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Quon Language: Surface Algebras and Fourier Duality

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Abstract: Quon language is a 3D picture language that we can apply to simulate mathematical concepts. We introduce the surface algebras as an extension of the notion of planar algebras to higher genus surface. We prove that there is a unique one-parameter extension. The 2D defects on the surfaces are quons, and surface tangles are transformations. We use quon language to simulate graphic states that appear in quantum information, and to simulate interesting quantities in modular tensor categories. This simulation relates the pictorial Fourier duality of surface tangles and the algebraic Fourier duality induced by the S matrix of the modular tensor category. The pictorial Fourier duality also coincides with the graphic duality on the sphere. For each pair of dual graphs, we obtain an algebraic identity related to the S matrix. These identities include well-known ones, such as the Verlinde formula; partially known ones, such as the 6j-symbol self-duality; and completely new ones.

1. Introduction

Quon language is a 3D picture language that we can apply to simulate mathematical concepts [LWJ17,JL18]. It was designed to answer a question in quantum information, where the underling symmetry is the group $\mathbb{Z}_2$ for qubits and $\mathbb{Z}_d$ for qudits. One can consider the quon language as a topological quantum field theory (TQFT) [Ati88,Wit88, RT91,TV92,Tur16] in the 3D space with lower dimensional defects, and a quon as a 2D defect on the boundary of the 3D TQFT. The underlying symmetry of the 3D picture language can be generalized to more general quantum symmetries captured by subfactor theory [JS97,EK98].

Jones introduced subfactor planar algebras as a topological axiomatization of the standard invariants of subfactors [Jon99]. One can consider a planar algebra as a 2D topological quantum field theory (TQFT) on the plane with line defects. A subfactor planar algebra is always spherical, so the theory also extends to a sphere.

Table 1. Fourier duality on MTC and 1-quons

MTC	Quon
Simple objects (Irr)	Ortho-normal-basis
Multiplication	Multiplication
Fusion	Convolution
S matrix	SFT $\mathfrak{F}_S$
Full subcategories $\mathscr{C}_K$	Biprojections P_K
Müger's center $\mathscr{C}_{\hat{K}}$	$P_{\hat{K}}$
$\hat{\hat{K}} = K$	$\mathfrak{F}_S^2(P_K) = P_K$
$\dim \mathscr{C}_{\hat{K}}$	$\mathcal{S}(P_K)$
$\dim \mathscr{C}_K \dim \mathscr{C}_{\hat{K}} = \dim C$	$\mathcal{S}(P_K)\mathcal{S}(P_{\hat{K}}) = \delta^2$

A vector in the planar algebra of a subfactor is a morphism in the bi-module category associated with the subfactor. From this point of view, a morphism is usually represented as a disc with m boundary points on the top and n boundary points at the bottom, and considered as a transformation with m inputs and n outputs,

In the 3D quon language, we consider these morphisms in planar algebras as quons, and consider planar tangles as transformations. This interpretation is similar to the original definition of Jones, which turns out to be more compatible with the notions in quantum information. A planar tangle has multiple input discs and one output disc. So it represents a transformation from multiple quons to one quon.

In quantum information, we usually consider multiple qubits and their transformations. To simulate multiple quon transformations, we generalize planar tangles to spherical tangles with multiple input discs and output discs. When we compose such tangles, we will obtain higher genus-surfaces. So we further extend the notion of planar algebras to higher genus surfaces, that we call surface algebras. The theory of planar algebras becomes the local theory of surface algebras.

There is a freedom to define the partition function of a sphere in this extension, denoted by ζ. We show that the partition function of the genus-g surface is ζ^{1-g}, which detects the topological non-triviality. We prove that any non-degenerate spherical planar algebra has a unique extension of to a surface algebra for any non-zero ζ. Therefore a subfactor not only defines a spherical planar algebra, but also a surface algebra parameterized by ζ. The fruitful theory of subfactors provides many interesting examples.

In this paper, we take the subfactor to be the quantum double of a unitary modular tensor category $\mathscr{C}$, also known as the Drinfeld double [Dri86]. Then the 2-box space of the planar algebra of the subfactor is isomorphic to $L^2(Irr)$, where Irr denotes the set of irreducible objects of $\mathscr{C}$. Xu and the author proved that the associated subfactor planar algebra is unshaded [LX16]. Thus the 2-box space becomes the 4-box space of the unshaded planar algebra, denoted by $\mathscr{S}_4$. The unshaded condition is crucial to define the string Fourier transform (SFT) on one space. Moreover, we proved that the SFT on $\mathscr{S}_4$ is identical to the modular S transformation of the MTC $\mathscr{C}$. Both transformations have been considered as a generalization of the Fourier transform from different point of views. This identification relates the two different Fourier dualities for MTC and subfactors. We restrict the 1-quon space as $\mathscr{S}_4$, in order to study this pair of Fourier dualities. We list the correspondence for 1-quons in Fig. 1; see Sect. 4 for details.

Verlinde proposed that the S matrix diagonalizes the fusion in the framework of CFT, known as the Verlinde formula [Ver88]. The Fourier duality on 1-quons gives a conceptual proof addressing Verlinde's original observation given by the Lines 3–5 in Fig. 1.
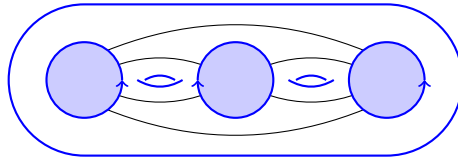
Fig. 1. Example: This is a genus-2 tangle with two input discs and one output disc. The genus-2 surface and the discs are drawn in thick blue lines. The strings are drawn in black

Jiang, Wu, and the author studied the Fourier analysis of the SFT on subfactors in [JLW16]. Through this identification, we obtain many inequalities for the S matrix, which will be discussed in a coming paper. It is particularly interesting that the ∞-1 Hausdorff–Young inequality for SFT gives an important inequality for the S matrix in unitary MTC proved by Terry Gannon [Gan05]:

$$\|\mathfrak{F}_S(\beta_Y)\|_\infty \leq \delta^{-1}\|\beta_Y\|_1 \tag{1}$$

$$\Rightarrow \left|\frac{S_X^Y}{S_X^0}\right| \leq \frac{S_0^Y}{S_0^0}. \tag{2}$$

The equality condition has been used by Müger to define the center of full subcategories in $\mathscr{C}$, known as Müger's center [Mug03b]. On the other hands, Bisch and Jones introduced biprojections for subfactors and planar algebras by studying intermediate subfactors [Bis94,BJ97]. The Lines 6–10 in Fig. 1 identifies full subcategories in $\mathscr{C}$ with biprojections in $\mathscr{S}_4$, and the several corresponding results between Müger's center and biprojections.

We extend the correspondence between $\mathfrak{F}_S$ and S from 1-quons to n-quons using surface algebras, see Theorem 6.1:

$$\begin{array}{ccc}
\text{surface tangles} & \xrightarrow{\quad Z \quad} & \text{graphic quons} \\
\vec{\mathfrak{F}}_S \Big\downarrow & & \vec{S} \Big\downarrow \\
\text{surface tangles} & \xrightarrow{\quad Z \quad} & \text{graphic quons}
\end{array}.$$

The left side is pictorial and $\mathfrak{F}_S$ could be considered as a global $90°$ rotation. The right side is algebraic and the S-matrix is generalization of the discrete Fourier transform. The partition function Z is a functor relating the pictorial Fourier duality and the algebraic Fourier duality.

In particular, the algebraic Fourier duality between the two qudit resource state $|GHZ\rangle$ and $|Max\rangle$ in quantum information turns out to be a pictorial Fourier duality in quon language [LWJ17], see Sect. 5.3 for details:

$$Max_{n,g} = \vec{\mathfrak{F}}_S(GHZ_{n,g}) \quad \Rightarrow \quad |Max\rangle_{n,g} = \vec{S}|GHZ\rangle_{n,g}. \tag{3}$$

Now this result also apply to unitary MTCs. Comparing the coefficients, we obtain the generalized Verlinde formula:

$$|Max\rangle_{n,g} = \vec{S}|GHZ\rangle_{n,g} \quad \Rightarrow \quad \dim(\vec{X},g) = \sum_{X \in Irr} (\prod_{i=1}^n S_{X_i}^X)(S_X^1)^{2-n-2g}. \tag{4}$$

The generalized Verlinde formula was first proved by Moore and Seiberg in CFT [MS89]. Here we prove it for any unitary MTC and any genus g. We refer the readers to an interesting discussion about various versions of the Verlinde formula on MathOverflow: https://mathoverflow.net/questions/151221/verlindes-formula.

Moreover, for each oriented graph Γ on the sphere, we define a surface tangle as a fat graph of Γ. Then its SFT becomes a fat graph of $\hat{\Gamma}$, where $\hat{\Gamma}$ is the dual graph of Γ forgetting the orientation. So the pictorial Fourier duality also coincide with the graphical duality. Therefore, we obtain one algebraic identity as the algebraic Fourier duality of quons from any graph Γ. We give some examples including well known ones, such as the Verlinde formula mentioned above; partially known ones; and completely new ones.

If the graph Γ is the tetrahedron, then the graphic self-duality of the tetrahedron gives an algebraic $6j$-symbol self-duality for unitary MTCs, see Sect. 6 for details:

$$\left|\begin{pmatrix} X_6 & X_5 & X_4 \\ X_3 & X_2 & X_1 \end{pmatrix}\right|^2 = \sum_{\vec{Y}\in Irr^6} \left(\prod_{k=1}^{6} S_{X_k}^{Y_k}\right) \left|\begin{pmatrix} Y_1 Y_2 Y_3 \\ Y_4 Y_5 Y_6 \end{pmatrix}\right|^2. \tag{5}$$

In the special case of quantum $SU(2)$, the identity for the 6j-symbol self-duality was discovered by Barrett in the study of quantum gravity [Bar03], based on an interesting identity of Robert [Rob95]. Then the identity was generalized to some other cases related to $SU(2)$ in [FNR07]. A general case for MTCs has been conjectured by Shamil Shakirov, which we answer positively here.

