Antisymmetric Characters and Fourier Duality

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Abstract: Inspired by the quantum McKay correspondence, we consider the classical ADE Lie theory as a quantum theory over $\mathfrak{sl}_2$. We introduce anti-symmetric characters for representations of quantum groups and investigate the Fourier duality to study the spectral theory. In the ADE Lie theory, there is a correspondence between the eigenvalues of the Coxeter element and the eigenvalues of the adjacency matrix. We formalize related notions and prove such a correspondence for representations of Verlinde algebras of quantum groups: this includes generalized Dynkin diagrams over any simple Lie algebra $\mathfrak{g}$ at any level k. This answers a recent comment of Terry Gannon on an old question posed by Victor Kac in 1994.

1. Introduction

In 1980, McKay found his correspondence between subgroups of $SU(2)$ and the affine ADE Dynkin diagrams [48]. In 1987, Cappelli, Itzykson and Zuber found a correspondence between ADE Dynkin diagrams and the modular invariant of quantum $\mathfrak{sl}_2$ [8], which was further formulated by Ocneanu as a correspondence to subgroups of quantum $\mathfrak{sl}_2$ [55]. This has been considered as a quantum analogue of the McKay Correspondence. We elaborate this idea in §2.

In ADE Lie theory, the action of the Coxeter element on the root system has periodicity equal to the Coxeter number $c_k = 2 + k$, for some $k \in \mathbb{N}^+$. Its eigenvalue is $e^{\frac{j2\pi i}{c_k}}$, for some $j \in \mathbb{Z}_{c_k}$, and j is called a Coxeter exponent. On the other hand, the eigenvalues of the adjacency matrix of the ADE Dynkin diagram are given by $e^{\frac{j\pi i}{c_k}} + e^{\frac{-j\pi i}{c_k}}$, where j is an exponent with the same multiplicity.

Kac asked the question, whether the Coxeter exponents for the ADE quivers can be generalized beyond $SU(2)$ theory in a talk given by Terry Gannon in 1994 at MIT [36]. Gannon commented in his lecture at the Shanks workshop at Vanderbilt University in September 2017, that if such a generalization exists, then the correspondence between

the Coxeter exponents and the spectrum of the adjacency matrices should work for all quivers (or NIM-reps) of module categories acted on by the representation category of quantum $\mathfrak{sl}_n$, and it may even be true over any simple Lie algebra $\mathfrak{g}$ at level k [23]. These quivers were called higher Dynkin diagrams by Ocneanu around 2000, see also [56].

The generalization of adjacency matrices and their spectrum are straightforward and well-understood. The generalizations of roots and the Coexter element have been considered by Di Francesco and Zuber for generalized Dynkin diagrams in [13] and by Ocneanu for higher Dynkin diagrams in [55] using different approaches. The spectrum of the adjacency matrices and the spectrum of the generalized Coxeter elements have been considered as two different generalizations of the Coxeter exponents.

In this paper, we study a generalization of the adjacency matrices and the Coxeter elements for any unital *-representation Π of the Verlinde algebra [63] of any simple Lie algebra $\mathfrak{g}$ at any level k, including generalized Dynkin diagrams and higher Dynkin diagrams as special cases. We prove a general correspondence between the spectrum of adjacency matrices and the spectrum of Coxeter elements for Π. The relation between the multiplicities of the two spectra are computed in Theorem 8.28. In particular, when $\mathfrak{g}$ is of type ADE and Π is graded by the center of $\mathfrak{g}$, we prove that the multiplicities of the two spectrums are the same. This answers Gannon's comment on Kac's question, by generalizing the equality of the multiplicities for Coxeter exponents associated with classical ADE Dynkin diagrams, see our dictionary after Theorem 8.28.

Many concepts in this paper were well studied in literature related to quantum McKay correspondence, from conformal field theory to category theory. We explain these concepts using classical Lie theory in an elementary and self-contained way, instead of using the fruitful approach of quantum groups and the McKay correspondence. We hope our approach will be helpful for many readers. We refer to §2 for further background and extensive, but definitely not encyclopedic, references.

We study the notions related to affine Lie algebras and quantum groups using Lie groups and their subgroups. We consider a quantum group as a simple Lie algebra $\mathfrak{g}$ with a natural number k, corresponding to the level. We construct its Verlinde algebra by *anti-symmetric characters* defined in §5. For each level k, the anti-symmetric characters are defined by the Weyl denominators on a domain $T_{k,0}$, a subset of the maximal torus of the corresponding Lie group. From the choice of the domain, we obtain a natural cutoff of the fusion rule of representations from the Lie algebra $\mathfrak{g}$ to the quantum group at level k, also known as the Wess–Zumino–Witten cutoff [51,66–68]. We attempt to understand the connection between the McKay correspondence and the quantum McKay correspondence with this approach. Another motivation for introducing the anti-symmetric characters is understanding the fusion rule and their generating functions in a closed form for the representations of two families of quantum subgroups constructed in [44].

Additive functions on Auslander–Reiten quivers were studied by Gabriel in [20]. For an ADE Dynkin diagram, the root system can be realized as additive functions on the Auslander–Reiten quiver, and the Coxeter element is given by a translation functor. The adjacency matrix of the ADE Dynkin diagram can be extended to a $\mathbb{Z}_2$-graded unital *-representation of the Verlinde algebra of $\mathfrak{sl}_2$ at level k, and $k = c_k - 2$. We study a generalization of related concepts for any (graded) unital *-representation Π of the Verlinde algebra of a simple Lie algebra $\mathfrak{g}$ at level k. We define the corresponding adjacency matrices, quantum Dynkin diagram, quantum root spaces, quantum Coxeter elements and quantum Coxeter exponents. See the end of §8 for a dictionary.

We study the spectral theory using the Fourier duality in §4. We apply the Fourier transform to diagonalize the actions of the adjacency matrices, and identify their simul-

taneous eigenvalues as elements in $T_{k,0}$ modulo the Weyl group action, which we call the *spectrum*. On the other hand, we apply the Fourier transform to diagonalize the actions of quantum Coxeter elements, and identify their simultaneous eigenvalues as elements in $T_{k,0}$. Moreover, we introduce the quantum Coxeter exponents for the elements in $T_{k,0}$. Then we can compare their multiplicities using these two identifications and obtain an equality in Theorem 8.28. This equality generalizes the known correspondence of Coxeter exponents for the ADE Dynkin diagrams over $\mathfrak{g} = \mathfrak{sl}_2$, and in this theorem we answer the Gannon's comment on Kac's question.

2. Background

It is well known that the simple Lie algebras are classified by Dynkin diagrams as their underlying symmetry. The construction from Dynkin diagrams to Lie algebras was given by Chevalley. The simply-laced Dynkin diagrams are the ADE Dynkin diagrams and one can obtain the others from an orbifold construction of the ADE ones.

In 1980, McKay found a one-to-one correspondence between subgroups of $SU(2)$ and the affine ADE Dynkin diagrams [48]. The affine Dynkin diagrams appeared as the quivers of the irreducible representations (irreps) of $SU(2)$ tensoring the standard representation. The Mckay correspondence relates subgroups of $SU(2)$ and ADE Lie theory.

Around 1968, Kac and Moody studied infinite dimensional Lie algebras, known as Kac–Moody Lie algebras, see [34,35,50]. The type A_{k+1} Dynkin diagrams appeared as the quivers of the semisimple irreps of the affine Lie algebra $\mathfrak{sl}(2)$ at level k. In 1983, Jones classified the indices of a subfactor $\mathcal{N} \subset \mathcal{M}$, an inclusion of von Neumann algebras with trivial center [31]:

$$\left\{ 4\cos^2 \frac{\pi}{2+k}, k = 1, 2, \cdots \right\} \cup [4, \infty].$$

For each k, he constructed a subfactor with index $4\cos^2 \frac{\pi}{2+k}$, whose principal graphs, namely the quiver of bimodules, is the A_{k+1} Dynkin diagram. The A_{k+1} Dynkin diagram also appeared as the quiver of semisimple irreps of the Drinfeld-Jimbo quantum group $U_q sl_2, q = e^{\frac{\pi i}{2+k}}$ [15,30]. The correspondence between the two representation theories is given in [32]. Wassermann found another conceptual connection between subfactor theory and representation theory of quantum groups in conformal field theory [64]. The correspondence between representations of affine Lie algebras and representations of quantum groups is given by Kazhdan and Lusztig in [39]. The A_{k+1} Dynkin diagram appeared as a cutoff of A_∞. This is a general phenomenon also known as the Wess–Zumino–Witten cutoff [51,66–68]. See the book [2] of Bakalov and Kirillov Jr. for general situations and further connections.

In 1987, Cappelli, Itzykson and Zuber classified the modular invariants of quantum $\mathfrak{sl}_2$ at level k by ADE Dynkin diagrams with Coxeter number $2+k$. The diagonals of the modular invariants matched the multiplicities of the Coxeter exponents, which was first observed by Kac for the E_6 case, see further discussion in [8]. A connection between the diagonals of the modular invariants and the spectrum of the adjacency matrices of representations of the Verlinde algebra of quantum $\mathfrak{sl}_n$ has been studied by Di Francesco and Zuber in [13]. In 1988, Ocneanu outlined the surprising A_n, D_{2n}, E_6, E_8 classification of the standard invariants of subfactors with index less than 4 in [52]. The proof of the classification is given in [3,24,28,29,38]. This ADE classification is also a classification of subfactors due to Popa's reconstruction theorem [60]. See the survey paper

[33] for more details. In 1994, Ocneanu proposed a connection between subfactors and extended Turaev-Viro TQFT [53]. Given a subfactor $\mathcal{N} \subset \mathcal{M}$, if the $\mathcal{N} - \mathcal{N}$ bimodule category is isomorphic to the representation category of quantum $\mathfrak{sl}_2$, which defines a Turaev-Viro TQFT [62], then the subfactor defines an extended TQFT. Ocneanu considered this subfactor (or the $\mathcal{M} - \mathcal{M}$ bimodule category) as a subgroup of quantum $\mathfrak{sl}_2$ and the $\mathcal{N} - \mathcal{M}$ module category as a module of quantum $\mathfrak{sl}_2$, see [54]. All ADE Dynkin diagrams appeared as the quiver of the generating $\mathcal{N} - \mathcal{N}$ bimodule, corresponding to the fundamental representation of quantum $\mathfrak{sl}_2$, acting on irreducible $\mathcal{N} - \mathcal{M}$ bimodules. The correspondence between ADE subfactors and modular invariants is given by Böckenhauer, Evans and Kawahigashi based on the α-induction [5,6,46,54,69]. A corresponding categorical formalization of module categories and the classification has been done by Kirillov and Ostrik with independent proofs in [40,57]. The ADE classification related to quantum $\mathfrak{sl}_2$ has become known as the quantum McKay correspondence.

Inspired by Chevalley's construction and McKay correspondence, one can consider the ADE Lie theory as a mathematical theory over $\mathfrak{sl}_2$. It turns out to be natural to study the notions in Lie theory over quantum $\mathfrak{sl}_2$ and other quantum groups. In this direction, Zuber introduced generalized Dynkin diagrams over $\mathfrak{sl}_n$ at level k and a generalization of Coxeter elements and Coxeter exponents in [70,71]. Many examples of generalized Dynkin diagrams appeared in conformal field theory, see the book of Di Francesco, Mathieu, and Sénéchal [12]. Ocneanu reformulated the generalized Dynkin diagrams as the quivers of modules of $\mathfrak{sl}_n$ at level k and proposed a classification for $\mathfrak{sl}_2$, $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ in [55], and he called these quivers higher Dynkin diagrams [56]. Another approach is given by Etingof and Khovanov in terms of integer modules over the Verlinde algebras of quantum groups [17].

Zuber's motivation for studying his generalized Dynkin diagrams arises from conformal field theory [70,71]. Xu constructed type E quantum subgroups of quantum groups through conformal inclusions in chiral conformal field theory [69], and the quantum subgroup is implemented as the (bi-)module category of commutative Frobenius algebras in the unitary modular tensor category of the quantum group. He also computed the corresponding quivers, namely higher Dynkin diagrams, for small rank quantum groups. The analogous construction for orbifolds, namely type D module categories, was given in [4] and the remaining E_7 modular invariant and module category was constructed in [6]. In general, it requires the modular invariant and 6j symbols to construct irreducible modules and to compute the quivers as shown in [5,6,54]. However, it remains a difficult problem to compute the 6j symbols in a closed form. It is also challenging to compute the corresponding quivers in closed forms when the quantum groups have large rank.

The first author introduced a new type of Schur-Weyl duality for families of type E quantum subgroups in [44]. This provides new methods to compute the quivers of module categories without computing the modular invariant and quantum 6j symbols. See further discussion in [45]. It would be interesting to compute the quantum Coxeter exponents and multiplicities for these families of examples.

For the $\mathfrak{sl}_3$ case, Di Francesco and Zuber investigated the McKay correspondence both in the classical and quantum sense in [13]. Gannon classified the modular invariants for quantum $\mathfrak{sl}_3$ in [21]. Ocneanu proposed in [55] a classification of unitary module categories of quantum $\mathfrak{sl}_3$, relying on the existence of a cell system, and asserted the non-existence of such a cell system for a particular NIM-rep of the list given by Di Francesco–Zuber in [13]. The existence of the Ocneanu cell systems for $\mathfrak{sl}_3$ was computed by Evans and Pugh in [16] and by Coquereaux, Schieber and Isasi in [9]; the non-existence of the cell system for the previously mentioned particular NIM-rep was also detailed in [9].

Recently Evans and Pugh classified modular invariants and unitary module categories for $SO(3)_{2m}$ in [18].

Following the quantum McKay correspondence, Ocneanu proposed a generalization of the Lie theory in [55] and gave a course "higher representation theory" at Harvard in the 2017 fall term. For Ocneanu's blueprint shown in his course, we refer readers to the lecture notes in [56]. Some motivating examples including higher Dynkin diagrams, higher roots, higher Coxeter elements over $\mathfrak{sl}_3$ were discussed in [56], see also hyper roots over $\mathfrak{sl}_3$ in [11]. There are different approaches to generalize Lie theory, which we do not discuss in this paper.

