# THE GROTHENDIECK RING OF A FAMILY OF SPHERICAL CATEGORIES 

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#### Abstract

The first author constructed a $q$-parameterized spherical category $\mathscr{C}$ over $\mathbb{C}(q)$ in [3], whose simple objects are labelled by all Young diagrams. In this paper, we compute closedform expressions for the fusion rule of $\mathscr{C}$, using Littlewood-Richardson coefficients, as well as the characters (including a generating function), using symmetric functions with infinite variables.


## 1. Introduction

Jones introduced planar algebras in 2 inspired by subfactor theory and knot theory. The topological notion of (spherical) planar algebra is parallel to the algebraic notion of (spherical) pivotal, monoidal category. The first author investigated the skein-theoretical classification of planar algebras and discovered a continuous family of unshaded planar algebras over $\mathbb{C}$ from the classification of Yang-Baxter relation planar algebras in [3]. This family was constructed in terms of $q$-parameterized generators and relations in linear skein theory. This family could be regarded as a planar algebra $\mathscr{C}$ over $\mathbb{C}(q)$ with a generic parameter $q$. The canonical idempotent category of this unshaded planar algebra $\mathscr{C}$ is a $\mathbb{Z}_{2}$-graded spherical monoidal category. It was shown in [3] that the Grothendick ring $\mathcal{G}$ of $\mathscr{C}$ has simple objects $X_{\lambda}$ labelled by all Young diagrams $\lambda$ with an explicit construction of a minimal idempotent $\tilde{y}_{\lambda}$ in $\mathscr{C}$ which represents $X_{\lambda}$.

The main purpose of this paper is computing the fusion rule of $\mathcal{G}$,

$$
\begin{equation*}
X_{\mu} X_{\nu}=\sum_{\lambda} R_{\mu, \nu}^{\lambda} X_{\lambda} \tag{1}
\end{equation*}
$$

where $R_{\mu, \nu}^{\lambda} \in \mathbb{N}$ is called the fusion coefficient for Young diagrams $\mu, \nu, \lambda$. We compute the fusion coefficient $R_{\mu, \nu}^{\lambda}$ in a closed-form expression in Theorem 4.23 ,

$$
\begin{equation*}
R_{\mu, \nu}^{\lambda}=\sum_{\alpha, \beta, \gamma} c_{\alpha, \beta}^{\mu} c_{\beta^{\prime}, \gamma}^{\nu} c_{\alpha, \gamma}^{\lambda}, \tag{2}
\end{equation*}
$$

where $\gamma^{\prime}$ is the Young diagram dual to $\gamma$ and $c_{\text {:, }}$, is the Littlewood-Richardson coefficient.
The paper is organized as follows. In $\S 2$, we recall some basic properties of $\mathscr{C}$ and its type $A$ Hecke subalgebra $H$. We prove in Theorem 2.8 that the Grothendieck ring $\mathcal{G}$ of $\mathscr{C}$ is the polynomial ring freely generated by the fundamental representations $\left\{X_{1^{n}}: n \in \mathbb{N}\right\}$, where $1^{n}$ is the Young diagram with one column and $n$ cells. In particular, $\mathcal{G}$ is commutative.

In 43 , we compute the fusion rule of $\mathcal{G}$ with respect to the fundamental representations in Theorem 3.11 .

$$
\begin{equation*}
X_{\left(1^{r}\right)} X_{\mu}=\sum_{i=0}^{r} \sum_{\nu \in \mu-i} \sum_{\lambda \in \nu+1^{r-i}} X_{\lambda} . \tag{3}
\end{equation*}
$$

The multiplicity of $X_{\lambda}$ is the number of ways of constructing $\lambda$ from $\mu$ by removing $i$ cells, no two in the same column, and then adding $r-i$ cells, no two in the same row. The proof follows from an
explicit construction of the basis of $\operatorname{hom}\left(X_{\left(1^{r}\right)} X_{\mu}, X_{\lambda}\right)$ in $\mathscr{C}$ through the linear skein theory of the Yang-Baxter relation planar algebra.

In $\S 4$, we compute fusion rules of $\mathcal{G}$. In principle, one can compute the fusion rule of $\mathcal{G}$ recursively using the fusion rule of fundamental representations. However, the complexity grows exponentially w.r.t. the size of the Young diagrams. We observe that $\mathcal{G}$ is isomorphic to the ring $\Lambda$ of symmetric polynomial with infinite variables. We establish a ring isomorphism $\Phi: \mathcal{G} \rightarrow \Lambda$ in Definition 4.19 and consider $\Phi\left(X_{\lambda}\right)$ as the character of the simple object $X_{\lambda}$ of $\mathcal{G}$. We prove in Theorem 4.20 that

$$
\begin{equation*}
\Phi\left(X_{\lambda}\right)=\sum_{\mu}(-1)^{|\mu|} s_{\lambda / 2 \mu} \tag{4}
\end{equation*}
$$

where $s_{\lambda / 2 \mu}$ is a skew-Schur polynomial. Moreover, we compute the generating function of the characters in closed form in Theorem 4.22 ,

$$
\begin{equation*}
\sum_{\lambda} s_{\lambda}(x) \Phi\left(X_{\lambda}\right)(y)=\prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i} x_{j}} \prod_{i, j} \frac{1}{1-x_{i} y_{j}} \tag{5}
\end{equation*}
$$

Using the generating function, we compute the fusion coefficient in a closed form, namely Equation (11), in Theorem 4.23. Our computational tools on the characters and the generating function come from the theory of symmetric functions [4] which we recall in $\$ 4$