We obtain a sequence of new algebraic self-dual identities from a sequence of self-dual graphs, see Sect. 6 for details:

$$\left|\begin{pmatrix} X_{2n} & X_{2n-1} \cdots X_n \\ X_n & X_{n-1} \cdots X_1 \end{pmatrix}\right|^2 = \sum_{\vec{Y}\in Irr^{2n}} \left(\prod_{k=1}^{2n} S_{X_k}^{Y_k}\right) \left|\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_n \\ Y_{n+1} Y_{n+2} \cdots Y_{2n} \end{pmatrix}\right|^2. \tag{6}$$

2. Surface Algebras

Jones introduced planar algebras in [Jon99], and proved that for each extremal, finite index subfactor $\mathcal{N} \subset \mathcal{M}$, one can construct a shaded, spherical planar algebra. Shaded means that the regions of planar tangles have an alternating shading colored by $\mathcal{N}$ and $\mathcal{M}$. Spherical means that the planar tangles can be defined on the sphere.

In this section, we are going to extend spherical planar algebras from the sphere to higher genus surfaces, which are the boundaries of 3-manifolds in the 3D space. The theory of spherical planar algebras become the genus-0 case.

To simply the notation, we only define the single color case and the ground field is $\mathbb{C}$. One can generalize these definitions to multi-color cases over a general field.

2.1. Surface tangles. If we consider a planar tangle as a spherical tangle by one point compactification of the plane, then the complement of the planar tangle becomes a disc on the sphere. The induced orientation of the boundary of the output disc will be changed. Thus we use anti-clockwise and clockwise orientations of boundary of discs to indicate input and output respectively.

The composition of planar tangles is still a planar tangle. In this case, the number of output disc is always one. If we allow spherical tangles to have multiple input discs and

output discs, then we will obtain tangles on higher genus surfaces when we compose these spherical tangles. We give a generalization of planar tangles in [Jon99] to surface tangles:

Definition 2.1. A genus-g tangle, for $g \in \mathbb{N}$, is a 3-manifold in the 3D space whose boundary is a genus-g surface. The surface consists of a finite (possibly empty) set of smooth closed discs $\mathcal{D}(T)$. For each disc $D \in \mathcal{D}(T)$, its boundary ∂D of is an oriented circle with a number of marked points. There is also a finite set of disjoint smoothly embedded curves called strings, which are either closed curves, or the end points of the strings are different marked points of discs. Each marked point is an end-point of some string, which meets the boundary of the corresponding disc transversally.

The connected components of the complement of the strings and discs are called regions. The connected component of the boundary of a disc, minus its marked points, will be called the intervals of that disc. To each disc there is a distinguished interval on its boundary. The distinguished interval is marked by an arrow $\rightarrow$, which also indicates the orientation.

A surface tangle is a disjoint union of finitely many higher-genus tangles. See Fig. 1 for an example of a surface tangle.

Remark. One can impose additional data to color the regions and the strings. In sub-factor theory, an alternating shading of the regions is preferred. Therefore the number of boundary points of each disc is even. In tensor categories, the strings are colored by simple objects. In 2-categories, one has multiple colors for regions and strings. In these cases, the boundary condition ∂D will be colored too.

Notation 2.2. *We consider the 3D topological isotopy by orientation-preserving diffeomorphisms in the 3D space.*

Notation 2.3. *Let $\partial \mathcal{D}$ be the set of boundary conditions of discs, i.e., the equivalent classes of ∂D modulo 3D topological isotopy. We say a disc D is an input (respectively, output) disc, if the orientation of ∂D is anti-clockwise (respectively, clockwise). Let $\mathcal{D}_I$ and $\mathcal{D}_O$ be the sets of input discs and output discs respectively.*

Notation 2.4. *Let r be a reflection by a plane in the 3D space.*

The reflection r is unique up to topological isotopy in the 3D space. Moreover r maps a surface tangle to a surface tangle and reverses the orientation of the boundary of discs. Thus r switches $\mathcal{D}_I$ and $\mathcal{D}_O$.

We define another two elementary operations for surface tangles,

(1) Tensor: taking a disjoint union of two surface tangles.
(2) Contraction: gluing two discs of a surface tangle whose boundaries are mirror images.

Definition 2.5. If a surface tangle T is a disjoint union of two surface tangles T_1 and T_2, then we call T a tensor product of T_1 and T_2.

Definition 2.6. Suppose T_1 is a surface tangle, D_i is an input disc of T_1 and D_o is an output disc of T_1, such that $\partial D_i = r(\partial D_o)$ in $\partial \mathcal{D}$. Suppose T_2 is a genus-0 tangle with one input disc D_i' and one out put disc D_o', so that D_i' and D_o' are identical to D_o and D_i respectively. Moreover, the two discs are the intersection of T_1 and T_2. Let T be the union T_1 and T_2 forgetting the data on the two discs at the intersection. Then T is a surface tangle. We call T a contraction of T_1 at D_i and D_o.

Modulo topological isotopy in the 3D space, there are inequivalent tensors and contractions. For example,

$$T = \qquad , \ T' = \qquad , \ T_1 = \qquad .$$
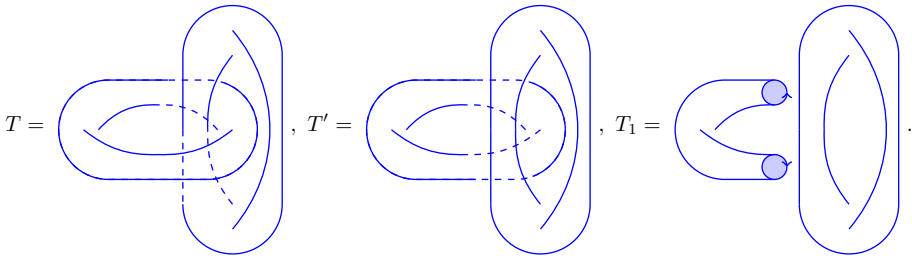
Both surface tangles T and T' are disjoint union of the torus, but they are not isotopic. Moreover, both of them are contractions of the surface tangle T_1.

We consider a planar tangle as a genus-0 surface tangle with one output disc. For example, the Fourier transform on subfactors was introduced by Ocneanu in terms of paragroups [Ocn89]. In planar algebras, it turns out to be a one-string rotation of the diagram, called the string Fourier transform (SFT), denoted by $\mathfrak{F}_S$. The planar tangle of the SFT has one input disc and one output disc. In the case that each disc has four marked points, the planar tangle is given by

We represent this planar tangle as a genus-0 surface tangle with its input disc on the top and its output disc at the bottom,

$$\mathfrak{F}_S = \qquad\qquad\qquad\qquad\qquad\qquad\qquad (7)$$

We ignore the blue thick lines indicating the shape of the surface, if there is no confusion as above.

The composition of planar tangles can be decomposed as a contraction and a tensor of genus-0 surface tangles.

2.2. Surface algebras.
We define surface algebras as finite dimensional representations of surface tangles whose target spaces are indexed by the boundary condition $\partial \mathcal{D}$:
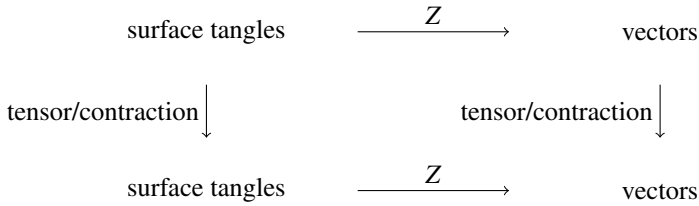
Definition 2.7. A surface algebra $\mathscr{S}_\bullet$ is a representation Z of surface tangles on the tensor products of a family of finite dimensional vector spaces $\{\mathscr{S}_i\}_{i \in \partial \mathcal{D}}$, having the following axioms:

(1) Boundary condition: For a surface tangle T, $Z(T)$ is a vector in $\bigotimes\limits_{D \in \mathcal{D}(T)} \mathscr{S}_{\partial D}$.

(1') Second boundary condition: If T has no discs, then $Z(T)$ is a scalar in the ground field.

(2) Duality: For any $i \in \partial \mathcal{D}$, $\mathscr{S}_{r(i)}$ is the dual space of $\mathscr{S}_i$.

(3) Isotopy invariance: The representation Z is well-defined up to isotopy in the 3D space.
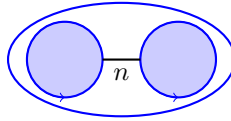
Fig. 2. The genus-0 tangle for the bilinear form B_n

(4) Naturality: The following commutative diagram holds:

$$
\begin{array}{ccc}
\text{surface tangles} & \xrightarrow{\quad Z \quad} & \text{vectors} \\
\text{tensor/contraction} \downarrow & & \text{tensor/contraction} \downarrow \\
\text{surface tangles} & \xrightarrow{\quad Z \quad} & \text{vectors}
\end{array}
$$

We also call $Z(T)$ the *partition function* of T from the statistic point of view.

Definition 2.8. The partition function of a sphere is called the 2D sphere value, denoted by ζ. The partition function of a closed string in a sphere is $\delta\zeta$. We call δ the 1D circle value.

If we restrict the representation Z to genus-0 tangles with one output disc, then we recover unital, finite dimensional, spherical planar algebras. Moreover, δ is the statistical dimension of the planar algebra.

Remark. The spherical condition for planar algebra is defined based on the evaluable condition, namely the 0-box space is one-dimensional [Jon12]. The spherical condition of surface algebras on the sphere does not require this one-dimensional condition. Typical examples of such planar algebras are graph planar algebras [Jon00].

Definition 2.9. We say a surface algebra is an extension of a planar algebra, if the restriction of its partition function Z on the planar tangles is the partition function of the planar algebra.

Remark. If the regions and strings of surface tangles are colored, then the index set $\mathbb{N}$ will be replaced by all permissible colors of the boundary of a disc.

Remark. If one considers surface algebras as 2D TQFT with line defects, then it is better to consider the discs of surface tangles as holes. However, we emphasize that these surfaces are boundaries of 3-manifolds, thus the notion of discs is more reasonable.

Notation 2.10. *For an input disc D, the boundary condition ∂D only depends on the number of marked points n. Thus we denote $\mathscr{S}_{\partial D}$ by $\mathscr{S}_n$ and its dual space by $\mathscr{S}_n^*$.*

We can consider $Z(T)$ as a multi-linear transformation on the vector space $\{\mathscr{S}_n\}_{n\in\mathbb{N}}$ from input discs to output discs.

Let us extend some notions from planar algebra to surface algebras.

Notation 2.11. *We use a thick string labelled by a number n to indicate n parallel strings.*

Notation 2.12. *The genus-0 tangle in Fig. 2 defines a bilinear form B_n on $\mathscr{S}_n \otimes \mathscr{S}_n$.*
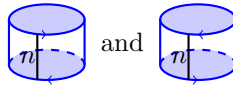
Fig. 3. The tangles for D_n and D_n^{-1}

Definition 2.13. A surface algebra is called *non-degenerate*, if the bilinear form B_n is non-degenerate for all n.

