

# CLASSIFICATION OF GROTHENDIECK RINGS OF COMPLEX FUSION CATEGORIES OF MULTIPLICITY ONE UP TO RANK SIX

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ABSTRACT. This paper classifies the Grothendieck rings of complex fusion categories of multiplicity one up to rank six. Among 72 possible fusion rings, 25 ones are filtered out by using categorification criteria. Each of the remaining 47 fusion rings admits a unitary complex categorification. We found 6 new Grothendieck rings, categorized by applying a localization approach of the Pentagon Equation.

## 1. INTRODUCTION

A complex fusion category is a  $\mathbb{C}$ -linear semisimple rigid tensor category with finitely many simple objects and finite dimensional spaces of morphisms, such that the neutral object is simple [8]. The Grothendieck ring of a fusion category is a fusion ring, first introduced (and called based ring) in [22]. A (complex/unitary) Grothendieck ring is a fusion ring admitting a categorification into a (complex/unitary) fusion category. One of the main challenges of the subject is to decide which fusion rings are Grothendieck rings; some ones (for example mentioned in this paper) are not. In theory, a fusion ring is a Grothendieck ring if and only if its Pentagon Equation (PE) admit a solution, but in practice, this direct approach is not workable without specific strategies. In this paper, we use two strategies: several criteria (necessary conditions) in §2 to rule out some fusion rings directly, and a localization of the PE to categorify the remaining ones, in §4. The notion of fusion ring is purely combinatorial and easy to list, and §3 provides the list of all the fusion rings of multiplicity one up to rank six, obtained by brute-force computation [26, 33] (see also the work of an independent group [32]), there are 72 ones, and (as we will show) exactly 47 of them are complex Grothendieck rings<sup>1</sup>, all of them are unitary. The main result is the following classification (proved in Subsection 3.7):

**Theorem 1.1.** *The complex Grothendieck rings of multiplicity one up to rank six are given by the following:*

- *known fusion categories (see the references below):*
  - $\text{Vec}(G)$  with finite group  $G = C_n$  ( $n \leq 6$ ),  $C_2^2$ ,  $S_3$ ,
  - $\text{Rep}(G)$  with finite group  $G = S_3, S_4, D_n$  ( $4 \leq n \leq 7$ ),  $D_9, Q_8, C_3 \rtimes C_4, C_3 \rtimes S_3$ ,
  - near-group  $C_n + 0$ ,  $n \leq 5$  (also called Tambara-Yamagami  $TY(C_n)$ ), see [12, 34],
  - $\text{SU}(2)_n$  ( $n \leq 5$ ),  $\text{PSU}(2)_n$  ( $3 \leq n \leq 11$ ),  $\text{SO}(3)_2, \text{SO}(5)_2$ , see [1, 3, 14],
  - even part of a 1-supertransitive subfactor of index  $3 + 2\sqrt{2}$ , see [18],
  - products of two above,

where  $C_n, D_n, Q_n, S_n$  are respectively the usual notations for cyclic, dihedral, quaternion, symmetric groups. Note that  $\text{PSU}(2)_k = \text{SU}(2)_k / C_2 \simeq$  even part of  $TLJ A_{k+1}$  subfactor.

- *new fusion categories: all with non self-adjoint objects, so none modular by Theorem 2.16; all weakly integral with  $\text{FPdim} < 84$ , so all weakly group-theoretical by [9]; some come from the new zesting construction [5]. In the following table, # counts the number of Grothendieck rings:*

#	FPdim	rank	type	zesting of
3	8	6	$[1, 1, 1, 1, \sqrt{2}, \sqrt{2}]$	$\text{Vec}(C_2) \otimes \text{SU}(2)_2$
1	12	5	$[1, 1, \sqrt{3}, \sqrt{3}, 2]$	$\text{SO}(3)_2$
1	20	6	$[1, 1, 2, 2, \sqrt{5}, \sqrt{5}]$	$\text{SO}(5)_2$
1	24	5	$[1, 1, 2, 3, 3]$	

Partial classifications exist in the literature [2, 13]. Note that §4 computes some categorifications for each new ring, but does not state whether the zested ones are among them. Finally, §5 gives observations and questions.

<sup>1</sup>Below  $n_r$  and  $m_r$  are the numbers of fusion rings and complex Grothendieck rings, of multiplicity one and rank  $r$ , see also [27, 28].

$r$	1	2	3	4	5	6
$n_r$	1	2	4	10	16	39
$m_r$	1	2	4	9	10	21

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## 2. LIST OF CATEGORIFICATION CRITERIA

This section lists all the categorification criteria applied in this paper, we checked them on every fusion ring (when possible), and *a posteriori* it turns out that a strict subset of criteria cover all the exclusions of this specific classification (see Subsection 3.7), but it is still good to mention the complement subset as additional data. The first criterion holds for unitary categorification, the next one for pivotal complex categorification, the next two ones for (general) complex categorification, and the next two ones for every categorification (over every field). Finally, the last two ones are specific to the modular or quadratic case.

**2.1. Schur product criterion.** Let  $\mathcal{F}$  be a commutative fusion ring. Let  $\Lambda = (\lambda_{i,j})$  be the table coming from the simultaneous diagonalization of its fusion matrices, with  $\lambda_{i,1} = \max_j(|\lambda_{i,j}|)$ . Here is the commutative Schur product criterion [21, Corollary 8.5]:

**Theorem 2.1.** *If  $\mathcal{F}$  admits a unitary categorification then for all triples  $(j_1, j_2, j_3)$  we have*

$$\sum_i \frac{\lambda_{i,j_1} \lambda_{i,j_2} \lambda_{i,j_3}}{\lambda_{i,1}} \geq 0.$$

Note that Theorem 2.1 is the corollary of a (less tractable) noncommutative version [21, Proposition 8.3].

**2.2. Drinfeld center criterion.** Let  $\mathcal{F}$  be a commutative fusion ring of basis  $(b_i)$ . Let  $(X_i)$  be the corresponding fusion matrices. Let  $A$  be  $\sum_i X_i X_i^*$ , and  $(c_j)$  its eigenvalues (a commutative reformulation of the formal codegrees in [24]). The fusion matrices commute over each other and are normal (because  $X_i^* = X_{i^*}$ ), so are simultaneously diagonalizable, say as  $(\lambda_{i,j})$ , called the character table of  $\mathcal{F}$ . Then  $c_j = \sum_i |\lambda_{i,j}|^2$ .

**Lemma 2.2.** *If  $\mathcal{F}$  admits a complex pivotal categorification  $\mathcal{C}$  then there exists  $j$  such that the categorical dimension of  $\mathcal{C}$  equals the formal codegree  $c_j$ .*

*Proof.* Let  $a$  be a pivotal structure on  $\mathcal{C}$ . By [7, Proposition 4.7.12], the dimension function  $\dim_a$  on the objects of  $\mathcal{C}$  induces a character  $\chi$  on its Grothendieck ring  $\mathcal{F}$ , which then must be given by a column of the character table, i.e. there is  $j$  such that  $\chi(b_i) = \lambda_{i,j}$ . Then the categorical dimension of  $\mathcal{C}$  must be  $c_j$ .  $\square$

**Theorem 2.3** (Pivotal version of Drinfeld center criterion). *If  $\mathcal{F}$  admits a complex pivotal categorification  $\mathcal{C}$  then there exists  $j$  such that for all  $i$ ,  $c_j/c_i$  is an algebraic integer.*

*Proof.* The result follows by Lemma 2.2 and [24, Corollary 2.14].  $\square$

Now  $\max_j(c_j) = \text{FPdim}(\mathcal{F})$ , say  $c_1$ . It is the categorical dimension in the pseudo-unitary case (by definition), so:

**Theorem 2.4** (Pseudo-unitary version of Drinfeld center criterion). *If  $\mathcal{F}$  admits a complex pseudo-unitary categorification  $\mathcal{C}$  then for all  $i$ ,  $c_1/c_i$  is an algebraic integer.*

If  $\mathcal{F}$  is the Grothendieck ring of  $\text{Rep}(G)$  with  $G$  a finite group, then the numbers  $c_1/c_j$  are exactly the sizes of the conjugacy classes of  $G$ .