In this paper, we compute the fusion rule of $\mathscr{C}$ over $\mathbb{C}(q)$. Unitary fusion categories $\mathscr{C}^{N, k, \ell}$, $N, k, \ell \in \mathbb{N}$, were constructed in 3 as quotients of $\mathscr{C}$ over $\mathbb{C}$ at $q=e^{\frac{\pi i}{2 N+2}}$. In particular, $\mathscr{C}^{N, 0,1}$ is an exceptional quantum subgroup of $S U(N)_{N+2}$, conjectured to be isomorphic to the exceptional quantum subgroup constructed by Xu in [7] in 1998 through the $\alpha$-induction of the conformal inclusion $S U(N)_{N+2} \subseteq S U(N(N+1) / 2)_{1}$. Xu asked the question to compute the fusion rules of these exceptional quantum subgroups [6], which we will compute in the future. From this point of view, $\mathscr{C}$ can be regarded as the parameterization of a family of exceptional quantum subgroups. It was conjecture in [3] that there is a continuous family of monoidal categories parameterizing the exceptional quantum subgroups from the $\alpha$-induction of any family of conformal inclusions of quantum groups. We believe that our methods in this paper also apply to the other continuous families of monoidal categories, if they exist.

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## 2. Yang-Baxter relation planar algebras and spherical categories

The first author constructed the following continuous family of Yang-Baxter relation planar algebras $\mathscr{C}$ in terms of generators and relations in linear skein theory in 3.

Definition 2.1. Let $\mathscr{C}_{\bullet}$ be the unshaded planar algebra over $\mathbb{C}(q)$ with circle parameter

$$
\bigcirc=\delta=q+q^{-1}
$$

which is generated by $R={ }_{R} \not$ with the following relations:

$$
\begin{align*}
& { }^{2} \neq i \neq 2  \tag{6}\\
& \text { R } \bigcirc=0 \text {, }  \tag{7}\\
& \left.{ }_{R}^{R}\right\rangle=\left\lvert\,-\frac{1}{\delta} \circlearrowright\right., \tag{8}
\end{align*}
$$

The vector space $\mathscr{C}_{n}$ consists of linear sums of $R$-labelled planar diagrams with $2 n$ boundary points modulo the above relations. Consider a disc with $2 n$ boundary points numbered by $\{1,2, \ldots, 2 n\}$ clockwise. A pairing $p$ of $\{1,2, \ldots, 2 n\}$ is a bijection on $\{1,2, \ldots, 2 n\}$, such that $p^{2}$ is the identity and $p(i) \neq i, \forall 1 \leq i \leq 2 n$. We call $\{i, p(i)\}$ a pair of the paring $p$. Let $P_{n}$ be the set of pairings of $2 n$ boundary points.

We can construct a diagram in the disc which connects the $n$ pairs of boundary points by $n$ strings with a minimal number of crossings. (The minimal condition is equivalent to that any two strings either intersect at one point transversally or do not intersect.) Such diagrams have been used by Brauer to construct the Brauer algebras. We label each crossing of the diagram by the generator $R$, then we obtain an element in $\mathscr{C}_{n}$, denoted by $\hat{p}$. Note that there are four choices to label $R$ at each crossing, and the corresponding elements in $\mathscr{C}_{n}$ differ by a phase due to Relation (6). We fix a choice at the beginning to define $\hat{p}$.

Proposition 2.2. The set $\mathcal{B}_{n}=\left\{\hat{p}: p \in P_{n}\right\}$ is a basis of the vector space $\mathscr{C}_{n}$ over $\mathbb{C}(q)$.
Proof. Applying the Yang-Baxter relation, any element in $\mathscr{C}_{n}$ is a linear sum of such $\hat{p}$ 's. On the other hand, $\operatorname{dim} \mathscr{C}_{n}=(2 n-1)!$ ! by Corollary 6.6 in [3], and $\#\left\{\hat{p}: p \in P_{n}\right\}=(2 n-1)!!$. Therefore, $\left\{\hat{p}: p \in P_{n}\right\}$ is a basis of $\mathscr{C}_{n}$.

Remark 2.3. Note that $A_{R}^{R /}$ and $A_{A}^{R}$ correspond to the same pairing. When we define the element $\hat{p}$, we fix a choice. Either $A_{R}^{R}$ or with the 14 lower terms

$$
\begin{aligned}
& |\cup, \cup|,| | 1, \Upsilon / \bigcap, \bigcap,
\end{aligned}
$$

$$
\begin{aligned}
& \underbrace{R}_{R}, ~ A_{R}^{R}, ~ R,
\end{aligned}
$$

form a basis of $\mathscr{C}_{3}$.
A planar algebra canonically has a vertical multiplication, a horizontal tensor product and a Markov trace, see [2]. For a diagram in $\mathscr{C}_{n}$, we draw the first $n$ boundary points on the top and the last $n$ boundary points at the bottom. Then $\mathscr{C}_{n}$ forms an algebra whose multiplication is gluing the diagrams vertically. The tensor product $\otimes: \mathscr{C}_{n} \otimes \mathscr{C}_{m} \rightarrow \mathscr{C}_{n+m}$ is a horizontal union of two diagrams.

In particular, the algebra $\mathscr{C}_{n}$ can be embedded in $\mathscr{C}_{n+1}$ by adding a through string on the right. So $\mathscr{C}_{\bullet}$ is a filtered algebra.