If the surface algebra is non-degenerate, then the bilinear form B_n induces an iso-morphism D_n from the vector space $\mathscr{S}_n$ to its dual space $\mathscr{S}_n^*$. From this point of view, the tangles for the map D_n and its inverse D_n^{-1} are given in Fig. 3. So we can identity the vector space $\mathscr{S}_n$ with its dual using these duality maps, denoted by D for short.

Definition 2.14. Suppose $*$ is an anti-linear involution on $\mathscr{S}_n$, $n \in \mathbb{N}$. Then $R(x) := D(x^*)$ is an anti-linear isomorphism from $\mathscr{S}_n$ to $\mathscr{S}_n^*$. We still denote its inverse and the linear extension on the tenor power by R. Then

$$\langle y, x \rangle := B_n(x, y^*) = R(y)(x) \tag{8}$$

is an inner product on $\mathscr{S}_n$.

Remark. The bilinear form in planar algebras is $\frac{1}{\zeta} B_n$.

Definition 2.15. A surface algebra is called a surface $*$-algebra, if it has an anti-linear involution, such that for any surface tangle T,

$$Z(r(T)) = R(Z(T)). \tag{9}$$

Definition 2.16. A surface $*$-algebra $\mathscr{S}_\bullet$ is called (semi-)positive, if the inner product $\langle \cdot, \cdot \rangle$ is (semi-)positive.

Note that positivity implies non-degeneracy.

For a positive surface algebra $\mathscr{S}_\bullet$, the vector space $\mathscr{S}_n$ is a Hilbert space. Moreover, the map R is the Riesz representation. Thus, we can consider a positive surface algebra as a Hilbert space representation of surface tangles satisfying an additional commutative diagram:

$$
\begin{array}{ccc}
\text{surface tangles} & \xrightarrow{\quad Z \quad} & \text{vectors} \\
{\scriptstyle\text{reflection}}\Big\downarrow & {\scriptstyle\text{Riesz representation}}\Big\downarrow & \\
\text{surface tangles} & \xrightarrow{\quad Z \quad} & \text{vectors}
\end{array}
\tag{10}
$$

2.3. Labelled tangles. For a surface tangle, we can partially fill its discs by a vector with compatible boundary condition. We consider the result as a labelled tangle. Let us extend the representation Z of surface tangles to labelled tangles.

Definition 2.17. Suppose $\mathscr{S}_\bullet$ is a surface algebra and T is a surface tangle. Let S be a subset of $\mathcal{D}(T)$ and v be a vector in $\bigotimes\limits_{D \in S} \mathscr{S}_{r(\partial D)}$. We call the pair T and v a labelled tangle, denoted by $T \circ_S v$, or $T(v)$ for short, in the sense that the discs in S are labelled by the vector v. We call it fully labelled, if all discs are labelled. We define the partition function of the labelled tangle $T(v)$ by

$$Z(T(v)) := v(Z(T)), \tag{11}$$

where $v \in \bigotimes\limits_{D \in S} \mathscr{S}_{r(\partial D)}$ is considered as a partial linear functional on $\bigotimes\limits_{D \in \mathcal{D}(T)} \mathscr{S}_{\partial D}$.

Definition 2.18. We define the reflection on a labelled tangle $T(v)$ by

$$r(T(v)) = r(T)(R(v)). \tag{12}$$

Suppose $\mathscr{S}_\bullet$ is a surface algebra and T is a surface tangle. Then $Z(T) \in \bigotimes\limits_{D \in \mathcal{D}(T)} \mathscr{S}_{\partial D}$.

Let S be a subset of $\mathcal{D}(T)$. Then each vector v in $\bigotimes\limits_{D \in S} \mathscr{S}_{r(\partial D)}$ is a partial linear functional on $\bigotimes\limits_{D \in \mathcal{D}(T)} \mathscr{S}_{\partial D}$. Moreover, $v(Z(T))$ is a vector in $\bigotimes\limits_{D \in \mathcal{D}(T) \backslash S} \mathscr{S}_{r(\partial D)}$, corresponding to the unlabelled discs of T.

Theorem 2.19. *For a surface algebra $\mathscr{S}_\bullet$, the extended representation Z of labelled tangles satisfies all axioms in Definition* 2.7.

Proof. Axioms (1') and (3) follow from the corresponding axioms for surface tangles.

Axiom (1) and (4) follow from the corresponding axioms for surface tangles and Eq. (11).

Axiom (2) follows the corresponding axioms for surface tangles and Eq. (12).

We give a proof for the tensor in axiom (4) in details. The others are similar. Suppose $T_1(v_1)$ and $T_2(v_2)$ are labelled tangles, then their disjoint union $T_1(v_1) \otimes T_2(v_2) = (T_1 \otimes T_2)(v_1 \otimes v_2)$ is a labelled tangle. So

$$\begin{aligned}
&Z(T_1(v_1) \otimes T_2(v_2)) \\
&= Z((T_1 \otimes T_2)(v_1 \otimes v_2)) \\
&= (v_1 \otimes v_2)(Z(T_1 \otimes T_2)) \\
&= (v_1 \otimes v_2)(Z(T_1) \otimes Z(T_2)) \\
&= v_1(Z(T_1)) \otimes v_2(Z(T_2)) \\
&= Z(T_1(v_1)) \otimes Z(T_2(v_2)).
\end{aligned}$$

$\square$

Let $T(v)$ be a a labelled tangle containing $T_1(v_1)$ as a sub labelled tangle. In other words, there is a labelled tangle $T_2(v_2)$, such that $T(v)$ is a multiple contractions between $T_1(v_1)$ and $T_2(v_2)$. We denote it by $T(v) = T_1(v_1) \circ_S T_2(v_2)$, where S indicates the unlabelled discs are that glued. If $T_3(v_3)$ is a labelled tangle which has the same partition function as $T_1(v_1)$. Then we can identify their unlabelled discs. If we replace $T_1(v_1)$ by $T_3(v_3)$ in $T(v)$, then we obtain a new labelled tangle $T_3(v_3) \circ_S T_2(v_2)$. By Theorem 2.19, we have that

Corollary 2.20. *If $Z(T_1(v_1)) = Z(T_3(v_3))$, then $Z(T_1(v_1) \circ_S T_2(v_2)) = Z(T_3(v_3) \circ_S T_2(v_2))$.*

Since the replacement of $T_1(v_1)$ by $T_3(v_3)$ will not affect the partition function, we call it a *relation* of labelled tangles, denoted by $T_1(v_1) = T_3(v_3)$.

Definition 2.21. Suppose $\mathscr{S}$ is a non-degenerated surface algebra, and T is a labelled tangles. If $Z(T) = 0$, then we call T a relation of $\mathscr{S}$, and denoted by $T = 0$.

The following genus-0 tangle I_n has one input disc and one output disc:

If $\mathscr{S}_\bullet$ is non-degenerate, then I_n defines the identity map on $\mathscr{S}_n$.

For any vector v in $\mathscr{S}_n$, we obtain a labelled tangle $I_n(v)$ by filling v in the input disc. Then $Z(I_n(v)) = v$. So the vector v can be considered as a labelled tangle, denoted by $v = I_n(v)$. Its pictorial representation is

This construction can be generalized to any vector in the tensor product of $\{\mathscr{S}_n\}_{n\in\mathbb{N}}$ and their dual spaces. Therefore, we can identify vectors and labelled tangles by each other in a surface algebra.

The vector spaces $\mathscr{S}_n$ and $\mathscr{S}_n^*$ are dual to each other. Let $\{\alpha_k\}$ be a basis of $\mathscr{S}_n$ and $\{\beta_k\}$ be its dual basis. Then we have that

$$Z(I_n) = \sum_k \alpha_k \otimes \beta_k. \tag{13}$$

The right hand side is independent of the choice of basis.

This defines a relation for labelled tangles that we call the *joint relation*.

Proposition 2.22 (Joint relation). *Suppose $\mathscr{S}_\bullet$ is a non-degenerate surface algebra, then we have the join relation for labelled tangels:*

$$\tag{14}$$

Consequently, if $\dim(\mathscr{S}_0) = 1$, then for the genus-g surface S_g, we have

$$Z(S_g) = \zeta^{1-g}. \tag{15}$$

Proof. Since $\mathscr{S}_\bullet$ is non-degenerated, by Eq. (13), the partition function of both sides of Eq. (14) are equal. So the Joint relation (14) holds.

Furthermore, if $\dim(\mathscr{S}_0) = 1$, then

$$\tag{16}$$

Applying Relation (16), we can remove a genus from a surface by multiplying a scalar ζ^{-1}, so $Z(S_g) = \zeta^{-1}Z(S_{g-1})$. Recall that $Z(S_0) = \zeta^1$, so $Z(S_g) = \zeta^{1-g}$. □

Remark. Due to the joint relation, one can regard the partition function Z as a symmetric monoidal functor from the category of cobordisms with line defects to the category of vector spaces.

2.4. Unique extension.

Theorem 2.23. *For any $\zeta \neq 0$, any non-degenerate, unital, finite dimensional, spherical planar algebra $\mathscr{P}_\bullet$ has an unique extension to a non-degenerate surface algebra $\mathscr{S}_\bullet$ with 2D sphere value ζ.*

In other words, the joint relation and the local relations defined by the planar algebra are consistent and the 2D sphere value ζ is a freedom.

Proof. Since $\mathscr{P}_\bullet$ is non-degenerate and $\zeta \neq 0$, its extension $\mathscr{S}_\bullet$ is non-degenerate. Moreover, the inner product on $\mathscr{S}_\bullet$ is ζ times the inner product on $\mathscr{P}_\bullet$. The anti-linear isomorphism $D_n : \mathscr{S}_n \to \mathscr{S}_n^*$ is defined by the Riesz representation.

The interior 3-manifold of a fully labelled surface tangle T is contractable to a graph G_T, homotopic to a planar graph. Moreover, the graph G_T is unique up to the contraction move which contracts an adjacent pair of an m-valent vertex and an n-valent vertex to an $(m + n - 2)$-valent vertex:
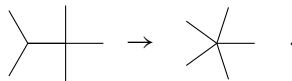
We consider T as a small neighborhood of G_T. We can decompose T into fully labelled genus-0 tangles by applying the joint relation (14) to all edges of G_T. Thus the partition function $Z(T)$ is determined by the value of Z on fully labelled genus-0 tangles. Therefore the extension is unique for a fixed ζ.

