Fusion bialgebras and Fourier analysis
Analytic obstructions for unitary categorification

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A B S T R A C T

We introduce fusion bialgebras and their duals and systematically study their Fourier analysis. As an application, we discover new efficient analytic obstructions on the unitary categorification of fusion rings. We prove the Hausdorff-Young inequality, uncertainty principles for fusion bialgebras and their duals. We show that the Schur product property, Young's inequality and the sum-set estimate hold for fusion bialgebras, but not always on their duals. If the fusion ring is the Grothendieck ring of a unitary fusion category, then these inequalities hold on the duals. Therefore, these inequalities are analytic obstructions of categorification. We classify simple integral fusion rings of Frobenius type up to rank 8 and of Frobenius-Perron dimension less than 4080. We find 34 ones, 4 of which are group-like and 28 of which can be eliminated by applying the Schur product property on the dual. In gen-

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eral, these inequalities are obstructions to subfactorize fusion bialgebras.

Contents

1. Introduction

Lusztig introduced fusion rings in [26]. Etingof, Nikshych and Ostrik studied fusion categories [9] as a categorification of fusion rings, see also [8,6,30,31]. A central question is whether a fusion ring can be unitarily categorified, namely it is the Grothendieck ring of a unitary fusion category.

Jones introduced subfactor planar algebras as an axiomatization of the standard invariant of a subfactor in [16]. Planar algebras and fusion categories have close connections. There are various ways to construct one from the other. For example, if $N \subset N \rtimes G$ is the group crossed product subfactor of a finite group G, then the 2-box space $\mathscr{P}_{2,+}$ of its planar algebra captures the unitary fusion category $Vec(G)$ and its Fourier dual $\mathscr{P}_{2,-}$ captures the unitary fusion category $\mathrm{Rep}(G)$. The Grothendieck ring of a unitary fusion category can be realized as the 2-box space of a subfactor planar algebra using the quantum double construction, such that the ring multiplication is implemented by the convolution of 2-boxes [28,22].

Recently, Jiang, the first author and the third author formalized and proved numbers of quantum inequalities for subfactor planar algebras [21,14,13,25] inspired by Fourier analysis. These inequalities automatically hold for the Grothendieck rings of unitary fusion categories $\mathscr{C}$ as explained in [22], through the well-known quantum double construction from unitary fusion categories to subfactors, see e.g. [28]. Moreover, the Fourier dual of a subfactor is still a subfactor. So these inequalities also hold on the Fourier dual of the Grothendieck ring, which can be regarded as representations of the Grothendieck ring.

This paper is inspired by three questions:

- Vaughan Jones [17]: What are the applications of these inequalities on subfactors to other areas?
- Zhenghan Wang [39]: Are these inequalities obstructions of categorification?
- Pavel Etingof [5]: Do the inequalities on Grothendieck rings hold on fusion rings?

In this paper, we prove that these quantum inequalities on subfactor planar algebras hold on fusion rings and partially, but not all, on the Fourier dual of fusion rings. Therefore, the inequalities that fail on the dual of the fusion rings are new analytic obstructions for unitary categorification of fusion rings. For examples, the quantum Schur product theorem [[21], Theorem 4.1] holds on the Fourier dual of Grothendieck rings, but not on the Fourier dual of fusion rings. It turns out to be a surprisingly efficient obstruction of unitary categorification of fusion rings. Moreover, it is easy to check the Schur product property on the dual of a commutative fusion ring in practice. In this way, we find many fusion rings which admit no unitary categorification, due to the Schur product property, and which cannot be ruled out by previous obstructions.

In §2, we introduce fusion bialgebras as a generalization of fusion rings and their duals over the field $\mathbb{C}$. The definition of fusion bialgebras is inspired by the 2-box spaces $\mathscr{P}_{2,\pm}$ of subfactor planar algebras. We show that if $\mathscr{P}_{2,+}$ is commutative, then it is a fusion bialgebra. If a fusion bialgebra arises in this way, then we say that it is subfactorizable. We classify fusion bialgebras up to dimension three. The classification of the two dimensional subfactorizable fusion bialgebras is equivalent to the remarkable classification of the Jones index of subfactors [15]. It remains challenging to classify three dimensional subfactorizable fusion bialgebras.

In §3-§6, we systematically study quantum Fourier analysis on fusion bialgebras. We show that the Hausdorff-Young inequalities, uncertainty principles hold for fusion bialgebras and their duals; Young's inequalities and the sum-set estimate hold for fusion bialgebras, but not necessarily on their duals. We characterize their extremizers in §6. In fact, for the dual of a fusion bialgebra, Young's inequality implies Schur product property, and Schur product property implies the sum-set estimate. Therefore, Young's inequality is also an obstruction to unitary categorify a fusion ring or to subfactorize a fusion bialgebra, and the sum-set estimate is a potential obstruction. It is worth mentioning that the Schur product property (or Young's inequality) holds on arbitrary n-box

space of the Temperley-Lieb-Jones planar algebra if and only if it is a subfactor planar algebras, namely the circle parameter is the square root of the Jones index [15].

In §8, we reformulate Schur product property (on the dual) in terms of irreducible representations of the fusion ring/algebra, especially in terms of the character table for the commutative case. In the family of fusion algebras of rank 3 with every object self-dual, we observe that about 30% of over 10000 samples do not have the Schur product property (on the dual). So they cannot be subfactorized. We consider families of rank 4 or 5 fusion rings, and we compare (visually) Schur product criterion and Ostrik's criterion [32, Theorem 2.21].

Next, we give a classification of simple integral fusion rings of Frobenius type with the following bounds of Frobenius-Perron dimensions (with FPdim $\neq p^a q^b, pqr$, by [10]).

rank	≤ 5	6	7	8	9	10	all
FPdim <	1000000	150000	15000	4080	504	240	132

First, given a Frobenius-Perron dimension, we classify all possible types (the list of dimensions of the "simple objects"). Secondly, we classify the fusion matrices for a given type. We derive several inequalities from Fourier analysis on fusion rings which bound the fusion coefficients using the dimensions. These inequalities are efficient in the second step of the classification. For some specific types, the use of these inequalities reduced drastically the computation time (from 50 hours to 5 seconds). We end up with 34 simple integral fusion rings in the classification (all commutative), 4 of which are group-like and 28 of which cannot be unitarily categorified by showing that the Schur product property (on the dual) does not hold. It remains 2 ones. None of these 28+2 ones can be ruled out by already known methods.

Question 1.1. Do the remaining two fusion rings admit a unitary categorification?

It has two motivations, first the categorification of a simple integral non group-like fusion ring would be non weakly-group-theoretical and so would provide a positive answer to Etingof-Nikshych-Ostrik [10, Question 2], next there is no known non group-like examples of irreducible finite index maximal depth 2 subfactor [33, Problem 4.12], but its fusion category would be unitary, simple, integral (and of Frobenius type, assuming Kaplansky's 6th conjecture [19]).

In summary, Fourier analysis on subfactors provides efficient analytic obstructions of unitary categorification or of subfactorization.

2. Fusion bialgebras

In this section, we introduce fusion bialgebras which capture fusion algebras of fusion rings over $\mathbb{C}$ and their duals, namely representations. The definition of fusion bialgebras is motivated by a connection between subfactor planar algebras and unitary fusion

categories based on the quantum double construction. Its algebraic aspects have been discussed in [22]. In this paper, we investigate its analytic aspects and study Fourier analysis on fusion bialgebras.

The fusion bialgebra has a second multiplication $\diamond$ and involution $\#$ on the fusion algebra. Several basic results on fusion rings, see for example [6], can be generalized to fusion bialgebras. Many examples of fusion bialgebras come from subfactor theory, and we say that they can be subfactorized. It is natural to ask whether a fusion bialgebra can be subfactorized. The question for the two dimensional case is equivalent to the classification of the Jones index. If a fusion ring has a unitary categorification, then the corresponding fusion bialgebra has a subfactorization. We introduce analytic obstructions of subfactorization from Fourier analysis on subfactors, so they are also obstructions of unitary categorification. We discuss their applications in §8.

2.1. Definitions

Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ be the set of all natural numbers. Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers.

Definition 2.1. Let $\mathcal{B}$ be a unital *-algebra over the complex field $\mathbb{C}$. We say $\mathcal{B}$ has a $\mathbb{R}_{\geq 0}$-basis $B = \{x_1 = 1_{\mathcal{B}}, x_2, \ldots, x_m\}, m \in \mathbb{Z}_{\geq 1}$, if

(1) $\{x_1, \ldots, x_m\}$ is a linear basis over $\mathbb{C}$;
(2) $x_j x_k = \sum\limits_{s=1}^{m} N_{j,k}^s x_s, \ N_{j,k}^s \in \mathbb{R}_{\geq 0}$;
(3) there exists an involution $*$ on $\{1, 2, \ldots, m\}$ such that $x_k^* := x_{k^*}$ and $N_{j,k}^1 = \delta_{j,k^*}$.

We write the identity $1_{\mathcal{B}}$ as 1 for short, if there is no confusion. When $N_{j,k}^s \in \mathbb{N}$, B gives a fusion ring, and $\mathcal{B}$ is called a fusion algebra. The *-algebra $\mathcal{B}$ with a $\mathbb{R}_{\geq 0}$-basis B can be considered as a fusion algebra over the field $\mathbb{C}$.

Definition 2.2. For a unital *-algebra $\mathcal{B}$ with a $\mathbb{R}_{\geq 0}$-basis B, we define a linear functional $\tau : \mathcal{B} \to \mathbb{C}$ by $\tau(x_j) = \delta_{j,1}$.

Then $\tau(x_j x_k) = N_{j,k}^1 = \delta_{j,k^*}$ and $\tau(xy) = \tau(yx)$ for any $x, y \in \mathcal{B}$. Moreover

$$N_{j,k}^{s^*} = \tau(x_j x_k x_s) = \tau(x_s x_j x_k) = N_{s,j}^{k^*} = \tau(x_k x_s x_j) = N_{k,s}^{j^*}. \tag{1}$$

Note that $x_{k^*} x_{j^*} = (x_j x_k)^*$. We obtain Frobenius reciprocity

$$N_{j,k}^s = N_{k^*,j^*}^{s^*} = N_{j^*,s}^k. \tag{2}$$

Therefore τ is a faithful tracial state on the *-algebra $\mathcal{B}$. Following the Gelfand-Naimark-Segal construction, we obtain a Hilbert space $\mathcal{H} = L^2(\mathcal{B}, \tau)$ with the inner product

$$\langle x, y \rangle = \tau(y^* x),$$

and a unital *-representation π of the $*$-algebra $\mathcal{B}$ on $\mathcal{H}$. Moreover B forms an orthonormal basis of $\mathcal{H}$. On this basis, we obtain a representation $\pi_B : \mathcal{B} \to M_m(\mathbb{C})$. In particular,

$$\pi_B(x_j)_{k,s} = N_{j,k}^s.$$

We denote the matrix $\pi_B(x_j)$ by L_j. Then

$$L_j L_k = \sum_{s=1}^{m} N_{j,k}^s L_s,$$

and

$$L_j^* = L_{j^*}.$$

Remark 2.3. Under the Gelfand-Naimark-Segal construction, the $*$-algebra $\mathcal{B}$ forms a C^*-algebra, which is also a von Neumann algebra. In this paper, we only consider the finite dimensional case, so we do not distinguish C^*-algebras and von Neumann algebras.

Definition 2.4. For a unital *-algebra $\mathcal{B}$ with a $\mathbb{R}_{\geq 0}$-basis B, we define a linear functional $d : \mathcal{B} \to \mathbb{C}$ by setting $d(x_j)$ to be the operator norm $\|L_j\|_{\infty}$ of L_j, below denoted by $\|x_j\|_{\infty, \mathcal{B}}$.

Recall the Perron-Frobenius theorem for matrices:

Theorem 2.5 *(Perron-Frobenius theorem, [9] Theorem 8.1). Let A be a square matrix with nonnegative entries.*

(1) *A has a nonnegative real eigenvalue. The largest nonnegative real eigenvalue $\lambda(A)$ of A dominates absolute values of all other eigenvalues of A.*
(2) *If A has strictly positive entries then $\lambda(A)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries.*
(3) *If A has an eigenvector f with strictly positive entries, then the corresponding eigenvalue is $\lambda(A)$.*

Proposition 2.6. *Let $\mathcal{B}$ be a unital *-algebra with a $\mathbb{R}_{\geq 0}$-basis B. Then*

$$d(x_j)d(x_k) = \sum_{s=1}^{m} N_{j,k}^s d(x_s), \quad d(x_j) = d(x_{j^*}) \geq 1.$$

Proof. The right multiplication of x_j on the orthonormal basis B defines a matrix R_j. Then $R = \sum_{j=1}^{m} R_j$ has strictly positive entries. Let $v = \sum_{j=1}^{m} \lambda_j x_j$ be the simple positive

eigenvector of the right action R. By Theorem 2.5, we can normalize v, such that $\lambda_1 = 1$ and $\lambda_j > 0$. As $L_j v$ is also a positive eigenvector, we have that $L_j v = \|L_j\|_\infty v = d(x_j) v$ by Theorem 2.5. Since $L_j L_k v = d(x_k) L_j v = d(x_k) d(x_j) v$, we obtain that

$$d(x_j) d(x_k) = \sum_{s=1}^m N_{j,k}^s d(x_s).$$

Note that $\sum_{j=1}^m d(x_j) x_j$ is an eigenvector for R by the equation above, we see that $\lambda_k = d(x_k)$ for any $1 \le k \le m$.

Note that $L_j^* = L_{j^*}$, we have $d(x_j) = d(x_{j^*})$, and $d(x_j)^2 = d(x_j) d(x_{j^*}) \ge 1$. Finally, we see that $d(x_j) \ge 1$. $\square$

Definition 2.7 *(An alternative C*-algebra $\mathcal{A}$).* We define an abelian C*-algebra $\mathcal{A}$ with the basis B, a multiplication $\diamond$ and an involution $\#$,

$$x_j \diamond x_k = \delta_{j,k} d(x_j)^{-1} x_j,$$
$$(x_j)^\# = x_j.$$

The C*-norm on $\mathcal{A}$ is given by $\|x\|_{\infty,\mathcal{A}} = \max_{1 \le j \le m} \dfrac{|d(x \diamond x_j)|}{d(x_j)}$ for any $x \in \mathcal{A}$.

Proposition 2.8. *The linear functional d is a faithful state on $\mathcal{A}$.*

Proof. Note that $\{d(x_j) x_j\}$ are orthogonal minimal projections of $\mathcal{A}$. By Proposition 2.6, $d(x_j) \ge 1$, so d is faithful. $\square$

Definition 2.9. For any $1 \le t \le \infty$, the t-norms on $\mathcal{A}$ and $\mathcal{B}$ are defined as follows:

$$\|x\|_{t,\mathcal{A}} = d(|x|^t)^{1/t},\ x \in \mathcal{A}, \quad \|x\|_{t,\mathcal{B}} = \tau(|x|^t)^{1/t},\ x \in \mathcal{B},\ |x| = (x^* x)^{1/2} \quad 1 \le t < \infty$$

and

$$\|x\|_{\infty,\mathcal{A}} = \max_{1 \le j \le m} \frac{|d(x \diamond x_j)|}{d(x_j)}, \quad \|x\|_{\infty,\mathcal{B}} = \sup_{\|y\|_{2,\mathcal{B}}=1} \|xy\|_{2,\mathcal{B}},$$

Remark 2.10. For any $x \in \mathcal{A}$, $|x| = (x^\# \diamond x)^{1/2}$.

Definition 2.11. [Fourier transform] Let $\mathcal{A}$, $\mathcal{B}$ be *-algebra with the same basis B, but different multiplications and involutions defined in Definition 2.1 and 2.7. The Fourier transform $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ is a linear map defined by

$$\mathfrak{F}(x_j) = x_j,\ \forall j.$$

Note that $\mathfrak{F}$ is a bijection.

Proposition 2.12 *(Plancherel's formula). The Fourier transform $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ is a unitary transformation:*

$$\|\mathfrak{F}(x)\|_{2,\mathcal{B}} = \|x\|_{2,\mathcal{A}},$$

i.e.

$$\tau(\mathfrak{F}(x)^*\mathfrak{F}(x)) = d(x^\# \diamond x).$$

Proof. We only have to check the equation for the basis B. For any $1 \le j, k \le m$, we have

$$d(x_j \diamond x_k) = \delta_{j,k} d(x_k)^{-1} d(x_k) = \delta_{j,k} = \tau(x_{j^*} x_k) = \tau(x_j^* x_k).$$

Then the proposition is true. $\quad\square$

Under the Fourier transform, the multiplication on $\mathcal{B}$ induces the convolution on $\mathcal{A}$. We denote the convolution of $x, y \in \mathcal{A}$ by

$$x * y := \mathfrak{F}^{-1}(\mathfrak{F}(x)\mathfrak{F}(y)).$$

The C*-algebras $\mathcal{A}$ and $\mathcal{B}$ share the same vector spaces, but have different multiplications, convolutions and traces. These traces are non-commutative analogues of measures.

We axiomatize the quintuple $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ as a *fusion bialgebra* in the following definition. To distinguish the multiplications and convolutions on $\mathcal{A}$ and $\mathcal{B}$, we keep the notations as above.

Definition 2.13 *(Fusion bialgebras).* Suppose $\mathcal{A}$ and $\mathcal{B}$ are two finite dimensional C*-algebras with faithful traces d and τ respectively, $\mathcal{A}$ is commutative, and $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ is a unitary transformation preserving 2-norms (i.e. $\tau(\mathfrak{F}(x)^*\mathfrak{F}(y)) = d(x^\# \diamond y)$ for any $x, y \in \mathcal{A}$). We call the quintuple $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ a fusion bialgebra, if the following conditions hold:

(1) Schur Product: For operators $x, y \ge 0$ in $\mathcal{A}$, $x * y := \mathfrak{F}^{-1}(\mathfrak{F}(x)\mathfrak{F}(y)) \ge 0$ in $\mathcal{A}$.
(2) Modular Conjugation: The map $J(x) := \mathfrak{F}^{-1}(\mathfrak{F}(x)^*)$ is an anti-linear, *-isomorphism on $\mathcal{A}$.
(3) Jones Projection: The operator $\mathfrak{F}^{-1}(1)$ is a positive multiple of a minimal projection in $\mathcal{A}$.

Furthermore, if $\mathfrak{F}^{-1}(1)$ is a minimal projection and $d(\mathfrak{F}^{-1}(1)) = 1$, then we call the fusion bialgebra canonical.

Remark 2.14. One can reformulate the definition of fusion bialgebras using the quintuple $(\mathcal{A}, *, J, d, \tau)$.

Remark 2.15. We show that subfactors provide fruitful fusion bialgebras in §2.2. One can compare the three conditions in Definition 2.13 with the corresponding concepts in subfactor theory.

Proposition 2.16 *(Gauge transformation). Given a fusion bialgebra* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, *then* $(\mathcal{A}, \mathcal{B}, \lambda_1^{\frac{1}{2}}\lambda_2^{-\frac{1}{2}}\mathfrak{F}, \lambda_1 d, \lambda_2 \tau)$ *is also a fusion bialgebra, with* $\lambda_1, \lambda_2 > 0$. *Therefore, any fusion bialgebra is equivalent to a canonical one up to a gauge transformation.*

Proof. It follows from the definition of the fusion bialgebra in Definition 2.13. $\quad\square$

Theorem 2.17. *If* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *is a fusion bialgebra, then* $\mathcal{B}$ *has a unique* $\mathbb{R}_{\geq 0}$-*basis* $B = \{x_1 = 1, x_2, \ldots, x_m\}$, *such that* $\mathfrak{F}^{-1}(x_j)$ *are multiples of minimal projections of* $\mathcal{A}$. *Moreover, B is invariant under the gauge transformation. Conversely, any* C^*-*algebra* $\mathcal{B}$ *with a* $\mathbb{R}_{\geq 0}$-*basis B can be extended to a canonical fusion bialgebra, such that* $\mathfrak{F}^{-1}(x_j)$ *are multiples of minimal projections of* $\mathcal{A}$.

Proof. By the above arguments, if a C*-algebra $\mathcal{B}$ has a $\mathbb{R}_{\geq 0}$-basis $B = \{x_1 = 1, x_2, \ldots, x_m\}$, then we obtain a canonical fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$.

On the other hand, suppose $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a fusion bialgebra. Let P_j, $1 \leq j \leq m$, be the minimal projections of $\mathcal{A}$, and $\mathfrak{F}^{-1}(1) = \delta_\mathcal{B} P_1$, for some $\delta_\mathcal{B} > 0$. The modular conjugation J is a *-isomorphism, so $J(P_j) = P_{j^*}$, for some $1 \leq j^* \leq m$. Then $\mathfrak{F}(P_j) = \mathfrak{F}(P_{j^*})^*$ and $J(P_1) = P_1$. Moreover,

$$d(P_j) = d(P_j^\# \diamond P_j) = \tau(\mathfrak{F}(P_j)^* \mathfrak{F}(P_j))$$
$$= \tau(\mathfrak{F}(P_{j^*})^* \mathfrak{F}(P_{j^*})) = d(P_{j^*}) \,.$$

By the Schur Product property,

$$P_j * P_k = \sum_{s=1}^m \tilde{N}_{j,k}^s P_s \,, \tag{3}$$

for some $\tilde{N}_{j,k}^s \in \mathbb{R}_{\geq 0}$. Since the functional d is faithful, $d(P_j) > 0$. Taking the inner product with P_1 on both sides of Equation (3), we have that

$$\tilde{N}_{j,k^*}^1 = \frac{d(P_1 \diamond (P_j * P_{k^*}))}{d(P_1)} = \frac{\tau(\mathfrak{F}(P_1)^* \mathfrak{F}(P_j * P_{k^*}))}{d(P_1)}$$
$$= \frac{1}{d(P_1)\delta_\mathcal{B}}\tau(\mathfrak{F}(P_j * P_{k^*})) = \frac{1}{d(P_1)\delta_\mathcal{B}}\tau(\mathfrak{F}(P_j)\mathfrak{F}(P_{k^*}))$$
$$= \frac{1}{d(P_1)\delta_\mathcal{B}}\tau(\mathfrak{F}(P_j)\mathfrak{F}(P_k)^*) = \frac{1}{d(P_1)\delta_\mathcal{B}}d(P_j \diamond P_k)$$
$$= \frac{d(P_j)\delta_{j,k}}{d(P_1)\delta_\mathcal{B}} \,.$$

In particular, $\tilde{N}^1_{1,1} = \delta_{\mathcal{B}}^{-1}$. Take

$$x_j = \delta_{\mathcal{B}}^{\frac{1}{2}} (\tilde{N}^1_{j,j^*})^{-\frac{1}{2}} \mathfrak{F}(P_j) \,,$$

$$N^s_{j,k} = \delta_{\mathcal{B}}^{\frac{1}{2}} (\tilde{N}^1_{j,j^*})^{-\frac{1}{2}} (\tilde{N}^1_{k,k^*})^{-\frac{1}{2}} (\tilde{N}^1_{s,s^*})^{\frac{1}{2}} \tilde{N}^s_{j,k} \,.$$

Then

$$x_j x_k = \sum_{s=1}^{m} N^s_{j,k} x_s, \quad x_j^* = x_{j^*} \,, \quad N^s_{j,k} \geq 0 \quad N^1_{j,k} = \delta_{j,k^*} \,.$$

Therefore, $\{x_j\}_{1 \leq j \leq m}$ forms a $\mathbb{R}_{\geq 0}$-basis of $\mathcal{B}$. Moreover, it is the unique $\mathbb{R}_{\geq 0}$-basis of $\mathcal{B}$ such that $\mathfrak{F}^{-1}(x_j)$ are positive multiples of minimal projections in $\mathcal{A}$.

Furthermore, applying the gauge transformation, we obtain a canonical fusion bialgebra

$$(\mathcal{A}, \mathcal{B}, \check{\mathfrak{F}}, \check{d}, \check{\tau}) = \left(\mathcal{A}, \mathcal{B}, \delta_{\mathcal{B}} \mathfrak{F}, \frac{d}{d(P_1)}, \frac{\tau}{d(P_1) \delta_{\mathcal{B}}^2} \right).$$

In this fusion bialgebra, the minimal projections in $\mathcal{A}$ are still P_j, $1 \leq j \leq m$. Their convolution becomes

$$P_j * P_k = \delta_{\mathcal{B}} \sum_{s=1}^{m} \tilde{N}^s_{j,k} P_s \,.$$

The corresponding x_j becomes

$$(\delta_{\mathcal{B}} \tilde{N}^1_{j,j^*})^{-\frac{1}{2}} \delta_{\mathcal{B}} \mathfrak{F}(P_j) = x_j \,.$$

Therefore, the $\mathbb{R}_{\geq 0}$-basis B is invariant under the gauge transformation. □

Definition 2.18 *(Frobenius-Perron dimension)*. For a fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, $\mathfrak{F}^{-1}(1)$ is a multiple of a minimal projection P_1 in $\mathcal{A}$. We define $\mu := \frac{d(1_{\mathcal{A}})}{d(P_1)}$ as the Frobenius-Perron dimension of the fusion bialgebra.

Remark 2.19. Note that the Frobenius-Perron dimension μ is invariant under the gauge transformations. When the fusion bialgebra is canonical, let $B = \{x_1 = 1, x_2, \ldots, x_m\}$ be the basis of $\mathcal{B}$ in Theorem 2.17. Denote $\mathfrak{F}^{-1}(x_j) \in \mathcal{A}$ by x_j (as in Definition 2.11). Then $\mu = \sum_{j=1}^{m} d(x_j)^2$. This coincides with the definition of the Frobenius-Perron dimension of a fusion ring.

Remark 2.20. For a canonical fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, one can consider τ as a Haar measure and $\mu d \circ \mathfrak{F}^{-1}$ as a Dirac measure on $\mathcal{B}$. On the dual side, one can consider $\mu^{-1} d$ as a Haar measure and $\tau \circ \mathfrak{F}$ as a Dirac measure on $\mathcal{A}$.

Proposition 2.21. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion ring. Then for any $x, y, z \in \mathcal{A}$, we have*

$$d((x * y) \diamond z) = \overline{d((J(z) * x^{\#}) \diamond J(y))}$$

Proof. We have

$$\begin{aligned}
d((x * y) \diamond z) &= \tau(\mathfrak{F}(x * y)\mathfrak{F}(z^{\#})^*) = \tau(\mathfrak{F}(x)\mathfrak{F}(y)\mathfrak{F}(z^{\#})^*) \\
&= \tau(\mathfrak{F}(z^{\#})^*\mathfrak{F}(x)\mathfrak{F}(y)) = \tau(\mathfrak{F}(J(z^{\#}))\mathfrak{F}(x)\mathfrak{F}(J(y))^*) \\
&= \tau(\mathfrak{F}(J(z^{\#}) * x)\mathfrak{F}(J(y))^*) = d((J(z^{\#}) * x) \diamond J(y)^{\#}) \\
&= \overline{d((J(z) * x^{\#}) \diamond J(y))}
\end{aligned}$$

This completes the proof of the proposition. $\quad\square$

2.2. Examples

Example 2.22. When the basis B forms a group under the multiplication of $\mathcal{B}$, the C*-algebra $\mathcal{B}$ is the group algebra, $\mathcal{H}$ is its left regular representation Hilbert space, and τ is the normalized trace. On the other side, the C*-algebra $\mathcal{A}$ is $L^\infty(B)$ and d is the unnormalized Haar measure.

Example 2.23. When the basis B forms a fusion ring, the C*-algebra $\mathcal{B}$ is the fusion algebra. The quintuple $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a canonical fusion bialgebra.

