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# Uncertainty principles for Kac algebras

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In this paper, we introduce the notation of bi-shift of biprojections in subfactor theory to unimodular Kac algebras. We characterize the minimizers of the Hirschman-Beckner uncertainty principle and the Donoho-Stark uncertainty principle for unimodular Kac algebras with biprojections and prove Hardy's uncertainty principle in terms of the minimizers. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4983755]

## I. INTRODUCTION

Uncertainty principles for locally compact abelian groups were studied by Hardy,[16] Hirschman,[17] Beckner,[2] Donoho and Stark,[10] Smith,[23] Tao,[25] etc. In 2008, Alagic and Russell[1] proved the Donoho-Stark uncertainty principle for compact groups. In 2004, Özaydm and Przebinda[22] characterized the minimizers of the Hirschman-Beckner uncertainty principle and the Donoho-Stark uncertainty principle for locally compact abelian groups. The uncertainty principles have important applications in the theory of compressed sensing.[5]

Kac algebras were introduced independently by Vainerman and Kac[28–30] and Enock and Schwartz,[11–13] which generalized locally compact groups and their duals. Furthermore, Kustermans and Vaes introduced locally compact quantum groups.[19] Recently Crann and Kalantar proved the Hirschman-Beckner uncertainty principle and the Donoho-Stark uncertainty principle for unimodular locally compact quantum groups.[8]

Subfactor theory also provides a natural framework to study quantum symmetry. The group symmetry is captured by the subfactor arisen from the group crossed product construction. Ocneanu first pointed out the one-to-one correspondence between finite dimensional Kac algebras and finite-index, depth-two, irreducible subfactors. This correspondence was proved by Szymanski.[24] Enock and Nest generalized the correspondence to infinite dimensional compact (or discrete) type Kac algebras and infinite-index, depth-two, irreducible subfactors.[15] In general, a subfactor provides a pair of non-commutative spaces dual to each other and a Fourier transform $\mathcal{F}$ between them. It appears to be natural to study Fourier analysis for subfactors.

In Ref. 18, Jiang and the authors studied uncertainty principles for finite index subfactors in terms of planar algebras. We proved the Hirschman-Beckner uncertainty principle and the Donoho-Stark uncertainty principle for finite index subfactors. Furthermore, we introduced bi-shifts of biprojections[3,4,32] and use them to characterize the minimizers of the two uncertainty principles.

Moreover, we formalized Hardy's uncertainty principle using the minimizers of the Hirschman-Beckner uncertainty principle and proved it for finite index subfactors. The case for finite-index, depth-two, irreducible subfactors covers the results for finite dimensional Kac algebras. The quantum group community wondered whether the methods in Ref. 18 work for infinite-dimensional cases. That is the motivation of this paper.

In this paper, we introduce notions in subfactor theory to unimodular Kac algebras, such as biprojections and bi-shifts of biprojections. For example, the identity of a compact quantum group

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is a biprojection. The Fourier transform of a biprojection is a biprojection. We characterize the minimizers of the Hirschman-Beckner uncertainty principle and the Donoho-Stark uncertainty principle for unimodular Kac algebras containing biprojections. Furthermore, we prove Hardy's uncertainty principle for such Kac algebras. Our proofs utilize the ideas in subfactor theory[18] and the methods for locally compact quantum groups.[19]

**Main Theorem 1** (Proposition 3.6, Theorem 3.15). *Let $\mathbb{G}$ be a unimodular Kac algebra. For any nonzero $w$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, the following statements are equivalent:*

*(1)* $H(|w|^2) + H(|\mathcal{F}(w)|^2) = -4\|w\|_2^2 \log \|w\|_2$;
*(2)* $\mathcal{S}(w)\mathcal{S}(\mathcal{F}(w)) = 1$;
*(3) $w$ is an extremal bi-partial isometry;*
*(4) $w$ is a bi-shift of a biprojection.*

Conditions (1) and (2) are inequalities, in general, namely, the Hirschman-Beckner uncertainty principle and the Donoho-Stark uncertainty principle. When $\mathbb{G}$ has biprojections, the above four conditions characterize the minimizers of the Hirschman-Beckner uncertainty principle. In terms of these minimizers, we obtain Hardy's uncertainty principle for unimodular Kac algebras.

**Main Theorem 2** (Hardy's uncertainty principle, Theorem 3.18). *Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose that a non-zero $w$ in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ satisfies the conditions in Theorem 1. For any $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, if $|x^*| \le C|w^*|$ and $|\mathcal{F}(x)^*| \le C'|\mathcal{F}(w)^*|$, for some constants $C > 0$ and $C' > 0$, then $x$ is a scalar multiple of $w$.*

## II. PRELIMINARIES

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful tracial weight $\varphi$.

A closed densely defined operator $x$ affiliated with $\mathcal{M}$ is called $\varphi$-measurable if for all $\epsilon > 0$ there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset \mathcal{D}(x)$, and $\varphi(1-p) \le \epsilon$, where $\mathcal{D}(x)$ is the domain of $x$. Denote by $\widetilde{\mathcal{M}}$ the set of $\varphi$-measurable closed densely defined operators. Then $\widetilde{\mathcal{M}}$ is $*$-algebra with respect to a strong sum, strong product, and adjoint operation. If $x$ is a positive self-adjoint $\varphi$-measurable operator, then $x^\alpha \log x$ is $\varphi$-measurable for any $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$, where $\Re\alpha$ is the real part of $\alpha$.

The sets

$$N(\varepsilon, \varepsilon') = \{x \in \widetilde{\mathcal{M}} \mid \exists \text{ a projection } p \in \mathcal{M} : p\mathcal{H} \subseteq \mathcal{D}(x), \|xp\| \le \varepsilon, \varphi(1-p) \le \varepsilon'\},$$

where $\epsilon, \epsilon' > 0$, form a basis for the neighborhoods of 0 for a topology on $\widetilde{\mathcal{M}}$ that turns $\widetilde{\mathcal{M}}$ into a topological vector space. Now $\widetilde{\mathcal{M}}$ is a complete Hausdorff topological $*$-algebra and $\mathcal{M}$ is a dense subset of $\widetilde{\mathcal{M}}$.

