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Classification of Thurston relation subfactor planar algebras

Corey Jones, Zhengwei Liu, and Yunxiang Ren

Abstract. Bisch and Jones proposed the classification of planar algebras by simple generators and relations. They investigated with the second author the classification of planar algebras generated by 2-boxes. In this paper, we classify singly-generated Thurston-relation planar algebras, defined as subfactor planar algebras generated by a 3-box satisfying a relation proposed by Dylan Thurston. Our main result shows that such subfactor planar algebras are either the E_6 subfactor planar algebras or belong to a two-parameter family of planar algebras arising from the representations of type A quantum groups. We introduce a new method for determining positivity of the Markov trace of planar algebras in this family.

Mathematics Subject Classification (2010). 46L37.

Keywords. Subfactors, planar algebras, Thurston relation, skein theory.

Contents

1. Introduction

We classify singly-generated Thurston-relation subfactor planar algebras, using a new method for determining positivity of the Markov trace for planar algebras in this family.

Background. With his seminal index theorem, Vaughan Jones initiated the modern theory of subfactors [12]. The standard invariant of a subfactor is the lattice of higher relative commutants of the Jones tower. A deep theorem of Popa shows that the standard invariant completely classifies strongly amenable subfactors of the hyperfinite II_1 factor [28]. Popa introduced standard λ-lattices as an axiomatization of the standard invariant [29]. Ocneanu also introduced paragroups for characterizations of finite depth subfactors [24].

Vaughan Jones introduced subfactor planar algebras as an alternative axiomatization of the standard invariant of a subfactor [14]. Planar algebras are described by a sequence of finite dimensional vector spaces $\mathscr{P}_{m,\pm}$ (which we call the m-box spaces), $m \in \mathbb{N}$, together with an natural action of the operad of planar tangles. This perspective displays the implicit topological properties of subfactors, and reveals deep connections to topological quantum field theory [2, 25].

Motivated by Conway's linear skein relation in knot theory, Vaughan Jones introduced a more general framework of skein theories for planar algebras. A skein theory is a collection of generators and relations, together with an evaluation algorithm. Skein theories are familiar in the paradigm of quantum link invariants.

- The generators of a planar algebra $\mathscr{P}_\bullet$ correspond to string crossings, skein relations correspond to Reidemeister moves together with possible linear relations, such as the Kauffman bracket and the HOMFLY-PT skein relation [17, 11].
- A braided tangle diagram is given by the image of a planar tangle acting on generators.
- The vectors in $\mathscr{P}_{m,\pm}$ are given by linear combinations of braided tangle diagrams in a disc with $2m$ boundary points. Skein relations of braids induce equalities of vectors in $\mathscr{P}_{m,\pm}$.

Crucially, an evaluation algorithm is required such that the each vector in $\mathscr{P}_{0,\pm}$ can be evaluated to an element in the ground field using prescribed skein relations. Furthermore, the relations must be consistent in the sense that the evaluation of a vector in $\mathscr{P}_{0,\pm}$ should be the same modulo different sequences of skein relations.

The Yang-Baxter relation as a linearization of the Reidemeister moves was introduced in [20]. In addition, there are skein theories arising from the discharging method in graph theory. For example, Landau introduced the exchange relation for 2-boxes [19] generalizing Bisch's exchange relation of a biprojection [5], see other relations in [27, 22]. Bigelow gave skein theory for ADE planar algebras using half-braidings [3]. This idea was generalized to the Jellyfish relation [10].

A skein theory uniquely determines a subfactor planar algebra, therefore one can ask for a skein theoretic classification of subfactor planar algebras. From this perspective of skein theory, the simplest planar algebras are the Temperley–Lieb–Jones planar algebras which have neither generators nor relations and are determined by the circle parameter. Planar algebras generated by 1-boxes were completely analyzed by Vaughan Jones [14]. Bisch and Jones proposed the classification of planar algebras by simple generators and relations in [7, 8] and consider the dimensions as complexity. The smallness of dimensions forces a simple skein theory in their classication. Motivated by their work, the second author investigated the classification of planar algebras by skein theory in [21], where commute relation planar algebras were completely analyzed without dimension restrictions. Exchange relation planar algebras with two 2-box generators were classified there. Based on the subsequent work of Bisch, Jones and the second author [8, 9], a classification of singly generated Yang-Baxter relation planar algebra was achieved in [20], where a new one-parameter family of planar algebras was constructed.

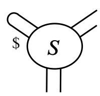
Main results. In this paper, we study planar algebras generated by a single 3-box. There is a known two parameter family of planar algebras $\mathscr{P}_\bullet^H(q,r)$ related to type A quantum groups, see Example 2.5 in [14]. We classify all q,r for which $\mathscr{P}_\bullet^H(q,r)$ has a semi positive definite partition function, (see Theorem 5.3). Its semisimple quotient $\widetilde{\mathscr{P}}_\bullet^H(q,r)$ is a subfactor planar algebra. The corresponding subfactors are known as Jones-Wenzl subfactors [13, 33]. The subfactor planar algebras were constructed by Xu [34]. The skein theory of $\mathscr{P}_\bullet^H(q,r)$ is inherited from the HOMFLY-PT skein relations [11, 30]. Dylan Thurston provided an intrinsic skein theory of $\mathscr{P}_\bullet^H(q,r)$ designed for 6-valent planar graphs [32], which we call the Thurston relation.

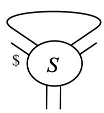
Definition 1.1 (Thurston relation [32]). We say a 3-box S satisfies the Thurston relation if the four following axioms are safisfied:

- *chirality*:

$$S - \rho(S) = \text{(lower terms)}; \tag{1}$$

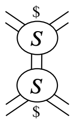
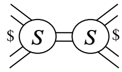
- $1 \to 0$ *move*:



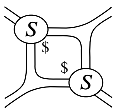
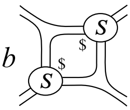
, 

$= \text{(lower terms)};$ (2)

- *unshaded* $2 \leftrightarrow 2$ *move*:

 $= a$  $+$ (lower terms), $\quad a \neq 0;$ (3)

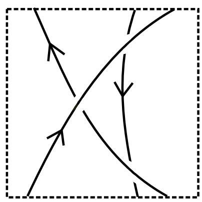
- *shaded* $2 \leftrightarrow 2$ *move*:

 $= b$  $+$ (lower terms), $\quad b \neq 0;$ (4)

where lower terms are a linear combination of diagrams with fewer generators and the coefficients in equations (1)–(4) are called *parameters* of the Thurston relation.

Dylan Thurston found $n!$ standard forms in the n-box space $\mathscr{P}^H_{n,+}$. His evaluation algorithm implies that the $n!$ standard forms form a generating set of the vector space $\mathscr{P}^H_{n,+}$. Therefore the dimension of $\mathscr{P}^H_{n,+}$ is bounded by $n!$.

It is clear from the definition that the generator R of $\mathscr{P}^H_\bullet(q,r)$,

$R =$  ,

satisfies the Thurston relation, but it is natural to wonder if these are the only possibilities. In this paper, we classify subfactor planar algebras generated by a non-Templeley–Lieb 3-box with a Thurston relation, which we call singly-generated Thurston-relation planar algebras (TRPA). Our main theorem is the following:

Theorem 1.2 (main theorem). *Any singly-generated Thurston-relation planar algebra is either an E_6 subfactor planar algebra or $\widetilde{\mathscr{P}}^H_\bullet(q,r)$, where $r = q^N$ for some $N \in \mathbb{N}$, $N \geq 3$, and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$, $l \geq 3$, or $q \geq 1$.*

We remark that the appearance of E_6 subfactor planar algebras in our classification is exceptional, since it is not included in a parameterized family.

A similar phenomena had appeared in the classification of subfactor planar algebras generated by a single 2-box. The $\mathbb{Z}_3$ group subfactor planar algebra was neither a Bisch–Jones planar algebra [6] nor a Birman–Murakami–Wenzl planar algebra [4, 23] in the classification [7, 8, 9]. Surprisingly it turns out to be the first example in another family of subfactor planar algebras discovered in [20]. Inspired by this phenomenon, we propose the following conjecture:

Conjecture. *There exists a parametrized skein theory for a family of subfactor planar algebras generated by a single* 3*-box, such that the* E_6 *subfactor planar algebras are in this family.*

The paper is organized as follows. In §2 , we recall the definitions of planar algebras, HOMFLY-PT skein theory, and Dylan Thurston's skein theory. In §3 , we classify singly generated *TRPA* for the generic case, namely $\dim \mathscr{P}_{4,\pm} = 4! = 24$. We set up five formal variables for the Thurston relation using 3-box relations. We prove that only two variables survive after considering the consistency of skein relations in $\mathscr{P}_{4,\pm}$ (theorems 3.10 and 3.11). Then we identify the two-parameter family with $\mathscr{P}_\bullet^H(q,r)$ (Theorem 3.13). Technically we simplify the computation by working on the reduced planar algebra with respect to the second Jones-Wenzl idempotent f_2. The reduced planar algebra has smaller m-box spaces and their generic dimensions are $1, 0, 1, 2, 9, \ldots$. In §4 , we classify singly generated *TRPA* for the reduced case, namely $\dim \mathscr{P}_{4,\pm} \leq 23$. We obtain subfactor planar algebra $\widetilde{\mathscr{P}}^H(q,r)$ subject to an equation of q and r as well as the E_6 subfactor planar algebras. In §5, we classify all q, r and involutions on $\mathscr{P}_\bullet^H(q,r)$ for which $\mathscr{P}_\bullet^H(q,r)$ has a positive semi-definite Markov trace, and thus the semisimple quotient $\widetilde{\mathscr{P}}_\bullet^H(q,r)$ is a subfactor planar algebra. Therefore, we complete our classification, Theorem 1.2.

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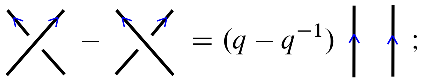
2. Preliminaries

As emphasized in the introduction, skein theory provides an important perspective from which to understand a planar algebra for many reasons. Skein theories are important starting points for the construction and classification of planar algebras.

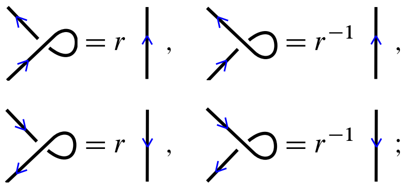
In this paper we will study subfactor planar algebras generated by a 3-box. In this section, we recall HOMFLY-PT planar algebras, and some properties of the Thurston relation. We refer the readers to [14] for the definition and properties of planar algebras. One can also find other interesting examples and skein theories there.