Theorem 2.24. *Suppose $\mathcal{N} \subset \mathcal{M}$ is a finite-index subfactor and $\mathscr{P}_\bullet$ is its planar algebra [16]. If $\mathscr{P}_{2,+}$ is abelian, then $(\mathscr{P}_{2,+}, \mathscr{P}_{2,-}, \mathfrak{F}_s, tr_{2,+}, tr_{2,-})$ is a fusion bialgebra, and μ is the Jones index. Moreover, we obtain a canonical one $(\mathscr{P}_{2,+}, \mathscr{P}_{2,-}, \mathfrak{F}, d, \tau)$, such that $d = \mu tr_{2,+}$ is the unnormalized trace of $\mathscr{P}_{2,+}$, $\tau = tr_{2,-}$ is the normalized trace of $\mathscr{P}_{2,-}$, and $\mathfrak{F} = \mu^{1/2}\mathfrak{F}_s = \delta\mathfrak{F}_s$, where $\mathfrak{F}_s : \mathscr{P}_{2,+} \to \mathscr{P}_{2,-}$ is the string Fourier transform.*

Proof. Let P_j, $j = 1, 2, \ldots, m$ be the minimal projections of $\mathscr{P}_{2,+}$ and P_1 be the Jones projection. Let Tr be the unnormalized trace of $\mathscr{P}_{2,+}$, namely $Tr(P_1) = 1$. Take $x_j = \frac{1}{\sqrt{Tr(P_i)}}\mathfrak{F}(P_j)$ and $x_{j^*} = \frac{1}{\sqrt{Tr(P_i)}}\mathfrak{F}(\overline{P_j})$, where $\overline{P_j}$ is the contragredient of P_j. Then

$$x_j x_k = N_{j,k}^s x_s,$$

x_1 is the identity, $x_k^* = x_{k^*}$, $N_{j,k}^s \geq 0$, and $N_{j,k}^1 = \delta_{j,k^*}$. $\quad\square$

Remark 2.25. On the 2-box space $\mathscr{P}_{2,\pm}$ of a subfactor planar algebra, the Fourier transform is a 90° rotation and the contragredient is a 180° rotation, see e.g. §2.1 in [21].

Definition 2.26 (*Subfactorization*). We call $(\mathscr{P}_{2,+}, \mathscr{P}_{2,-}, \mathfrak{F}_s, tr_{2,+}, tr_{2,-})$ the fusion bialgebra of the subfactor $\mathcal{N} \subset \mathcal{M}$. We say a fusion bialgebra $(\mathcal{A}, \mathcal{B})$ can be subfactorized,

if it comes from a subfactor $\mathcal{N} \subset \mathcal{M}$ in this way. We call $\mathcal{N} \subset \mathcal{M}$ a subfactorization of the fusion bialgebra.

2.3. Classifications

In this section, we classify fusion bialgebras up to dimension three. By the gauge transformation, it is enough to classify canonical fusion bialgebras, which reduces to classify the $\mathbb{R}_{\geq 0}$-basis of C*-algebra by Theorem 2.17. Recall in Theorem 2.24 that $(\mathscr{P}_{2,+}, \mathscr{P}_{2,-}, \mathfrak{F}_s, tr_{2,+}, tr_{2,-})$ of a subfactor planar algebra is a fusion bialgebra, if $\mathscr{P}_{2,+}$ is abelian. We refer the readers to [1–3,23,36,4] for known examples of three dimensional fusion bialgebras from 2-box spaces of subfactors planar algebras. In these examples, different subfactor planar algebras produce different fusion bialgebras.

In general, without assuming $\mathscr{P}_{2,+}$ to be abelian, $(\mathscr{P}_{2,+}, \mathscr{P}_{2,-}, \mathfrak{F}_s, tr_{2,+}, tr_{2,-})$ has been studied as *the structure of 2-boxes of a planar algebra*, see Definition 2.25 in [21]. One may ask when a subfactor planar algebra (generated by its 2-boxes) is determined by its structure of 2-boxes, equivalently by its fusion bialgebra when $\mathscr{P}_{2,+}$ is abelian. A positive answer is given in Theorem 2.26 in [21] for exchange relation planar algebras: exchange relation planar algebras are classified by its structure of 2-boxes. Classifying fusion bialgebras is a key step to classify exchange relation planar algebras. On the other hand, it would be interesting to find different subfactors planar algebras generated by 2-boxes with the same fusion bialgebras (or structures of 2-boxes).

Proposition 2.27 (*Rank-two classification*). *Two dimensional canonical fusion bialgebras are classified by the Frobenius-Perron dimension $\mu \geq 2$. Moreover, they can be subfactorized if and only if μ is a Jones index.*

Proof. If $\{x_1, x_2\}$ is a $\mathbb{R}_{\geq 0}$-basis, then $x_2^* = x_2$. By Proposition 2.6, $d_2 := d(x_2) \geq 1$, and

$$x_2^2 = x_1 + \frac{d_2^2 - 1}{d_2} x_2 \ .$$

So $\mu \geq 2$. Conversely, when $\mu \geq 2$, we obtain a $\mathbb{R}_{\geq 0}$-basis in this way.

Furthermore, when μ is a Jones index, the canonical fusion bialgebra can be subfactorized by the Temperley-Lieb-Jones subfactors with index μ. $\square$

Suppose $\{x_1 = 1, x_2, x_3\}$ is the $\mathbb{R}_{\geq 0}$-basis of a three-dimensional C*-algebra $\mathcal{B}$. Then $\mathcal{B}$ is commutative. Take $d_2 = d(x_2)$ and $d_3 = d(x_3)$. There are two different cases: $x_2^* = x_2$ or $x_2^* = x_3$.

Proposition 2.28 (*Rank-three classification, type I*). *In the case $x_2^* = x_2$, three dimensional canonical fusion bialgebras are classified by three parameters d_2, d_3, a, such that $d_2, d_3 \geq 1$, $0 \leq a \leq 1$, $b = 1 - a$, $d_2^2 - 1 - ad_3^2 \geq 0$, $d_3^2 - 1 - bd_2^2 \geq 0$. Moreover,*

$$x_2 x_2 = x_1 + \frac{d_2^2 - 1 - a d_3^2}{d_2} x_2 + a d_3 x_3 \ ,$$

$$x_2 x_3 = a d_3 x_2 + b d_2 x_3 \ ,$$

$$x_3 x_3 = x_1 + b d_2 x_2 + \frac{d_3^2 - 1 - b d_2^2}{d_3} x_3 \ .$$

Proof. Take parameters a, b, such that

$$x_2 x_3 = a d_3 x_2 + b d_2 x_3 \ .$$

Then $a, b \geq 0$. Computing d on both sides, we have that $a + b = 1$. So $a \leq 1$. By Equation (1), $N_{2,2}^3 = N_{2,3}^2$. Hence

$$x_2 x_2 = x_1 + \frac{d_2^2 - 1 - a d_3^2}{d_2} x_2 + a d_3 x_3 \ ,$$

by computing d on both sides. Similarly $N_{3,3}^2 = N_{2,3}^3$, and

$$x_3 x_3 = x_1 + b d_2 x_2 + \frac{d_3^2 - 1 - b d_2^2}{d_3} x_3 \ .$$

As the coefficients are non-negative, we have that $d_2^2 - 1 - a d_3^2 \geq 0$ and $d_3^2 - 1 - b d_2^2 \geq 0$.

Conversely, with the above parameters, the multiplication is associative and $(x_j x_k)^* = x_k^* x_j^*$ by a direct computation. Therefore, we obtain the classification. $\square$

Proposition 2.29 (Rank-three classification, type II). *In the case $x_2^* = x_3$, three dimensional canonical fusion bialgebras are classified by one parameter $\mu \geq 3$. Moreover, $d_2 = d_3 = \sqrt{\frac{\mu - 1}{2}}$,*

$$x_2 x_2 = \frac{d_2^2 - 1}{2 d_2} x_2 + \frac{d_2^2 + 1}{2 d_2} x_3 \ ,$$

$$x_2 x_3 = x_1 + \frac{d_2^2 - 1}{2 d_2} (x_2 + x_3) \ ,$$

$$x_3 x_3 = \frac{d_2^2 + 1}{2 d_2} x_2 + \frac{d_2^2 - 1}{2 d_2} x_3 \ .$$

Proof. As $x_2^* = x_3$, we have that $d_2 = d_3 = \sqrt{\frac{\mu - 1}{2}} \geq 1$ and $x_2 x_3$ is self-adjoint. So

$$x_2 x_3 = x_1 + \lambda (x_2 + x_3) \ ,$$

for some $\lambda \geq 0$. Computing d on both sides, we have that $\lambda = \frac{d_2^2 - 1}{2 d_2}$. By Equation (1), $N_{2,2}^2 = N_{2,3}^3$. So

$$x_2 x_2 = \frac{d_2^2 - 1}{2d_2} x_2 + \frac{d_2^2 + 1}{2d_2} x_3 \ ,$$

by computing d on both sides. Similarly $N_{3,3}^3 = N_{2,3}^2$, and

$$x_3 x_3 = \frac{d_2^2 + 1}{2d_2} x_2 + \frac{d_2^2 - 1}{2d_2} x_3 \ .$$

The coefficients are non-negative.

Conversely, with the above parameters, the multiplication is associative and $(x_j x_k)^* = x_k^* x_j^*$ by a direct computation. Therefore, we obtain the classification. $\quad\square$

The one-parameter family of three dimensional canonical fusion bialgebras in the above classification can be realized as the 2-box spaces of a one-parameter family of planar algebras constructed in [23]. For each $d_2 \geq 1$, there are a complex-conjugate pair of planar algebras to realize the fusion bialgebra as the 2-box spaces. So such a realization may not be unique. Moreover, these planar algebras arise from subfactors if and only if $\mu = \cot^2(\frac{\pi}{2N+2})$ for some $N \in \mathbb{Z}_+$. Inspired by this observation, we conjecture that:

Conjecture 2.30. *In the case II, the one-parameter family of three dimensional fusion bialgebras can be subfactorized if and only if $\mu = \cot^2(\frac{\pi}{2N+2})$.*

2.4. Duality

Definition 2.31. For a fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, we define the quintuple $(\mathcal{B}, \mathcal{A}, \widetilde{\mathfrak{F}}, \tau, d)$ as its Fourier dual, where $\widetilde{\mathfrak{F}} = \#\mathfrak{F}^{-1}*$.

Remark 2.32. To be compatible with the examples from subfactor theory, this is the natural Fourier dual, not $(\mathcal{B}, \mathcal{A}, \mathfrak{F}^{-1}, \tau, d)$.

Definition 2.33 *(Contragredient).* For any $x \in \mathcal{A}$, we define its contragredient as

$$\overline{x} := \widetilde{\mathfrak{F}} \mathfrak{F}(x) \ .$$

For any $y \in \mathcal{B}$, we define its contragredient as

$$\overline{y} := \mathfrak{F} \widetilde{\mathfrak{F}}(y) \ .$$

When $\mathcal{B}$ is commutative, it is natural to ask whether the dual $(\mathcal{B}, \mathcal{A}, \mathfrak{F}^{-1}, \tau, d)$ is also a fusion bialgebra. We need to check the three conditions in Definition 2.13. The conditions (2) and (3) always hold on the dual, but condition (1) may not hold.

Proposition 2.34 *(Dual modular conjugation).* *For a fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, the map $J_{\mathcal{B}}(x) := \widetilde{\mathfrak{F}}^{-1}(\widetilde{\mathfrak{F}}(x)^\#)$ is an anti-linear, *-isomorphism on $\mathcal{B}$.*

Proof. Note that the map $J_\mathcal{B}$ is anti-linear and

$$J_\mathcal{B}(x_j) = \widetilde{\mathfrak{F}}^{-1}(\widetilde{\mathfrak{F}}(x_j)^\#) = \mathfrak{F}(\mathfrak{F}^{-1}(x_j^*)^\#)^* = x_j,$$

so $J_\mathcal{B}(x_j x_k) = x_j x_k$, and $J_\mathcal{B}$ is an anti-linear *-isomorphism on $\mathcal{B}$. □

Proposition 2.35 *(Dual Jones projection). For a fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, $\mathfrak{F}(1_\mathcal{A})$ is a positive multiple of a central, minimal projection $e_\mathcal{B}$ in $\mathcal{B}$, where $1_\mathcal{A}$ is the identity of $\mathcal{A}$. Moreover, $\mu = \frac{\tau(1_\mathcal{B})}{\tau(e_\mathcal{B})}$.*

Proof. Since the gauge transformation only changes the global scaler, without loss of generality, we assume that $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a canonical fusion bialgebra. Then

$$\mathfrak{F}(1_\mathcal{A}) = \sum_{j=1}^m \mathfrak{F}(P_j) = \sum_{j=1}^m d(x_j) x_j \ .$$

Note that $d(x_j) = d(x_{j*}) \geq 0$ and $x_j^* = x_{j*}$, so $\mathfrak{F}(1_\mathcal{A}) = \mathfrak{F}(1_\mathcal{A})^*$. By Equation (1) and Proposition 2.6,

$$\mathfrak{F}(1_\mathcal{A}) x_k = \sum_{j=1}^m d(x_j) x_j x_k = \sum_{j,s=1}^m d(x_j) N_{j,k}^s x_s$$

$$= \sum_{j,s=1}^m d(x_{j*}) N_{k,s*}^{j^*} x_s = \sum_{s=1}^m d(x_k x_{s*}) x_s$$

$$= d(x_k) \sum_{s=1}^m d(x_s) x_s = d(x_k) \mathfrak{F}(1_\mathcal{A}) \ .$$

So $\mathfrak{F}(1_\mathcal{A}) * \mathfrak{F}(1_\mathcal{A}) = \mu \mathfrak{F}(1_\mathcal{A})$ and

$$e_\mathcal{B} = \mu^{-1} \widetilde{\mathfrak{F}}(1_\mathcal{A}) = \mu^{-1} \mathfrak{F}(1_\mathcal{A}) = \mu^{-1} \sum_{j=1}^m d(x_j) x_j$$

is a central, minimal projection. Moreover,

$$\tau(e_\mathcal{B}) = \mu^{-1} \sum_{j=1}^m d(x_j) \tau(x_j) = \mu^{-1}, \quad \tau(1_\mathcal{B}) = \tau(x_1) = 1 \ .$$

We have $\frac{\tau(1)}{\tau(e_\mathcal{B})} = \mu$. □

2.5. Self duality

In this subsection, we will give the definition of the self-dual fusion bialgebra and study the S-matrix associated to it.

Definition 2.36. Two fusion bialgebras $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ and $(\mathcal{A}', \mathcal{B}', \mathfrak{F}', d', \tau')$ are called isomorphic, if there are *-isomorphisms $\Phi_\mathcal{A} : \mathcal{A} \to \mathcal{A}'$ and $\Phi_\mathcal{B} : \mathcal{B} \to \mathcal{B}'$, such that $\Phi_\mathcal{B}\mathfrak{F} = \mathfrak{F}'\Phi_\mathcal{A}$, $d = d'\Phi_\mathcal{A}$ and $\tau = \tau'\Phi_\mathcal{B}$.

Definition 2.37. A fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is called self-dual, if its dual $(\mathcal{B}, \mathcal{A}, \widetilde{\mathfrak{F}}, \tau, d)$ is a fusion bialgebra and they are isomorphic. Furthermore, it is called symmetrically self-dual, if $\Phi_\mathcal{B}\Phi_\mathcal{A} = 1$ on $\mathcal{A}$.

The maps $\Phi_\mathcal{A}, \Phi_\mathcal{B}$ implementing the self-duality may not be unique, even for finite abelian groups.

Proposition 2.38. *Suppose $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a self-dual canonical fusion bialgebra with a $\mathbb{R}_{\geq 0}$-basis $B = \{x_1 = 1_\mathcal{B}, x_2, \ldots, x_m\}$ of $\mathcal{B}$, then*

$$\overline{\sum_j \lambda_j x_j} = \sum_j \lambda_j x_{j^*}.$$

Consequently, the contragredient maps on $\mathcal{A}$ and $\mathcal{B}$ are anti--isomorphisms.*

Proof. The statements follow from the fact that the contragredient map is linear and

$$\overline{x_j} = \mathfrak{F}\widetilde{\mathfrak{F}}(x_j) = \mathfrak{F}\mathfrak{F}^{-1}(x_j^*)^\# = x_{j^*} = x_j^*, \ \forall \ 1 \leq j \leq m. \quad \square$$

Definition 2.39. Suppose $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a self-dual canonical fusion bialgebra with a $\mathbb{R}_{\geq 0}$-basis $B = \{x_1 = 1_\mathcal{B}, x_2, \ldots, x_m\}$ of $\mathcal{B}$, we define the S-matrix S as an $m \times m$ matrix with entries S_j^k, such that

$$\mathfrak{F}\Phi_\mathcal{B}(x_j) = \sum_{k=1}^m S_j^k x_k.$$

Proposition 2.40. *For a self-dual canonical fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, $\mathfrak{F}\Phi_\mathcal{B}$ is a unitary transformation on $L^2(\mathcal{B}, \tau)$, and the S matrix is a unitary.*

Proof. Both $\mathfrak{F}$ and $\Phi_\mathcal{B}$ are unitary transformations, so the composition is a unitary on $L^2(\mathcal{B}, \tau)$. Recall that B is an orthonormal basis of $L^2(\mathcal{B}, \tau)$, we have S is a unitary matrix. $\square$

Proposition 2.41. *A self-dual canonical fusion bialgebra is symmetrically self-dual if and only if $S_j^k = S_k^j$. In this case,*

$$\mathfrak{F}\Phi_\mathcal{B} = \Phi_\mathcal{A}\widetilde{\mathfrak{F}}.$$

Proof. For a self-dual canonical fusion bialgebra, we have that

$$\Phi_{\mathcal{B}}\mathfrak{F}\Phi_{\mathcal{B}}\mathfrak{F} = \widetilde{\mathfrak{F}}\Phi_{\mathcal{A}}\Phi_{\mathcal{B}}\mathfrak{F} \ .$$

By Propositions 2.38 and 2.40, the fusion bialgebra is symmetrically self-dual if and only if $(\Phi_{\mathcal{B}}\mathfrak{F})^2 = \widetilde{\mathfrak{F}}\mathfrak{F}$ if and only if $(S^2)_j^k = \delta_{j,k^*}$ if and only if $S_j^k = S_k^j$. In this case,

$$\Phi_{\mathcal{B}}\mathfrak{F}\Phi_{\mathcal{B}}\mathfrak{F} = \Phi_{\mathcal{B}}\Phi_{\mathcal{A}}\widetilde{\mathfrak{F}}\mathfrak{F} \ .$$

So $\mathfrak{F}\Phi_{\mathcal{B}} = \Phi_{\mathcal{A}}\widetilde{\mathfrak{F}}$. $\quad\square$

Remark 2.42. For the group case, S is a bicharacter, see [24] for the discussion on self-duality and symmetrically self-duality.

Theorem 2.43 *(Verlinde formula). For a self-dual canonical fusion bialgebra* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$,

$$SL_j S^* = D_j, \quad 1 \leq j \leq m,$$

where $L_j x = x_j x$ *for* $x \in \mathcal{B}$ *and* $D_j x_k = \frac{S_j^k}{d(x_k)} x_k$.

Proof. Assume that $B = \{x_1 = 1, x_2, \ldots, x_m\}$ is a basis of $\mathcal{B}$. We have

$$\begin{aligned}
\mathfrak{F}\Phi_{\mathcal{B}} L_j (\mathfrak{F}\Phi_{\mathcal{B}})^{-1}(x_k)) &= \mathfrak{F}\Phi_{\mathcal{B}}(x_j (\mathfrak{F}\Phi_{\mathcal{B}})^{-1}(x_k)) \\
&= \mathfrak{F}(\Phi_{\mathcal{B}}(x_j) \diamond \Phi_{\mathcal{B}}((\mathfrak{F}\Phi_{\mathcal{B}})^{-1}(x_k))) \\
&= \mathfrak{F}(\mathfrak{F}^{-1}\mathfrak{F}\Phi_{\mathcal{B}}(x_j) \diamond \mathfrak{F}^{-1}(x_k)) \\
&= \sum_{s=1}^{m} S_j^s \mathfrak{F}(\mathfrak{F}^{-1}(x_s) \diamond \mathfrak{F}^{-1}(x_k)) \\
&= \frac{S_j^k}{d(x_k)} x_k
\end{aligned}$$

This completes the proof of the theorem. $\quad\square$

3. Schur product property

In this section, we will study Schur product property for the dual of a fusion bialgebra.

Definition 3.1. For a fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, the multiplication $\diamond$ on $\mathcal{A}$ induces a convolution $*_{\mathcal{B}}$ on $\mathcal{B}$: $\forall x, y \in \mathcal{B}$,

$$x *_{\mathcal{B}} y := \widetilde{\mathfrak{F}}^{-1}(\widetilde{\mathfrak{F}}(x) \diamond \widetilde{\mathfrak{F}}(y)) = (\mathfrak{F}(\mathfrak{F}^{-1}(y^*) \diamond \mathfrak{F}^{-1}(x^*)))^* \ . \tag{4}$$

We say $\mathcal{B}$ has the Schur product property, if $x *_{\mathcal{B}} y \geq 0$, for any $x, y \geq 0$ in $\mathcal{B}$.

Proposition 3.2 *(Frobenius reciprocity). Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Then for any $x, y, z \in \mathcal{B}$, we have*

$$\tau((x *_{\mathcal{B}} y)z) = \overline{\tau((J_{\mathcal{B}}(x) *_{\mathcal{B}} z^*)y^*)}$$

Proof. We have

$$
\begin{aligned}
\tau((x *_{\mathcal{B}} y)z) &= \tau((\mathfrak{F}(\mathfrak{F}^{-1}(y^*) \diamond \mathfrak{F}^{-1}(x^*)))^* z) \\
&= d(\mathfrak{F}^{-1}(y^*)^{\#} \diamond \mathfrak{F}^{-1}(x^*)^{\#} \diamond \mathfrak{F}^{-1}(z)) \\
&= \tau(y\mathfrak{F}(\mathfrak{F}^{-1}(x^*)^{\#} \diamond \mathfrak{F}^{-1}(z))) \\
&= \overline{\tau(\mathfrak{F}(\mathfrak{F}^{-1}(J_{\mathcal{B}}(x)^*) \diamond \mathfrak{F}^{-1}(z^{**}))^* y^*)} \\
&= \overline{\tau((J_{\mathcal{B}}(x) *_{\mathcal{B}} z^*)y^*)}
\end{aligned}
$$

This completes the proof of the proposition. □

Proposition 3.3. *Suppose the fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ can be subfactorized. Then the Schur product property holds on the dual.*

Proof. It follows from the quantum Schur product theorem on subfactors, Theorem 4.1 in [21]. □

Proposition 3.4. *Suppose the fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is self-dual. Then the Schur product property holds on the dual.*

Proof. It follows from the definition of self-dual fusion bialgebras. □

We define a linear map $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ such that

$$\Delta(x_j) = \frac{1}{d(x_j)} x_j \otimes x_j, \quad \Delta(x^*) = \Delta(x)^*, \quad x \in \mathcal{B}.$$

Then Δ is a $*$-preserving map. We say Δ is positive if $\Delta(x) \geq 0$ for any $x \geq 0$.

Proposition 3.5. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra and suppose Δ is positive. Then the Schur product property holds for $(\mathcal{B}, \mathcal{A}, \widetilde{\mathfrak{F}}, \tau, d)$*

Proof. We denote by ι the identity map. Note that for any $x = \sum_{j=1}^{m} \lambda_j x_j, y = \sum_{j=1}^{m} \lambda'_j x_j \in \mathcal{B}$, we have

$$(\iota \otimes \tau)(\Delta(x)(1 \otimes \overline{y})) = (\iota \otimes \tau) \left(\sum_{j=1}^{m} \frac{\lambda_j}{d(x_j)} x_j \otimes x_j \left(1 \otimes \sum_{j=1}^{m} \lambda'_j x_{j^*} \right) \right)$$

$$= (\iota \otimes \tau) \left(\sum_{j,k=1}^{m} \frac{\lambda_j \lambda'_k}{d(x_j)} x_j \otimes x_j x_{k^*} \right)$$

$$= \sum_{j,k=1}^{m} \frac{\lambda_j \lambda'_k}{d(x_j)} x_j \tau(x_j x_{k^*})$$

$$= \sum_{j=1}^{m} \frac{\lambda_j \lambda'_j}{d(x_j)} x_j = \overline{x *_{\mathcal{B}} y}.$$

Let $\overline{y} = y_1 y_1^*$ and $x \geq 0$. Then $(1 \otimes y_1^*) \Delta(x)(1 \otimes y_1) \geq 0$. Hence

$$(\iota \otimes \tau) \left((1 \otimes y_1^*) \Delta(x)(1 \otimes y_1) \right) \geq 0,$$

i.e. $x *_{\mathcal{B}} y \geq 0$. $\square$

Proposition 3.6. *For a fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$, the Schur product property holds on $\mathcal{B}$, if and only if*

$$d((J(x) * x) \diamond (J(y) * y) \diamond (J(z) * z)) \geq 0, \ \forall x, y, z \in \mathcal{A}.$$

Proof. By the Schur product property on $\mathcal{A}$,

$$(J(x) * x)^* = J(x)^{\#} * x^* = J(x^{\#}) * x^*, \ \forall x \in \mathcal{A}.$$

Note that $\mathfrak{F}(J(x) * x) = |\mathfrak{F}(x)|^2 \geq 0$, and any positive operator in $\mathcal{B}$ is of such form. Therefore, by Proposition 2.12 and Equation (4),

$$d((J(x) * x) \diamond (J(y) * y) \diamond (J(z) * z)) \geq 0, \ \forall x, y, z \in \mathcal{A},$$

$$\Longleftrightarrow \tau(\mathfrak{F}(J(x^{\#}) * (x^{\#})^* \mathfrak{F}((J(y) * y) \diamond (J(z) * z)) \geq 0, \ \forall x, y, z \in \mathcal{A},$$

$$\Longleftrightarrow \tau(|\mathfrak{F}(x^*)|^2 (|\mathfrak{F}(y)|^2 *_{\mathcal{B}} |\mathfrak{F}(z)|^2)^*) \geq 0 \ \forall x, y, z \in \mathcal{A},$$

$$\Longleftrightarrow |\mathfrak{F}(y)|^2 *_{\mathcal{B}} |\mathfrak{F}(z)|^2 \geq 0 \ \forall y, z \in \mathcal{A},$$

if and only if the Schur product property holds on $\mathcal{B}$. $\square$

The Schur product property may not hold on the dual, even for a 3-dimensional fusion bialgebra. We give a counterexample. For this reason, Young's inequality do not hold on the dual as well, see §5 for further discussions. As a preparation, we first construct the minimal projections in $\mathcal{B}$.

Proposition 3.7. *For the canonical fusion bialgebra $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ in Proposition 2.28, the minimal projections of $\mathcal{B}$ are given by*

$$Q_1 = \mu^{-1}(x_1 + d_2 x_2 + d_3 x_3) \,,$$

$$Q_2 = \nu_2^{-1}(x_1 - \frac{\lambda_2}{d_2} x_2 - \frac{1 - \lambda_2}{d_3} x_3) \,,$$

$$Q_3 = \nu_3^{-1}(x_1 - \frac{\lambda_3}{d_2} x_2 - \frac{1 - \lambda_3}{d_3} x_3) \,,$$

where λ_2, λ_3 are the solutions of

$$\lambda_2 + \lambda_3 = a d_3^2 - b d_2^2 + 1 \,,$$

$$\lambda_2 \lambda_3 = -b d_2^2 \,;$$

and $\nu_j = 1 + \frac{\lambda_j^2}{d_2^2} + \frac{(1 - \lambda_j)^2}{d_3^2}$.