For any positive self-adjoint operator $x$ affiliated with $\mathcal{M}$, we put

$$\varphi(x) = \sup_{n \in \mathbb{N}} \varphi\left(\int_0^n t de_t\right),$$

where $x = \int_0^\infty t de_t$ is the spectral decomposition of $x$. Then for $p \in [1, \infty)$, the noncommutative $L^p$ space $L^p(\mathcal{M})$ with respect to $\varphi$ is given by

$$L^p(\mathcal{M}) = \{x \text{ densely defined, closed, affiliated with } \mathcal{M} \mid \varphi(|x|^p) < \infty\}.$$

The $p$-norm $\|x\|_p$ of $x$ in $L^p(\mathcal{M})$ is given by $\|x\|_p = \varphi(|x|^p)^{1/p}$. We have that $L^p(\mathcal{M}) \subseteq \widetilde{\mathcal{M}}$. For more details on noncommutative $L^p$ space, we refer to Refs. 26 and 27.

Throughout the paper, we will use the results in Ref. 19 frequently. Let us recall the definition of locally compact quantum groups.

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful weight $\varphi$. Then $\mathfrak{N}_\varphi = \{x \in \mathcal{M} | \varphi(x^*x) < \infty\}$, $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^* \mathfrak{N}_\varphi$, and $\mathfrak{M}_\varphi^+ = \{x \ge 0 | x \in \mathfrak{M}_\varphi\}$. Denote by $\mathcal{H}_\varphi$ the Hilbert space by taking the closure of $\mathfrak{N}_\varphi$. The map $\Lambda_\varphi : \mathfrak{N}_\varphi \mapsto \mathcal{H}_\varphi$ is the inclusion map. We may use $\Lambda$ instead of $\Lambda_\varphi$ if there is no confusion.

A locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ consists of the following:

(1) a von Neumann algebra $\mathcal{M}$,
(2) a normal, unital, *-homomorphism $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$ such that $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$,
(3) a normal, semi-finite, faithful weight $\varphi$ such that $(\iota \otimes \varphi)\Delta(x) = \varphi(x)1$, $\forall x \in \mathfrak{M}_\varphi^+$; a normal, semi-finite, faithful weight $\psi$ such that $(\psi \otimes \iota)\Delta(x) = \psi(x)1$, $\forall x \in \mathfrak{M}_\psi^+$,

where $\overline{\otimes}$ denotes the von Neumann algebra tensor product and $\iota$ denotes the identity map. The normal, unital, *-homomorphism $\Delta$ is a comultiplication of $\mathcal{M}$, $\varphi$ is the left Haar weight, and $\psi$ is the right Haar weight.

We assume that $\mathcal{M}$ acts on $\mathcal{H}_\varphi$. There exists a unique unitary operator $W \in \mathcal{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$ which is known as the multiplicative unitary defined by

$$W^*(\Lambda_\varphi(a) \otimes \Lambda_\varphi(b)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(b)(a \otimes 1)), \quad a, b \in \mathfrak{N}_\varphi.$$

Moreover for any $x \in \mathcal{M}$, $\Delta(x) = W^*(1 \otimes x)W$.

For the locally compact quantum group $\mathbb{G}$, there exist an antipode $S$, a scaling automorphism group $\tau$, and a unitary antipode $R$ and there also exists a dual locally compact quantum group $\hat{\mathbb{G}} = (\hat{\mathcal{M}}, \hat{\Delta}, \hat{\varphi}, \hat{\psi})$ of $\mathbb{G}$. The antipode, the scaling group, and the unitary antipode of $\hat{\mathbb{G}}$ will be denoted by $\hat{S}$, $\hat{\tau}$, and $\hat{R}$, respectively. We refer to Refs. 19 and 20 for more details.

For any $\omega \in \mathcal{M}_*$, $\lambda(\omega) = (\omega \otimes \iota)(W)$ is the Fourier representation of $\omega$, where $\mathcal{M}_*$ is the Banach space of all bounded normal functionals on $\mathcal{M}$. For any $\omega$, $\theta$ in $\mathcal{M}_*$, the convolution $\omega * \theta$ is given by

$$\omega * \theta = (\omega \otimes \theta)\Delta.$$

In Ref. 21, Wang and the authors defined the convolution $x * y$ of $x \in L^p(\mathbb{G})$ and $L^q(\mathbb{G})$ for $1 \le p, q \le 2$. If the left Haar weights $\varphi$, $\hat{\varphi}$ of $\mathbb{G}$ and $\hat{\mathbb{G}}$, respectively, are tracial weights, we have that the convolution is well defined for $1 \le p, q \le \infty$ by the results in Ref. 21.

For any locally compact quantum group $\mathbb{G}$, the Fourier transform $\mathcal{F}_p : L^p(\mathbb{G}) \to L^q(\hat{\mathbb{G}})$ is well defined. (See Refs. 6 and 7 for the definition of Fourier transforms and Ref. 31 for the definition of the Fourier transform for algebraic quantum groups.) For any $x$ in $L^1(\mathbb{G})$, we denote by $x\varphi$ the bounded linear functional on $L^\infty(\mathbb{G})$ given by $(x\varphi)(y) = \varphi(yx)$ for any $y$ in $L^\infty(\mathbb{G})$. Recall that a projection $p$ in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ is a biprojection if $\mathcal{F}_1(p\varphi)$ is a multiple of a projection in $L^\infty(\hat{\mathbb{G}})$ (see Ref. 21 for more properties of biprojections).

## III. MAIN RESULTS

In this section, we will focus on a unimodular Kac algebra $\mathbb{G}$, which is a locally compact quantum group subject to the condition $\varphi = \psi$ that is tracial. (See Ref. 14 for more details.) We denote $L^\infty(\mathbb{G})$ by $\mathcal{M}$. The Fourier transform $\mathcal{F}_p$ from $L^p(\mathbb{G})$ to $L^q(\hat{\mathbb{G}})$ is given by $x \mapsto \lambda(x\varphi)$ for any $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. For a unimodular Kac algebra $\mathbb{G}$, we will denote by $\mathcal{F}$ the Fourier transform for simplicity.