In general, if  $\mathcal{F}$  admits a complex pseudo-unitary categorification  $\mathcal{C}$  (so spherical), then by [24, Theorem 2.13] the numbers  $c_1/c_j$  are exactly the FPdim of the simple objects of the Drinfeld center which contains the trivial object in  $\mathcal{C}$  under the forgetful functor.

Note that Theorem 2.4 admits the following conjectural stronger version extending Theorem 3.7 of Isaacs' book [15].

**Conjecture 2.5** (Isaacs criterion). *If  $\mathcal{F}$  is a complex pseudo-unitary commutative Grothendieck ring, then  $\frac{\lambda_{i,j} c_1}{\lambda_{i,1} c_j}$  is an algebraic integer for all  $i, j$ .*

Note that it should admit more general versions. As observed by P. Etingof [10] (and then [11]), it is related to Kaplansky's 6th conjecture (generalized to fusion categories), because:

**Proposition 2.6.** *Conjecture 2.5 implies that  $\mathcal{F}$  is of Frobenius type (i.e.  $\frac{c_1}{\lambda_{i,1}}$  is an algebraic integer).*

*Proof.* First,  $\lambda_{i,j}$  is an algebraic integer, and Conjecture 2.5 states that  $\frac{\lambda_{i,j} c_1}{\lambda_{i,1} c_j}$  is an algebraic integer too, then

$$\sum_j \left( \frac{\lambda_{i,j} c_1}{\lambda_{i,1} c_j} \right) \overline{\lambda_{i',j}} = \frac{c_1}{\lambda_{i,1}} \sum_j \frac{1}{c_j} \lambda_{i,j} \overline{\lambda_{i',j}} = \frac{c_1}{\lambda_{i,1}} \delta_{i,i'},$$

is also an algebraic integer, which means Frobenius type. Note that the last equality (called Schur orthogonality relation) comes from [25, Lemma 2.3] and the fact that a finite dimensional isometry is unitary.  $\square$

Conjecture 2.5 is true for the multiplicity one up to rank six case. There exists another criterion (also proved by V. Ostrik [24, Theorem 2.21]) using the formal codegrees. It is good to mention its commutative version here (as it is short to state):

**Theorem 2.7.** *If  $2 \sum_j 1/c_j^2 > 1 + 1/c_1$ , then  $\mathcal{F}$  admits no pseudo-unitary complex categorification.*

Note that up to rank six it applies at multiplicity at least two (so not the case concerned by this paper). It was visually compared in [21] with the criterion of §2.1.

**2.3. d-number criterion.** Let  $\mathcal{F}$  be a *commutative* fusion ring, and let  $(c_j)$  be its formal codegrees as defined in Subsection 2.2.

**Definition 2.8** ([25], Definition 1.1). *An algebraic integer  $\alpha$  is called a d-number if the ideal it generates in the ring of algebraic integers is invariant under the action of the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ .*

**Theorem 2.9** ([25], Theorem 1.2). *The formal codegrees of a complex (multi-)fusion category are d-numbers.*

In particular, if  $\mathcal{F}$  admits a complex categorification then the formal codegrees  $(c_i)$  are d-numbers. Here is a practical way to check whether a number is a d-number.

**Lemma 2.10** ([25], Lemma 2.7). *An algebraic integer  $\alpha$  is a d-number if and only if its minimal polynomial  $p(x) = x^n + a_1x^{n-1} + \dots + a_n$  (where  $a_i \in \mathbb{Z}$ ) satisfies that  $(a_n)^i$  divides  $(a_i)^n$  for all  $i$ .*

**2.4. Extended cyclotomic criterion.** The following theorem is a slight extension of the usual cyclotomic criterion (on the simple object  $\text{FPdim}$ ) of a fusion ring to all the entries of its formal character table, but in the commutative case only.

**Theorem 2.11.** *Let  $\mathcal{F}$  be a commutative fusion ring. If there is a fusion matrix such that the splitting field of its minimal polynomial is a non-abelian extension of  $\mathbb{Q}$ , then  $\mathcal{F}$  admits no complex categorification.*

*Proof.* First of all, a field extension of  $\mathbb{Q}$  is abelian if and only if it is cyclotomic [17]. So the statement says that the eigenvalues of the fusion matrices of a commutative complex fusion category are cyclotomic integers. Next, a fusion ring is commutative if and only if its irreducible representations are one-dimensional, so the result follows by [8, Theorem 8.51].  $\square$

**Question 2.12.** *Does Theorem 2.11 extend to noncommutative fusion rings?*

**2.5. Lagrange criterion.** Let mention here the generalization of Lagrange's theorem from finite groups to finite tensor categories. It will be used as a criterion of general categorification (over any field).

**Theorem 2.13** ([7], Theorem 7.17.6). *Let  $\mathcal{D}$  be a finite tensor category, and  $\mathcal{C} \subset \mathcal{D}$  be a tensor subcategory. Then the ratio  $\text{FPdim}(\mathcal{D})/\text{FPdim}(\mathcal{C})$  is an algebraic integer.*

**2.6. Zero spectrum criterion.** Here is a general categorification obstruction over every field (and see Remark 4.3). It corresponds to the existence of an equation of the form  $xy = 0$  with  $x, y \neq 0$ .

**Theorem 2.14** ([19]). *For a fusion ring  $\mathcal{F}$ , if there are indices  $i_j \in I$ ,  $1 \leq j \leq 9$ , such that  $N_{i_4, i_1}^{i_6}, N_{i_5, i_4}^{i_2}, N_{i_5, i_6}^{i_3}, N_{i_7, i_9}^{i_1}, N_{i_2, i_7}^{i_8}, N_{i_8, i_9}^{i_3}$  are non-zero, and*

$$\begin{aligned}
(1) \quad & \sum_k N_{i_4, i_7}^k N_{i_5^*, i_8}^k N_{i_6, i_9^*}^k = 0; \\
(2) \quad & N_{i_2, i_1}^{i_3} = 1; \\
(3) \quad & \sum_k N_{i_5, i_4}^k N_{i_3, i_1^*}^k = 1 \text{ or } \sum_k N_{i_2, i_4}^k N_{i_3, i_6^*}^k = 1 \text{ or } \sum_k N_{i_5^*, i_2}^k N_{i_6, i_1^*}^k = 1, \\
(4) \quad & \sum_k N_{i_2, i_7}^k N_{i_3, i_9^*}^k = 1 \text{ or } \sum_k N_{i_8, i_7}^k N_{i_3, i_1^*}^k = 1 \text{ or } \sum_k N_{i_2^*, i_8}^k N_{i_1, i_9^*}^k = 1,
\end{aligned}$$

*then  $\mathcal{F}$  cannot be categorified, i.e.  $\mathcal{F}$  is not the Grothendieck ring of a fusion category, over any field.*

**2.7. Quadratic fusion rings.** Let  $\mathcal{F}$  be a fusion ring with basis  $B = \{b_1, \dots, b_r\}$ . Let  $G$  be the group of invertible elements  $b_i$  of  $B$  (i.e.  $\text{FPdim}(b_i) = 1$ ). The fusion ring  $\mathcal{F}$  is called *pointed* if  $B = G$ , *near-group* if  $|B \setminus G| = 1$ , and more generally *quadratic* if the action of  $G$  on  $B \setminus G$  is transitive. A fusion category with a quadratic Grothendieck ring is called (in the literature) a *quadratic category* or a *generalized near-group category*.