2.1. HOMFLY-PT planar algebras. The HOMFLY-PT polynomial is a link invariant which is defined using the following skein relations:

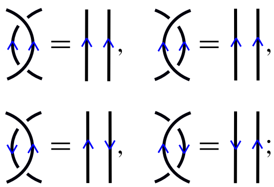
- *Hecke relation*:



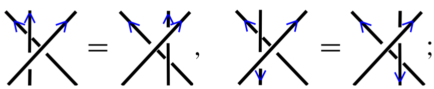
- *Reidemeister moves* I:



- *Reidemeister moves* II:



- *Reidemeister moves* III:

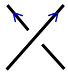


- *circle parameter*:

- $r - r^{-1} = \delta(q - q^{-1})$.

Remark 1. When $q = \pm 1$, we have $r = \pm 1$. Then the skein relation is determined by the circle parameter δ. When $q \neq \pm 1$, $\delta = \frac{r - r^{-1}}{q - q^{-1}}$.

Let σ_i, $i \geq 1$, be the diagram obtained by adding $i-1$ oriented (from bottom to top) through-strings on the left of



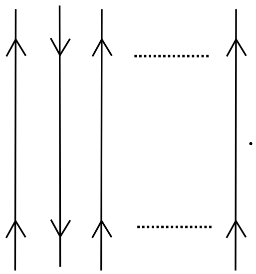
The Hecke algebra of type A is a (unital) filtered algebra $H_\bullet$. The algebra H_n is generated by σ_i, $1 \leq i \leq n-1$ and H_n is identified as a subalgebra of H_{n+1} by adding an oriented through string on the right. Over the field $\mathbb{C}(q,r)$, the equivalence classes of minimal idempotents of H_n are indexed by Young-diagrams with n cells. The trace formula of minimal idempotents is as follows:

Theorem 2.1 ([31, 1]). *Let λ be a Young diagram and m_λ the minimal idempotent corresponding to λ, then*

$$\mathrm{Tr}(m_\lambda) = \prod_{i,j} \frac{rq^{c(i,j)} - r^{-1}q^{-c(i,j)}}{q^{h_{(i,j)}} - q^{-h_{(i,j)}}},$$

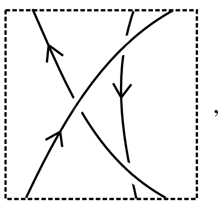
where $c(i,j) = j - i$ is the content of the cell (i,j) in λ and $h_{(i,j)}$ is its hook length.

V. Jones studied the planar algebras $\mathscr{P}^H_\bullet(q,r)$ associated with HOMFLY-PT skein relation [14]. Its n-box space consists of HOMFLY-PT diagrams which have $2n$ boundary points and alternating orientations on the boundary as follows:



Moreover, he proved that $\mathscr{P}^H_\bullet(q,r)$ is generated by a 3-box:

Theorem 2.2 (Jones). *The planar algebra $\mathscr{P}^H_\bullet(q,r)$ is generated by*



denoted by R.

When $r = q^N$ for some $N \in \mathbb{N}$ and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$ or $q \geq 1$, $\mathscr{P}_\bullet^H(q,r)$ admits an involution $*$ such that the Markov trace is positive semi-definite. Therefore, the semisimple quotient $\widetilde{\mathscr{P}}_\bullet^H(q,r)$ is a subfactor planar algebra, which can be constructed from the representation category of quantum group $U_q\mathfrak{sl}(N)$ [34]. (When $q = 1$, we have $r = 1$ and $\delta = N$, corresponding to the representation category of the Lie group $\mathrm{SU}(N)$.)

Remark 2. When $q = \pm 1, r = \pm 1$, the planar algebra in Theorem 2.2 is determined by the circle parameter δ. Therefore, we use the notation $\mathscr{P}_\bullet^H(1,1,\delta)$ for the planar algebra.

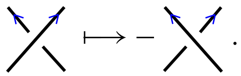
We prove that these are the only possibilities such that $\mathscr{P}_\bullet^H(q,r)$ has positivity in Theorem 5.3. First, we identify the isomorphism classes of $\mathscr{P}_\bullet^H(q,r)$:

Proposition 2.3. *The eight planar algebras*

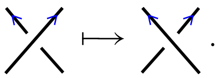
$$\mathscr{P}_\bullet^H(q,r), \quad \mathscr{P}_\bullet^H(q,-r^{-1}), \quad \mathscr{P}_\bullet^H(-q^{-1},r), \quad \mathscr{P}_\bullet^H(-q^{-1},-r^{-1}),$$
$$\mathscr{P}_\bullet^H(-q,r^{-1}), \quad \mathscr{P}_\bullet^H(-q,-r), \quad \mathscr{P}_\bullet^H(q^{-1},r^{-1}), \quad \mathscr{P}_\bullet^H(q^{-1},-r)$$

are isomorphic.

Proof. The planar algebras $\mathscr{P}_\bullet^H(q,r)$ and $\mathscr{P}_\bullet^H(-q^{-1},r)$ are isomorphic, as the generators satisfy the same skein relations. The isomorphism between $\mathscr{P}_\bullet^H(q,r)$ and $\mathscr{P}_\bullet^H(q,-r^{-1})$ is induced by sending

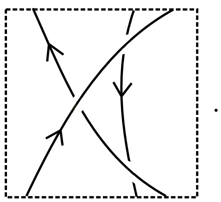


The isomorphism between $\mathscr{P}_\bullet^H(q,r)$ and $\mathscr{P}_\bullet^H(q^{-1},r^{-1})$ is induced by sending



From the three isomorphisms, the eight planar algebras in the statement are isomorphic. □

2.2. Thurston relations. Recall that $\mathscr{P}_\bullet^H(q,r)$ is generated by the 3-box

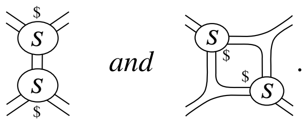


One can evaluate a vector in $\mathscr{P}^H_{0,\pm}(q,r)$ using the HOMFLY-PT skein relation. Dylan Thurston provides an intrinsic skein relations for the 3-box generator as shown in Definition 1.1, which we call the Thurston relation. He also provides an intrinsic evaluation algorithm in [32]. One can generalize the Thurston relation and his evaluation algorithm to multiple 3-box generators. We recall some results of Dylan Thurston:

Theorem 2.4 (Thurston [32]). *Suppose $\mathscr{P}_\bullet$ admits the Thurston relation. Then $\mathscr{P}_{n,\pm}$ is spanned by the standard forms and* $\dim \mathscr{P}_{n,\pm} \leq n!$.

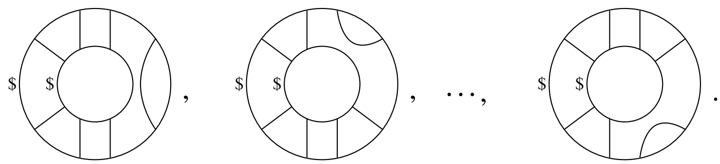
Corollary 2.5 (Thurston [32]). *In the generic case, namely $\mathscr{P}_{4,\pm} = 24$, we have the basis of the* 4*-box space given by the standard form as follows*:

- 14 *Temperley–Lieb diagrams*;
- 8 *diagrams in the annular consequences, which we denote by* AC;
- 2 *diagrams with two generators*:



Moreover, one can replace them by the other diagrams with two generators using the Thurston relation (3) *and* (4).

Remark 3. The annular consequences AC consists of diagrams obtained by the applying the following eight annular tangles to the generator S:



The main purpose of this paper is to classify all subfactor planar algebras generated by a 3-box S satisfying the Thurston relation. We give the classification for the generic case in §3 and for the reduced case in §4.

3. Generic case

Let $\mathscr{P}_\bullet$ be a singly-generated Thurston-relation planar algebra (TRPA). In this section, we classify such singly-generated TRPA for the generic case, namely $\dim(\mathscr{P}_{4,\pm}) = 24$. Most results in this section also work for case $\dim(\mathscr{P}_{4,\pm}) = 23$.

When the Jones index δ^2 is at most 4, the E_6 and $E_6^{(1)}$ subfactor planar algebras are the only subfactor planar algebras with Jones index δ^2 at most 4 generated by a 3-box. The dimensions of their 4-box spaces are 21 and 22 respectively. So we only need to consider the case $\delta^2 > 4$. By Theorem 2.4, $\dim(\mathscr{P}_{3,\pm}) \leq 6$. When $\delta^2 > 4$, the 5 Temperley–Lieb diagrams are linearly independent. The generator S is non-Temperley–Lieb, so $\dim(\mathscr{P}_{3,\pm}) = 6$.

3.1. Generators

Notation 3.1. For a subfactor planar algebra $\mathscr{P}_\bullet$, we use the following notations: e_n is the n^{th} Jones projection; f_n is the n^{th} Jones-Wenzl idempotent; and $\mathscr{I}_{n,\pm}$ is the basic construction ideal in $\mathscr{P}_{n,\pm}$.

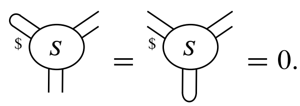
Since $\dim(\mathscr{P}_{3,+}) = 6$, there exist two minimal idempotents P and Q in $\mathscr{P}_{3,+}/\mathscr{I}_{3,+}$ and $P + Q = f_3$. Since $\mathrm{Tr}(f_3) \neq 0$, we assume that $\mathrm{Tr}(Q) \neq 0$. We take $S = \gamma P - Q$, where $\gamma = \frac{\mathrm{Tr}(P)}{\mathrm{Tr}(Q)}$, as the generator for $\mathscr{P}_\bullet$. Then S satisfies the following relations:

(1) S is an eigenvector of the (2-click) rotation ρ. i.e.,

$$(S) = \omega S, \quad \omega^3 = 1. \tag{5}$$

Here ω is called the rotation eigenvalue for the generator S.

(2) S is totally uncappable, i.e.,



(6)

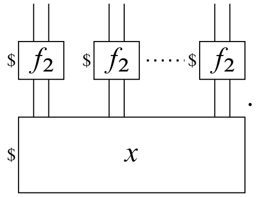
(3) S satisfies a quadratic relation:

$$S^2 = (\gamma - 1)S + \gamma f_3. \tag{7}$$

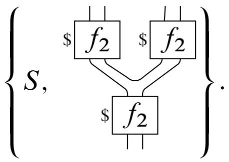
Note that equation (6) is the $1 \to 0$ move in the Thurston relation as in Definition 1.1.