The planar algebra $\mathscr{C}_{\bullet}$ has a has a type $A$ Hecke subalgebra $H_{\bullet}$ generated by

$$
\alpha=\frac{q-q^{-1}}{2}| |+\frac{q-q^{-1}}{2 i} \bigcup^{\circlearrowright}+\frac{q+q^{-1}}{2} R^{2} / .
$$

The generic type $A$ Hecke algebra has two parameters $q$ and $r$. Here $q r=\sqrt{-1}$. The planar algebra has a Markov trace by gluing the top and the bottom from the right. The Markov trace of $\mathscr{C}_{\bullet}$ extends the Markov trace of the Hecke algebra in [1].

The idempotent $e=\frac{1}{\delta} \bigcup$ in $\mathscr{C}_{2}$ is called the Jones idempotent ${ }^{1}$. The two-sided ideal of $\mathscr{C}_{n}$ generated by $e$ is denoted by $\mathscr{I}_{n}$, called the basic construction ideal. The complement of the maximal idempotent of $\mathscr{I}_{n}$ is denoted by $s_{n}$. Then, $s_{n}$ is central in $\mathscr{C}_{n}$ and

$$
\begin{align*}
x s_{n} & =0, \forall x \in \mathscr{I}_{n}  \tag{10}\\
s_{n}\left(s_{m} \otimes s_{n-m}\right) & =s_{n}, \forall m \leq n . \tag{11}
\end{align*}
$$

The following proposition is a consequence of Theorem 6.5 in [3]:
Proposition 2.4. For any $n \geq 0$,

$$
\begin{align*}
H_{n} & \cong s_{n} H_{n}=s_{n} \mathscr{C}_{n}  \tag{12}\\
\mathscr{C}_{n} & =\mathscr{I}_{n} \oplus H_{n}=\mathscr{I}_{n} \oplus s_{n} H_{n} \tag{13}
\end{align*}
$$

For each Young diagram $\lambda$, let $|\lambda|=n$ be the number of cells of $\lambda$. A minimal idempotent $y_{\lambda}$ in $\operatorname{hom}_{\mathscr{C}}\left(X^{n}, X^{n}\right)$ was constructed in Section 2.5 in [3]. These $y_{\lambda}$ 's are representatives of the equivalent minimal idempotents of $H_{n}$. Furthermore, $\tilde{y}_{\lambda}=s_{n} y_{\lambda}$ are representatives of the equivalent minimal idempotents of $s_{n} H_{n}$. We refer the readers to [3] for the explicit construction of $s_{n}, y_{\lambda}$ and $\tilde{y}_{\lambda}$. Following the construction in Theorem 6.5 in [3], we have that

$$
\begin{equation*}
y_{\left(1^{n}\right)}=\tilde{y}_{\left(1^{n}\right)} \tag{14}
\end{equation*}
$$

where $\left(1^{n}\right)$ is the Young diagram with one column and $n$ cells. The Young diagram with one row and $n$ cells is denoted by $(n)$. We write them as $1^{n}$ and $n$ for short, if there is no risk of confusion.

We recall the above properties of $\mathscr{C}$, which we will apply in this paper. We refer the readers to [3] for the construction of $s_{n}, y_{\lambda}$ and $\tilde{y}_{\lambda}$, which we do not repeat here.
Proposition 2.5. Note that $\operatorname{hom}_{H}\left(y_{\mu} \otimes y_{\nu}, y_{\lambda}\right) \subseteq H_{n} \subseteq \mathscr{C}_{n}$, when $|\mu|+|\nu|=|\lambda|=n$. We have that

$$
s_{n} \operatorname{hom}_{H}\left(y_{\mu} \otimes y_{\nu}, y_{\lambda}\right)=\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)
$$

Proof. Note that $s_{n} y_{\mu} \otimes y_{\nu}=\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}$ and $s_{n} y_{\lambda}=\tilde{y}_{\lambda}$, so

$$
s_{n} \operatorname{hom}_{H}\left(y_{\mu} \otimes y_{\nu}, y_{\lambda}\right) \subseteq \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)
$$

On the other hand, by Equation 13 , for any element $x$ in $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right) \subseteq \mathscr{C}_{n}$, we have a unique decomposition $x=y+z$, such that $y \in \mathscr{I}_{n}$ and $z \in H_{n}$. Note that

$$
\begin{aligned}
& y_{\lambda} x\left(y_{\mu} \otimes y_{\nu}\right)=x, \\
& y_{\lambda} y\left(y_{\mu} \otimes y_{\nu}\right) \in \mathscr{I}_{n}, \\
& y_{\lambda} z\left(y_{\mu} \otimes y_{\nu}\right) \in H_{n} .
\end{aligned}
$$

[^0]So $y_{\lambda} z\left(y_{\mu} \otimes y_{\nu}\right)=z$, and $z \in \operatorname{hom}_{H}\left(y_{\mu} \otimes y_{\nu}, y_{\lambda}\right)$. Moreover, $s_{n} z=s_{n} x=x$. So

$$
s_{n} \operatorname{hom}_{H}\left(y_{\mu} \otimes y_{\nu}, y_{\lambda}\right)=\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)
$$

From the semi-simple, spherical, unshaded planar algebra $\mathscr{C}$, we obtain an $\mathbb{N}$-graded monoidal category $\mathscr{C}^{q, 0}$ whose degree $n$ objects are idempotents in $\mathscr{C}_{n}$. In subfactor theory, we usually consider the Jones idempotent to be equivalent to the unit in $\mathscr{C}_{0}$, namely the empty diagram $\emptyset$. The isometries between the two idempotents are given by the the diagrams $\cap$ and $\cup$. Modulo this relation, we obtain a $\mathbb{Z}_{2}$-graded spherical category $\mathscr{C}^{q, 1}$, the canonical one associated with the spherical planar algebra $\mathscr{C}$.