**Theorem 2.15.** *A categorification  $\mathcal{C}$  of a quadratic fusion ring must admit a spherical (so pivotal) structure.*

*Proof.* By [35, Theorem IV.3.6.],  $\mathcal{C}$  must be  $\varphi$ -pseudo-unitary (i.e. pseudo-unitary up to Galois automorphism), and then spherical (so pivotal) by [6, Proposition 2.16].  $\square$

Theorem 2.15 will be used to exclude from (general) complex categorification some quadratic fusion rings already excluded from pivotal complex categorification by Theorem 2.3.

**2.8. Modular categorification criterion.** Let mention the following result allowing us to see that some fusion rings admits no (complex) modular categorification.

**Theorem 2.16.** *Let  $\mathcal{F}$  be a non-pointed weakly integral fusion ring of rank up to seven. If it is not a product, and has non self-adjoint objects, then it admits no complex modular categorification.*

*Proof.* It is an immediate consequence of [3, Theorem 1.2], because in the Ising and metaplectic categories, respectively Grothendieck equivalent to  $\text{PSU}(2)_3$  and  $\text{SO}(N)_2$  (with  $N > 1$  odd) the objects are self-adjoint (see §4.6).  $\square$

### 3. THE LIST OF FUSION RINGS OF MULTIPLICITY ONE UP TO RANK SIX

This section provides the full list of 72 fusion rings of multiplicity one up to rank 6, together with additional data as whether it is a complex Grothendieck ring, a fusion category model (when known), and properties (as quadratic).

**3.1. Notations.** If a fusion ring does not check the Schur product (resp. Drinfeld center, d-number, extended cyclotomic, Lagrange, zero spectrum) criterion Theorem 2.1 (resp. 2.3, 2.9, 2.11, 2.13, 2.14) then it will be qualified as *non-Schur* (resp. *non-Drinfeld*, *non-d-number*, *non-cyclo*, *non-Lagrange*, *non-Czero*), and so ruled out from unitary (resp. pivotal complex, complex, complex, any, any) categorification. Otherwise it is Schur (resp. Drinfeld, d-number, cyclo, Lagrange, Czero) by default, except for the non-cyclo ones, on which the Drinfeld criteria, d-number and Lagrange were not tested. The *type* of a fusion ring is the list of  $\text{FPdim}$  of the basic elements. Each commutative fusion ring is provided by its formal codegrees (with multiplicities) in decreasing order (exact form if quadratic, otherwise numerical with equation), then the reader can easily checked the Drinfeld criterion in cyclo case. The non-Czero ones are also provided by indices  $(i_1, \dots, i_9)$  applying on Theorem 2.14. Let  $\alpha_r$  denotes the number  $2 \cos(\pi/r)$ , so  $\alpha_3 = 1$ ,  $\alpha_4 = \sqrt{2}$ ,  $\alpha_5 = (1 + \sqrt{5})/2$ ,  $\alpha_6 = \sqrt{3}$ . For each rank the fusion rings are numbered with  $\mathbb{N}^{\circ}$ . Finally, the fusion rings which admit a complex categorification are marked with  $\star$ .

#### 3.2. Rank 2.

- $\text{FPdim}$  2, type  $[1, 1]$ , one fusion ring ( $\mathbb{N}^{\circ}1$ ):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(2, 2)]$ .
- ★ Properties: pointed, simple.
- Model:  $\text{Vec}(C_2)$ .

- $\text{FPdim}$   $\alpha_5 + 2 \simeq 3.618$ , type  $[1, \alpha_5]$ , one fusion ring ( $\mathbb{N}^{\circ}2$ ):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

- Formal codegrees:  $[((5 + \sqrt{5})/2, 1), ((5 - \sqrt{5})/2, 1)]$ .
- ★ Properties: near-group  $C_1 + 1$ , simple.
- Model:  $\text{PSU}(2)_3$ .

#### 3.3. Rank 3.

- $\text{FPdim}$  3, type  $[1, 1, 1]$ , one fusion ring ( $\mathbb{N}^{\circ}1$ ):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(3, 3)]$ .
- ★ Properties: pointed, simple.
- Model:  $\text{Vec}(C_3)$ .

- $\text{FPdim}$  4, type  $[1, 1, \alpha_4]$ , one fusion ring ( $\mathbb{N}^{\circ}2$ ):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(4, 2), (2, 1)]$ .
- ★ Properties: near-group  $C_2 + 0$ .
- Model:  $\text{SU}(2)_2$ .

- $\text{FPdim}$  6, type  $[1, 1, 2]$ , one fusion ring ( $\mathbb{N}^{\circ}3$ ):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Formal codegrees:  $[(6, 1), (3, 1), (2, 1)]$ .
- ★ Properties: near-group  $C_2 + 1$ .
- Model:  $\text{Rep}(S_3)$ ,  $\text{PSU}(2)_4$ .

- FPdim  $\alpha_7^4 - \alpha_7^2 + 1 \simeq 9.296$ , type  $[1, \alpha_7, \alpha_7^2 - 1]$ , one fusion ring ( $\mathbb{N}^{\#4}$ ):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Formal codegrees  $\simeq [(9.296, 1), (2.863, 1), (1.841, 1)]$ , roots of  $x^3 - 14x^2 + 49x - 49$ .
- ★ Properties: simple.
- Model:  $\text{PSU}(2)_5$ .

### 3.4. Rank 4.

- FPdim 4, type  $[1, 1, 1, 1]$ , two fusion rings ( $\mathbb{N}^{\#1,2}$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(4, 4)]$ .
- ★ Properties: pointed.
- Model:  $\text{Vec}(C_4)$ .

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(4, 4)]$ .
- ★ Properties: pointed.
- Model:  $\text{Vec}(C_2^2)$ .

- FPdim 6, type  $[1, 1, 1, \alpha_6]$ , one fusion ring ( $\mathbb{N}^{\#3}$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(6, 2), (3, 2)]$ .
- ★ Properties: near-group  $C_3 + 0$ .
- Model:  $TY(C_3)$ .

- FPdim  $2\alpha_5 + 4 \simeq 7.236$ , type  $[1, 1, \alpha_5, \alpha_5]$ , one fusion ring ( $\mathbb{N}^{\#4}$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(5 + \sqrt{5}, 2), (5 - \sqrt{5}, 2)]$ .
- ★ Properties: quadratic with  $G = C_2$ .
- Model:  $\text{SU}(2)_3$ , Bisch-Haagerup  $BH_1$ ,  $\text{Vec}(C_2) \otimes \text{PSU}(2)_3$

- FPdim  $(13 + \sqrt{13})/2 \simeq 8.302$ , type  $[1, 1, 1, (\sqrt{13} + 1)/2]$ , one fusion ring ( $\mathbb{N}^{\#5}$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- Formal codegrees:  $[((13 + \sqrt{13})/2, 1), ((13 - \sqrt{13})/2, 1), (3, 2)]$ .
- Properties: near-group  $C_3 + 1$ , non-Lagrange, non-d-number, non-Drinfeld.
- Note: the only exclusion up to rank four (and multiplicity one).

- FPdim 10, type  $[1, 1, 2, 2]$ , one fusion ring ( $\mathbb{N}^{\#6}$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[(10, 1), (5, 2), (2, 1)]$ .
- ★ Properties: integral.
- Model:  $\text{Rep}(D_5)$ .

