat{\Lambda}(\lambda(x_{\beta,m}\varphi))\rangle = \langle(\log \hat{d})\hat{J}\mathcal{F}_2(\xi), \hat{J}\mathcal{F}_2(\xi)\rangle.$$

Therefore for any $\xi \in L^2(\mathbb{G})$, we have the Hirschman–Beckner uncertainty principle. □

Remark 5.16. Suppose $\mathbb{G}$ is a unimodular Kac algebra. By the results in [20] and Remark 5.14, we have that

$$H(\xi) + H(\mathcal{F}_2(\xi)) \geq 0, \quad \xi \in L^2(\mathbb{G}),$$

and by the argument in Proposition 3.3 in [20], we have

$$\mathcal{S}_r(\xi)\mathcal{S}_r(\mathcal{F}_2(\xi)) \geq 1, \quad \xi \in L^2(\mathbb{G}).$$

Corollary 5.17. *Suppose $\mathbb{G}$ is a compact quantum group or a discrete quantum group. If $\xi(\omega) \in \mathcal{D}(d^{-1/2})$ and $\hat{\Lambda}(\lambda(\omega)) \in \mathcal{D}(\hat{d}^{-1/2})$, then*

$$H(\xi(\omega)) + H(\hat{\Lambda}(\lambda(\omega))) \geq -\log\|d^{-1/2}\xi(\omega)\|^2 - \log\|\hat{d}^{-1/2}\hat{\Lambda}(\lambda(\omega))\|^2.$$

Let $\xi \in L^2(\mathbb{G})$ be such that $\xi \in \mathcal{D}(d^{-1/2})$ and $\mathcal{F}_2(\xi) \in \mathcal{D}(\hat{d}^{-1/2})$. Then

$$H(\xi) + H(\mathcal{F}_2(\xi)) \geq -\log\|d^{-1/2}\xi\|^2 - \log\|\hat{d}^{-1/2}\mathcal{F}_2(\xi)\|^2.$$

Proof. This is followed by Jensen's inequality. □

There is an alternative way to define an entropy of $\xi \in L^2(\mathbb{G})$.

Definition 5.18. Suppose $\mathbb{G}$ is a locally compact quantum group. Let $\xi \in L^2(\mathbb{G})$ be such that $\langle|\log|\Phi_2^{-1}(\xi)|^2 - \log d|J_\varphi\xi, J_\varphi\xi\rangle < \infty$. Then we can define a modified entropy of ξ as

$$H_0(\xi) = \langle(\log|\Phi_2^{-1}(\xi)|^2 - \log d)J_\varphi\xi, J_\varphi\xi\rangle.$$

Corollary 5.19. *Suppose $\mathbb{G}$ is a compact quantum group or a discrete quantum group. Let $\xi \in L^2(\mathbb{G})$ be such that $H_0(\xi)$, $H_0(\mathcal{F}_2(\xi))$ are finite. Then*

$$H_0(\xi) + H_0(\mathcal{F}_2(\xi)) \geq 0.$$

Proof. By a similar argument in Proposition 5.13 and Theorem 5.15, we see the Corollary is true. □

Remark 5.20. Suppose that $\mathbb{G}$ is a compact quantum group or a discrete quantum group. Let $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ with $\|\xi(\omega)\| = 1$ be a minimizer of the Donoho–Stark uncertainty principle. Then $\xi(\omega)$ is a minimizer of the uncertainty principle in Corollary 5.19, i.e.

$$H_0(\xi(\omega)) + H_0(\hat{\Lambda}(\lambda(\omega))) = 0.$$

In fact, suppose $\omega = \mu v \varphi$ and $\lambda(\omega) = \hat{\mu}\hat{v}$, where $\mu, \hat{\mu} > 0$, v, $\hat{v}$ are partial isometries. We have

$$\varphi(|v|) = \mu^{-2}, \hat{\varphi}(|\hat{v}|) = \hat{\mu}^{-2}, \hat{\mu} = \mu^{-1}$$
$$H_0(\xi(\omega)) = \langle (\mu^2 \log \mu^2)|v| J_\varphi \Lambda_\varphi(v), J_\varphi \Lambda_\varphi(v) \rangle,$$

and

$$H_0(\hat{\Lambda}(\lambda(\omega))) = \langle (\hat{\mu}^2 \log \hat{\mu}^2)|\hat{v}| \hat{J}\hat{\Lambda}(\hat{v}), \hat{J}\hat{\Lambda}(\hat{v}) \rangle.$$

Then

$$H_0(\xi(\omega)) + H_0(\hat{\Lambda}(\lambda(\omega))) = \mu^2 \varphi(|v|) \log \mu^2 + \hat{\mu}^2 \hat{\varphi}(|\hat{v}|) \log \hat{\mu}^2 = \log \mu^2 + \log \hat{\mu}^2 = 0.$$

In general, ω might not be a minimizer of the uncertainty principle in Theorem 5.15.

Next we will give a Rényi entropic uncertainty principle for locally compact quantum groups. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. For any $0 < t < 1$, we define $L^t(\phi)$ to be the set of all densely defined closed operators x on $\mathcal{H}_\varphi$ with the polar decomposition $x = v_x|x|$ such that $v_x \in \mathcal{M}$ and $|x|^t \in L^1(\phi)$.