For any $\varphi$-measurable element $x$ in $\widetilde{\mathcal{M}}$, the von Neumann entropy $H(|x|^2)$ is defined by

$$H(|x|^2) = -\varphi(x^* x \log x^* x).$$

*Proposition 3.1. Let $\mathbb{G}$ be a unimodular Kac algebra. Then for any $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, we have*

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \ge -4\|x\|_2^2 \log \|x\|_2.$$

*Proof.* By Lemma 18 in Ref. 26, we have that $\alpha \mapsto |x|^\alpha$ is differentiable for $\alpha > 0$. Now differentiating the Hausdorff-Young inequality[7]

$$\|\mathcal{F}(x)\|_q \le \|x\|_p, \quad x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G}), \quad p \in [1, 2], \quad \frac{1}{p} + \frac{1}{q} = 1,$$

with respect to $p$ and plug $p = 2$ into the result inequality, we can obtain that

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \ge -4\|x\|_2^2 \log \|x\|_2. \qquad \square$$

*Remark 3.2. For the proof of Proposition 3.1, we refer to Ref. 9 for the commutative case.*

For any $x \in \widetilde{\mathcal{M}}$, let $\mathcal{S}(x) = \varphi(\mathcal{R}(x))$, where $\mathcal{R}(x)$ is the range projection of $x$.

*Proposition 3.3. Let $\mathbb{G}$ be a unimodular Kac algebra. Then for any nonzero $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, we have*

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq 1.$$

*Proof.* We present two proofs here.

1. Note that for any $a, b \in (0, \infty)$, one has that $a - a \log a \leq b - a \log b$. Then by functional calculus, we obtain that for any $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$,
$$|x|^2 - |x|^2 \log |x|^2 \leq \frac{1}{\varphi(\mathcal{R}(|x|))}\mathcal{R}(|x|) - |x|^2 \log \frac{1}{\varphi(\mathcal{R}(|x|))}.$$
Suppose that $\|x\|_2 = 1$ and applying $\varphi$ to both sides of the inequality above, we have $H(|x|^2) \leq \log \varphi(\mathcal{R}(|x|))$. Since $\varphi(\mathcal{R}(|x|)) = \varphi(\mathcal{R}(x))$, we have the inequality $\log \mathcal{S}(x) \geq H(|x|^2)$, when $\|x\|_2 = 1$. By Proposition 3.1, we see that the proposition is true.
2. We assume that $\mathcal{S}(x), \mathcal{S}(\mathcal{F}(x)) < \infty$. Then by Hölder's inequality, we have
$$\begin{aligned}\|\mathcal{F}(x)\|_\infty \leq \|x\|_1 &\leq \|\mathcal{R}(x)\|_2\|x\|_2 \\ &= \mathcal{S}(x)^{1/2}\|\mathcal{F}(x)\|_2 \\ &\leq \mathcal{S}(x)^{1/2}\mathcal{S}(\mathcal{F}(x))^{1/2}\|\mathcal{F}(x)\|_\infty.\end{aligned}$$
Therefore $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq 1$. □

*Definition 3.4. An element $x$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is said to be extremal if $\|\mathcal{F}(x)\|_\infty = \|x\|_1$. We say that a nonzero element $x$ is an (extremal) bi-partial isometry if $x$ and $\mathcal{F}(x)$ are multiples of (extremal) partial isometries.*

*Proposition 3.5. Let $\mathbb{G}$ be a unimodular Kac algebra. If $x$ is extremal, then $x^*$ and $R(x)$ are extremal.*

*Proof.* By Proposition 2.4 in Ref. 20, we have

$$\begin{aligned}\|\mathcal{F}(x^*)\|_\infty &= \|\lambda(x^*\varphi)\|_\infty = \|\lambda(x^*\varphi)^*\|_\infty \\ &= \|\lambda(\overline{x^*\varphi}R)\|_\infty = \|\lambda(x\varphi R)\|_\infty \\ &= \|\hat{R}(\lambda(x\varphi))\|_\infty = \|\lambda(x\varphi)\|_\infty, \\ \|\mathcal{F}(R(x))\|_\infty &= \|\lambda(R(x)\varphi)\|_\infty = \|\lambda(x\varphi R)\|_\infty \\ &= \|\hat{R}(\lambda(x\varphi))\|_\infty = \|\lambda(x\varphi)\|_\infty,\end{aligned}$$

and

$$\varphi(|x|) = \varphi(|x^*|) = \varphi(R(|x|)) = \varphi(|R(x)|).$$

Therefore $x^*$ and $R(x)$ are extremal. □

*Proposition 3.6. Let $\mathbb{G}$ be a unimodular Kac algebra. For any nonzero $x$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, the following statements are equivalent:*

*(1)* $H(|x|^2) + H(|\mathcal{F}(x)|^2) = -4\|x\|_2^2 \log \|x\|_2$;
*(2)* $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = 1$;
*(3) $x$ is an extremal bi-partial isometry.*

*Proof.* "(1) ⇒ (3)." We assume that $\|x\|_2 = 1$. Now we follow the proof in Ref. 18. First, we define a complex function $F(z)$ for $z = \sigma + it$, $\frac{1}{2} < \sigma < 1$, as

$$F(z) = \hat{\varphi}(\mathcal{F}(w_x|x|^{2z})|\mathcal{F}(x)|^{2z}w^*_{\mathcal{F}(x)}),$$

where $w_x$ means the partial isometry in the polar decomposition of $x$. Note that $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, we see that $\mathcal{F}(w_x|x|^{2z})$ is well defined.