Gabriel constructed the root category using the quiver representations of ADE Dynkin diagrams in [19]. Using the root category, Ringle constructed the positive part of the corresponding Lie algebras in his seminal work [61]. Happel gave another construction of the root category using 2-periodic derived categories of quiver representations in [26,27]. Remarkably, Peng and Xiao constructed simple Lie algebras in [58] and symmetrizable Kac–Moody Lie algebras using these derived categories in [59]. Dorey constructed the ADE root system and the Coxeter element using the corresponding NIM-rep data of quantum $\mathfrak{sl}_2$ in [14]. Ocneanu proposed a construction of the root category using the module category over quantum $\mathfrak{sl}_2$ in [55,56], see further discussion by Kirillov and Thind in [41,42]. Furthermore, Kirillov and Thind constructed the corresponding derived category in [43]. It will be interesting to generalize the construction of the Lie algebras for quivers over a quantum group $\mathfrak{g}$ beyond $\mathfrak{sl}_2$. It would require a generalization of Lie bracket and the Jacob identity.

Around 1986, Kac observed a correspondence between the diagonal of an $\mathfrak{sl}_2$ modular invariant and the Coxeter exponents of E_6, which turned out to be a general phenomenon for all $\mathfrak{sl}_2$ modular invariants as discussed in [7,8,37]. The diagonal of a modular invariant is also a natural generalization of the Coxeter exponent. It is natural to ask for a generalization of this correspondence over a simple Lie algebra $\mathfrak{g}$ at level k as well. Kac and Gannon asked whether for any module category of the representation category of quantum $\mathfrak{sl}_n$, there is a correspondence between the diagonal of its modular invariant and the spectrum of its adjacency matrices.

3. Preliminaries

Let $\mathfrak{g}$ be a simple complex **Lie algebra**, that is the complexification of the Lie algebra $\mathfrak{k}$ of a simply-connected compact **Lie group** K. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$, and let T be the **maximal torus**, a Lie subgroup of K whose Lie algebra is $\mathfrak{t}$. Then $\mathfrak{h} = \mathfrak{t} + i\,\mathfrak{t}$ is a **Cartan subalgebra** of $\mathfrak{g}$. Denote by $\langle \cdot, \cdot \rangle$ an inner product on $\mathfrak{g}$ such that it is invariant under the adjoint action of K and taking real values on $\mathfrak{k}$. Let ℓ be the **rank** of $\mathfrak{g}$.

Denote the set of **roots** of $\mathfrak{g}$ by Δ. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the set of **simple roots** in Δ. We denote by Q the **root lattice**,

$$Q = \left\{ \sum_{j=1}^{\ell} k_j \alpha_j : k_j \in \mathbb{Z} \right\}.$$

The set of **positive roots** is denoted by Δ_+. Let θ be the **highest positive root**, namely $\theta + \alpha$ is not a root for any simple root α. We assume that the inner product $\langle \cdot, \cdot \rangle$ is normalized such that $\langle \theta, \theta \rangle = 2$. For any root $\alpha \in \Delta$, let $\alpha^\vee := \dfrac{2}{\langle \alpha, \alpha \rangle} \alpha$ be the **coroot** of α for $\mathfrak{g}$.

Denote by $Q^\vee$ the **coroot lattice** of $\mathfrak{g}$. Let $\theta = \sum_{j=1}^{\ell} a_j \alpha_j$ and $\theta^\vee = \sum_{j=1}^{\ell} a_j^\vee \alpha_j$.
Then $h = 1 + \sum_{j=1}^{\ell} a_j$ is the **Coxeter number** of $\mathfrak{g}$ and $h^\vee = 1 + \sum_{j=1}^{\ell} a_j^\vee$ is the **dual Coxeter number**.

Let $\Omega = \{\omega_1, \ldots, \omega_\ell\}$ be the set of **fundamental weights**, such that $\langle \omega_j, H_{\alpha_i} \rangle = \delta_{j,i}$ for any $1 \leq j, i \leq \ell$. Let P be the **weight lattice** of $\mathfrak{g}$. The root lattice Q is a subgroup of P. It is known that

$$Z(\mathfrak{g}) := P/Q$$

is isomorphic to the **center** $Z(K)$ of the Lie group K. Denote its order by $n_z := |Z(\mathfrak{g})|$.
Let $\mathcal{C}$ be the **closed fundamental Weyl chamber** of P, i.e.

$$\mathcal{C} = \left\{ \lambda = \sum_{j=1}^{\ell} k_j \omega_j : k_j \in \mathbb{N}, 1 \leq j \leq \ell \right\}.$$

For any $\lambda \in \mathcal{C}$, we denote by $\mathcal{W}(\lambda)$ the **weight diagram** of the irreducible representation with the highest weight λ. The **Weyl vector** ρ is the sum of fundamental weights, $\rho = \sum_{j=1}^{\ell} \omega_j$. Another expression of ρ is half the sum of positive roots,

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha . \tag{1}$$

For any $\alpha \in \Delta$, we denote by r_α the **reflection** $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for any $\lambda \in P$.
Let W be the **Weyl group** of $\mathfrak{g}$ generated by the reflections $r_\alpha(\lambda)$. For any $r \in W$, we denote by $\det(r)$ the sign of r.

For any $\lambda \in \mathcal{C}$, we denote V_λ the **irreducible representation** of K with highest weight λ. Let χ_λ be the character of the Lie group K associated to V_λ. Suppose $V_\mu \otimes V_\lambda \cong \bigoplus_{\nu \in \mathcal{C}} N_{\mu,\lambda}^\nu V_\nu$, where $N_{\mu,\lambda}^\nu \in \mathbb{N}$ is called a **fusion coefficient**. Then

$$\chi_\mu \chi_\lambda = \sum_{\nu \in \mathcal{C}} N_{\mu,\lambda}^\nu \chi_\nu. \tag{2}$$

For any $\lambda \in \mathcal{C}$ and $H \in \mathfrak{t}$, the **Weyl character formula** states

$$\chi_\lambda(e^H) = \frac{\sum_{r \in W} \det(r) e^{i \langle r(\lambda+\rho)-\rho, H \rangle}}{\prod_{\alpha \in \Delta_+} \left(1 - e^{-i \langle \alpha, H \rangle} \right)} = \frac{\sum_{r \in W} \det(r) e^{i \langle r(\lambda+\rho), H \rangle}}{\prod_{\alpha \in \Delta_+} \left(e^{i \langle \alpha, H \rangle/2} - e^{-i \langle \alpha, H \rangle/2} \right)} .$$

Moreover, **the Weyl denominator formula** states

$$\sum_{r \in W} \det(r) e^{i \langle r(\rho), H \rangle} = e^{i \langle \rho, H \rangle} \prod_{\alpha \in \Delta_+} \left(1 - e^{-i \langle \alpha, H \rangle} \right) = \prod_{\alpha \in \Delta_+} \left(e^{i \langle \alpha, H \rangle/2} - e^{-i \langle \alpha, H \rangle/2} \right), \tag{3}$$

see e.g. Theorem 10.4 and Equation 10.4.4 in [35]. The Weyl character formula can be rewritten as

$$\chi_\lambda(e^H) = \frac{\sum_{r \in W} \det(r) e^{i \langle r(\lambda+\rho), H \rangle}}{\sum_{r \in W} \det(r) e^{i \langle r(\rho), H \rangle}} .$$

Now let us recall the notions related to a level $k \in \mathbb{N}$ of $\mathfrak{g}$. Let $c_k = h^\vee + k$ be the **altitude**, which plays the role of **quantum Coxeter number** over $\mathfrak{g}$ at level k in this paper. Define

$$Q_h := \{r(\theta) | r \in W\};$$
$$Q_k^\vee := c_k Q^\vee.$$

Then $Q^\vee = \mathrm{span}_{\mathbb{Z}} Q_h$.

For any $\lambda \in P$, define a **translation** ϑ_λ on P by

$$\vartheta_\lambda(\mu) = \lambda + \mu, \quad \mu \in P.$$

Then the set $\{\vartheta_\lambda\}_{\lambda \in P}$ is a free abelian group using composition. For a level $k \in \mathbb{N}$, the **affine Weyl group** $\widehat{W}$ is generated by the Weyl group W and the translation $\vartheta_{c_k \theta}$. The translation subgroup W_t of $\widehat{W}$ is given by

$$W_t := \{\vartheta_\alpha \mid \alpha \in Q_k^\vee\}.$$

Then $\widehat{W} = W_t \rtimes W$. For any $r \in \widehat{W}$, we denote by $\det(r)$ the sign of r in $\widehat{W}/W_t \cong W$.

The **alcove** $\mathcal{C}_k$ is defined as

$$\mathcal{C}_k = \left\{ \lambda = \sum_{j=1}^{\ell} k_j \omega_j \in \mathcal{C} : k_j \in \mathbb{N}^+, j = 1, \ldots, \ell, \langle \lambda, H_\theta \rangle < c_k \right\} \subset \mathcal{C}.$$

The finite dimensional irreducible representations (with non-vanishing quantum dimension) of the quantum group $\mathfrak{g}$ at level k are denoted by $\{\tilde{V}_\lambda \mid \lambda \in \mathcal{C}_k\}$, and $\tilde{V}_\lambda$ corresponds to $V_{\lambda - \rho}$. They form a unitary fusion category, after a semisimplification of the representation category of the quantum group. Their fusion rule, known as the Wess–Zumino–Witten cutoff, can be computed using the Kac-Walton algorithm [35,65].

$$\tilde{V}_\lambda \otimes \tilde{V}_\mu = \sum_{\nu \in \mathcal{C}_k} \tilde{N}_{\lambda,\mu}^\nu \tilde{V}_\nu = \sum_{\substack{\nu \in \mathcal{C}_k, r \in \widehat{W} \\ r(\nu) - \rho \in \mathcal{C}}} \det(r) N_{\lambda-\rho,\mu-\rho}^{r(\nu)-\rho} \tilde{V}_\nu, \tag{4}$$

The fusion algebra is known as the **Verlinde algebra** of the quantum group [63]. These irreps form a $Z(\mathfrak{g})$-graded **fusion ring** [47], such that $\tilde{V}_\lambda$ is graded by $\lambda - \rho$ in $Z(\mathfrak{g})$.

For ADE Dynkin diagrams, Gabriel constructed the root system using additive functions on the Auslander–Reiten quiver [20], see also [26]. We briefly recall this construction and formalize related concepts in a general situation in §8. Take an ADE Dynkin diagram G as a bipartite graph and let $\varepsilon : G_v \to \mathbb{Z}_2$ be a $\mathbb{Z}_2$ grading of the vertices G_v of G. The vertices of the **Auslander–Reiten quiver** is $\Gamma = \{(i, x) \in \mathbb{Z}_{2c_k} \times G_v : i + \varepsilon(x) = 0 \in \mathbb{Z}_2\}$. A function f on Γ is called **additive**, if $f(i, x) \in \mathbb{Z}$ and

$$f(i, x) + f(i + 2, x) = \sum_{y \in N(x)} f(i + 1, y), \ \forall (i, x) \in \Gamma, \tag{5}$$

where $N(x)$ is the set of vertices adjacent to x. Let $\mathcal{H}$ be the space of additive functions with the discrete measure on Γ, $\mathcal{P}_{\mathcal{H}}$ be the orthogonal projection from $L^2(\Gamma)$ onto $\mathcal{H}$, and $\delta_{i,x}$ be the delta function at (i, x). Then $2c_k \mathcal{P}_{\mathcal{H}}(\delta_{i,x})$ is an additive function. Moreover,

$$\{\sqrt{2c_k} \mathcal{P}_{\mathcal{H}}(\delta_{i,x}) : (i, x) \in \Gamma\}$$

forms the root system of G. Furthermore the translation ϑ on Γ, $\vartheta(i, x) = (i + 2, x)$, induces a Coxeter transformation on the root system. This construction was rediscovered and used by Dorey in [14] using a NIM-rep of quantum $\mathfrak{sl}_2$. An explicit description, for several root systems, of this periodic quiver was given in [10].

4. Fourier Duality

In Lie theory, the exponential map $H \mapsto e^{2\pi H}$ is a group isomorphism $\mathfrak{t}/Q^\vee \cong T$. For any $\lambda \in P$, the Fourier transform $\hat{\ }$: $\lambda \mapsto \hat{\lambda}$, given by

$$\hat{\lambda}(e^{2\pi H}) = e^{2\pi i \langle \lambda, H \rangle} , \tag{6}$$

is well defined on $e^{2\pi H} \in T$. The map $\hat{\ }$ is a group isomorphism from P to the dual of the abelian group T. The weight lattice P is the dual of the coroot lattice $Q^\vee$:

(1) $\lambda \in P$ iff $\langle \lambda, H \rangle \in \mathbb{Z}, \forall H \in Q^\vee$;
(2) $H \in Q^\vee$ iff $\langle \lambda, H \rangle \in \mathbb{Z}, \forall \lambda \in P$.

Definition 4.1. Suppose A is a sub lattice of P and P/A is finite. We define the dual lattice of A in $\mathfrak{t}$ as

$$\mathfrak{t}_A = \{H \in \mathfrak{t} | \langle \alpha, H \rangle \in \mathbb{Z}, \forall \alpha \in A\}.$$

Define the corresponding subgroup of T as

$$T_A = \left\{ e^{2\pi H} \in T : H \in \mathfrak{t}_A \right\}.$$

Then $T_A \cong \mathfrak{t}_A/Q^\vee$. By the duality of lattices, we have the following result:

Proposition 4.2. *For any $\alpha \in \mathfrak{t}$, $\alpha \in A$ iff $\langle \alpha, H \rangle \in \mathbb{Z}$, $\forall H \in \mathfrak{t}_A$.*

Theorem 4.3. *The map $\hat{\ }$ induces a group isomorphism from P/A to the dual of the abelian group T_A.*

Proof. If $\lambda \in A$, then by the definition of T_A, we have $\hat{\lambda}(e^H) = 1$, for any $e^H \in T_A$. So $\hat{\ }$ is well-defined from P/A to the dual of T_A. Conversely, for any $\lambda \in P$, if $\hat{\lambda}(e^{2\pi H}) = 1$, $\forall H \in \mathfrak{t}_A$, then $\langle \lambda, H \rangle \in \mathbb{Z}$. By Proposition 4.2, $\lambda \in A$. So $\hat{\ }$ is injective. Suppose that f is a character of T_A. Then $f(e^H) = e^{i\langle \lambda, H \rangle}$, for some $\lambda \in \mathfrak{t}$. For any $H \in 2\pi Q^\vee$, we have $e^H = 1 \in T_A$. So $f(e^H) = 1$ and $\langle \lambda, H \rangle \in 2\pi\mathbb{Z}$. By the duality between the lattices P and $Q^\vee$, we have $\lambda \in P$. So the map $\hat{\ }$ is surjective. Therefore, $\hat{\ }$ is a group isomorphism from P/A to the dual of the abelian group T_A. $\square$

For any $k \in \mathbb{N}$, consider the case $A = Q_k^\vee = c_k Q^\vee$.