Definition 5.21. Suppose $\mathcal{M}$ is von Neumann algebra with a normal semifinite faithful weight φ. For any $x \in L^t(\phi)$, $t \in (0,1) \cup (1,\infty)$, the Rényi entropy of x is defined to be

$$h_t(x) = \frac{t}{1-t} \log \|x\|_{t,\phi}.$$

Proposition 5.22 *(Rényi entropic uncertainty principle). Suppose $\mathbb{G}$ is a locally compact quantum group. For any $x \in L^t(\mathbb{G})$, $1/t + 1/t' = 1$, $1 \leq t \leq 2$, we have*

$$h_{t/2}(|\Phi_t^{-1}(x)|^2) + h_{t'/2}(|\hat{\Phi}_{t'}^{-1}(\mathcal{F}_t(x))|^2) \geq 0.$$

Proof. By the Hausdorff–Young inequality, we have

$$\log \|\mathcal{F}_t(x)\|_{t'} \leq \log \|x\|_t.$$

Hence

$$\begin{aligned}
&h_{t/2}(|\Phi_t^{-1}(x)|^2) + h_{t'/2}(|\hat{\Phi}_{t'}^{-1}(\mathcal{F}(x))|^2)\\
&= \frac{1}{1-t/2}\log \||\Phi_t^{-1}(x)|^2\|_{t/2}^{t/2} + \frac{1}{1-t'/2}\log \||\hat{\Phi}_{t'}^{-1}(\mathcal{F}_t(x))|^2\|_{t'/2}^{t'/2}\\
&= \frac{t}{1-t/2}\log \|x\|_t + \frac{t'}{1-t'/2}\log \|\mathcal{F}_t(x)\|_{t'}\\
&= \frac{2t}{2-t}\log \|x\|_t + \frac{2t}{t-2}\log \|\mathcal{F}_t(x)\|_{t'}\\
&\geq \frac{2t}{2-t}0 = 0. \qquad \square
\end{aligned}$$

Proposition 5.23. *Suppose $\mathcal{M} = \oplus_j \mathcal{M}_j$ is a von Neumann algebra with a normal semifinite faithful weight φ such that $\varphi|_{\mathcal{M}_j}$ is bounded. Let $\omega \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$ such that $\|\xi(\omega)\|_2 = 1$. Then*

$$\lim_{t\to 2^-} h_{t/2}(|\Phi_t^{-1}(\xi_t(\omega))|^2) = H(\xi(\omega)) + \langle(\log d)J_\varphi\xi(\omega), J_\varphi\xi(\omega)\rangle.$$

Proof.

$$\begin{aligned}
&\lim_{t\to 2^-} h_{t/2}(|\Phi_t^{-1}(\xi_t(\omega))|^2)\\
&= \lim_{t\to 2^-} \frac{1}{1-t/2}(\log \|\xi_t(\omega)\|_t^t - \log \|\xi(\omega)\|_2^2)\\
&= -\frac{d}{dt}\log \|\xi_t(\omega)\|_t^t\Big|_{t=2^-}\\
&= -\frac{1}{\|\xi_t(\omega)\|_t}\frac{d}{dt}\|\xi_t(\omega)\|_t^t\Big|_{t=2^-}\\
&= -\langle \log|\Phi_2^{-1}(\xi(\omega))|^2 J_\varphi\xi(\omega), J_\varphi\xi(\omega)\rangle + \langle(\log d)J_\varphi\xi(\omega), J_\varphi\xi(\omega)\rangle. \qquad \square
\end{aligned}$$

Remark 5.24. In Proposition 5.23, let $x \in \mathfrak{N}_\varphi$ be such that $\|\Lambda_\varphi(x)\| = 1$. Then we similarly have

$$\lim_{t\to 2^+} h_{t/2}(|xd^{1/2}|^2) = H(\Lambda_\varphi(x)) + \langle(\log d)J_\varphi\Lambda_\varphi(x), J_\varphi\Lambda_\varphi(x)\rangle.$$

6. Hardy's uncertainty principle

In this section, we show that minimizers of the Donoho–Stark uncertainty principle are bi-shifts of group-like projections and then prove Hardy's uncertainty principle for locally compact quantum groups with group-like projections.

Proposition 6.1. *Suppose $\mathbb{G}$ is a locally compact quantum group and v is a minimizer of the Donoho–Stark uncertainty principle. Then $\frac{1}{\varphi(|v|)}(\varphi|v| \otimes \iota)\Delta(|v|)$ is a group-like projection.*

Proof. Using the argument in Proposition 4.7 in [19] and Proposition 4.9, we have that $\frac{1}{\varphi(|v|)}(\varphi|v| \otimes \iota)\Delta(|v|)$ is a group-like projection. □

We will recall the definition of the left shift of a group-like projection first.

Definition 6.2. Suppose $\mathbb{G}$ is a locally compact quantum group and there exists a group-like projection B in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. A projection x in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ is called a left shift of the group-like projection B if $\varphi(x) = \varphi(B)$ and

$$\Delta(x)(1 \otimes B) = x \otimes B, \quad \Delta(B)(1 \otimes x) = R(x) \otimes x.$$

In [19], Liu, Wang, Wu showed that for a left shift x of a group-like projection B in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ and $t \in \mathbb{R}$,

$$\tau_t(x) = x, \quad \sigma_t^\varphi(x) = x, \quad x\delta^{it} = \mu_x^{it} x,$$

where $\mu_x > 0$.

Proposition 6.3. *Suppose $\mathbb{G}$ is a locally compact quantum group and $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ is a projection. Then x is a minimizer of the Donoho–Stark uncertainty principle if and only if x is a left shift of a group-like projection.*

Proof. Suppose that x is a left shift of a group-like projection. By Proposition 4.4, Corollary 4.14, Proposition 4.17 in [19], we have

$$\mathcal{S}_r(x\varphi)\mathcal{S}_r(\lambda(x\varphi)) = 1, \quad \sigma_t(x) = x, \quad \hat{\sigma}_t(|\lambda(x\varphi)|) = |\lambda(x\varphi)|.$$

By Proposition 4.7, we have that x is a minimizer of the Donoho–Stark uncertainty principle.