By Hölder's inequality and the Hausdorff-Young inequality,[7] we have

$$|F(\sigma + it)| \leq \|\mathcal{F}(w_x|x|^{2z})\|_{\frac{1}{1-\sigma}} \||\mathcal{F}(x)|^{2z}\|_{\frac{1}{\sigma}} \leq \||x|^{2\sigma}\|_{\frac{1}{\sigma}} \||\mathcal{F}(x)|^{2\sigma}\|_{\frac{1}{\sigma}} = 1.$$

This implies that $F(z)$ is bounded on $\frac{1}{2} < \sigma < 1$. Again by Lemma 18 in Ref. 26, we can follow the proof of Theorem 6.4 in Ref. 18 directly to obtain that

$$\hat{\varphi}(\mathcal{F}(x|x|)|\mathcal{F}(x)|\mathcal{F}(x)^*) = 1.$$

Now we see that

$$\begin{aligned} 1 &= \hat{\varphi}(\mathcal{F}(x|x|)|\mathcal{F}(x)|\mathcal{F}(x)^*) \\ &= (x|x|\varphi \otimes (|\mathcal{F}(x)|\mathcal{F}(x)^*)\hat{\varphi})(W) \\ &= (w_x|x|^2\varphi \otimes (|\mathcal{F}(x)|^2 w^*_{\mathcal{F}(x)})\hat{\varphi})(W) \\ &= (|x|^2\varphi \otimes (|\mathcal{F}(x)|^2)\hat{\varphi})((1 \otimes w^*_{\mathcal{F}(x)})W(w_x \otimes 1)) \\ &\leq (|x|^2\varphi \otimes (|\mathcal{F}(x)|^2)\hat{\varphi})(1 \otimes 1) = 1. \end{aligned} \tag{1}$$

Let $p = w_x^* w_x$ and $q = w^*_{\mathcal{F}(x)} w_{\mathcal{F}(x)}$. Since the equality holds in inequality (1), we have that

$$(p \otimes w^*_{\mathcal{F}(x)})W(w_x \otimes q) = p \otimes q.$$

Applying $|x|\varphi \otimes \iota$ to both sides of the equation above, we obtain that

$$w^*_{\mathcal{F}(x)}\mathcal{F}(x)q = \varphi(|x|)q,$$

i.e., $\mathcal{F}(x) = \varphi(|x|)w_{\mathcal{F}(x)}$. Similarly, we can obtain that $x = \hat{\varphi}(|\mathcal{F}(x)|)w_x$. Now we see that $x$ is an extremal bi-partial isometry.

"(3) $\Rightarrow$ (2)." Suppose $x$ is an extremal bi-partial isometry. Following the second proof in Proposition 3.3, we have

$$\begin{aligned} \|\mathcal{F}(x)\|_\infty = \|x\|_1 &= \|\mathcal{R}(x)\|_2\|x\|_2 \\ &= \varphi(\mathcal{R}(x))^{1/2}\|\mathcal{F}(x)\|_2 \\ &= \varphi(\mathcal{R}(x))^{1/2}\hat{\varphi}(\mathcal{R}(\mathcal{F}(x)))^{1/2}\|\mathcal{F}(x)\|_\infty. \end{aligned}$$

Hence $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = 1$.

"(2) $\Rightarrow$ (1)." Since (2) is stronger than (1), we see that (2) implies (1). □

*Definition 3.7. Let $\mathbb{G}$ be a unimodular Kac algebra with a biprojection $B$ in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. A projection $x$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is called a left shift of a biprojection $B$ if $\varphi(x) = \varphi(B)$ and $x * B = \varphi(B)x$. A projection $x$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is called a right shift of a biprojection $B$ if $\varphi(x) = \varphi(B)$ and $B * x = \varphi(B)x$.*

*Proposition 3.8. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose that there is a biprojection $B$ in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ and $x$ is a right (or left) shift of a biprojection $B$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. Then $x$ is an extremal bi-partial isometry.*

*Proof.* By Proposition 3.6, it suffices to show that $x$ is a minimizer of the uncertainty principle. Since $B * x = \varphi(B)x$, we have $\mathcal{F}(B)\mathcal{F}(x) = \varphi(B)\mathcal{F}(x)$, i.e., $\mathcal{R}(\mathcal{F}(x)) \leq \mathcal{R}(\mathcal{F}(B))$.
By Proposition 3.3, we have $\varphi(x)\hat{\varphi}(\mathcal{R}(\mathcal{F}(x))) \geq 1$ and

$$1 = \varphi(B)\hat{\varphi}(\mathcal{R}(\mathcal{F}(B))) \geq \varphi(x)\hat{\varphi}(\mathcal{R}(\mathcal{F}(x))) \geq 1.$$

Now we have $\mathcal{R}(\mathcal{F}(x)) = \mathcal{R}(\mathcal{F}(B))$. Hence $x$ is a minimizer of the uncertainty principle. □

*Definition 3.9. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose there exists a biprojection $B$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, we denote by $\widetilde{B}$ the range projection of $\mathcal{F}(B)$. A nonzero element $x$ in $L^\infty(\mathbb{G})$ is said* *to be a bi-shift of a biprojection $B$ if there exist a right shift $B_g$ of the biprojection $B$, a right shift $\widetilde{B}_h$ of the biprojection $\widetilde{B}$, and an element $y$ in $L^\infty(\mathbb{G})$ such that*

$$x = \widehat{\mathcal{F}}(\widetilde{B}_h) * (B_g y).$$

Now we will prove that the bi-shift of a biprojection described as above is a minimizer of the uncertainty principle. To see this, we need the following lemma.

*Lemma 3.10. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose $x$, $y$, and $\mathcal{R}(x), \mathcal{R}(y)$ are in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then*

$$(x * y)(x * y)^* \leq \|\mathcal{R}(x^*)\|_2^2 (xx^*) * (yy^*),$$

*and*

$$\mathcal{R}(x * y) \leq \mathcal{R}(\mathcal{R}(x) * \mathcal{R}(y)).$$

*Proof.* First, we assume that $x$ and $y$ are positive. Then $x \leq \|x\| \mathcal{R}(x)$ and $y \leq \|y\| \mathcal{R}(y)$. Now by computing the convolution,[21] we obtain that

$$\begin{aligned} x * y &= ((x\varphi)R \otimes \iota)(\Delta(y)) \\ &= ((x^{1/2}\varphi x^{1/2})R \otimes \iota)(\Delta(y)) \\ &\leq \|y\|((x^{1/2}\varphi x^{1/2})R \otimes \iota)(\Delta(\mathcal{R}(y))) \\ &= \|y\| x * \mathcal{R}(y) \\ &= \|y\|(\iota \otimes \mathcal{R}(y)\varphi R)(\Delta(x)) \\ &\leq \|x\|\|y\| \mathcal{R}(x) * \mathcal{R}(y). \end{aligned}$$