Definition 4.4. The lattice $\mathfrak{t}_k$ in $\mathfrak{t}$ is

$$\mathfrak{t}_k = \{H \in \mathfrak{t} | \langle \alpha, H \rangle \in \mathbb{Z}, \forall \alpha \in Q_k^\vee\}.$$

The subgroup T_k of T is

$$T_k = \left\{ e^{2\pi H} : H \in \mathfrak{t}_k \right\} \subset T.$$

We call the quotient group $P_k := P/Q_k^\vee$ a weight torus.

Corollary 4.5. *The map* ˆ *induces a group isomorphism from P_k to the dual of the abelian group T_k.*

Corollary 4.6. *The order of the group T_k is*

$$|T_k| = |P_k| = |P/Q| \cdot |Q/Q^\vee| \cdot |Q^\vee/Q_k^\vee| = n_z c_k^\ell |Q/Q^\vee|.$$

Remark 4.7. When $\mathfrak{g} = \mathfrak{sl}_n$, $Q_k^\vee = c_k Q$. So $|T_k| = |P_k| = n(n+k)^{n-1}$.

Let $L^2(P_k)$ be the complex L^2 functions on P_k with counting measure. Let $L^2(T_k)$ be the complex L^2 functions on T_k with Haar measure. Recall that P_k and T_k are dual to each other. For any $f \in L^2(P_k)$, its Fourier transform $\mathcal{F}(f)$ in $L^2(T_k)$ is

$$\mathcal{F}(f)(e^H) = \sum_{\lambda \in P_k} f(\lambda) e^{i\langle \lambda, H \rangle}, \quad e^H \in T_k.$$

Then $\mathcal{F}$ is a unitary transformation:

$$\langle \mathcal{F}(f), \mathcal{F}(f') \rangle = \langle f, f' \rangle, \quad f, f' \in L^2(P_k).$$

For any $g \in L^2(T_k)$, the inverse Fourier transform $\mathcal{F}^{-1}(g)$ of f is

$$\mathcal{F}^{-1}(g)(\lambda) = \frac{1}{|T_k|} \sum_{e^H \in T_k} g(e^H) e^{-i\langle \lambda, H \rangle}, \quad \lambda \in P_k.$$

For any $f \in L^2(P_k)$ and $r \in W$, we define $r(f)$ as $r(f)(\lambda) = f(r^{-1}(\lambda))$ for any $\lambda \in P_k$.

Definition 4.8. Let $L^2(P_k)^W$ be the space of all anti-symmetric functions on P_k:

$$L^2(P_k)^W = \{f \in L^2(P_k) : r(f) = \det(r)f, \ \forall r \in W\}.$$

For any $g \in L^2(T_k)$ and $r \in W$, we define $r(g)$ as $r(g)(e^H) = g(e^{r^{-1}(H)})$ for any $e^H \in T_k$.

Definition 4.9. Let $L^2(T_k)^W$ be the space of all anti-symmetric functions on T_k:

$$L^2(T_k)^W = \{g \in L^2(T_k) : r(g) = \det(r)g, \ \forall r \in W\}.$$

Proposition 4.10. *We have*

$$\mathcal{F}(L^2(P_k)^W) = L^2(T_k)^W.$$

Proof. Suppose f is an anti-symmetric function in $L^2(P_k)$. Then for any $e^H \in T_k$ and $r \in W$,

$$r(\mathcal{F}(f))(e^H) = \sum_{\lambda \in P_k} f(\lambda) e^{i\langle \lambda, r^{-1}(H) \rangle} = \sum_{\lambda \in P_k} f(\lambda) e^{i\langle r(\lambda), H \rangle}$$

$$= \sum_{\lambda \in P_k} f(r^{-1}(\lambda)) e^{i\langle \lambda, H \rangle} = \sum_{\lambda \in P_k} \det(r) f(\lambda) e^{i\langle \lambda, H \rangle}$$

$$= \det(r) \mathcal{F}(f)(e^H).$$

Hence $\mathcal{F}(f)$ is anti-symmetric.

Conversely, if g is an anti-symmetric function in $L^2(T_k)$, then $\mathcal{F}^{-1}(g)$ is anti-symmetric by a similar computation. Therefore, $\mathcal{F}$ is a unitary transformation from $L^2(P_k)^W$ to $L^2(T_k)^W$. $\square$

5. Anti-symmetric k-Characters

In this section, we introduce anti-symmetric characters for a simple Lie algebra $\mathfrak{g}$ and a level k. We construct a C^*-algebra of those characters which represents the Verlinde algebra of the corresponding quantum group. Therefore, we call it the *k-character Verlinde algebra of $\mathfrak{g}$ at level k*. We reformulate some well-known properties of the Verlinde algebra and its S matrix in term of anti-symmetric characters. To be self-contained, we prove these results using basic properties of Lie algebras.

Definition 5.1. We say $\lambda \in P_k$ is in a mirror, if it is in $M(P_k) := \{\lambda \in P_k : r_\alpha(\lambda) = \lambda$, for some $\alpha \in \Delta\}$. We say $e^H \in T_k$ is in a mirror if it is in $M(T_k) := \{e^H \in T_k : e^{r_\alpha(H)} = e^H$, for some $\alpha \in \Delta\}$. Furthermore,

$$P_{k,0} = P_k \backslash M(P_k);$$
$$T_{k,0} = T_k \backslash M(T_k).$$

Note that the Weyl group W action fixes $M(T_k)$ and $M(P_k)$, so it also fixes $P_{k,0}$ and $T_{k,0}$. Moreover, the action of W is transitive on each orbit in $P_{k,0}$ and $T_{k,0}$. Recall that the alcove C_k is defined as

$$C_k = \left\{ \lambda = \sum_{j=1}^{\ell} k_j \omega_j \in C : k_j \in \mathbb{N}^+, j = 1, \ldots, \ell, \langle \lambda, H_\theta \rangle < c_k \right\} \subset C.$$

In affine Lie algebras, it is known that C_k is a fundamental domain of $P_{k,0}$ under the action of W.

Remark 5.2. If the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$, then $|C_k|$ is the binomial coefficient

$$|C_k| = \binom{c_k - 1}{n - 1}.$$

Definition 5.3. (*k-characters*) For any $\lambda \in P_k$, $e^H \in T_k$, we define $\widehat{\chi}_\lambda$ at λ by

$$\widehat{\chi}_\lambda = \mathcal{F}\left(\sum_{r \in W} \det(r) \delta_{r(\lambda)} \right),$$

where δ_λ is 1 on λ and 0 elsewhere. Then

$$\widehat{\chi}_\lambda(e^H) = \sum_{r \in W} \det(r) e^{i \langle r(\lambda), H \rangle}.$$

We define the k-character $\widetilde{\chi}_\lambda$ to be the restriction of $\widehat{\chi}_\lambda$ on $T_{k,0}$.

Proposition 5.4. *The function $\widehat{\chi}$ is anti-symmetric on P_k and T_k, namely, for any $\lambda \in P_k$, $r \in W$,*

$$\widehat{\chi}_{r(\lambda)} = det(r)\widehat{\chi}_\lambda,$$
$$r(\widehat{\chi}_\lambda) = det(r)\widehat{\chi}_\lambda.$$

Consequently, $\widehat{\chi}_\lambda$ is supported in $T_{k,0}$.

Proof. By the definition of $\widehat{\chi}$, it is anti-symmetric on P_k. By Proposition 4.10, $\widehat{\chi}$ is anti-symmetric on T_k.   □

Corollary 5.5. *If λ is in a mirror, then $\widehat{\chi}_\lambda = 0$. For any $\lambda \in P_k$, if e^H is in a mirror, then $\widehat{\chi}_\lambda(e^H) = 0$.*

Proof. By Proposition 5.4, if $r_\alpha(\lambda) = \lambda$ for some $\alpha \in \Delta$, then

$$\widehat{\chi}_\lambda = \widehat{\chi}_{r_\alpha(\lambda)} = \det(r_\alpha)\widehat{\chi}_\lambda = -\widehat{\chi}_\lambda,$$

and we obtain $\widehat{\chi}_\lambda = 0$. For any $\lambda \in P_k$ and $e^{r_\alpha(H)} = e^H$ for some $\alpha \in \Delta$, we have

$$\widehat{\chi}_\lambda(e^H) = \widehat{\chi}_\lambda(e^{r_\alpha(H)}) = -\widehat{\chi}_\lambda(e^H),$$

and hence $\widehat{\chi}_\lambda(e^H) = 0$. $\qquad\square$

Theorem 5.6. *The set $\{|W|^{-\frac{1}{2}}\widehat{\chi}_\lambda\}_{\lambda \in \mathcal{C}_k}$ forms an orthonormal basis (ONB) of $L^2(T_k)^W$. In particular,*

$$\dim L^2(T_k)^W = |\mathcal{C}_k|.$$

Proof. Note that $\{|W|^{-1/2}\sum_{r \in W}\det(r)\delta_{r(\lambda)}\}_{\lambda \in \mathcal{C}_k}$ form an orthonormal basis of $L^2(P_k)^W$. By the definition of $\widehat{\chi}$ and Proposition 4.10, $\{|W|^{-\frac{1}{2}}\widehat{\chi}_\lambda\}_{\lambda \in \mathcal{C}_k}$ forms an orthonormal basis of $L^2(T_k)^W$. $\qquad\square$

Definition 5.7. Define the space $L^2(T_{k,0})^W$ as anti-symmetric functions on $T_{k,0}$:

$$L^2(T_{k,0})^W = \{g \in L^2(T_{k,0}) : r(g) = \det(r)g, \ \forall r \in W\}.$$

By anti-symmetry, any function in $L^2(T_k)^W$ is supported in $T_{k,0}$. So $L^2(T_k)^W \cong L^2(T_{k,0})^W$.

Corollary 5.8. *The set of multiples of k-characters $\{|W|^{-\frac{1}{2}}\widetilde{\chi}_\lambda\}_{\lambda \in \mathcal{C}_k}$ forms an orthonormal basis of $L^2(T_{k,0})^W$.*

Fact 5.9. *Recall that ρ is the Weyl vector. By the Weyl denominator formula* (3)*, for any $e^H \in T_k$, $\widehat{\chi}_\rho(e^H) \neq 0$ iff $e^H \in T_{k,0}$. Equivalently, $\widetilde{\chi}_\rho$ is invertible in $L^2(T_{k,0})$.*

Recall that $\widetilde{\chi}$ is the restriction of $\widehat{\chi}$ on $T_{k,0}$.

Definition 5.10. We define the multiplication $\star$ of $\widetilde{\chi}_\lambda$ and $\widetilde{\chi}_\mu$ for any $\lambda, \mu \in P_k$ to be

$$\widetilde{\chi}_\lambda \star \widetilde{\chi}_\mu = \frac{\widetilde{\chi}_\lambda \widetilde{\chi}_\mu}{\widetilde{\chi}_\rho}.$$

Then $\widetilde{\chi}_\rho$ is the identity under this multiplication.

Recall that the fusion coefficients $\tilde{N}^\nu_{\lambda,\mu}$ of the representations $\{\tilde{V}_\lambda : \lambda \in \mathcal{C}_k\}$ of $\mathfrak{g}$ at level k is given in Eq. (4).

Definition 5.11. For $\lambda, \mu, \nu \in \mathcal{C}_k$, we define the fusion coefficient $\tilde{N}^\nu_{\lambda,\mu}$ as

$$\tilde{N}^\nu_{\lambda,\mu} = \sum_{\substack{\nu'-\rho \in \mathcal{C}, r \in W \\ r(\nu')-\nu \in Q^\vee_k}} \det(r) N^{\nu'-\rho}_{\lambda-\rho,\mu-\rho}. \tag{7}$$

By Eq. (4), the fusion rule of representations of quantum $\mathfrak{g}$ at level k is given by

$$V_{\lambda-\rho} \otimes V_{\mu-\rho} = \bigoplus_{\nu \in \mathcal{C}_k} \tilde{N}^{\nu}_{\lambda,\mu} V_{\nu-\rho}.$$

Consequently, $\tilde{N}^{\nu}_{\lambda,\mu} \in \mathbb{N}$.

Theorem 5.12. *For any $\lambda, \mu, \nu \in \mathcal{C}_k$,*

$$\tilde{\chi}_{\lambda} \star \tilde{\chi}_{\mu} = \sum_{\nu \in \mathcal{C}_k} \tilde{N}^{\nu}_{\lambda,\mu} \tilde{\chi}_{\nu}.$$

Equivalently,

$$\tilde{N}^{\nu}_{\lambda,\mu} = \frac{1}{|W|} \langle \tilde{\chi}_{\lambda} \star \tilde{\chi}_{\mu}, \tilde{\chi}_{\nu} \rangle.$$

Proof. Note that $\dfrac{\tilde{\chi}_{\lambda}}{\tilde{\chi}_{\rho}} = \chi_{\lambda-\rho}$ on $T_{k,0}$. Then in $L^2(T_{k,0})$, one has

$$\begin{aligned}
\tilde{\chi}_{\lambda} \star \tilde{\chi}_{\mu} &= \frac{\tilde{\chi}_{\lambda}\tilde{\chi}_{\mu}}{\tilde{\chi}_{\rho}} = \chi_{\lambda-\rho}\chi_{\mu-\rho}\tilde{\chi}_{\rho} \\
&= \sum_{\nu'-\rho \in \mathcal{C}} N^{\nu'-\rho}_{\lambda-\rho,\mu-\rho}\chi_{\nu'-\rho}\tilde{\chi}_{\rho} \\
&= \sum_{\nu'-\rho \in \mathcal{C}} N^{\nu'-\rho}_{\lambda-\rho,\mu-\rho}\tilde{\chi}_{\nu'} \\
&= \sum_{\nu \in \mathcal{C}_k} \tilde{N}^{\nu}_{\lambda,\mu}\tilde{\chi}_{\nu}.
\end{aligned}$$

The last equality uses the anti-symmetry established in Proposition 5.4, $\tilde{\chi}_{\nu'} = \det(r)\tilde{\chi}_{\nu}$, when $r \in W$ and $r(\nu') - \nu \in Q^{\vee}_k$. $\qquad\square$

Recall that $\alpha_1, \cdots, \alpha_{\ell}$ are simple roots in Δ. Then $\{-\alpha_1, \cdots, -\alpha_{\ell}\}$ is also a set of simple roots. So there is an element $\kappa \in W$, such that $\kappa(\rho) = -\rho$. Moreover, $\kappa(\mathcal{C}_k) = -\mathcal{C}_k$.