Proof. Note that $\mu = 1 + d_2^2 + d_3^2$. By Proposition 2.35,

$$Q_1 = e_{\mathcal{B}} = \mu^{-1}(x_1 + d_2 x_2 + d_3 x_3) \,.$$

For $j = 2, 3$, $d(Q_1 Q_j) = 0$, so

$$Q_2 = \nu_2^{-1}(x_1 - \frac{\lambda_2}{d_2} x_2 - \frac{1 - \lambda_2}{d_3} x_3) \,,$$

$$Q_3 = \nu_3^{-1}(x_1 - \frac{\lambda_3}{d_2} x_2 - \frac{1 - \lambda_3}{d_3} x_3) \,,$$

for some $\nu_2, \nu_3 > 0$. As $Q_j^2 = Q_j$, we have that $\nu_j = 1 + \frac{\lambda_j^2}{d_2^2} + \frac{(1 - \lambda_j)^2}{d_3^2}$. Furthermore, $Q_2 Q_3 = 0$, so

$$x_1 - \frac{\lambda_2 + \lambda_3}{d_2} x_2 - \frac{2 - \lambda_2 - \lambda_3}{d_3} x_3 + \frac{\lambda_2 \lambda_3}{d_2^2} x_2^2 + \frac{\lambda_2 + \lambda_3 - 2\lambda_2 \lambda_3}{d_2 d_3} x_2 x_3$$

$$+ \frac{(1 - \lambda_2)(1 - \lambda_3)}{d_3^2} x_3^2 = 0.$$

The coefficient of x_j is 0, for $1 \le j \le 3$. So

$$1 + \frac{\lambda_2 \lambda_3}{d_2^2} + \frac{(1 - \lambda_2)(1 - \lambda_3)}{d_3^2} = 0 \,,$$

$$-\frac{\lambda_2 + \lambda_3}{d_2} + \frac{\lambda_2 \lambda_3}{d_2^2} \frac{d_2^2 - 1 - a d_3^2}{d_2} + \frac{\lambda_2 + \lambda_3 - 2\lambda_2 \lambda_3}{d_2 d_3} a d_3 + \frac{(1 - \lambda_2)(1 - \lambda_3)}{d_3^2} b d_2 = 0 \,,$$

$$-\frac{2 - \lambda_2 - \lambda_3}{d_3} + \frac{\lambda_2 \lambda_3}{d_2^2} a d_3 + \frac{\lambda_2 + \lambda_3 - 2\lambda_2 \lambda_3}{d_2 d_3} b d_2 + \frac{(1 - \lambda_2)(1 - \lambda_3)}{d_3^2} \frac{d_3^2 - 1 - b d_2^2}{d_3} = 0 \,.$$

Take

$$\omega_1 = \lambda_2 + \lambda_3 \ ,$$
$$\omega_2 = \lambda_2 \lambda_3 \ .$$

Then

$$1 + \frac{\omega_2}{d_2^2} + \frac{1 - \omega_1 + \omega_2}{d_3^2} = 0 \ ,$$

$$-\frac{\omega_1}{d_2} + \frac{\omega_2}{d_2^2}\frac{d_2^2 - 1 - ad_3^2}{d_2} + \frac{\omega_1 - 2\omega_2}{d_2 d_3} ad_3 + \frac{1 - \omega_1 + \omega_2}{d_3^2} bd_2 = 0 \ ,$$

$$-\frac{2 - \omega_1}{d_3} + \frac{\omega_2}{d_2^2} ad_3 + \frac{\omega_1 - 2\omega_2}{d_2 d_3} bd_2 + \frac{1 - \omega_1 + \omega_2}{d_3^2}\frac{d_3^2 - 1 - bd_2^2}{d_3} = 0 \ .$$

Solving the linear system, we have that

$$\omega_1 = ad_3^2 - bd_2^2 + 1 \ ;$$
$$\omega_2 = -bd_2^2 \ .$$

Therefore, λ_2, λ_3 are the solutions of

$$\lambda_2 + \lambda_3 = ad_3^2 - bd_2^2 + 1 \ ,$$
$$\lambda_2 \lambda_3 = -bd_2^2 \ . \quad \square$$

Theorem 3.8. *For the three dimensional canonical fusion bialgebras $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ parameterized by d_2, d_3, a in Proposition 2.28, the Schur product property does not hold on the dual in general, for example, $d_2 = 1000, d_3 = 500, a = 0.750001$.*

Proof. Fix $0 < a < 1$, and $b = 1 - a$. Take $d_2 \to \infty$ and $d_3^2 - 1 = bd_2^2$, then $\lambda_3 \to b^{-1}, \lambda_2 d_2^{-2} \to -b^2$. Moreover,

$$d_2^{-2} d\left(\left(\mathfrak{F}^{-1}(\nu_2 Q_2)\right)^3\right)$$

$$= d_2^{-2}\left(1 - \left(\frac{\lambda_2}{d_2}\right)^3 d_2^{-2} d_2 - \left(\frac{1 - \lambda_2}{d_3}\right)^3 d_3^{-2} d_3\right)$$

$$= d_2^{-2}\left(1 - \left(\lambda_2 d_2^{-2}\right)^3 d_2^2 - \frac{(1 - \lambda_2)^3}{d_3^4}\right)$$

$$= -\left(\lambda_2 d_2^{-2}\right)^3 + d_2^{-2}\left(1 - \frac{(1 - \lambda_2)^3}{(1 + bd_2^2)^2}\right)$$

$$\to b^6 - b^4 < 0$$

By Proposition 3.7, the Schur product property does not hold in general on the dual. Numerically, one can take $d_2 = 1000, d_3 = 500, a = 0.750001$, then $d((\mathfrak{F}^{-1}(\nu_2 Q_2))^3) < 0$. $\quad \square$

Remark 3.9. Subsection 8.3 provides a complementary approach for the study of this family of rank 3 fusion bialgebras, leading to visualize the areas of parameters where Schur product property (on the dual) does not hold, and to a character table whose matrix (function of the fusion coefficients) is equal to the inverse of the one underlying Theorem 3.7 (function of the Frobenius-Perron dimensions).

4. Hausdorff-Young inequality and uncertainty principles

In this section, we will recall some inequalities for general von Neumann algebras first and then we will prove the Hausdorff-Young inequalities and uncertainty principles for fusion bialgebras.

Proposition 4.1 *(Hölder's inequality, see for example Proposition 4.3, 4.5 in [14]). Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful tracial state τ and $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. Then for any $x \in L^p(\mathcal{M})$, $y \in L^q(\mathcal{M})$, we have*

$$\|xy\|_1 \leq \|x\|_p \|y\|_q$$

Moreover

(1) for $1 < p < \infty$, $\|xy\|_1 = \|x\|_p \|y\|_q$ if and only if $\frac{|x|^p}{\|x\|_p^p} = \frac{|y^*|^q}{\|y\|_q^q}$;
(2) for $p = \infty$, $\|xy\|_1 \leq \|x\|_\infty \|y\|_1$ if and only if the spectral projection of $|x|$ corresponding to $\|x\|_\infty$ contains the projection $\mathcal{R}(y)$ as subprojection, where $\mathcal{R}(y)$ is the range projection of y.

Corollary 4.2. *Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful tracial state τ and $x \in \mathcal{M}$. Then $\|x\|_2^2 = \|x\|_\infty \|x\|_1$ if and only if x is a multiple of a partial isometry.*

Proposition 4.3 *(Interpolation theorem, see for example Theorem 1.2 in [20]). Let $\mathcal{M}, \mathcal{N}$ be finite von Neumann algebras with normal faithful states τ_1, τ_2. Suppose $T : \mathcal{M} \to \mathcal{N}$ is a linear map. If*

$$\|Tx\|_{p_1, \tau_2} \leq K_1 \|x\|_{q_1, \tau_1}, \ and \ \|Tx\|_{q_1, \tau_2} \leq K_2 \|x\|_{q_2, \tau_1},$$

then

$$\|Tx\|_{p_\theta, \tau_2} \leq K_1^{1-\theta} K_2^\theta \|x\|_{q_\theta, \tau_1},$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, $0 \leq \theta \leq 1$.

4.1. Hausdorff-Young inequality

Proposition 4.4. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Then*

$$\|\mathfrak{F}(x)\|_{\infty, \mathcal{B}} \leq \|x\|_{1, \mathcal{A}}, \quad x \in \mathcal{A}$$

and

$$\|\widetilde{\mathfrak{F}}(x)\|_{\infty, \mathcal{A}} \leq \|x\|_{1, \mathcal{B}}, \quad x \in \mathcal{B}.$$

Proof. Let $x = \sum_{j=1}^{m} \lambda_j \mathfrak{F}^{-1}(x_j) \in \mathcal{A}$. Then

$$\|\mathfrak{F}(x)\|_{\infty, \mathcal{B}} = \left\| \sum_{j=1}^{m} \lambda_j x_j \right\|_{\infty, \mathcal{B}} \leq \sum_{j=1}^{m} |\lambda_j| \, \|x_j\|_{\infty, \mathcal{B}}$$

$$= \sum_{j=1}^{m} |\lambda_j| d(x_j) = \|x\|_{1, \mathcal{A}}.$$

This proves the first inequality.

For the second inequality, we let $x = \sum_{j=1}^{m} \lambda_j x_j$. Then $\|\widetilde{\mathfrak{F}}(x)\|_{\infty, \mathcal{A}} = \max\limits_{1 \leq j \leq m} \dfrac{|\lambda_j|}{d(x_j)}$. For any k such that $\lambda_k \neq 0$, we have

$$\tau \left(\left| \sum_{j=1}^{m} \lambda_j x_j \right| \right) \geq \frac{\tau \left(\overline{\lambda_k} x_{k^*} \sum_{j=1}^{m} \lambda_j x_j \right)}{|\lambda_k| d(x_{k^*})} = \frac{|\lambda_k|}{d(x_k)}.$$

Hence $\|\widetilde{\mathfrak{F}}(x)\|_{\infty, \mathcal{A}} \leq \|x\|_{1, \mathcal{B}}$. This completes the proof of the proposition. $\quad\square$

Theorem 4.5 *(Hausdorff-Young inequality). Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Then for any $1 \leq p \leq 2$, $1/p + 1/q = 1$, we have*

$$\|\mathfrak{F}(x)\|_{q, \mathcal{B}} \leq \|x\|_{p, \mathcal{A}}, \quad x \in \mathcal{A}$$

and

$$\|\widetilde{\mathfrak{F}}(x)\|_{q, \mathcal{A}} \leq \|x\|_{p, \mathcal{B}}, \quad x \in \mathcal{B}.$$

Proof. It follows from Proposition 2.12, 4.4 and Proposition 4.3. $\quad\square$

We divide the first quadrant into three regions R_T, R_F, R_{TF}. Recall that $\mu = \sum_{j=1}^{m} d(x_j)^2$ is the Frobenius-Perron dimension of $\mathcal{B}$. Let K be a function on $[0, 1]^2$ given by
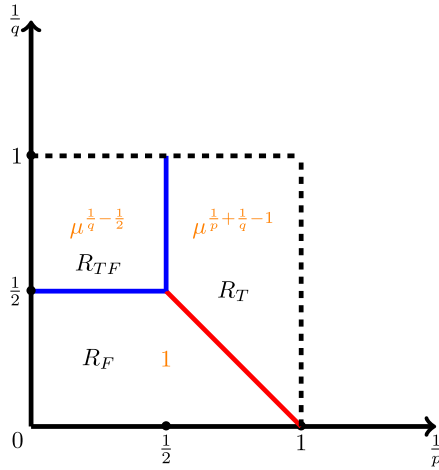
Fig. 1. The norms of the Fourier Transform.

$$K(1/p, 1/q) = \begin{cases} 1 & \text{for} \quad (1/p, 1/q) \in R_F, \\ \mu^{1/p+1/q-1} & \text{for} \quad (1/p, 1/q) \in R_T, \\ \mu^{1/q-1/2} & \text{for} \quad (1/p, 1/q) \in R_{TF}, \end{cases} \tag{5}$$

as illustrated in Fig. 1.

Theorem 4.6. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra and* $x \in \mathcal{B}$ *Then for any* $1 \le p, q \le \infty$, *we have*

$$K(1/p, 1/q)^{-1} \|x\|_{p,\mathcal{B}} \le \|\widetilde{\mathfrak{F}}(x)\|_{q,\mathcal{A}} \le K(1/p, 1/q) \|x\|_{p,\mathcal{B}}$$

Proof. It follows from the proof of Theorem 3.13 in [25]. We leave the details to the readers. □

4.2. Uncertainty principles

We will prove the Donoho-Stark uncertainty principle, Hirschman-Beckner uncertainty principle and Rényi entropic uncertainty principle for fusion bialgebras. For any $x \in \mathcal{A}$, we let $\mathcal{R}(x)$ be the range projection of x and $\mathcal{S}(x) = d(\mathcal{R}(x))$. For any $x \in \mathcal{B}$, $\mathcal{S}(x) = \tau(\mathcal{R}(x))$.

Lemma 4.7. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then we have*

$$\mathcal{S}(x) = \mathcal{S}(x^{\#}) = \mathcal{S}(J(x)), \quad x \in \mathcal{A}$$

and

$$\mathcal{S}(x) = \mathcal{S}(x^*) = \mathcal{S}(J_{\mathcal{B}}(x)), \quad x \in \mathcal{B}.$$

Theorem 4.8 *(Donoho-Stark uncertainty principle). Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $0 \neq x \in \mathcal{A}$, *we have*

$$\mathcal{S}(x)\mathcal{S}(\mathfrak{F}(x)) \geq 1;$$

for any $0 \neq x \in \mathcal{B}$, *we have*

$$\mathcal{S}(x)\mathcal{S}(\widetilde{\mathfrak{F}}(x)) \geq 1;$$

Proof. The second inequality is the reformulation of the first one. We only have to prove the first one. In fact,

$$
\begin{aligned}
\|\mathfrak{F}(x)\|_{\infty,\mathcal{B}} &\leq \|x\|_{1,\mathcal{A}} \leq \|\mathcal{R}(x)\|_{2,\mathcal{A}}\|x\|_{2,\mathcal{A}} \quad \text{Proposition 4.4, 4.1} \\
&= \mathcal{S}(x)^{1/2}\|\mathfrak{F}(x)\|_{2,\mathcal{B}} \\
&\leq \mathcal{S}(x)^{1/2}\|\mathcal{R}(\mathfrak{F}(x))\|_{2,\mathcal{B}}\|\mathfrak{F}(x)\|_{\infty,\mathcal{B}} \quad \text{Proposition 4.1} \\
&= \mathcal{S}(x)^{1/2}\mathcal{S}(\mathfrak{F}(x))^{1/2}\|\mathfrak{F}(x)\|_{\infty,\mathcal{B}},
\end{aligned}
$$

i.e. $\mathcal{S}(x)\mathcal{S}(\mathfrak{F}(x)) \geq 1$. This completes the proof of the theorem. $\square$

For any $x \in \mathcal{B}$, the von Neumann entropy $H(|x|^2)$ is defined by $H(|x|^2) = -\tau(x^*x \log x^*x)$ and for any $x \in \mathcal{A}$ the von Neumann entropy is defined by $H(|x|^2) = -d\left((x^{\#} \diamond x) \log(x^{\#} \diamond x)\right)$.

Theorem 4.9 *(Hirschman-Beckner uncertainty principle). Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x \in \mathcal{A}$, *we have*

$$H(|x|^2) + H(|\mathfrak{F}(x)|^2) \geq -4\|x\|_{2,\mathcal{A}}^2 \log \|x\|_{2,\mathcal{A}}.$$

Proof. We assume that $x \neq 0$. Let $f(p) = \log \|\mathfrak{F}(x)\|_{p,\mathcal{B}} - \log \|x\|_{q,\mathcal{A}}$, where $p \geq 2$ and $1/p + 1/q = 1$. By using the computations in the proof of Theorem 5.5 in [14], we have

$$\frac{d}{dp}\|\mathfrak{F}(x)\|_{p,\mathcal{B}}^p \bigg|_{p=2} = -\frac{1}{2}H(|\mathfrak{F}(x)|^2)$$

and

$$\frac{d}{dp} \log \|\mathfrak{F}(x)\|_{p,\mathcal{B}} \bigg|_{p=2} = -\frac{1}{4} \log \|\mathfrak{F}(x)\|_{2,\mathcal{B}}^2 - \frac{1}{4}\frac{H(|\mathfrak{F}(x)|^2)}{\|x\|_{2,\mathcal{A}}^2}.$$

We obtain that

$$f'(2) = -\frac{1}{2} \log \|x\|_{2,\mathcal{A}}^2 - \frac{1}{4\|x\|_{2,\mathcal{A}}^2}(H(|\mathfrak{F}(x)|^2) + H(|x|^2)).$$

By Proposition 2.12, we have that $f(2) = 0$. By Theorem 4.5, we have $f(p) \leq 0$ for $p \geq 2$. Hence $f'(2) \leq 0$ and

$$H(|\mathfrak{F}(x)|^2) + H(|x|^2) \geq -4\|x\|_{2,\mathcal{A}}^2 \log \|x\|_{2,\mathcal{A}}. \quad \square$$

Remark 4.10. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. The Hirschman-Beckner uncertainty principle is also true for $x \in \mathcal{B}$ with respect to the Fourier transform $\widetilde{\widetilde{\mathfrak{F}}}$.

We give a second proof of Theorem 4.8:

Proof. By using Theorem 4.9 and the inequality $\log \mathcal{S}(x) \geq H(|x|^2)$, for any $\|x\|_{2,\mathcal{A}} = 1$, we see that Theorem 4.8 is true. $\quad \square$

For any $x \in \mathcal{A}$ or $\mathcal{B}$ and $t \in (0,1) \cup (1, \infty)$, the Rényi entropy $H_t(x)$ is defined by

$$H_t(x) = \frac{t}{1-t}\|x\|_t.$$

Then $H_t(x)$ are decreasing function (see Lemma 4.3 in [25]) with respect to t for $\|x\|_{\infty,\mathcal{A}} \leq 1$ and $\|x\|_{\infty,\mathcal{B}} \leq 1$ respectively.

Theorem 4.11 (*Rényi entropic uncertainty principles*). *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra, $1 \leq t, s \leq \infty$. Then for any $x \in \mathcal{A}$ with $\|x\|_{2,\mathcal{A}} = 1$, we have*

$$(1/t - 1/2)H_{t/2}(|\mathfrak{F}(x)|^2) + (1/2 - 1/s)H_{s/2}(|x|^2) \geq -\log K(1/t, 1/s).$$

Proof. The proof is similar to the proof of Proposition 4.1 in [25], using Theorem 4.6. $\quad \square$

5. Young's inequality

In this section, we study Young's inequality for the dual of fusion bialgebra and the connections between Young's inequality and the Schur product property.

Proposition 5.1. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Then for any $x, y \in \mathcal{A}$, we have*

$$\|x * y\|_{\infty,\mathcal{A}} \leq \|x\|_{\infty,\mathcal{A}}\|y\|_{1,\mathcal{A}}.$$

Proof. For any $x = \sum_{j=1}^m \lambda_j \mathfrak{F}^{-1}(x_j)$ and $y = \sum_{j=1}^m \lambda_j' \mathfrak{F}^{-1}(x_j)$, we have

$$\|x * y\|_{\infty,\mathcal{A}} = \left\| \sum_{j,k=1}^m \lambda_j \lambda_k' \mathfrak{F}^{-1}(x_j x_k) \right\|_{\infty,\mathcal{A}} = \left\| \sum_{j,k,s=1}^m \lambda_j \lambda_k' N_{j,k}^s \mathfrak{F}^{-1}(x_s) \right\|_{\infty,\mathcal{A}}$$

$$= \max_{1 \leq s \leq m} \frac{\left| \sum_{j,k=1}^m \lambda_j \lambda_k' N_{j,k}^s \right|}{d(x_s)} \leq \max_{1 \leq s \leq m} \frac{\sum_{j,k=1}^m \left| \lambda_j \lambda_k' N_{j,k}^s \right|}{d(x_s)}$$

$$\leq \max_{1 \leq j \leq m} \frac{|\lambda_j|}{d(x_j)} \max_{1 \leq s \leq m} \frac{\sum_{j,k=1}^{m} \left| d(x_j) \lambda_k' N_{j,k}^s \right|}{d(x_s)}$$

$$= \|x\|_{\infty, \mathcal{A}} \sum_{k=1}^{m} |\lambda_k'| d(x_k) = \|x\|_{\infty, \mathcal{A}} \|y\|_{1, \mathcal{A}}.$$

This completes the proof of the proposition. $\square$

Remark 5.2. It would be natural to ask whether the following Young's inequality for the dual $(\mathcal{B}, \mathcal{A}, \widetilde{\mathfrak{F}}, \tau, d)$

$$\|x *_{\mathcal{B}} y\|_{\infty, \mathcal{B}} \leq \|x\|_{\infty, \mathcal{B}} \|y\|_{1, \mathcal{B}} \tag{6}$$

holds in general, but it does not, because we will see that it implies the Schur product property on the dual, which does not hold on many examples provided by Theorem 3.8, Subsections 8.3 and 8.4.

Proposition 5.3. *Suppose that $\mathfrak{F}^{-1}(x) > 0$, $x \in \mathcal{B}$, we actually have that Inequality (6) is true. Hence*

$$\|x *_{\mathcal{B}} y\|_{\infty, \mathcal{B}} \leq 4\|x\|_{\infty, \mathcal{B}} \|y\|_{1, \mathcal{B}}. \tag{7}$$

Proof. Let $x = \sum_{j=1}^{m} \lambda_j \mathfrak{F}(x_j)$ with $\lambda_j \geq 0$ and $y = \sum_{j=1}^{m} \lambda_j' \mathfrak{F}(x_j)$. Then $\|x\|_{\infty, \mathcal{B}} = \sum_{j=1}^{m} \lambda_j d(x_j)$ and

$$\|x *_{\mathcal{B}} y\|_{\infty, \mathcal{B}} = \left\| \sum_{j=1}^{m} \lambda_j \lambda_j' d(x_j)^{-1} x_j \right\|_{\infty, \mathcal{B}} \leq \sum_{j=1}^{m} |\lambda_j \lambda_j'|$$

$$= \sum_{j=1}^{m} \lambda_j d(x_j) \frac{|\lambda_j'|}{d(x_j)} \leq \max_{1 \leq j \leq m} \frac{|\lambda_j'|}{d(x_j)} \sum_{j=1}^{m} \lambda_j d(x_j)$$

$$= \|\mathfrak{F}^{-1}(y)\|_{\infty, \mathcal{A}} \|x\|_{\infty, \mathcal{B}} \leq \|y\|_{1, \mathcal{B}} \|x\|_{\infty, \mathcal{B}} \quad \text{Proposition 4.4}$$

Inequality (7) follows directly by the fact that any element is a linear combination of four positive elements. $\square$

Proposition 5.4. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra with $\mathcal{B}$ commutative. Then the Schur product property for the dual $(\mathcal{B}, \mathcal{A}, \widetilde{\mathfrak{F}}, \tau, d)$ implies inequality (6).*

Proof. By Definition 2.13, $(\mathcal{B}, \mathcal{A}, \widetilde{\mathfrak{F}}, \tau, d)$ is a fusion bialgebra. The proposition follows from Proposition 5.1. $\square$

Proposition 5.5. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. If $\|\Delta\| \leq 1$ then Inequality (6) holds.*

Proof. As in the proof of Proposition 3.5, for any $x, y \in \mathcal{B}$, we have that $\overline{x *_{\mathcal{B}} y} = (\iota \otimes \tau)(\Delta(x)(1 \otimes \overline{y}))$. Then

$$
\begin{aligned}
\|(\iota \otimes \tau)(\Delta(x)(1 \otimes y))\|_{\infty,\mathcal{B}} &= \sup_{\|z\|_{1,\mathcal{B}}=1} |(\tau \otimes \tau)(\Delta(x)(z \otimes \overline{y}))| \\
&\leq \sup_{\|z\|_{1,\mathcal{B}}=1} \|\Delta(x)\|_{\infty,\mathcal{B}} \|z\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}} \\
&\leq \|x\|_{\infty,\mathcal{B}} \|y\|_{1,\mathcal{B}}.
\end{aligned}
$$

This completes the proof. $\square$

Proposition 5.6. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Then for any $x, y \in \mathcal{A}$, we have*

$$
\|x * y\|_{1,\mathcal{A}} = \|x\|_{1,\mathcal{A}} \|y\|_{1,\mathcal{A}}.
$$

Proof. Suppose $x = \sum_{j=1}^{m} \lambda_j \mathfrak{F}^{-1}(x_j)$ and $y = \sum_{j=1}^{m} \lambda_j' \mathfrak{F}^{-1}(x_j)$. Then

$$
\begin{aligned}
\|x * y\|_{1,\mathcal{A}} &= \left\| \sum_{j=1,k=1,s=1}^{m} \lambda_j \lambda_k' N_{j,k}^s \mathfrak{F}^{-1}(x_s) \right\|_{1,\mathcal{A}} = \sum_{j=1,k=1,s=1}^{m} |\lambda_j \lambda_k'| N_{j,k}^s d(x_s) \\
&= \sum_{j=1,k=1}^{m} |\lambda_j \lambda_k'| d(x_j) d(x_k) = \sum_{j=1}^{m} |\lambda_j| d(x_j) \sum_{k=1}^{m} |\lambda_k'| d(x_k) \\
&= \|x\|_{1,\mathcal{A}} \|y\|_{1,\mathcal{A}},
\end{aligned}
$$

i.e. $\|x * y\|_{1,\mathcal{A}} = \|x\|_{1,\mathcal{A}} \|y\|_{1,\mathcal{A}}$. $\square$

Proposition 5.7. *The following two statements are equivalent for $C > 0$:*

(1) *For any $x, y \in \mathcal{B}$, $\|x *_{\mathcal{B}} y\|_{\infty,\mathcal{B}} \leq C \|x\|_{1,\mathcal{B}} \|y\|_{\infty,\mathcal{B}}$.*
(2) *For any $x, y \in \mathcal{B}$, $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} \leq C \|x\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}}$.*

Proof. $(1) \Rightarrow (2)$:

$$
\begin{aligned}
\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} &= \sup_{\|z\|_{\infty}=1} \tau((x *_{\mathcal{B}} y)z) \\
&= \sup_{\|\tilde{z}\|_{\infty}=1} \tau((J_{\mathcal{B}}(x) *_{\mathcal{B}} z^*)y^*) \quad \text{Proposition 3.2} \\
&\leq \sup_{\|\tilde{z}\|_{\infty}=1} \|J_{\mathcal{B}}(x) *_{\mathcal{B}} \tilde{z}^*\|_{\infty,\mathcal{B}} \|y\|_{1,\mathcal{B}} \\
&\leq C \|x\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}}.
\end{aligned}
$$

$(2) \Rightarrow (1)$:

$$\|x *_{\mathcal{B}} y\|_{\infty,\mathcal{B}} = \sup_{\|z\|_1=1} \tau((x *_{\mathcal{B}} y)z)$$

$$= \sup_{\|\tilde{z}\|_1=1} \tau((J_{\mathcal{B}}(x) *_{\mathcal{B}} z^*)y^*) \quad \text{Proposition 3.2}$$

$$\leq \sup_{\|\tilde{z}\|_1=1} \|J_{\mathcal{B}}(x) *_{\mathcal{B}} \tilde{z}^*\|_{1,\mathcal{B}} \|y\|_{\infty,\mathcal{B}}$$

$$\leq C\|x\|_{1,\mathcal{B}} \|y\|_{\infty,\mathcal{B}}. \quad \square$$

Proposition 5.8. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. The following statements:*

(1) the Schur product property holds on the dual.
(2) $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} = \|x\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}}$, *for any* $x \geq 0, y \geq 0$ *in* $\mathcal{B}$;
(3) $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} \leq \|x\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}}$, *for any* x, y *in* $\mathcal{B}$;
(4) $\|x *_{\mathcal{B}} y\|_{\infty,\mathcal{B}} \leq \|x\|_{1,\mathcal{B}} \|y\|_{\infty,\mathcal{B}}$, *for any* x, y *in* $\mathcal{B}$;

satisfy that $(4) \Leftrightarrow (3) \Rightarrow (1) \Leftrightarrow (2)$.

Proof. Suppose $x = \sum_{j=1}^m \lambda_j x_j \geq 0$ and $y = \sum_{j=1}^m \lambda'_j x_j \geq 0$ in $\mathcal{B}$. Then

$$\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} \geq |\tau(x *_{\mathcal{B}} y)| = \lambda_1 \lambda'_1 = \tau(x)\tau(y) = \|x\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}}.$$

$(1) \Rightarrow (2)$: By the Schur product property, we have $x *_{\mathcal{B}} y \geq 0$ and $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} = \tau(x *_{\mathcal{B}} y)$. This implies (2).