Notation 2.6. Let $\mathcal{G}$ be the Grothendieck ring of $\mathscr{C}^{q, 1}$. It has a basis $X_{\lambda}$ corresponding to the minimal idempotents $\tilde{y}_{\lambda}$ of $s_{n} \mathscr{C}_{n}, n \in \mathbb{N}$, for all Young diagrams $\lambda$.

Let $1^{r}$ be the Young diagram with one column and $r$ cells. In particular, $X=X_{1}$, corresponds to the Young diagram with one cell. Then the identity map $1_{X}$ is a through string, and

$$
\begin{aligned}
& \left(\cup \otimes 1_{X}\right)\left(1_{X} \otimes \cap\right)=1_{X}, \\
& \left(1_{X} \otimes \cup\right)\left(\cap \otimes 1_{X}\right)=1_{X} .
\end{aligned}
$$

The morphism space $\operatorname{hom}_{\mathscr{C}}\left(X^{n}, X^{m}\right)$ consists of linear combinations of $R$-labelled planar diagrams in $\mathscr{C}$ with $n$ boundary points on the top and $m$ boundary points at the bottom. For Young diagrams $\mu, \nu, \lambda$, the morphisms of $\mathscr{C}^{q, 1}$ are given by

$$
\operatorname{hom}_{\mathscr{C}}\left(X_{\mu} \otimes X_{\nu}, X_{\lambda}\right)=\tilde{y}_{\lambda}\left(\operatorname{hom}_{\mathscr{C}}\left(X^{|\mu|+|\nu|}, X^{|\lambda|}\right)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}\right)\right.
$$

Notation 2.7. Let $Y_{\lambda}$ be the element of $\mathcal{G}$ corresponding to the idempotent $y_{\lambda}$. Note that $\tilde{y}_{\lambda}=s_{|\lambda|} y_{\lambda}$, so $y_{\lambda}-\tilde{y}_{\lambda}$ is an idempotent in $\mathscr{I}_{|\lambda|}$. Therefore,

$$
\begin{equation*}
Y_{\lambda}=X_{\lambda}+\sum_{|\mu|<|\lambda|} n_{\lambda, \mu} X_{\mu}, \text { for some } n_{\lambda, \mu} \in \mathbb{N} \tag{15}
\end{equation*}
$$

We call $n_{\lambda, \mu}$ the extended constants. Then we can solve for the $X_{\lambda}$ recursively in terms of the $Y_{\lambda}$, and

$$
\begin{equation*}
X_{\lambda}=Y_{\lambda}+\sum_{|\mu|<|\lambda|} z_{\lambda, \mu} Y_{\mu}, \text { for some } z_{\lambda, \mu} \in \mathbb{Z} \tag{16}
\end{equation*}
$$

We call $z_{\lambda, \mu}$ the inverse extended constants. By Equation (14), for any $n \geq 0$,

$$
\begin{equation*}
X_{1^{n}}=Y_{1^{n}} \tag{17}
\end{equation*}
$$

Theorem 2.8. The Grothendieck ring $\mathcal{G}$ of $\mathscr{C}$ is the polynomial ring in the generators $\left\{X_{1^{n}}: n>0\right\}$. In particular, $\mathcal{G}$ is commutative.
Proof. Note that $\left\{X_{\lambda}\right\}$ forms a basis of the Grothendieck ring $\mathcal{G}$. By Equations (15) and (16), \{Y $\}$ also forms a basis of $\mathcal{G}$. It is known that the set $\left\{Y_{\lambda}\right\}$ is a basis of the polynomial ring in the generators $\left\{Y_{1^{n}}: n>0\right\}$. By Equation 17, $\mathcal{G}$ is the polynomial ring in the generators $\left\{X_{1^{n}}: n>0\right\}$.

Based on the algebraic structure on $\mathscr{C}_{n} \cong \operatorname{hom}_{\mathscr{C}}\left(X^{n}, X^{n}\right)$, We decompose the partitions $P_{n}$ into two subset sets $I_{n}$ and $T_{n}$. A pairing is in $I_{n}$ if there is a pair among the first $n$ points. On the other hand, a pairing $p$ in $T_{n}$ pairs the first $n$ points with the last $n$ points. For any pairing $p \in T_{n}$, we can identify $p$ with an element $p^{\prime}$ in the permutation group $S_{n}$, such that $p^{\prime}(i)=2 n+1-p(i)$, for $1 \leq i \leq n$.

Proposition 2.9. For a pairing $p \in P_{n}$, we have that $p \in I_{n}$ iff $\hat{p} \in \mathscr{I}_{n}$. Moreover, $\left\{s_{n} \hat{p}: p \in T_{n}\right\}$ is a basis of $s_{n} H_{n}$.

Proof. Obviously if $p \in P_{n}$, then $\hat{p} \in \mathscr{I}_{n}$ and $s_{n} \hat{p}=0$. By Equation (13), $\left\{s_{n} \hat{p}: p \in T_{n}\right\}$ is a spanning set of $s_{n} H_{n}$. By Equation (12), $\operatorname{dim} s_{n} H_{n}=\operatorname{dim} H_{n}=\# T_{n}$, so $\left\{s_{n} \hat{p}: p \in T_{n}\right\}$ is a basis of $s_{n} H_{n}$, and for any $p \in T_{n}, s_{n} \hat{p} \neq 0$, namely $\hat{p} \notin \mathscr{I}_{n}$.