- FPdim  $5\alpha_5^2 \simeq 13.090$ , type  $[1, \alpha_5, \alpha_5, \alpha_5 + 1]$ , one fusion ring ( $\mathbb{N}^{\#7}$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- Formal codegrees:  $[((15 + 5\sqrt{5})/2, 1), (5, 2), ((15 - 5\sqrt{5})/2, 1)]$ .
- ★ Properties: perfect (the first non-simple one).
- Model:  $\text{PSU}(2)_3^{\otimes 2}$ .

- FPdim  $8 + 4\alpha_4 \simeq 13.657$ , type  $[1, 1, \alpha_4 + 1, \alpha_4 + 1]$ , two fusion rings ( $\mathbb{N}8,9$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

– Formal codegrees:  $[(8 + 4\sqrt{2}, 1), (4, 2), (8 - 4\sqrt{2}, 1)]$ .

★ Properties: quadratic ( $C_2, 1, 1$ ).

– Model: PSU(2)<sub>6</sub>, even part the 1-supertransitive subfactor of index  $3 + 2\alpha_4$  without non-self-adjoint objects.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees:  $[(8 + 4\sqrt{2}, 1), (4, 2), (8 - 4\sqrt{2}, 1)]$ .

★ Properties: quadratic with  $G = C_2$ .

– Model: even part of the 1-supertransitive subfactor of index  $3 + 2\alpha_4$  with non-self-adjoint objects [18].

- FPdim  $\alpha_9^4 + 2\alpha_9 + 3 \simeq 19.234$ , type  $[1, \alpha_9, \alpha_9^2 - 1, \alpha_9 + 1]$ , one fusion ring ( $\mathbb{N}10$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees  $\simeq [(19.234, 1), (5.445, 1), (3, 1), (2.31996, 1)]$ , roots of  $(x^3 - 27x^2 + 162x - 243)(x - 3)$ .

★ Properties: simple.

– Model: PSU(2)<sub>7</sub>.

### 3.5. Rank 5.

- FPdim 5, type  $[1, 1, 1, 1, 1]$ , one fusion ring ( $\mathbb{N}1$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

– Formal codegrees:  $[(5, 5)]$ .

★ Properties: simple.

– Model: Vec( $C_5$ ).

- FPdim 8, type  $[1, 1, 1, 1, 2]$ , two fusion rings ( $\mathbb{N}2,3$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

– Formal codegrees:  $[(8, 2), (4, 3)]$ .

★ Properties: near-group  $C_4 + 0$ .

– Model: TY( $C_4$ ).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

– Formal codegrees:  $[(8, 2), (4, 3)]$ .

★ Properties: near-group  $C_2^2 + 0$ .

– Model: Rep( $D_4$ ), Rep( $Q_8$ ).

- FPdim  $(\sqrt{17} + 17)/2 \simeq 10.562$ , type  $[1, 1, 1, 1, (\sqrt{17} + 1)/2]$ , two fusion rings ( $\mathbb{N}4,5$ ):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees:  $[((17 + \sqrt{17})/2, 1), ((17 - \sqrt{17})/2, 1), (4, 3)]$ .

– Properties: near-group  $C_4 + 1$ , non-Lagrange, non-d-number, non-Drinfeld.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees:  $[((17 + \sqrt{17})/2, 1), ((17 - \sqrt{17})/2, 1), (4, 3)]$ .

– Properties: near-group  $C_2^2 + 1$ , non-Lagrange, non-d-number, non-Drinfeld.















- FPdim  $\alpha_{13}^{10} - 7\alpha_{13}^8 + 17\alpha_{13}^6 - 16\alpha_{13}^4 + 6\alpha_{13}^2 + 3 \simeq 56.747$ , type  $[1, \alpha_{13}, \alpha_{13}^2 - 1, \alpha_{13}^3 - 2\alpha_{13}, \alpha_{13}^4 - 3\alpha_{13}^2 + 1, \alpha_{13}^5 - 4\alpha_{13}^3 + 3\alpha_{13}]$ , one fusion ring (№37):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees  $\simeq [(56.747, 1), (15.049, 1), (7.391, 1), (4.799, 1), (3.717, 1), (3.298, 1)]$  roots of  $x^6 - 91x^5 + 2366x^4 - 26364x^3 + 142805x^2 - 371293x + 371293$ .

★ Properties: simple.

– Model: PSU(2)<sub>11</sub>.

- FPdim  $(65 + 17\sqrt{13})/2 \simeq 63.147$ , type  $[1, (\sqrt{13} + 3)/2, (\sqrt{13} + 3)/2, (\sqrt{13} + 3)/2, (\sqrt{13} + 3)/2, (\sqrt{13} + 5)/2]$ , two fusion rings (№38,39):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees:  $[((65 + 17\sqrt{13})/2, 1), (9, 4), ((65 - 17\sqrt{13})/2, 1)]$ .

– Properties: simple, non-Schur, cyclo but not of Frobenius type, non-d-number, non-Drinfeld, non-Czero (1, 1, 3, 1, 1, 1, 3, 4, 1).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

– Formal codegrees:  $[((65 + 17\sqrt{13})/2, 1), (9, 4), ((65 - 17\sqrt{13})/2, 1)]$ .

– Properties: simple, non-Schur, cyclo but not of Frobenius type, non-d-number, non-Drinfeld.

**3.7. Proof of Theorem 1.1.** A fusion ring is given by its fusion coefficients  $N_{i,j}^k \in \mathbb{Z}_{\geq 0}$ , so at rank  $r$  and multiplicity  $m$ , there are  $r^3$  variables and so  $(m+1)^{r^3}$  possibilities to check (e.g.  $2^{216}$  ones if  $r=6$  and  $m=1$ ). Fortunately, we can drastically reduce the number of variables by using the axioms of fusion rings reformulated in term of  $N_{i,j}^k$ :

- Associativity:  $\sum_s N_{ij}^s N_{sk}^t = \sum_s N_{jk}^s N_{is}^t$ ,
- Neutral:  $N_{1i}^j = N_{i1}^j = \delta_{ij}$ ,
- Dual:  $N_{i^*,k}^1 = N_{k,i^*}^1 = \delta_{i,k}$ ,
- Frobenius reciprocity:  $N_{ij}^k = N_{i^*k}^j = N_{kj^*}^i$

where  $i \mapsto i^*$  is a dual structure fixed beforehand (there are few ones up to equivalence). Then we can get the 72 fusion rings listed above in a reasonable time. The SageMath code (too long for this paper) is available in [26], together with the data. Now, let us provide the list of fusion rings which do not pass a given criterion:

	criterion type	№ at rank 4	№ at rank 5	№ at rank 6
non-Schur	unitary		9, 13	22, 24, 25, 26, 32, 33, 35, 36, 38, 39
non-Drinfeld	complex pivotal	5	4, 5, 9, 13	11, 16, 17, 18, 19, 22, 24, 25, 26, 28, 29, 30, 38, 39
non-d-number	complex	5	4, 5, 9, 13	11, 24, 28, 29, 38, 39
non-cyclo	complex		14, 15	32, 33, 35, 36
non-Lagrange	general	5	4, 5, 9	11, 24, 28, 29
non-Czero	general		9, 13	25, 26, 32, 33, 35, 36, 38

Observe that all the excluded fusion rings are non-Drinfeld, non-d-number or non-cyclo, which means that the corresponding three criteria are sufficient for proving Theorem 1.1 (see their SageMath codes in Section 6). We still mentioned the three other criteria because they should be useful in the future (for example, see Subsection 5.2). The №16, 17, 18, 19, 22, 30 at rank 6 are only excluded from pivotal complex categorification (non-Drinfeld) but they are all quadratic, so excluded from (general) complex categorification by Theorem 2.15. The non-excluded fusion rings are exactly the Grothendieck rings listed in Theorem 1.1, the new ones being categorified in Section 4.  $\square$

#### 4. UNITARY CATEGORIFICATION OF THE NEW COMPLEX GROTHENDIECK RINGS

In this section, we consider the new complex Grothendieck rings of multiplicity one up to rank 6, and provide some unitary solutions of their Pentagon Equation (PE).