Now we prove the existence of such extension. We need to prove that the partition function $Z(T)$ is well-defined.

Let $\{\alpha_k\}$ be a basis of $\mathscr{S}_n$ and $\{\beta_k\}$ be its dual basis. Let $\{\alpha_{k'}\}$ be a basis of $\mathscr{S}_m$ and $\{\beta_{k'}\}$ be its dual basis. By basic linear algebra, for any $f \in \mathscr{S}_{n+m}$, we have that

$$\tag{17}$$

Therefore, for a fixed G_T, $Z(T)$ is well-defined up to isotopy.

By basic linear algebra, for any $\alpha \in \mathscr{P}_n$ and $\beta \in \mathscr{P}_n^*$, we have that

$$\tag{18}$$

Thus, $Z(T)$ is invariant under the contraction move and it is independent of the choice the G_T. Therefore $Z(T)$ is well-defined for fully labelled surface tangles.

Applying the joint relation to a fully labelled tangle is equivalent to applying the inverse of the contraction move to the graph. Thus the joint relation is a relation for Z. Therefore, we obtain an extension from $\mathscr{P}_\bullet$ to $\mathscr{S}_\bullet$. $\square$

Consequently, the general constructions of spherical planar algebras can be extended to surface algebras. For example,

Corollary 2.24. *Suppose a surface algebra* $(\mathscr{S}_\bullet)_k$ *is an extension of a planar algebra* $(\mathscr{P}_\bullet)_k$ *with sphere value* ζ_k, *for* $k = 1, 2$. *Then* $(\mathscr{S}_\bullet)_1 \otimes (\mathscr{S}_\bullet)_2$ *is an extension of* $(\mathscr{P}_\bullet)_1 \otimes (\mathscr{P}_\bullet)_2$ *with sphere value* $\zeta_1 \zeta_2$.

Theorem 2.25. *Suppose a surface algebra* $\mathscr{S}_\bullet$ *is an extension of a subfactor planar algebra* $\mathscr{P}_\bullet$ *with sphere value* ζ. *Then* $\mathscr{S}_\bullet$ *is positive, if and only if* $\zeta > 0$.

Proof. We consider the genus-0 labelled tangle with one disc as a hemisphere. The sphere is a composition of an unlabelled hemisphere and its mirror image, so $\zeta > 0$ is necessary. Conversely, if $\zeta > 0$, then the partition function of $\mathscr{S}_\bullet$ is positive on the sphere. By the joint relation (14), any labelled tangles is a sum of disjoint unions of hemispheres. The positivity follows. $\square$

3. Jones-Wassermann Subfactors

The Jones-Wassermann subfactor was first studied in the framework of conformal nets [LR95, Was98, Xu00, KLM01]. Motivated by the reconstruction program from modular tensor categories (MTC), (cf. [Tur16]), to conformal field theory (CFT), Xu and the author have constructed m-interval Jones-Wassermann subfactors for modular tensor categories, and proved that these subfactors are symmetrically self-dual, called the modular self-duality for MTC [LX16]. This is a resource of a large family of unshaded planar algebras, where the input data is a modular tensor category.

For readers' convenience, we briefly recall the correspondence between notions in planar algebras and notions in category theory. Each irreducible, finite index subfactor $\mathcal{N} \subset \mathcal{M}$ defines a subfactor planar algebra $\mathscr{P}_\bullet$ [Jon99]. A subfactor planar algebra has an alternating shading colored by $\mathcal{N}$ and $\mathcal{M}$. In category theory, the $\mathcal{N}$–$\mathcal{N}$ (or $\mathcal{M}$–$\mathcal{M}$) bimodule category associated with the subfactor is a spherical category $\mathscr{C}_\mathcal{N}$ (or $\mathscr{C}_\mathcal{M}$) [BW99, Mug03a]. Moreover, the factor $\mathcal{M}$ defines a Frobenius algebra γ in $\mathscr{C}_N$ which induces the Morita equivalence between the two monoidal categories $\mathscr{C}_\mathcal{N}$ and $\mathscr{C}_\mathcal{M}$. The n-box space $\mathscr{P}_{n,+}$ of the planar algebra is isomorphic to $\hom_{\mathscr{C}_\mathcal{N}}(1, \gamma^n)$.

Furthermore, if we can lift the shading, then the planar algebra is called unshaded. In this case, the associated category is a $\mathbb{Z}_2$ graded spherical category $\mathscr{D}$, so that the generating, irreducible $\mathcal{N}$–$\mathcal{M}$ bimodule M defines an odd, symmetrically self-dual, irreducible object τ in $\mathscr{D}$; $\tau \otimes \tau = \gamma$; $\mathscr{C}_\mathcal{N}$ and $\mathscr{C}_\mathcal{M}$ are isomorphic to the even part of $\mathscr{D}$; and the $2n$-box space of the unshaded planar algebra is isomorphic to $\hom_\mathscr{D}(1, \tau^{2n}) \cong \hom_{\mathscr{C}_\mathcal{N}}(1, \gamma^n)$.

An irreducible object is call symmetrically self-dual, if it is self-dual with Frobenius-Schur indicator 1. The corresponding subfactor is called symmetrically self-dual, see [LMP] for further discussions and examples.

We recall some notations and results in [LX16]. Let $\mathscr{C}$ be a unitary modular tensor category and Irr be the set of irreducible objects of $\mathscr{C}$. For an object X, its dual object

is denoted by $\overline{X}$. Its quantum dimension is $d(X)$. Let $\mu = \sum_{X \in Irr} d(X)^2$ be the global dimension of $\mathscr{C}$.

Let $\mathscr{S}_\bullet$ be the unshaded planar algebra of the 2-interval Jones-Wassermann subfactor for $\mathscr{C}$, and $\mathscr{D}$ be the associated $\mathbb{Z}_2$ graded spherical category. Then the even part of $\mathscr{D}$ is a subcategory of $\mathscr{C} \otimes \mathscr{C}^{op}$. Since we begin with unshaded planar algebras in this paper, the vector space $\mathscr{S}_{2n}$ of the unshaded planar algebra is $\mathscr{S}_n$ of the subfactor planar algebra in [LX16].

In [LX16], we considered $\mathscr{C} \otimes \mathscr{C}$ instead of $\mathscr{C} \otimes \mathscr{C}^{op}$, which is necessary in studying the m-interval Jones-Wassermann subfactor for all $m \geq 1$. In this paper, we only deal with the case $m = 2$. It is more convenient to work on $\mathscr{C} \otimes \mathscr{C}^{op}$. The opposite map $\mathscr{C} \to \mathscr{C}^{op}$ here corresponds to the map $\theta_\mathscr{C}$ on $\mathscr{C}$ in [LX16]. The object X^{op} in $\mathscr{C}^{op}$ corresponds to the dual object of X in $\mathscr{C}$, denoted by $\overline{X}$.

We also defined a map θ_2 on $\mathscr{C}$ in [LX16]. For a morphism a in $hom_\mathscr{C}(1, X_1 \otimes \cdots \otimes X_n)$, $X_1, \cdots, X_n \in Irr$, $\theta_\mathscr{C}(a)$ is a morphism in $hom_\mathscr{C}(1, \overline{X_n} \otimes \cdots \otimes \overline{X_1})$,

$$\theta_2(a) = \quad \overline{X_1} \qquad \overline{X_2} \qquad \overline{X_3} \qquad \overline{X_4} \quad , \tag{19}$$

where $n = 4$ in this example. A vector $a_0 \otimes a_1^{op}$ in $\mathscr{C} \otimes \mathscr{C}^{op}$, for $a_0, a_1 \in \mathscr{C}$, correspondes to the vector $a_0 \otimes \theta_2(a_1)$ in $\mathscr{C} \otimes \mathscr{C}$.

Let τ be the generating, odd, symmetrically self-dual, irreducible object in $\mathscr{D}$. Then $\mathscr{S}_n = hom_\mathscr{D}(1, \tau^n)$. By parity, $\mathscr{S}_n$ is zero for odd n. Moreover, $\tau^2 = \gamma$ is an even object, where

$$\gamma = \bigoplus_{X \in Irr} X \otimes X^{op}, \tag{20}$$

and

$$\mathscr{S}_{2n} = \hom_\mathscr{D}(1, \tau^{2n}) = \hom_\mathscr{D}(1, \gamma^n) = \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma^n). \tag{21}$$

In particular, when the Grothendieck ring of $\mathscr{C}$ is the group $\mathbb{Z}_d$, then $\mathscr{D}$ is a Tambara-Yamagami category [TY98]. More precisely, $\mathscr{D}$ is $\mathbb{Z}_2$ graded. Its the even objects are invertible objects with fusion ring $\mathbb{Z}_d$. It has one odd object τ, so that τ^2 is a direct sum of all invertible objects.

Recall that δ is the value of a closed circle in $\mathscr{S}_\bullet$, namely $d(\tau)$, then the Jones index δ^2 is the global dimension μ of the MTC $\mathscr{C}$,

$$\delta^2 = d(\tau^2) = d(\gamma) = \sum_{X \in Irr} d(X)^2 = \mu. \tag{22}$$

Note that $\mathscr{S}_4 = \hom_\mathscr{D}(1, \tau^4) = \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma^2)$. By Frobenius reciprocity, we can identify the $\mathscr{S}_4$ as
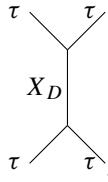
$$\hom_\mathscr{D}(\tau^2, \tau^2) = \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(\gamma, \gamma) = \bigoplus_{X \in Irr} \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(X_D, X_D) \cong L^2(Irr), \tag{23}$$

where $X_D = X \otimes X^{op}$. The space $\mathscr{S}_4$ has been considered as the 1-quon space for quantum information [LWJ17].
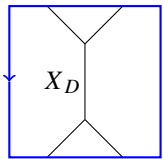
We represent the identity map 1_{X_D} on X_D in the category $\mathscr{C} \otimes \mathscr{C}^{op}$ as

$$X_D \bigg|.$$

While identifying it as a morphism in the even part of $\mathscr{D}$, we use the following pictorial representation:

where the trivalent vertices are isometries from τ^2 to X_D and backwards. As a vector in $\mathscr{S}_4$, we use the following pictorial representation:

Take

$$\beta_X = d(X)^{-1} 1_{X_D}, \tag{24}$$

where 1_{X_D} is the identity map in $\hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(X_D, X_D)$. Then $\{\beta_X\}_{X \in Irr}$ form an ONB of the 1-quon space, called the *quantum coordinate* [LX16].