We define $T_{i, n-i}$ to be a subset of $P_{n}$ as follows: $T_{0, n}=T_{n, 0}=T_{n}$, and for any $1 \leq i \leq n-1$,

$$
T_{i, n-i}=\left\{p \in P_{n}: 2 n+1-j \leq p(j) \leq 2 n, \forall 1 \leq j \leq i ; n+1 \leq p(j) \leq 2 n-i, \forall i+1 \leq j \leq n\right\}
$$

We can consider $T_{i, n-i}$ as $T_{i} \times T_{n-i}$. Note that $T_{i, n-i} \subseteq T_{n}$.
Notation 2.10. For any Young diagram $\lambda,|\lambda|=n$, we express $\tilde{y}_{\lambda}$ in terms of the basis $\mathcal{B}_{n}$,

$$
\tilde{y}_{\lambda}=\sum_{p \in P_{n}} c_{p} \hat{p}, c_{p} \in \mathbb{C}(q)
$$

We define

$$
\begin{equation*}
\tilde{y}_{\lambda, i}=\sum_{p \in T_{i, n-i}} c_{p} \hat{p} \tag{18}
\end{equation*}
$$

Lemma 2.11. For any Young diagram $\lambda$, we have that $\tilde{y}_{\lambda, 0} \tilde{y}_{\lambda}=\tilde{y}_{\lambda} \tilde{y}_{\lambda, 0}=\tilde{y}_{\lambda}$.
Proof. Note that $s_{n} \hat{p}=0, \forall p \in I_{n}$, and $\tilde{y}_{\lambda}=s_{n} \tilde{y}_{\lambda}$. So

$$
\tilde{y}_{\lambda} \tilde{y}_{\lambda, 0}=\tilde{y}_{\lambda} s_{n} \sum_{p \in T_{n}} c_{p} \hat{p}=\tilde{y}_{\lambda} s_{n} \sum_{p \in P_{n}} c_{p} \hat{p}=\tilde{y}_{\lambda} s_{n} \tilde{y}_{\lambda}=\tilde{y}_{\lambda} .
$$

Similarly, $\tilde{y}_{\lambda, 0} \tilde{y}_{\lambda}=\tilde{y}_{\lambda}$.
Lemma 2.12. For any $0 \leq i \leq n$, we have that

$$
\tilde{y}_{1^{n}, i}\left(\tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{n-i}}\right)=c \tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{n-i}}
$$

for some $c \neq 0$ in $\mathbb{C}(q)$, and $\lim _{q \rightarrow 1} c=\binom{n}{i}^{-1}$.
Proof. By Proposition 2.4, $\tilde{y}_{1^{k}}$ is a minimal central projection in $\mathscr{C}_{k}$, so for any $p \in T_{i, n-i}$, $\hat{p}\left(\tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{n-i}}\right)$ is a multiple of $\tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{n-i}}$. Therefore,

$$
\tilde{y}_{1^{n}, i}\left(\tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{n-i}}\right)=c \tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{n-i}},
$$

for some $c \in \mathbb{C}(q)$. Moreover, $\tilde{y}_{1^{n}, i} \tilde{y}_{1^{n}}=c \tilde{y}_{1^{n}}$. We need to show that $c \neq 0$. For any $p \in T_{n}$, we consider $p$ as a permutation. Without loss of generality, we assume that the strings of $\hat{p}$ move vertically and the generator $R$ 's of $\hat{p}$ are all labelled on the left side of the crossings. Note that $R \tilde{y}_{1^{2}}=-\tilde{y}_{1^{2}}$, so

$$
\hat{p} \tilde{y}_{1^{n}}=(-1)^{|p|} \tilde{y}_{1^{n}},
$$

where $|p|$ is the number of the crossing $R$ 's in $\hat{p}$. We express $\tilde{y}_{1^{n}}, y_{1^{n}}$ and $y_{1^{i}} \otimes y_{1^{n-i}}$ in terms of the basis $\mathcal{B}_{n}$ as

$$
\begin{aligned}
& \tilde{y}_{1^{n}}=\sum_{p \in P_{n}} c_{p} \hat{p}, \\
& y_{1^{n}}=\sum_{p \in P_{n}} c_{p}^{\prime} \hat{p} .
\end{aligned}
$$

Then

$$
c=\sum_{p \in T_{i, n}-i} \frac{(-1)^{|p|}}{n!} c_{p} .
$$

Recall that $\tilde{y}_{\lambda}=s_{n} y_{\lambda}$, so by Proposition 2.9 ,

$$
c_{p}=c_{p}^{\prime}, \forall p \in T_{n}
$$

When $q \rightarrow 1$, the Hecke algebra $H$ specializes to the symmetric group algebra; the generator $\alpha$ becomes the symmetric braiding; $\alpha-R \rightarrow 0$; and $n!y_{1^{n}}$ becomes the alternating sum of permutations of the symmetric groups $S_{n}$. So for any $p \in T_{n}$,

$$
\begin{equation*}
\lim _{q \rightarrow 1} c_{p}^{\prime}=\frac{(-1)^{|p|}}{n!} \tag{19}
\end{equation*}
$$

Then

$$
\begin{equation*}
\lim _{q \rightarrow 1} c=\lim _{q \rightarrow 1} \sum_{p \in T_{i, n-1}} \frac{(-1)^{|p|}}{n!} c_{p}=\sum_{p \in T_{i, n-i}} \frac{1}{n!}=\binom{n}{i}^{-1} . \tag{20}
\end{equation*}
$$

Therefore, $C \neq 0$ in $\mathbb{C}(q)$.