Now we focus on the f_2-cutdown of $\mathscr{P}_{n,+}$ instead of the entire subfactor planar algebra. Technically, this reduces the dimension of the n-box space and simplify the computation. Elements in the n-box cutdown space will be elements $x \in \mathscr{P}_{n,+}$

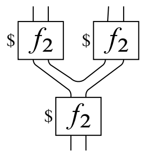
of the following form



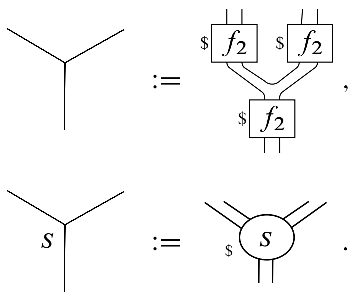
Then the f_2-cutdown of $\mathscr{P}_{3,+}$ has a basis



Notation 3.2. We simplify the notations in the f_2-cutdown by rewriting f_2 as a single string labeled by f_2. (We ignore the label if there is no confusion.) In this setting, we express S and



as follows:

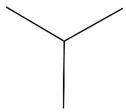
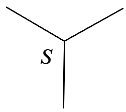


where the position of S indicates the position of the dollar sign \$.

The even part of a planar algebra is a monoidal category. One can consider the even part of a singly-generated Thurston-relation planar algebra as a monoidal category generated by two trivalent vertices as above. The Thurston relation looks similar to the H-I relation, also known as $6j$-symbols, in a monoidal category. The $6j$-symbol in a monoidal category are the coefficients of the change of basis matrix in the morphism spaces, and a monoidal category is determined by the $6j$-symbol up to monoidal equivalence, however it seems hopeless to determine the $6j$-symbol in general. Even for the example $\mathcal{P}_{\bullet}^{H}(q,r)$ appearing in our classification, it is difficult to compute the $6j$-symbol.

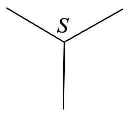
The Thurston relation provides an evaluation algorithm which only requires partial information about the $6j$-symbol for two objects, together with the data for the dual planar algebra. This combination, rather than considering only one shading, appears to be powerful, and determines the planar algebra completely.

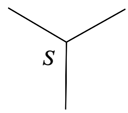
3.2. Relations in 3-boxes. Now let us set up the formal variables as coefficients in the Thurston relation of

 and 

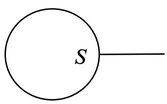
in $\mathcal{P}_{3,+}$ additional to relations (5)–(7). We solve for the possible coefficients in terms of four formal variables.

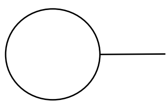
Lemma 3.3. *We have the following skein relations in the f_2 cut-down of $\mathcal{P}_{\bullet}$ in terms of δ, ω, γ and one new parameter ε:*

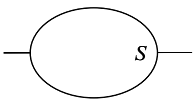
(i) 

$= \omega$ 

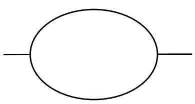
;

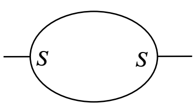
(ii) 

$=$  $= 0$;

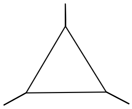
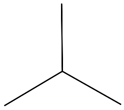
(iii) 

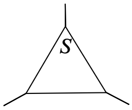
$= 0$;

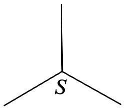
(iv)  $= \dfrac{\delta^2 - 2}{\delta}$;

(v) 

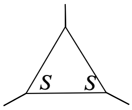
$= \gamma \dfrac{\delta^3 - 2\delta}{\delta^2 - 1}$;

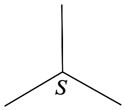
(vi)  $= \dfrac{\delta^2 - 3}{\delta}$  ;

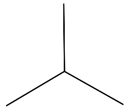
(vii) 

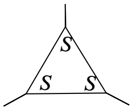
$= -\dfrac{1}{\delta}$ 

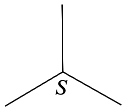
;

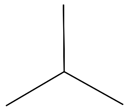
(viii) 

$= (\gamma - 1)$ 

$- \gamma \dfrac{\delta}{\delta^2 - 1}$  ;

(ix) 

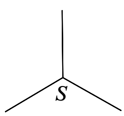
$= \delta_{\omega,1}\varepsilon$ 

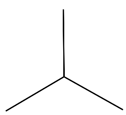
$+ \gamma(\gamma - 1)\dfrac{\delta^2}{\delta^2 - 1}$  ,

where $\varepsilon \in \mathbb{C}$ is a formal variable.

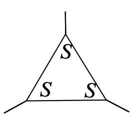
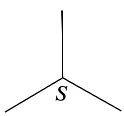
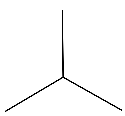
Proof. The relations (i)–(viii) follow from relations (6) and (7) in the planar algebra $\mathscr{P}_\bullet$.

For the relation (ix), since

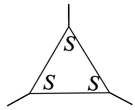


and 

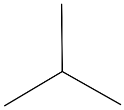
form an orthogonal basis of the f_2-cutdown of $\mathscr{P}_{3,+}$, we assume that

 $= a$  $+ b$  . (8)

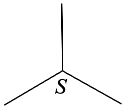
Note that both



and

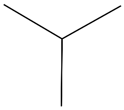


are invariant under 1-click rotation, while



is invariant under 1-click rotation if and only if $\omega = 1$, thus $a = \delta_{\omega,1}\varepsilon$, where $\delta_{\omega,1}$ is the Kronecker delta.

For the coefficient b, we multiply

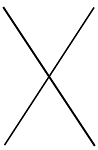


from the bottom of both sides of equation (8). Since the f_2-cutdown of $\mathscr{P}_{1,+}$ is one-dimensional, applying relation (viii), we have

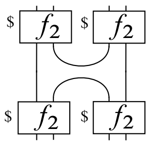
$$b = \gamma(\gamma - 1)\frac{\delta^2}{\delta^2 - 1}. \qquad \square$$

3.3. Relations in 4-boxes. We proceed to discuss the Thurston relation in the f_2-cutdown of $\mathscr{P}_{4,+}$.

Notation 3.4. We define

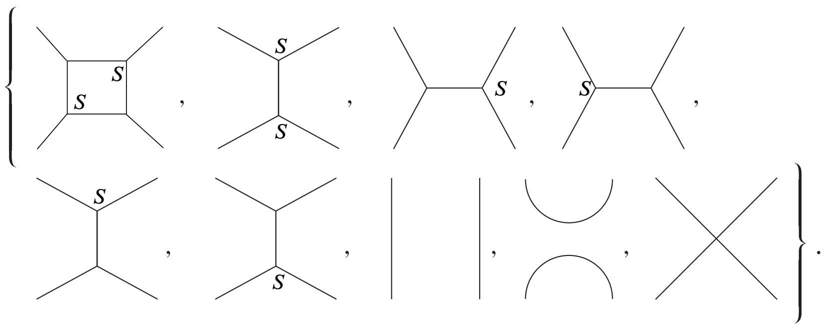


:=

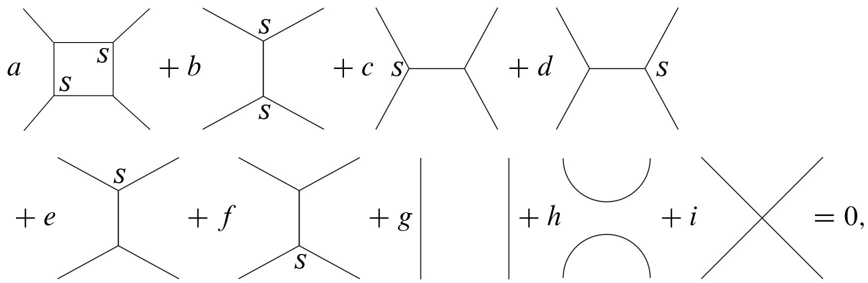


.

Lemma 3.5. *The following set B is a basis of the f_2-cutdown of $\mathscr{P}_{4,+}$,*

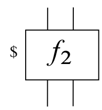


Proof. Note that the f_2-cutdown of any standard form of $\mathscr{P}_{4,+}$ in Corollary 2.5 is in the linear span of B, thus the f_2-cutdown of $\mathscr{P}_{4,+}$ is spanned by B. Suppose

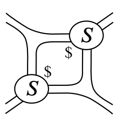


(9)

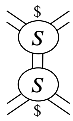
for some $a, b, c, d, e, f, g, h, i \in \mathbb{C}$. Recall that each string in equation (9) represents



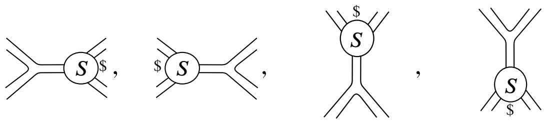
in $\mathscr{P}_\bullet$. We consider it as an equation in $\mathscr{P}_{4,+}$ and rewrite it in terms of the basis of $\mathscr{P}_{4,+}$ in Corollary 2.5. The coefficients of



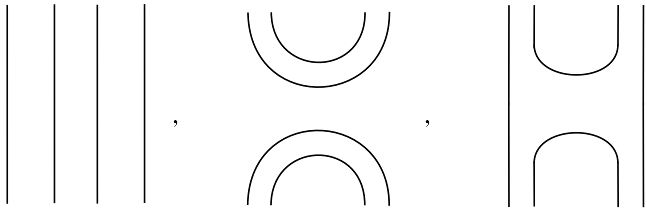
and



are a and b respectively, thus $a = b = 0$. Furthermore, the coefficients of

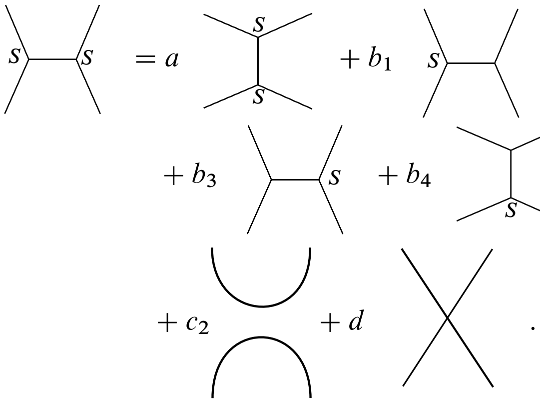


are c, d, e, and f respectively, thus $c = d = e = f = 0$. Finally, the coefficients of the Temperley–Lieb diagrams



in $\mathcal{P}_{4,+}$ are g, h, and i respectively, thus $g = h = i = 0$. Therefore B is linearly independent, and B is a basis. □

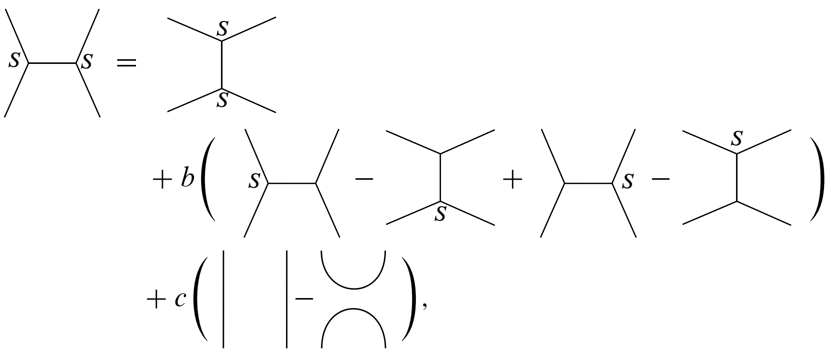
Since B is a basis, we have the unshaded $2 \leftrightarrow 2$ move (3) for S in terms of formal variables:



(10)

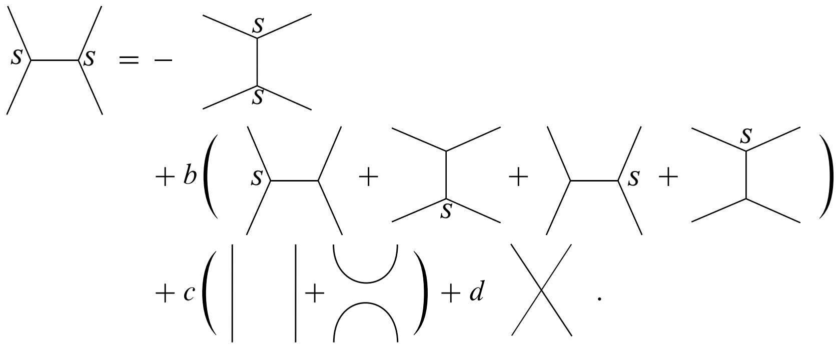
Using the rotational symmetry, we can immediately simplify the formal variables.