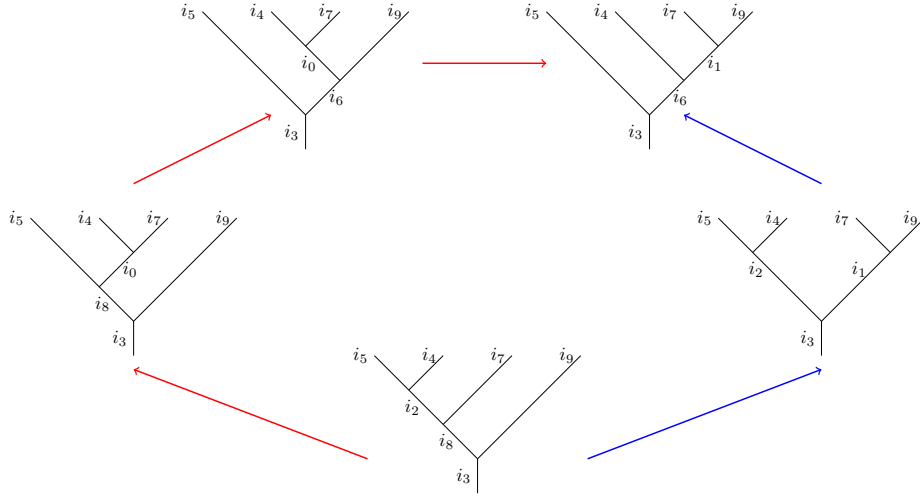


FIGURE 1. Pentagon Equation in multiplicity one

4.1. **The Pentagon Equation in multiplicity one.** In this subsection, every hom-space  $\text{hom}_{\mathcal{C}}(X_i \otimes X_j, X_k)$  is assumed to be of dimension  $N_{i,j}^k \leq 1$ , so that every morphism in it is completely determined by  $i, j$  and  $k$ , up to a multiplicative constant, which makes the PE much easier to deal with. The F-symbols are defined as follows:

$$(5) \quad \begin{array}{c} i_1 \\ \diagdown \quad \diagup \\ i_3 \quad i_2 \quad i_4 \\ \diagup \quad \diagdown \\ i_5 \end{array} = \sum_{i_6} \begin{pmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{pmatrix}_F \begin{array}{c} i_1 \\ \diagdown \quad \diagup \\ i_6 \quad i_2 \quad i_4 \\ \diagup \quad \diagdown \\ i_5 \end{array}.$$

They satisfy the PE, Figure 1 being a pictorial representation [36], with the following algebraic reformulation:

$$(6) \quad \begin{pmatrix} i_2 & i_7 & i_8 \\ i_9 & i_3 & i_1 \end{pmatrix}_F \begin{pmatrix} i_5 & i_4 & i_2 \\ i_1 & i_3 & i_6 \end{pmatrix}_F = \sum_{i_0} \begin{pmatrix} i_5 & i_4 & i_2 \\ i_7 & i_8 & i_0 \end{pmatrix}_F \begin{pmatrix} i_5 & i_0 & i_8 \\ i_9 & i_3 & i_6 \end{pmatrix}_F \begin{pmatrix} i_4 & i_7 & i_0 \\ i_9 & i_6 & i_1 \end{pmatrix}_F$$

Let  $d_i := \dim_{\mathcal{C}}(X_i)$ . By using the following notation:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{pmatrix} := d_{i_6}^{-1} \begin{pmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{pmatrix}_F$$

the PE becomes:

$$(7) \quad \begin{pmatrix} i_2 & i_7 & i_8 \\ i_9 & i_3 & i_1 \end{pmatrix} \begin{pmatrix} i_5 & i_4 & i_2 \\ i_1 & i_3 & i_6 \end{pmatrix} = \sum_{i_0} d_{i_0} \begin{pmatrix} i_5 & i_4 & i_2 \\ i_7 & i_8 & i_0 \end{pmatrix} \begin{pmatrix} i_5 & i_0 & i_8 \\ i_9 & i_3 & i_6 \end{pmatrix} \begin{pmatrix} i_4 & i_7 & i_0 \\ i_9 & i_6 & i_1 \end{pmatrix}.$$

**Proposition 4.1** (Pivotal axioms). *Let  $\mathcal{C}$  be a fusion category with above PE. If there are roots of unity  $(t_i)$  such that:*

- $t_1 = 1$ ,
- $t_{i^*} = t_i^{-1}$ ,
- $t_i^{-1} t_j^{-1} t_k = d_{i^*} d_{j^*} d_k \begin{pmatrix} i & j & k \\ k^* & 1 & i^* \end{pmatrix} \begin{pmatrix} j & k^* & i^* \\ i & 1 & j^* \end{pmatrix} \begin{pmatrix} k^* & i & j^* \\ j & 1 & k \end{pmatrix}$ ,  $\forall i, j, k$  with  $N_{i,j}^k \neq 0$ ,

then  $\mathcal{C}$  is pivotal, and  $(t_i)$  are called the pivotal coefficients. If moreover all  $t_i = \pm 1$  then  $\mathcal{C}$  is spherical.

*Proof.* It is a reformulation of [36, Proposition 4.16 (1)(2)], where  $F_{d;n,m}^{a,b,c} = \begin{pmatrix} a & b & m \\ c & d & n \end{pmatrix}_F = d_n \begin{pmatrix} a & b & m \\ c & d & n \end{pmatrix}$ .  $\square$

**Corollary 4.2.** *A solution of the PE under below (a),(b),(c) gives a pseudo-unitary categorification of the fusion ring.*

- (a)  $d_i = \text{FPdim}(X_i)$ ,
- (b) *evaluation when one object is trivial:*

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ 1 & i_3 & i_2 \end{pmatrix} = d_{i_2}^{-1/2} d_{i_3}^{-1/2},$$





4.2. **Rank 5, FPdim 12 and type**  $[1, 1, \sqrt{3}, \sqrt{3}, 2]$ . Let consider the fusion matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

There are 12 real variables:  $[1, 2, 2, 2, 4, 3]$ ,  $[1, 2, 2, 3, 1, 2]$ ,  $[1, 4, 4, 4, 1, 4]$ ,  $[1, 4, 4, 4, 4, 4]$ ,  $[2, 2, 4, 2, 4, 4]$ ,  $[2, 2, 4, 3, 4, 4]$ ,  $[2, 2, 4, 4, 2, 3]$ ,  $[2, 2, 4, 4, 3, 2]$ ,  $[2, 3, 4, 3, 4, 4]$ ,  $[2, 3, 4, 4, 2, 2]$ ,  $[2, 4, 3, 4, 4, 2]$ ,  $[4, 4, 4, 4, 4, 4]$ ; 4 complex (non-real) variables:  $[1, 2, 2, 3, 4, 2]$ ,  $[1, 2, 2, 4, 2, 4]$ ,  $[1, 2, 2, 4, 3, 4]$ ,  $[2, 2, 4, 4, 2, 2]$ , and then their reflection. Then we consider 20 variables, ordered as above. Here a some solutions of the PE (where  $I$  is the imaginary unit):

$$\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{3^{3/4}}{6}, \frac{3^{3/4}}{6}, -\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, -\frac{3^{3/4}}{6}, -\frac{\sqrt{3}}{6}, -\frac{3^{3/4}}{6}, 0, -\frac{1}{6r^2}, \frac{\sqrt{3}}{6r}, -\frac{\sqrt{3}}{6r}, -\frac{I}{2}, -2r^2, r, -r, \frac{I}{2}\right),$$

with 8 variations by (pointwise) multiplying by  $(s_1, 1, 1, 1, s_2, s_1 s_2, 1, -s_1, -s_1 s_2, 1, -s_2, 1, 1, 1, -s_1, s_3, 1, 1, -s_1, -s_3)$  with  $s_i \in \{-1, 1\}$ . The unitary case corresponds to  $|r|^4 = 1/12$ .