Suppose that x is a minimizer of the Donoho–Stark uncertainty principle and x is a projection. Let $B = \frac{1}{\varphi(x)}(\varphi x \otimes \iota)\Delta(x)$. Then B is a group-like projection by Proposition 6.1 and $\varphi(B) = \psi(B)$ by the results in [19]. We have $\varphi(B) = \varphi(x)$. Note that

$$\begin{aligned}
&(\psi \otimes \varphi)(|\Delta(B)(1 \otimes x) - R(x) \otimes x|^2) \\
&= (\psi \otimes \varphi)((1 \otimes x)\Delta(B)(1 \otimes x)) + (\psi \otimes \varphi)(R(x) \otimes x) \\
&\quad - 2\Re(\psi \otimes \varphi)((1 \otimes x)\Delta(B)(R(x) \otimes x)) \\
&= \psi(B)\varphi(x) + \varphi(x)^2 - 2\Re(\psi \otimes \varphi)(\Delta(B)(R(x) \otimes x))
\end{aligned}$$

and

$$\begin{aligned}(\psi\otimes\varphi)(\Delta(x)(R(x)\otimes x)) &= R(x)\psi S^{-1}((\iota\otimes\varphi)(1\otimes B)\Delta(x))\\ &= (\varphi x)((\iota\otimes\varphi)(1\otimes B)\Delta(x))\\ &= \varphi(B((\varphi x\otimes\iota)\Delta(x)))\\ &= \varphi(x)\varphi(B) = \varphi(x)^2.\end{aligned}$$

We obtain that $\Delta(B)(1\otimes x) = R(x)\otimes x$. Since

$$\begin{aligned}&(\varphi\otimes\varphi)(|\Delta(x)(1\otimes B) - x\otimes B|^2)\\ &= (\varphi\otimes\varphi)(\Delta(x)(1\otimes B)) + \varphi(x)\varphi(B) - 2\Re(\varphi\otimes\varphi)((x\otimes 1)\Delta(x)B)\\ &= \varphi S^{-1}((\iota\otimes\varphi)((1\otimes x)\Delta(B))) - \varphi(x)^2 \quad \text{Propositions 4.9, 4.11}\\ &= \varphi S^{-1}(R(x))\varphi(x) - \varphi(x)^2 = 0,\end{aligned}$$

we see that $\Delta(x)(1\otimes B) = x\otimes B$. Therefore x is a left shift of the group-like projection B. □

We will update the definition of a bi-shift of a biprojection in [19]. Note that by the results in [15], we see that there are eight forms to define a bi-shift of a biprojection. But we only take one of them as our definition of a bi-shift of a group-like projection here.

Definition 6.4. Suppose $\mathbb{G}$ is a locally compact quantum group and there exists a group-like projection B in $L^1(\mathbb{G})\cap L^\infty(\mathbb{G})$. Denote by $\tilde{B}$ the range projection of $\lambda(B\varphi)$ in $L^\infty(\mathbb{G})$. A nonzero element x in $L^1(\mathbb{G})\cap L^\infty(\mathbb{G})$ is said to be a bi-shift of a group-like projection B if there exist a left shift B_g of the group-like projection B and a left shift $\tilde{B}_h$ of the group-like projection $\tilde{B}$ and an element $y\in L^\infty(\mathbb{G})$ such that

$$x\varphi = (yB_g\varphi) * (\hat{\lambda}(\tilde{B}_h\hat{\varphi})\varphi).$$

Proposition 6.5. *Suppose $\mathbb{G}$ is a locally compact quantum group and $x\in L^1(\mathbb{G})\cap L^\infty(\mathbb{G})$. Then x is a minimizer of the Donoho–Stark uncertainty principle if and only if x is a bi-shift of a group-like projection.*

Proof. Suppose that x is a minimizer of the Donoho–Stark uncertainty principle and x is a partial isometry. Then $|x|(=B_g)$ is a left shift of the group-like projection $B = \frac{1}{\varphi(|x|)}(\varphi|x|\otimes\iota)\Delta(|x|)$ and $\frac{1}{\varphi(|x|)}|\lambda(x\varphi)|(=\tilde{B}_h)$ is a left shift of the group-like projection $\mathcal{R}(\lambda(B\varphi))(=\tilde{B})$. Then

$$\begin{aligned}\lambda(x\varphi * (\hat{\lambda}(|\lambda(x\varphi)|\hat{\varphi}))) &= \lambda(x\varphi)\lambda(\hat{\lambda}(|\lambda(x\varphi)|\hat{\varphi}))\\ &= \lambda(x\varphi)|\lambda(x\varphi)|\end{aligned}$$

$$= \frac{1}{\varphi(|x|)}\lambda(x\varphi),$$

i.e.

$$x\varphi = x\varphi * (\frac{1}{\varphi(|x|)}(\hat{\lambda}(|\lambda(x\varphi)|\hat{\varphi})) = xB_g\varphi * \hat{\lambda}(\tilde{B}_h\hat{\varphi})\varphi.$$

Hence $x = xB_g * \hat{\lambda}(\tilde{B}_h\hat{\varphi})$ is a bi-shift of the group-like projection B.

Suppose that x is a bi-shift of a group-like projection B. By Proposition 4.17 in [19], we have that $|\hat{\lambda}(\tilde{B}_h\hat{\varphi})| = B$. For any $a \in L^\infty(\mathbb{G})$, we have that

$$\begin{aligned}(x\varphi)(B_g a) &= (\varphi\otimes\varphi)(\Delta(B_g a)(yB_g \otimes \hat{\lambda}(\tilde{B}_h\hat{\varphi})))\\ &= (\varphi\otimes\varphi)((1\otimes B)\Delta(B_g)\Delta(a)(yB_g \otimes \hat{\lambda}(\tilde{B}_h\hat{\varphi})))\\ &= (\varphi\otimes\varphi)((B_g\otimes B)\Delta(a)(yB_g \otimes \hat{\lambda}(\tilde{B}_h\hat{\varphi})))\\ &= (\varphi\otimes\varphi)(\Delta(a)(yB_g \otimes \hat{\lambda}(\tilde{B}_h\hat{\varphi})))\\ &= (x\varphi)(a),\end{aligned}$$