Lemma 3.6. *Either* $a = 1$ *and*



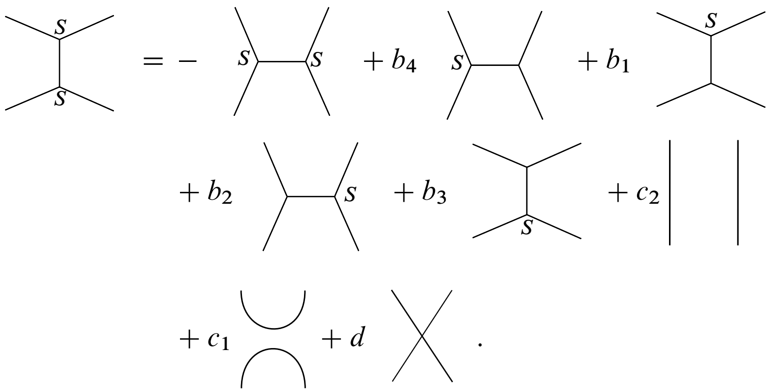
(11)

or $a = -1$ *and*



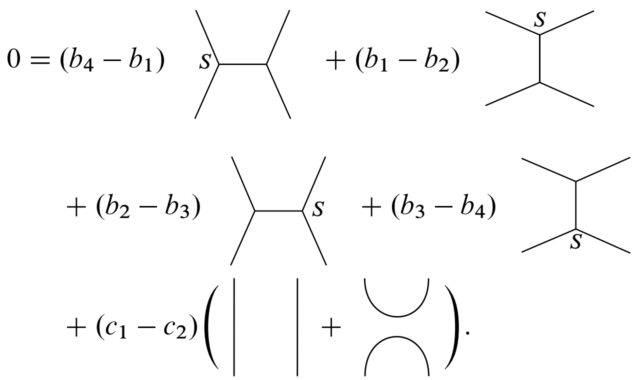
(12)

Proof. If $a = -1$, then we apply rotation by $\frac{\pi}{2}$ to equation (10) and obtain



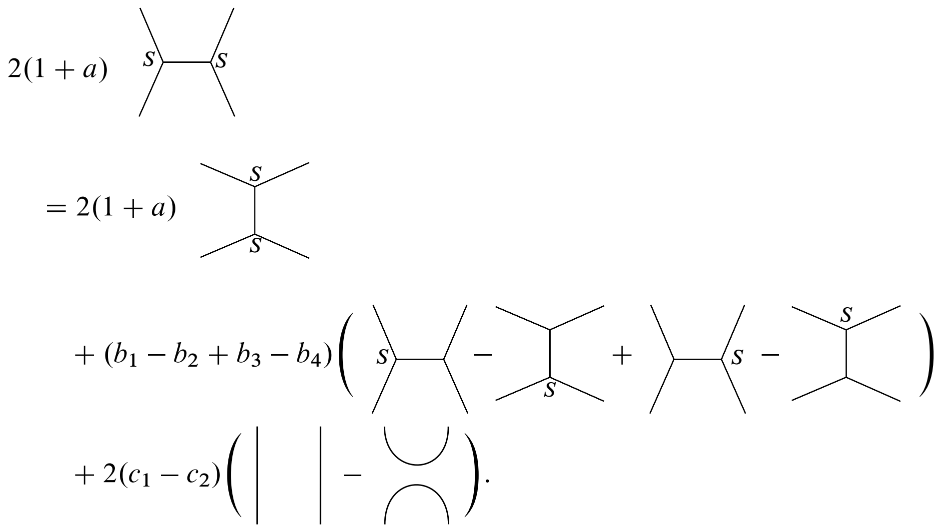
(13)

Subtracting equations (10) from (13) yields



Thus $b_1 = b_2 = b_3 = b_4$ and $c_1 = c_2$ and we obtain equation (12).

If $a \neq -1$, then we apply rotations by $\frac{k\pi}{2}$, $k = 0, 1, 2, 3$, to equation (10) and take the alternating sum of the four resultant equations. We have



Taking the quotient by $2(1 + a)$, we obtain equation (11). □

Furthermore, we solve for the coefficients b, c, d in Lemma 3.6 in terms of δ, γ, ω.

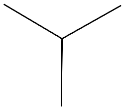
Lemma 3.7. *If* $a = 1$, *then*

$$\begin{cases} b = -(\gamma - 1)\dfrac{\delta}{\delta^2 - 2 + \omega + \omega^{-1}}, \\ c = -\gamma\dfrac{\delta}{\delta^2 - 1}. \end{cases} \tag{14}$$

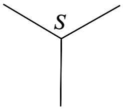
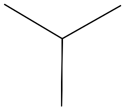
If $a = -1$, *then*

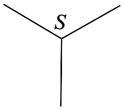
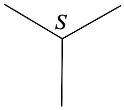
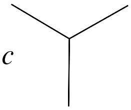
$$\begin{cases} b = \dfrac{(\gamma - 1)\delta}{\delta^2 - 2 - \omega - \omega^{-1}}, \\ c = \gamma\delta\Big(2\dfrac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1} - \dfrac{1}{\delta^2 - 1}\Big), \\ d = -2\gamma\dfrac{\delta^2}{\delta^4 - 3\delta^2 + 1}. \end{cases} \tag{15}$$

Proof. When $a = 1$, applying

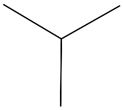
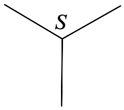


to the bottom on both sides of equation (11), we obtain

$(\gamma - 1)$  $-\gamma \frac{\delta}{\delta^2 - 1}$ 

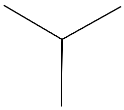
$= b\Big((\omega + \omega^{-1})\big(-\frac{1}{\delta}\big)$  $- \frac{\delta^2 - 2}{\delta}$  $\Big) + c$  .

By comparing the coefficients of

 and  ,

we solve for b, c as in equation (14).

When $a = -1$, we apply



to the bottom on both sides of equation (12). By a similar computation, we solve for b, c, d as in equation (15). □

Corollary 3.8. *The formal varible ϵ in Lemma 3.3 satisfies the following equations*:

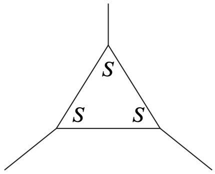
(1) *if $a = 1$, then*

$$\frac{\gamma\delta(\delta^2 - 3)}{\delta^2 - 1} - \frac{2(\gamma - 1)^2\delta}{\delta^2 - 2 + \omega + \omega^{-1}} = \delta_{\omega,1}\epsilon; \tag{16}$$

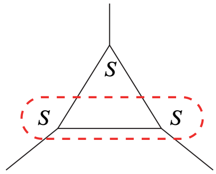
(2) *if $a = -1$, then*

$$\gamma\delta\Big(2\frac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1} - 1\Big) + \frac{2(\gamma - 1)\delta}{\delta^2 - 2 - \omega - \omega^{-1}} = \delta_{\omega,1}\epsilon. \tag{17}$$

Proof. By relation (i) in Lemma 3.3, we have

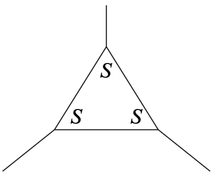


$=$

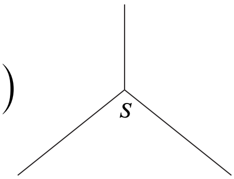


.

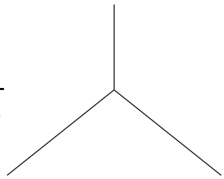
Suppose $a = 1$. By applying the unshaded $2 \leftrightarrow 2$ move, namely equation (11), in the dotted circle and relations (i)–(viii) in Lemma 3.3, we obtain



$$= \left(\frac{\gamma\delta(\delta^2 - 3)}{\delta^2 - 1} - \frac{2(\gamma - 1)^2\delta}{\delta^2 - 2 + \omega + \omega^{-1}} \right)$$



$$+ \gamma(\gamma - 1)\frac{\delta^2}{\delta^2 - 1}$$



.

By relation (ix) in Lemma 3.3, we have that equation (16) holds.

Suppose $a = -1$. It follows similarly that equation (17) holds. □

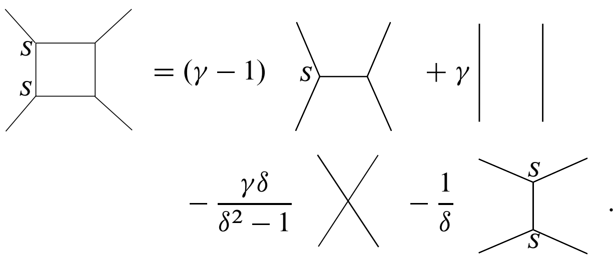
Theorem 3.9. *Suppose $\mathscr{P}_\bullet$ is a planar algebra generated by a non-trivial 3-box satisfying the Thurston relation, then $\mathscr{P}_\bullet$ is determined by $(\delta, \gamma, \omega, a, a')$, where δ^2 is the index, γ the ratio of the trace of the two orthogonal minimal idempotents, ω is the rotation eigenvalue and a, a' are the signs in the unshaded and shaded $2 \leftrightarrow 2$ moves.*

Proof. Suppose $\mathscr{P}_\bullet$ and $\mathscr{P}'_\bullet$ are two planar algebras and there exist orthogonal minimal idempotents $P, Q \in \mathscr{P}_\bullet$ and $P', Q' \in \mathscr{P}'_\bullet$. Then we can construct uncappable rotation eigenvectors S and S' as in section 3 satisfying the same quadratic relation. Since the planar algebra is generated by a 3-box, we define a map $\phi : \mathscr{P}_3 \to \mathscr{P}'_3$ by sending S to S'. Since all the coefficients in the skein theory are determined by $(\delta, \gamma, \omega, a, a')$, ϕ extends to a planar algebra isomorphism. □

3.4. Classification. In this subsection, we prove our classification result (Theorem 1.2) for the generic case. In Theorem 3.9, we show that $\mathscr{P}_\bullet$ is determined by $\delta > 2, \gamma, \omega, a$ and a'. First we prove that $\omega = 1$ and $a, a' = 1$, so $\mathscr{P}_\bullet$ is determined by δ, γ. Then we identify the planar algebra $\mathscr{P}_\bullet$ parameterized by δ and γ with $\mathscr{P}_\bullet^H(q, r)$.