**Remark 4.4** (Resolution mode). *We listed all the possible equations, up to symmetry, by a straightforward code. Note that there is no equation containing all above variables together, the maximum number of variables in a single equation is 7 here (where a complex variable and its reflection count for one). There are exactly 59, 53, 39, 64, 44, 13, 4 (non-trivial) such equations with 1, 2, ..., 7 variables respectively. We first solved the equations with less than 4 variables, which provided 16 variations, but only 8 variations survived after checking the rest of the equations.*

**Remark 4.5** (Variation). *In this paper, a solution is called a variation of an other one, if they are equal up to (pointwise) signs. We did not check whether they define the same fusion category (gauge equivalent).*

4.3. **Rank 6, FPdim 8 and type**  $[1, 1, 1, 1, \sqrt{2}, \sqrt{2}]$ . Consider the three first such fusion rings, in the same order than in §3.6.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

There are 4 real variables  $[1, 1, 3, 3, 1, 2]$ ,  $[1, 4, 4, 4, 3, 5]$ ,  $[1, 5, 5, 5, 3, 4]$ ,  $[3, 4, 5, 4, 3, 4]$ ; 3 complex (non-real) variables  $[1, 1, 3, 4, 5, 5]$ ,  $[1, 1, 3, 5, 4, 4]$ ,  $[1, 4, 4, 5, 1, 4]$ , and then their reflection. So we need to consider 10 variables. Here are some solutions of the equations.

$$\left(-1, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -Ir, r, \frac{1+I}{2}, \frac{I\sqrt{2}}{2r}, \frac{\sqrt{2}}{2r}, \frac{1-I}{2}\right),$$

together with 8 variations given by (pointwise) multiplying by  $(1, s_1, -s_1, 1, s_2, 1, s_3, -s_2, 1, -s_3)$  with  $s_i \in \{-1, 1\}$ . The unitary case corresponds to  $|r|^4 = 1/2$ .

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

There are 5 real variables  $[1, 1, 3, 3, 1, 2]$ ,  $[1, 4, 5, 4, 3, 5]$ ,  $[1, 5, 4, 5, 3, 4]$ ,  $[3, 4, 4, 4, 3, 4]$ ,  $[3, 5, 5, 5, 3, 5]$ ; 4 complex (non-real) variables  $[1, 1, 3, 4, 5, 5]$ ,  $[1, 1, 3, 5, 4, 4]$ ,  $[1, 4, 5, 5, 1, 4]$ ,  $[1, 4, 5, 5, 2, 4]$ , and then their reflection. So we need to consider 13 variables. Here are some solutions of the equations:

$$\left(-1, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2r_1}, \frac{I\sqrt{2}}{2r_1}, \frac{1}{2r_2}, \frac{I}{2r_2}, r_1, -Ir_1, r_2, -Ir_2\right),$$

together with 4 variations given by (pointwise) multiplying by  $(1, s_1, -s_1, 1, 1, 1, s_2, 1, s_2, 1, -s_2, 1, s_2)$  with  $s_i \in \{-1, 1\}$ . The unitary case corresponds to  $|r_1|^4 = |r_2|^2 = 1/2$ .

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

There are 5 real variables  $[1, 4, 5, 4, 1, 4]$ ,  $[1, 4, 5, 5, 2, 5]$ ,  $[1, 4, 5, 5, 3, 5]$ ,  $[2, 4, 4, 5, 2, 4]$ ,  $[3, 4, 4, 5, 3, 4]$ ; 4 complex (non-real) variables  $[1, 2, 3, 3, 1, 2]$ ,  $[1, 2, 3, 4, 5, 4]$ ,  $[1, 2, 3, 5, 4, 5]$ ,  $[2, 4, 4, 5, 3, 4]$ , and then their reflection. So we need to consider 13 variables. Here are some solutions of the equations:

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{4r_1^4 r_2^2}, \frac{1}{2r_1 r_2}, \frac{\sqrt{2}}{2r_1}, \frac{1}{2r_2}, 4r_1^4 r_2^2, \sqrt{2}r_1 r_2, r_1, r_2\right),$$

together with 4 variations given by (pointwise) multiplying by  $(1, s_1, -s_1, s_2, -s_2, 1, -s_2, 1, 1, 1, -s_2, 1, 1)$  with  $s_i \in \{-1, 1\}$ . The unitary case corresponds to  $|r_1|^4 = |r_2|^2 = 1/2$ .

**4.4. Rank 5, FPdim 24 and type  $[1, 1, 2, 3, 3]$ .** Consider the fusion matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

There are 16 real variables  $[1, 2, 2, 2, 1, 2]$ ,  $[1, 2, 2, 2, 2, 2]$ ,  $[1, 3, 3, 3, 2, 4]$ ,  $[1, 3, 3, 4, 1, 3]$ ,  $[2, 2, 2, 2, 2, 2]$ ,  $[2, 2, 2, 3, 3, 3]$ ,  $[2, 2, 2, 3, 3, 4]$ ,  $[2, 2, 2, 3, 4, 4]$ ,  $[2, 2, 2, 4, 4, 4]$ ,  $[2, 3, 3, 3, 2, 4]$ ,  $[2, 3, 3, 4, 2, 3]$ ,  $[2, 3, 4, 3, 2, 3]$ ,  $[2, 3, 4, 3, 3, 3]$ ,  $[2, 3, 4, 4, 3, 4]$ ,  $[2, 3, 4, 4, 4, 4]$ ,  $[2, 4, 3, 4, 3, 4]$ ; 16 complex variables  $[1, 2, 2, 3, 3, 4]$ ,  $[1, 2, 2, 3, 4, 4]$ ,  $[1, 3, 3, 3, 3, 4]$ ,  $[1, 3, 3, 4, 2, 3]$ ,  $[1, 3, 3, 4, 3, 3]$ ,  $[2, 3, 3, 3, 2, 3]$ ,  $[2, 3, 3, 3, 3, 3]$ ,  $[2, 3, 3, 3, 3, 4]$ ,  $[2, 3, 3, 3, 4, 3]$ ,  $[2, 3, 3, 4, 3, 3]$ ,  $[2, 3, 3, 4, 3, 4]$ ,  $[2, 3, 3, 4, 4, 4]$ ,  $[3, 3, 3, 3, 3, 3]$ ,  $[3, 3, 3, 3, 3, 4]$ ,  $[3, 3, 3, 4, 4, 4]$ ,  $[3, 3, 4, 3, 3, 3]$ , and then their reflection. So we need to consider 48 variables. The PE has 1053 equations and Krull dimension three.