## 3. Fusion Rules of Fundamental Representations for the Generic Case

In this section, we compute the fusion rule for $X_{1^{n}} \otimes$ in the Grothendieck ring $\mathcal{G}$, and construct a basis for the hom space. We apply this to study the characters of the simple objects using symmetric functions. Recall that $\left\{X_{\lambda}\right\}$ indexed by Young diagrams are a basis of $\mathcal{G}$. The structure constants of the multiplication $R_{\mu, \nu}^{\lambda}$ are defined by:

$$
X_{\mu} X_{\nu}=\sum_{\lambda} R_{\mu, \nu}^{\lambda} X_{\lambda}
$$

Notation 3.1. We define the morphism $\cup_{n}$ in $\operatorname{hom}_{\mathscr{C}}\left(X^{2 n}, \emptyset\right)$ as

where the label $n$ in the first picture indicates the number of parallel strings.
By Proposition 9.2 in [3], the dual object (or the $180^{\circ}$ rotation) of $\tilde{y}_{1^{n}}$ is $\tilde{y}_{n}$. In particular, $\cup_{n}\left(\tilde{y}_{1^{n}} \otimes \tilde{y}_{n}\right)$ is a non-zero morphism in $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{n} \otimes \tilde{y}_{1^{n}}, \emptyset\right)$.

Proposition 3.2. The dual object of $\tilde{y}_{\lambda}$ is $\tilde{y}_{\lambda^{\prime}}$, where $\lambda^{\prime}$ is the reflection of $\lambda$ in the diagonal, called the Young diagram dual to $\lambda$.

Proof. This is a consequence of Proposition 9.6 in 3. We give a quick proof here. The duality map $\lambda \rightarrow \lambda^{\prime}$ is a $\mathbb{Z}_{2}$ automorphism of the principal graph of the planar algebra $\mathscr{C}$, which is Young's lattice. This $\mathbb{Z}_{2}$ fixes the the Young diagrams $\emptyset$ and 1 , and switches $1^{2}$ and 2 . Therefore it has to be the reflection in the diagonal.
Notation 3.3. For any Young diagram $\lambda$, we define the following sets of Young diagrams:
(1) $\lambda-1^{n}$ are Young diagrams that removes $n$ cells from $\lambda$, and no two cells in the same row;
(2) $\lambda+1^{n}$ are Young diagrams that adds $n$ cells to $\lambda$, and no two cells in the same row;
(3) $\lambda-n$ are Young diagrams that removes $n$ cells from $\lambda$, and no two cells in the same column;
(4) $\lambda+n$ are Young diagrams that adds $n$ cells to $\lambda$, and no two cells in the same column.

The following result is well-known for the type $A$ Hecke algebra. It can be derived from the fusion rule of fundamental representations of (quantum) $S U(N)$, as $N \rightarrow \infty$. The fusion rule can be characterized by Schur polynomials.

Lemma 3.4. Suppose $\lambda$ and $\mu$ are Young diagrams. If $n=|\mu|-|\lambda| \geq 0$, then

$$
\operatorname{dim} \operatorname{hom}_{H}\left(y_{\lambda} \otimes y_{1^{n}}, y_{\mu}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda+1^{n} \\
0, \forall \mu \notin \lambda+1^{n}
\end{array}\right.
$$

We give an explicit construction of a non-zero morphism $\rho$ in $\operatorname{hom}_{H}\left(y_{\lambda} \otimes y_{1^{n}}, y_{\mu}\right)$.
Lemma 3.5. Suppose $\lambda$ and $\mu$ are Young diagrams. If $n=|\mu|-|\lambda| \geq 0$, then

$$
\begin{aligned}
\operatorname{dim}_{\operatorname{hom}_{\mathscr{C}}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\mu}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda+1^{n} \\
0, \forall \mu \notin \lambda+1^{n}
\end{array}\right. \\
\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{n}, \tilde{y}_{\mu}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda+n \\
0, \forall \mu \notin \lambda+n
\end{array}\right.
\end{aligned}
$$

If $n=|\lambda|-|\mu| \geq 0$, then

$$
\begin{aligned}
\operatorname{dim}_{\operatorname{hom}_{\mathscr{C}}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\mu}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda-n \\
0, \forall \mu \notin \lambda-n
\end{array}\right. \\
\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{n}, \tilde{y}_{\mu}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda-1^{n} \\
0, \forall \mu \notin \lambda-1^{n}
\end{array}\right.
\end{aligned}
$$

Proof. If $n=|\mu|-|\lambda| \geq 0$, then by Equation (12), Proposition 2.5 and Lemma (3.4), we have

$$
\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\mu}\right)=\operatorname{dim} \operatorname{hom}_{H}\left(y_{\lambda} \otimes y_{1^{n}}, y_{\mu}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda+1^{n} \\
0, \forall \mu \notin \lambda+1^{n}
\end{array}\right.
$$

The planar algebra $\mathscr{C}$ has a $\mathbb{Z}_{2}$ automorphism $\Omega$ mapping the generator $R$ to $-R$. By Proposition 9.5 in [3], the idempotent $\Omega\left(\tilde{y}_{\lambda}\right)$ is equivalent to $\tilde{y}_{\lambda^{\prime}}$. (The dual Young diagram $\lambda^{\prime}$ is denoted by $\Omega(\lambda)$ in 3].)