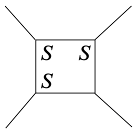
Theorem 3.10. *Suppose $\mathscr{P}_\bullet$ is a singly-generated Thurston-relation planar algebra and* $\dim(\mathscr{P}_{4,+}) = 24$ *with parameters $(\delta, \gamma, \omega, a, a')$ as in Theorem 3.9. Then the rotation eigenvalue $\omega = 1$.*

Proof. A direct computation by Lemma 3.3 shows that



(18)

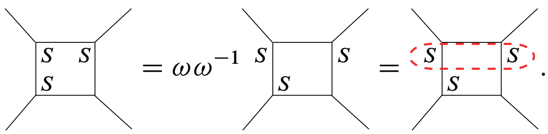
Suppose $a = 1$. We have two different approaches to rewrite the element



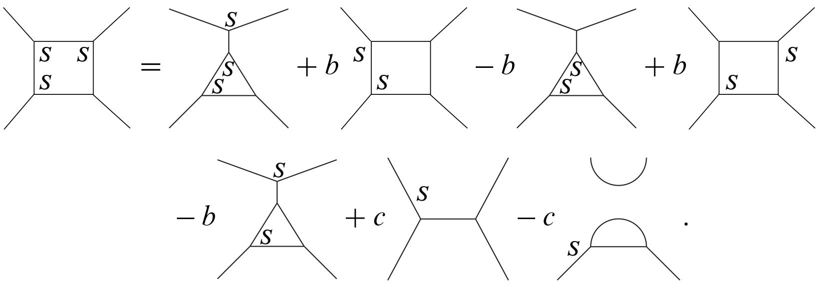
(19)

as a linear combination of the basis B as stated in Lemma 3.5.

Approach 1. By relation (i) in Lemma 3.3, we have

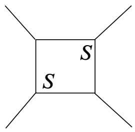


By applying equation (11) in the dotted circle, we obtain the following equation:



(20)

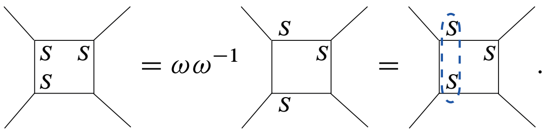
The right hand side of equation (20) can be expressed as a linear combination in terms of the basis B using Lemma 3.3 and equation (18). By a direct computation, the coefficient of



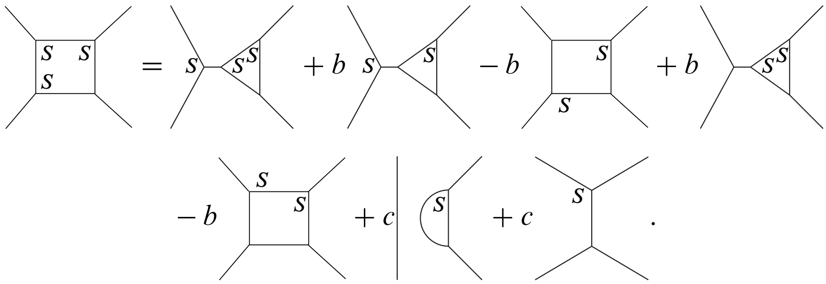
(21)

in the linear expression is $b\omega^{-1}$.

Approach 2. By relation (i) in Lemma 3.3, we also have



By applying equation (12) in the dotted circle, we obtain the following equation:



(22)

The right hand side of equation (22) can be exprresed as a linear combination in terms of the basis B using Lemma 3.3 and equation (18). By a direct computation the coefficient of (21) in the linear expression is $b\omega$.

Therefore by equating the coefficients of (21) in Approach 1 and Approach 2, we obtain the following equation,

$$b\omega = b\omega^{-1}. \tag{23}$$

If $\omega \neq 1$, then ω is $e^{\frac{2\pi\sqrt{-1}}{3}}$ or $e^{\frac{4\pi\sqrt{-1}}{3}}$. Therefore we have that $b = 0$. Combined with equation (14), we have

$$-(\gamma - 1)\frac{\delta}{\delta^2 - 3} = 0.$$

This implies that $\gamma = 1$ and thus equation (16) yields to

$$\frac{\delta(\delta^2 - 3)}{\delta^2 - 1} = 0.$$

It follows that the above equation does not have a solution when $\delta > 2$ and thus it leads to a contradiction. Thus $\omega = 1$.

Now suppose $a = -1$. Similar to the case in which $a = 1$, we can rewrite the element (19) as a linear comination of the basis B in two different approaches. By comparing the coefficients of (21) in the linear combinations, we obtain that $b = 0$ if $\omega \neq 1$. Combined with equation (12), we have

$$(\gamma - 1)\frac{\delta}{\delta^2 - 1} = 0.$$

This implies that $\gamma = 1$ and thus equation (17) yields to

$$\delta\Big(2\frac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1} - 1\Big) = 0.$$

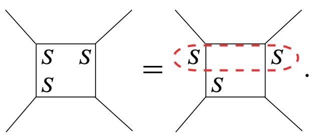
It follows that the above equation does not have a solution when $\delta > 2$ and thus it leads to a contradiction. This concludes that $\omega = 1$. $\square$

Theorem 3.11. *If $\mathscr{P}_\bullet$ is a singly-generated Thurston-relation planar algebra and* $\dim(\mathscr{P}_{4,+}) = 24$ *with parameters* $(\delta, \gamma, 1, a, a')$ *in Theorem* 3.9*, then* $a = a' = 1$.

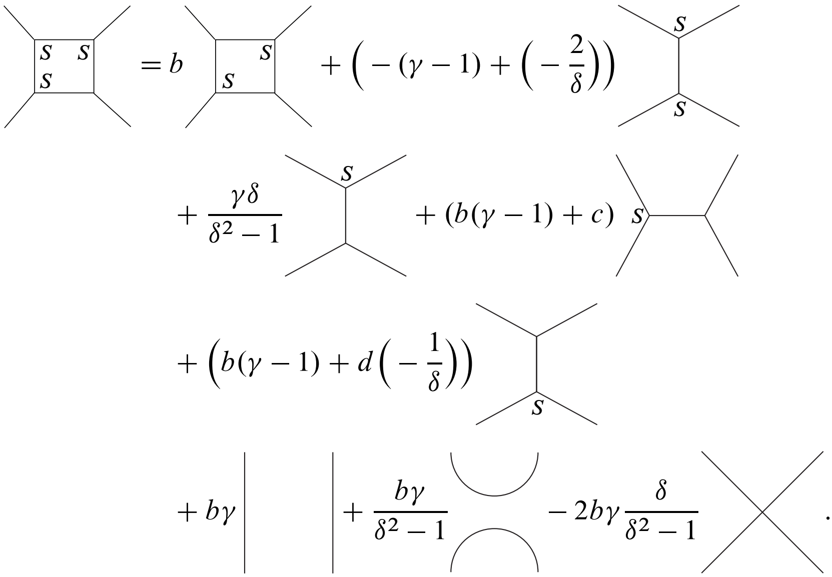
Proof. We prove that a must take the value 1 and it follows from the same proof in the f_2 cut-down in the dual planar algebra that a' must also take the value 1.

Suppose $a = -1$. Similar to the proof of Theorem 3.10, we rewrite the element (21) in two different approaches.

Approach 1. By relation (i) in Lemma 3.3, we have

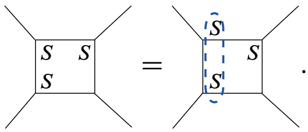


By applying equation (12) in the dotted circle, equation (18) and relations (i)–(ix) in Lemma 3.3, we obtain the following equation:

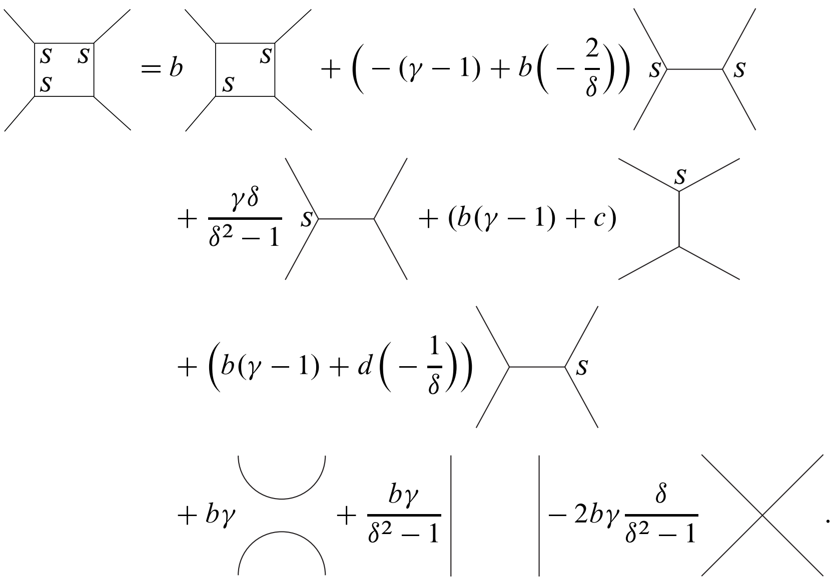


(24)

Approach 2. By relation (i) in Lemma 3.3, we also have

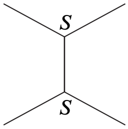


By applying equation (12) in the dotted circle, equation (18) and relation (i)–(ix) in Lemma 3.3, we obtain the following equation:



(25)

Note that the f_2-cutdown of $\mathscr{P}_{4,+}$ has the basis B as in Lemma 3.5. From equation (24), we deduce that the coefficient of

 (26)

in the linear expression of (19) with respect to the basis B is

$$-(\gamma-1)+b\Big(-\frac{2}{\delta}\Big).$$

While from equation (25), we deduce that the coefficient of (26) in the linear expression of (19) with respect to the basis B is

$$(\gamma-1)-b\Big(-\frac{2}{\delta}\Big),$$

since $a=-1$. This implies that

$$-(\gamma-1)+b\Big(-\frac{2}{\delta}\Big)=0. \tag{27}$$

Note that

$$b=\frac{(\gamma-1)\delta}{\delta^2-4},$$

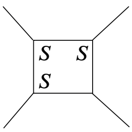
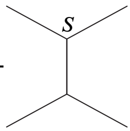
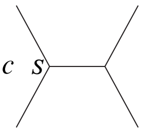
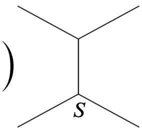
when $\omega=1$ by equation (15) in Lemma 3.7. Therefore, the above equation yields to

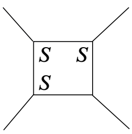
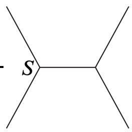
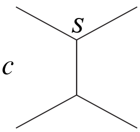
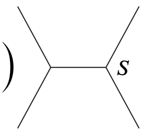
$$-(\gamma-1)+\frac{(\gamma-1)\delta}{\delta^2-4}\Big(-\frac{2}{\delta}\Big)=0,$$

$$(\gamma-1)\Big(\frac{\delta^2-2}{\delta^2-4}\Big)=0.$$

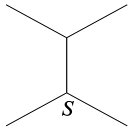
Since $\delta>2$, we have that $\gamma=1$ and thereby $b=0$.