$(2) \Rightarrow (1)$: This implies that $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} = \tau(x *_{\mathcal{B}} y)$ for $x, y \geq 0$. However this implies that $x *_{\mathcal{B}} y \geq 0$, i.e. the Schur product property holds.

$(3) \Rightarrow (1)$: (3) implies that $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} = \tau(x *_{\mathcal{B}} y)$ for $x, y \geq 0$. Thus the Schur product property holds.

$(3) \Leftrightarrow (4)$: It follows from Proposition 5.7. $\quad \square$

Proposition 5.9. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x, y \in \mathcal{A}$, $1 \leq p \leq \infty$, *we have*

$$\|x * y\|_{p,\mathcal{A}} \leq \|x\|_{p,\mathcal{A}} \|y\|_{1,\mathcal{A}}.$$

Proof. It follows from Proposition 5.1, 5.6 and Proposition 4.3. $\quad \square$

Proposition 5.10. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x, y \in \mathcal{A}$, $1 \leq p \leq \infty$, $1/p + 1/q = 1$, *we have*

$$\|x * y\|_{\infty,\mathcal{A}} \leq \|x\|_{p,\mathcal{A}} \|y\|_{q,\mathcal{A}}.$$

Proof. We have

$$
\begin{aligned}
\|x * y\|_{\infty,\mathcal{A}} &= \sup_{\|z\|_{1,\mathcal{A}}=1} |d((x*y) \diamond z)| \\
&= \sup_{\|z\|_{1,\mathcal{A}}=1} \left| d((J(z)*x^{\#}) \diamond J(y)) \right| \quad \text{Proposition 2.21} \\
&\leq \sup_{\|z\|_{1,\mathcal{A}}=1} \|J(z)*x^{\#}\|_{p,\mathcal{A}} \|J(y)\|_{q,\mathcal{A}} \quad \text{Proposition 4.1} \\
&\leq \|x^{\#}\|_{p,\mathcal{A}} \|y\|_{q,\mathcal{A}} \quad \text{Proposition 5.9} \\
&= \|x\|_{p,\mathcal{A}} \|y\|_{q,\mathcal{A}}.
\end{aligned}
$$

This completes the proof of the proposition. □

Theorem 5.11 (*Young's inequality*). *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x, y \in \mathcal{A}$, $1 \leq p, q, r \leq \infty$, $1/p + 1/q = 1 + 1/r$, *we have*

$$
\|x * y\|_{r,\mathcal{A}} \leq \|x\|_{p,\mathcal{A}} \|y\|_{q,\mathcal{A}}.
$$

Proof. It follows from Propositions 5.9, 5.10 and Proposition 4.3. □

Proposition 5.12. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x, y \in \mathcal{B}$, $2 \leq r \leq \infty$, $1 \leq p, q \leq 2$, $1/p + 1/q = 1 + 1/r$, *we have*

$$
\|x *_{\mathcal{B}} y\|_{r,\mathcal{B}} \leq \|x\|_{p,\mathcal{B}} \|y\|_{q,\mathcal{B}}.
$$

Proof. For any $x, y \in \mathcal{B}$, $1/r + 1/r' = 1$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, we have

$$
\begin{aligned}
\|x *_{\mathcal{B}} y\|_{r,\mathcal{B}} &= \|\mathfrak{F}(\widetilde{\mathfrak{F}}(x) \diamond \widetilde{\mathfrak{F}}(y))\|_{r,\mathcal{B}} \\
&\leq \|\widetilde{\mathfrak{F}}(x) \diamond \widetilde{\mathfrak{F}}(y)\|_{r',\mathcal{A}} \quad \text{Proposition 4.5} \\
&= \|\widetilde{\mathfrak{F}}(x)\|_{p',\mathcal{A}} \|\widetilde{\mathfrak{F}}(y)\|_{q',\mathcal{A}} \quad \text{Proposition 4.1} \\
&\leq \|x\|_{p,\mathcal{B}} \|y\|_{q,\mathcal{B}}. \quad \text{Proposition 4.5}
\end{aligned}
$$

This completes the proof of the proposition. □

Proposition 5.13. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a self-dual fusion bialgebra. Then Young's inequality holds on the dual.*

Proof. It is directly from the definition. □

Proposition 5.14. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then the following are equivalent:*

(1) $\|x *_{\mathcal{B}} y\|_{r,\mathcal{B}} \leq \|x\|_{p,\mathcal{B}} \|y\|_{q,\mathcal{B}}$, $1 \leq p,q,r \leq \infty$, $1/p + 1/q = 1 + 1/r$ *for any* $x, y \in \mathcal{B}$;
(2) $\|x *_{\mathcal{B}} y\|_{1,\mathcal{B}} \leq \|x\|_{1,\mathcal{B}} \|y\|_{1,\mathcal{B}}$ *for any* $x, y \in \mathcal{B}$;
(3) $\|x *_{\mathcal{B}} y\|_{\infty,\mathcal{B}} \leq \|x\|_{\infty,\mathcal{B}} \|y\|_{1,\mathcal{B}}$ *for any* $x, y \in \mathcal{B}$.

We say the dual has Young's property if one of the above statements is true.

Proof. It follows the similar proof of Proposition 5.7 and Proposition 5.9 and 5.10. □

Remark 5.15. By Proposition 5.8, we have that for the dual, Young's property implies Schur product property.

Proposition 5.16. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x, y \in \mathcal{A}$*, we have*

$$\mathcal{R}(x * y) \leq \mathcal{R}(\mathcal{R}(x) * \mathcal{R}(y)).$$

In particular, $\mathcal{R}(x * y) = \mathcal{R}(\mathcal{R}(x) * \mathcal{R}(y))$ *if* $x \geq 0$*,* $y \geq 0$*.*

Proof. It follows from the Schur product property. □

Remark 5.17. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Suppose that the dual has Schur product property. Then $\mathcal{R}(x *_{\mathcal{B}} y) = \mathcal{R}(\mathcal{R}(x) *_{\mathcal{B}} \mathcal{R}(y))$ if $x > 0$, $y > 0$ in $\mathcal{B}$.

Proposition 5.18. *Young's property holds on a fusion bialgebra* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *which can be subfactorized.*

Proof. It follows from Theorem 4.13 in [14] for subfactors. □

Theorem 5.19 *(Sum set estimate). Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then for any* $x, y \in \mathcal{A}$*, we have*

$$\mathcal{S}(\mathcal{R}(x) * \mathcal{R}(y)) \geq \max\{\mathcal{S}(x), \mathcal{S}(y)\}.$$

Proof. We have that

$$\begin{aligned}
\mathcal{S}(x)\mathcal{S}(y) &= \|\mathcal{R}(x)\|_{1,\mathcal{A}} \|\mathcal{R}(y)\|_{1,\mathcal{A}} \\
&= \|\mathcal{R}(x) * \mathcal{R}(y)\|_{1,\mathcal{A}} \quad \text{Proposition 5.6} \\
&\leq \|\mathcal{R}(\mathcal{R}(x) * \mathcal{R}(y))\|_{2,\mathcal{A}} \|\mathcal{R}(x) * \mathcal{R}(y)\|_{2,\mathcal{A}} \quad \text{Proposition 4.1} \\
&\leq \mathcal{S}(\mathcal{R}(x) * \mathcal{R}(y))^{1/2} \|\mathcal{R}(x)\|_{1,\mathcal{A}} \|\mathcal{R}(y)\|_{2,\mathcal{A}} \quad \text{Proposition 5.9} \\
&= \mathcal{S}(\mathcal{R}(x) * \mathcal{R}(y))^{1/2} \mathcal{S}(x)\mathcal{S}(y)^{1/2}.
\end{aligned}$$

Hence $\mathcal{S}(\mathcal{R}(x) * \mathcal{R}(y)) \geq \mathcal{S}(y)$. □

Remark 5.20. We thank Pavel Etingof for noticing us another proof of Theorem 5.19 from an algebraic point of view [5].

Theorem 5.21 *(Sum set estimate). Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Suppose that the dual has the Schur product property. Then for any $x, y \in \mathcal{B}$, we have*

$$\mathcal{S}(\mathcal{R}(x) *_\mathcal{B} \mathcal{R}(y)) \geq \max\{\mathcal{S}(x), \mathcal{S}(y)\}.$$

Proof. By Proposition 5.12 and 5.8, the proof is similar to the one of Theorem 5.19. □

6. Fusion subalgebras and bishifts of biprojections

In this section, we define fusion subalgebras, biprojections and bishifts of biprojections for fusion bialgebras. We prove a correspondence between fusion subalgebras and biprojections. We prove partially that bishifts of biprojections are the extremizers of the inequalities proved in the previous sections.

Definition 6.1 *(Fusion subalgebra).* Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. A subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ is a fusion subalgebra if $(\mathcal{A}_0, \mathfrak{F}(\mathcal{A}_0), \mathfrak{F}, d, \tau)$ is a fusion bialgebra.

Definition 6.2 *(Biprojection).* Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. We say $x \in \mathfrak{A}$ is a biprojection if x is projection and $\mathfrak{F}(x)$ is a multiple of a projection in $\mathcal{B}$.

Proposition 6.3. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra and P a biprojection. Then there is a fusion subalgebra $\mathcal{A}_0$ such that the range of P is $\mathcal{A}_0$.*

Proof. We write $\mathfrak{F}(P) = \sum_{j=1}^m \lambda_j x_j$. By the fact that P is a projection and $\mathfrak{F}(P)$ is a multiple of a projection, we obtain that $\lambda_j = 0$ or $\lambda_j = d(x_j)$, and

$$\mathfrak{F}(P)^2 = \lambda \mathfrak{F}(P), \quad \mathfrak{F}(P)^* = \frac{\overline{\lambda}}{\lambda} \mathfrak{F}(P). \tag{8}$$

Solving the Equation (8), we obtain that

$$\lambda \overline{\lambda_{j^*}} = \overline{\lambda} \lambda_j \quad , \quad \lambda \lambda_s = \sum_{j,k=1}^m \lambda_j \lambda_k N_{j,k}^s. \tag{9}$$

Let

$$\mathcal{A}_0 = \mathrm{span}\{\mathfrak{F}^{-1}(x_j) : \lambda_j \neq 0\}$$

and

$$I_{\mathcal{A}_0} = \{1 \leq j \leq m : \lambda_j \neq 0\} \subset \{1, \ldots, m\}.$$

Then $\mathfrak{F}(P) = \sum_{j \in I_{\mathcal{A}_0}} d(x_j) x_j$. By Equations (9), we have that

$$\lambda d(x_s) = \sum_{j,k \in I_{\mathcal{A}_0}} d(x_j) d(x_k) N_{j,k}^s,$$

and

$$\lambda = \sum_{j,k \in I_{\mathcal{A}_0}} \lambda_j \lambda_k \delta_{j^*,k} = \sum_{j \in I_{\mathcal{A}_0}} |\lambda_j|^2 = \sum_{j \in I_{\mathcal{A}_0}} d(x_j)^2 > 0.$$

Let $\mu_{\mathcal{A}_0} = \sum_{j \in I_{\mathcal{A}_0}} d(x_j)^2$. We have

$$d(x_s) = \mu_{\mathcal{A}_0}^{-1} \sum_{j,k \in I_{\mathcal{A}_0}} d(x_j) d(x_k) N_{j,k}^s, \quad \forall s \in I_{\mathcal{A}_0}. \tag{10}$$

By Equation (9) and (10), we have that the involution $*$ is invariant on $I_{\mathcal{A}_0}$ and

$$\begin{aligned}
\mu_{\mathcal{A}_0} d(x_s) &= \sum_{j,k \in I_{\mathcal{A}_0}} d(x_j) d(x_k) N_{j,k}^s \\
&= \sum_{j \in I_{\mathcal{A}_0}} d(x_j) \sum_{k \in I_{\mathcal{A}_0}} d(x_{k^*}) N_{s^*,j}^{k^*} \\
&\leq \sum_{j \in I_{\mathcal{A}_0}} d(x_j) \sum_{k=1}^m d(x_{k^*}) N_{s^*,j}^{k^*} \\
&= \sum_{j \in I_{\mathcal{A}_0}} d(x_j) d(x_{s^*}) d(x_j) \\
&= \mu_{\mathcal{A}_0} d(x_s),
\end{aligned}$$

i.e. $N_{s,j}^k = 0$ for any $k \notin I_{\mathcal{A}_0}$. Therefore $x_j x_k = \sum_{s \in I_{\mathcal{A}_0}} N_{j,k}^s x_s$ for any $j,k \in I_{\mathcal{A}_0}$, i.e. $\mathfrak{F}(\mathcal{A}_0)$ is a $*$-algebra and $(\mathcal{A}_0, \mathfrak{F}(\mathcal{A}_0), \mathfrak{F}, d, \tau)$ is a fusion bialgebra. $\square$

Proposition 6.4. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra and $(\mathcal{A}_0, \mathfrak{F}(\mathcal{A}_0), \mathfrak{F}, d, \tau)$ is a fusion subalgebra. Then there is a biprojection P such that the range of P is $\mathcal{A}_0$.*

Proof. Let $\{y_1, \ldots, y_{m'}\}$ be a $\mathbb{R}_{\geq 0}$-basis of $\mathfrak{F}^{-1}(\mathcal{A}_0)$ such that $y_1 = 1$, $y_j \in \mathcal{B}$ and $y_j y_k = \sum_{s=1}^{m'} M_{j,k}^s y_s$, where $M_{j,k}^s \in \mathbb{N}$ and $M_{j,k}^1 = \delta_{y_j^*, y_k}$. Suppose that $y_j = \sum_{k=1}^m C_{j,k} x_k$, $C_{j,k} \in \mathbb{Z}$. Then

$$1 = M_{j^*,j}^1 = \tau(y_j^* y_j) = \sum_{k=1}^m C_{j,k}^2.$$

Hence $y_j = x_{m_j}$ for some $1 \leq m_j \leq m$ and $M_{j,k}^s = N_{m_j,m_k}^{m_s}$ for $1 \leq j,k,s \leq m'$. Let $P = \sum_{j=1}^{m'} d(y_j) \mathfrak{F}^{-1}(y_j)$. Then

$$\mathfrak{F}(P)^2 = \mu_{\mathcal{A}_0}\mathfrak{F}(P), \quad \mathfrak{F}(P)^* = \mathfrak{F}(P).$$

Then P is a biprojection. $\quad\square$

Theorem 6.5. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then there is a bijection between the set of fusion subalgebras and the set of biprojections.*

Proof. It follows from Proposition 6.3 and 6.4. $\quad\square$

Definition 6.6 *(Left and right shifts).* Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra and B a biprojection. A projection $P \in \mathcal{B}$ is a shift of $\mathcal{R}(\mathfrak{F}(B))$ if $\tau(P) = \tau(\mathcal{R}(\mathfrak{F}(B)))$ and $P *_{\mathcal{B}} \mathcal{R}(\mathfrak{F}(B)) = \tau(\mathcal{R}(\mathfrak{F}(B)))P$. A projection $P \in \mathcal{A}$ is a left shift of B if $d(B) = d(P)$ and $P * B = d(B)P$; A projection $P \in \mathcal{A}$ is a right shift of B if $d(B) = d(P)$ and $B * P = d(B)P$.

Lemma 6.7. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra and B a biprojection. Then*

(1) $\mathcal{R}(\mathfrak{F}(B))$ *is a shift of* $\mathcal{R}(\mathfrak{F}(B))$*, B is a left (right) shift of B;*
(2) $J_{\mathcal{B}}(P)$ *is a shift of* $\mathcal{R}(\mathfrak{F}(B))$ *when P is a shift of* $\mathcal{R}(\mathfrak{F}(B))$*;*
(3) $J(P)$ *is a left (right) shift of B when P is a right (left) shift of B;*
(4) $\mathcal{S}(P)\mathcal{S}(\mathfrak{F}(P)) = 1$ *when P is a left (right) shift of B or* $\mathcal{S}(\mathfrak{F}^{-1}(P))\mathcal{S}(P) = 1$ *when P is a shift of* $\mathcal{R}(\mathfrak{F}(B))$*;*
(5) $\mathcal{R}(\mathfrak{F}^{-1}(P)) = B$ *if P is a left (right) shift of* $\mathcal{R}(\mathfrak{F}(B))$ *and* $\mathcal{R}(\mathfrak{F}(P)) = \mathcal{R}(\mathfrak{F}(B))$ *if P is a shift of B.*

Proof. (1) By Proposition 6.3, we have $\mathcal{R}(\mathfrak{F}(B)) = d(B)^{-1}\mathfrak{F}(B)$,

$$B * B = d(B)B, \quad \mathcal{R}(\mathfrak{F}(B)) *_{\mathcal{B}} \mathcal{R}(\mathfrak{F}(B)) = d(B)^{-1}\mathcal{R}(\mathfrak{F}(B))$$

and

$$\mathcal{S}(B)\mathcal{S}(\mathfrak{F}(B)) = \tau(\mathcal{R}(\mathfrak{F}(B)))d(B) = \frac{1}{d(B)}d(B) = 1.$$

It indicates that $\mathcal{R}(\mathfrak{F}(B))$ is a shift of $\mathcal{R}(\mathfrak{F}(B))$ and B is a left (right) shift of B
    (2) and (3) can be followed by the property of J and $J_{\mathcal{B}}$.
    (4) Suppose P is a shift of $\mathcal{R}(\mathfrak{F}(B))$. Then $\mathcal{R}(\mathfrak{F}^{-1}(P)) \leq B$ and

$$1 \leq \mathcal{S}(P)\mathcal{S}(\mathfrak{F}^{-1}(P)) = \tau(P)d(\mathcal{R}(\mathfrak{F}^{-1}(P))) \leq \mathcal{S}(\mathfrak{F}(B))d(B) = 1,$$

i.e. $\mathcal{S}(P)\mathcal{S}(\mathfrak{F}^{-1}(P)) = 1$ and $\mathcal{R}(\mathfrak{F}^{-1}(P)) = B$.
    Suppose P is a right shift of B. Then $\mathcal{R}(\mathfrak{F}(P)) \leq \mathcal{R}(\mathfrak{F}(B))$ and

$$1 \leq \mathcal{S}(P)\mathcal{S}(\mathfrak{F}(P)) \leq \mathcal{S}(B)\mathcal{S}(\mathfrak{F}(B)) = 1,$$

i.e. $\mathcal{S}(P)\mathcal{S}(\mathfrak{F}(P)) = 1$. Hence $\mathcal{R}(\mathfrak{F}(P)) = \mathcal{R}(\mathfrak{F}(B))$.

Suppose P is a left shift of B. Then $J(P)$ is a right shift of B. By Lemma 4.7, we have $\mathcal{S}(P)\mathcal{S}(\mathfrak{F}(P)) = 1$. $\square$

Definition 6.8 *(Bishift of biprojection)*. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra and B a biprojection. A nonzero element x is a bishift of the biprojection B if there is $y \in \mathcal{A}$, a shift $\widetilde{B}_g$ of $\mathcal{R}(\mathfrak{F}(B))$ and a right shift B_h of B such that $x = \mathfrak{F}^{-1}(\widetilde{B}_g) * (y \diamond B_h)$.

Lemma 6.9. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra and* B *a biprojection. Suppose* $x = \mathfrak{F}^{-1}(\widetilde{B}_g) * (y \diamond B_h)$ *is a bishift of a biprojection* B. *Then* $\mathcal{R}(x) = \widetilde{B}_g$, $\mathcal{R}(\mathfrak{F}(x)) = B_h$ *and*

$$\mathcal{S}(x)\mathcal{S}(\mathfrak{F}(x)) = 1.$$

Proof. By Proposition 5.16, we have

$$\mathcal{R}(x) \leq \mathcal{R}(\mathcal{R}(\mathfrak{F}^{-1}(B_g)) * \mathcal{R}(y \diamond B_h)) \leq \mathcal{R}(B * B_h) = B_h$$

Then we obtain that

$$1 \leq \mathcal{S}(x)\mathcal{S}(\mathfrak{F}(x)) \leq \mathcal{S}(B_h)\mathcal{S}(\widetilde{B}_g) = \mathcal{S}(B)\mathcal{S}(\mathfrak{F}(B)) = 1.$$

Hence the inequalities above are equalities, i.e. $\mathcal{R}(x) = B_h$, $\mathcal{R}(\mathfrak{F}(x)) = \widetilde{B}_g$. $\square$

Definition 6.10. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. An element $x \in \mathcal{B}$ is said to be extremal if $\|\mathfrak{F}^{-1}(x)\|_{\infty,\mathcal{A}} = \|x\|_{1,\mathcal{B}}$. An element $x \in \mathcal{A}$ is said to be extremal if $\|\mathfrak{F}(x)\|_{\infty,\mathcal{B}} = \|x\|_{1,\mathcal{A}}$.

Definition 6.11. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. An element $x \in \mathcal{A}$ is a bi-partial isometry if x and $\mathfrak{F}(x)$ are multiples of partial isometries. An element $x \in \mathcal{A}$ is an extremal bi-partial isometry if x is a bi-partial isometry and x, $\mathfrak{F}(x)$ are extremal.

Theorem 6.12. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Then the following statements are equivalent:*

(1) $H(|x|^2) + H(|\mathfrak{F}(x)|^2) = -4\|x\|_{2,\mathcal{A}}^2 \log \|x\|_{2,\mathcal{A}}$;
(2) $\mathcal{S}(x)\mathcal{S}(\mathfrak{F}(x)) = 1$;
(3) x *is an extremal bi-partial isometry.*

Proof. The arguments are similar to the one of Theorem 6.4 in [14], since only the Hausdorff-Young inequality is involved. $\square$

Proposition 6.13. *Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra and* w *an extremal bi-partial isometry. Suppose that* w *is a projection. Then* $\tilde{w}$ *is a right shift of a biprojection.*

Proof. Let $w = \sum_{j \in J} d(x_j) \mathfrak{F}^{-1}(x_j)$. Then

$$\|w\|_{\infty,\mathcal{A}} = 1, \quad \|w\|_{2,\mathcal{A}} = \|\mathfrak{F}(w)\|_{2,\mathcal{B}} = \left(\sum_{j \in J} d(x_j)^2 \right)^{1/2}, \quad \|w\|_{1,\mathcal{A}} = \sum_{j \in J} d(x_j)^2. \quad (11)$$

By the assumption, we have

$$\|\mathfrak{F}(w)\|_{1,\mathcal{B}} = \|w\|_{\infty,\mathcal{A}} = 1, \quad \|\mathfrak{F}(w)\|_{\infty,\mathcal{B}} = \|w\|_{1,\mathcal{A}} = \sum_{j \in J} d(x_j)^2. \quad (12)$$

Let $P = \mathfrak{F}(w)\mathfrak{F}(w)^*$. Then P is a multiple of a projection in $\mathcal{B}$ and

$$\|P\|_{\infty,\mathcal{B}} = \left(\sum_{j \in J} d(x_j)^2 \right)^2, \quad \|P\|_{1,\mathcal{B}} = \|w\|_{2,\mathcal{A}}^2 = \sum_{j \in J} d(x_j)^2, \quad \|P\|_{2,\mathcal{B}} = \left(\sum_{j \in J} d(x_j)^2 \right)^{3/2} \quad (13)$$

We will show that $\mathfrak{F}^{-1}(P)$ is a multiple of partial isometry. By Corollary 4.2, we have to check $\|\mathfrak{F}^{-1}(P)\|_{2,\mathcal{A}}^2 = \|\mathfrak{F}^{-1}(P)\|_{\infty,\mathcal{A}} \|\mathfrak{F}^{-1}(P)\|_{1,\mathcal{A}}$. In fact,

$$\left(\sum_{j \in J} d(x_j)^2 \right)^3 = \|\mathfrak{F}^{-1}(P)\|_{2,\mathcal{A}}^2$$

$$\leq \|\mathfrak{F}^{-1}(P)\|_{\infty,\mathcal{A}} \|\mathfrak{F}^{-1}(P)\|_{1,\mathcal{A}} \quad \text{Equation (13) and Proposition 4.1}$$

$$\leq \|P\|_{1,\mathcal{B}} \|w * J(w)\|_{1,\mathcal{A}} \quad \text{Proposition 4.4}$$

$$= \left(\sum_{j \in J} d(x_j)^2 \right) \|w\|_{1,\mathcal{A}}^2 \quad \text{Proposition 5.6}$$

$$= \left(\sum_{j \in J} d(x_j)^2 \right)^3,$$

i.e. $\|\mathfrak{F}^{-1}(P)\|_{2,\mathcal{A}}^2 = \|\mathfrak{F}^{-1}(P)\|_{\infty,\mathcal{A}} \|\mathfrak{F}^{-1}(P)\|_{1,\mathcal{A}}$ and $\mathfrak{F}^{-1}(P)$ is a multiple of a partial isometry. By Schur product property, we have that $\mathfrak{F}^{-1}(P) > 0$ and $\mathfrak{F}^{-1}(P)$ is a multiple of a projection. Hence

$$\|\mathfrak{F}^{-1}(P)\|_{\infty,\mathcal{A}} = \sum_{j \in J} d(x_j)^2,$$

$$\mathcal{R}(\mathfrak{F}^{-1}(P)) = \left(\sum_{j \in J} d(x_j)^2 \right)^{-1} \mathfrak{F}^{-1}(P), \quad (14)$$

$$d(\mathcal{R}(\mathfrak{F}^{-1}(P))) = \sum_{j \in J} d(x_j)^2$$

and $\left(\sum_{j\in J} d(x_j)^2\right)^{-1} \mathfrak{F}^{-1}(P)$ is a biprojection

By Equation (14), we have

$$R(\mathfrak{F}^{-1}(P)) * w = \left(\sum_{j\in J} d(x_j)^2\right)^{-1} \mathfrak{F}^{-1}(P) * w = \left(\sum_{j\in J} d(x_j)^2\right)^{-1} \mathfrak{F}^{-1}(P\mathfrak{F}(w))$$

$$= \|w\|_{\infty,\mathcal{B}}^2 \left(\sum_{j\in J} d(x_j)^2\right)^{-1} w = \left(\sum_{j\in J} d(x_j)^2\right) w$$

$$= d(R(\mathfrak{F}^{-1}(P)))w = \|w\|_{1,\mathcal{A}} w.$$

Hence w is a right shift of $R(\mathfrak{F}^{-1}(P))$. $\quad\square$

Corollary 6.14. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Then a left shift of a biprojection is a right shift of a biprojection.*

Proof. It follows from Lemma 6.7 and Proposition 6.13. $\quad\square$

Question 6.15. Are the minimizers of the Donoho-Stark uncertainty principle bishifts of biprojections?

Theorem 6.16. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. Suppose that the dual has Young's property. Then the minimizers of the Donoho-Stark uncertainty principle are bishifts of biprojections.*

Proof. The proof is similar to the proof of Proposition 6.13. We leave the details to the reader. $\quad\square$

Theorem 6.17 *(Exact inverse sum set theorem). Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra and P, Q projections in $\mathcal{A}$. Then the following are equivalent:*

(1) $\mathcal{S}(P * Q) = \mathcal{S}(P)$;
(2) $\frac{1}{\mathcal{S}(Q)} P * Q$ *is a projection;*
(3) *there is a biprojection B such that $Q \leq B_h$ and $P = R(x * B)$ for some $x \neq 0$ in $\mathcal{A}$, where B_h is a right shift of B.*

Proof. (1) $\Rightarrow$ (2) By Theorem 5.19, we have that $\mathcal{S}(P * Q) \geq \mathcal{S}(P)$. By the assumption and the proof of Theorem 5.19, we have

$$\|P * Q\|_{1,\mathcal{A}} = \|R(P * Q)\|_{2,\mathcal{A}} \|P * Q\|_{2,\mathcal{A}}, \quad \|P * Q\|_{2,\mathcal{A}} = \|P\|_{2,\mathcal{A}} \|Q\|_{1,\mathcal{A}}. \quad (15)$$

By Proposition 4.1, we have that

$$P * Q = \lambda \mathcal{R}(P * Q)$$

and

$$\lambda \mathcal{S}(P * Q)^{1/2} = \|P * Q\|_{2,\mathcal{A}} = \mathcal{S}(P)^{1/2} \mathcal{S}(Q).$$

Therefore $\lambda = \mathcal{S}(Q)$ and $\frac{1}{\mathcal{S}(Q)} P * Q$ is a projection.