In particular, $n^{\prime}=1^{n}$. Note that $\mu \in \lambda+n$ iff $\mu^{\prime} \in \lambda^{\prime}+1^{n}$. So

$$
\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{n}, \tilde{y}_{\mu}\right)=\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda^{\prime}} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\mu^{\prime}}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda+n \\
0, \forall \mu \notin \lambda+n
\end{array}\right.
$$

By Proposition 9.2 in [3], the dual object (or $180^{\circ}$ rotation) of $\tilde{y}_{1^{n}}$ is $\tilde{y}_{n}$. If $n=|\lambda|-|\mu| \geq 0$, then by Frobenius reciprocity,

$$
\begin{aligned}
& \operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\mu}\right)=\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda}, \tilde{y}_{\mu} \otimes \tilde{y}_{n}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda-n \\
0, \forall \mu \notin \lambda-n
\end{array}\right. \\
& \operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\lambda} \otimes \tilde{y}_{n}, \tilde{y}_{\mu}\right)=\operatorname{dim}_{\operatorname{hom}}^{\mathscr{C}}\left(\tilde{y}_{\lambda}, \tilde{y}_{\mu} \otimes \tilde{y}_{1^{n}}\right)=\left\{\begin{array}{l}
1, \forall \mu \in \lambda-1^{n} \\
0, \forall \mu \notin \lambda-1^{n}
\end{array}\right.
\end{aligned}
$$

Notation 3.6. Suppose $a, b, c \in \mathbb{N}$, and $n=a+b+c$. Let $p_{a, b, c} \in P_{n}$ be the pairing

$$
p_{a, b, c}(k)= \begin{cases}(2 n+1-k), & \forall 1 \leq k \leq a \text { or } 2 n-a<k \leq 2 n  \tag{21}\\ (2 a+2 b+1-k), & \forall a<k \leq a+2 b \\ (2 n+2 b+1-k), & \forall n+b<k \leq 2 n-a\end{cases}
$$

We can identify $\hat{p}_{a, b, c} \in \mathscr{C}_{n}$ as a morphism in $\operatorname{hom}_{\mathscr{C}}\left(X^{a+b} \otimes X^{b+c}, X^{a+c}\right)$, illustrated as

where $a, b, c$ in the first picture indicate the number of parallel strings.
Notation 3.7. Suppose $\mu$ is a Young diagram, $|\mu|=a+b$. Take Young diagrams $\nu \in \mu-b$ and $\lambda \in \nu+1^{c}$. By Lemma 3.5, there are non-zero morphisms $\rho_{1, \nu} \in \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\nu} \otimes \tilde{y}_{b}\right)$ and $\rho_{2, \nu} \in \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\nu} \otimes \tilde{y}_{1^{c}}, \tilde{y}_{\lambda}\right)$. We construct a morphism $\rho_{\mu, \nu, \lambda}^{\prime} \in \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}, \tilde{y}_{\lambda}\right)$ as

$$
\begin{equation*}
\rho_{\mu, \nu, \lambda}^{\prime}:=\rho_{2, \nu} \hat{p}_{a, b, c}\left(\rho_{1, \nu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \tag{22}
\end{equation*}
$$

We identify $\tilde{y}_{b+c}$ with a morphism in $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{1^{b+c}}, \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right)$ and construct a morphism $\rho_{\mu, \nu, \lambda}^{\prime} \in$ $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b+c}}, \tilde{y}_{\lambda}\right)$ :

$$
\begin{equation*}
\rho_{\mu, \nu, \lambda}:=\rho_{\mu, \nu, \lambda}^{\prime}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{b+c}\right)=\rho_{2, \nu} \hat{p}_{a, b, c}\left(\rho_{1, \nu} \otimes \tilde{y}_{b+c}\right) . \tag{23}
\end{equation*}
$$

Their pictorial representations are



Lemma 3.8. Suppose $a, b, c \in \mathbb{N}$ and $r=b+c$. For any Young diagrams $\mu$ and $\lambda,|\mu|=a+b,|\lambda|=$ $a+c$, the elements $\left\{\rho_{\mu, \nu, \lambda}^{\prime}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ are linearly independent in home $\mathscr{C}^{\prime}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}, \tilde{y}_{\lambda}\right)$.
Proof. By Frobenius reciprocity, for any $\nu,\left(\tilde{y}_{\nu} \otimes \cup_{b}\right)\left(\rho_{\mu, \nu} \otimes \tilde{y}_{1^{b}}\right) \neq 0$ in $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}}, \tilde{y}_{\nu}\right)$. As $\mathscr{C}$ is semi-simple, there is a morphism $\rho_{3, \nu} \in \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\nu}, \tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}}\right)$, such that

$$
\left(\tilde{y}_{\nu} \otimes \cup_{b}\right)\left(\rho_{\mu, \nu} \otimes \tilde{y}_{1^{b}}\right) \rho_{3, \nu}=\tilde{y}_{\nu}
$$

If

$$
\sum_{\nu \in \mu-b, \nu \in \lambda-1^{c}} c_{\nu} \rho_{\mu, \nu, \lambda}^{\prime}=0, c_{\nu} \in \mathbb{C}(q)
$$

then for any $\nu^{\prime} \in \mu-b, \nu^{\prime} \in \lambda-1^{c}$,

$$
\rho_{3, \nu^{\prime}} \sum_{\nu \in \mu-b, \nu \in \lambda-1^{c}} \rho_{\mu, \nu, \lambda}^{\prime}=c_{\nu^{\prime}} \rho_{2, \nu^{\prime}}=0
$$

So $c_{\nu^{\prime}}=0$. Therefore, $\left\{\rho_{\mu, \nu, \lambda}^{\prime}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ are linearly independent.