Notation 3.1. *We denote the bra–ket notation for the 1-quon* $\sum_{X \in Irr} c_X \beta_X$ *by* $\sum_{X \in Irr} c_X |X\rangle$.

The modular transformation S of a MTC is originally defined by a hopf link.[1] In general, the SFT will change the shading of diagrams in a subfactor planar algebra. It is crucial that the planar algebra $\mathscr{S}_\bullet$ of the Jones-Wassermann subfactor is unshaded, so that the SFT is defined on each $\mathscr{S}_n$, $n \geq 0$. Furthermore, Xu and the author proved that the action of $\mathfrak{F}_S$ on the quantum coordinate of the 1-quon space is the S matrix in [LX16]:

Proposition 3.2. *On the ONB* $\{\beta_X\}_{X \in Irr}$ *of* $\mathscr{S}_4$, *the SFT* $\mathfrak{F}_S$ *is the modular S matrix, i.e.,*

$$\mathfrak{F}_S(|X\rangle) = \sum_{Y \in Irr} S_X^Y |Y\rangle. \tag{25}$$

---

[1] The entries of the S matrix is defined by the value of a Hopf link in a MTC, usually denoted by $S_{X,Y}$. Here we write it as S_X^Y while considering it as a matrix on 1-quons.

In $\mathscr{S}_\bullet$, we have the following pictorial representation:

$$d(X)^{-1} \quad \boxed{X_D} \quad = \sum_{Y \in Irr} S_X^Y d(Y)^{-1} \quad \boxed{Y_D} \quad . \tag{26}$$

In $\mathscr{D}$, we have the following pictorial representation:

$$d(X)^{-1} \quad \tau \quad \tau \quad \boxed{X_D} \quad \tau \quad \tau \quad = \sum_{Y \in Irr} S_X^Y d(Y)^{-1} \quad \tau \quad \tau \quad Y_D \quad \tau \quad \tau \quad . \tag{27}$$

Note that this relation does not hold in $\mathscr{C} \otimes \mathscr{C}^{op}$, as the SFT is not defined on $\mathscr{C} \otimes \mathscr{C}^{op}$.

Let Irr^n be the nth tensor power of Irr. Its element is given by $\vec{X} := X_1 \otimes \cdots \otimes X_n$. Then $d(\vec{X}) = \prod_{j=1}^{n} d(X_j)$. Let $ONB(\vec{X})$ be an orthonormal basis of $\hom_{\mathscr{C}}(1, \vec{X})$.

Proposition 3.3. *The partition function of the following planar tangle with $2n$ boundary points*

$$\tag{28}$$

is given by the vector $\delta^{\frac{n}{2}} \mu_n$ in $\mathscr{S}_{2n} = \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma^n)$, where

$$\mu_n = \delta^{1-n} \sum_{\vec{X} \in Irr^n} d(\vec{X})^{\frac{1}{2}} \sum_{\alpha \in ONB(\vec{X})} \alpha \otimes \alpha^{op}. \tag{29}$$

Proof. Following the construction in [LX16], for $\vec{X} \in Irr^n$, take $ONB(\vec{X})$ to be an orthonormal basis of $\hom_{\mathscr{C}}(1, \vec{X})$. Then $B = \{a_0 \otimes \theta_2(a_1) | a_0, a_1 \in ONB(\vec{X}), \vec{X} \in Irr^n\}$ form an orthonormal basis of $\hom_{\mathscr{C} \otimes \mathscr{C}}(1, \gamma^n)$. The partition function of the planar tangle (28) has been computed as a vector $\delta^{\frac{n}{2}} \mu_n$ in $\hom_{\mathscr{C} \otimes \mathscr{C}}(1, \gamma^n)$,

$$\mu_n = \sum_{\alpha \in B} \overline{L(\alpha)} \alpha, \tag{30}$$

where L is simplified as follows for the 2-interval case,

$$L(a_0 \otimes \theta_2(a_1)) = \mu^{\frac{1-n}{2}} d(\vec{X})^{\frac{1}{2}} \qquad \theta_2(a_1) \qquad a_0 \qquad . \tag{31}$$

By Eq. (19),

$$L(a_0 \otimes \theta_2(a_1)) = \mu^{\frac{1-n}{2}} d(\vec{X})^{\frac{1}{2}} \langle a_1, a_0 \rangle, \tag{32}$$

where $\langle a_0, a_1 \rangle$ is the inner product in $\mathscr{C}$. Therefore

$$\mu_n = \delta^{1-n} \sum_{\vec{X} \in Irr^n} d(\vec{X})^{\frac{1}{2}} \sum_{\alpha \in ONB(\vec{X})} \alpha \otimes \theta_2(\alpha). \tag{33}$$

Identifying μ_n as a vector in $\mathscr{S}_{2n} = \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma^n)$, still denoted by μ_n, we obtain Eq. (29). □

Note that μ_1 is a morphism in $\hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma)$. By Frobenius reciprocity, $\mu_3 \in \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma^3)$ also defines a morphism in $\hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(\gamma^2, \gamma)$. The triple (γ, μ_1, μ_3) defines the Frobenius algebra in $\mathscr{C} \otimes \mathscr{C}^{op}$ associated to the 2-interval Jones-Wasserman subfactor. Its dual category is known as the quantum double.

In general, when we identify the string labelled by the Frobenius algebra γ as a pair of parallel strings labelled by τ and $\bar{\tau}$ with a shading in the middle. This is a classical identification in subfactor theory. The alternating shading is essential in the study of subfactor planar algebras. Here we can lift the shading by the modular self-duality of MTCs, and the both strings are labelled by the symmetrically self-dual object τ in $\mathscr{D}$. Therefore the SFT is defined on $\mathscr{D}$, which is used in a crucial way in Sect. 6.

4. Fourier Duality on 1-Quons

A quon x in $\mathscr{S}_4$ is represented by a labelled tangle which has one output disc with 4 points on the boundary. We modify shape of the disc as a square and represent x as follows:

The outside region belongs to the output disc, when we consider it as a genus-0 labelled tangle.

For quons $x, y \in \mathscr{S}_4$, we can compose the square-like labelled tangles vertically or horizontally:
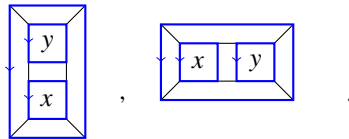
Both operations define associative multiplications on $\mathscr{S}_4$. We call the vertical composition the multiplication of x and y, denoted by xy. We call the horizontal composition the convolution of x and y, denoted by $x * y$.[2]

Furthermore, the SFT is given by the following 90° rotation

It intertwines the two multiplications,

$$\mathfrak{F}_S(xy) = \mathfrak{F}_S(x) * \mathfrak{F}_S(y). \tag{34}$$

---

[2] The horizontal multiplication is usually called the coproduct on subfactor planar algebras.

This is a corner stone of the pictorial Fourier duality.

Let us consider the 1-quon space $\mathscr{S}_4 \cong L^2(Irr)$ as functions on the quantum coordinates. Then we have the following formulas for the multiplication and the convolution.

Proposition 4.1 (Multiplication). *For* $X, Y \in Irr$,

$$|X\rangle |Y\rangle = \delta_{X,Y} d(X)^{-1} |X\rangle. \tag{35}$$

Proof. It follows from Eq. (24). □

Proposition 4.2 (Convolution). *For* $X, Y \in Irr$,

$$|X\rangle * |Y\rangle = \delta^{-1} \sum_{W \in Irr} N_{X,Y}^W |W\rangle, \tag{36}$$

where $N_{X,Y}^W = \dim \hom_{\mathscr{C}}(W, X \otimes Y)$.

Proof. It follows from Eq. (3.3). □

The matrix $N_X = N_{X,-}^-$ is called the adjacent matrix or the fusion. Verlinde first proposed that the modular transformation S diagonalizes the fusion [Ver88]. The Fourier duality of 1-quons gives a conceptual explanation of this result.

Theorem 4.3 (Verlinde formula). *For any* $X \in Irr$,

$$\delta^{-1} S N_X S^{-1} = \sum_{Y \in Irr} S_X^Y d(Y)^{-1} \delta_Y, \tag{37}$$

where δ_Y *is the projection onto* $\mathbb{C}\beta_Y$.

Proof. By Eqs. (35), (36), (34) and (25)

$$\begin{aligned} \delta^{-1} S N_X S^{-1}(S |W\rangle) &= S(|X\rangle * |W\rangle) \\ &= (S|X\rangle)(S|W\rangle) \\ &= \sum_{Y \in Irr} S_X^Y d(Y)^{-1} \delta_Y(S|W\rangle). \end{aligned}$$

Since $\{S|W\rangle\}_{W \in Irr}$ form an ONB of $\mathscr{S}_4$, we obtain Eq. (37). □

Now we give another application of the Fourier duality on 1-quons. The set Irr of irreducible objects of $\mathscr{C}$ forms a fusion ring under the direct sum $\oplus$ and the tensor $\otimes$. For any subset $K \subset Irr$, we define its indicator function as

$$P_K = \sum_{X \in K} 1_{X_D}. \tag{38}$$

Then P_K is a projection in $\mathscr{S}_4 \cong L^2(Irr)$. This is a bijection between subsets of Irr and projections in $\mathscr{S}_4$.

Proposition 4.4 (Theorem 4.12 in [Liu16]). *Suppose P is a projection in $\mathscr{S}_4$ under the multiplication. If the range projection of $P * P$ is a subprojection of P, then P is a biprojection.*

Notation 4.5. *Let us define* $SUB_{\otimes} = \{K \subset Irr | K$ *is closed under* $\otimes\}$.

Theorem 4.6. *Take $K \subset Irr$, then $K \in SUB_\otimes$ iff P_K is a biprojection. Consequently, if K is closed under $\otimes$, the it is closed under the dual.*

Proof. By Eq. (36), if P_K is a biprojection, then K is closed under the tensor and the dual. Conversely, if K is closed under $\otimes$, by Eq. (36), the range projection of $P_K * P_K$ is a subprojection of P_K. By Proposition 4.4, P_K is a biprojection. $\square$

Definition 4.7. For a subset K of Irr, let $\overline{K}$ be the smallest subset in $SUB_\otimes$ containing K.