Now equations (24) and (25) reduce to the following equations:

 $= \dfrac{\delta}{\delta^2-1}$  $+\,c$  $+\,d\Big(-\dfrac{1}{\delta}\Big)$  , (28)

 $= \dfrac{\delta}{\delta^2-1}$  $+\,c$  $+\,d\Big(-\dfrac{1}{\delta}\Big)$  . (29)

By comparing the coefficients of



in the two expressions as in equations (28) and (29), we obtain that

$$d\left(-\frac{1}{\delta}\right) = 0.$$

Note that

$$d = -2\gamma \frac{\delta^2}{\delta^4 - 3\delta^2 + 1}$$

by equation (15) in Lemma 3.7. However, this implies $\delta = 0$ since $\gamma = 1$ and this is a contradiction. □

Remark 4. In the reduced case, one can always assume that (26) is in the basis of $\mathscr{P}_{4,+}$ and (21) is not. Therefore we still obtain equation (27) since the coefficients of (26) in the linear expression of (19) are the same obtained from two different approaches. Thus the above proof works for the reduced case in which dim($\mathscr{P}_{4,+}$)=23.

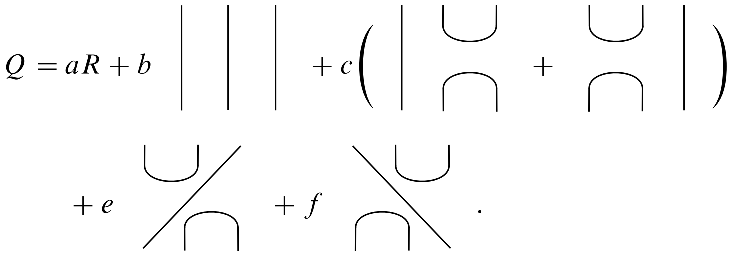
These two theorems show that a singly-generated Thurston-relation planar algebra with parameters $(\delta, \gamma, \omega, a, a')$ satisfies $\omega = a = a' = 1$. Thus the planar algebra is parameterized by (δ, γ). With the following lemma and theorem, we identify any singly-generated Thurston-relation planar algebra as $\mathscr{P}_\bullet^H(q, r)$ for some (q, r). Note that the Jones-Wenzl idempotent f_3 is a sum of two minimal idempotents P and Q in $\mathscr{P}_{3,+}^H(q, r)$. In the following lemma, we compute the trace formula for the two minimal idempotents.

Lemma 3.12. *The two minimal idempotents* $P, Q \in \mathscr{P}_{3,+}^H(q, r)$ *with* $P + Q = f_3$ *statisfy the following trace formulas*:

$$\mathrm{Tr}(P) = \frac{(r - r^{-1})(rq - r^{-1}q^{-1})(rq^{-2} - r^{-1}q^2)}{(q + q^{-1})(q - q^{-1})^3},$$

$$\mathrm{Tr}(Q) = \frac{(r - r^{-1})(rq^2 - r^{-1}q^{-2})(rq^{-1} - r^{-1}q)}{(q + q^{-1})(q - q^{-1})^3}.$$

Proof. Note that the 5 Temperley–Lieb diagrams and the generator R of $\mathcal{P}_\bullet^H(q,r)$ as stated in Theorem 2.2 form a basis of $\mathcal{P}^H_{3,+}(q,r)$. We assume that



Since Q is a minimal subprojection of the Jones-Wenzl idempotent f_3, i.e., $e_k Q = Qe_k = 0$ for $k = 1, 2$ where e_k is the Jones proejctions, by comparing the coefficients with respect to $\mathcal{B}$ of $e_k Q = Qe_k = 0$ for $k = 1, 2$ we obtain the following equations:

$$\begin{cases} ar + c + e\dfrac{r - r^{-1}}{q - q^{-1}} = 0, \\ b + c\dfrac{r - r^{-1}}{q - q^{-1}} + f = 0, \\ ar^{-1} + c + f\dfrac{r - r^{-1}}{q - q^{-1}} = 0, \\ a(q - q^{-1}) + b + c\dfrac{r - r^{-1}}{q - q^{-1}} + e = 0. \end{cases}$$

The above linear system have a solution in terms of a, b, r and q:

$$\begin{cases} c = a\dfrac{r^{-1}(q - q^{-1})^2}{r^2 + r^{-2} - q^2 - q^{-2}} - b\dfrac{(r - r^{-1})(q - q^{-1})}{r^2 + r^{-2} - q^2 - q^{-2}}, \\ e = -\dfrac{q - q^{-1}}{r - r^{-1}}\Big(ar + ar^{-1}\dfrac{(q - q^{-1})^2}{r^2 + r^{-2} - q^2 - q^{-2}} - b\dfrac{(r - r^{-1})(q - q^{-1})}{r^2 + r^{-2} - q^2 - q^{-2}}\Big), \\ f = -\dfrac{q - q^{-1}}{r - r^{-1}}\Big(ar^{-1}\dfrac{(r - r^{-1})^2}{r^2 + r^{-2} - q^2 - q^{-2}} + b\dfrac{(r - r^{-1})(q - q^{-1})}{r^2 + r^{-2} - q^2 - q^{-2}}\Big). \end{cases}$$

Since Q is an idempotent, so we set up equations by comparing the coefficients of Q^2 and Q and find two solutions:

$$\begin{cases} a = \dfrac{1}{q + q^{-1}}, \\ b = \dfrac{q^{-1}}{q + q^{-1}}; \end{cases} \quad \text{or} \quad \begin{cases} a = -\dfrac{1}{q + q^{-1}}, \\ b = \dfrac{q}{q + q^{-1}}. \end{cases} \tag{30}$$

Without loss of generality, we choose the first solution in equation (30) to determine the coefficient of the linear expression of the minimal idempotent Q. Then we have

$$\begin{aligned}
\mathrm{Tr}(Q) =& ar\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^2 + b\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^3 + c2\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^2 + (e+f)\frac{r-r^{-1}}{q-q^{-1}} \\
=& \frac{r}{q+q^{-1}}\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^2 + \frac{q^{-1}}{q+q^{-1}}\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^3 \\
&+ \frac{q-q^{-1}}{q+q^{-1}}\frac{r^{-1}q-r^{-1}q}{r^2+r^{-2}-q^2-q^{-2}}2\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^2 \\
&+ \Big(-\frac{r+r^{-1}}{q+q^{-1}} - 2\frac{q-q^{-1}}{q+q^{-1}}\frac{r^{-1}q-r^{-1}q}{r^2+r^{-2}-q^2-q^{-2}}\Big) \\
=& \frac{r}{q+q^{-1}}\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^2 + \frac{q^{-1}}{q+q^{-1}}\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^3 \\
&- \frac{r+r^{-1}}{q+q^{-1}} + 2\frac{q-q^{-1}}{q+q^{-1}}\frac{r^{-1}q-r^{-1}q}{r^2+r^{-2}-q^2-q^{-2}}\Big(\Big(\frac{r-r^{-1}}{q-q^{-1}}\Big)^2-1\Big) \\
=& \frac{1}{(q+q^{-1})(q-q^{-1})^3}(r(r-r^{-1})^2(q-q^{-1}) + q^{-1}(q-q^{-1})^3 \\
&\qquad - (r+r^{-1})(q-q^{-1})^3 \\
&\qquad + 2(r^{-1}q-rq^{-1})(q-q^{-1})^2) \\
=& \frac{(r-r^{-1})(r^2q+r^{-2}q^{-1}-q^3-q^{-3})}{(q^2-q^{-2})(q-q^{-1})} \\
=& \frac{(r-r^{-1})(rq^2-r^{-1}q^{-2})(rq^{-1}-r^{-1}q)}{(q+q^{-1})(q-q^{-1})^3}
\end{aligned}$$

Applying a similar computation, we see that $P = f_3 - Q$ is an idempotent with

$$\mathrm{Tr}(P) = \frac{(r-r^{-1})(rq-r^{-1}q^{-1})(rq^{-2}-r^{-1}q^2)}{(q+q^{-1})(q-q^{-1})^3}.$$

We remark that the second solution in equation (30) simply switches P and Q. □

Now we identify the two-parameter family of singly-generated TRPA with the HOMFLY-PT planar algebras $\mathscr{P}^H_\bullet(q,r)$.

Theorem 3.13. *Suppose $\mathscr{P}_\bullet$ is a singly-generated Thurston-relation planar algebra and* $\dim \mathscr{P}_{4,\pm} = 24$. *Then it is isomorphic to the semisimple quotient of $\mathscr{P}^H_\bullet(q,r)$ for some (q,r).*

Proof. Suppose $\mathscr{P}_\bullet$ is a singly-generated Thurston-relation planar algebra as stated in the theorem. We find (q,r) such that $\mathscr{P}^H_\bullet(q,r)$ has the same parameter (δ,γ).