**Remark 4.6.** *In general, the dimension of the affine variety defined by an ideal  $I$  in a polynomial ring  $R$  is the Krull dimension of  $R/I$ . So here, it is the dimension  $d$  of the variety of solutions of the PE. But SageMath needs dimension zero to provide explicit solutions. By fixing  $d$  variables appropriately, we get a non-empty subvariety of dimension zero.*

So, by fixing three variables appropriately, we got some unitary solutions:

- for the 16 real variables (with  $\epsilon \in \{-1, 1\}$ ):

$$\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, 0, -\epsilon \frac{\sqrt{3}}{6}, \epsilon \frac{\sqrt{3}}{6}, -\epsilon \frac{\sqrt{3}}{6}, \epsilon \frac{\sqrt{3}}{6}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}\right),$$

- for the 16 complex variables:

$$\left((1-I)\frac{\sqrt{3}}{6}, (1-I)\frac{\sqrt{3}}{6}, -\frac{1}{3}, -\frac{I}{3}, \frac{1}{3}, 0, -\frac{\sqrt{3}}{6}, \frac{1}{6}, I\frac{\sqrt{3}}{6}, -\frac{I}{6}, -I\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, -\frac{1}{6}, -\frac{I}{6}, \frac{1}{6}, -\frac{I}{6}\right),$$

and then their complex-conjugate (for the reflections).

**4.5. Rank 6, FPdim 20 and type  $[1, 1, 2, 2, \sqrt{5}, \sqrt{5}]$ .** Consider the fusion matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

There are 29 real variables  $[1, 2, 2, 2, 1, 2]$ ,  $[1, 2, 2, 2, 3, 2]$ ,  $[1, 2, 2, 3, 2, 3]$ ,  $[1, 2, 2, 3, 3, 3]$ ,  $[1, 3, 3, 3, 1, 3]$ ,  $[1, 3, 3, 3, 2, 3]$ ,  $[1, 4, 4, 4, 2, 5]$ ,  $[1, 4, 4, 4, 3, 5]$ ,  $[1, 4, 4, 5, 1, 4]$ ,  $[2, 2, 3, 3, 2, 2]$ ,  $[2, 2, 3, 4, 4, 4]$ ,  $[2, 2, 3, 4, 5, 5]$ ,  $[2, 2, 3, 5, 4, 4]$ ,  $[2, 2, 3, 5, 5, 5]$ ,  $[2, 3, 3, 3, 2, 3]$ ,  $[2, 3, 3, 4, 4, 4]$ ,  $[2, 3, 3, 4, 5, 4]$ ,  $[2, 3, 3, 5, 4, 5]$ ,  $[2, 3, 3, 5, 5, 5]$ ,  $[2, 4, 4, 4, 2, 5]$ ,  $[2, 4, 4, 4, 3, 5]$ ,  $[2, 4, 4, 5, 2, 4]$ ,  $[2, 4, 5, 4, 2, 4]$ ,  $[2, 4, 5, 4, 3, 4]$ ,  $[2, 4, 5, 5, 3, 5]$ ,  $[2, 5, 4, 5, 3, 5]$ ,  $[3, 4, 4, 4, 3, 5]$ ,  $[3, 4, 4, 5, 3, 4]$ ,  $[3, 4, 5, 4, 3, 4]$ ; 16 complex variables  $[1, 2, 2, 4, 4, 5]$ ,  $[1, 2, 2, 4, 5, 5]$ ,  $[1, 3, 3, 4, 4, 5]$ ,  $[1, 3, 3, 4, 5, 5]$ ,  $[1, 4, 4, 5, 2, 4]$ ,  $[1, 4, 4, 5, 3, 4]$ ,  $[2, 2, 3, 3, 2, 3]$ ,  $[2, 2, 3, 4, 4, 5]$ ,  $[2, 2, 3, 5, 4, 5]$ ,  $[2, 3, 3, 4, 4, 5]$ ,  $[2, 3, 3, 4, 5, 5]$ ,  $[2, 4, 4, 4, 2, 4]$ ,  $[2, 4, 4, 4, 3, 4]$ ,  $[2, 4, 4, 5, 3, 4]$ ,  $[2, 4, 4, 5, 3, 5]$ ,  $[3, 4, 4, 4, 3, 4]$ , and then their reflection. So we need to consider 61 variables. The polynomial ring modulo the ideal generated by the (1231) equations is of Krull dimension two (see Remark 4.6). So, by fixing two variables appropriately, we got some unitary solutions:

- for the 29 real variables:

$$\left(\frac{1}{2}, -\frac{1}{2}, -\epsilon_1 \frac{1}{2}, \epsilon_1 \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -a, -a, a, 0, \epsilon_2 b, \epsilon_2 b, -\epsilon_2 b, -\epsilon_2 b, 0, \epsilon_2 b, -\epsilon_2 b, \epsilon_2 b, -\epsilon_2 b, c, -d, -c, -c, d, -d, d, c, -c, -c\right),$$

- for the 16 complex variables:

$$\left(e, e, -\epsilon_1 \epsilon_2 e, -\epsilon_1 \epsilon_2 e, -a, -a, \frac{1}{2}, b, -b, -b, b, \epsilon_3 f I, -\epsilon_2 \epsilon_3 g I, d, \epsilon_2 \epsilon_3 g I, -\epsilon_3 f I\right),$$

and then their complex-conjugate (for the reflections), with  $\epsilon_i \in \{-1, 1\}$  and

$$(a, b, c, d, e, f, g) = \left(\frac{\sqrt{5}}{5}, 80^{-1/4}, \frac{5 + \sqrt{5}}{20}, \frac{5 - \sqrt{5}}{20}, 20^{-1/4}, \sqrt{\frac{5 - \sqrt{5}}{40}}, \sqrt{\frac{5 + \sqrt{5}}{40}}\right).$$

**4.6. Models by zesting construction.** E.C. Rowell pointed out to us a new construction called *zesting* [5], providing models for some of the new Grothendieck rings mentioned in this section. The metaplectic categories are those Grothendieck equivalent to  $\mathrm{SO}(N)_2$  with  $N = 2n + 1 \geq 1$ . At fixed  $n$ , it is of multiplicity one, rank  $n + 4$ , type  $[[1, 2], [2, n], [\sqrt{N}, 2]]$ . Let  $z$  be the non-trivial object of FPdim 1,  $(y_i)$  those of FPdim 2, and  $x_1, x_2$  those of FPdim  $\sqrt{N}$ . As mentioned in [1, §3], the (commutative) fusion rules are the following:

- (1)  $zy_i = y_i, zx_1 = x_2, zx_2 = x_1, z^2 = 1,$
- (2)  $x_i^2 = 1 + \sum_i y_i,$
- (3)  $x_1x_2 = z + \sum_i y_i,$
- (4)  $y_iy_j = y_{\min(i+j, N-i-j)} + y_{|i-j|},$  for  $i \neq j, y_i^2 = 1 + z + y_{\min(2i, N-2i)}.$

**Theorem 4.7** (Twisted metaplectic categories). *The exchange of 1 and  $z$  in (2) and (3) above produces a new family of complex Grothendieck rings.*

*Proof.* The result follows from the new zesting construction [5, Proposition 6.3] using the braiding of  $\mathrm{SO}(N)_2$  and its  $C_2$ -grading, which twists the associativity by a 3-cocycle. E.C. Rowell provided more details about that in [29].  $\square$

The cases  $n = 1, 2$  correspond to the new (unitary) Grothendieck rings of §4.2 and §4.5 respectively (and  $n = 0$  to  $\mathrm{Vec}(C_4)$ ). Note that in the same way, the ones of §4.3 are zestings of  $\mathrm{Vec}(C_2) \otimes \mathrm{SU}(2)_2$ . Finally, the one of §4.4 cannot be a zesting, because there is no grading. But observe that we can produce two new families of fusion rings, the first one by adding  $n(x_1 + x_2)$  to the right hand-side of (2) and (3) above (which recovers  $\mathrm{Rep}(S_4)$  when  $n = 1$ ), and the second one by twisting the first one as in Theorem 4.7, which (for  $n = 1$ ) would provide a model for the new (unitary) Grothendieck ring of §4.4.