The Hausdorff–Young inequality for subfactor planar algebras has been proved in [JLW16]. Applying the ∞-1 Hausdorff–Young inequality to the vector $\beta_Y = d(Y)^{-1}1_{Y_D}$, we have

$$\|\mathfrak{F}_S(\beta_Y)\|_\infty \leq \delta^{-1}\|\beta_Y\|_1. \tag{39}$$

By Proposition 3.2,

$$\left|\frac{S_X^Y}{S_X^0}\right| \leq \frac{S_0^Y}{S_0^0}. \tag{40}$$

We recover this important inequality for unitary MTC proved by Terry Gannon [Gan05].

Definition 4.8. For a subset K of Irr, we define

$$\hat{K} = \left\{X \in Irr \;\middle|\; \frac{S_X^Y}{S_X^0} = \frac{S_0^Y}{S_0^0}, \; \forall\, Y \in K\right\}. \tag{41}$$

Proposition 4.9 (Theorem 4.12 in [Liu16]). *Suppose A is a positive operator in $\mathscr{S}_4$ under the multiplication. Let P be the smallest biprojection containing the range projection of A. Then the spectral projection of $\mathfrak{F}_S(P)$ at its norm is a biprojection and it is a multiple of $\mathfrak{F}_S(P)$.*

Theorem 4.10. *For a subset K of Irr, we have $\hat{K} \in SUB_\otimes$, $\hat{\hat{K}} = \overline{K}$ and $\hat{\hat{K}} = \overline{K}$.*

Proof. By Proposition 3.2,

$$\mathfrak{F}_S(P_K) = \sum_{X \in Irr} \sum_{Y \in K} S_X^Y d(Y) d(X)^{-1} 1_{X_D}$$

$$= \sum_{X \in Irr} \sum_{Y \in K} \frac{S_X^Y}{S_X^0} S_0^Y 1_{X_D}.$$

By the Hausdorff–Young inequality,

$$\left|\sum_{Y \in K} \frac{S_X^Y}{S_X^0} S_0^Y\right| \leq \sum_{Y \in K} \frac{S_0^Y}{S_0^0} S_0^Y = \delta^{-1}\|P_K\|_1. \tag{42}$$

Moreover, the equality holds iff $X \in \hat{K}$. Therefore, $\|\mathfrak{F}_S(P_K)\| = \delta^{-1}\|P_K\|_1$ and the spectral projection of $\mathfrak{F}_S(P_K)$ at $\|\mathfrak{F}_S(P_K)\|$ is $P_{\hat{K}}$.

By Proposition 4.9, $P_{\hat{K}}$ is a biprojection and it is a multiple of $\mathfrak{F}_S(P_{\overline{K}})$. So $\hat{K} \in SUB_\otimes$ and $\hat{K} = \overline{\hat{K}}$. Moreover, $\mathfrak{F}_S(P_{\hat{K}})$ is a multiple of $\mathfrak{F}_S^2(P_{\overline{K}})$ and $\mathfrak{F}_S^2(P_{\overline{K}}) = P_{\overline{K}}$, so $\hat{\hat{K}} = \overline{K}$. □

Definition 4.11. Suppose $K \in SUB_\otimes$. Then $K = \overline{K}$ and $\hat{\hat{K}} = K$. We call $\hat{K}$ and $P_{\hat{K}}$ the Fourier duals of K and P_K respectively.

Notation 4.12. *When $K \in SUB_\otimes$, we define $\mathscr{C}_K$ to be the full fusion subcategories of $\mathscr{C}$ whose simple objects are given by K.*

This is a bijection between $SUB_\otimes$ and full fusion subcategories of $\mathscr{C}$. By Theorem 4.6, we obtain a bijections between full fusion subcategories $\{\mathscr{C}_K\}$ of $\mathscr{C}$ and biprojections $\{P_K\}$ of $\mathscr{S}_4$.

Let $\mathcal{S}(x)$ be the trace of the range projection of x. Let $\dim \mathscr{C}_k$ be the global dimension of $\mathscr{C}_K$. Then

$$\dim \mathscr{C}_K := \sum_{X \in K} d(X)^2 = \mathcal{S}(P_K). \tag{43}$$

Moreover,

$$\dim \mathscr{C}_K \dim \mathscr{C}_{\hat{K}} = \mathcal{S}(P_K)\mathcal{S}(\mathfrak{F}_S(P_K)) = \delta^2. \tag{44}$$

In general, we have the Donoho-Stark uncertainty principle $\mathcal{S}(x)\mathcal{S}(\mathfrak{F}_S(x)) \geq \delta^2$, for any non-zero $x \in \mathscr{S}_4$, see [JLW16].

In particular, $\mathscr{C}_{\hat{K}}$ is known as Müger's center of $\mathscr{C}_K$ [Mug03b]. Using the bijection between $\mathscr{C}_{\hat{K}}$ and $\{P_K\}$, we find a correspondence between the property of Müger's center and that of biprojections. We summarize the results in this section in the Table 1.

5. Graphic Quons

5.1. Definitions. In this section, we extend the unshaded subfactor planar algebra $\mathscr{S}_\bullet$ to a surface algebra by Theorem 2.23, still denoted by $\mathscr{S}_\bullet$. We consider $\zeta := Z(S_0)$ as a free variable. We study n-quons through the surface algebra, particularly the ones represented by surface tangles.

Recall that $\mathscr{S}_4$ is the space of 1-quons. Take its nth tensor power $(\mathscr{S}_4)^n$ to be the space of n-quons. Let us denote $Q_n^m := \hom((\mathscr{S}_4)^m, (\mathscr{S}_4)^n)$ to be the space of transformations from m-quons to n-quons. We ignore the index when it is zero. For $\vec{X} = X_1 \otimes \cdots \otimes X_n \in Irr^n$, we define $\beta_{\vec{X}} = \beta_{X_1} \otimes \cdots \otimes \beta_{X_n}$. Then $\{\beta_{\vec{X}}\}_{\vec{X} \in Irr^n}$ form an ONB of Q_n.

Notation 5.1. *We denote the bra–ket notation for the n-quon $\beta_{\vec{X}}$ by $|\vec{X}\rangle$ and $|\vec{X}\rangle = |X_1 \cdots X_n\rangle$. The bra–ket notation for a transformation in Q_n^m is given by*

$$\sum_{\vec{Y} \in Irr^m} \sum_{\vec{X} \in Irr^n} c_{\vec{X}}^{\vec{Y}} |\vec{X}\rangle \langle \vec{Y}|.$$

By the commutative diagram (10), when we reverse the orientation of a disc of a surface tangle, we switch $\langle X|$ and $|X\rangle$ in its partition function. One can consider it as the Frobenius reciprocity. When we use the bra–ket notation for n-quons, we have an order for the tensor. Thus we also order the discs for surface tangles from 1 to n. The choice of the order is identical to the action of a permutation on the tensors.

Notation 5.2. *Let LT_n^m be the set of labelled surface tangles with m input discs and n output discs, so that each disc has four boundary points.*

Then the partition function Z is a surjective map from LT_n^m to $\mathscr{S}_4$.

Definition 5.3. For a genus-g labelled tangle T in LT_n^m, we define the normalized quon $|T\rangle$ by

$$|T\rangle := Z(S_g)^{-1}Z(T). \tag{45}$$

By Theorem 2.23, the extension from spherical planar algebra to surface algebras is unique up to the choice of $\zeta = Z(S_0)$.

Proposition 5.4. *The normalized quon $|T\rangle$ is independent of the choice of ζ.*

Proof. It follows from the joint relation (14). □

Definition 5.5. Let T_n^m be the subset LT_n^m consisting of surface tangles. We call $GQ_n^m := Z(T_n^m)$ the space of graphic quon transformations and $GQ_n := Z(T_n)$ the space of graphic n-quons.

5.2. From graphs to graphic quons. Let G_n be the set of oriented graphs on a surface whose the edges are ordered from 1 to n. For $\Gamma \in G_n$, let us construct a surface tangle $T_\Gamma \in T_n$: We replace each edge of Γ by a pair of parallel strings, and replace the arrow indicating the orientation by an output disc with four marked points; we replace the neighborhood of an n-valent vertex of Γ by a planar diagram with $2n$ boundary points as follows:
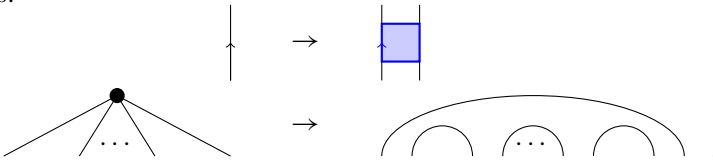
Moreover, we obtain a graphic quon $|T_\Gamma\rangle$.

For example, we take an oriented tetrahedron and order the edges by 1–6 as shown in Fig. 4. We denote this graph by Γ_6. Then we obtain a surface tangle T_{Γ_6} as follows:
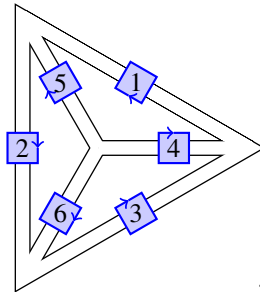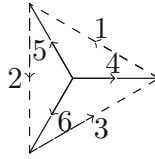
Fig. 4. Tetrahedron Γ_6: The first three edges are outside and the last three edges are inside. They are order by the angle from $0°$ to $360°$. The orientation of the first three edges are anti-clockwise. The orientation of the last three edges towards outside. We consider the tetrahedron as a graph on the sphere. Dashed lines indicate that the first three edges are at the back of the sphere

For $\vec{X} \in Irr^6$, the coefficient of $|\vec{X}\rangle$ in $|T_{\Gamma_6}\rangle$ is

$$\left|\begin{pmatrix} X_1 X_2 X_3 \\ X_4 X_5 X_6 \end{pmatrix}\right|^2 :=$$ (46)

where $\beta_j = d(X_j)^{-1} 1_{X_j \otimes X_j^{op}}$, $1 \leq j \leq 6$. By Proposition 3.3, we have

$$\left|\begin{pmatrix} X_1 X_2 X_3 \\ X_4 X_5 X_6 \end{pmatrix}\right|^2 = \sum_{\vec{\alpha} \in B(\vec{X})}$$ (47)

where $\vec{\alpha} = \alpha_0 \otimes \alpha_1 \otimes \alpha_2 \otimes \alpha_3$ and $B(\vec{X})$ is an orthonormal basis of $\hom_{\mathscr{C}}(X_4 \otimes X_5 \otimes X_6, 1) \otimes \hom_{\mathscr{C}}(X_1, X_4 \otimes X_3) \otimes \hom_{\mathscr{C}}(X_2, X_5 \otimes X_1) \otimes \hom_{\mathscr{C}}(X_3, X_6 \otimes X_2)$.