Case 1: $\gamma = ((\delta+2)(\delta-1))/((\delta-2)(\delta+1))$. In this case, we show that that $\mathscr{P}_{\bullet}^{H}(1,1)$ with a circle parameter δ is a solution. First note that this planar algebra has the desired δ. Therefore, we only need to show that the ratio of two traces of the two minimal idempotents in $\mathscr{P}_{3}^{H}(1,1)$ equals to γ. By Lemma 3.12, we know that $\mathscr{P}_{3}^{H}(1,1)$ with circle parameter δ has two minimal idempotents P, Q with

$$\begin{aligned} \mathrm{Tr}(P) &= \frac{\delta(\delta+2)(\delta-1)}{2}, \\ \mathrm{Tr}(Q) &= \frac{\delta(\delta-2)(\delta+1)}{2}. \end{aligned}$$

The ratio equals to

$$\frac{(\delta+2)(\delta-1)}{(\delta-2)(\delta+1)}.$$

Therefore, $\mathscr{P}_{\bullet}^{H}(1,1)$ with circle parameter δ gives a solution.

Case 2: $\gamma \neq ((\delta+2)(\delta-1))/((\delta-2)(\delta+1))$. Let (q,r) be the solution of

$$q^2+q^{-2} = 2(\delta^2-2)/\Big(\delta^2-\Big(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}\Big)^2\Big)-2, \tag{31}$$

$$r-r^{-1} = \delta(q-q^{-1}). \tag{32}$$

(Note that the assumption $\gamma \neq \frac{(\delta+2)(\delta-1)}{(\delta-2)(\delta+1)}$ implies that $q \neq \pm 1$.) From equation (31), we see that

$$\begin{aligned} \delta^2(q^2+q^{-2}-2)+2 &= \Big(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1})\Big)^2-2, \\ \delta^2(q-q^{-1})^2+4 &= \Big(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1})\Big)^2, \\ (r-r^{-1})^2+4 &= \Big(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1})\Big)^2. \end{aligned} \tag{33}$$

From equation (33), we obtain

$$r+r^{-1} = \pm\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1}).$$

Then we have the following equations:

$$r^2 - r^{-2} = \pm(\delta^2 - 2)\frac{\gamma - 1}{\gamma + 1}(q^2 - q^{-2}), \tag{34}$$

$$r^2 + r^{-2} = \delta^2(q^2 + q^{-2} - 2) + 2. \tag{35}$$

Recall that $q \neq \pm 1$, we define A, B by the formulas in Lemma 3.12

$$A = \frac{(r - r^{-1})(rq - r^{-1}q^{-1})(rq^{-2} - r^{-1}q^2)}{(q + q^{-1})(q - q^{-1})^3},$$

$$B = \frac{(r - r^{-1})(rq^2 - r^{-1}q^{-2})(rq^{-1} - r^{-1}q)}{(q + q^{-1})(q - q^{-1})^3}.$$

Then we have

$$A + B = \frac{(r - r^{-1})((r^2 + r^{-2})(q + q^{-1}) + 2(q^3 + q^{-3}))}{(q + q^{-1})(q - q^{-1})^3},$$

$$A - B = \frac{(r - r^{-1})(r^2 - r^{-2})(q - q^{-1})}{(q + q^{-1})(q - q^{-1})^3}.$$

Combined with the equations (32), (34), and (35), we have the following two solutions:

$$\begin{cases} A = (\delta^3 - 2\delta)\dfrac{1}{1+\gamma}, \\ B = (\delta^3 - 2\delta)\dfrac{\gamma}{1+\gamma}; \end{cases} \quad \text{or} \quad \begin{cases} A = (\delta^3 - 2\delta)\dfrac{\gamma}{1+\gamma}, \\ B = (\delta^3 - 2\delta)\dfrac{1}{1+\gamma}. \end{cases}$$

In both cases, the singly-generated Thurston-relation planar algebra with (δ, γ) and $\mathscr{P}^H_\bullet(q, r)$ have the same skein theory. Therefore, $\mathscr{P}_\bullet$ is isomorphic to the semisimple quotient of the planar algebra $\mathscr{P}^H_\bullet(q, r)$. □

Remark 5. There are eight solutions (q, r) in the above theorem. Proposition 2.3 shows that the eight corresponding planar algebras are isomorphic.

4. Reduced case

In this section, we will classify subfactor planar algebras $\mathscr{P}_\bullet$ generated by a 3-box with the Thurston relation for the reduced case, namely $\mathscr{P}_{4,\pm} \leq 23$.

4.1. The case for at most 22 dimensional 4-box space. We classify the singly-generated TRPA with $\dim(\mathscr{P}_{4,+}) \leq 22$.

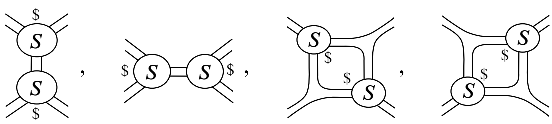
Proposition 4.1. *Suppose $\mathscr{P}_\bullet$ is a singly-generated TRPA. Then* $\dim \mathscr{P}_{4,\pm} \leq 22$ *if and only if $\mathscr{P}_\bullet$ is either the E_6 or $E_6^{(1)}$ subfactor planar algebra.*

Proof. If $\delta > 2$, it is shown in [15] that the 14 Temperley–Lieb diagrams and the 8 elements in the annular consequence AC are linearly independent. Since $\dim(\mathscr{P}_{4,+}) \leq 22$, the annular multiplicity sequence has to be $0^2 10$. By Theorem 5.1.11 in [16], there is no such subfactor planar algebras. If $\delta \leq 2$, then the 2-supertransitive subfactor planar algbera has principal graph E_6 or $E_6^{(1)}$.

Conversely, E_6 or $E_6^{(1)}$ subfactor planar algebras are generated by a 3-box, and the 4-box space $\mathscr{P}_{4,+}$ is spanned by Temperley–Lieb diagrams and the annular consequence AC. So the generator statisfies the Thurston relation. Therefore, E_6 or $E_6^{(1)}$ subfactor planar algebras are sinlgy-generated TRPA. Moreover, the dimensons of their 4-box spaces are 21 and 22 respectively. $\square$

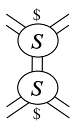
Thus the main Theorem 1.2 holds for the reduced case in which $\dim \mathscr{P}_{4,\pm} \leq 22$ by noting that $E_6^{(1)}$ is the HOMFLY-PT planar algebra $\mathscr{P}_\bullet^H(q,r)$ with $q = \exp^{\frac{\sqrt{-1}\pi}{6}}$ and $r = \sqrt{-1}$.

4.2. The case for 23 dimensional 4-box space. In this section, we classify subfactor planar algebras $\mathscr{P}_\bullet$ generated by a 3-box S with the Thurston relation, such that $\dim(\mathscr{P}_{4,\pm}) = 23$. In this case, we have $\delta > 2$ and $\mathscr{P}_{3,+} = 6$. By the result of Vaughan Jones [15], the 14 Temperley–Lieb diagrams and the 8 diagrams in the annular consequence are linearly independent. Then one of the diagram with two generators

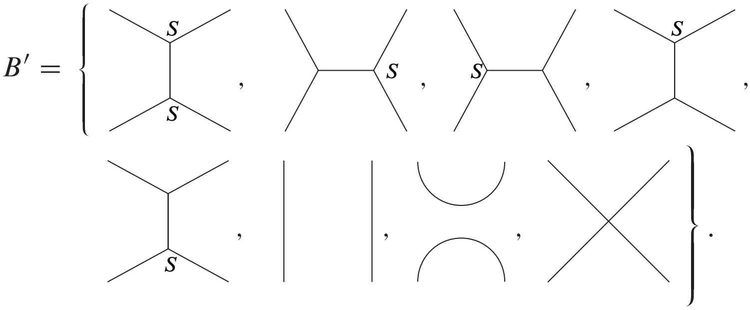


is linearly independent with 22 lower terms. Otherwise $\mathscr{P}_{4,\pm} = 22$.

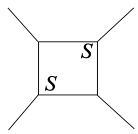
Up to rotation and the duality of the shading, we can assume that



and the 22 lower terms form a basis. Similarly to Lemma 3.5, the basis for the f_2-cutdown of $\mathscr{P}_{4,+}$ is given by



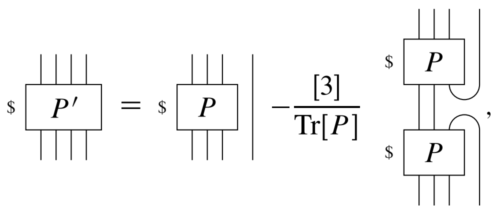
Note that all results in §3 work for the case $\dim(\mathscr{P}_{4,\pm}) = 23$, except for Theorem 3.10. There we used the fact that



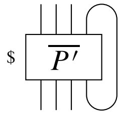
is linearly independent with B', which is no longer true for the reduced case here. Now we give a different proof of $\omega = 1$ for the reduced case. Consequently the main Theorem 1.2 holds for the reduced case $\dim \mathscr{P}_{4,\pm} \leq 23$.

Theorem 4.2. *If $\mathscr{P}_\bullet$ is a singly-generated Thurston-relation subfactor planar algebra with parameters $(\delta, \gamma, \omega, a, a')$ and $\dim(\mathscr{P}_{4,\pm}) = 23$, then $\omega = 1$.*

Proof. By Proposition 4.1, we have that $\delta > 2$ and thus $\mathscr{P}_\bullet$ has annnular multiplicity $0^2 11$. Since $\dim(\mathscr{P}_{4,+}) = 23$, we know that there exists one vertex of the principal graph for $\mathscr{P}_\bullet$ at depth 3 whose degree is at most 2. Let P be the corresponding minimal projection in $\mathscr{P}_{3,+}$ and $Q = f_3 - P$. We define $P' \in \mathscr{P}_{4,+}$ to be the following projection:



where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ is the n^{th} quantum number (where q is defined so that $\delta = [2] = q + q^{-1}$ and $q > 1$). Let $\overline{P'}$ be the contragredient of P' and $\bigcap_4 \overline{P'}$ be the following element



.

It follows that $\bigcap_4 \overline{P'}$ belongs to the subspace in $\mathscr{P}_{3,+}$ spanned by $\{S, f_3\}$ and thus let $\mathrm{Coeff}_{\in\bigcap_4 \overline{P'}}(S)$ be the coefficient of S in the linear expression of $\bigcap_4 \overline{P'}$ with respect to $\{S, f_3\}$. By Theorem 3.3 in [26], we have that

$$(\gamma - 1) - \frac{\sigma + \sigma^{-1}}{[3]} = -(1+\gamma)\frac{[4]}{[3]}(\mathrm{Coeff}_{\in\bigcap_4 \overline{P'}}(S)), \tag{36}$$

where $\sigma^2 = \omega$.