i.e. $\mathcal{R}(x\varphi) \leq B_g$. On the other hand, we have that $\lambda(x\varphi) = \lambda(yB_g\varphi)\tilde{B}_h$ and $\mathcal{R}_r(\lambda(x\varphi)) \leq \tilde{B}_h$. Hence $\mathcal{S}_r(x\varphi)\mathcal{S}_r(\lambda(x\varphi)) \leq \varphi(B_g)\hat{\varphi}(\tilde{B}_h) = 1$. By Theorem 4.2, the Donoho–Stark uncertainty principle, we see that

$$\mathcal{S}_r(x\varphi)\mathcal{S}_r(\lambda(x\varphi)) = 1, \quad \mathcal{R}(x\varphi) = B_g, \quad \mathcal{R}_r(\lambda(x\varphi)) = \tilde{B}_h.$$

By Proposition 4.4 and Proposition 4.5, we have that x is a bi-partial isometry. Note that $\hat{\sigma}_t(\tilde{B}_h) = \tilde{B}_h$. By Proposition 4.7, we have that x is a minimizer of the Donoho–Stark uncertainty principle. □

Theorem 6.6 *(Hardy's uncertainty principle). Suppose $\mathbb{G}$ is a locally compact quantum group with a bi-shift w of a group-like projection. Let $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ be such that*

$$|x^*| \leq C|w^*|, \quad |\lambda(x\varphi)| \leq C'|\lambda(w\varphi)|,$$

for some $C, C' > 0$. Then x is a multiple of w.

Proof. We assume that w is a partial isometry. Consider the element x^*w in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Since $|x^*| \leq C|w^*|$, x^*w is nonzero. We have that $\mathcal{S}_r(x^*w\varphi) \leq \varphi(|w|)$. To estimate $\mathcal{S}_r(\lambda(x^*w\varphi))$, we will show

$$\lambda(x^*w\varphi) = (\hat{\varphi}\lambda(x\varphi)^* \otimes \iota)\hat{\Delta}(\lambda(w\varphi)).$$

This is because

$$\begin{aligned}\hat{\Lambda}((\hat{\varphi}\lambda(x\varphi)^*\otimes\iota)\hat{\Delta}(\lambda(w\varphi))) &= \Lambda_\varphi(\hat{\lambda}(\lambda(x\varphi)\hat{\varphi})^*\hat{\lambda}(\lambda(w\varphi)\hat{\varphi})) \\ &= \Lambda_\varphi(x^*w)=\hat{\Lambda}(\lambda(x^*w\varphi)).\end{aligned}$$

By Proposition 1.24 in [16] and $|\lambda(x\varphi)|\leq C'|\lambda(w\varphi)|$, we have that

$$\begin{aligned}&|(\hat{\varphi}\lambda(x\varphi)^*\otimes\iota)\hat{\Delta}(\lambda(w\varphi))|^2 \\ &\leq \|\hat{\varphi}(|\lambda(x\varphi)|^2)1\|(\hat{\varphi}\otimes\iota)((\frac{|\lambda(w\varphi)|}{\varphi(|w|)}\otimes 1)\hat{\Delta}(|\lambda(w\varphi)|^2)(\frac{|\lambda(w\varphi)|}{\varphi(|w|)}\otimes 1)) \\ &= \varphi(x^*x)(\hat{\varphi}\otimes\iota)(\hat{\Delta}(|\lambda(w\varphi)|^2)(\frac{|\lambda(w\varphi)|}{\varphi(|w|)}\otimes 1)) \quad \text{Propositions 6.5, 4.7} \\ &= \varphi(x^*x)(\hat{\varphi}\otimes\iota)(\hat{\Delta}(|\lambda(w\varphi)|)(|\lambda(w\varphi)|\otimes 1)).\end{aligned}$$

By Proposition 4.10, we see that

$$\mathcal{S}_r(\lambda(x^*w\varphi))\leq\mathcal{S}_r((\hat{\varphi}\otimes\iota)(\Delta(|\lambda(w\varphi)|)(|\lambda(w\varphi)|\otimes 1)))=\hat{\varphi}(\frac{|\lambda(w\varphi)|}{\varphi(|w|)})=\mathcal{S}_r(\lambda(w\varphi)).$$

By Theorem 4.2, we have that

$$\mathcal{S}_r(x^*w\varphi)\mathcal{S}_r(\lambda(x^*w\varphi))=1.$$

and

$$\mathcal{R}(x^*w\varphi)=|w|,\quad \mathcal{R}_r(\lambda(x^*w\varphi))=\tilde{B},$$

where $\tilde{B}=\mathcal{R}((\hat{\varphi}\otimes\iota)(\Delta(|\lambda(w\varphi)|)(|\lambda(w\varphi)|\otimes 1)))$. Now we will show that $(\varphi\otimes\iota)((w^*x\otimes 1)\Delta(|w|))=\varphi(w^*x)B$, where $B=\frac{1}{\varphi(|w|)}(\varphi|w|\otimes\iota)\Delta(|w|)$. By Proposition 1.24 in [16], we have that

$$\begin{aligned}|(\varphi\otimes\iota)((w^*x\otimes 1)\Delta(|w|))|^2 &\leq \|\varphi(w^*xx^*w)1\|(\varphi\otimes\iota)((|w|\otimes\iota)\Delta(|w|)) \\ &= \|x\|^2\varphi(|w|)^2B.\end{aligned}$$