The value of the tetrahedron in Eq. (47) is well-known as the $6j$ symbol in $\mathscr{C}$. The corresponding value in $\mathscr{C}^{op}$ is its complex conjugate. Thus, the coefficient becomes the absolute square of the $6j$ symbol summing over an orthonormal basis. The sum is a good quantity to understand the global property of 6j symbols, as it is independent of the choice of the orthonormal basis.

So we obtain a 6-quon in terms of $6j$-symbol squares,

$$|T_{\Gamma_6}\rangle = \sum_{\vec{X} \in Irr^6} \left|\begin{pmatrix} X_1 X_2 X_3 \\ X_4 X_5 X_6 \end{pmatrix}\right|^2 |\vec{X}\rangle . \tag{48}$$

For a general MTC $\mathscr{C}$, it could be difficult to compute the coefficients of these graphic quons. Actually the closed form of $6j$ symbols are only known for a few examples. We can manipulate these graphic quons in a pictorial way by their graphic definition, even though we do not know the algebraic closed forms of their coefficients.

We can also define $|T_\Gamma\rangle$ for Γ in $\mathscr{C} \otimes \mathscr{C}^{op}$ directly similar to Eq. (47): Each k-valent vertex of Γ is replaced by the rotationally invariant morphism $\delta^{\frac{k}{2}}\mu_k \in \hom_{\mathscr{C} \otimes \mathscr{C}^{op}}(1, \gamma^k)$ in Proposition (3.3). Each edge is an output disc with two marked points. The target space of each output disc is $\hom(\gamma, \gamma) \cong L^2(Irr)$.

Definition 5.6. An non-zero n-quon is called positive, if all coefficients are non-negative.

Proposition 5.7. *For any $\Gamma \in G_n$, $|T_\Gamma\rangle$ is positive. Equivalently, its dual $\langle T_\Gamma|$ is a positive linear functional on the tensor power of $L^2(Irr)$.*

Proof. For any $\vec{X} \in Irr^n$, we label the jth edge of Γ by $\beta(X_j) = d(X)^{-1}1_X \otimes 1_{X^{op}}$. We label each k-valent vertex of Γ by $\delta^{\frac{k}{2}}\mu_k$. Since μ_k is a positive linear sum of $\alpha \otimes \alpha^{op}$ in Eq. (3.3). It is enough to show that for each choice of $\alpha \otimes \alpha^{op}$, the value is non-negative. For each choice, we obtain a fully surface labelled tangle in $\mathscr{C}$ and its opposite in $\mathscr{C}^{op}$. Thus the value is multiplication of a complex conjugate pair, which is non-negative. Therefore $\langle T_\Gamma | \vec{X} \rangle \geq 0$.

If the graph Γ is connected, then $|T_\Gamma\rangle$ is usually entangled for any bipartite partition. So we call $\langle T_\Gamma|$ a *topologically entangled measurement* on quons.

Definition 5.8. For $\Gamma \in G_n$, we define $\overline{\Gamma} \in G_n$ by reversing the orientations of all edges of Γ.

Proposition 5.9. *For any $\Gamma \in G_n$,*

$$|T_{\overline{\Gamma}}\rangle = |T_\Gamma\rangle . \tag{49}$$

Proof. For any $\vec{X} \in Irr^n$,

$$\langle \vec{X} | T_{\overline{\Gamma}} \rangle = \langle \overline{\vec{X}} | T_\Gamma \rangle = \overline{\langle \vec{X} | T_\Gamma \rangle} = \langle \vec{X} | T_\Gamma \rangle ,$$

where $\overline{\vec{X}} = \overline{X_1} \otimes \cdots \otimes \overline{X_n}$ and the last equality follows from Proposition 5.7. So $|T_{\overline{\Gamma}}\rangle = |T_\Gamma\rangle$. $\square$
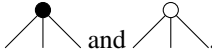
5.3. GHZ and Max. Greenberg, Horne and Zeilinger introduced a multipartite resource state for quantum information, called the GHZ state, denoted by $|GHZ\rangle$ [GHZ89]. In [JLW], Jaffe, Wozniakowski and the author find another resource state following topological intuition, called $|Max\rangle$. They both generalize the Bell state. For the 3-qubit case,

$$|GHZ\rangle = 2^{-1/2}(|000\rangle + |111\rangle),$$
$$|Max\rangle = 2^{-1}(|000\rangle + |011\rangle + |101\rangle + |110\rangle).$$

We observe that $|GHZ\rangle$ and $|Max\rangle$ are Fourier duals of each other:
$$|Max\rangle = (F \otimes F \otimes F)^{\pm 1}|GHZ\rangle, \tag{50}$$

where F is the discrete Fourier transform.

In tensor networks, the $|GHZ\rangle$ and $|Max\rangle$ are represented as two trivalent vertices:

 and . They have been considered as two fundamental tensors in [Laf03], see also [LWJ17, Bia17, CK17].

It is shown in [LWJ17] that $|GHZ\rangle$ and $|Max\rangle$ are graphic quons, and the corresponding surface tangles are given by
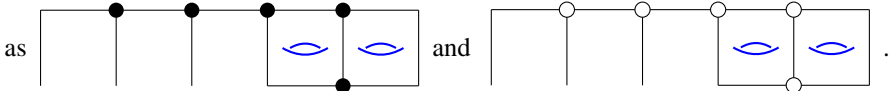
 and .
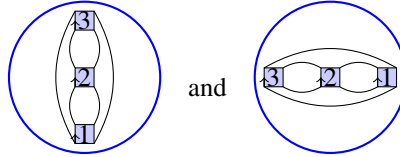
Inspired by this observation, we generalize $|GHZ\rangle$ and $|Max\rangle$ to n-quons on genus-g surfaces for the MTC $\mathscr{C}$.

Definition 5.10. Let us define the genus-g tangles $GHZ_{n,g}$ and $Max_{n,g}$ in T_n as follows:

$$GHZ_{n,g} = \quad \text{} \qquad (51)$$

$$Max_{n,g} = \quad \text{} \qquad (52)$$

Here we draw the tangles for $n = 3$, $g = 2$. The readers can figure out the general case. The corresponding tensor network notations could be generalized (up to a scalar)

as and .

Remark. From tensor network to quons language, we fat a string to a cuboid. The relations of the two Frobenius algebras becomes topological isotopy in two orthogonal directions, indicated by black and white.

Proposition 5.11. *For n, $g \geq 0$,*

$$|GHZ\rangle_{n,g} = \sum_{X \in Irr} d(X)^{2-n-2g} \overbrace{|XX \cdots X\rangle}^{n \text{ entries}}. \qquad (53)$$

Proof. By the joint relation (14), the coefficient of $|\vec{X}\rangle$ in $|GHZ\rangle_{n,g}$ is given by

$$Z(S_0)^{-1} \sum_{\vec{Y} \in Irr^g}$$

$$(54)$$

Since $\beta_X = d(X)^{-1} 1_{X_D}$ and 1_{X_D} is a minimal projection, the coefficient is nonzero only when $|\vec{X}\rangle = |XX \cdots X\rangle$, for some $X \in Irr$. In this case, the coefficient is $d(X)^{2-n-2g}$. $\quad \square$

For $\vec{X} \in Irr^n$, let $\dim(\vec{X}, g)$ be the dimension of the vector space consisting of vectors in genus-g surface with boundary points $X_1, X_2, \ldots, X_n$ in $\mathscr{C}$. Then

$$\dim(\vec{X}, 0) = \dim \hom_{\mathscr{C}}(1, \vec{X}),$$
$$\dim(\vec{X}, g) = \sum_{\vec{Y} \in Irr^g} \dim \hom_{\mathscr{C}}(1, \vec{X} \otimes \vec{Y} \otimes \theta_1(\vec{Y})),$$

where $\theta_1(\vec{Y}) = \overline{Y_g} \otimes \cdots \otimes \overline{Y_1}$.

Proposition 5.12. *For any* $n, g \geq 0$,

$$|Max\rangle_{n,g} = \delta^{2-n-2g} \sum_{\vec{X} \in Irr^n} \dim(\vec{X}, g)|\vec{X}\rangle. \tag{55}$$

Proof. By the joint relation (14), the coefficient of $|\vec{X}\rangle$ in $|Max\rangle_{n,g}$ is given by

$$Z(S_0)^{-1} \sum_{\vec{Y} \in Irr^g}$$

$$(56)$$

Appying Eq. (3.3), the coefficient is

$$\delta^{2-n-2g} \sum_{\vec{Y} \in Irr^g} \sum_{\alpha_1, \alpha_2 \in ONB(\vec{X} \otimes \vec{Y} \otimes \theta_1(\vec{Y}))} \langle \alpha_1 \otimes \alpha_1^{op}, \alpha_2 \otimes \alpha_2^{op} \rangle$$

$$= \delta^{2-n-2g} \sum_{\vec{Y} \in Irr^g} \dim \hom_{\mathscr{C}}(1, \vec{X} \otimes \vec{Y} \otimes \theta_1(\vec{Y}))$$

$$= \delta^{2-n-2g} \dim(\vec{X}, g).$$

$\square$

Definition 5.13. Let us define the generating function for $|GHZ\rangle$ and $|Max\rangle$,

$$|GHZ\rangle_n(z) = \sum_{g=0}^{\infty} |GHZ\rangle_{n,g} z^g, \tag{57}$$

$$|Max\rangle_n(z) = \sum_{g=0}^{\infty} |Max\rangle_{n,g} z^g. \tag{58}$$

Proposition 5.14. *For $n \geq 0$,*

$$|GHZ\rangle_n(z) = \sum_{X \in Irr} \frac{d(X)^{4-n}}{d(X)^2 - z} \overbrace{|XX \dots X\rangle}^{n \text{ entries}}. \tag{59}$$

The coefficients of $|GHZ\rangle_n(z)$ are all rational functions. It is less obvious that the coefficients of $|Max\rangle_n(z)$ are also rational functions. We prove this in Theorem 6.5.

6. Fourier Duality

In this section, we study the Fourier duality on graphic quons. Without loss of generality, we only consider surface tangles in T_n, i.e., all discs are output discs. Then their partition functions are graphic quons in GQ_n.