If $\omega \neq 1$, then $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$ or $e^{\frac{4\pi\sqrt{-1}}{3}}$. We have that $\sigma + \sigma^{-1} = \pm 1$. We determine $\mathrm{Coeff}_{\in\bigcap_{n+1} \bar{P}'}(S)$ in equation (36) for two cases:

Case 1: $\overline{P'} = P'$. Then we have

$$\bigcap\nolimits_4 \overline{P'} = \frac{\mathrm{Tr}(P')}{\mathrm{Tr}(P)} P.$$

By the definition of P', we know that $\mathrm{Tr}(P') = [2]\,\mathrm{Tr}(P) - [3]$. Since

$$\mathrm{Tr}(P) = [4]\frac{1}{1+\gamma} \quad \text{and} \quad P = \frac{1}{1+\gamma}S + \frac{1}{1+\gamma}f_3,$$

we have that

$$\mathrm{Coeff}_{\in\bigcap_4 \overline{P'}}(S) = \left([2] - \frac{[3]}{[4]}(1+\gamma)\right)\frac{1}{1+\gamma}.$$

Therefore, equation (36) yields to

$$(\gamma - 1) \pm \frac{1}{[3]} = -\frac{[4][2]}{[3]} + 1 + \gamma.$$

From the definition of quantum numbers and $\delta = [2]$, we have

$$\delta^4 - 4\delta^2 + 2 \pm 1 = 0. \tag{37}$$

But that equation (37) has no solution δ such that $\delta > 2$ and thus it is a contradiction. So $\omega = 1$.

Case 2: $\overline{P'} \neq P'$. Then we have

$$\bigcap_4 \overline{P'} = \frac{\mathrm{Tr}(P')}{\mathrm{Tr}(Q)} Q.$$

By the definition of P', we know that $\mathrm{Tr}(P') = [2]\,\mathrm{Tr}(P) - [3]$. Since

$$\mathrm{Tr}(P) = [4]\frac{1}{1+\gamma}, \quad \mathrm{Tr}(Q) = [4]\frac{\gamma}{1+\gamma}$$

and

$$Q = -\frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} f_3,$$

we have that

$$\mathrm{Coeff}_{\in \bigcap_4 \overline{P'}}(S) = -\Big([2] - \frac{[3]}{[4]}(1+\gamma)\Big)\frac{1}{\gamma(1+\gamma)}.$$

Therefore, equation (36) yields to

$$(\gamma - 1) \pm \frac{1}{[3]} = \frac{1}{\gamma}\Big(\frac{[2][4]}{[3]} - (1+\gamma)\Big).$$

From the definition of quantum numbers, we have

$$[3]\gamma^2 \pm \gamma - [5] = 0. \tag{38}$$

This is an equation for δ and γ. When $\omega \neq 1$, we have another equation for δ and γ in Corollary 3.8.

Subcase 1. When $a = 1$, by equations (38) and (16), we have

$$\begin{cases} [3]\gamma^2 \pm \gamma - [5] = 0, \\ \dfrac{\gamma\delta(\delta^2 - 3)}{\delta^2 - 1} - \dfrac{2(\gamma - 1)^2\delta}{\delta^2 - 3} = 0. \end{cases}$$

We use Mathematica to solve numerical solutions for (δ, γ). For all solutions, the value of δ is far below 2 and this contradicts that $\delta > 2$.

Subcase 2. When $a = -1$, by equations (38) and (17), we have

$$\begin{cases} [3]\gamma^2 \pm \gamma - [5] = 0, \\ \gamma\delta\Big(2\dfrac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1} - 1\Big) + \dfrac{2(\gamma - 1)\delta}{\delta^2 - 1} = 0. \end{cases} \tag{39}$$

The linear system (39) does not have a solution in which the value of δ is greater than 2 by checking the numerical solutions using Mathematica and this leads to a contradiction.

We conclude that $\omega = 1$ for the reduced case in which $\dim(\mathscr{P}_{4,+}) = 23$. □

Theorem 4.3. *Suppose $\mathscr{P}_\bullet$ is a singly-generated Thurston-relation planar algebra and $\dim \mathscr{P}_{4,\pm} = 23$, then it is isomorphic to the semisimple quotient of $\widetilde{\mathscr{P}}_\bullet^H(q,r)$, where $r = q^N$ for some q, r.*

Proof. It follows from theorems 4.2 and 3.11 and the proof of Theorem 3.13. □

Remark 6. We notice that in the case of 24 dimensional 4-box spaces, our proof did not need the full strength of the assumption that we had a subfactor planar algebra. In particular, we did not need that the canonical inner product on the box spaces induces a positive definite inner product on the hom spaces in any essential way, and it would have been sufficient in principal to assume that this inner product was simply non-degenerate and $\delta > 2$. For the cases in this section, when the 4-box space has dimension less than 24, we appealed to triple point obstructions and classification of subfactors, which are theorems for *subfactor planar algebras*, and thus we used the subfactor assumption in an essential way.

5. Positivity

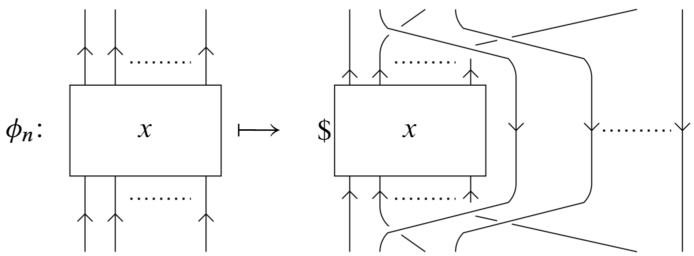
Convention: We say a planar algebra has positivity, if it has a positive semi-definite Markov trace with respect to an involution.

In this section, we determine the positivity of $\mathscr{P}_\bullet^H(q,r)$: We classify q, r in $\mathbb{C}$ and involutions on $\mathscr{P}_\bullet^H(q,r)$, so that the Markov trace is positive semi-definite.

Definition 5.1. We define the map

$$\phi_n \colon H_n(q,r) \longrightarrow \mathscr{P}^H_{2n,+}(q,r),$$

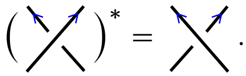
for $x \in H_n(q,r)$ defined $\phi_n(x)$ as follows:



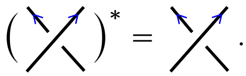
Proposition 5.2. *The map ϕ_n is an algebra homomorphism from $H_n(q,r)$ to $\mathcal{P}^H_{2n,+}(q,r)$ preserving the normalized Markov trace.*

Proof. This follows from the HOMFLY-PT skein relations. □

Recall that when $r = q^N$ for some $N \in \mathbb{N}$ and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$, $\mathcal{P}^H_\bullet(q,r)$ admits an involution $*$ such that



When $q \geq 1$, $\mathcal{P}^H_\bullet(q,r)$ admits an involution $*$ such that



In both case, the Markov trace is semi positive-definite. Therefore the semisimple quotient of $\mathcal{P}^H_\bullet(q,r)$ is a subfactor planar algebra, denoted by $\widetilde{\mathcal{P}}^H_\bullet(q,r)$.

We show that they are the only values of (q,r) and involution such that the positivity holds.

Theorem 5.3. *The planar algebra $\mathcal{P}^H_\bullet(q,r)$ has positivity if and only if $r = q^N$ for some $N \in \mathbb{N}$, and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$ or $q \geq 1$, and the involution is unique.*

Proof. If $\mathcal{P}^H_\bullet(q,r)$ has positivity, then the idempotents P and Q in Lemma 3.12 are projections, therefore the generator S is self-adjoint. Since $\mathcal{P}^H_\bullet(q,r)$ is generated by S, the involution $*$ is uniquely determined.

In theorems 3.13 and 4.3, we show that a singly-generated Thurston-relation planar algebra with parameters (δ,γ) and $\delta > 2$ is isomorphic to $\mathcal{P}^H_\bullet(q,r)$ for some (q,r). By Lemma 2.3, we can assume that $\Re q \geq 0$, $\Im q \geq 0$. Note that $\delta > 2$ and $\gamma > 0$, then by equation (31), we have

$$q + q^{-1} = \sqrt{2(\delta^2-2)/\Big(\delta^2 - \Big(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}\Big)^2\Big)}. \tag{40}$$

Note that the term in the square root is positive, so $q + q^{-1} > 0$. We have $q = e^{i\theta}$ with $0 \leq \theta \leq \pi/2$ or $q \geq 1$.

Let $[n]$ denote the Young diagram with 1 row and n columns and $[1^n]$ denote the Young diagram with n rows and 1 column.

Case 1: $q > 1$. By Lemma 2.3, we can assume that $\Re r \geq 0$. By

$$\frac{r - r^{-1}}{q - q^{-1}} = \delta > 2,$$

we have that $r > 1$. If $r = q^N$, for some $N \in \mathbb{N}$, then we know that $\mathscr{P}_\bullet^H(q, r)$ has positivity. Otherwise, $q^N < r < q^{N+1}$. Then the idempotent $m_{[N+2]}$ is well-defined and $\mathrm{Tr}(m_{[N+2]}) < 0$. By Proposition 5.2, $\phi_{N+2}(m_{[N+2]})$ is an idempotent with a negative trace. So $\mathscr{P}_\bullet^H(q, r)$ does not have positivity.

Case 2: $q = e^{i\theta}$ and $q \neq 1$. By Lemma 2.3, we can assume that $\Im r \geq 0$. By $\dfrac{r - r^{-1}}{q - q^{-1}} = \delta > 2$, we have $r = e^{i\alpha}$, for some $\theta < \alpha < \pi - \theta$.

Subcase 1. If $N\theta < \alpha < (N+1)\theta$, for some $N \in \mathbb{N}$, then the idempotent $m_{[N+2]}$ is well-defined and $\mathrm{Tr}(m_{[N+2]}) < 0$. By Proposition 5.2, $\mathscr{P}_\bullet^H(q, r)$ does not have positivity.

Subcase 2. If $\alpha = N\theta$ and $\frac{\pi}{N+l+1} < \theta < \frac{\pi}{N+l}$, for some $N, l \in \mathbb{N}$, then the idempotent $m_{[1^{l+1}]}$ is well-defined and $\mathrm{Tr}(m_{1^{[l+1]}}) < 0$. By Proposition 5.2, $\mathscr{P}_\bullet^H(q, r)$ does not have positivity.

Subcase 3. If $\alpha = N\theta$ and $\theta = \frac{\pi}{N+l}$, for some $N, l \in \mathbb{N}$, then we know that $\mathscr{P}_\bullet^H(q, r)$ has positivity.

Case 3: $q = 1$. By $r - r^{-1} = \delta(q - q^{-1})$, we have $r = 1$. By a similar argument in Case 1, one can show that $\delta = N$, for some $N \in \mathbb{N}$. In this case, we know that $\mathscr{P}_\bullet^H(q, r, \delta) = \mathscr{P}_\bullet^H(1, 1, N)$ has positivity. $\square$

By Proposition 4.1 and theorems 3.13, 4.3, and 5.3, we obtain our classification result Theorem 1.2.

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