Hence we have that

$$\begin{aligned}(\varphi\otimes\iota)((w^*x\otimes 1)\Delta(|w|)) &= (\varphi\otimes\iota)((w^*x\otimes 1)\Delta(|w|))B \\ &= (\varphi\otimes\iota)((w^*x\otimes 1)\Delta(|w|)(1\otimes B)) \\ &= (\varphi\otimes\iota)((w^*x\otimes 1)(|w|\otimes B)) \\ &= \varphi(w^*x)B \\ &= \frac{\varphi(w^*x)}{\varphi(|w|)}(\varphi\otimes\iota)((|w|\otimes 1)\Delta(|w|)).\end{aligned}$$

Applying the map Λ_φ, we obtain

$$\Lambda_\varphi((\varphi \otimes \iota)((w^*x \otimes 1)\Delta(|w|))) = \Lambda_\varphi(\frac{\varphi(w^*x)}{\varphi(|w|)}(\varphi \otimes \iota)((|w| \otimes 1)\Delta(|w|))).$$

Therefore

$$\lambda(x^*w\varphi)^*\lambda(|w|\varphi) = \frac{\varphi(w^*x)}{\varphi(|w|)}\lambda(|w|\varphi)^*\lambda(|w|\varphi).$$

Now we obtain that

$$x^*w = \frac{\varphi(w^*x)}{\varphi(|w|)}w^*w,$$

and

$$x = \frac{\varphi(x^*w)}{\varphi(|w|)}w,$$

i.e. x is a multiple of w. $\square$

Remark 6.7. Note that Hardy's uncertainty principle for unimodular Kac algebras proved in [20] imply the uniqueness of a bi-shift of a biprojection, but Hardy's uncertainty principle for locally compact quantum groups given here does not imply the uniqueness (see also Theorem 4.12).

7. Young's inequality revisited

In [19], Liu, Wang and Wu shows that for any $x \in L^t(\mathbb{G})$ and $y \in L^s(\mathbb{G})$, $1 \le t, s \le 2$, the convolution $x * \rho_{-i/t'}(y)$ is well defined by using Cauchy sequences and

$$\hat{\Phi}_{r'}^{-1}\mathcal{F}_r(x * \rho_{-i/t'}(y)) = \hat{\Phi}_{t'}^{-1}\mathcal{F}_t(x)\hat{\Phi}_{s'}^{-1}\mathcal{F}_s(y),$$

where $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$, $\frac{1}{t} + \frac{1}{t'} = 1$. In this definition, we have that $\hat{\Phi}_{t'}^{-1}\mathcal{F}_t(x)\hat{\Phi}_{s'}^{-1}\mathcal{F}_s(y) \in L^{r'}(\hat{\mathbb{G}})$ for $2 \le r' \le \infty$. When $1 \le r' \le 2$, we can give a new definition of the convolution of $x \in L^t(\mathbb{G})$ and $y \in L^s(\mathbb{G})$, $1 \le t, s \le 2$, by

$$x * \rho_{-i/t'}(y) = \hat{\mathcal{F}}_{r'}\hat{\Phi}_{r'}(\hat{\Phi}_{t'}^{-1}\mathcal{F}_t(x)\hat{\Phi}_{s'}^{-1}\mathcal{F}_s(y)).$$

Note that for this definition we could have written the convolution as $x * y$, but it will not coincide with the case when $2 \le r' \le \infty$.

Combining Theorem 3.4 in [19] and the definition above we have Young's inequality for any $1 \le t, s \le 2$ as follows:

Theorem 7.1. *Suppose $\mathbb{G}$ is a locally compact quantum group. For any $x \in L^t(\mathbb{G})$, $y \in L^s(\mathbb{G})$, $1 \le t, s \le 2$, we have*

$$\|x * \rho_{-i/t'}(y)\|_r \le \|x\|_t \|y\|_s,$$

where $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$ *and* $\frac{1}{t} + \frac{1}{t'} = 1$.

In [17], Kustermans and Vaes defined the subspace $L^1(\mathbb{G})^\#$ of $L^1(\mathbb{G})$ as

$$L^1(\mathbb{G})^\# = \{\omega \in L^1(\mathbb{G}) | \exists \rho \in L^1(\mathbb{G}) : \rho(x) = \overline{\omega}(S(x)), \text{ for all } x \in \mathcal{D}(S)\}.$$

For all $\omega \in L^1(\mathbb{G})^\#$, we define $\omega^\#(x) = \overline{\omega}(S(x))$ for all x in $\mathcal{D}(S)$. It is known that $\lambda(\omega^\#) = \lambda(\omega)^*$ for all $\omega \in L^1(\mathbb{G})^\#$. Now we define subspaces $L^t(\mathbb{G})^\#$ of $L^t(\mathbb{G})$, $1 \le t \le 2$, as

$$L^t(\mathbb{G})^\# = \{x \in L^t(\mathbb{G}) | \exists x_0 \in L^t(\mathbb{G}) : \Phi^{-1}_{\frac{t}{t-1}} \mathcal{F}_t(x_0) = (\Phi^{-1}_{\frac{t}{t-1}} \mathcal{F}_t(x))^*\}.$$

Note that $L^2(\mathbb{G}) = L^2(\mathbb{G})^\#$. For any $x \in L^t(\mathbb{G})^\#$, we define $x^\# = x_0$ where x_0 is the one described in the definition. The definition here coincides with the definition given in [17] when $t = 1$ by Proposition 2.4 in [17]. We define a norm $\| \cdot \|_{t,\#}$ on $L^t(\mathbb{G})^\#$ by

$$\|x\|_{t,\#} = \max\{\|x\|_t, \|x^\#\|_t\}.$$

Proposition 7.2. *Suppose* $\mathbb{G}$ *is a locally compact quantum group. Let* $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ *be such that* $\omega \in \mathcal{D}(\tau^*_{(-1/t'+1/2)i}\delta^*_{-i/t'})$ *for any* $1 \le t \le 2$, *where* $1/t' + 1/t = 1$. *Then*

$$\xi_t(\omega)^\# = \xi_t(\tau^*_{(1/t'-1/2)i}\delta^*_{i/t'}(\overline{\omega}R)).$$