Remark 5.13. It is known that $\varepsilon(\kappa) = (-1)^{|\Delta_+|}$. If the Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}_n$, we have for any $\lambda \in \mathcal{C}_k$,

$$\varepsilon(\kappa) = (-1)^{\frac{n(n-1)}{2}}.$$

Definition 5.14. Define an involution $* : \lambda \mapsto \kappa(-\lambda)$ on $\mathcal{C}_k$. Then it is well defined on P_k, and $\rho^* = \rho$.

For any $\lambda \in P_k$, by the definition of $\tilde{\chi}$ and its anti-symmetry, we have

$$\tilde{\chi}_{\lambda^*} = \tilde{\chi}_{\kappa(-\lambda)} = \varepsilon(\kappa)\tilde{\chi}_{-\lambda} = \varepsilon(\kappa)\overline{\tilde{\chi}_{\lambda}}. \tag{8}$$

The involution $*$ on $\mathcal{C}_k$ induces an involution on $L^2(T_{k,0})^W$:

$$\tilde{\chi}^*_{\lambda} := \tilde{\chi}_{\lambda^*}, \ \lambda \in \mathcal{C}_k.$$

Proposition 5.15. *For any* $\lambda, \mu \in \mathcal{C}_k$, $\tilde{N}^{\rho}_{\lambda,\mu} = 1$ *iff* $\lambda = \mu^*$. *Moreover, * induces an anti-isomorphism on* $L^2(T_{k,0})^W$ *with multiplication* $\star$.

Proof. By Corollary 5.8 and Theorem 5.12, for any $\lambda, \mu \in \mathcal{C}_k$,

$$\tilde{N}^{\rho}_{\lambda,\mu} = |W|^{-1}\langle \tilde{\chi}_\lambda \star \tilde{\chi}_\mu, \tilde{\chi}_\rho \rangle = |W|^{-1}\langle \tilde{\chi}_\lambda, \tilde{\chi}_\rho \star \tilde{\chi}^*_\mu \rangle = |W|^{-1}\langle \tilde{\chi}_\lambda, \tilde{\chi}^*_\mu \rangle = \delta_{\lambda,\mu^*}.$$

Moreover,

$$(\tilde{\chi}_\lambda \star \tilde{\chi}_\mu)^* = \varepsilon(\kappa)\overline{\tilde{\chi}_\lambda \star \tilde{\chi}_\mu} = \varepsilon(\kappa)\frac{\overline{\tilde{\chi}_\lambda \tilde{\chi}_\mu}}{\overline{\tilde{\chi}_\rho}} = \frac{\tilde{\chi}^*_\lambda \tilde{\chi}^*_\mu}{\tilde{\chi}_\rho} = \tilde{\chi}^*_\mu \star \tilde{\chi}^*_\lambda.$$

$\square$

Corollary 5.16. *For any* $\lambda, \mu, \nu \in \mathcal{C}_k$, $\tilde{N}^{\nu}_{\lambda,\mu} = \tilde{N}^{\nu^*}_{\lambda^*,\mu^*}$.

Recall that the Weyl character χ_λ is graded by λ in $Z(\mathfrak{g}) = P/Q$. Therefore, the coefficient $N^{\nu}_{\mu,\lambda}$ is non-zero only if the grading matches, namely $\mu + \lambda - \nu \in Q$.

Definition 5.17. We define the grading of $\tilde{\chi}_\lambda$ to be $\lambda - \rho$ in $Z(\mathfrak{g}) = P/Q$.

By Definition 5.11, $\tilde{N}^{\nu}_{\mu,\lambda}$ is non-zero only if the grading matches. So the grading is additive under the multiplication $\star$.

Definition 5.18. Let R_k denote the $Z(\mathfrak{g})$-graded fusion ring with basis $\{\tilde{\chi}_\lambda\}_{\lambda \in \mathcal{C}_k}$.

Proposition 5.19. *For any* $g, g' \in L^2(T_{k,0})^W$, $\lambda \in \mathcal{C}_k$, *we have*

$$\langle \tilde{\chi}_\lambda \star g, g' \rangle = \langle g, \tilde{\chi}^*_\lambda \star g' \rangle.$$

Proof. By Eq. (8), we have

$$\langle \tilde{\chi}_\lambda \star g, g' \rangle = \langle \frac{\tilde{\chi}_\lambda}{\tilde{\chi}_\rho} g, g' \rangle = \langle g, \overline{\left(\frac{\tilde{\chi}_\lambda}{\tilde{\chi}_\rho}\right)} g' \rangle = \langle g, \frac{\tilde{\chi}_{\lambda^*}}{\tilde{\chi}_\rho} g' \rangle = \langle g, \tilde{\chi}_{\lambda^*} \star g' \rangle.$$

$\square$

As a consequence, $L^2(T_{k,0})^W$ is a faithful $Z(\mathfrak{g})$-graded, unital *-representation of the *-algebra $L^2(T_{k,0})^W$. So $L^2(T_{k,0})^W$ is an abelian $Z(\mathfrak{g})$-graded, unital C^*-algebra with the multiplication $\star$, and involution $*$.

Definition 5.20. We call the $Z(\mathfrak{g})$-graded, unital C^*-algebra $L^2(T_{k,0})^W$ the k-character Verlinde algebra.

Remark 5.21. By Theorem 5.12, one sees that $L^2(T_{k,0})^W$ is isomorphic to the Verlinde algebra of quantum $\mathfrak{g}$ at level k, and R_k is the $Z(\mathfrak{g})$-graded fusion ring. Therefore, one can consider the anti-symmetric k-characters as the characters of the corresponding irreps. The grading is the usual one for irreps of quantum groups. Theorems 5.6 and 5.12 correspond to the unitarity of the S matrix and the Verlinde formula respectively. (See [1] for a construction of the corresponding fusion category.)

6. GUS-Representations

In this section, we recall some well-known, elementary properties of generalizations of Dynkin diagrams over $\mathfrak{g}$ at level k. Such notions have been well studied by many people, as referred in §2 by slight different conditions for different settings. Taking into account the already existing closely related terminological variants that refer to slightly distinct concepts generalizing Dynkin diagrams, we decided to use the words *quantum Dynkin diagrams* in our own work.

For our purpose, we consider the general case for a ($Z(\mathfrak{g})$-graded,) unital, *-representations Π of R_ℓ and we prove our main results about the correspondence of *quantum Coxeter exponents* for Π in §8. The results for different kinds of generalized Dynkin diagrams appear to be special cases. In particular, we define *quantum Dynkin diagrams* in this section, such that the multiplicities of Coxeter exponents in the two different generalizations are identical for such Dynkin diagrams.

A major difference between Di Francesco-Zuber generalized Dynkin diagram and Ocneanu's higher Dynkin diagram is that the later one requires a categorification. The quantum Dynkin diagrams in this paper are slightly different from the generalized Dynkin diagrams. The quantum Dynkin diagram requires a grading in the definition, and it requires the adjacent matrix for the fundamental representations to be matrices of natural numbers, but not for all irreducible representations. We define quantum Dynkin diagrams in this way to minimize the assumptions such that the main result Theorem 8.28 holds. Because of the difference among the three generalizations of Dynkin diagrams, we used different terminologies, so that there would be no confusions.

Recall that the fusion ring R_k of $\mathfrak{g}$ at level k has a $Z(\mathfrak{g}) = P/Q$ grading, and $\widetilde{\chi}_\mu \in R_k$ is graded by $\mu - \rho \in Z(\mathfrak{g})$. Moreover, $\{\widetilde{\chi}_\mu\}_{\mu \in \mathcal{C}_k}$ is a basis of R_k and $\{\widetilde{\chi}_{\omega+\rho}\}_{\omega \in \Omega}$ are generators of the fusion ring R_k.

Definition 6.1 (*GUS-rep.*) For a finite dimensional Hilbert space $\mathcal{H}$, a representation $\Pi : R_k \to \hom(\mathcal{H})$ is called a unital, *-representation, abbreviated as US-rep, if for any $\widetilde{\chi}_\mu \in R_k$, $\lambda \in Z(\mathfrak{g})$, the following properties are satisfied:

(1) $\Pi(\widetilde{\chi}_\rho) = I$;
(2) $\langle \Pi(\widetilde{\chi}_\mu)v, v' \rangle = \langle v, \Pi(\widetilde{\chi}_\mu^*)v' \rangle$ for any $v, v' \in \mathcal{H}$.

Furthermore, if $\mathcal{H} = \bigoplus_{\lambda \in Z(\mathfrak{g})} \mathcal{H}_\lambda$ is $Z(\mathfrak{g})$-graded, and

(3) $\Pi(\widetilde{\chi}_\mu)\mathcal{H}_\lambda \subseteq \mathcal{H}_{\lambda+\mu-\rho}$, $\forall \lambda \in Z(\mathfrak{g})$,

then Π is called a $Z(\mathfrak{g})$-graded, unital, *-representation, abbreviated as GUS-rep. Equivalently, Π is a ($Z(\mathfrak{g})$-graded,) unital, *-representation of the k-character Verlinde algebra.

Definition 6.2. For a d-dimensional US-rep Π, an ONB B of $\mathcal{H}$ is called a $\mathbb{K}$ basis, $\mathbb{K} = \mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{N}$, if $\Pi_\omega := \Pi(\widetilde{\chi}_{\omega+\rho}) \in M_d(\mathbb{K})$, for any $\omega \in \Omega$. If Π is a GUS-rep, $B = \bigsqcup_{\lambda \in Z(\mathfrak{g})} B_\lambda$ and each B_λ is an ONB of $\mathcal{H}_\lambda$. then B is called $Z(\mathfrak{g})$-graded.

Remark 6.3. If $\Pi(\widetilde{\chi}_\mu) \in M_d(\mathbb{N})$, for any $\mu \in \mathcal{C}_k$, then Π is called a NIM-rep, see e.g. Definition 3 in [22].

A quiver G consists of a set G_v of vertices, a set G_e of oriented edges, a function $s : G_e \mapsto G_v$ giving the start of the edge and another function $t : G_e \mapsto G_v$ giving the target of the edge.

Definition 6.4. We call a quiver G $\mathfrak{g}$-graded, if there is a grading map ε, $\varepsilon : G_v \to Z(\mathfrak{g})$ and $\varepsilon : G_e \mapsto \Omega$, such that for any $e \in G_e$, it is true that $\varepsilon(s(e)) + \varepsilon(e) = \varepsilon(t(e))$ in $Z(\mathfrak{g})$. Moreover, the adjacency matrix Γ_ω, $\omega \in \Omega$ is defined as a matrix acting on G_v, whose v_1, v_2 entry is the number of edges graded by ω from the vertex v_2 to the vertex v_1.

Remark 6.5. Similar notions without grading have been studied by Etingof and Khovanov in [17],

Definition 6.6 (*Quantum Dynkin diagrams*). Let G be a $\mathfrak{g}$-graded quiver and $k \in \mathbb{N}$. We call G a quantum Dynkin diagram over $\mathfrak{g}$ at level k, if there is a $Z(\mathfrak{g})$-graded unital *-homomorphism $\Pi_G : R_k \to M_{G_v}(\mathbb{Z})$, such that

$$\Gamma_\omega = \Pi_\omega, \ \forall\, \omega \in \Omega.$$

Furthermore, we say G is a $\mathfrak{g}$-graded NIM-rep if $\Pi(\widetilde{\chi}_\lambda) \in M_{G_v}(\mathbb{N})$, $\forall\, \lambda \in \mathcal{C}_k$. We call G simple, if G is a connected quiver.

Proposition 6.7. *If G is a quantum Dynkin diagram, then the *-homomorphism Π_G is unique.*

Proof. For any $\omega \in \Omega$, the adjacency matrix $\Pi_\omega = \Gamma_\omega$ is determined by G by definition. Since $\{\widetilde{\chi}_{\omega+\rho} | \omega \in \Omega\}$ generate the ring of R_k, Π is uniquely determined by G. $\qquad\square$

The multiplication $\star$ of $\widetilde{\chi}_\lambda$, $\lambda \in \mathcal{C}_k$, on $L^2(T_{k,0})^W$ defines a GUS-rep Π_A of R_k. Recall that $\{\frac{1}{|W|}\widetilde{\chi}_\mu\}_{\mu \in \mathcal{C}_k}$ is an orthonormal basis of $L^2(T_{k,0})^W$. Acting on this basis, we have the regular representation

$$\Pi_A(\widetilde{\chi}_\lambda)\widetilde{\chi}_\mu := \widetilde{\chi}_\lambda \star \widetilde{\chi}_\mu = \sum_{v \in \mathcal{C}_k} \tilde{N}^v_{\lambda,\mu} \widetilde{\chi}_v.$$

In particular $\Pi_A(\widetilde{\chi}_\lambda)_{v,\mu} = \tilde{N}^v_{\lambda,\mu}$.

The fusion graph of the Verlinde algebra with respect to the fundamental representations is a quantum Dynkin diagram, which is usually referred as the type A graph in different formulations:

Definition 6.8 (*Type A quantum Dynkin diagrams*). For any simple Lie algebra $\mathfrak{g}$ and level k, we define the type A quantum Dynkin diagram $\mathcal{A}_k(\mathfrak{g})$ as follows:

(1) The vertices of $\mathcal{A}_k(\mathfrak{g})$ is $\{\widetilde{\chi}_\lambda : \lambda \in \mathcal{C}_k\}$ graded by $Z(\mathfrak{g})$.
(2) For any $\omega \in \Omega$, the multiplicity of the edge from μ to v graded by ω is $\tilde{N}^v_{\omega+\rho,\mu}$.