(2) $\Rightarrow$ (1) By the assumption and Proposition 5.6, we have

$$\mathcal{S}(P * Q) = \mathcal{S}\left(\frac{1}{\mathcal{S}(Q)} P * Q\right) = \frac{1}{\mathcal{S}(Q)} d(P * Q) = \frac{\mathcal{S}(P)\mathcal{S}(Q)}{\mathcal{S}(Q)} = \mathcal{S}(P).$$

(2) $\Rightarrow$ (3) Let $P_1 = \mathcal{R}(P * Q)$. Then by Proposition 2.21, we have

$$d((P_1 * J(Q)) \diamond P) = d((J(P) * P_1) \diamond Q) = d((J(Q) * J(P)) \diamond J(P_1))$$
$$= d((P * Q) \diamond P_1) = d(P * Q)$$
$$= d(P)d(Q) = d(P_1 * J(Q)).$$

Hence $\mathcal{R}(P_1 * J(Q)) \le P$. But by Theorem 5.19, we have

$$d(\mathcal{R}(P_1 * J(Q))) = \mathcal{S}(P_1 * J(Q)) \ge \mathcal{S}(P_1) = \mathcal{S}(P).$$

Then

$$\frac{1}{\mathcal{S}(J(Q))} P_1 * J(Q) = \mathcal{R}(P_1 * J(Q)) = P.$$

Expanding the expression, we have

$$\frac{1}{\mathcal{S}(Q)^2} P * Q * J(Q) = P.$$

Note that $\|\mathfrak{F}(Q)\|_{\infty,\mathcal{B}} \le \|Q\|_{1,\mathcal{A}} = \mathcal{S}(Q)$. Let

$$\mathfrak{F}(B) = \lim_{n \to \infty} \frac{1}{\mathcal{S}(Q)^{2n}} (\mathfrak{F}(Q)\mathfrak{F}(Q)^*)^n. \tag{16}$$

Then $\mathfrak{F}(B)$ is the spectral projection of $\mathcal{S}(Q)^{-2}\mathfrak{F}(Q)\mathfrak{F}(Q)^*$ corresponding to $\frac{\|\mathfrak{F}(Q)\|_{\infty,\mathcal{B}}^2}{\mathcal{S}(Q)^2}$. Moreover $B > 0$, $P * B = P$, $P = \mathcal{R}(P * B)$, $B \ne 0$ and

$$\|\mathfrak{F}(B)\|_{\infty,\mathcal{B}} = 1 = \|B\|_{1,\mathcal{A}}, \quad \|B\|_{\infty,\mathcal{A}} \le \|\mathfrak{F}(B)\|_{1,\mathcal{B}} = \|\mathfrak{F}(B)\|_{2,\mathcal{B}}^2 = \|B\|_{2,\mathcal{A}}^2 \tag{17}$$

Hence

$$\|B\|_{\infty,\mathcal{A}} \|B\|_{1,\mathcal{A}} \le \|B\|_{2,\mathcal{A}}^2 \le \|B\|_{\infty,\mathcal{A}} \|B\|_{1,\mathcal{A}}.$$

By Corollary 4.2, we have that B is a multiple of a partial isometry and then B is a multiple of a projection. Therefore $\|B\|_{2,\mathcal{A}}^{-2}B$ is a biprojection.

Let $Q_1 = \mathcal{R}(B * Q)$. Then

$$
\begin{aligned}
\mathcal{R}(Q_1 * J(Q_1)) &= \mathcal{R}(\mathcal{R}(B * Q)J(\mathcal{R}(B * Q))) \\
&= \mathcal{R}(\mathcal{R}(B * Q)\mathcal{R}(J(Q) * B)) \\
&= \mathcal{R}(B * Q * J(Q) * B) \quad \text{Proposition 5.16} \\
&= \mathcal{R}(B).
\end{aligned}
$$

Hence $\mathcal{S}(Q_1 * J(Q_1)) = \mathcal{S}(B)$. On the other hand, by Theorem 5.19, we have

$$
\mathcal{S}(Q_1 * J(Q_1)) \geq \mathcal{S}(Q_1) = \mathcal{S}(B * Q) \geq \mathcal{S}(B).
$$

Now we obtain that $\mathcal{S}(B * Q) = \mathcal{S}(B)$. By "(1) $\Rightarrow$ (2)", we have $Q_1 = \frac{1}{\mathcal{S}(Q)}\mathcal{R}(B) * Q$. Note that

$$
d(Q_1) = \mathcal{S}(Q_1) = \mathcal{S}(B) = d(\mathcal{R}(B))
$$

and

$$
\mathcal{R}(B) * Q_1 = \|B\|_{2,\mathcal{A}}^{-2}B * Q_1 = d(\mathcal{R}(B))Q_1.
$$

We see Q_1 is a right shift of $\mathcal{R}(B)$.

Now we have to check $Q \leq Q_1$. Note that $\mathfrak{F}^{-1}(1_\mathcal{B}) \leq \|B\|_{\infty,\diamond}B$. Then by Proposition 5.16, we have $Q \leq Q_1$.

(3) $\Rightarrow$ (1) By the assumption and Proposition 5.16, we have

$$
P * B = \mathcal{R}(x * B * B) = P.
$$

By Proposition 6.13, we have

$$
Q * J(Q) \leq B_h * J(B_h) = \mathcal{S}(B)B.
$$

Then by Proposition 5.16 and Theorem 5.19, we have

$$
\begin{aligned}
\mathcal{S}(P) &\leq \mathcal{S}(P * Q) = \mathcal{S}(\mathcal{R}(P * Q)) \\
&\leq \mathcal{S}(\mathcal{R}(P * Q) * J(Q)) = \mathcal{S}(P * Q * J(Q)) \\
&\leq \mathcal{S}(P * B) = \mathcal{S}(P).
\end{aligned}
$$

Hence $\mathcal{S}(P * Q) = \mathcal{S}(P)$. $\quad\square$

Remark 6.18. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be a fusion bialgebra. If the Schur product property holds on the dual, the results in Theorem 6.17 are true for projections in $\mathcal{B}$.

Following the proofs in [13], one can obtain the following theorems:

Theorem 6.19 (*Extremizers of Young's inequality*)*. Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialge-bra. Suppose the dual has Young's property. Then the following are equivalent:*

(1) $\|x * y\|_{r,\mathcal{A}} = \|x\|_{t,\mathcal{A}}\|y\|_{s,\mathcal{A}}$ *for some* $1 < r, t, s < \infty$ *such that* $1/r + 1 = 1/t + 1/s$;
(2) $\|x * y\|_{r,\mathcal{A}} = \|x\|_{t,\mathcal{A}}\|y\|_{s,\mathcal{A}}$ *for any* $1 \le r, t, s \le \infty$ *such that* $1/r + 1 = 1/t + 1/s$;
(3) x, y *are bishifts of biprojection such that* $\mathcal{R}(\mathfrak{F}(x)) = \mathcal{R}(\mathfrak{F}(y))$.

Theorem 6.20 (*Extremizers of the Hausdorff-Young inequality*)*. Let* $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ *be a fusion bialgebra. Suppose the dual has Young's property, then the following are equivalent:*

(1) $\|x\|_{\frac{t}{t-1},\mathcal{B}} = \|x\|_{t,\mathcal{A}}$ *for some* $1 < t < 2$;
(2) $\|x\|_{\frac{t}{t-1},\mathcal{B}} = \|x\|_{t,\mathcal{A}}$ *for any* $1 \le t \le 2$;
(3) x *is a bishift of a biprojection.*

7. Quantum Schur product theorem on unitary fusion categories

In this section, we reformulate the quantum Schur product theorem (Theorem 4.1 in [21]) in categorical language. Planar algebras can be regarded as a topological axiomatization of pivotal categories (or 2-category in general). Subfactor planar algebras satisfy particular conditions designed for subfactor theory, see Page 9-13 of [16] for Jones' original motivation. A subfactor planar algebra is equivalent to a rigid C^*-tensor category with a Frobenius *-algebra. The correspondence between subfactor planar algebras and unitary fusion categories was discussed by Müger, particularly for Frobenius algebras in [27] and for the quantum double in [28].

Let $\mathscr{D}$ be a unitary fusion category, (or a rigid C^*-tensor category in general). Let (γ, m, η) be a Frobenius *-algebra of $\mathscr{D}$, γ is an object of $\mathscr{D}$, $m \in \hom_{\mathscr{D}}(\gamma \otimes \gamma, \gamma)$, $\eta \in \hom_{\mathscr{D}}(1, \gamma)$, where 1 is the unit object of $\mathscr{D}$, such that (γ, m, η) is a monoid object and (γ, m^*, η^*) is a comonoid object. Let $\cup_{\gamma} = \eta^* m$ be the evaluation map and $\cap_{\gamma} = m^* \eta$ be the co-evaluation map. Then $\cup_{\gamma}^* = \cap_{\gamma}$.

We construct a quintuple $(\mathcal{A}, *, J, d, \tau)$ from the Frobenius algebra: Take the C^* algebra

$$\mathcal{A} = \hom_{\mathscr{D}}(\gamma, \gamma),$$

with the ordinary multiplication and adjoint operation. For $x, y \in \mathcal{A}$, their convolution is

$$x * y = m(x \otimes y)m^*.$$

The modular conjugation J is the restriction of the dual map of $\mathscr{D}$ on $\mathcal{A}$. The Haar measure is

$$d(x) = \cup_\gamma (x \otimes 1_\gamma) \cap_\gamma,$$

where $1_\cdot$ is the identity map on the object $\cdot$. The Dirac measure is

$$\tau(x) = \eta^* x \eta.$$

We reformulate the quantum Schur product theorem on subfactor planar algebras and its proof as follows:

Theorem 7.1 *(Theorem 4.1 in [21]). Given a Frobenius algebra (γ, m, η) of a rigid C^*-tensor category, for any $x, y \in \mathcal{A} := \hom_{\mathscr{D}}(\gamma, \gamma)$, $x, y > 0$, we have that*

$$x * y := m(x \otimes y)m^* > 0.$$

Proof. Let $\sqrt{x}$ and $\sqrt{y}$ be the positive square roots of the positive operators x and y respectively. Then

$$x * y = ((\sqrt{x} \otimes \sqrt{y})m^*)^*((\sqrt{x} \otimes \sqrt{y})m^*) \geq 0.$$

Note that d is a faithful state, so $d(x), d(y) > 0$. Moreover, $d(x * y)$ is a positive multiple of $d(x)d(y)$, so $d(x * y) > 0$ and $x * y > 0$. $\square$

By the associativity of m and $J(m) = m$, the vector space $\hom_{\mathscr{D}}(\gamma, \gamma)$ forms another C^*-algebra $\mathcal{B}$, with a multiplication $*$ and involution J. The identity map induces a unitary transformation $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$, due to the Plancherel's formula,

$$\tau(x * J(x)) = \cup_\gamma (x \otimes J(x)) \cap_\gamma = \cup_\gamma (x^* x \otimes 1_\gamma) \cap_\gamma = d(x^* x).$$

Proposition 7.2. *When $\mathcal{A}$ is commutative, the quintuple $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a canonical fusion bialgebra.*

Proof. The Schur product property follows from Theorem 7.1. The modulo conjugation property holds, as the duality map is an anti-linear *-isomorphism. The Jones projection property holds, as $\mathfrak{F}(1_1)$ is the identity of $\mathcal{B}$. Moreover, 1_1 is a minimal central projection and $d(1_1) = 1$, so $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ is a canonical fusion bialgebra. $\square$

Following the well-known correspondence between subfactor planar algebras and a rigid C^* tensor category with a Frobenius *-algebra, we reformulate the subfactorization of a fusion bialgebra as follows:

Definition 7.3. A fusion bialgebra is subfactorizable if and only if it is the quintuple $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ arisen from a Frobenius *-algebra in a rigid C^* tensor category constructed above.

There is another way to construct the dual $\mathcal{B}$ using the dual of $\mathscr{D}$ w.r.t. the Frobenius algebra γ, which is compatible with the Fourier duality of subfactor planar algebras. The dual $\hat{\mathscr{D}}$ of $\mathscr{D}$ w.r.t. the Frobenius algebra (γ, m, η) is defined as the $\gamma - \gamma$ bimodule category over $\mathscr{D}$, with the unit object γ. The dual Frobenius *-algebra of (γ, m, η) is $(\hat{\gamma}, \hat{m}, \hat{\eta})$, $\hat{\gamma} = \gamma \otimes \gamma$, $\hat{m} = 1_\gamma \otimes \cup_\gamma \otimes 1_\gamma$, $\hat{\eta} = m^*$. Then the quintuple from the Frobenius algebra $\hat{\gamma}$ of $\hat{\mathscr{D}}$ is dual to the quintuple from the Frobenius algebra γ of $\mathscr{D}$. In particular, the C^*-algebra $\mathcal{B}$ can be implemented by $\hom_{\hat{\mathscr{D}}}(\hat{\gamma}, \hat{\gamma})$ with ordinary multiplication and adjoint operation.

Let $\mathscr{C}$ be a unitary fusion category. Take $\mathscr{D} = \mathscr{C} \boxtimes \overline{\mathscr{C}}$, then $\mathscr{D}$ has a canonical Frobenius algebra (γ, m, η). Here $\gamma = \bigoplus_{j=1}^{m} X_j \otimes \overline{X_j}$, where $\{X_1, X_2, \ldots, X_m\}$ is the set of irreducible (or simple) objects of $\mathscr{C}$, X_1 is the unit and $\overline{X_j} = X_{j^*}$;

$$m = \text{FPdim}(\mathscr{C})^{1/4} \bigoplus_{j,k,s=1}^{m} (\text{FPdim}(X_j)\text{FPdim}(X_k)\text{FPdim}(X_s))^{1/2} \sum_{\alpha \in ONB(X_j, X_k; X_s)} \alpha \boxtimes \overline{\alpha},$$

where $\text{FPdim}(X_j)$ is the Frobenius-Perron dimension of X_j,

$$\text{FPdim}(\mathscr{C}) = \sum_{j=1}^{m} \text{FPdim}(X_j)^2$$

is the Frobenius-Perron dimension of $\mathscr{C}$ and $ONB(X_j, X_k; X_s)$ is an orthonormal basis of $\hom_{\mathscr{C}}(X_j \otimes X_k, X_s)$; and $\text{FPdim}(\mathscr{C})^{1/4}\eta \in \hom_{\mathscr{D}}(1, \gamma)$ is the canonical inclusion (in particular, $\hat{\gamma}$ is the image of the unit of $\mathscr{C}$ under the action of the adjoint functor of the forgetful functor from $Z(\mathscr{C})$ to $\mathscr{C}$). Its dual $\hat{\mathscr{D}}$ is isomorphic to the Drinfeld center $Z(\mathscr{C})$ of $\mathscr{C}$ as a fusion category. This construction is well-known as the quantum double construction. Consequently,

Proposition 7.4. *Let $\mathcal{R}$ be the Grothendieck ring of a unitary fusion category $\mathscr{C}$. Then the canonical fusion bialgebra associated to the fusion ring $\mathcal{R}$ is isomorphic to the one $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ associated to the canonical Frobenius algebra γ of $\mathscr{C} \otimes \overline{\mathscr{C}}$ in the quantum double construction. So it is subfactorizable.*

Proof. Following the notations above, we take $x_j := \text{FPdim}(X)^{-1} 1_{X_j} \boxtimes 1_{\overline{X_j}}$. Then

$$d(x_j) = PF(X_j)$$
$$x_j \diamond x_k = \delta_{j,k} d(x_j)^{-1} x_j \; ;$$

$$(x_j)^{\#} = x_j \ ;$$

$$\tau(x_j) = \delta_{1,j} \ ;$$

$$J(x_j) = x_{j^*} \ ;$$

$$x_j * x_k = \sum_{k \in Irr} \dim |Z\rangle \ ,$$

where $\diamond$ and $\#$ are the ordinary multiplication and adjoint operator on the commutative C^*-algebra $\mathcal{A} = \hom_{\mathscr{C}}(\gamma, \gamma)$ respectively, $\delta_{j,k}$ is the Kronecker delta and $N_{j,k}^s = \hom_{\mathscr{C}}(X_j \otimes X_k, X_s)$. Therefore, the fusion bialgebra associated to the Grothendieck ring is isomorphic to the fusion bialgebra arisen from the canonical Frobenius algebra (γ, m, η) of $\mathscr{C} \boxtimes \overline{\mathscr{C}}$ in the quantum double construction. So it is subfactorizable. $\quad \square$

Remark 7.5. To encode the fusion rule of $\mathscr{C}$ as the convolution on $\mathcal{A}$ exactly, our normalization of the Frobenius algebra (γ, m, η) is slightly different from the usual one identical to planar tangles in planar algebras, see e.g. Equation (18), Propositions 4.1 and 4.2 in [22].

Proposition 7.6. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be the canonical fusion bialgebra associated with the Grothendieck ring $\mathcal{R}$ of the unitary fusion category $\mathscr{C}$. Then the Schur product property holds on $\mathcal{B}$, the dual of the fusion ring $\mathcal{R}$.*

Proof. Applying the quantum Schur product theorem, Theorem 7.1, to the Frobenius algebra $(\hat{\gamma}, \hat{m}, \hat{\eta})$ of the Drinfeld center $Z(\mathscr{C})$, we obtain the Schur product property on $\mathcal{B}$. $\quad \square$

We obtain an equivalent statement on $\mathcal{A}$ as follows (see another equivalent statement in Proposition 8.3):

Proposition 7.7. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ (or $(\mathcal{A}, *, J, d, \tau)$ as in Remark 2.14) be the canonical fusion bialgebra associated with the Grothendieck ring $\mathcal{R}$ of the unitary fusion category $\mathscr{C}$. Then*

$$d((J(x) * x) \diamond (J(y) * y) \diamond (J(z) * z)) \geq 0, \ \forall x, y, z \in \mathcal{A}.$$

Proof. It follows from Propositions 3.6 and 7.6.

We give a second proof without passing through the Drinfeld center $Z(\mathscr{C})$.

$$d((J(x) * x) \diamond (J(y) * y) \diamond (J(z) * z))$$
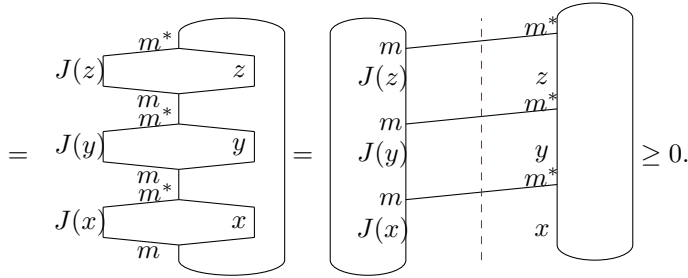
The last inequality follows from reflection positivity of the horizontal reflection, namely the dual functor on $\mathscr{C}$. $\quad\square$

Remark 7.8. Let us mention [11] which contains a reformulation of our first proof together with a discussion on some integrality properties of the numbers appearing in the Schur product criterion.

In particular, if Grothendieck ring $\mathcal{R}$ is commutative, then $\mathcal{B}$ is commutative. There is a one-to-one correspondence between minimal projections P_j in $\mathcal{B}$ and characters χ_j of $\mathcal{R}$, $j = 1, 2, \ldots, m$:

$$P_j = \sum_{k=1}^{n} d(x_k)^{-1} \chi_j(x_k) x_{k^*}.$$

Take

$$P_j *_B P_k = \sum_{s=1}^{m} \hat{N}_{j,k}^s P_s,$$

then $\hat{N}_{j,k}^s \geq 0$, due to the Schur product property on $\mathcal{B}$.

The dual of the fusion ring $\mathcal{R}$ is independent of its categorification. The Schur product property may not hold on the dual of a fusion ring in general. Therefore, the Schur product property is an analytic obstruction of unitary categorification of fusion rings. We discuss its applications in §8. Similarly, Young's inequality and sumset estimates are also analytic obstructions of unitary categorification of fusion rings.

8. Applications and conclusions

In this section, we show that the Schur product property on the dual is an analytic obstruction for the unitary categorification of fusion rings. Furthermore, this obstruction is very efficient to rule out the fusion rings of high ranks (we apply it on simple integral fusion rings). The inequalities for the fusion coefficients (Proposition 8.1) in the next subsection are essential for finding new fusion rings more efficiently.

8.1. Upper bounds on the fusion coefficients

In this subsection, we obtain inequalities for fusion rings from the inequalities proved in previous sections.

Proposition 8.1. *Let $\mathfrak{A}$ be a fusion ring. Then*

(1) $\sum_{\ell=1}^{m} \left(N_{j,k}^{\ell} \right)^2 \leq \min\{d(x_j)^2, d(x_k)^2\};$
(2) $N_{j,k}^{\ell} \leq d(x_\ell)d(x_j)^{\frac{2-t}{t}} d(x_k)^{\frac{t-2}{t}}$ *for any $t \geq 1$;*
(3) $N_{j,k}^{\ell} \leq \min\{d(x_j), d(x_k), d(x_\ell)\};$
(4) $\sum_{s=1}^{m} N_{j_1,j_2}^{s} N_{j_3,j_4}^{s} \leq \min_{j \neq j' \in \{j_1,j_2,j_3,j_4\}} d(x_j)d(x_{j'}).$

Proof. Let $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ be the fusion bialgebra arising from the fusion ring $\mathfrak{A}$. By Theorem 5.11, we have for any $1/r + 1 = 1/p + 1/q$,

$$\left\| \sum_{\ell=1}^{m} N_{j,k}^{\ell} \mathfrak{F}^{-1}(x_\ell) \right\|_{r,\mathcal{A}} = \|\mathfrak{F}^{-1}(x_j) * \mathfrak{F}^{-1}(x_k)\|_{r,\mathcal{A}} \leq \|\mathfrak{F}^{-1}(x_j)\|_{p,\mathcal{A}} \|\mathfrak{F}^{-1}(x_k)\|_{q,\mathcal{A}}.$$

If $r < \infty$, then we obtain that

$$\left(\sum_{\ell=1}^{m} d(x_\ell)^{2-r} \left(N_{j,k}^{\ell} \right)^r \right)^{1/r} \leq d(x_j)^{\frac{2-p}{p}} d(x_k)^{\frac{2-q}{q}}. \tag{18}$$

If $r = \infty$, then we have

$$\max_{1 \leq \ell \leq m} \frac{N_{j,k}^{\ell}}{d(x_\ell)} \leq d(x_j)^{\frac{2-p}{p}} d(x_k)^{\frac{2-q}{q}}. \tag{19}$$

In Inequality (18), let $r = 2$, $p = 1$, $q = 2$, we have $\sum_{\ell=1}^{m} \left(N_{j,k}^{\ell} \right)^2 \leq d(x_j)^2$; let $r = 2$, $p = 2$, $q = 1$, we have $\sum_{\ell=1}^{m} \left(N_{j,k}^{\ell} \right)^2 \leq d(x_k)^2$. Hence

$$\sum_{\ell=1}^{m} \left(N_{j,k}^{\ell} \right)^2 \leq \min\{d(x_j)^2, d(x_k)^2\}.$$

This proves (1).

In Inequality (19), let $p = t$ and $q = \frac{t}{t-1}$ for any $t \geq 1$. Then

$$N_{j,k}^{\ell} \leq d(x_\ell)d(x_j)^{\frac{2-t}{t}} d(x_k)^{\frac{t-2}{t}}.$$

This shows (2) is true.

Take $p = q = 2$ in Inequality (19), we have $N_{j,k}^\ell \leq d(x_\ell)$. By Equation (1), we have

$$N_{j,k}^\ell \leq \min\{d(x_j), d(x_k), d(x_\ell)\}.$$

This indicates that (4) is true.

By Theorem 5.11 again, we have

$$\left\|\sum_{t=1}^m \sum_{s=1}^m N_{j,k}^s N_{s,\ell}^t \mathfrak{F}^{-1}(x_t)\right\|_{\infty,\mathcal{A}} = \|\mathfrak{F}^{-1}(x_j) * \mathfrak{F}^{-1}(x_k) * \mathfrak{F}^{-1}(x_\ell)\|_{\infty,\mathcal{A}}$$

$$\leq \|\mathfrak{F}^{-1}(x_j)\|_{1,\mathcal{A}} \|\mathfrak{F}^{-1}(x_k)\|_{2,\mathcal{A}} \|\mathfrak{F}^{-1}(x_\ell)\|_{2,\mathcal{A}}.$$

Then

$$\frac{\sum_{s=1}^m N_{j,k}^s N_{s,\ell}^t}{d(x_t)} \leq d(x_j).$$

We have $\sum_{s=1}^m N_{j,k}^s N_{s,\ell}^t \leq d(x_t)d(x_j)$. By Equation (1), we have

$$\sum_{s=1}^m N_{j,k}^s N_{\ell,t}^s \leq d(x_t)d(x_j).$$

Note that j, k, ℓ, t can be interchanged, we see (5) is true. □

Proposition 8.2. *Let $\mathfrak{A}$ be a fusion ring. Suppose that the fusion bialgebra arising from $\mathfrak{A}$ is self-dual. Let S be the S-matrix associated to $\mathfrak{A}$. Then we have the following inequalities:*

(1) $\sum_{j=1}^m d(x_j)^{\frac{t-2}{t-1}} |S_{kj}|^{\frac{t}{t-1}} \leq d(x_k)^{\frac{2-t}{t-1}} d(\mathfrak{A})^{\frac{t-2}{2t-2}}$, $1 < t \leq 2$;
(2) $d(x_j)^{t-2} d(\mathfrak{A})^{1-t/2} \leq \sum_{k=1}^m |S_{jk}|^t d(x_k)^{2-t}$, $1 < t \leq 2$;
(3) $|S_{jk}| \leq d(x_j)d(x_k)d(\mathfrak{A})^{-1/2}$,

where $d(\mathfrak{A})$ is the Frobenius-Perron dimension of $\mathfrak{A}$.

Proof. It follows from the Hausdorff-Young inequalities. □

8.2. Schur product property reformulated

In this subsection we reformulate Schur product property (on the dual) using the irreducible complex representation of the fusion algebra, which in the commutative case, becomes a purely combinatorial property of the character table.

Note that Proposition 3.3 states that if the fusion ring $\mathfrak{A}$ is the Grothendieck ring of unitary fusion category, then Schur product property holds on the dual of $\mathfrak{A}$, so it can be seen as a criterion for unitary categorification.

Proposition 8.3 *(Non-commutative Schur product criterion). The Schur product property holds on the dual of a fusion ring/algebra $\mathfrak{A}$ with basis $\{x_1 = 1, \ldots, x_r\}$ if and only if for all triple of irreducible unital *-representations $(\pi_s, V_s)_{s=1,2,3}$ of the fusion ring/algebra $\mathfrak{A}$ over $\mathbb{C}$, and for all $v_s \in V_s$, we have*

$$\sum_i \frac{1}{d(x_i)} \prod_{s=1}^{3} (v_s^* \pi_s(x_i) v_s) \geq 0. \tag{20}$$

Proof. Let $\mathfrak{A}$ be a fusion ring/algebra with basis $\{x_1 = 1, \ldots, x_r\}$ and $(\mathcal{A}, \mathcal{B}, \mathfrak{F}, d, \tau)$ the fusion bialgebra arising from $\mathfrak{A}$. By Proposition 3.6 and the fact that d is multiplicative, the Schur product property holds on $\mathcal{B}$ if and only if

$$d\left(\prod_{s=1}^{3} X_s *_{\mathcal{A}} X_s^*\right) \geq 0$$

for all $X_s \in \mathcal{A}$. Now, $X_s = \sum_i \alpha_{s,i} x_i$, so it is equivalent to

$$\sum_i \frac{1}{d(x_i)} \prod_{s=1}^{3} \left(\sum_{j,k} \alpha_{s,k} \overline{\alpha}_{s,j^*} N^i_{k,j^*}\right) \geq 0$$

for all $\alpha_{s,i} \in \mathbb{C}$. Now let M_i be the matrix (N^i_{k,j^*}) which is also $(N^k_{i,j})$ by Frobenius reciprocity, so that M_i is the fusion matrix of x_i. Let u_s be the vector $(\alpha_{s,i})$. Then

$$\sum_{j,k} \alpha_{s,k} \overline{\alpha}_{s,j^*} N^i_{k,j^*} = u_s^* M_i u_s.$$

Then the criterion is equivalent to have

$$\sum_i \frac{1}{d(x_i)} \prod_{s=1}^{3} (u_s^* M_i u_s) \geq 0. \tag{21}$$

for all $u_s \in \mathbb{C}^r$. Recall that the map $\pi : x_i \to M_i$ is a unital *-representation of $\mathfrak{A}$. So Equation (20) implies Equation (21). On the other hand, π is faithful, so Equation (21) implies Equation (20). $\square$

Assume that the fusion ring/algebra $\mathfrak{A}$ is commutative, then for all i, $x_i x_{i^*} = x_{i^*} x_i$, so that the fusion matrices M_i are normal (so diagonalizable) and commuting, so they are simultaneously diagonalizable, i.e. there is an invertible matrix P such that $P^{-1} M_i P = diag(\lambda_{i,1}, \ldots, \lambda_{i,r})$, so that the maps $\pi_j : M_i \mapsto \lambda_{i,j}$ completely characterize the irreducible complex representations π_j of $\mathfrak{A}$. We can assume that $\pi_1 = d$, so that $\lambda_{i,1} = d(x_i) = \|M_i\|$.