In particular, if $\omega \in \mathcal{D}(\delta^*_{-i/2})$, *we have that*

$$\xi(\omega)^\# = \xi(\delta^*_{i/2}(\overline{\omega}R)).$$

Proof. We check the equation in the Hilsum space $L^t(\phi)$,

$$\begin{aligned}
(\Phi_t^{-1}\mathcal{F}_t(\xi_t(\tau^*_{(1/t'-1/2)i}\delta^*_{i/t'}(\overline{\omega}R))))^* &= (\lambda(\tau^*_{(1/t'-1/2)i}\delta^*_{i/t'}(\overline{\omega}R))\hat{d}^{1/t'})^* \\
&= (\hat{\sigma}_{-i/t'}(\lambda(\omega^\#))\hat{d}^{1/t'})^* \\
&= (\hat{d}^{1/t'}\lambda(\omega)^*)^* \\
&= \lambda(\omega)\hat{d}^{1/t'} \\
&= \Phi_t^{-1}\mathcal{F}_t(\xi_t(\omega)). \qquad \square
\end{aligned}$$

Proposition 7.3. *Suppose* $\mathbb{G}$ *is a locally compact quantum group. Let* $x \in L^t(\mathbb{G})^\#$, $y \in L^s(\mathbb{G})$ *and* $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$, *where* $1 \le r, t, s \le 2$. *Then*

$$(x * \rho_{-i/t'}(y))^\# = y^\# * \rho_{-i/s'}(x^\#).$$

Proof. By Proposition 3.7 in [19], we have

$$\begin{aligned}\Phi_{r'}^{-1}\mathcal{F}_r((x*\rho_{-i/t'}(y))^{\#}) &= (\Phi_{r'}^{-1}\mathcal{F}_r(x*\rho_{-i/t'}(y)))^* \\ &= (\Phi_{t'}^{-1}\mathcal{F}_t(x)\Phi_{s'}^{-1}\mathcal{F}_s(y))^* \\ &= \Phi_{s'}^{-1}\mathcal{F}_s(y)^*\Phi_{t'}^{-1}\mathcal{F}_t(x)^* \\ &= \Phi_{s'}^{-1}\mathcal{F}_s(y^{\#})\Phi_{t'}^{-1}\mathcal{F}_t(x^{\#}) \\ &= \Phi_{r'}^{-1}\mathcal{F}_r(y^{\#}*\rho_{-i/s'}(x^{\#})),\end{aligned}$$

i.e. $(x*\rho_{-i/t'}(y))^{\#} = y^{\#}*\rho_{-i/s'}(x^{\#})$. □

Proposition 7.4. *Suppose* $\mathbb{G}$ *is a locally compact quantum group. Let* $x \in L^t(\mathbb{G})^{\#}$, $y \in L^s(\mathbb{G})^{\#}$ *and* $\frac{1}{r}+1=\frac{1}{t}+\frac{1}{s}$, $1\le t,s\le 2$. *If* $2\le r\le\infty$, *then*

$$\|x^{\#}*\rho_{-i/t'}(y^{\#})\|_r \le \|x\|_t\|y\|_s.$$

If $1\le r\le 2$, *then*

$$\|x*\rho_{-i/t'}(y)\|_{r,\#} \le \|x\|_{t,\#}\|y\|_{s,\#}.$$

Proof. For $2\le r\le\infty$, we have

$$\begin{aligned}\|x^{\#}*\rho_{-i/t'}(y^{\#})\|_r &= \|\hat{\mathcal{F}}_{r'}\hat{\Phi}_{r'}(\hat{\Phi}_{t'}^{-1}\mathcal{F}_t(x^{\#})\Phi_{s'}^{-1}\mathcal{F}_s(y^{\#}))\|_r \\ &\le \|\hat{\Phi}_{t'}^{-1}\mathcal{F}_t(x^{\#})\Phi_{s'}^{-1}\mathcal{F}_s(y^{\#})\|_{r',\phi} \\ &= \|\Phi_{s'}^{-1}\mathcal{F}_s(y)\hat{\Phi}_{t'}^{-1}\mathcal{F}_t(x)\|_{r',\phi} \\ &\le \|\mathcal{F}_s(y)\|_{s'}\|\mathcal{F}_t(x)\|_{t'} \\ &\le \|y\|_s\|x\|_t.\end{aligned}$$

For $1\le r\le 2$, we have

$$\begin{aligned}\|x*\rho_{-i/t'}(y)\|_{r,\#} &= \max\{\|x*\rho_{-i/t'}(y)\|_r, \|(x*\rho_{-i/t'}(y))^{\#}\|_r\} \\ &= \max\{\|x*\rho_{-i/t'}(y)\|_r, \|y^{\#}*\rho_{-i/s'}(x^{\#})\|_r\} \\ &\le \max\{\|x\|_t\|y\|_s, \|x^{\#}\|_t\|y^{\#}\|_s\} \\ &\le \|x\|_{t,\#}\|y\|_{s,\#}. \qquad \square\end{aligned}$$

Lemma 7.5. Suppose that $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight φ. Let $x\in L^1(\mathcal{M})\cap L^\infty(\mathcal{M})$ be such that $\|\Lambda_\varphi(x)\|^2=\|x\|_\infty\|x\varphi\|$. Then x is a multiple of a partial isometry and $\sigma_t^\varphi(|x|)=|x|$ for any $t\in\mathbb{R}$.