We do not need the well-known construction of NIM-reps from module categories in our approach in this paper. We recall this construction here for readers who might be interested in this connection. For further literature, we refer the readers to references mentioned in §2. Let $\mathscr{C}$ be the unitary modular tensor category obtained by the semisimplification of the representation category of quantum $\mathfrak{g}$ at level k. (The category $\mathscr{C}$ can also be realized as the representation category of projective positive energy representations of the corresponding loop groups or vertex operator algebras.) Let $\mathscr{M}$ be a module category of $\mathscr{C}$. We recall the following well-known construction of a quiver $G_\mathscr{M}$, or a NIM-rep, from the action of $\mathscr{C}$ on $\mathscr{M}$:

(1) The vertices of $G_\mathscr{M}$ are (representatives of) irreducible modules in $\mathscr{M}$, denoted by $Irr_\mathscr{M}$.

(2) For any $\omega \in \Omega$, and irreducible modules m_1, m_2 in $\mathcal{M}$, the multiplicity of the edge from m_1 to m_2 graded by ω is $\dim \hom_{\mathcal{M}}(V_\omega \otimes m_1, m_2)$.

Proposition 6.9. *For any $\lambda \in \mathcal{C}_k$, and $m_1 \in Irr_{\mathcal{M}}$, we define*

$$\Pi_{\mathcal{M}}(\widetilde{\chi}_\lambda) m_1 = \sum_{m_2 \in Irr_{\mathcal{M}}} \dim \hom_{\mathcal{M}}(V_{\lambda-\rho} \otimes m_1, m_2) m_2.$$

Then $\Pi_{\mathcal{M}}$ is a US-rep and $Irr_{\mathcal{M}}$ is a NIM-rep basis.

Proof. Since V_0 is the trivial representation, $\Pi_{\mathcal{M}}(\widetilde{\chi}_\rho)$ is the identity. The associativity of the action of $\mathcal{C}$ on $\mathcal{M}$ implies that Π is a representation. By the Frobenius reciprocity

$$\dim \hom_{\mathcal{M}}(V_{\lambda-\rho} \otimes m_1, m_2) = \dim \hom_{\mathcal{M}}(m_1, V_{\lambda^*-\rho} \otimes m_2),$$

we have that Π is a US-rep. Furthermore, $\dim \hom_{\mathcal{M}}(V_{\lambda-\rho} \otimes m_1, m_2) \in \mathbb{N}$, so $Irr_{\mathcal{M}}$ is a NIM-rep basis. □

In the rest of the paper, we deal with the general case for any US-rep or GUS-rep of R_k for any simple Lie algebra $\mathfrak{g}$ at any level $k \in \mathbb{N}$. In particular, the results for the general case hold for quantum Dynkin diagrams over $\mathfrak{g}$ at level k. This provides a general theory in the study of quantum Dynkin diagrams.

7. Spectrum of Representations

In this section, we investigate the spectral theory of a GUS-rep Π of R_k defined in Definition 6.1. We give an explicit construction of the spectrum of the regular representation Π_A in T. We prove that the spectrum for a GUS-rep Π is contained in the spectrum of the type A quantum Dynkin diagram. The results in this section are well-known properties of the spectrum of the Verlinde algebra [63] and representations of C^*-algebras.

Definition 7.1. Recall that the Weyl group W acts on each orbit of $T_{k,0}$ transitively. Define $\mathbf{Spec}_k :\cong T_{k,0}/W$.

For any $e^H \in T_{k,0}$, let δ_{e^H} be the delta function at e^H. For any $\lambda \in \mathcal{C}_k$, define

$$\Lambda_{e^H} = \sum_{r \in W} \det(r) \delta_{e^{r(H)}}. \tag{9}$$

Then $\Lambda_{e^H} \in L^2(T_{0,k})^W$. Moreover,

$$\Pi_A(\widetilde{\chi}_\lambda) \Lambda_{e^H} = \widetilde{\chi}_\lambda \star \Lambda_{e^H} = \frac{\widetilde{\chi}_\lambda \Lambda_{e^H}}{\widetilde{\chi}_\rho} = \chi_{\lambda-\rho}(e^H) \Lambda_{e^H}.$$

So Λ_{e^H} is a common eigenvector of $\Pi_A(\widetilde{\chi}_\lambda)$ with eigenvalue $\chi_{\lambda-\rho}(e^H)$.

Note that $\Lambda_{e^{r(H)}} = \det(r) \Lambda(e^H)$, so $\mathbb{C}(\Lambda_{e^H})$ is a well-defined eigenspace for $e^H \in \mathbf{Spec}_k$. The Weyl character $\chi_{\lambda-\rho}$ is symmetric, so it is also well-defined on $e^H \in T_{k,0}/W$.

Lemma 7.2. *For any $e^H, e^{H'} \in T_{k,0}$, the following are equivalent:*

(1) $e^H = e^{r(H')}$, for some $r \in W$;
(2) $\chi_\lambda(e^H) = \chi_\lambda(e^{H'})$, for any $\lambda \in \mathcal{C}$;
(3) $\chi_\omega(e^H) = \chi_\omega(e^{H'})$, for any $\omega \in \Omega$.

Proof. $(1) \to (2)$: It follows from the fact that Weyl characters are symmetric under the Weyl group action.

$(2) \to (1)$: The Weyl characters $\{\chi_\lambda\}_{\lambda \in \mathcal{C}}$ are a basis of symmetric functions on the maximal torus T with respect to the Weyl group W action. So any symmetric function has the same value on e^H and $e^{H'}$. By the Stone-Weierstrass theorem, $e^H = e^{r(H')}$, for some $r \in W$.

$(2) \leftrightarrow (3)$: It follows from the fact that $\{\chi_\omega | \omega \in \Omega\}$ generate the ring of Weyl characters. $\qquad \square$

The adjacency matrices of the Verlinde algebra commute with each other, and they can be diagonalized simultaneously, this is equivalent to the Verlinde formula [63]. Their common eigenvectors are captured by the modular S matrix. For a common eigenvector, we call the corresponding eigenvalues of the adjacency matrices a **simultaneous eigenvalue**. Now we represent these simultaneous eigenvalues using $e^H \in \mathbf{Spec}_k$:

Theorem 7.3. *When $\Pi = \Pi_A$, for any $e^H \in \mathbf{Spec}_k$, $\mathbb{C}(\Lambda_{e^H})$ is the common eigenspace of the adjacency matrices Π_ω, $\omega \in \Omega$, with eigenvalue $\chi_\omega(e^H)$. Any simultaneous eigenvalue of the adjacency matrices is of this form and it has multiplicity one.*

Proof. We have shown that for any $e^H \in \mathbf{Spec}_k$, $\mathbb{C}(\Lambda_{e^H})$ is the common eigenspace of the adjacency matrices Π_ω, $\omega \in \Omega$, with eigenvalue $\chi_\omega(e^H)$. By Lemma 7.2, if e^H and $e^{H'}$ are different in $\mathbf{Spec}_k$, then the corresponding eigenvalues are different.

Note that $\dim(L^2(P_k)^W) = \frac{|W_{k,0}|}{|W|}$ and $\dim(L^2(T_k)^W) = \frac{|T_{k,0}|}{|W|}$. By Proposition 4.10, their dimensions are the same. So

$$|\mathbf{Spec}_k| = \frac{|T_{k,0}|}{|W|} = \frac{|W_{k,0}|}{|W|} = |\mathcal{C}_k|.$$

Therefore, each eigenvalue corresponding to $e^H \in \mathbf{Spec}_k$ has multiplicity one, and they are all eigenvalues. $\qquad \square$

Theorem 7.4. *Suppose Π is a US-rep of R_k over $\mathfrak{g}$ at level k. Then for any common eigenvector v of the adjacency matrices Π_ω, $\omega \in \Omega$, there is $e^H \in \mathbf{Spec}_k$ such that*

$$\Pi_\omega v = \chi_\omega(e^H) v.$$

Proof. Since $L^2(T_{k,0})^W$ is an abelian C^* algebra, all its one dimensional representations are sub-representations of the regular representation Π_A. So any simultaneous eigenvalue of Π is always a simultaneous eigenvalue of Π_A. By Theorem 7.3, the statement holds. $\qquad \square$

The following result is well-known in different formulations, see e.g. [12,17]:

Fact 7.5. *Quantum Dynkin diagrams over $\mathfrak{sl}_2$ at level $k \in \mathbb{N}$ are ADE Dynkin diagrams with Coxeter number $2 + k$.*

Proof. If G is an ADE Dynkin diagram, then they are $\mathbb{Z}_2$-graded quivers of modules of quantum $\mathfrak{sl}_2$. So they are quantum Dynkin diagrams over $\mathfrak{sl}_2$ at level k, and $k + 2$ is the Coxeter number of G.

If G be a quantum Dynkin diagram over $\mathfrak{sl}_2$ at level k, then Π_1 is the adjacency matrix of the bipartite graph G. By Theorem 7.4, any eigenvalue of Π_1 is $e^{\frac{\pi i}{k+2}t} - e^{-\frac{\pi i}{k+2}t}$, for some $t \in \mathbb{Z}_{2(k+2)}$, $t \neq 0$ and $t \neq k + 2$. So $\|\Pi_1\| < 2$. Therefore, G is an ADE Dynkin diagram, a well known result. Moreover, R_k is determined by Π_1, so k is determined. $\square$

Definition 7.6. Suppose Π is a US-rep of R_k over $\mathfrak{g}$ at level k. For a common eigenvector v of the adjacency matrices, we call the corresponding $e^H \in \mathbf{Spec}_k$ in Theorem 7.4 the spectrum of v, denoted by $\mathrm{sp}(v)$.

Definition 7.7. Suppose Π is a US-rep of R_k over $\mathfrak{g}$ at level k. Let B be a common eigenbasis of the adjacency matrices. We define the spectrum of Π to be $\{\mathrm{sp}(v), v \in B\}$, a subset of $\mathbf{Spec}_k$. We define $B(e^H)$ to be the subset of B with spectrum e^H. The multiplicity of the spectrum $e^H \in \mathbf{Spec}_k$ is the order $|B(e^H)|$, denoted by $m_\Pi(e^H)$.

Since the Weyl Character is symmetric under the action of W, we can lift the spectrum from $\mathbf{Spec}_k$ to $T_{k,0}$. We identify $\mathbf{Spec}_k$ as a fundamental domain in $T_{k,0}$, still denoted by $\mathbf{Spec}_k$. Then for any $e^H \in T_{k,0}, r \in W$, we have $m_\Pi(e^H) = m_\Pi(e^{r(H)})$.

Notation 7.8. *Furthermore, if V and Π are $Z(\mathfrak{g})$ graded, then for any $v \in L^2(B)$, we have the decomposition*

$$v = \sum_{\lambda \in Z(\mathfrak{g})} v_\lambda,$$

where $v_\lambda \in V_\lambda$, called a λ-graded vector.

Definition 7.9. Suppose Π is a GUS-rep of R_k over $\mathfrak{g}$ at level k. We define a group action of T_Q on $L^2(B)$: For any $e^{H'} \in T_Q$ and any λ-graded vector v_λ,

$$e^{H'} \circ v_\lambda := e^{i \langle \lambda, H' \rangle} v_\lambda.$$

Theorem 7.10. *Suppose Π is a GUS-rep of R_k over $\mathfrak{g}$ at level k. For any $\omega \in \Omega$, $e^H \in T_k$ and $e^{H'} \in T_Q$, the following are equivalent:*

(1) $\Pi_\omega(v) = \chi_\omega(e^H)v$;
(2) $\Pi_\omega(e^{H'} \circ v) = \chi_\omega(e^{H-H'})e^{H'} \circ v$.

Consequently, $m_\Pi(e^H) = m_\Pi(e^{H-H'})$.

Proof. Suppose $v = \sum\limits_{\lambda \in Z(\mathfrak{g})} v_\lambda$, where v_λ is graded by λ. If $\Pi_\omega(v) = \chi_\omega(e^H)v$, then

$$\chi_\omega(e^H)v = \Pi_\omega v = \sum_{\lambda \in Z(\mathfrak{g})} \Pi_\omega v_\lambda.$$

So $\Pi_\omega v_\lambda = \chi_\omega(e^H)v_{\lambda+\omega}$. For any $e^{H'} \in T_Q$,

$$\Pi_\omega(e^{H'} \circ v) = \sum_{\lambda \in Z(\mathfrak{g})} e^{i \langle \lambda, H' \rangle} \Pi_\omega v_\lambda = \sum_{\lambda \in Z(\mathfrak{g})} e^{i \langle \lambda, H' \rangle} \chi_\omega(e^H) v_{\lambda+\omega} = \chi_\omega(e^H) e^{i \langle -\omega, H' \rangle}(e^{H'} \circ v).$$

For any $\lambda \in P_k, r \in W$, we have $r(\lambda) - \lambda \in Q_k^\vee$. For any $e^{H'} \in T_Q$,

$$\widetilde{\chi}_\lambda(e^{H+H'}) = \widetilde{\chi}_\lambda(e^H) e^{i \langle \lambda, H' \rangle}.$$

So,

$$\chi_\omega(e^H) e^{i \langle \omega, H' \rangle} = \frac{\widetilde{\chi}_{\omega+\rho}(e^H) e^{i \langle \omega+\rho, H' \rangle}}{\widetilde{\chi}_\rho(e^H) e^{i \langle \rho, H' \rangle}} = \frac{\widetilde{\chi}_{\omega+\rho}(e^{H+H'})}{\widetilde{\chi}_\rho(e^{H+H'})} = \chi_\omega(e^{H+H'}).$$

Therefore,

$$\Pi_\omega(e^{H'} \circ v) = \chi_\omega(e^{H-H'})e^{H'} \circ v.$$

$\square$

8. Quantum Coxeter Exponents

As mentioned in the introduction and §2, Gabriel constructed the root system using the quiver representations of ADE Dynkin diagrams in [19], and using additive functions on the Auslander–Reiten quiver in [20]. This construction is recovered by Dorey [14] in terms of NIM-reps of $\mathfrak{g} = \mathfrak{sl}_2$. Ocneanu proposed a generalization of roots, called higher roots, with an inner product formula for higher Dynkin diagrams over other types of Lie algebras $\mathfrak{g}$ in [55], also called hyper roots in [11] when $\mathfrak{g} = \mathfrak{sl}_3$.