Therefore,

$$\mathcal{R}(x * y) \leq \mathcal{R}(\mathcal{R}(x) * \mathcal{R}(y)).$$

When $x$, $y$ are in the general case, we will show that

$$(x * y)(x * y)^* \leq \|\mathcal{R}(x^*)\|_2^2 (xx^*) * (yy^*). \tag{2}$$

If this inequality (2) is true, then we can see that the second inequality in the lemma is proved. By Lemma 9.5 in Ref. 19 and $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G}) \subset \mathfrak{N}_\varphi$, we have

$$\begin{aligned} &R((xx^*) * (yy^*)) \\ &\quad = R((xx^*\varphi)R \otimes \iota)(\Delta(yy^*)) \\ &\quad = (\iota \otimes \omega_{\Lambda(x),\Lambda(x)})\,(\Delta(R(y)^* R(y)) \\ &\quad \geq \frac{1}{\|\mathcal{R}(x^*)\|_2^2}((\iota \otimes \omega_{\Lambda(x),\Lambda(\mathcal{R}(x^*))})\Delta(R(y)))^*(\iota \otimes \omega_{\Lambda(x),\Lambda(\mathcal{R}(x^*))})\Delta(R(y)) \\ &\quad = \frac{1}{\|\mathcal{R}(x^*)\|_2^2}(R(x * y))^* R(x * y) \\ &\quad = \frac{1}{\|\mathcal{R}(x^*)\|_2^2} R((x * y)(x * y)^*), \end{aligned}$$

i.e.,

$$(x * y)(x * y)^* \leq \|\mathcal{R}(x^*)\|_2^2 (xx^*) * (yy^*). \qquad \square$$

*Proposition 3.11. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose $x$* is *the bi-shift of the biprojection $B$ as in Definition 3.9. Then $\mathcal{R}(x) = B_g$ and $\mathcal{R}(\mathcal{F}(x)) = \widetilde{B}_h$. Moreover, $x$ is a minimizer of the uncertainty principles.*

*Proof.* Note that $x = \widehat{\mathcal{F}}(\widetilde{B}_h) * (B_g y)$, we then have $\mathcal{F}(x) = \widetilde{B}_h \mathcal{F}(B_g y)$. This implies that $\mathcal{R}(\mathcal{F}(x)) \leq \widetilde{B}_h$. From the fact that $\widetilde{B}_h$ is a right shift of the biprojection $\widetilde{B}$, we see that $\hat{\varphi}(\widetilde{B}_h) = \hat{\varphi}(\widetilde{B})$.

On the other hand, we have $\mathcal{R}(\widehat{\mathcal{F}}(\widetilde{B}_h)) = \mathcal{R}(\widehat{\mathcal{F}}(\widetilde{B})) = \mathcal{R}(B) = B$ and by Lemma 3.10

$$\begin{aligned}\mathcal{R}(x) &\leq \mathcal{R}(\mathcal{R}(\widehat{\mathcal{F}}(\widetilde{B}_h)) * \mathcal{R}(B_g y)))\\ &\leq \mathcal{R}(B * B_g) = B_g.\end{aligned}$$

Now by Proposition 3.3, we see that

$$\begin{aligned}1 &\leq \varphi(\mathcal{R}(x))\hat{\varphi}(\mathcal{R}(\mathcal{F}(x))) \leq \varphi(B_g)\hat{\varphi}(\widetilde{B})\\ &= \varphi(B)\hat{\varphi}(\widetilde{B}) = 1.\end{aligned}$$

Therefore all inequalities above must be equalities and $\mathcal{R}(x) = B_g$ and $\mathcal{R}(\mathcal{F}(x)) = \widetilde{B}_h$. Moreover, $x$ is a minimizer of the uncertainty principles. □

*Proposition 3.12. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose $w$* is *a partial isometry in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ and $\mathcal{F}(w)$ is extremal. Then $w$ is an extremal bi-partial isometry.*

*Proof.* By Hölder's inequality, we have that $x$ is a multiple of a partial isometry if and only if $\|x\|_2^2 = \|x\|_\infty \|x\|_1$. To see that $\mathcal{F}(w)$ is a multiple of a partial isometry, it is enough to check that

$$\|\mathcal{F}(w)\|_2^2 = \|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1.$$

Since $\mathcal{F}(w)$ is extremal, we have

$$\|w\|_\infty = \|\widehat{\mathcal{F}}(\mathcal{F}(w))\|_\infty = \|\mathcal{F}(w)\|_1.$$

Now by Hölder's inequality and Hausdorff-Young inequality,[7] we obtain

$$\begin{aligned}\|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1 &\geq \|\mathcal{F}(w)\|_2^2 = \|w\|_2^2\\ &= \|w\|_\infty \|w\|_1\\ &\geq \|\mathcal{F}(w)\|_1 \|\mathcal{F}(w)\|_\infty.\end{aligned}$$

Hence $\|\mathcal{F}(w)\|_2^2 = \|\mathcal{F}(w)\|_\infty \|\mathcal{F}(w)\|_1$ and $\|\mathcal{F}(w)\|_\infty = \|w\|_1$. Now we see that $w$ is an extremal bi-partial isometry. □

**Theorem 3.13.** *Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose there is an extremal bi-partial isometry $w$ in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. Then*

$$(w * R(w)^*)(w^* * R(w)) = \|w\|_2^2 (ww^*) * (R(w)^* R(w)).$$

*Moreover* $\frac{1}{\|w\|_2^2} w * R(w)^*$ *is a partial isometry and* $\|w\|_1 = \frac{1}{\|w\|_2} \|w * R(w)^*\|_1$.