Definition 8.4. The matrix $\Lambda(\mathfrak{A}) := (\lambda_{i,j})$ is called the *character table* of the commutative fusion ring $\mathfrak{A}$.

Note that for a finite group G, if $\mathfrak{A}_G$ is the Grothendieck ring of the unitary fusion category $\mathrm{Rep}(G)$, then $\Lambda(\mathfrak{A}_G)$ is the usual character table of G.

Corollary 8.5 *(Commutative Schur product criterion). The Schur product property holds on the dual of a commutative fusion ring/algebra $\mathfrak{A}$ with character table $\Lambda = (\lambda_{i,j})$ if and only if for all triple (j_1, j_2, j_3)*

$$\sum_i \frac{\lambda_{i,j_1} \lambda_{i,j_2} \lambda_{i,j_3}}{\lambda_{i,1}} \geq 0.$$

Proof. Immediate from Proposition 8.3, because here the irreducible representations are one-dimensional, so that there we have $v_s^* \pi_s(M_i) v_s = \|v_s\|^2 \pi_s(M_i)$. $\square$

Remark 8.6. The formula in [38, Theorem 7.2.1] looks similar to the one in Corollary 8.5 (applied to the group case), because the underlying planar diagrams are the same. However, the results are different as the underlying planar algebras are so: the subfactor planar algebra of a finite group G for the first formula, and for the second one, the dual of the quantum double of $Vec(G)$, namely the Drinfeld center of $Vec(G)$.

In order to test the efficiency of Schur product criterion, we wrote a code computing the character table of a commutative fusion ring/algebra and checking whether Schur product property holds (on the dual) using Corollary 8.5. The next two subsections present the first results.

8.3. Fusion algebras of small rank

Ostrik [32] already classified the pivotal fusion category of rank 3. In this section we would like to show how efficient is Schur product criterion in this case. We will next consider two families of rank 4 fusion rings/algebras found by David Penneys and his collaborators[1][35], and finally look to a family of rank 5 fusion rings/algebras.

Recall [32, Proposition 3.1] that a fusion ring $\mathfrak{A}$ of rank 3 and basis $\{x_1 = 1, x_2, x_3\}$ satisfies either $x_2^* = x_3$ and then is $\mathbb{C}C_3$, or $x_i^* = x_i$ and then is of the following form (extended to fusion algebras):

$$\begin{cases} x_2 x_2 = x_1 + p x_2 + m x_3, \\ x_2 x_3 = m x_2 + n x_3, \\ x_3 x_3 = x_1 + n x_2 + q x_3, \end{cases}$$

---

[1] At the 2014 AMS MRC on Mathematics of Quantum Phases of Matter and Quantum Information.

with $m, n, p, q \in \mathbb{R}_{\geq 0}$ and $m^2 + n^2 = 1 + mq + np$ (given by associativity). Note that $x_3 x_2 = x_2 x_3$ by Frobenius reciprocity, so that the fusion algebra is commutative. We can assume (up to equivalence) that $m \leq n$, and then $n > 0$ (because if $n = 0$ then $m = 0$ and the above associativity relation becomes $0 = 1$, contradiction), so that $p = (m^2 + n^2 - 1 - mq)/n$; and it is a fusion ring if and only if in addition $m, n, p, q \in \mathbb{Z}_{\geq 0}$ and n divides $(m^2 - 1 - mq)$. Recall [32, Section 4.5] that it admits a pivotal categorification if and only if $(m, n, q) = (0, 1, 0), (0, 1, 1), (0, 1, 2), (1, 1, 1)$, and there is a unitary model for all of them.

Let M_i be the fusion matrix of x_i, written below (with $p = (m^2 + n^2 - 1 - mr)/n$):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & p & m \\ 0 & m & n \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & m & n \\ 1 & n & q \end{pmatrix}$$

Let χ_i be the characteristic polynomial of M_i:

$$\chi_2(x) = x^3 - (p+n)x^2 + (pn - 1 - m^2)x + n$$
$$\chi_3(x) = x^3 - (q+m)x^2 + (qm - 1 - n^2)x + m$$

The matrix M_i is self-adjoint thus its eigenvalues (and so the roots of χ_i) are real. By using [12, Theorem A.4], we can deduce the following character table:

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{b_2}{3} + 2r_2 c_2 & \frac{b_2}{3} - r_2(c_2 - \sqrt{3}s_2) & \frac{b_2}{3} - r_2(c_2 + \sqrt{3}s_2) \\ \frac{b_3}{3} + 2r_3 c_3 & \frac{b_3}{3} - r_3(c_3 + \sqrt{3}s_3) & \frac{b_3}{3} - r_3(c_3 - \sqrt{3}s_3) \end{bmatrix}$$

with $c_i = \cos(\frac{\varphi_i}{3})$, $s_i = \sin(\frac{\varphi_i}{3})$, $\varphi_i = \arccos\left(\frac{q_i/2}{(p_i/3)^{3/2}}\right)$, $r_i = \sqrt{\frac{p_i}{3}}$, $p_i = \frac{b_i^2}{3} - a_i$, $q_i = \frac{2b_i^3}{27} - \frac{b_i a_i}{3} - d_i$, $a_2 = pn - 1 - m^2$, $b_2 = p + n$, $d_2 = n$, $a_3 = qm - 1 - n^2$, $b_3 = q + m$ and $d_3 = m$.

We observe that about 30% of over 10000 samples can be ruled out by Schur's criterion.[2] Note that Ostrik used the inequality in [32, Theorem 2.21] to rule out some fusion rings. See Fig. 2 to visualize the efficiency of Schur product criterion and Ostrik's criterion for this family. Note that Ostrik's criterion works for the fusion rings only (not algebras[3]) and is no more efficient for higher ranks, whereas Schur product criterion does (see Subsection 8.4).

---

[2] It is nontrivial to characterize the set of all the triples (m, n, q) for which Schur product property (on the dual) does not hold. Using the above character table together with Theorem 8.5 and computer assistance, for $q, n, m \in \mathbb{Z}$, $0 \leq q \leq 30$, $1 \leq n \leq 30$ and $0 \leq m \leq n$, there are exactly 14509 fusion bialgebras (resp. 542 fusion rings), and among them, 4757 (resp. 198) ones can be ruled out from subfactorization (resp. unitary categorification) by Schur product criterion.

[3] Consider the (δ_1, δ_2)-Bisch-Jones subfactor, its 2-box space provides a fusion algebra in this family with

$$(m, n, q) = (0, (\delta_2^2 - 1)^{\frac{1}{2}}, \delta_2(\delta_1^2 - 2)(\delta_1^2 - 1)^{-\frac{1}{2}}),$$
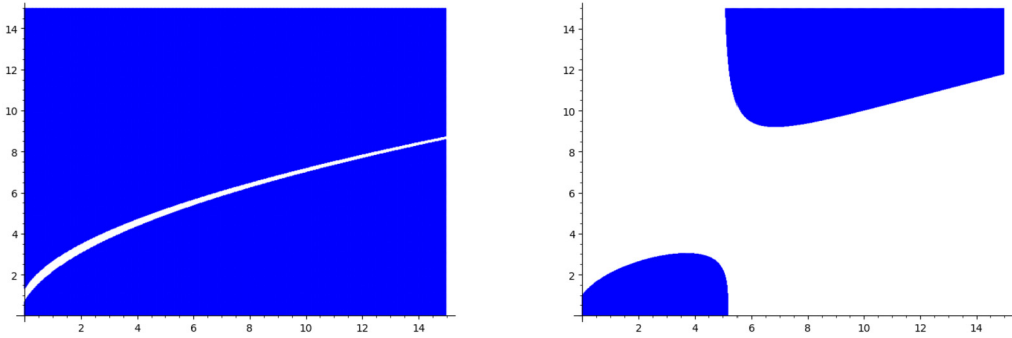
Fig. 2. Rank 3: for $q = 5$, the set of (m, n) such that Schur product property on the dual (resp. Ostrik's inequality) does not hold is (numerically) given by the right (resp. left) figure (where, for clarity, neither $m \leq n$ nor $m^2 + n^2 - 1 - mq \geq 0$ is assumed). About the right one, there are two areas, one (at the bottom) is finite, the other infinite; moreover, the projection of these two areas on the m-axis overlap around $m = q$. Each area corresponds to the application of Theorem 8.5 on one column. The form appears for all the samples of q we tried, so it is not hard to believe that it is the generic form, and in particular that Schur product property (on the dual) does not hold if $q + 1 \leq m \leq n$ and $n \geq 2q + 2$, with $m, n, q \in \mathbb{R}_{\geq 0}$ (so that the corresponding fusion bialgebras admit no subfactorization); it should be provable using the given character table (we did not make the computation).
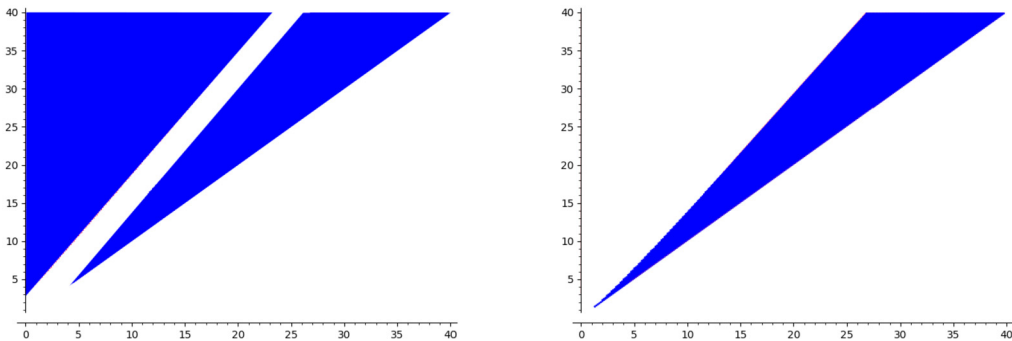
Fig. 3. Rank 4, family of fusion algebras $K_3(b, d)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & d-b & b & 0 \\ 0 & b & d & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & d & 1 \\ 1 & d & d+b & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Schur product property on the dual (resp. Ostrik's inequality) does not hold for (b, d) in the right (resp. left) figure. The area $b > d$ is not considered because $d - b \geq 0$.

Then, let us mention two families (denoted K_3 and K_4) of fusion algebras of rank 4 with self-adjoint objects provided by David Penneys and his collaborators [35]. Visualize the obstructions on Figs. 3 and 4.

---

which is often in the colored area of the figure for Ostrik's criterion, for example if $(\delta_1, \delta_2) = (\sqrt{2}, 10/3)$ then $(m, n, q) = (0, \sqrt{91}/3, 5)$. Let us also mention here that for a fusion ring, *subfactorizable* is strictly weaker than *unitarily categorifiable*, because if $(\delta_1^2, \delta_2^2) = (6 + 2\sqrt{6}, 2)$ then $(m, n, q) = (0, 1, 4)$, which is ruled out from pivotal categorification by Ostrik's paper [32].
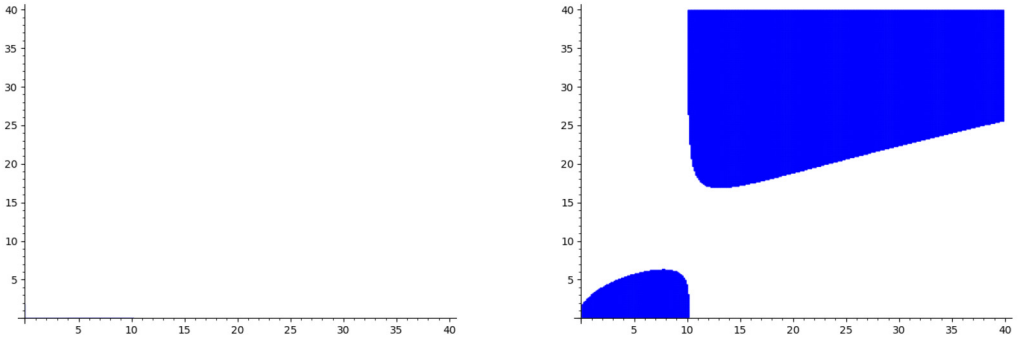
Fig. 4. Rank 4, family $K_4(a, b, d, g)$ with $bd > 0$ and $a = (b^2 + d^2 - 2 - bg)/d$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & a & b & 1 \\ 0 & b & d & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & d & 0 \\ 1 & d & g & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Same convention as above (to simplify, a is not assumed non-negative) with $g = 10$. It should be the generic shape for fixed g. Ostrik's inequality always holds, so the left figure is empty. About the figure for Schur product criterion on the right, the structure is similar to Fig. 2, one finite area on the bottom, one infinite area, and the projection of both on the b-axis should overlap around $b = g$.

Finally, let us consider the family of fusion rings of rank 5 with exactly three self-adjoint simple objects. By Frobenius reciprocity, the fusion rules must be as follows (with 16 parameters):

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & a & k & f & j \\ 1 & a & a & b & c \\ 0 & d & f & g & h \\ 0 & e & j & i & l \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & a & a & d & e \\ 0 & k & a & f & j \\ 0 & f & b & g & i \\ 0 & j & c & h & l \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & b & f & g & i \\ 0 & f & d & g & h \\ 1 & g & g & m & o \\ 0 & i & h & o & p \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & c & j & h & l \\ 0 & j & e & i & l \\ 0 & h & i & o & p \\ 1 & l & l & p & n \end{pmatrix}$$

such that $n_{i,j}^k \in \mathbb{Z}_{\geq 0}$, and $\sum_s n_{i,j}^s n_{s,k}^t = \sum_s n_{j,k}^s n_{i,s}^t$ (associativity). We found (up to equivalence) exactly 47 ones at multiplicity ≤ 4 (by brute-force computation), 4 of which are simple. The Schur product property on the dual (resp. Ostrik's inequality) does not hold on exactly 6 (resp. 1) among the 47 ones, and on exactly 2 (resp. 1) among the 4 simple ones. Schur product criterion may be more efficient at higher multiplicity. Here are the two simple ones on which the Schur product property on the dual (and Ostrik's inequality) holds (note that they are also of Frobenius type).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Note that $(d(b_1), d(b_2), d(b_3), d(b_4), d(b_5)) = (1, \alpha, \alpha, \beta, \gamma)$ with $\alpha = 1 + 2\cos(2\pi/7) \simeq 2.2469$, $\beta = 1 - 2\cos(6\pi/7) \simeq 2.8019$, $\gamma = \alpha + \beta - 1 \simeq 4.0489$, so that FPdim $\simeq 36.650$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 4 & 4 \end{pmatrix}$$

Note that $(d(b_1), d(b_2), d(b_3), d(b_4), d(b_5)) = (1, \alpha, \alpha, \beta, \gamma)$ with $\alpha = 3 + \sqrt{6} \simeq 5.4494$, $\beta = 4 + 2\sqrt{6} \simeq 8.8989$, $\gamma = 5 + 2\sqrt{6} \simeq 9.8989$, so that FPdim $= 120 + 48\sqrt{6} \simeq 237.57$.

8.4. Simple integral fusion rings (high rank)

A fusion category (resp. ring) is called *simple* if it has no nontrivial proper fusion subcategories (resp. subrings). Here is a result of Etingof, Nikshych and Ostrik [10, Proposition 9.11]:

Proposition 8.7. *A weakly group-theoretical simple fusion category has the Grothendieck ring of* $\mathrm{Rep}(G)$*, with G a finite simple group.*

A fusion category (resp. ring) is called *integral* if the Frobenius-Perron dimension of every (simple) object is an integer. Here is the strong version of [10, Question 2]:

Question 8.8. Is there an integral fusion category which is not weakly group-theoretical?

Then it seems relevant to look for integral simple fusion rings which are not Grothendieck rings of any $\mathrm{Rep}(G)$ with G finite (simple) group, because according to Proposition 8.7, the categorification of one of them would provide a positive answer to Question 8.8.

Definition 8.9. Let $\mathfrak{A}$ be a fusion ring of basis $\{x_1 = 1, \ldots, x_r\}$ with $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_r)$. Let us call r its *rank*, $\sum_i d(x_i)^2$ its *Frobenius-Perron dimension* (or FPdim($\mathfrak{A}$)), and $[d(x_1), d(x_2), \ldots, d(x_r)]$ its *type*, which will also be written by $[[n_1, m_1], [n_2, m_2], \ldots, [n_s, m_s]]$ where m_i is the number of x_j with $d(x_j) = n_i$, $\sum_i m_i = r$ and $1 = n_1 < n_2 < \cdots < n_s$.

Recall that the Grothendieck ring of $\mathrm{Rep}(G)$ is simple if and only if G is simple because every fusion subring is given by $\mathrm{Rep}(G/N)$ with N a normal subgroup. Now a non-abelian simple group is perfect (i.e. $[G, G]$ generates G), and there is also a way to characterize the perfect groups at the fusion ring level:

Proposition 8.10. *A finite group G is perfect if and only if the type of the Grothendieck ring of* $\mathrm{Rep}(G)$ *satisfies $m_1 = 1$ (i.e. every one-dimensional representation must be trivial).*

Proof. Let G be a perfect group and let π be a one-dimensional representation of G. By assumption, every $g \in G$ is a product of commutators, but $\pi(G)$ is abelian (because π is one-dimensional), so that $\pi(g) = \pi(1)$. It follows that π is trivial.

Now assume that every one-dimensional representation is trivial, and consider the quotient map $p : G \to Z$ with $Z = G/\langle [G,G] \rangle$ which is abelian. Then p induces a representation π of G with $\pi(G)$ abelian, so that π is a direct sum of one-dimensional representations. It follows by assumption that $Z = \pi(G)$ is trivial, which means that G is perfect. $\square$

This proposition leads us to call *perfect* a fusion ring with $m_1 = 1$. Note that a non-perfect simple fusion ring is given by a prime order cyclic group. The fusion ring $\mathfrak{A}$ is called of *Frobenius type* if $\frac{\mathrm{FPdim}(\mathfrak{A})}{d(x_i)}$ is an algebraic integer for all i, and if $\mathfrak{A}$ is integral, this means that $d(x_i)$ divides $\mathrm{FPdim}(\mathfrak{A})$. Kaplansky's 6th conjecture [19] states that for every finite dimensional semisimple Hopf algebra H over $\mathbb{C}$, the integral fusion category $\mathrm{Rep}(H)$ is of Frobenius type. If in addition H has a $*$-structure (i.e. is a Kac algebra), then $\mathrm{Rep}(H)$ is unitary. For a *first step* in the proof of this conjecture, see [18, Theorem 2]. Note that there exist simple integral fusion rings which are not of Frobenius type (see Subsection A.2). The integral simple (and perfect) fusion rings of Frobenius type are classified in the following cases (with $\mathrm{FPdim} \neq p^a q^b, pqr$, by [10]), with computer assistance, significantly boosted by Proposition 8.1.[4]

rank	≤ 5	6	7	8	9	10	all
FPdim $<$	1000000	150000	15000	4080	504	240	132

We found exactly 34 ones, and each of them is commutative; which leads to:

Question 8.11. Is there a non-commutative simple integral fusion ring (of Frobenius type)?

Let us first summarize these results of the computer search: four of them are Grothendieck rings of $\mathrm{Rep}(G)$ with G a non-abelian finite simple group, 28 (among the remaining 30) are ruled out (from unitary categorification) by Schur product property (on the dual), and none can be ruled out by already known obstructions (as Ostrik's inequality); the existence of a unitary categorification is unknown for each of the remaining two. Here are the results in details, where $\#$ counts the number of fusion rings, whereas $\#$Schur counts those checking the *commutative* Schur product criterion (all the fusion matrices are available in Appendix):

---

[4] In preparing this paper, we first had a classification up to some smaller dimensions, given by a previous version of the computer program. Then the use of inequalities in Proposition 8.1 boosted the computation, allowing us to extend the bounds significantly.

#	rank	FPdim	type	#Schur	group
1	5	60	[[1,1],[3,2],[4,1],[5,1]]	1	PSL(2,5)
1	6	168	[[1,1],[3,2],[6,1],[7,1],[8,1]]	1	PSL(2,7)
2	7	210	[[1,1],[5,3],[6,1],[7,2]]	1	
2	7	360	[[1,1],[5,2],[8,2],[9,1],[10,1]]	1	PSL(2,9)
4	7	7980	[[1,1],[19,1],[20,1],[21,1],[42,2],[57,1]]	0	
15	8	660	[[1,1],[5,2],[10,2],[11,1],[12,2]]	2	PSL(2,11)
5	8	990	[[1,1],[9,1],[10,1],[11,4],[18,1]]	0	
2	8	1260	[[1,1],[6,1],[7,2],[10,1],[15,1],[20,2]]	0	
2	8	1320	[[1,1],[6,2],[10,1],[11,1],[15,2],[24,1]]	0	

Question 8.12. Are there only finitely many simple integral fusion rings of a given rank (assuming Frobenius type and perfect)? Is the above list the full classification at rank ≤ 6? If the Schur product property (on the dual) is assumed to hold, is it full at rank ≤ 8?

Let us write here the fusion matrices and character tables for the first fusion ring ruled out written above, and for the two which were not.

First the simple integral fusion ring of rank 7, FPdim 210, type $[[1,1],[5,3],[6,1],[7,2]]$ and fusion matrices:

$$
\begin{smallmatrix}
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1
\end{smallmatrix}
\quad
\begin{smallmatrix}
0&1&0&0&0&0&0\\
1&1&0&1&0&1&1\\
0&0&1&0&1&1&1\\
0&1&0&0&1&1&1\\
0&0&1&1&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1
\end{smallmatrix}
\quad
\begin{smallmatrix}
0&0&1&0&0&0&0\\
0&0&1&0&1&1&1\\
1&1&1&0&0&1&1\\
0&0&0&1&1&1&1\\
0&1&0&1&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1
\end{smallmatrix}
\quad
\begin{smallmatrix}
0&0&0&1&0&0&0\\
0&1&0&0&1&1&1\\
0&0&0&1&1&1&1\\
1&0&1&1&0&1&1\\
0&1&1&0&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1
\end{smallmatrix}
\quad
\begin{smallmatrix}
0&0&0&0&1&0&0\\
0&0&1&1&0&1&1\\
0&1&0&1&1&1&1\\
0&1&1&0&1&1&1\\
1&1&1&1&1&1&1\\
0&1&1&1&1&2&1\\
0&1&1&1&1&1&2
\end{smallmatrix}
\quad
\begin{smallmatrix}
0&0&0&0&0&1&0\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&2&1\\
1&1&1&1&2&0&3\\
0&1&1&1&1&3&1
\end{smallmatrix}
\quad
\begin{smallmatrix}
0&0&0&0&0&0&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&1\\
0&1&1&1&1&1&2\\
0&1&1&1&1&3&1\\
1&1&1&1&2&1&2
\end{smallmatrix}
$$

Its character table is the following:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & -1 & -\zeta_7 - \zeta_7^6 & -\zeta_7^5 - \zeta_7^2 & -\zeta_7^4 - \zeta_7^3 & 0 & 0 \\
5 & -1 & -\zeta_7^5 - \zeta_7^2 & -\zeta_7^4 - \zeta_7^3 & -\zeta_7 - \zeta_7^6 & 0 & 0 \\
5 & -1 & -\zeta_7^4 - \zeta_7^3 & -\zeta_7 - \zeta_7^6 & -\zeta_7^5 - \zeta_7^2 & 0 & 0 \\
6 & 0 & -1 & -1 & -1 & 1 & 1 \\
7 & 1 & 0 & 0 & 0 & 0 & -3 \\
7 & 1 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
$$

It is possible to see why it was ruled out by Schur product criterion by observing this character table (in particular its last column) together with Corollary 8.5:

$$
\frac{1^3}{1} + \frac{0^3}{5} + \frac{0^3}{5} + \frac{0^3}{5} + \frac{1^3}{6} + \frac{(-3)^3}{7} + \frac{2^3}{7} = -\frac{65}{42} < 0.
$$

Remark 8.13. Here we applied Corollary 8.5 by using three times the same block (i.e. irreducible representation, or column here), but it is not always possible. For example, the simple fusion ring of type $[[1,1],[5,2],[8,2],[9,1],[10,1]]$ (the one not given by PSL(2,9)) required two blocks to be ruled out.