In §8.1, we explain how additive functions, the i.e., root space, Coxeter elements and exponents can be generalized for a US-rep Π over $\mathfrak{g}$ at level k. We prove an inner product formula for additive functions in Theorem 8.9, which has a similar form as Ocneanu's inner product formula for higher roots. In §8.2, we prove the one to one correspondence between the spectrum of Π and the spectrum of the generalized Coxeter elements associated with Π. Furthermore, we prove an identity between their multiplicities, when Π is a GUS-rep over $\mathfrak{g}$ and $\mathfrak{g}$ is of type ADE. This generalizes the identity of multiplicities of Coxeter exponents for the classical ADE Dynkin diagrams and answers Gannon's comment on Kac's question. The identity of multiplicities does not hold when $\mathfrak{g}$ is not of type ADE, see Theorem 8.28.

8.1. Quantum Root System. Suppose Π is a US-rep of R_k over $\mathfrak{g}$ at level k, and B is a common eigenbasis of Π_ω, $\omega \in \Omega$. We define the quantum root system $\mathcal{S}_{B,0}$ in 8.15, when Π and B are $Z(\mathfrak{g})$-graded. First we lift the spectrum of Π from $\mathbf{Spec}_k$ to $T_{k,0}$ and define the corresponding eigenspace:

Definition 8.1. We define the lifted eigenspace $\mathcal{E}_\pi$ as

$$\mathcal{E}_\pi := \{\tilde{g} \in L^2(T_k) \otimes \mathcal{H} : (\chi_\omega \otimes I - I \otimes \Pi_\omega)\tilde{g} = 0, \ \forall \ \omega \in \Omega\}.$$

Proposition 8.2. *The set* $B_\pi := \{\delta_{e^H} \otimes v : e^H \in T_{k,0}, v \in B(e^H)\}$ *is an ONB of* $\mathcal{E}_\pi$.

Proof. Note that the set

$$\{\delta_{e^H} \otimes v : e^H \in T_k, v \in B\}$$

is an ONB of $L^2(T_k) \otimes \mathcal{H}$, so for any $\tilde{g} \in L^2(T_k) \otimes \mathcal{H}$, we have the decomposition

$$\tilde{g} = \sum_{e^H \in T_k} \sum_{v \in B} \beta_{e^H, v} \delta_{e^H} \otimes v,$$

for some $\beta_{e^H, v} \in \mathbb{C}$. If $\tilde{g} \in \mathcal{E}_\pi$, namely $(\chi_\omega \otimes I)\tilde{g} = (I \otimes \Pi_\omega)\tilde{g}$, for any $\omega \in \Omega$, then

$$\sum_{e^H \in T_k} \sum_{v \in B} \chi_\omega(e^H)\beta_{e^H, v} \delta_{e^H} \otimes v = \sum_{e^H \in T_k} \sum_{v \in B} \chi_\omega(sp(v))\beta_{e^H, v} \delta_{e^H} \otimes v.$$

If $\beta_{e^H, v} \neq 0$, then $\chi_\omega(e^H) = \chi_\omega(sp(v))$, $\forall \ \omega \in \Omega$. By Lemma 7.2, there is an $r \in W$, such that $sp(v) = e^{r(H)}$, equivalently $v \in B(e^H)$.

On the other hand, for any $v \in B(e^H)$, we have

$$(\chi_\omega \otimes I - I \otimes \Pi_\omega)(\delta_{e^H} \otimes v) = (\chi_\omega(e^H) - \chi_\omega(e^H))(\delta_{e^H} \otimes v) = 0,$$

so $\delta_{e^H} \otimes v \in \mathcal{E}_\pi$. Therefore, B_π is an ONB of $\mathcal{E}_\pi$. $\qquad\qquad\square$

Definition 8.3. For any $f \in L^2(P_k)$, we define

$$(\Gamma_\lambda f)(\mu) = \sum_{\nu \in \mathcal{W}(\lambda)} m_\lambda(\nu) f(\mu + \nu), \forall j \in P_k,$$

where $\mathcal{W}(\lambda)$ is the weight diagram of $\lambda \in \mathcal{C}$, and m_λ is the multiplicity function.

Lemma 8.4. *Let $\mathcal{F} : L^2(P_k) \to L^2(T_k)$ be the Fourier transform in Eq.* (6). *Then for any $\omega \in \Omega$, we have*

$$\mathcal{F}\Gamma_\omega = \chi_\omega \mathcal{F}.$$

Proof. For any $\lambda \in P_k$,

$$\mathcal{F}\Gamma_\omega(\delta_\lambda) = \sum_{e^H \in T_k} \sum_{\nu \in \omega} m_\omega(\nu) e^{i\langle \lambda+\nu, H \rangle} \delta_{e^H} = \sum_{e^H \in T_k} \chi_\omega(e^H) e^{i\langle \lambda, H \rangle} \delta_{e^H} = \chi_\omega \mathcal{F}(\delta_\lambda).$$

$\square$

Definition 8.5. For a US-rep Π, we define the full quantum root space $\mathscr{R}_\pi$ as

$$\mathscr{R}_\pi := \{\tilde{f} \in L^2(P_k) \otimes \mathcal{H} : (\Gamma_\omega \otimes I - I \otimes \Pi_\omega)\tilde{f} = 0, \forall \omega \in \Omega\}.$$

Furthermore, if Π is $Z(\mathfrak{g})$-graded, we define the quantum root space $\mathscr{R}_{\pi,0}$ as

$$\mathscr{R}_{\pi,0} := \{\tilde{f} \in \bigoplus_{\lambda \in P_k} \mathbb{C}\delta_\lambda \otimes \mathcal{H}_{-\lambda} : (\Gamma_\omega \otimes I - I \otimes \Pi_\omega)\tilde{f} = 0, \forall \omega \in \Omega\}.$$

Remark 8.6. When G is an ADE Dynkin diagram, Π_G is a GUS-rep graded by $\mathbb{Z}_2 = Z(\mathfrak{sl}_2)$, and $\mathscr{R}_{\pi,0}$ is the space of additive functions on the corresponding Auslander–Reiten quiver, isomorphic to the root space.

Notation 8.7. *Let $\mathcal{P}_{\mathscr{E}_\pi}$ be the orthogonal projection from $L^2(T_k) \otimes \mathcal{H}$ to $\mathscr{E}_\pi$. Let $\mathcal{P}_{\mathscr{R}_\pi}$ be the orthogonal projection from $L^2(P_k) \otimes \mathcal{H}$ to $\mathscr{R}_\pi$.*

Proposition 8.8. *The Fourier transform $\mathcal{F} \otimes I$ is a unitary transformation from $\mathscr{R}_\pi$ to $\mathscr{E}_\pi$, namely*

$$(\mathcal{F} \otimes I)\mathcal{P}_{\mathscr{R}_\pi} = \mathcal{P}_{\mathscr{E}_\pi}(\mathcal{F} \otimes I).$$

Proof. It follows from Lemma 8.4 and Definitions 8.1 and 8.5. $\square$

Now we give the inner product formula on $\mathscr{R}_\pi$ using Fourier duality between $\mathscr{R}_\pi$ and $\mathscr{E}_\pi$.

Theorem 8.9. *For any $\lambda, \mu \in P_k$ and $v_1, v_2 \in \mathcal{H}$, we have*

$$\langle \mathcal{P}_{\mathscr{R}_\pi}(\delta_\lambda \otimes v_1), \mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v_2) \rangle = \frac{1}{|T_k|} \sum_{r \in W} det(r)\langle v_1, \Pi(\widetilde{\chi}_{\mu-\lambda+r(\rho)})v_2 \rangle. \quad (10)$$

Proof. For any $e^H \in \mathbf{Spec}_k$, $B(e^H)$ is an eigenbasis of the eigenspace in $L^2(T_{k,0})^W$ with spectrum e^H. Then

$$\mathcal{P}_{\mathscr{E}_\pi}(\mathcal{F}\delta_\lambda \otimes v_1) = \mathcal{P}_{\mathscr{E}_\pi}\left(\sum_{e^H \in T_k} e^{i\langle \lambda, H\rangle}\delta_{e^H} \otimes v_1\right) = \sum_{e^H \in T_{k,0}}\sum_{v \in B(e^H)} e^{i\langle \lambda, H\rangle}\langle v, v_1\rangle \delta_{e^H} \otimes v.$$

By Proposition 8.8,

$$
\begin{aligned}
&\langle \mathcal{P}_{\mathscr{R}_\pi}(\delta_\lambda \otimes v_1), \mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v_2)\rangle\\
&= \langle \mathcal{P}_{\mathscr{E}_\pi}(\mathcal{F}\delta_\lambda \otimes v_1), \mathcal{P}_{\mathscr{E}_\pi}(\mathcal{F}\delta_\mu \otimes v_2)\rangle\\
&= \frac{1}{|T_k|}\sum_{e^H \in T_{k,0}}\sum_{v \in B(e^H)} \overline{e^{i\langle \lambda, H\rangle}\langle v, v_1\rangle}e^{i\langle \mu, H\rangle}\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{e^H \in T_{k,0}}\sum_{v \in B(e^H)} e^{i\langle \mu-\lambda, H\rangle}\langle v_1, v\rangle\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{e^H \in \mathbf{Spec}_k}\sum_{v \in B(e^H)}\sum_{r' \in W} e^{i\langle \mu-\lambda, r'(H)\rangle}\langle v_1, v\rangle\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{r \in W}\sum_{e^H \in \mathbf{Spec}_k}\sum_{v \in B(e^H)}\sum_{r' \in W} \det(r)\frac{e^{i\langle \mu-\lambda+r(\rho), r'(H)\rangle}}{\widetilde{\chi}_\rho(e^{r'(H)})}\langle v_1, v\rangle\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{r \in W}\sum_{e^H \in \mathbf{Spec}_k}\sum_{v \in B(e^H)}\sum_{r' \in W} \det(rr')\frac{e^{i\langle r'(\mu-\lambda+r(\rho)), H\rangle}}{\widetilde{\chi}_\rho(e^{H})}\langle v_1, v\rangle\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{r \in W}\sum_{e^H \in \mathbf{Spec}_k}\sum_{v \in B(e^H)} \det(r)\frac{\widetilde{\chi}_{\mu-\lambda+r(\rho)}(e^{H})}{\widetilde{\chi}_\rho(e^{H})}\langle v_1, v\rangle\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{r \in W}\sum_{e^H \in \mathbf{Spec}_k}\sum_{v \in B(e^H)} \det(r)\langle v_1, \Pi(\widetilde{\chi}_{\mu-\lambda+r(\rho)})v\rangle\langle v, v_2\rangle\\
&= \frac{1}{|T_k|}\sum_{r \in W} \det(r)\langle v_1, \Pi(\widetilde{\chi}_{\mu-\lambda+r(\rho)})v_2\rangle.
\end{aligned}
$$

$\square$

Remark 8.10. Ocneanu called $\mathscr{R}_\pi$ the space of biharmonic functions and outlined a proof of the inner product formula (10) for higher Dynkin diagrams over $\mathfrak{sl}_n$ in the course [56]. Our statement and proof are different from the one outlined by Ocneanu.

Corollary 8.11. *For any $\mu \in P_k$ and $v \in \mathcal{H}$, we have*

$$\mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v) = \frac{1}{|T_k|}\sum_{\lambda \in P_k}\sum_{r \in W} \det(r)\delta_\lambda \otimes \Pi(\widetilde{\chi}_{\mu-\lambda+r(\rho)})v.$$

Proof. It follows from checking the inner product with $\delta_\lambda \otimes v_1$ on both sides using Theorem 8.9. $\square$

Remark 8.12. If B is a $\mathbb{Z}$-basis, then the functions $\{|T_k|\mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v) \in \mathscr{R}_\pi : \mu \in P_k, \ v \in B\}$ have integer coefficients on the ONB $\{\delta_\lambda : \lambda \in P_k\} \otimes B$. We consider them as a generalization of additive functions on the Auslander–Reiten quivers given in Eq. (5).

Corollary 8.13. *When Π is a GUS-rep, let $\mathcal{P}_0$ be the orthogonal projection from $L^2(P_k) \times \mathcal{H}$ onto the neutral subspace $\bigoplus\limits_{\lambda \in P_k} \mathbb{C}\delta_\lambda \otimes \mathcal{H}_{-\lambda}$. Then*

$$\mathcal{P}_{\mathscr{R}_\pi}\mathcal{P}_0 = \mathcal{P}_0\mathcal{P}_{\mathscr{R}_\pi},$$

and it is the projection onto $\mathscr{R}_{\pi,0}$.

Proof. When Π is a GUS-rep, by Corollary 8.11, for any $\mu \in P_k$ and $v \in \mathcal{H}_{-\mu}$, we have that

$$\mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v) = \mathcal{P}_0\mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v).$$

Therefore, $\mathcal{P}_{\mathscr{R}_\pi}\mathcal{P}_0 = \mathcal{P}_0\mathcal{P}_{\mathscr{R}_\pi}\mathcal{P}_0$. Both $\mathcal{P}_{\mathscr{R}_\pi}$ and $\mathcal{P}_0$ are projections, so

$$\mathcal{P}_{\mathscr{R}_\pi}\mathcal{P}_0 = \mathcal{P}_0\mathcal{P}_{\mathscr{R}_\pi}.$$

By definition, it is the projection onto $\mathscr{R}_{\pi,0}$. $\square$

Therefore, we also call $\mathscr{R}_{\pi,0}$ the neutral subspace of $\mathscr{R}_\pi$, as its total grading is 0 in $Z(\mathfrak{g})$.