Proof. Note that $\varphi(|x|) \leq \|x\varphi\| < \infty$, we have that $|x|^{1/2} \in \mathfrak{N}_\varphi$. Then

$$\begin{aligned}\varphi(|x|^2) &= \langle |x|\Lambda_\varphi(|x|^{1/2}), \Lambda_\varphi(|x|^{1/2})\rangle \\ &\leq \||x|\Lambda_\varphi(|x|^{1/2})\|\|\Lambda_\varphi(|x|^{1/2})\| \\ &\leq \|x\|_\infty \varphi(|x|) \\ &\leq \|x\|_\infty \|x\varphi\|.\end{aligned}$$

By the assumption, we obtain that

$$|x|\Lambda_\varphi(|x|^{1/2}) = \|x\|_\infty \Lambda_\varphi(|x|^{1/2}), \quad \|x\varphi\| = \varphi(|x|).$$

Then $|x| = \|x\|_\infty \mathcal{R}(|x|)$ and by Proposition 4.5, we have that x is a partial isometry such that $\sigma_t^\varphi(|x|) = |x|$ for any $t \in \mathbb{R}$. □

In [21], Liu and Wu completely characterize the extremal pairs of Young's inequality for unimodular Kac algebras. But in general, it is quite difficult to characterize the extremal pairs of Young's inequality for locally compact quantum groups.

Proposition 7.6. *Suppose $\mathbb{G}$ is a locally compact quantum group. Let $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ be such that $\|\omega * \xi(\omega)^\#\|_2 = \|\omega\|\|\xi(\omega)\|$. Then ω is a minimizer of the Donoho–Stark uncertainty principle.*

Proof. Note that

$$\begin{aligned}\|\omega * \xi(\omega)^\#\|_2^2 &= \|\lambda(\omega)\Phi_2^{-1}\mathcal{F}_2(\xi(\omega))^*\|_{2,\phi}^2 \\ &= \|\lambda(\omega)\hat{d}^{1/2}\lambda(\omega)^*\|_{2,\phi}^2 \\ &= \int \lambda(\omega)\hat{d}^{1/2}\lambda(\omega)^*\lambda(\omega)\hat{d}^{1/2}\lambda(\omega)^* d\phi \\ &\leq \|\lambda(\omega)^*\lambda(\omega)\hat{d}^{1/2}\lambda(\omega)^*\|_{2,\phi}\|\hat{d}^{1/2}\lambda(\omega)^*\|_{2,\phi} \\ &\leq \|\lambda(\omega)\|_\infty^2\|\xi(\omega)\|_2^2 \\ &\leq \|\omega\|^2\|\xi(\omega)\|^2.\end{aligned}$$

Then we have that all inequalities above must be equalities by the assumption. This is to say,

$$\lambda(\omega)^*\lambda(\omega)\hat{d}^{1/2}\lambda(\omega)^* = \|\lambda(\omega)\|_\infty^2\hat{d}^{1/2}\lambda(\omega)^*, \quad \|\lambda(\omega)\|_\infty = \|\omega\|_1.$$

Let $p_2 = \mathcal{R}(\hat{d}^{1/2}\lambda(\omega)^*)$ and $p_\infty = \mathcal{R}(\lambda(\omega)^*)$. Then $\lambda(\omega)^*\lambda(\omega)p_2 = \|\lambda(\omega)\|_\infty^2 p_2$ and hence $p_2 \leq p_\infty$. Note that

$$\hat{\varphi}(p_2) = \frac{1}{\|\lambda(\omega)\|_\infty^2}\hat{\varphi}(|\lambda(\omega)|p_2|\lambda(\omega)|) \leq \frac{\|\xi(\omega)\|_2^2}{\|\lambda(\omega)\|_\infty^2}.$$

By Proposition 4.4, we have that

$$\hat{\varphi}(p_2) = \mathcal{S}_r(\hat{\Lambda}(\lambda(\omega))) \geq \frac{\|\hat{\Lambda}(\lambda(\omega))\|_2^2}{\|\lambda(\omega)\|_\infty^2}.$$

Combining the two inequalities above, we have that

$$\hat{\varphi}(p_2) = \frac{\|\hat{\Lambda}(\lambda(\omega))\|_2^2}{\|\lambda(\omega)\|_\infty^2}$$

and $p_2 = p_\infty$. By Proposition 4.7, we have that $\lambda(\omega)$ is a multiple of a partial isometry and $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$. Now we have to show that $\omega = v\varphi$ is such that v is a multiple of a partial isometry. Note that $\lambda(\omega) \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, we let $v = \hat{\lambda}(\lambda(\omega)\hat{\varphi})$. Then

$$\Lambda_\varphi(v) = \Lambda_\varphi(\hat{\lambda}(\lambda(\omega)\hat{\varphi})) = \hat{\Lambda}(\lambda(\omega)) = \xi(\omega),$$

i.e. $\omega = v\varphi$. We will check that $\|\Lambda_\varphi(v)\|^2 = \|v\|_\infty \|v\varphi\|$.

$$\begin{aligned}\|v\|_\infty\|v\varphi\| &\geq \|\Lambda_\varphi(v)\|^2 = \|\hat{\Lambda}(\lambda(\omega))\|^2 \\ &= \|\lambda(\omega)\|_\infty\|\lambda(\omega)\hat{\varphi}\| \\ &= \|\omega\|\|\lambda(\omega)\hat{\varphi}\| \\ &\geq \|v\varphi\|\|v\|_\infty,\end{aligned}$$

i.e. $\|\Lambda_\varphi(v)\|^2 = \|v\|_\infty\|v\varphi\|$. By Lemma 7.5, we have that v is a multiple of a partial isometry and $\sigma_t(|v|) = |v|$. Hence ω is a minimizer of the Donoho–Stark uncertainty principle by Proposition 4.7. □

Remark 7.7. Suppose $\mathbb{G}$ is a locally compact quantum group. Let $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ be such that $\|\xi(\omega)^\# * \rho_{-i/2}(\omega)\|_2 = \|\xi(\omega)\|\|\omega\|$. We can not prove that ω is a minimizer of the Donoho–Stark uncertainty principle. Actually we can show that $\lambda(\omega)$ is a multiple of a partial isometry and $\|\lambda(\omega)\|_\infty = \|\omega\|_1$.