*Proof.* By Lemma 9.5 in Ref. 19, we have

$$\begin{aligned}&R((ww^*) * (R(w)^* R(w)))\\ &\quad = R((ww^* \varphi R \otimes \iota)(\Delta(R(w)^* R(w))))\\ &\quad = (\iota \otimes (\omega_{\Lambda(w),\Lambda(w)})(\Delta(ww^*)))\\ &\quad \geq \frac{1}{\|w\|_2^2}((\iota \otimes \omega_{\Lambda(w),\Lambda(|w|)})\Delta(w^*))^*((\iota \otimes \omega_{\Lambda(w),\Lambda(|w|)})\Delta(w^*))\\ &\quad = \frac{1}{\|w\|_2^2}(R(w\varphi R \otimes \iota)(\Delta(R(w^*))))^*(R(w\varphi R \otimes \iota)(\Delta(R(w^*))))\\ &\quad = \frac{1}{\|w\|_2^2} R(w * R(w^*))^* R(w * R(w^*))\\ &\quad = \frac{1}{\|w\|_2^2} R((w * R(w^*))(w^* * R(w))),\end{aligned}$$

i.e.,

$$(w * R(w)^*)(w^* * R(w)) \leq \|w\|_2^2 (ww^*) * (R(w)^* R(w)). \tag{3}$$

We will show that the traces of both sides are equal. For the right hand side, we have

$$\begin{aligned}\varphi((ww^*) * (R(w)^*R(w))) &= \varphi(ww^*)\varphi(R(w)^*R(w)) \\ &= \|w\|_2^2\|R(w)\|_2^2 = \|w\|_2^4.\end{aligned} \tag{4}$$

On the other hand, since $w$ is an extremal bi-partial isometry, we let $w = \widehat{\mathcal{F}}(x)$ for $x$ in $L^1(\widehat{\mathbb{G}})$. Then we have that

$$\mathcal{F}(w * R(w)^*) = \mathcal{F}(w)\mathcal{F}(R(w)^*) = xx^*.$$

Therefore $w * R(w)^* = \widehat{\mathcal{F}}(xx^*)$ and

$$\begin{aligned}\varphi((w * R(w)^*)(w^* * R(w))) &= \varphi(\widehat{\mathcal{F}}(xx^*)\widehat{\mathcal{F}}(xx^*)^*) \\ &= \hat{\varphi}(xx^*xx^*).\end{aligned}$$

Note that $x$ is a multiple of a partial isometry. We assume that $x = \mu x_0$ for some $\mu \in \mathbb{C}$ and a partial isometry $x_0$. Then $(xx^*)^2 = |\mu|^4|x_0|$. Since $w$ is a minimizer of the uncertainty principle, we have $\varphi(|w|)\hat{\varphi}(|x_0|) = 1$, i.e., $\hat{\varphi}(|x_0|) = \frac{1}{\|w\|_2^2}$. Meanwhile we have $\|w\|_2 = \|x\|_2$. Now we can obtain that $\|w\|_2^2 = |\mu|^2\frac{1}{\|w\|_2^2}$ and $|\mu| = \|w\|_2^2$.

Hence $\hat{\varphi}((xx^*)^2) = |\mu|^4\frac{1}{\|w\|_2^2} = \|w\|_2^6$, i.e., the trace of the left hand side of inequality (3) is $\|w\|_2^6$. By Eq. (4), we have that the trace of the right hand side of inequality (3) is $\|w\|_2^6$. This implies that

$$(w * R(w)^*)(w^* * R(w)) = \|w\|_2^2(ww^*) * (R(w)^*R(w)).$$

Now we show that $w * R(w)^*$ is a multiple of a partial isometry. By Hölder's inequality we have

$$\|w\|_2^6 = \|w * R(w)^*\|_2^2 \leq \|w * R(w)^*\|_\infty\|w * R(w)^*\|_1.$$

By the Hausdorff-Young inequality,[7] we obtain

$$\|w * R(w)^*\|_\infty = \|\widehat{\mathcal{F}}(xx^*)\|_\infty \leq \|xx^*\|_1 = \|x\|_2^2 = \|w\|_2^2$$

and by Young's inequality, we have

$$\|w * R(w)^*\|_1 \leq \|w\|_1\|R(w)^*\|_1 = \|w\|_1^2 = \|w\|_2^4.$$

Hence all equalities of the inequalities above hold and

$$\|w * R(w)^*\|_2^2 = \|w * R(w)^*\|_\infty\|w * R(w)^*\|_1.$$

Finally we see that $\frac{1}{\|w\|_2^2}w * R(w)^*$ is a partial isometry and

$$\|w\|_1 = \|w\|_2^2 = \|\frac{1}{\|w\|_2^2}w * R(w)^*\|_1. \qquad \square$$

Corollary 6.12 in Ref. 18 is a useful tool to find an extremal bi-partial isometry in a given element. However, that result is not true in general. Instead, we have the following result for unimodular Kac algebras:

*Proposition 3.14. Let* $\mathbb{G}$ *be a unimodular Kac algebra. Suppose* $w \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ *such that* $\|w * R(w^*)\|_\infty = \|w\|_2^2$, $\|w\|_2^2$ *is a point spectrum of* $w * R(w^*)$, *and* $Q$ *is the spectral projection of* $|w * R(w^*)|$ *with spectrum* $\|w\|_2^2$. *Then* $Q$ *is a biprojection.*

*Proof.* We assume that $\|w\|_2 = 1$. Take $x = (w^* * R(w))(w * R(w^*))$. Then $\|x\|_\infty = 1$. Note that $\lim_{k\to\infty} x^k = Q$ in the strong operator topology and $Q$ is a projection. By Hölder's inequality and Young's inequality, we have that

$$\|x^k\|_1 \leq \|w * R(w^*)\|_1 \leq \|w\|_1^2, \quad k = 1, 2, \ldots. \tag{5}$$

Note that $x^k \leq x$, and by dominant convergence theorem,

$$\lim_{k\to\infty}\|x^k - Q\|_1 = 0.$$

By the Hausdorff-Young inequality,[7] we obtain that

$$\lim_{k\to\infty} \|\mathcal{F}(x^k) - \mathcal{F}(Q)\|_\infty = 0.$$

Note that $\mathcal{F}(x) = (\mathcal{F}(w^*)\mathcal{F}(w^*)^*) * (\mathcal{F}(w)\mathcal{F}(w)^*) \geq 0$ and $\mathcal{F}(x^k) = \mathcal{F}(x)^{*(k)} \geq 0$. Thus

$$\mathcal{F}(Q) = \lim_{k\to\infty} \mathcal{F}(x^k) \geq 0.$$

Moreover,

$$\|\mathcal{F}(x^k)\|_1 = \|((\mathcal{F}(w^*)\mathcal{F}(w^*)^*) * (\mathcal{F}(w)\mathcal{F}(w)^*))^{*(k)}\|_1 = \|w\|_2^{4k} = 1.$$

By the Hausdorff-Young inequality and Eq. (5), $\|\mathcal{F}(x^k)\|_\infty \leq \|w\|_1^2$. So

$$\|\mathcal{F}(Q)\|_1 \leq \lim_{k\to\infty} \|F(x^k)\|_1 = 1.$$

By the Hausdorff-Young inequality, $1 = \|Q\|_\infty \leq \|\mathcal{F}(Q)\|_1 = 1$. So $\mathcal{F}(Q)$ is extremal.