Corollary 8.14. *For any $\mu \in P_k$ and $v \in \mathcal{H}$, we have*

$$\|\mathcal{P}_{\mathscr{R}_\pi}(\delta_\mu \otimes v)\|_2^2 = \frac{|W|}{|T_k|}\|v\|_2^2.$$

Proof. It follows from Theorem 8.9 and that $\det(r)\Pi_{r(\rho)} = \Pi_\rho$ is the identity. $\square$

Definition 8.15. We define the quantum root sphere as

$$\mathcal{S}_\pi := \{\sqrt{|T_k|}\mathcal{P}_{\mathscr{R}_\pi}(\delta_\lambda \otimes v) : \lambda \in P_k, v \in \mathcal{H}, \|v\|_2 = 1\}.$$

For an ONB B of $\mathcal{H}$, we define the full quantum root system as

$$\mathcal{S}_B := \{\sqrt{|T_k|}\mathcal{P}_{\mathscr{R}_\pi}(\delta_\lambda \otimes v) : \lambda \in P_k, v \in B, \|v\|_2 = 1\}.$$

When Π and B are $Z(\mathfrak{g})$-graded, we define the quantum root system as

$$\mathcal{S}_{B,0} := \{\sqrt{|T_k|}\mathcal{P}_{\mathscr{R}_\pi}(\delta_\lambda \otimes v) : \lambda \in P_k, v \in B_{-\lambda}, \|v\|_2 = 1\}.$$

(Recall that $B_{-\lambda}$ is defined in 6.2 as the graded basis.)

By Corollary 8.14, any vector in $\mathcal{S}_\pi$ has length $\sqrt{|W|}$. By Theorem 8.9, the inner product of vectors in $\mathcal{S}_B$ is determined by the adjacency matrices. Moreover, the (full) quantum root space is spanned by the (full) quantum root system.

Remark 8.16. When G is an ADE Dynkin diagram, $\mathcal{S}_{B,0}$ is a standard realization of the root system by additive functions, see [20,26]

8.2. Quantum Coxeter Exponents. Coxeter elements and Coxeter exponents can be defined through additive functions on the Auslander–Reiten of ADE Dynkin diagrams. In this section, we generalize them for the quantum root space of a US-rep Π over $\mathfrak{g}$ at level k. We identify a quantum Coxeter exponent with a spectrum e^H of Π, and compare their multiplicities in Theorem 8.28. We prove that two multiplicities are the same for quantum Dynkin diagrams over $\mathfrak{g}$ at level k, when $\mathfrak{g}$ is a type ADE Lie algebra, but not the same, when $\mathfrak{g}$ is not of type ADE. This gives a positive answer to the Gannon's commnent on Kac's question when $\mathfrak{g}$ is of type ADE, and a negative answer when $\mathfrak{g}$ is not of type ADE in our approach. There may be other approaches, such that the identity between different generalizations of Coxeter exponents and their multiplicities holds for some generalizations of Dynkin diagrams over all types of quantum groups.

For any $\mu \in P_k$, the translation $\vartheta_\mu : \lambda \mapsto \lambda + \mu$ on P_k induces a dual action on $L^2(P_k)$, still denoted by ϑ_μ:

$$\vartheta_\mu(\delta_\lambda) = \delta_{\lambda-\mu}, \ \forall \lambda \in P_k.$$

We define the corresponding translation on $L^2(P_k) \otimes \mathcal{H}$ as

$$\tilde{\vartheta}_\mu := \vartheta_\mu \otimes I.$$

Then $\tilde{\vartheta}_\mu$ is well-defined on $\mathscr{R}_\pi$ as well. Moreover, $\{\tilde{\vartheta}_\mu\}_{\mu \in P_k}$ is a finite abelian group.

Proposition 8.17. *The quantum root sphere $\mathcal{S}_\pi$ and quantum root system $\mathcal{S}_B$ are translation invariant by $\tilde{\vartheta}_\mu$, $\mu \in P_k$.*

Proof. It follows from Definition 8.15. □

Theorem 8.18. *The set $B_\pi = \{\delta_{e^H} \otimes v : e^H \in T_{k,0}, v \in B(e^H)\}$ is a common eigenbasis of $\{\mathcal{F}\vartheta_\mu\mathcal{F}^{-1} \otimes I\}_{\mu \in P_k}$ and $\{I \otimes \Pi_\omega\}_{\omega \in \Omega}$ in $\mathscr{E}_\pi$.*

Proof. By Proposition 8.2, B_π is a basis of $\mathscr{E}_\pi$. By Fourier duality,

$$\vartheta_\mu \mathcal{F}^{-1}\delta_{e^H} = e^{i\langle -\mu, H \rangle}\mathcal{F}^{-1}\delta_{e^H}, \ \forall \mu \in P_k, \ e^H \in T_{k,0}.$$

So

$$(\mathcal{F}\vartheta_\mu\mathcal{F}^{-1} \otimes I)(\delta_{e^H} \otimes v) = e^{i\langle -\mu, H \rangle}\delta_{e^H} \otimes v.$$

On the other hand,

$$(I \otimes \Pi_\omega)(\delta_{e^H} \otimes v) = \chi_\omega(e^H)\delta_{e^H} \otimes v.$$

Therefore, B_π is a common eigenbasis. □

Definition 8.19. By Theorem 8.18, if $\tilde{f}$ is a common eigenvector of the translations, then

$$\tilde{\vartheta}_\mu \tilde{f} = e^{i\langle -\mu, H \rangle}\tilde{f}, \ \forall \mu \in P_k,$$

for some $e^H \in T_{k,0}$. We call e^H the spectrum of $\tilde{f}$, denoted by $sp_\vartheta(\tilde{f}) = e^H$.

Definition 8.20. Suppose Π is a US-rep and A is a subgroup of P_k. We define the multiplicity m_A of e^H to be dimension of the common eigenspace in $\mathscr{R}_\pi$ of the translations $\{\vartheta_\mu\}_{\mu \in A}$ with spectrum e^H, namely

$$m_A(e^H) := \dim \left\{ \tilde{f} \in \mathscr{R}_\pi : \vartheta_\mu \tilde{f} = e^{i\langle -\mu, H \rangle}\tilde{f}, \ \forall \mu \in A \right\}.$$

For a subgroup A of $P_k = P/Q_k^\vee$, we consider it as an intermediate subgroup of $Q_k^\vee \subset P$. By Definition 4.1,

$$\mathfrak{t}_A = \{H \in \mathfrak{t} : \langle \alpha, H \rangle \in \mathbb{Z}, \; \forall \, \alpha \in A\},$$

$$T_A = \left\{ e^{2\pi H} \in T : H \in \mathfrak{t}_A \right\} \subseteq T_k.$$

Theorem 8.21. *Suppose Π is a US-rep and A is a subgroup of P_k. For any $e^H \in T_{k,0}$,*

$$m_A(e^H) = \sum_{e^{H'} \in T_A} m_\Pi(e^{H+H'}). \tag{11}$$

In particular,

$$m_P(e^H) = m_\Pi(e^H) = m_P(e^{r(H)}), \; \forall \, r \in W.$$

Proof. Note that for any $e^{H_1} \in T_{k,0}$ and $v \in B(e^{H_1})$, the following are equivalent:

(1) $\tilde{\vartheta}_\mu(\delta_{e^{H_1}} \otimes v) = e^{i\langle -\mu, H \rangle} \delta_{e^{H_1}} \otimes v, \; \forall \, \mu \in A$;
(2) $\tilde{\vartheta}_\mu(e^{H_1}) = \tilde{\vartheta}_\mu(e^H), \; \forall \, \mu \in A$;
(3) $\tilde{\vartheta}_\mu(e^{H_1-H}) = 1, \; \forall \, \mu \in A$;
(4) $e^{H_1-H} \in T_A$,

where (3) $\iff$ (4) follows from Theorem 4.3. By Theorem 8.18, we obtain Eq. (11). Furthermore, if $A = P$, then T_P is the trivial group. So

$$m_P(e^H) = m_\Pi(e^H) = m_\Pi(e^{r(H)}) = m_P(e^{r(H)}), \; \forall \, r \in W.$$

$\square$

If Π is a GUS-rep, then $\mathscr{R}_{\pi,0}$ is the neutral subspace of $\mathscr{R}_\pi$, and $\mathscr{R}_{\pi,0}$ is translation invariant under the action of $\tilde{\vartheta}_\mu$, $\mu \in Q$.

Definition 8.22. Suppose Π is a GUS-rep, and A is a subgroup of $Q/Q_k^\vee$. We define

$$m_{A,0}(e^H) = \dim\{\tilde{f} \in \mathscr{R}_{\pi,0} : \vartheta_\mu \tilde{f} = e^{i\langle -\mu, H \rangle} \tilde{f}, \; \forall \, \mu \in A\}.$$

Corollary 8.23. *Moreover, $m_A(e^H) = m_A(e^{H+H'})$ for any $e^{H'} \in T_{Q^\vee}$.*

Proof. It follows from Theorems 8.21 and 7.10. $\square$

Theorem 8.24. *Suppose Π is a GUS-rep, and A is a subgroup of $Q/Q_k^\vee$. For any $e^H \in T_{k,0}$, and $e^{H'} \in T_Q$,*

$$m_A(e^H) = m_A(e^{H+H'}) = n_z m_{A,0}(e^H).$$

In particular,

$$m_{Q,0}(e^H) = m_\Pi(e^H).$$

Proof. By Theorem 7.10 and Theorem 8.21, we have that

$$m_\Pi(e^H) = m_\Pi(e^{H+H'});$$
$$m_A(e^H) = m_A(e^{H+H'}).$$

Note that $\mathscr{R}_\pi = \bigsqcup_{\lambda \in Z(\mathfrak{g})} \vartheta_\lambda \mathscr{R}_{\pi,0}$, so

$$m_A(e^H) = n_z m_{A,0}(e^H).$$

When $A = Q$, by Theorem 8.21,

$$m_Q(e^H) = \sum_{e^{H'} \in T_Q} m_\Pi(e^{H+H'}) = |T_Q| m_\Pi(e^H) = n_z m_\Pi(e^H).$$

Therefore,

$$m_{Q,0}(e^H) = m_\Pi(e^H).$$

$\square$

Recall that Q_h is the set of highest positive roots of $\mathfrak{g}$. For any $\eta \in Q_h$, its order in the group P_k is the quantum Coxeter number c_k. So the translation $\tilde{\vartheta}_\eta$ on $\mathscr{R}_\pi$ (or $\mathscr{R}_{\pi,0}$) has periodicity c_k.

Definition 8.25. For any $\eta \in Q_h$, we call the translation $\tilde{\vartheta}_\eta$ on $\mathscr{R}_\pi$ a quantum Coxeter element.

Definition 8.26. We define the exponent map $\Phi : T_{k,0} \times Q_h \to \mathbb{Z}_{c_k}$ as

$$\Phi(e^H, \eta) = \frac{\langle \eta, H \rangle}{2\pi}.$$

Equivalently,

$$e^{\frac{2\pi i}{c_k} \Phi(e^H, \eta)} = e^{i\langle \eta, H \rangle}.$$

We call $\Phi(e^H, \eta)$ the quantum Coxeter exponent of the quantum Coxeter element η at spectrum $e^H \in T_{k,0}$. We call the map $\Phi(e^H, \cdot) : Q_h \to \mathbb{Z}_{c_k}$ the quantum Coxeter exponent at the spectrum e^H.

Since $Q_h \subset Q$, $\mathcal{S}_{B,0}$ is invariant under the action of quantum Coxeter elements $\{\vartheta_\eta\}_{\eta \in Q_h} = \{\vartheta_{r(\theta)}\}_{r \in W}$. For the $\mathfrak{sl}_2$ case, $\vartheta_{-\theta}$ is also a Coxeter element on the root system. This induces a $\mathbb{Z}_2$ symmetry on the eigenvalue of the Coxeter element. In general, the eigenvalue of the quantum Coxeter elements has a Weyl group W symmetry.

Definition 8.27. Suppose Π is a US-rep. For any $e^H \in T_{k,0}$, we define the multiplicity of the quantum Coxeter exponent $\Phi(e^H, \cdot)$ for the action of quantum Coxeter elements on $\mathscr{R}_\pi$ as

$$m_\Phi(e^H) := \dim \left\{ \tilde{f} \in \mathscr{R}_\pi : \tilde{\vartheta}_\eta \tilde{f} = e^{i\langle \eta, H \rangle} \tilde{f}, \ \forall \eta \in Q_h \right\}.$$

When Π is a GUS-rep, we define the multiplicity for the action on $\mathscr{R}_{\pi,0}$ as

$$m_{\Phi,0}(e^H) := \dim \left\{ \tilde{f} \in \mathscr{R}_{\pi,0} : \tilde{\vartheta}_\eta \tilde{f} = e^{i\langle \eta, H \rangle} \tilde{f}, \ \forall \eta \in Q_h \right\}.$$

Theorem 8.28. *Suppose* Π *is a US-rep over quantum* $\mathfrak{g}$ *at level* k. *For any* $e^H \in T_{k,0}$,

$$m_{\Phi}(e^H) = m_{Q^\vee}(e^H) = \sum_{e^{H'} \in T_{Q^\vee}} m_{\Pi}(e^{H+H'}).$$

When Π *is a GUS-rep,*

$$m_{\Phi,0}(e^H) = m_{Q^\vee,0}(e^H) = \sum_{e^{H'} \in T_{Q^\vee}/T_Q} m_{\Pi}(e^{H+H'}).$$

In particular, if $\mathfrak{g}$ *is an ADE Lie algebra, then* $T_Q = T_{Q^\vee}$. *We have that*

$$m_{\Phi}(e^H) = m_{\Pi}(e^H).$$

Proof. Since $Q^\vee$ is generated by Q_h, for any $e^H \in T_k, m_{\Phi}(e^H) = m_{Q^\vee,0}(e^H)$. By Theorem 8.24, $m_{Q^\vee,0}(e^H) = \dfrac{1}{n_z} m_{Q^\vee}(e^H)$. By Theorems 8.21, $m_{Q^\vee}(e^H) = \displaystyle\sum_{e^{H'} \in T_{Q^\vee}} m_{\Pi}(e^{H+H'})$.