Next, the fusion matrices of the simple integral fusion ring of same type as above, for which Schur product property (on the dual) holds:

$$
\begin{array}{llllll}
\begin{smallmatrix}
1\,0\,0\,0\,0\,0\,0\\
0\,1\,0\,0\,0\,0\,0\\
0\,0\,1\,0\,0\,0\,0\\
0\,0\,0\,1\,0\,0\,0\\
0\,0\,0\,0\,1\,0\,0\\
0\,0\,0\,0\,0\,1\,0\\
0\,0\,0\,0\,0\,0\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,1\,0\,0\,0\,0\,0\\
1\,1\,0\,1\,0\,1\,1\\
0\,0\,1\,0\,1\,1\,1\\
0\,1\,0\,0\,1\,1\,1\\
0\,0\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,1\,0\,0\,0\,0\\
0\,0\,1\,0\,1\,1\,1\\
1\,1\,1\,0\,1\,1\,0\\
0\,0\,0\,1\,1\,1\,1\\
0\,1\,0\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,1\,0\,0\,0\\
0\,1\,0\,0\,1\,1\,1\\
0\,0\,0\,1\,1\,1\,1\\
1\,0\,1\,1\,0\,1\,1\\
0\,1\,1\,0\,1\,1\,1\\
1\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,1\,0\,0\\
0\,0\,1\,1\,1\,1\,1\\
0\,1\,0\,1\,1\,1\,1\\
0\,1\,1\,0\,1\,1\,1\\
1\,1\,1\,1\,1\,2\,1\\
0\,1\,1\,1\,1\,2\,1\\
0\,1\,1\,1\,1\,1\,2
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,0\,1\,0\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,2\,1\\
1\,1\,1\,1\,2\,1\,2\\
0\,1\,1\,1\,1\,2\,2
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,0\,0\,1\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,1\,1\,1\,2\\
0\,1\,1\,1\,1\,1\,2\\
0\,1\,1\,1\,1\,2\,2\\
1\,1\,1\,1\,2\,2\,1
\end{smallmatrix}
\end{array}
$$

Let us call $\mathfrak{F}_{210}$ the corresponding fusion ring (mentioned after [33, Problem 4.12]). Its character table is:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1\\
5 & -1 & -\zeta_7 - \zeta_7^6 & -\zeta_7^5 - \zeta_7^2 & -\zeta_7^4 - \zeta_7^3 & 0 & 0\\
5 & -1 & -\zeta_7^5 - \zeta_7^2 & -\zeta_7^4 - \zeta_7^3 & -\zeta_7 - \zeta_7^6 & 0 & 0\\
5 & -1 & -\zeta_7^4 - \zeta_7^3 & -\zeta_7 - \zeta_7^6 & -\zeta_7^5 - \zeta_7^2 & 0 & 0\\
6 & 0 & -1 & -1 & -1 & 1 & 1\\
7 & 1 & 0 & 0 & 0 & \zeta_5 + \zeta_5^4 & \zeta_5^2 + \zeta_5^3\\
7 & 1 & 0 & 0 & 0 & \zeta_5^2 + \zeta_5^3 & \zeta_5 + \zeta_5^4
\end{bmatrix}
$$

Finally, the fusion matrices of the only simple integral fusion ring (not given by a group) of rank 8, FPdim 660, type $[[1, 1], [5, 2], [10, 2], [11, 1], [12, 2]]$ on which Schur product property (on the dual) holds:

$$
\begin{array}{llllllll}
\begin{smallmatrix}
1\,0\,0\,0\,0\,0\,0\,0\\
0\,1\,0\,0\,0\,0\,0\,0\\
0\,0\,1\,0\,0\,0\,0\,0\\
0\,0\,0\,1\,0\,0\,0\,0\\
0\,0\,0\,0\,1\,0\,0\,0\\
0\,0\,0\,0\,0\,1\,0\,0\\
0\,0\,0\,0\,0\,0\,1\,0\\
0\,0\,0\,0\,0\,0\,0\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,1\,0\,0\,0\,0\,0\,0\\
0\,0\,1\,1\,1\,0\,0\,0\\
1\,0\,0\,0\,0\,0\,1\,1\\
0\,0\,1\,0\,1\,1\,1\,1\\
0\,0\,1\,1\,0\,1\,1\,1\\
0\,0\,0\,1\,1\,0\,1\,1\\
0\,1\,0\,1\,1\,1\,1\,1\\
0\,1\,0\,1\,1\,1\,1\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,1\,0\,0\,0\,0\,0\\
1\,0\,0\,0\,0\,0\,1\,1\\
0\,1\,0\,1\,1\,0\,0\,0\\
0\,1\,0\,1\,1\,1\,0\,0\\
0\,1\,0\,1\,0\,1\,1\,1\\
0\,1\,0\,1\,0\,1\,1\,1\\
0\,0\,1\,1\,1\,1\,1\,1\\
0\,0\,1\,1\,1\,1\,1\,1
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,1\,0\,0\,0\,0\\
0\,0\,1\,0\,1\,1\,1\,1\\
0\,1\,0\,1\,1\,1\,0\,0\\
1\,0\,0\,3\,1\,1\,2\,2\\
0\,1\,1\,1\,1\,2\,2\,2\\
1\,0\,0\,1\,3\,1\,2\,2\\
0\,1\,1\,2\,2\,2\,2\,2\\
0\,1\,1\,2\,2\,2\,2\,2
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,1\,0\,0\,0\\
0\,0\,1\,1\,0\,1\,1\,1\\
0\,1\,0\,1\,0\,1\,1\,1\\
0\,1\,1\,1\,1\,2\,2\,2\\
1\,0\,0\,1\,3\,1\,2\,2\\
0\,1\,1\,2\,1\,2\,2\,2\\
0\,1\,1\,2\,2\,2\,2\,2\\
0\,1\,1\,2\,2\,2\,2\,2
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,0\,1\,0\,0\\
0\,0\,0\,1\,1\,0\,1\,1\\
0\,0\,1\,1\,1\,1\,1\,1\\
0\,0\,1\,1\,2\,1\,2\,2\\
0\,1\,1\,2\,1\,2\,2\,2\\
1\,1\,1\,2\,2\,2\,3\,2\\
0\,1\,1\,2\,2\,3\,2\,3\\
0\,1\,1\,2\,2\,2\,3\,3
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,0\,0\,1\,0\\
0\,1\,0\,1\,1\,1\,1\,1\\
0\,0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,2\,2\,2\,2\,2\\
0\,1\,1\,2\,2\,2\,2\,2\\
1\,1\,1\,2\,2\,2\,3\,3\\
0\,1\,1\,2\,2\,3\,2\,3\\
0\,1\,1\,2\,2\,3\,3\,2
\end{smallmatrix} &
\begin{smallmatrix}
0\,0\,0\,0\,0\,0\,0\,1\\
0\,1\,0\,1\,1\,1\,1\,1\\
0\,0\,1\,1\,1\,1\,1\,1\\
0\,1\,1\,2\,2\,2\,2\,2\\
0\,1\,1\,2\,2\,2\,2\,2\\
0\,1\,1\,2\,2\,2\,3\,3\\
0\,1\,1\,2\,2\,3\,3\,3\\
1\,1\,1\,2\,2\,3\,3\,2
\end{smallmatrix}
\end{array}
$$

Let us call $\mathfrak{F}_{660}$ the corresponding fusion ring. Its character table is:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
5 & 0 & -1 & 1 & -1 & 0 & \zeta_{11} + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9 & \zeta_{11}^2 + \zeta_{11}^6 + \zeta_{11}^7 + \zeta_{11}^8 + \zeta_{11}^{10}\\
5 & 0 & -1 & 1 & -1 & 0 & \zeta_{11}^2 + \zeta_{11}^6 + \zeta_{11}^7 + \zeta_{11}^8 + \zeta_{11}^{10} & \zeta_{11} + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9\\
10 & 0 & 1 + \sqrt{3} & 0 & 1 - \sqrt{3} & 0 & -1 & -1\\
10 & 0 & 1 - \sqrt{3} & 0 & 1 + \sqrt{3} & 0 & -1 & -1\\
11 & 1 & -1 & -1 & -1 & 1 & 0 & 0\\
12 & \zeta_5 + \zeta_5^4 & 0 & 0 & 0 & \zeta_5^2 + \zeta_5^3 & 1 & 1\\
12 & \zeta_5^2 + \zeta_5^3 & 0 & 0 & 0 & \zeta_5 + \zeta_5^4 & 1 & 1
\end{bmatrix}
$$

Question 8.14. Do $\mathfrak{F}_{210}$ or $\mathfrak{F}_{660}$ admit a unitary categorification?

Subsection A.3 of Appendix mentions 2561 extra perfect integral fusion rings of rank ≤ 10. Among them, 7 ones are simple and 9 ones are noncommutative (none both). In the commutative case, 2072 ones can be ruled out from unitary categorification by Corollary 8.5 (more than 80%).

Acknowledgments

Appendix A

A.1. List of simple integral fusion rings of Frobenius type

We provide here the fusion matrices for the 34 simple integral fusion rings mentioned above (all commutative) together with what we know about them. Those ruled out have no additional data.

- Rank 5, FPdim 60, one of type $[[1,1],[3,2],[4,1],[5,1]]$, given by the group $PSL(2,5)$:

$$
\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0 \\ 1&1&0&0&1 \\ 0&0&0&1&1 \\ 0&0&1&1&1 \\ 0&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&1 \\ 1&0&1&0&1 \\ 0&1&0&1&1 \\ 0&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0 \\ 0&0&1&1&1 \\ 0&1&0&1&1 \\ 1&1&1&1&1 \\ 0&1&1&1&2 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1 \\ 0&1&1&1&1 \\ 0&1&1&1&1 \\ 0&1&1&1&2 \\ 1&1&1&2&2 \end{pmatrix}
$$

- Rank 6 and FPdim 168, one of type $[[1,1],[3,2],[6,1],[7,1],[8,1]]$, given by $PSL(2,7)$:

$$
\begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 1&0&0&0&0&1 \\ 0&0&1&0&1&1 \\ 0&0&0&1&1&1 \\ 0&1&0&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&1&0&0&0 \\ 1&0&0&0&0&1 \\ 0&1&0&1&0&0 \\ 0&1&0&0&1&1 \\ 0&0&0&1&1&1 \\ 0&0&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&1&0&1&1 \\ 0&1&0&0&1&1 \\ 1&0&0&2&1&2 \\ 0&1&1&1&2&2 \\ 0&1&1&2&2&2 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&1&1&1&2&2 \\ 1&1&1&2&2&2 \\ 0&1&1&2&2&3 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&1&1&1 \\ 0&1&1&2&2&2 \\ 0&1&1&2&2&3 \\ 1&1&1&2&3&3 \end{pmatrix}
$$

- Rank 7 and FPdim 210, two of type $[[1,1],[5,3],[6,1],[7,2]]$:

$$
\begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0&0&0 \\ 1&1&0&1&0&1&0 \\ 0&0&1&0&1&1&1 \\ 0&1&0&0&1&1&1 \\ 0&0&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&1&0&0&0&0 \\ 0&0&1&0&1&1&1 \\ 1&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0&0&0 \\ 0&1&0&0&1&1&1 \\ 0&0&0&1&1&1&1 \\ 1&0&1&0&0&1&1 \\ 0&1&1&0&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1&0&0 \\ 0&0&1&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&0&1&1&1 \\ 1&1&1&1&1&1&1 \\ 0&1&1&1&1&2&1 \\ 0&1&1&1&1&2&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&1&0 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&2&1 \\ 1&1&1&1&2&0&3 \\ 0&1&1&1&1&3&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&0&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&2 \\ 0&1&1&1&1&3&1 \\ 1&1&1&1&2&1&2 \end{pmatrix}
$$

The one satisfying Schur product property (on the dual):

$$
\begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0&0&0 \\ 1&1&0&1&0&1&0 \\ 0&0&1&0&1&1&1 \\ 0&1&0&0&1&1&1 \\ 0&0&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&1&0&0&0&0 \\ 0&0&1&0&1&1&1 \\ 1&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0&0&0 \\ 0&1&0&0&1&1&1 \\ 0&0&0&1&1&1&1 \\ 1&0&1&0&0&1&1 \\ 0&1&1&0&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1&0&0 \\ 0&0&1&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&0&1&1&1 \\ 1&1&1&1&1&1&1 \\ 0&1&1&1&1&2&1 \\ 0&1&1&1&1&2&1 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&1&0 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&2&1 \\ 1&1&1&1&2&1&2 \\ 0&1&1&1&1&2&2 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&0&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&2 \\ 0&1&1&1&1&2&2 \\ 1&1&1&1&2&2&1 \end{pmatrix}
$$

- Rank 7 and FPdim 360, two of type $[[1,1],[5,2],[8,2],[9,1],[10,1]]$:

$$
\begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix},
\begin{pmatrix} 0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&1&1&1&1&1 \\ 0&0&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&0&1&1&1&2 \end{pmatrix},
\begin{pmatrix} 0&0&1&0&0&0&0 \\ 0&0&0&0&1&1&0 \\ 1&0&1&0&0&1&1 \\ 0&0&1&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&0&1&1&1&1&2 \end{pmatrix},
\begin{pmatrix} 0&0&0&1&0&0&0 \\ 0&0&1&1&1&1&1 \\ 0&0&1&1&1&1&1 \\ 1&1&1&2&1&2&2 \\ 0&1&1&2&0&2&2 \\ 0&1&1&2&2&2&2 \\ 0&1&1&2&2&2&2 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&1&0&0 \\ 0&0&1&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&2&0&2&2 \\ 1&1&1&0&3&1&2 \\ 0&1&2&1&2&2&2 \\ 0&1&1&2&2&2&2 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&1&0 \\ 0&1&1&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&2&1&2&2 \\ 0&1&2&1&2&2&2 \\ 1&1&2&2&2&2&2 \\ 0&1&1&2&2&2&3 \end{pmatrix},
\begin{pmatrix} 0&0&0&0&0&0&1 \\ 0&1&0&1&1&1&2 \\ 0&0&1&1&2&1&2 \\ 0&1&2&2&2&2&2 \\ 0&1&1&2&2&2&2 \\ 0&1&1&2&2&2&3 \\ 1&2&2&2&2&3&2 \end{pmatrix}
$$

The one given by the finite simple group PSL(2, 9):

```
1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1 0 0    0 0 0 0 0 1 0    0 0 0 0 0 0 1
0 1 0 0 0 0 0    1 1 0 0 0 1 1    0 0 0 1 1 1 0    0 0 1 1 1 1 1    0 0 1 1 1 1 1    0 1 1 1 1 1 1    0 1 0 1 1 1 2
0 0 1 0 0 0 0    0 0 0 1 1 1 0    1 0 1 0 0 1 1    0 1 0 1 1 1 1    0 1 0 1 1 1 1    0 1 1 1 1 1 1    0 0 1 1 1 1 2
0 0 0 1 0 0 0  , 0 0 1 1 1 1 1 , 0 1 0 1 1 1 1 , 1 1 1 2 1 1 2 , 0 1 1 1 1 2 2 , 0 1 1 1 2 2 2 , 0 1 1 2 2 2 2
0 0 0 0 1 0 0    0 0 1 1 1 1 1    0 1 0 1 1 1 1    0 1 1 1 1 2 2    1 1 1 1 2 1 2    0 1 1 2 1 2 2    0 1 1 2 2 2 2
0 0 0 0 0 1 0    0 1 1 1 1 1 1    0 1 1 1 1 1 1    0 1 1 2 2 2 2    0 1 1 2 1 2 2    1 1 1 2 2 2 2    0 1 1 2 2 2 3
0 0 0 0 0 0 1    0 1 0 1 1 1 2    0 0 1 1 1 1 2    0 1 1 2 2 2 2    0 1 1 2 2 2 2    0 1 1 2 2 2 3    1 2 2 2 2 3 2
```

• Rank 7 and FPdim 7980, four of type [[1, 1], [19, 1], [20, 1], [21, 1], [42, 2], [57, 1]]:

```
1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1   0  0    0 0 0 0 0   1  0    0 0 0 0 0   0  1
0 1 0 0 0 0 0    1 0 0 1 2 2 3    0 0 1 1 2 2 3    0 1 1 1 2 2 3    0 2 2 2 4   4  6    0 2 2 2 4   4  6    0 3 3 3 6   6  7
0 0 1 0 0 0 0    0 0 1 1 2 2 3    1 1 1 1 2 2 3    0 1 1 2 2 2 3    0 2 2 2 5   4  6    0 2 2 2 4   5  6    0 3 3 3 6   6  8
0 0 0 1 0 0 0  , 0 1 1 1 2 2 3 , 0 1 1 2 2 2 3 , 1 1 2 2 0 4 3 , 0 2 2 0 9   2  6 , , 0 2 2 4 2   7  6 , 0 3 3 3 6   6  9
0 0 0 0 1 0 0    0 2 2 2 4 4 6    0 2 2 2 5 4 6    0 2 2 0 9 2 6    1 4 5 9 2  15 12    0 4 4 2 15  6 12    0 6 6 6 12 12 18
0 0 0 0 0 1 0    0 2 2 2 4 4 6    0 2 2 2 4 5 6    0 2 2 4 2 7 6    0 4 4 2 15  6 12    1 4 5 7 6  12 12    0 6 6 6 12 12 18
0 0 0 0 0 0 1    0 3 3 3 6 6 7    0 3 3 3 6 6 8    0 3 3 3 6 6 9    0 6 6 6 12 12 18    0 6 6 6 12 12 18    1 7 8 9 18 18 22

1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1   0  0    0 0 0 0 0   1  0    0 0 0 0 0   0  1
0 1 0 0 0 0 0    1 0 0 1 2 2 3    0 0 1 1 2 2 3    0 1 1 1 2 2 3    0 2 2 2 4   4  6    0 2 2 2 4   4  6    0 3 3 3 6   6  7
0 0 1 0 0 0 0    0 0 1 1 2 2 3    1 1 1 1 2 2 3    0 1 1 2 2 2 3    0 2 2 2 5   4  6    0 2 2 2 4   5  6    0 3 3 3 6   6  8
0 0 0 1 0 0 0  , 0 1 1 1 2 2 3 , 0 1 1 2 2 2 3 , 1 1 2 2 0 4 3 , 0 2 2 0 7   4  6 , , 0 2 2 4 4   5  6 , 0 3 3 3 6   6  9
0 0 0 0 1 0 0    0 2 2 2 4 4 6    0 2 2 2 5 4 6    0 2 2 0 7 4 6    1 4 5 7 7  11 12    0 4 4 4 11  9 12    0 6 6 6 12 12 18
0 0 0 0 0 1 0    0 2 2 2 4 4 6    0 2 2 2 4 5 6    0 2 2 4 4 5 6    0 4 4 4 11  9 12    1 4 5 3 8  12 12    0 6 6 6 12 12 18
0 0 0 0 0 0 1    0 3 3 3 6 6 7    0 3 3 3 6 6 8    0 3 3 3 6 6 9    0 6 6 6 12 12 18    0 6 6 6 12 12 18    1 7 8 9 18 18 22

1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1   0  0    0 0 0 0 0   1  0    0 0 0 0 0   0  1
0 1 0 0 0 0 0    1 0 0 1 2 2 3    0 0 1 1 2 2 3    0 1 1 1 2 2 3    0 2 2 2 4   4  6    0 2 2 2 4   4  6    0 3 3 3 6   6  7
0 0 1 0 0 0 0    0 0 1 1 2 2 3    1 1 1 1 2 2 3    0 1 1 2 2 2 3    0 2 2 2 5   4  6    0 2 2 2 4   5  6    0 3 3 3 6   6  8
0 0 0 1 0 0 0  , 0 1 1 1 2 2 3 , 0 1 1 2 2 2 3 , 1 1 2 2 0 4 3 , 0 2 2 0 5   6  6 , , 0 2 2 4 6   3  6 , 0 3 3 3 6   6  9
0 0 0 0 1 0 0    0 2 2 2 4 4 6    0 2 2 2 5 4 6    0 2 2 0 5 6 6    1 4 5 5 8  11 12    0 4 4 6 11  8 12    0 6 6 6 12 12 18
0 0 0 0 0 1 0    0 2 2 2 4 4 6    0 2 2 2 4 5 6    0 2 2 4 6 3 6    0 4 4 6 11  8 12    1 4 5 3 8  12 12    0 6 6 6 12 12 18
0 0 0 0 0 0 1    0 3 3 3 6 6 7    0 3 3 3 6 6 8    0 3 3 3 6 6 9    0 6 6 6 12 12 18    0 6 6 6 12 12 18    1 7 8 9 18 18 22

1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1   0  0    0 0 0 0 0   1  0    0 0 0 0 0   0  1
0 1 0 0 0 0 0    1 0 0 1 2 2 3    0 0 1 1 2 2 3    0 1 1 1 2 2 3    0 2 2 2 4   4  6    0 2 2 2 4   4  6    0 3 3 3 6   6  7
0 0 1 0 0 0 0    0 0 1 1 2 2 3    1 1 1 1 2 2 3    0 1 1 2 2 2 3    0 2 2 2 5   4  6    0 2 2 2 4   5  6    0 3 3 3 6   6  8
0 0 0 1 0 0 0  , 0 1 1 1 2 2 3 , 0 1 1 2 2 2 3 , 1 1 2 2 0 4 3 , 0 2 2 0 3   8  6 , , 0 2 2 4 8   1  6 , 0 3 3 3 6   6  9
0 0 0 0 1 0 0    0 2 2 2 4 4 6    0 2 2 2 5 4 6    0 2 2 0 3 8 6    1 4 5 3 5  15 12    0 4 4 8 15  3 12    0 6 6 6 12 12 18
0 0 0 0 0 1 0    0 2 2 2 4 4 6    0 2 2 2 4 5 6    0 2 2 4 8 1 6    0 4 4 8 15  3 12    1 4 5 1 3  18 12    0 6 6 6 12 12 18
0 0 0 0 0 0 1    0 3 3 3 6 6 7    0 3 3 3 6 6 8    0 3 3 3 6 6 9    0 6 6 6 12 12 18    0 6 6 6 12 12 18    1 7 8 9 18 18 22
```

• Rank 8 and FPdim 660, fifteen of type [[1, 1], [5, 2], [10, 2], [11, 1], [12, 2]]:

```
1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 0 1 1 1 1    0 0 1 1 0 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 0 1 1 1 1    0 1 0 1 0 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 0 1 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 0 0 2 2 1 2 2    0 1 1 2 0 2 2 2    0 1 1 1 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 0 1 0 0 0  , 0 0 1 1 0 1 1 1 , 0 1 0 1 0 1 1 1 , 1 0 0 2 2 1 2 2 , 1 0 0 0 4 1 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 2
0 0 0 0 0 1 0 0    0 0 1 1 0 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 3 0 5    0 1 1 2 2 2 5 1
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 0 5    0 1 1 2 2 2 5 1
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 5 1    1 1 1 2 2 3 1 4

1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 0 1 1 1 1    0 0 1 1 0 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 0 1 1 1 1    0 1 0 1 0 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 1 0 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 0 0 2 2 1 2 2    0 1 1 2 0 2 2 2    0 1 1 1 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 1 0 0 0 0  , 0 0 1 1 0 1 1 1 , 0 1 0 1 0 1 1 1 , 0 1 1 2 0 2 2 2 , 1 0 0 0 4 1 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 2
0 0 0 0 1 0 0 0    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 3 1 4    0 1 1 2 2 2 4 2
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 1 4    0 1 1 2 2 2 4 2
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 4 2    1 1 1 2 2 3 2 3

1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 0 1 1 1 1    0 0 1 1 0 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 0 1 1 1 1    0 1 0 1 0 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 1 0 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 0 0 2 2 1 2 2    0 1 1 2 0 2 2 2    0 1 1 1 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 1 0 0 0 0  , 0 0 1 1 0 1 1 1 , 0 1 0 1 0 1 1 1 , 0 1 1 2 0 2 2 2 , 1 0 0 0 4 1 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 3
0 0 0 0 1 0 0 0    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 3 2 3    0 1 1 2 2 2 2 3
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 3 3    0 1 1 2 2 2 3 3
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 3 3    1 1 1 2 2 3 3 2

1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 0 1 1 1 1    0 0 1 1 0 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 0 1 1 1 1    0 1 0 1 0 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 1 0 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 0 0 3 1 1 2 2    0 1 1 1 1 2 2 2    0 1 1 1 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 1 0 0 0 0  , 0 0 1 1 0 1 1 1 , 0 1 0 1 0 1 1 1 , 0 1 1 1 2 2 2 2 , 1 0 0 3 1 3 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 3
0 0 0 0 1 0 0 0    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 2 3 2    0 1 1 2 2 2 2 3
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 1 4    0 1 1 2 2 2 5 1
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 5 1    1 1 1 2 2 3 1 4

1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 0 1 1 1 1    0 0 1 1 0 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 0 1 1 1 1    0 1 0 1 0 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 1 0 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 0 0 3 1 1 2 2    0 1 1 1 1 2 2 2    0 1 1 1 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 1 0 0 0 0  , 0 0 1 1 0 1 1 1 , 0 1 0 1 0 1 1 1 , 0 1 1 1 2 2 2 2 , 1 0 0 3 1 3 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 3
0 0 0 0 1 0 0 0    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 3 1 4    0 1 1 2 2 2 4 2
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 1 4    0 1 1 2 2 2 4 2
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 4 2    1 1 1 2 2 3 2 3

1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 1 0 1 1 1    0 0 1 0 1 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 1 0 1 1 1    0 1 0 0 1 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 1 0 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 1 0 3 0 0 2 2    0 0 0 3 0 2 2    0 1 1 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 1 0 0 0 0  , 0 0 1 0 1 1 1 1 , 0 1 0 0 1 1 1 1 , 0 0 0 3 0 2 2 2 , 1 1 1 0 3 1 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 2
0 0 0 0 1 0 0 0    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 3 0 5    0 1 1 2 2 2 5 1
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 0 5    0 1 1 2 2 2 5 1
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 5 1    1 1 1 2 2 3 1 4

1 0 0 0 0 0 0 0    0 1 0 0 0 0 0 0    0 0 1 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 1 0 0 0    0 0 0 0 0 1 0 0    0 0 0 0 0 0 1 0    0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0    0 0 1 1 1 0 0 0    1 0 0 0 0 0 1 1    0 0 1 1 0 1 1 1    0 0 1 0 1 1 1 1    0 0 0 1 1 1 1 1    0 1 0 1 1 1 1 1    0 1 0 1 1 1 1 1
0 0 1 0 0 0 0 0    1 0 0 0 0 0 1 1    0 1 0 1 1 0 0 0    0 1 0 1 0 1 1 1    0 1 0 0 1 1 1 1    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 0 1 1 1 1 1 1
0 0 1 0 0 0 0 0    0 0 1 0 1 1 1 1    0 1 0 0 1 1 1 1    1 1 0 3 0 0 2 2    0 0 0 3 0 2 2    0 1 1 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2
0 0 0 1 0 0 0 0  , 0 0 1 0 1 1 1 1 , 0 1 0 0 1 1 1 1 , 0 0 0 3 0 2 2 2 , 1 1 1 0 3 1 2 2 , 0 1 1 2 1 2 2 2 , 0 1 1 2 2 2 2 2 , 0 1 1 2 2 2 2 2
0 0 0 0 1 0 0 0    0 0 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 1 2 2 2    1 1 1 2 2 2 3 2    0 1 1 2 2 3 1 4    0 1 1 2 2 2 4 2
0 0 0 0 0 0 1 0    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 3 2    1 1 1 2 2 3 1 4    0 1 1 2 2 2 4 2
0 0 0 0 0 0 0 1    0 1 0 1 1 1 1 1    0 0 1 1 1 1 1 1    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 2    0 1 1 2 2 2 2 3    0 1 1 2 2 4 2    1 1 1 2 2 3 2 3
```

```
10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00110111  00101111  00011111  01011111  01011111
00100000  10000011  01011000  01010111  01001111  00011111  00111111  00111111
00010000  00110111  01010111  11103122  00030222  00021222  01122222  01122222
00001000, 00101111, 00030222, 00030222, 11103122, 01121222, 01122222, 01122222
00000100  00011111  00011111  01122222  01121222  11122222  01122232  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11122323  01122323
00000001  01011111  00111111  01122222  01122222  01122223  01122233  11122332

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00110111  00101111  00011111  01011111  01011111
00100000  10000011  01011000  01010111  01001111  00011111  00111111  00111111
00010000  00110111  01010111  11111222  00021222  01112222  01122222  01122222
00001000, 00101111, 01001111, 00021222, 11112122, 01121222, 01122222, 01122222
00000100  00011111  00011111  01122222  01122222  11122222  01122232  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11122305  01122251
00000001  01011111  00111111  01122222  01122222  01122223  01122251  11122314

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00110111  00101111  00011111  01011111  01011111
00100000  10000011  01011000  01010111  01001111  00011111  00111111  00111111
00010000  00110111  01010111  11111222  00021222  01112222  01122222  01122222
00001000, 00101111, 01001111, 00021222, 11112122, 01121222, 01122232, 01122223
00000100  00011111  00011111  01122222  01121222  11122222  01122232  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11123314  01122242
00000001  01011111  00111111  01122222  01122222  01122223  01122242  11122323

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00101111  00110111  00011111  01011111  01011111
00100000  10000011  01011000  01001111  01010111  00011111  00111111  00111111
00010000  00101111  01001111  01120222  10022122  01112222  01122222  01122222
00001000, 00110111, 01010111, 10022122, 01102222, 01121222, 01122233, 01122223
00000100  00011111  00011111  01122222  01121222  11122222  01122235  01122251
00000010  01011111  00111111  01122222  01122222  01122232  11123305  01122251
00000001  01011111  00111111  01122222  01122222  01122223  01122251  11122314

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00101111  00110111  00011111  01011111  01011111
00100000  10000011  01011000  01001111  01010111  00011111  00111111  00111111
00010000  00101111  01001111  01120222  10022122  01112222  01122222  01122222
00001000, 00110111, 01010111, 10022122, 01102222, 01121222, 01122232, 01122223
00000100  00011111  00011111  01122222  01121222  11122222  01122314  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11123314  01122242
00000001  01011111  00111111  01122222  01122222  01122223  01122242  11122323

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00101111  00110111  00011111  01011111  01011111
00100000  10000011  01011000  01001111  01010111  00011111  00111111  00111111
00010000  00101111  01001111  01120222  10022122  01112222  01122222  01122222
00001000, 00110111, 01010111, 10022122, 01102222, 01121222, 01122232, 01122223
00000100  00011111  00011111  01122222  01121222  11122222  01122314  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11123323  01122233
00000001  01011111  00111111  01122222  01122222  01122223  01122233  11122332
```