By Proposition 3.12 and Corollary 3.16, we see that $Q$ is a biprojection. □

**Theorem 3.15.** *Let $\mathbb{G}$ be a unimodular Kac algebra and $w \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then $w$ is an extremal bi-partial isometry if and only if $w$ is a bi-shift of a biprojection. Furthermore, if $w$ is a projection, then it is a left (or right) shift of a biprojection.*

*Proof.* Suppose $w$ is an extremal bi-partial isometry and $w$ is a partial isometry. Let

$$B = \frac{1}{\|w\|_2^4}(w * R(w)^*)(w^* * R(w)).$$

By Theorem 3.13, we have that $\frac{1}{\|w\|_2^2} w * R(w)^*$ is a partial isometry and hence $B$ is a projection.

Now we compute the Fourier transform of $B$,

$$\begin{aligned}
\mathcal{F}(B) &= \frac{1}{\|w\|_2^4}\mathcal{F}((w * R(w)^*)(w^* * R(w)))\\
&= \frac{1}{\|w\|_2^2}\mathcal{F}((ww^*) * (R(w)^*R(w)))\\
&= \frac{1}{\|w\|_2^2}\mathcal{F}(ww^*)\mathcal{F}(R(w)^*R(w))\\
&= \frac{1}{\|w\|_2^2}\mathcal{F}(ww^*)\mathcal{F}(ww^*)^*.
\end{aligned}$$

Hence it suffices to check that $\mathcal{F}(ww^*)$ is a multiple of partial isometry. First we observe that $\mathcal{F}(w)$ is an extremal bi-partial isometry. By Theorem 3.13, we have that $\mathcal{F}(w) * \hat{R}(\mathcal{F}(w)^*)$ is a multiple of partial isometry and

$$\mathcal{F}(w) * \hat{R}(\mathcal{F}(w)^*) = \mathcal{F}(w) * \mathcal{F}(w^*) = \mathcal{F}(ww^*).$$

Therefore $\mathcal{F}(B)$ is a multiple of a projection and $B$ is a biprojection.

Now we define $B_g = ww^*$, then $B_g$ is a projection. We are going to show that $B_g$ is a right shift of the biprojection $B$. By Theorem 3.13, we have that $\frac{1}{\|w\|_2^2} B_g * R(B_g) = B$. Computing the trace on both sides, we have $\frac{1}{\|w\|_2^2}\varphi(B_g)^2 = \varphi(B)$. Note that $\varphi(B_g) = \|w\|_2^2$, and we see that

$$\varphi(B) = \frac{1}{\|w\|_2^2}(\|w\|_2^2)^2 = \|w\|_2^2 = \varphi(B_g).$$

Recall that $\mathcal{F}(w)$ is an extremal bi-partial isometry. We have $\|\mathcal{F}(w)\|_\infty = \|w\|_1$, and $\frac{1}{\|w\|_2^2}\mathcal{F}(w)$ is a partial isometry. By Theorem 3.13, we see that

$$\frac{1}{\|\frac{1}{\|w\|_2^2}\mathcal{F}(w)\|_2^2}\frac{\mathcal{F}(w)}{\|w\|_2^2} * \frac{\hat{R}(\mathcal{F}(w)^*)}{\|w\|_2^2} = \frac{1}{\|w\|_2^2}\mathcal{F}(ww^*) = \frac{1}{\|w\|_2^2}\mathcal{F}(B_g)$$

is a partial isometry.

Hence we obtain that

$$\begin{aligned}\mathcal{F}(B_g) &= \frac{1}{\|w\|_2^4}\mathcal{F}(B_g)\mathcal{F}(B_g)^*\mathcal{F}(B_g)\\ &= \frac{1}{\|w\|_2^4}\mathcal{F}(B_g)\mathcal{F}(R(B_g))\mathcal{F}(B_g)\\ &= \frac{1}{\|w\|_2^4}\mathcal{F}(B_g * R(B_g) * B_g)\end{aligned}$$

and $\frac{1}{\|w\|_2^4}B_g * R(B_g) * B_g = B_g$. Then

$$B * B_g = \frac{1}{\|w\|_2^2}B_g * R(B_g) * B_g = \|w\|_2^2 B_g = \varphi(B_g)B_g.$$

Therefore $B_g$ is a right shift of the biprojection $B$.

Let $\widetilde{B}_h = \frac{1}{\|w\|_2^4}\mathcal{F}(w)\mathcal{F}(w)^*$. We have $\widehat{\mathcal{F}}(\widetilde{B}_h) = \frac{1}{\|w\|_2^4} w * R(w)^*$. Finally we will find a form of $w$ in terms of $B_g$ and $\widetilde{B}_h$,

$$\begin{aligned}\mathcal{F}(w) &= \frac{1}{\|w\|_2^4}\mathcal{F}(w)\mathcal{F}(w)^*\mathcal{F}(w)\\ &= \frac{1}{\|w\|_2^4}\mathcal{F}(w)\mathcal{F}(R(w)^*)\mathcal{F}(w)\\ &= \frac{1}{\|w\|_2^4}\mathcal{F}(w * R(w)^* * w).\end{aligned}$$

Then $w = \frac{1}{\|w\|_2^4} w * R(w)^* * w = \widehat{\mathcal{F}}(\widetilde{B}_h) * (B_g w)$. □

*Corollary 3.16. Let $\mathbb{G}$ be a unimodular Kac algebra. If $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ and $\mathcal{F}(x)$ are positive and $\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) = 1$, then $x$ is a biprojection.*

*Lemma 3.17. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose $B$ is a biprojection in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ and $\widetilde{B}$ is the range projection of $\mathcal{F}(B)$ in $L^1(\widehat{\mathbb{G}}) \cap L^\infty(\widehat{\mathbb{G}})$. If $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ such that $\mathcal{R}(x) = B$ and $\mathcal{R}(\mathcal{F}(x)) = \widetilde{B}$, then $x$ is a multiple of $B$.*