Now we show that bi-shifts of group-like projections are extremal operators for the Hausdorff–Young inequality.

Proposition 7.8. *Suppose that $\mathbb{G}$ is a locally compact quantum group. Let v be a minimizer of the Donoho–Stark uncertainty principle. Then for any $1 \leq t \leq 2$, $1/t + 1/t' = 1$,*

$$\|\mathcal{F}_t(\xi_t(v\varphi))\|_{t'} = \|\xi_t(v\varphi)\|_t.$$

Proof. Suppose that v is a partial isometry and $\lambda(v\varphi) = \hat{\mu}\hat{v}$, where $\hat{\mu} > 0$ and $\hat{v}$ is a partial isometry. By Theorem 4.2, we have

$$\hat{\varphi}(|\hat{v}|) = \varphi(|v|)^{-1} = \hat{\mu}^{-1}.$$

$$\begin{aligned}\|\mathcal{F}_t(\xi_t(v\varphi))\|_{t'} &= \|\iota^{t'}(\lambda(v\varphi))\|_{t'} \\ &= \hat{\varphi}(|\lambda(v\varphi)|^{t'})^{1/t'} \\ &= \hat{\mu}\hat{\mu}^{-1/t'} = \hat{\mu}^{1/t} \\ &= \varphi(|v|)^{1/t} = \|\xi_t(v\varphi)\|_t,\end{aligned}$$

i.e. $\|\mathcal{F}_t(\xi_t(v\varphi))\|_{t'} = \|\xi_t(v\varphi)\|_t$. □

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References

[1] G. Alagic, A. Russell, Uncertainty principles for compact groups, Illinois J. Math. 52 (4) (2008) 1315–1324.
[2] W. Beckner, Inequalities in Fourier analysis, Ann. of Math. 102 (1) (1975) 159–182.
[3] M. Caspers, The L^p-Fourier transform on locally compact quantum groups, J. Operator Theory 69 (2013) 161–193.
[4] A. Connes, On the spatial theory of von Neumann algebras, J. Funct. Anal. 35 (1980) 153–164.
[5] T. Cooney, A Hausdorff–Young inequality for locally compact quantum groups, Int. J. Math. 21 (12) (2012) 1619–1632.
[6] J. Crann, M. Kalantar, An uncertainty principle for unimodular quantum groups, J. Math. Phys. 55 (2014) 081704.
[7] E. Candes, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory 52 (2006) 489–509.
[8] M. Daws, Representing multipliers of the Fourier algebra on non commutative L^p spaces, Canad. J. Math. 63 (2011) 798–825.
[9] D.L. Donoho, P.B. Stark, Uncertainty principles and signal recovery, SIAM J. Appl. Math. 49 (1989) 906–931.
[10] J. Gilbert, Z. Rzesotnik, The norm of the Fourier transform on finite abelian groups, Ann. Inst. Fourier (Grenoble) 60 (4) (2010) 1317–1346.
[11] G.H. Hardy, A theorem concerning Fourier transforms, J. Lond. Math. Soc. (3) s1-8 (1933) 227–231.
[12] M. Hilsum, Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale, J. Funct. Anal. 40 (1981) 151–169.
[13] I.I. Hirschman, A note on entropy, Amer. J. Math. 79 (1957) 152–156.
[14] C. Jiang, Z. Liu, J. Wu, Noncommutative uncertainty principles, J. Funct. Anal. 270 (2016) 264–311.

[15] C. Jiang, Z. Liu, J. Wu, Block maps and Fourier analysis, Preprint, 2017.
[16] J. Kustermans, S. Vaes, Locally compact quantum groups, Ann. Sci. Éc. Norm. Supér. 33 (6) (2000) 837–934.
[17] J. Kustermans, S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting, Math. Scand. 92 (1) (2003) 68–92.
[18] M. Landstad, A. Van Daele, Compact and discrete subgroups of algebraic quantum groups I, arXiv:math/0702458v2, 2007.
[19] Z. Liu, S. Wang, J. Wu, Young's inequality for locally compact quantum groups, J. Operator Theory 77 (2017) 109–131.
[20] Z. Liu, J. Wu, Uncertainty principles for Kac algebras, J. Math. Phys. 58 (2017) 052102.
[21] Z. Liu, J. Wu, Extremal pairs of Young's inequality for Kac algebras, Preprint, 2016.
[22] M. Ohya, D. Petz, Quantum Entropy and Its Use, Springer-Verlag, Berlin Heidelberg, 2004.
[23] M. Özaydm, T. Przebinda, An entropy-based uncertainty principle for a locally compact abelian group, J. Funct. Anal. 215 (2004) 241–252.
[24] M. Reed, B. Simon, Methods of Modern Mathematical Physics, I: Functional Analysis, Academic Press, Inc, San Diego, California, 1972.
[25] K.T. Smith, The uncertainty principle on groups, SIAM J. Appl. Math. 50 (1990) 876–882.
[26] M. Takesaki, The Theory of Operator Algebras, I, II, III, Springer-Verlag, Berlin Heidelberg, 2002.
[27] S.L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys. 111 (4) (1987) 613–665.
[28] S.L. Woronowicz, Tannaka–Krein duality for compact matrix pseudogroups, Invent. Math. 93 (1) (1988) 35–76.
[29] S.L. Woronowicz, Differential calculus on compact matrix pseudo groups (quantum groups), Comm. Math. Phys. 122 (1) (1989) 125–179.
[30] S.L. Woronowicz, Compact quantum groups, in: Symétries Quantiques, Les Houches, 1995, North-Holland, Amsterdam, 1998, pp. 845–884.