The one given by the finite simple group PSL(2, 11):

```
10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00110111  00101111  00011111  01011111  01011111
00100000  10000011  01011000  01010111  01001111  00011111  00111111  00111111
00010000  00110111  01010111  11112122  00021222  01112222  01122222  01122222
00001000, 00101111, 01001111, 00021222, 11112122, 01121222, 01122222, 01122222
00000100  00011111  00011111  01112222  01121222  11122222  01122232  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11122323  01122233
00000001  01011111  00111111  01122222  01122222  01122223  01122233  11122332
```

The one (non group-like) satisfying Schur product property (on the dual):

```
10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00111000  10000011  00101111  00110111  00011111  01011111  01011111
00100000  10000011  01011000  01001111  01010111  00011111  00111111  00111111
00010000  00101111  01001111  10033122  00033122  01112222  01122222  01122222
00001000, 00110111, 01010111, 01111222, 10013122, 01121222, 01122222, 01122222
00000100  00011111  00011111  01112222  01122222  11122222  01122232  01122223
00000010  01011111  00111111  01122222  01122222  01122232  11123323  01122233
00000001  01011111  00111111  01122222  01122222  01122223  01122233  11122332
```

- Rank 8 and FPdim 990, five of type [[1, 1], [9, 1], [10, 1], [11, 4], [18, 1]]:

```
10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  14011110  11111112  01111112  01111112  01111112  01111112  00222223
00100000  00111112  11111112  01121112  01112112  01111112  01111112  02222223
00010000  11111112  01121112  11200232  11202222  11221112  11132102  02222224
00001000, 01111112, 01111212, 11020222, 11220122, 01121112, 01111222, 02222224
00000100  01111112  01111212  01122112  01121212  11212022  01111222  02222224
00000010  01111112  01111112  01132102  01122112  01122112  11201222  02222224
00000001  00222223  02222223  02222224  02222224  02222224  02222224  13344445

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  14011110  00111112  01111112  01111112  01111112  01111112  00222223
00100000  00111112  11111112  01121112  01112112  01111212  01111112  02222223
00010000  11111112  01121112  11211122  01112112  01111222  01121112  02222224
00001000, 01111112, 01111212, 01121212, 11221112, 11121122, 01111222, 02222224
00000100  01111112  01111122  01112112  01121212  11211212  01122112  02222224
00000010  01111112  01111122  01121212  01111222  01122112  11212112  02222224
00000001  00222223  02222223  02222224  02222224  02222224  02222224  13344445

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  14011110  00111112  01111112  01111112  01111112  01111112  00222223
00100000  00111112  11111112  01121112  01112112  01111212  01111112  02222224
00010000  11111112  01121112  11211122  01112122  01112112  01111212  02222224
00001000, 01111112, 01111212, 01111212, 11211122, 01112112, 01121112, 02222224
00000100  01111112  01111112  01111122  11220022  11210312  01122112  02222224
00000010  01111112  01111122  01112202  01122112  01122112  11201132  02222224
00000001  00222223  02222223  02222224  02222224  02222224  02222224  13344445

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  14011110  00111112  01111112  01111112  01111112  01111112  00222223
00100000  00111112  11111112  01121112  01112112  01111212  01111112  02222223
00010000  01111112  01121112  11211122  01112122  01112112  01122202  02222224
00001000, 01111112, 01112112, 01111222, 11221112, 01121122, 01121212, 02222224
00000100  01111112  01111212  01111212  01121212  11212112  01121212  02222224
00000010  01111112  01111122  01122202  01121212  01122112  11201132  02222224
00000001  00222223  02222223  02222224  02222224  02222224  02222224  13344445

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  14011110  00111112  01111112  01111112  01111112  01111112  00222223
00100000  00111112  11111112  01121112  01112112  01111212  01111112  02222223
00010000  01111112  01121112  11211122  01112122  01112112  01122202  02222224
00001000, 01111112, 01112112, 01111222, 11211122, 01121122, 01121212, 02222224
00000100  01111112  01111212  01111222  11210212  11221112  01121212  02222224
00000010  01111112  01111122  01112212  01121212  01122112  11211122  02222224
00000001  00222223  02222223  02222224  02222224  02222224  02222224  13344445
```

- Rank 8 and FPdim 1260, two of type $[[1,1],[6,1],[7,2],[10,1],[15,1],[20,2]]$:

```
10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  10000101  00100101  00010101  00000021  01110202  00002032  01111221
00100000  00100101  11010101  00110101  00001021  01110302  00002042  01111222
00010000  00010101  00110101  11100101  00001021  01110302  00002042  01111222
00001000  00000021  00001021  00001021  10112301  00003042  02220414  02222445
00000100  01110202  01110302  01110302  00003042  12330604  00004094  02222445
00000010  00002032  00002042  00002042  02220414  00004094  13441927  02224476
00000001  01111221  01111222  01111222  01111243  02224445  02224476  11223567

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  10000101  00100101  00010101  00000021  01110202  00002023  01111230
00100000  00100101  11010101  00110101  00001021  01110302  00002033  01111231
00010000  00010101  00110101  11100101  00001021  01110302  00002033  01111231
00001000  00000021  00001021  00001021  10112301  00003042  02220423  01111234
00000100  01110202  01110302  01110302  00003042  12330604  00004076  02222463
00000010  00002023  00002033  00002033  02220423  00004076  12332729  03333692
00000001  01111230  01111231  01111231  01111234  02222463  03333692  10114 3 2 13
```

- Rank 8 and FPdim 1320, two of type $[[1,1],[6,2],[10,1],[11,1],[15,2],[24,1]]$:

```
10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00011010  10001001  00100111  01100111  00111102  00011211  01011123
00100000  10001001  00011100  01000111  01100111  00111121  01011012  00111213
00010000  00100111  00011100  10011112  00012112  01111123  01111213  01122334
00001000  01100111  01000111  00012112  10022112  01111223  01111223  01122335
00000100  00111102  00111121  01111123  01111223  11112225  00222324  02133456
00000010  00011211  01011012  01111213  01111223  11112225  00222324  02133456
00000001  01011123  00111213  01122334  01122335  01233546  02133456  13345 6 6 11

10000000  01000000  00100000  00010000  00001000  00000100  00000010  00000001
01000000  00011100  10001001  00010000  00001000  00011121  00111012  01011213
00100000  10001001  00011100  01000111  01100111  00011121  01011012  00111123
00010000  00100111  00011100  10011112  00012112  01111123  01111213  01122334
00001000  01100111  01000111  00012112  10022112  01111223  01111223  01122335
00000100  00011121  00011121  01111123  01111223  11112225  02022324  01233456
00000010  00111012  01011012  01111213  01111223  02022324  11112225  01233456
00000001  01011213  00111123  01122334  01122335  01233546  01233456  13345 6 6 11
```

A.2. Examples of simple integral fusion rings not of Frobenius type

There is a simple integral fusion ring of rank 6, FPdim 143 and type $[[1,1],[4,2],[5,1],[6,1],[7,1]]$ which is not of Frobenius type. Here are its fusion matrices:

$$
\begin{array}{llllll}
\begin{smallmatrix}
1&0&0&0&0&0\\
0&1&0&0&0&0\\
0&0&1&0&0&0\\
0&0&0&1&0&0\\
0&0&0&0&1&0\\
0&0&0&0&0&1
\end{smallmatrix},&
\begin{smallmatrix}
0&1&0&0&0&0\\
1&0&1&1&1&0\\
0&1&0&1&0&1\\
0&1&1&1&0&1\\
0&1&0&0&1&2\\
0&0&1&1&2&1
\end{smallmatrix},&
\begin{smallmatrix}
0&0&1&0&0&0\\
0&1&0&1&0&1\\
1&0&2&0&0&1\\
0&1&0&2&1&0\\
0&0&0&1&2&1\\
0&1&1&0&1&2
\end{smallmatrix},&
\begin{smallmatrix}
0&0&0&1&0&0\\
0&1&1&1&0&1\\
0&1&0&2&1&0\\
1&1&2&1&0&1\\
0&0&1&0&2&2\\
0&1&0&1&2&2
\end{smallmatrix},&
\begin{smallmatrix}
0&0&0&0&1&0\\
0&1&0&0&1&2\\
0&0&0&1&2&1\\
0&0&1&0&2&2\\
1&1&2&2&1&1\\
0&2&1&2&1&2
\end{smallmatrix},&
\begin{smallmatrix}
0&0&0&0&0&1\\
0&0&1&1&2&1\\
0&1&1&0&1&2\\
0&1&0&1&2&2\\
0&2&1&2&1&2\\
1&1&2&2&2&2
\end{smallmatrix}
\end{array}
$$

Note that $143 = 11\cdot 13$, so it admits no categorification because by [7], any fusion category of Frobenius-Perron dimension pq (with p, q different odd primes) is group-theoretical, whereas by [10], a (weakly) group-theoretical fusion category is of Frobenius type (or alternatively, cannot be both simple and non group-like).

There are 21 simple integral fusion rings not of Frobenius type, of rank ≤ 7 and FPdim ≤ 1500 (with FPdim $\neq p^a q^b, pqr$, by [10]), and exactly 4 ones (below) pass the Schur product criterion:

- rank 6, FPdim $924 = 2^2 \cdot 3 \cdot 7 \cdot 11$, type $[[1,1],[7,1],[8,1],[12,1],[15,1],[21,1]]$ and fusion matrices:

$$
\begin{array}{llllll}
\begin{smallmatrix}
1&0&0&0&0&0\\
0&1&0&0&0&0\\
0&0&1&0&0&0\\
0&0&0&1&0&0\\
0&0&0&0&1&0\\
0&0&0&0&0&1
\end{smallmatrix},&
\begin{smallmatrix}
0&1&0&0&0&0\\
1&0&0&1&1&1\\
0&0&1&1&1&1\\
0&1&1&1&1&2\\
0&1&1&1&1&3\\
0&1&1&2&3&3
\end{smallmatrix},&
\begin{smallmatrix}
0&0&1&0&0&0\\
0&0&1&1&1&1\\
1&1&1&1&1&1\\
0&1&1&2&1&2\\
0&1&1&1&2&3\\
0&1&1&2&3&4
\end{smallmatrix},&
\begin{smallmatrix}
0&0&0&1&0&0\\
0&1&1&1&1&2\\
0&1&1&2&1&2\\
1&1&2&1&3&3\\
0&1&1&3&3&4\\
0&2&2&3&4&6
\end{smallmatrix},&
\begin{smallmatrix}
0&0&0&0&1&0\\
0&1&1&1&1&3\\
0&1&1&1&2&3\\
0&1&1&3&3&4\\
1&1&2&3&4&5\\
0&3&3&4&5&7
\end{smallmatrix},&
\begin{smallmatrix}
0&0&0&0&0&1\\
0&1&1&2&3&3\\
0&1&1&2&3&4\\
0&2&2&3&4&6\\
0&3&3&4&5&7\\
1&3&4&6&7&10
\end{smallmatrix}
\end{array}
$$

- rank 6, FPdim $1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$, type $[[1,1],[9,1],[10,1],[11,1],[21,1],[24,1]]$ and fusion matrices:

```
1 0 0 0 0 0    0 1 0 0 0 0    0 0 1 0 0 0    0 0 0 1 0 0    0 0 0 0 1 0    0 0 0 0 0 1
0 1 0 0 0 0    1 0 0 1 1 2    0 0 1 1 1 2    0 1 1 1 1 2    0 1 1 1 1 2    0 2 2 2 4 3
0 0 1 0 0 0    0 0 1 1 1 2    1 1 0 0 2 2    0 1 0 1 2 2    0 1 2 2 3 4    0 2 2 2 4 4
0 0 0 1 0 0 ,  0 1 1 1 1 2 ,  0 1 0 1 2 2 ,  1 1 1 1 2 2 ,  0 1 2 2 4 4 ,  0 2 2 2 4 5
0 0 0 0 1 0    0 1 1 1 3 4    0 1 2 2 3 4    0 1 2 2 4 4    1 3 3 4 7 8    0 4 4 4 8 9
0 0 0 0 0 1    0 2 2 2 4 3    0 2 2 2 4 4    0 2 2 2 4 5    0 4 4 4 8 9    1 3 4 5 9 11
```

- rank 7, FPdim $560 = 2^4 \cdot 5 \cdot 7$, type $[[1,1],[6,1],[7,2],[10,2],[15,1]]$ and fusion matrices:

```
1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1 0 0    0 0 0 0 0 1 0    0 0 0 0 0 0 1
0 1 0 0 0 0 0    1 0 0 0 1 1 1    0 0 1 0 1 1 1    0 0 0 1 1 1 1    0 1 1 1 0 1 2    0 1 1 1 1 0 2    0 1 1 1 2 2 2
0 0 1 0 0 0 0    0 0 1 0 1 1 1    1 1 0 1 1 1 1    0 0 1 1 1 1 1    0 1 1 1 1 1 2    0 1 1 1 1 1 2    0 1 1 1 2 2 3
0 0 0 1 0 0 0 ,  0 0 0 1 1 1 1 ,  0 0 1 1 1 1 1 ,  0 1 1 1 1 1 1 ,  0 1 1 1 1 1 2 ,  0 1 1 1 1 1 2 ,  0 1 1 1 2 2 3
0 0 0 0 1 0 0    0 1 1 1 0 1 2    0 1 1 1 1 1 2    0 1 1 1 1 1 2    0 1 1 1 2 3 2    1 0 1 1 2 2 3    0 2 2 2 3 2 4
0 0 0 0 0 1 0    0 1 1 1 1 1 2    0 1 1 1 1 1 2    0 1 1 1 1 1 2    1 0 1 1 2 2 3    0 1 1 1 3 2 2    0 2 2 2 2 3 4
0 0 0 0 0 0 1    0 1 1 1 2 2 2    0 1 1 1 2 2 3    0 1 1 1 2 2 3    0 2 2 2 3 2 4    0 2 2 2 2 3 4    1 2 3 3 4 4 6
```

- rank 7, FPdim $798 = 2 \cdot 3 \cdot 7 \cdot 19$, type $[[1,1],[7,1],[8,1],[9,3],[21,1]]$ and fusion matrices:

```
1 0 0 0 0 0 0    0 1 0 0 0 0 0    0 0 1 0 0 0 0    0 0 0 1 0 0 0    0 0 0 0 1 0 0    0 0 0 0 0 1 0    0 0 0 0 0 0 1
0 1 0 0 0 0 0    1 0 0 1 1 1 1    0 0 1 1 1 1 1    0 1 1 1 1 1 1    0 1 1 1 1 1 1    0 1 1 1 1 1 1    0 1 1 1 1 1 5
0 0 1 0 0 0 0    0 0 1 1 1 1 1    1 1 1 1 1 1 1    0 1 1 2 1 1 1    0 1 1 1 2 1 1    0 1 1 1 1 2 1    0 1 1 1 1 1 6
0 0 0 1 0 0 0 ,  0 1 1 1 2 1 1 ,  0 1 1 2 1 1 1 ,  1 1 2 1 1 1 1 ,  0 1 1 1 2 2 1 ,  0 1 1 2 1 1 1 ,  0 1 1 1 1 1 7
0 0 0 0 1 0 0    0 1 1 1 1 1 1    0 1 1 1 2 1 1    0 1 1 1 2 2 1    1 1 2 2 1 1 1    0 1 1 2 1 2 1    0 1 1 1 1 1 7
0 0 0 0 0 1 0    0 1 1 1 1 1 1    0 1 1 1 1 2 1    0 1 1 2 2 1 1    0 1 1 2 1 2 1    1 1 2 1 2 1 1    0 1 1 1 1 1 7
0 0 0 0 0 0 1    0 1 1 1 1 1 5    0 1 1 1 1 1 6    0 1 1 1 1 1 7    0 1 1 1 1 1 7    0 1 1 1 1 1 7    1 5 6 7 7 7 8
```

A.3. Extra perfect fusion rings

The computer program uses several *necessary* conditions for a fusion ring $\mathfrak{A}$ to be simple, perfect, of Frobenius type, and categorifiable, mainly collected in [10,29]:

- FPdim($\mathfrak{A}$) not of the form $p^a q^b$ or pqr, with p, q, r prime,
- $d(x_2) \geq 3$ (in particular, $m_1 = 1$, i.e. $\mathfrak{A}$ is perfect).
- $s \geq 3$ (in particular rank $r \geq 3$),
- $n_{r+1} < n_r^2$ for all $r > 1$ (otherwise it trivially cannot be simple (and perfect)),
- (Frobenius type) n_i divides FPdim($\mathfrak{A}$) for all i (idem for the fusion subrings[5]),
- $gcd(n_2, \ldots, n_s) = 1$ (consequence of Frobenius type).

Now, all these necessary conditions together are not sufficient for having a simple fusion ring, so that the computer search provided also 319 new perfect non-simple fusion rings (4 of which being noncommutative), see Table 1. Each of them contains the Grothendieck ring of Rep(G) as a proper subring (with $G = \mathrm{PSL}(2,q)$, $q = 5, 7, 9$). None of them comes from a perfect group. Ostrik's inequality holds on all of them. In the commutative case, the Schur product property (on the dual) holds on exactly 114 ones.

Let us mention finally that we also found extra 2242 perfect integral fusion rings out the bounds of the above table (among them, 7 are simple, 5 are noncommutative, none both), see Table 2. In the commutative case, the Schur product property (on the dual) holds on exactly 480 of them (none of which is simple), and Ostrik's inequality holds on all of them.

---

[5] There are fusion rings of rank 9, FPdim 4620 and type $[[1,1],[4,2],[5,1],[6,1],[7,1],[11,1],[66,1]]$ (Frobenius type) with a simple fusion subring of rank 8, FPdim 264 and type $[[1,1],[4,2],[5,1],[6,1],[7,1],[11,1]]$ (not Frobenius type).

#	rank	FPdim	type	#Schur	note
1	7	7224	[[1,1],[3,2],[6,1],[7,1],[8,1],[84,1]]	1	$84 > 8^2$
16	8	360	[[1,1],[3,2],[4,1],[5,1],[10,3]]	6	
26	8	660	[[1,1],[3,2],[4,1],[5,1],[10,2],[20,1]]	5	
24	8	960	[[1,1],[3,2],[4,1],[5,1],[10,1],[20,2]]	14	
47	8	1260	[[1,1],[3,2],[4,1],[5,1],[20,3]]	7	
1	8	1440	[[1,1],[3,2],[4,1],[5,1],[10,1],[16,1],[32,1]]	1	
1	8	1680	[[1,1],[3,2],[4,1],[5,1],[12,1],[24,1],[30,1]]	1	
2	8	2160	[[1,1],[3,2],[4,1],[5,1],[10,1],[20,1],[40,1]]	2	
1	8	3120	[[1,1],[3,2],[4,1],[5,1],[10,1],[16,1],[52,1]]]	1	
60	8	3360	[[1,1],[3,2],[4,1],[5,1],[10,1],[40,2]]	32	
120	8	3696	[[1,1],[3,2],[6,1],[7,1],[8,1],[42,2]]	40	
2	8	3960	[[1,1],[5,2],[8,2],[9,1],[10,1],[60,1]]	1	
7	9	360	[[1,1],[3,2],[4,1],[5,4],[15,1]]	2	
11	9	420	[[1,1],[3,2],[4,1],[5,1],[6,2],[12,2]]	1	+4NC

Table 1
Perfect extra ones.

#	rank	FPdim	type	#Schur	note
1	7	28392	[[1,1],[3,2],[6,1],[7,1],[8,1],[168,1]]	1	$168 > 8^2$
4	8	4620	[[1,1],[4,2],[5,1],[6,1],[7,1],[11,1],[66,1]]	0	
1	8	5460	[[1,1],[3,2],[4,1],[5,1],[10,1],[20,1],[70,1]]	1	
8	8	5460	[[1,1],[6,1],[7,2],[10,2],[15,1],[70,1]]	1	
1	8	6960	[[1,1],[3,2],[4,1],[5,1],[10,1],[20,1],[80,1]]	1	
5	8	8160	[[1,1],[3,2],[4,1],[5,1],[10,1],[40,1],[80,1]]	5	
250	8	14280	[[1,1],[3,2],[6,1],[7,1],[8,1],[84,2]]	78	$84 > 8^2$
2	8	44310	[[1,1],[5,3],[6,1],[7,2],[210,1]]	1	$210 > 7^2$
1	9	1080	[[1,1],[5,2],[8,2],[9,1],[10,1],[12,1],[24,1]]	0	
22	9	1260	[[1,1],[3,2],[4,1],[5,1],[10,3],[30,1]]	10	
1	9	1320	[[1,1],[5,2],[6,2],[10,1],[11,1],[20,1],[24,1]]	0	simple
111	9	1344	[[1,1],[3,2],[6,1],[7,1],[8,1],[14,2],[28,1]]	26	
4	9	1512	[[1,1],[6,1],[7,1],[8,4],[21,1],[27,1]]	0	simple
52	9	1560	[[1,1],[3,2],[4,1],[5,1],[10,2],[20,1],[30,1]]	9	
56	9	2160	[[1,1],[3,2],[4,1],[5,1],[20,3],[30,1]]	17	
69	9	2160	[[1,1],[5,2],[8,2],[9,1],[10,1],[30,2]]	10	
1086	9	2520	[[1,1],[3,2],[6,1],[7,1],[8,1],[28,3]]	221	
5	9	2760	[[1,1],[3,2],[4,1],[5,1],[10,2],[30,1],[40,1]]	2	
13	9	3696	[[1,1],[3,2],[6,1],[7,1],[8,1],[14,2],[56,1]]	5	
16	9	3960	[[1,1],[3,2],[4,1],[5,1],[10,3],[60,1]]	6	
52	9	4200	[[1,1],[3,2],[4,1],[5,1],[24,1],[30,2],[42,1]]	4	
29	9	4260	[[1,1],[3,2],[4,1],[5,1],[10,2],[20,1],[60,1]]	6	
24	9	4560	[[1,1],[3,2],[4,1],[5,1],[10,1],[20,2],[60,1]]	14	
404	9	4872	[[1,1],[3,2],[6,1],[7,1],[8,1],[28,2],[56,1]]	56	
2	10	720	[[1,1],[4,2],[5,2],[9,1],[10,3],[16,1]]	0	simple
1	10	1200	[[1,1],[3,2],[4,3],[5,1],[8,1],[12,1],[30,1]]	0	
7	10	1260	[[1,1],[3,2],[4,1],[5,4],[15,1],[30,1]]	2	
12	10	1320	[[1,1],[3,2],[4,1],[5,1],[6,2],[12,2],[30,1]]	3	+3NC
3	10	1920	[[1,1],[3,2],[4,1],[5,1],[8,2],[16,1],[24,1],[30,1]]	1	+2NC

Table 2
Out of the bounds.

The fusion matrices of all the (2595) perfect integral fusion rings mentioned in this paper, together with the computer programs (written in SageMath [37]) and checks, are available in the second author's webpage [34].

References

[1] D. Bisch, V. Jones, Singly generated planar algebras of small dimension, Duke Math. J. 101 (2000) 41–75.

[2] D. Bisch, V. Jones, Singly generated planar algebras of small dimension, part II, Adv. Math. 175 (2003) 297–318.

[3] D. Bisch, V. Jones, Z. Liu, Singly generated planar algebras of small dimension, part III, Trans. Am. Math. Soc. 369 (2017) 2461–2476.

[4] J. Edge, Classification of spin models for Yang-Baxter planar algebras, preprint, https://arxiv.org/pdf/1902.08984.pdf.

[5] P. Etingof, Private discussions at MIT in 2019.

[6] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor Categories, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, 2015.

[7] P. Etingof, S. Gelaki, V. Ostrik, Classification of fusion categories of dimension pq, Int. Math. Res. Not. 57 (2004) 3041–3056.

[8] P. Etingof, M. Khovanov, Representations of tensor categories and Dynkin diagrams, Int. Math. Res. Not. 5 (1995) 235–247.

[9] P. Etingof, D. Nikshych, V. Ostrik, On fusion categories, Ann. Math. 162 (2005) 581–642.

[10] P. Etingof, D. Nikshych, V. Ostrik, Weakly group-theoretical and solvable fusion categories, Adv. Math. 226 (2011) 176–205.

[11] P. Etingof, D. Nikshych, V. Ostrik, On a necessary condition for unitary categorification of fusion rings, arXiv:2102.13239, 2021.

[12] S. Janson, Roots of polynomials of degrees 3 and 4, preprint, https://arxiv.org/pdf/1009.2373.pdf, 2010.

[13] C. Jiang, Z. Liu, J. Wu, Block maps and Fourier analysis, Sci. China Math. 62 (2019) 1585–1614.

[14] C. Jiang, Z. Liu, J. Wu, Noncommutative uncertainty principles, J. Funct. Anal. 270 (2016) 264–311.

[15] V.F.R. Jones, Index for subfactors, Invent. Math. 72 (1983) 1–25.

[16] V.F.R. Jones, Planar algebras, I, preprint, https://arxiv.org/pdf/math/9909027.pdf.

[17] V.F.R. Jones, Private discussions at Vanderbilt University in 2015.

[18] G.I. Kac, Certain arithmetic properties of ring groups, Funct. Anal. Appl. 6 (1972) 158–160.

[19] I. Kaplansky, Bialgebras, Lecture Notes in Mathematics, Department of Mathematics, University of Chicago, 1975, iv+57 pp.

[20] H. Kosaki, Applications of the complex interpolation method to a von Neumann algebra: noncommutative L^p-spaces, J. Funct. Anal. 56 (1984) 29–78.

[21] Z. Liu, Exchange relation planar algebras of small rank, Trans. Am. Math. Soc. 368 (2016) 8303–8348.

[22] Z. Liu, Quon language: surface algebras and Fourier duality, Commun. Math. Phys. 366 (2019) 865–894.

[23] Z. Liu, Yang-Baxter relation planar algebras, preprint, https://arxiv.org/pdf/1507.06030.pdf.

[24] Z. Liu, S. Morrison, D. Penneys, Lifting shadings on symmetrically self-dual subfactor planar algebras, in: Topological Phases of Matter and Quantum Computation, in: Contemporary Mathematics, vol. 747, Amer. Math. Soc., Providence, RI, 2020, pp. 51–61.

[25] Z. Liu, J. Wu, Non-commutative Rényi entropic uncertainty principles, Sci. China Math. 63 (11) (2020) 2287–2298.

[26] G. Lusztig, Leading coefficients of character values of Hecke algebras, Proc. Symp. Pure Math. 47 (1987) 235–262.

[27] M. Müger, From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories, J. Pure Appl. Algebra 180 (2003) 81–157.

[28] M. Müger, From subfactors to categories and topology II. The quantum double of tensor categories and subfactors, J. Pure Appl. Algebra 180 (2003) 159–219.

[29] S. Natale, Semisolvability of Semisimple Hopf Algebras of Low Dimension, vol. 874, American Mathematical Society (AMS), Providence, RI, 2007.

[30] V. Ostrik, Fusion categories of rank 2, Math. Res. Lett. 10 (2003) 177–183.

[31] V. Ostrik, Module categories, weak Hopf algebras and modular invariants, Transform. Groups 8 (2003) 177–206.

[32] V. Ostrik, Pivotal fusion categories of rank 3, Mosc. Math. J. 15 (2015) 373–396, 405.

[33] S. Palcoux, Ore's theorem for cyclic subfactor planar algebras and beyond, Pac. J. Math. 292 (1) (2018) 203–221.

[34] S. Palcoux, https://sites.google.com/view/sebastienpalcoux/fusion-rings, 2020.
[35] D. Penneys, Private discussion at the 2019 Oberwolfach workshop subfactor and applications.
[36] Y. Ren, Universal skein theory for group actions, Adv. Math. 356 (2019) 106804.
[37] The Sage Developers, SageMath, the Sage mathematics software system, Version 9.0, sagemath.org, 2020.
[38] J.P. Serre, Topics in Galois Theory, second edition, Research Notes in Mathematics, vol. 1, 2008, xvi+120 pp.
[39] Z. Wang, Private discussions at Harvard University in 2017.