*Proof.* By the assumption, we have $Bx = x$ and $\mathcal{F}(B)\mathcal{F}(x) = \varphi(B)\mathcal{F}(x)$, i.e., $B * x = \varphi(B)x$. Hence $B * Bx = \varphi(B)x$. Note that $B$ is biprojection, then $B$ is a group-like projection,[21] i.e.,

$$\Delta(B)(B \otimes 1) = \Delta(B)(1 \otimes B) = B \otimes B.$$

Now we have

$$\begin{aligned}\varphi(B)x = B * (Bx) &= (\varphi \otimes \iota)((B \otimes 1)\Delta(Bx))\\ &= (\varphi \otimes \iota)((1 \otimes B)\Delta(B)\Delta(x))\\ &= \varphi(Bx)B,\end{aligned}$$

i.e., $x$ is a multiple of $B$. □

**Theorem 3.18.** *[Hardy's uncertainty principle]. Suppose $\mathbb{G}$ is a unimodular Kac algebra and $w \in \mathbb{G}$ is a bi-shift of biprojection. For any $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, if $|x^*| \le C|w^*|$ and $|\mathcal{F}(x)^*| \le C'|\mathcal{F}(w)^*|$, for some constants $C > 0$ and $C' > 0$, then $x$ is a scalar multiple of $w$.*

*Proof.* Suppose $w \in L^\infty(\mathbb{G})$ is a bi-shift of a biprojection $B$. Let $\widetilde{B}$ be the range projection of $\mathcal{F}(B)$, and $B_g$ and $\widetilde{B}_h$ be right shifts of biprojections $B$ and $\widetilde{B}$, respectively, such that $\mathcal{R}(w) \le B_g$ and

$\mathcal{R}(\mathcal{F}(w)) \leq \widetilde{B}_h$. If $x$ satisfies the assumption, then $\mathcal{R}(x) \leq B_g$ and $\mathcal{R}(\mathcal{F}(x)) \leq \widetilde{B}_h$. By Theorem 1, we have that $\mathcal{R}(w) = \mathcal{R}(x) = B_g$ and $\mathcal{R}(\mathcal{F}(w)) = \mathcal{R}(\mathcal{F}(x)) = \widetilde{B}_h$.

We assume that $x \neq 0$. Then $x^* w$ and $w^* w$ are nonzero and

$$\begin{aligned}\mathcal{R}(\mathcal{F}(x^* w)) &= \mathcal{R}(\mathcal{F}(x^*) * \mathcal{F}(w))\\ &= \mathcal{R}\,(\hat{R}(\mathcal{F}(x))^* * (\mathcal{F}(w))\\ &\leq \mathcal{R}(\hat{R}(\widetilde{B}_h) * \widetilde{B}_h).\end{aligned}$$

By Theorem 3.13, $\hat{R}(\widetilde{B}_h) * \widetilde{B}_h$ is a multiple of a projection and

$$\mathcal{S}(\mathcal{F}(x^* w)) \leq \mathcal{S}(\hat{R}(\widetilde{B}_h) * \widetilde{B}_h) = \mathcal{S}(\widetilde{B}_h) = \mathcal{S}(\mathcal{F}(w)).$$

Then

$$1 \leq \mathcal{S}(x^* w)\mathcal{S}(\mathcal{F}(x^* w)) = \mathcal{S}(w^* x)\mathcal{S}(\mathcal{F}(x^* w)) \leq \mathcal{S}(w^*)\mathcal{S}(\mathcal{F}(w)) = 1.$$

Hence we have

$$\mathcal{S}(w^* x) = \mathcal{S}(w^*); \quad \mathcal{S}(\mathcal{F}(x^* w)) = \mathcal{S}(\mathcal{F}(w)) = \mathcal{S}(\hat{R}(\widetilde{B}_h) * \widetilde{B}_h).$$

Therefore

$$\mathcal{R}(w^* x) = \mathcal{R}(w^*), \quad \mathcal{R}(\mathcal{F}(x^* w)) = \mathcal{R}(\hat{R}(\widetilde{B}_h) * \widetilde{B}_h).$$

Hence $x^* w$ is a bi-shift of a biprojection. Similarly $w^* w$ is a bi-shift of a biprojection. Moreover,

$$\mathcal{R}(w^* x) = \mathcal{R}(w^* w), \quad \mathcal{R}(\mathcal{F}(x^* w)) = \mathcal{R}(\mathcal{F}(w^* w)).$$

By a similar argument, we have that $R(w^* w)^* * (x^* w)$ and $R(w^* w)^* * (w^* w)$ are bi-shifts of biprojections and

$$\begin{aligned}\mathcal{R}(R(w^* w)^* * (w^* x)) &= \mathcal{R}(R(w^* w)^* * (w^* w)),\\ \mathcal{R}(\mathcal{F}(R(w^* w)^* * (x^* w))) &= \mathcal{R}(\mathcal{F}(R(w^* w)^* * (w^* w))).\end{aligned} \tag{6}$$

By Theorem 3.13, we have that $R(w^* w)^* * (w^* w)$ is a multiple of a biprojection $Q$. By Lemma 3.17 and Eq. (6), we have that $R(w^* w)^* * (w^* x)$ is a multiple of biprojection $Q$ and then $R(w^* w)^* * (x^* w)$ is a multiple of biprojection $Q$. Observe that both $x$ and $w$ are multiples of $(Q * (w^* w))w^*$. Therefore $x$ is a scalar multiple of $w$. □

*Corollary 3.19. Let $\mathbb{G}$ be a unimodular Kac algebra. Suppose $B$ is a biprojection in $L^1(\mathbb{G})$ and $\widetilde{B}$ is the range projection of $\mathcal{F}(B)$ in $L^1(\widehat{\mathbb{G}})$. Let $B_g$ and $\widetilde{B}_h$ be right shifts of biprojections $B$ and $\widetilde{B}$, respectively. Then there is at most one element $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ up to a scalar such that the range projection of $x$ is contained in $B_g$ and the range projection of $\mathcal{F}(x)$ is contained in $\widetilde{B}_h$.*

*Remark 3.20. Therefore we can use the supports of $B_g$ and $\widetilde{B}_h$ to define a bi-shift of a biprojection. It is independent of the choice of $y$ in Definition 3.9.*


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