By Theorem 7.10, $\dfrac{1}{n_z} \displaystyle\sum_{e^{H'} \in T_{Q^\vee}} m_{\Pi}(e^{H+H'}) = \displaystyle\sum_{e^{H'} \in T_{Q^\vee}/T_Q} m_{\Pi}(e^{H+H'}).$ $\square$

Case for $\mathfrak{g} = \mathfrak{sl}_2$: When $\mathfrak{g} = \mathfrak{sl}_2$ and G is an ADE Dynkin diagram, we have the following correspondence:

(1) c_k is the Coxeter number;
(2) Π is a GUS-rep;
(3) Π_ω is the adjacency matrix of G, where ω is the fundamental weight;
(4) $T_{k,0} = \left\{ e^H = \begin{bmatrix} e^{\frac{j\pi i}{2c_k}} & 0 \\ 0 & e^{\frac{-j\pi i}{2c_k}} \end{bmatrix} : j = 1, 2, \cdots, c_k - 1, c_k + 1, c_k + 2, \cdots, 2c_k - 1 \right\}$;
(5) $\mathbf{Spec}_k = \left\{ \left(\begin{bmatrix} e^{\frac{j\pi i}{2c_k}} & 0 \\ 0 & e^{\frac{-j\pi i}{2c_k}} \end{bmatrix}, \begin{bmatrix} e^{\frac{-j\pi i}{2c_k}} & 0 \\ 0 & e^{\frac{j\pi i}{2c_k}} \end{bmatrix} \right) : j = 1, 2, \cdots, c_k - 1 \right\}$;
(6) The eigenvalue of the adjacency matrix Π_ω at e^H is $\chi_\omega(e^H) = e^{\frac{j\pi i}{2c_k}} + e^{\frac{-j\pi i}{2c_k}}$ with multiplicity $m_{\Pi}(e^H)$;
(7) $\mathcal{S}_{B,0}$ is the root system;
(8) $\mathscr{R}_{\pi,0}$ is the root space;
(9) $\{\tilde{\vartheta}_\eta : \eta \in Q_h\} = \{\pm\theta\}$ is an opposite pair of Coxeter elements;
(10) The eigenvalue of the Coxeter element $\tilde{\vartheta}_{\pm\theta}$ at e^H is $\chi_\omega(e^H) = e^{\frac{\pm j\pi i}{c_k}}$ with multiplicity $m_{Q^\vee,0}(e^H)$;
(11) $\Phi(\pm\theta) = \pm j$, where j is the Coxeter exponent, with multiplicity $m_{\Phi,0}(e^H)$.

The classical correspondence of Coxeter exponents in the ADE Lie theory is given by

$$m_{\Pi}(e^H) = m_{Q^\vee,0}(e^H) = m_{\Phi,0}(e^H).$$

Theorem 8.28 is a generalization of this correspondence for any GUS-rep of the Verlinde algebra of any ADE Lie algebra $\mathfrak{g}$ at any level k. We answer the recent comment posed

by Gannon at Vanderbilt University in 2017 on the question posed by Kac at MIT in 1994 positively.

Note that the Dynkin diagram of type T, a chain with a self-loop at the end, induces a US rep of the Verlinde algebra of $\mathfrak{g} = \mathfrak{sl}_2$, but not a GUS rep. The identity

$$m_\Phi(e^H) = m_\Pi(e^H)$$

of the multiplicities of Coxeter exponents of ADE Dynkin diagrams cannot be generalized to the tadpole diagram directly. A modified correspondence for any US-rep or GUS-rep of the Verlinde algebra of any simple Lie algebra $\mathfrak{g}$ at any level k is proved in Theorems 8.21 and 8.24 respectively.

When $\mathfrak{g} = \mathfrak{sl}_n$, we have $Q = Q^\vee$, and

$$T_{k,0} = \left\{ diag(p_1, p_2, \ldots, p_n) : \prod_{j=1}^{n} p_j = 1, \; p_j \neq p_i, \; p_j^{c_k} = p_i^{c_k} \right\}.$$

Let $v_1, v_2, \cdots, v_n$ be the weights corresponding to the ONB of the standard representation of $\mathfrak{sl}_n$. Then $\sum_{j=1}^{n} v_j = 0$. For any $e^H \in T_k$, $e^{i\langle v_j, H \rangle} = p_j$. Moreover, for any $r \in W \cong S_n$, its action on $\{v_1, v_2, \cdots, v_n\}$ is a permutation: $r(v_i) = v_{r(i)}$. Note that $\theta = v_1 - v_n$ The quantum Coxeter exponent is given by $\Phi(e^H, r(\theta)) = \frac{\langle r(\theta), H \rangle}{2\pi}$, and

$$e^{\frac{2\pi i}{c_k} \Phi(e^H, r)} = e^{i\langle \eta, H \rangle} = \frac{p_{r(1)}}{p_{r(n)}}.$$

9. Summary

Notations and Results: Let $\mathfrak{g}$ be a simple complex Lie algebra such that it is the complexification of the Lie algebra $\mathfrak{k}$ of a simply-connected compact Lie group K. Let $\mathfrak{t}$ be a Cartan subalgebra and $T = \{e^H | H \in \mathfrak{t}\}$ be the maximal torus in K. Let $h^\vee$ be the dual Coxeter number, Ω be the fundamental weights, ρ be the Weyl vector, P be the weight lattice and Q be the root lattice, W be the Weyl group, and $Q^\vee$ be the coroot lattice, which is generated by $Q_h := \{r(\theta) | r \in W\}$, where θ is a highest positive root. Take $Z(\mathfrak{g}) = P/Q$ and $n_z = |Z(\mathfrak{g})|$. The exponential map $H \mapsto e^{2\pi H}$ is a group isomorphism $\mathfrak{t}/Q^\vee \cong T$. Therefore, the map $\hat{} : \lambda \mapsto \hat{\lambda}$ given by

$$\hat{\lambda}(e^H) = e^{2\pi i \langle \lambda, H \rangle}, \; \lambda \in P,$$

is well defined on $e^H \in T$. The Fourier transform $\hat{}$ is a group isomorphism from P to the dual of the abelian group T.

For any level $k \in \mathbb{N}$, take the quantum Coxeter number $c_k = h^\vee + k$ and $Q_k^\vee := c_k Q^\vee$. We call $P_k = P/Q_k^\vee$ the weight torus. We define its Fourier dual T_k as a finite subgroup of T, which admits a Weyl group W action. Let $P_{k,0}$ and $T_{k,0}$ be the corresponding subsets of elements off the Weyl mirrors. Take an alcove $\mathcal{C}_k$ containing Ω and **Spec**$_k$ to be the fundamental domains of P_k and $T_{k,0}$ subject to the Weyl group action respectively.

Let $L^2(T_{k,0})^W$ be the Hilbert space of anti-symmetric functions with respect to the Weyl group action and the measure is a Haar measure. We introduce a multiplication $\star$ and an involution $*$ on $L^2(T_{k,0})^W$ and show that $L^2(T_{k,0})^W$ becomes an abelian C^*

algebra. We construct an orthonormal basis (ONB) $\{|W|^{-\frac{1}{2}}\tilde{\chi}_\lambda, \lambda \in \mathcal{C}_k\}$ of $L^2(T_{k,0})^W$, and prove that $\{\tilde{\chi}_\lambda, \lambda \in \mathcal{C}_k\}$ form a $Z(\mathfrak{g})$-graded fusion ring R_k, and the corresponding C^*-algebra is isomorphic to the Verlinde algebra of quantum $\mathfrak{g}$ at level k. Therefore, we call the anti-symmetric function $\tilde{\chi}_\lambda$ an k-character, and $L^2(T_{k,0})^W$ the k-character Verlinde algebra. In particular, the fusion coefficients of representations can be computed from the inner product of k-characters:

$$\tilde{N}_{\lambda,\mu}^\nu = \frac{1}{|W|} \langle \tilde{\chi}_\lambda \star \tilde{\chi}_\mu, \tilde{\chi}_\nu \rangle.$$

Suppose $\mathcal{H}$ is a finite dimensional Hilbert space and $\Pi : R_k \to \hom(\mathcal{H})$ is a unital, *-representation. We prove that for any common eigenvector $v \in \mathcal{H}$ of the matrices $\{\Pi_\omega := \Pi(\tilde{\chi}_{\omega+\rho})\}_{\omega \in \Omega}$, there is a $e^H \in \mathbf{Spec}_k$, such that

$$\Pi_\omega v = \chi_\omega(e^H)v, \omega \in \Omega,$$

where χ_ω is the Weyl character. Therefore, we call $\mathbf{Spec}_k$ the spectrum of R_k and e^H the spectrum of v. We define the multiplicity m_Π of e^H to be the dimension of the corresponding eigenspace in $\mathcal{H}$:

$$m_\Pi(e^H) = \dim\{v \in \mathcal{H} \mid \Pi_\omega v = \chi_\omega(e^H)v, \ \forall \ \omega \in \Omega\}.$$

We lift the spectrum from $\mathbf{Spec}_k$ to $T_{k,0}$ and define the corresponding eigenspace

$$\mathcal{E}_\pi := \{\tilde{g} \in L^2(T_k) \otimes \mathcal{H} \mid (\chi_\omega \otimes I - I \otimes \Pi_\omega)\tilde{g} = 0, \ \forall \ \omega \in \Omega\}.$$

The Fourier dual of $\mathcal{E}_\pi$ is

$$\mathcal{R}_\pi := \{\tilde{f} \in L^2(P_k) \otimes \mathcal{H} \mid (\Gamma_\omega \otimes I - I \otimes \Pi_\omega)\tilde{f} = 0, \ \forall \ \omega \in \Omega\},$$

generalizing additive functions on the Auslander–Reiten quiver. We call $\mathcal{R}_\pi$ the full quantum root space. Take the orthogonal projection $\mathcal{P}_{\mathcal{R}_\pi} : L^2(P_k) \otimes \mathcal{H} \to \mathcal{R}_\pi$. We prove the following inner product formula in Theorem 8.9: For any $\lambda, \mu \in P_k$ and $v_1, v_2 \in \mathcal{H}$, we have

$$\langle \mathcal{P}_{\mathcal{R}_\pi}(\delta_\lambda \otimes v_1), \mathcal{P}_{\mathcal{R}_\pi}(\delta_\mu \otimes v_2) \rangle = \frac{1}{|T_k|} \sum_{r \in W} \det(r) \langle v_1, \Pi(\tilde{\chi}_{\mu-\lambda+r(\rho)})v_2 \rangle.$$

Based on the inner product formula, we define the quantum root sphere and quantum root system.

Furthermore, if the representation Π is $Z(\mathfrak{g})$-graded, then the full quantum root space $\mathcal{R}_\pi$ is also $Z(\mathfrak{g})$-graded. Moreover, $\mathcal{R}_\pi$ decomposes into n_z copies of $\mathcal{R}_{\pi,0}$, called the quantum root space, where $\mathcal{R}_{\pi,0}$ is the neutral subspace.

Let $\vartheta_\mu, \mu \in P_k$ be the translation by μ on P_k and the induced action on $\mathcal{R}_\pi$ is denoted by $\tilde{\vartheta}_\mu$. Both $\mathcal{R}_\pi$ and $\mathcal{R}_{\pi,0}$ are invariant under the action of $\tilde{\vartheta}_\mu$, for any $\mu \in Q$. We prove that if $\tilde{f}$ is a common eigenvector of the translations, then

$$\tilde{\vartheta}_\mu \tilde{f} = e^{i\langle -\mu, H \rangle} \tilde{f}, \ \forall \ \mu \in P_k$$

for some $e^H \in T_{k,0}$. We call e^H the spectrum of $\tilde{f}$.

For any $\eta \in Q_h$, the order of η in the group P_k is the quantum Coxeter number c_k. So the translation $\tilde{\vartheta}_\eta$ has periodicity c_k, and we call it a quantum Coxeter element. In particular, $\mathscr{R}_{\pi,0}$ is invariant under the action of quantum Coxeter elements $\{\vartheta_\eta\}_{\eta \in Q_h}$.

We define the exponent map $\Phi : T_{k,0} \times Q_h \to \mathbb{Z}_{c_k}$, such that

$$e^{\frac{2\pi i}{c_k} \Phi(e^H, \eta)} = e^{i\langle \eta, H \rangle}.$$

We define the multiplicity of the quantum Coxeter exponent $\Phi(e^H, \cdot)$ for the action of quantum Coxeter elements on $\mathscr{R}_\pi$, and on $\mathscr{R}_{\pi,0}$ for the graded case respectively, as

$$m_\Phi(e^H) = \dim\{f \in \mathscr{R}_\pi \mid \tilde{\vartheta}_\eta f = e^{i\langle \eta, H \rangle} f, \ \forall \eta \in Q_h\};$$
$$m_{\Phi,0}(e^H) = \dim\{f \in \mathscr{R}_{\pi,0} \mid \tilde{\vartheta}_\eta f = e^{i\langle \eta, H \rangle} f, \ \forall \eta \in Q_h\}.$$

Then

$$m_\Phi(e^H) = m_{Q^\vee}(e^H) = \sum_{e^{H'} \in T_{Q^\vee}} m_\Pi(e^{H+H'}).$$

For any intermediate group A of $Q_k^\vee \subset P$, we define the Fourier dual of P/A as

$$T_A = \left\{ e^{2\pi H} \in T \mid H \in \mathfrak{t}_A \right\} \subset T_k,$$

where $\mathfrak{t}_A = \{H \in \mathfrak{t} \mid \langle \alpha, H \rangle \in \mathbb{Z}, \forall \alpha \in A\}$. We define the multiplicity m_A of e^H to be the dimension of the common eigenspace in $\mathscr{R}_\pi$ of the translations $\{\tilde{\vartheta}_\mu\}_{\mu \in A}$ with spectrum e^H. We prove our main theorem that, for any $e^H \in T_{k,0}$,

$$m_A(e^H) = \sum_{e^{H'} \in T_A} m_\Pi(e^{H+H'}).$$

In particular, $m_P(e^H) = m_\Pi(e^H)$.

Furthermore, if Π is $Z(\mathfrak{g})$-graded and $A \subseteq Q$, we define the multiplicity $m_{A,0}$ of e^H to be the dimension of the corresponding eigenspace in $\mathscr{R}_{\pi,0}$. Then $m_A = n_z m_{A,0}$ and $m_A(e^H) = m_A(e^{H+H'})$ for any $e^{H'} \in T_{Q^\vee}$. In particular,

$$m_{Q,0}(e^H) = m_\Pi(e^H).$$

Finally, we prove in Theorem 8.28 that

$$m_{\Phi,0}(e^H) = m_{Q^\vee,0}(e^H) = \sum_{e^{H'} \in T_{Q^\vee}/T_Q} m_\Pi(e^{H+H'}).$$

In particular, if $\mathfrak{g}$ is an ADE Lie algebra, then $T_Q = T_{Q^\vee}$. We have that

$$m_{\Phi,0}(e^H) = m_\Pi(e^H).$$

This generalizes the correspondence of Coxeter exponents with multiplicities in the *ADE* Lie theory and answers the Gannon's comment on Kac's question.

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