**Question 4.8.** *Are the fusion rings of these two new families, (unitary) Grothendieck rings?*

## 5. OBSERVATIONS AND QUESTIONS

This classification leads to many observations and questions, grouped in this section.

**5.1. All criteria passed and categorification.** Every fusion ring ruled out here was directly excluded by some of the criteria mentioned in Section 2 (without considering that of Subsection 2.8).

**Question 5.1.** *Is there a fusion ring of multiplicity one which passes all the criteria of Section 2 without being categorifiable?*

Note that without the multiplicity one assumption (but pivotal or characteristic zero), the above question already admits a negative answer in [19] with the fusion ring denoted  $\mathcal{F}_{210}$ , of multiplicity 2, rank 7, FPdim 210, type  $[1, 5, 5, 5, 6, 7, 7]$  and fusion matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 & 2 & 2 \end{bmatrix}$$

It corresponds to the case  $q = 6$  of the interpolated family of fusion rings of Lie type in [20], all of them being of multiplicity 2 or 3, and satisfying all the criteria of Section 2 (the existence of a categorification is open for all non prime-power  $q \neq 6$ ).

There are fusion rings of multiplicity one and rank 7 which pass all these criteria but not the one of Conjecture 2.5; one of them is of FPdim  $20 + 4\sqrt{5}$ , type  $[1, 1, 1, 1, 2, 1 + \sqrt{5}, 1 + \sqrt{5}]$  and fusion matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

**5.2. Categorification in positive characteristic.** Among the 25 fusion rings ruled out from complex categorification, 16 are also ruled out from any categorification (over any field) by Theorem 2.14 or 2.13. So, among the remaining 9 ones (№14, 15 at rank 5 and №16-19, 22, 30, 39 at rank 6):

**Question 5.2.** *Which ones admit a categorification over a field of positive characteristic?*

**Question 5.3.** *In general, is there a fusion ring without categorification in characteristic zero but positive?*

### 5.3. Unitary categorification.

**Corollary 5.4.** *A complex Grothendieck ring of multiplicity one up to rank six is unitary.*

**Question 5.5.** *Is there a complex Grothendieck ring of multiplicity one which is not unitary?*

Note that without the multiplicity one assumption, the above question already admits a negative answer in [30] providing a complex non pseudo-unitary Grothendieck ring of multiplicity 2, rank 6, FPdim  $9(3+3a_1-a_4) \simeq 74.6177$  (with  $a_k = 2 \cos(k\pi/9)$ ), type  $[1, 1+a_1, 1+a_1, 1+a_1, 1+2a_1-a_4, 2+2a_1-a_4]$  and fusion matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{bmatrix}$$

More generally (a particular case of [7, Question 4.8.3]):

**Question 5.6.** *Does every complex fusion category admits a pivotal structure? A spherical structure?*

Recall that in the list “unitary, pseudo-unitary,  $\varphi$ -pseudo-unitary, spherical, pivotal”, one implies its successor.

**5.4. Non-cyclotomic fusion categories.** On page 591 of [8], it is asked whether any (multi-)fusion category is defined over a cyclotomic field. This question was answer negatively in [23]. Now §4.2 and §4.5 mention non-cyclotomic solutions of their PE: they have F-symbols equal to  $\frac{\pm 3^{3/4}}{6}$ ,  $20^{-1/4}$  or  $80^{-1/4}$ . By Kronecker-Weber theorem and the following SageMath computation, all these numbers are non-cyclotomic.

```
sage: x=(3^(3/4))/6 # or 20^(-1/4) or 80^(-1/4)
sage: f=minpoly(x)
sage: K.<a> = f.splitting_field()
sage: K.is_abelian()
False
```

To be non-cyclotomic, a fusion category must have non-cyclotomic F-symbols for every choice of basis, whereas the solutions mention above come from a single choice.

**Question 5.7.** *Is the fusion category mentioned in §4.2 or §4.5 non-cyclotomic?*

### 5.5. Integral Grothendieck rings.

**Corollary 5.8.** *A weakly integral fusion ring of multiplicity one up to rank six is always unitarily categorifiable.*

There are integral fusion rings of multiplicity one without any categorification. Below is an example at rank 7, FPdim 42 and type  $[1, 1, 2, 3, 3, 3, 3]$  which is non-Czero  $(3, 3, 2, 2, 4, 3, 3, 3, 3)$ :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

**5.6. Simple Grothendieck rings.** A fusion ring is called *simple* if it has no proper non-trivial fusion subring. A Grothendieck ring is called simple if it is so as fusion ring.

**Corollary 5.9.** *A simple complex Grothendieck ring of multiplicity one up to rank six is given by the following:*

- $\text{Vec}(C_p)$ , with  $C_p$  the cyclic group of order  $p$  prime,
- $\text{PSU}(2)_{2n+1}$ , with  $n \geq 0$ .

**Question 5.10.** *Is there a simple complex Grothendieck ring of multiplicity one, not in above families?*

## 6. APPENDIX: SAGEMATH CODE

This section provides the SageMath code for the criteria of Theorems 2.3, 2.9 and 2.11 (the only criteria needed to prove Theorem 1.1). They apply in the commutative case only (as needed). Just apply the function *Checking* below to a fusion ring written as a list  $M$ , for example, the following computation shows that the fusion ring  $\mathbb{N}^{\#5}$  at rank 4 is non-Drinfeld and non-d-number (as written in Subsection 3.7).

```
sage: M=[[ [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]],
....:    [[0, 1, 0, 0], [0, 0, 1, 0], [1, 0, 0, 0], [0, 0, 0, 1]],
....:    [[0, 0, 1, 0], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 0, 1]],
....:    [[0, 0, 0, 1], [0, 0, 0, 1], [0, 0, 0, 1], [1, 1, 1, 1]]]
sage: Checking(M)
non-Drinfeld
non-d-number
```

Here is the code of the function Checking:

```

UCF=UniversalCyclotomicField()

def Checking(M):
    r=len(M)
    N=zero_matrix(QQ,r)
    for i in range(r):
        Mi=matrix(QQ,M[i])
        for j in range(i):
            Mj=matrix(QQ,M[j])
            if Mi*Mj!=Mj*Mi:
                return 'non-commutative'
        Ti=Mi.transpose()
        Ni=Mi*Ti
        N+=Ni
    f = N.minpoly()
    ff=N.charpoly()
    if not Cyclo(M):
        return 'non-cyclo'      # Extended cyclotomic criterion
    K.<a> = f.splitting_field()
    n = K.conductor()
    L=ff.roots(CyclotomicField(n))
    LL=[UCF(l[0]) for l in L]
    rL=[l[0].n() for l in L]
    mm=max(rL)
    for ii in range(len(L)):
        if mm==rL[ii]:
            dim=LL[ii]
            break
    c=0
    for x in LL:
        d=0
        for y in LL:
            yy=UCF(x/y)
            if '/' in list(str(yy)):
                d=1
                break
        if d==0:
            c=1
    if c==0:
        print('non-Drinfeld')      # Drinfeld center criterion
    for x in LL:
        p=list(UCF(x).minpoly())
        n=len(p)-1
        A=p[0]
        d=0
        for i in range(n+1):
            a=p[i]
            j=n-i
            y=UCF((a^n)/(A^j))
            if '/' in list(str(y)):
                d=1
                break
        if d==1:
            print('non-d-number')      # d-number criterion
            break

```

```

def Cyclo(M):
    r=len(M)
    for k in range(len(M)):
        N=matrix(QQ,M[k])
        f = N.minpoly()
        K.<a> = f.splitting_field()
        if not K.is_abelian():
            return false
    return true

```

**Statements.** *On behalf of all authors, the corresponding author states that there is no conflict of interest. The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.*

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