Recall that the SFT $\mathfrak{F}_S$ is a 90° rotation of the output disc. The corresponding genus-0 tangle is given by

$$\mathfrak{F}_S = \quad\quad\quad\quad\quad \tag{60}$$

The action of $\mathfrak{F}_S$ on the quantum coordinate $\{\beta_X\}_{X \in Irr}$ is identical to the S matrix of $\mathscr{C}$. We define the action of $\vec{\mathfrak{F}}_S$ on T_n as the action of $\mathfrak{F}_S$ on all output discs. We define the action of $\vec{S}$ on GQ_n as the nth tensor power of S.

Theorem 6.1. *For any unitary MTC $\mathscr{C}$, the following commutative diagram holds,*

$$
\begin{array}{ccc}
\textit{surface tangles} & \xrightarrow{\quad Z \quad} & \textit{graphic quons} \\
\vec{\mathfrak{F}}_S \downarrow & & \vec{S} \downarrow \\
\textit{surface tangles} & \xrightarrow{\quad Z \quad} & \textit{graphic quons}
\end{array}
$$

Proof. It follows from Proposition 3.2 and 2.23. □

In general, if we apply a global 90° rotation to a labelled surface tangle, then its partition function is acted by the conjugation of S. By Proposition 5.9, we have that

Corollary 6.2. *For any oriented graph on the surface $T \in G_n$,*

$$\vec{S}^2 |T_\Gamma\rangle = |T_\Gamma\rangle . \tag{61}$$

So we call the graphic quons $|T_\Gamma\rangle$ and $\vec{S} |T_\Gamma\rangle$ the Fourier dual of each other.

Remark. By Proposition 5.7, the Fourier dual pair of quons are both positive. It is an interesting phenomenon that the modular transformation S preserves this positivity. It is difficult to construct such positive Fourier duals algebraically.

Corollary 6.3. *Note that $Max_{n,g} = \vec{\mathfrak{F}}_S(GHZ_{n,g})$, for any $n, g \geq 0$, so*

$$|Max\rangle_{n,g} = \vec{S}|GHZ\rangle_{n,g}. \tag{62}$$

Theorem 6.4 (Verlinde formula). *For any unitary MTC $\mathscr{C}$ and any $n, g \geq 0$,*

$$\dim(\vec{X}, g) = \sum_{X \in Irr} (\prod_{i=1}^{n} S_{X_i}^X)(S_X^1)^{2-n-2g}. \tag{63}$$

Proof. Note that $|GHZ\rangle_{n,g}$ and $|Max\rangle_{n,g}$ are computed in Propositions 5.11 and 5.12, and $d(X) = \delta S_X^1$. The statement follows from comparing the coefficients on both sides of Eq. (62). □

The higher-genus Verlinde formula was first proved by Moore and Seiberg in the framework of CFT in [MS89]. Here we prove it for any unitary MTC as the Fourier duality of $|GHZ\rangle$ and $|Max\rangle$. The unitary condition is not necessary in the proof.

Theorem 6.5. *For any $n \geq 0$,*

$$|Max\rangle_n(z) = \sum_{\vec{X} \in Irr^n} \sum_{X \in Irr} (\prod_{i=1}^{n} S_{X_i}^X) \frac{d(X)^{4-n}}{d(X)^2 - z} |\vec{X}\rangle . \tag{64}$$
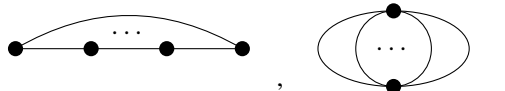
Proof. By Eq. (62), we have

$$|Max\rangle_n(z) = \vec{S}|GHZ\rangle_n(z). \tag{65}$$

By Proposition 5.14, the statement holds. □

It is interesting that the coefficients of $|Max\rangle_n(z)$, namely the generating functions of $\{\dim(\vec{X}, g)\}_{g \in \mathbb{N}}$, $\vec{X} \in Irr^n$, for all $n \geq 0$, live in a small dimensional vector space spanned by $\left\{ \frac{1}{d(X)^2 - z} \right\}_{X \in Irr}$.

Note that the genus-0 $|GHZ\rangle$ and $|Max\rangle$ can be defined through the cycle graph and the dipole graph,

The two graphs are dual to each other.

In general, for an oriented graph $\Gamma \in G_n$ on the sphere, we obtain a genus-0 tangle T_Γ. If we do not lift the shading, then the tangle T_Γ has an alternating shading, and all distinguished intervals of the discs are unshaded. When we apply $\vec{\mathfrak{F}}_S$ to T_Γ, all distinguished intervals become shaded. By contracting the unshaded regions to a point, we obtain an oriented graph $\hat{\Gamma}$ in G_n, such that $T_{\hat{\Gamma}} = \vec{\mathfrak{F}}_S(T_\Gamma)$. By Theorem 6.1, we have that

Theorem 6.6. *For any oriented graph $\Gamma \in G_n$ on the sphere,*

$$|T_{\hat{\Gamma}}\rangle = \vec{S}\,|T_\Gamma\rangle\,. \tag{66}$$

If we forget the orientation, then $\hat{\Gamma}$ is the dual graph of Γ on the sphere. Thus the graphic duality coincides with the Fourier duality of quons on the sphere. However, this is not true on surfaces. One needs further assumptions for graphs on surfaces: The faces are simply connected and the edges are contractable. We call such graphs *local*. Then Eq. (66) remains true for local graphs.

There are interesting graphs on surfaces that are not local. Actually the graphs for $|GHZ\rangle$ and $|Max\rangle$ on higher-genus surfaces are not local. So the quon language provides a natural extension of the graphic duality, which is compatible with the algebraic Fourier duality.

There are many interesting graphs on surfaces. The symmetry of (oriented) graphs leads to the symmetry of graphic quons. For examples, there are five platonic solids on the spheres: tetrahedron, cube, octahedron, dodecahedron and icosahedron. The number of edges are 6, 12, 12, 30, 30 respectively.

For the five platonic solids on the spheres, there are two dual pairs and one self-dual tetrahedron. We obtain three identities for the Fourier duality: a Fourier self-duality of the 6-quon associated with the tetrahedron; a Fourier duality between two 12-quons associated with the cube and the octahedron; a Fourier duality between two 30-quons associated with the dodecahedron and the icosahedron.

The self-duality of the tetrahedron gives a self-duality for 6j-symbols.

Theorem 6.7 (6j-symbol self-duality). *For any MTC $\mathscr{C}$, and any $\vec{X} \in Irr^6$,*

$$\left|\begin{pmatrix} X_6 & X_5 & X_4 \\ X_3 & X_2 & X_1 \end{pmatrix}\right|^2 = \sum_{\vec{Y} \in Irr^6} \left(\prod_{k=1}^{6} S_{X_k}^{Y_k}\right) \left|\begin{pmatrix} Y_1 Y_2 Y_3 \\ Y_4 Y_5 Y_6 \end{pmatrix}\right|^2. \tag{67}$$

Proof. We take the tetrahedron Γ_6 in Fig. 4. Its dual graph $\hat{\Gamma}_6$ is given by the second. The third is isotopic to the second by 180° rotation.

$$\tag{68}$$
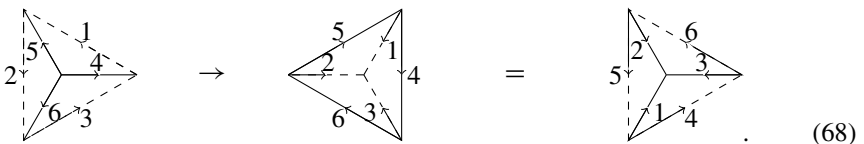
By Theorem 6.6, we have $|T_{6j}\rangle = \vec{S}\,|T_{\hat{6j}}\rangle$. Comparing the coefficients using Eq. (48), we obtain Eq. (67).

We also give a direct proof using Proposition 3.2: The corresponding Fourier duality on 6-quons is represented as follows,

$$\text{(69)}$$

where $\beta_j = d(X_j)^{-1} 1_{X_j \otimes X_j^{op}}$, and $\beta'_j = d(Y_j)^{-1} 1_{Y_j \otimes Y_j^{op}}$, $1 \leq j \leq 6$.

In the special case of quantum $SU(2)$, the identity for the 6j-symbol self-duality was discovered by Barrett in [Bar03], based on an interesting identity of J. Robert [Rob95]. Then the identity was generalized to some other cases related to $SU(2)$ in [FNR07].

To generalize the triangle to all regular polygons, our order of edges of the tetrahedron is slightly different from Barrett's choice. To recover Barrett's original formula, we take the following tetrahedron:

The first arrow is the graphic duality. The $=$ is a rotation. The last arrow is a vertical reflection. By Propositions 5.7 and 5.9, the 6-quons corresponding to the last two graphs are the same.

We can generalize the tetrahedron to a sequence of self-dual graphs on the sphere:
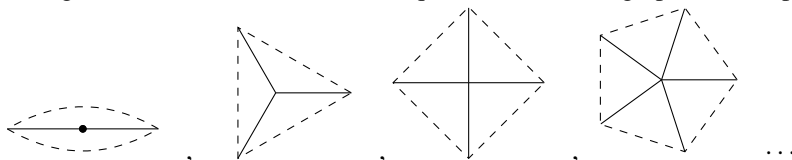
We order and orient the edges of each graph similar to Γ_6 in Fig. 4, and denote the oriented graph by Γ_{2n}, for $n \geq 2$. Then we obtain a $2n$-quon, denoted by

$$|T_{\Gamma_{2n}}\rangle = \sum_{\vec{X} \in Irr^{2n}} \left| \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ X_{n+1} X_{n+2} \cdots X_{2n} \end{pmatrix} \right|^2 |\vec{X}\rangle . \tag{70}$$

Theorem 6.8. *For any MTC $\mathscr{C}$, and any $\vec{X} \in Irr^{2n}$, $n \geq 2$,*

$$\left| \begin{pmatrix} X_{2n} & X_{2n-1} \cdots X_n \\ X_n & X_{n-1} & \cdots X_1 \end{pmatrix} \right|^2 = \sum_{\vec{Y} \in Irr^{2n}} \left(\prod_{k=1}^{2n} S_{X_k}^{Y_k} \right) \left| \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_n \\ Y_{n+1} Y_{n+2} \cdots Y_{2n} \end{pmatrix} \right|^2 . \tag{71}$$

Proof. The dual graph of Γ_{2n} is obtained similar to Eq. (68). The statement follows from Theorem 6.6. □

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