Lemma 3.9. Suppose $a, b, c \in \mathbb{N}$ and $r=b+c$. For any Young diagrams $\mu$ and $\lambda$ with $|\mu|=a+b$, $|\lambda|=a+c$, the morphisms $\left\{\rho_{\mu, \nu, \lambda}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ form a spanning set of hom $\mathscr{C}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{r}}, \tilde{y}_{\lambda}\right)$.
Proof. For any $p_{1} \in P_{a+b}, p_{2} \in P_{b+c}$ and $p_{3} \in P_{a+c}$. we define

$$
\begin{equation*}
x_{p_{1}, p_{2}, p_{3}}=\hat{p}_{3} \hat{p}_{a, b, c}\left(\hat{p}_{1} \otimes \hat{p}_{2}\right) \tag{24}
\end{equation*}
$$

By Proposition 2.2, $\left\{x_{p_{1}, p_{2}, p_{3}}: p_{1} \in P_{a+b}, p_{2} \in P_{b+c}, p_{3} \in P_{a+c}.\right\}$ is a spanning set of $\mathscr{C}_{n}$, because any pairing in $P_{n}$ can be implemented by some diagram $x_{p_{1}, p_{2}, p_{3}}$ with a minimal number of crossings. Note that $y_{1^{b+c}}$ is a central minimal idempotent in $\mathscr{C}_{b+c}$. By Equation $12, \tilde{y}_{1^{b+c}}$ is a central minimal idempotent in $H_{b+c}$.
 $\left.\tilde{y}_{1}{ }^{c}\right) \tilde{y}_{1^{b+c}}=\tilde{y}_{1^{b+c}}$. We define

$$
\tilde{x}_{p_{1}, p_{2}, p_{3}}=\tilde{y}_{\lambda} x_{p_{1}, p_{2}, p_{3}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b+c}}\right)
$$

Then $\left\{\tilde{x}_{p_{1}, p_{2}, p_{3}}: p_{1} \in P_{a+b}, p_{2} \in P_{b+c}, p_{3} \in P_{a+c}.\right\}$ is a spanning set of hom $\mathscr{C}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{r}}, \tilde{y}_{\lambda}\right)$. Recall that the $180^{\circ}$ rotation of $\tilde{y}_{1^{b}}$ is $\tilde{y}_{b}$. So

for some $\rho_{1, j} \in \operatorname{hom}_{\tilde{y}_{\mu}, \tilde{y}_{\nu} \otimes \tilde{y}_{b}}, \rho_{2, j} \in \operatorname{hom}_{\tilde{y}_{\nu} \otimes \tilde{y}_{1} c}, \tilde{y}_{\lambda}$, and $c_{j} \in \mathbb{C}(q)$. Precisely, the label $a$ is replaced by $s_{a}$ in the first equality by Equation (11). Then $s_{a}$ is replaced by $\tilde{y}_{\nu}$ in the second equality by Equation (12). Then we obtain the third equality by Lemma 3.5. Therefore, $\left\{\rho_{\mu, \nu, \lambda}: \nu \in \mu-b, \nu \in\right.$ $\left.\lambda-1^{c}\right\}$ is a spanning set of $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{r}}, \tilde{y}_{\lambda}\right)$.

Lemma 3.10. Suppose $a, b, c \in \mathbb{N}$ and $r=b+c$. For any Young diagrams $\mu$ and $\lambda$ with $|\mu|=a+b$, $|\lambda|=a+c$, the morphisms $\left\{\rho_{\mu, \nu, \lambda}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ are linearly independent over $\mathbb{C}(q)$.

Proof. Take $n=a+b+c$, and define

- $S_{1}=\{k \in \mathbb{N}: 1 \leq k \leq a+b\}$,
- $S_{2}=\{k \in \mathbb{N}: a+b<k \leq a+2 b+c\}$,
- $S_{3}=\{k \in \mathbb{N}: a+2 b+c<k \leq 2 n\}$,
- $S=\left\{p \in P_{n}: p\right.$ has no pair in $\left.S_{i}, i=1,2,3\right\}$.

Note that for any pairing $p \in S, p$ has $a$ pairs between $S_{1}$ and $S_{3} ; b$ pairs between $S_{1}$ and $S_{2}$; and $c$ pairs between $S_{2}$ and $S_{3}$. So we obtain a bijection $\iota: T_{a+b} \times T_{b+c} \times T_{a+c} \rightarrow S$ via

$$
\iota\left(p_{1}, p_{2}, p_{3}\right)=p_{a, b, c} \circ\left(p_{1} \otimes p_{2} \otimes p_{3}\right)
$$

where $p_{a, b, c}$ is defined in Equation (21), and $p_{1} \otimes p_{2} \otimes p_{3}$ is a permutation on $2 n$ points,

$$
\left(p_{1} \otimes p_{2} \otimes p_{3}\right)(k)= \begin{cases}p_{1}(k), & \forall 1 \leq k \leq a+b \\ p_{2}(k-a-b)+a+b, & \forall a+b<k \leq n+b \\ p_{3}(k-n-b)+n+b, & \forall n+b<k \leq 2 n\end{cases}
$$

For any $p \in S$, we can choose $\hat{p} \in \mathcal{B}_{n}$ as

$$
\hat{p}=x_{\iota^{-1}(p)}
$$

where $x_{p_{1}, p_{2}, p_{3}}$ is defined in Equation 24 .
Assume that

$$
\sum_{\nu} c_{\mu, \nu, \lambda} \rho_{\mu, \nu, \lambda}=0
$$

for some $c_{\mu, \nu, \lambda} \in \mathbb{C}(q)$.
Recall that $\rho^{\prime} \in \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}, \tilde{y}_{\lambda}\right)$ and $\rho \in \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b+c}}, \tilde{y}_{\lambda}\right)$ are defined in Equations (22) and 23). We identify the two hom spaces with subspaces of $\mathscr{C}_{n}$. By Proposition 2.2 ,

$$
\begin{aligned}
\rho_{\mu, \nu, \lambda} & =\sum_{p \in P_{n}} b_{\mu, \nu, \lambda}(p) \hat{p} \\
\rho_{\mu, \nu, \lambda}^{\prime}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b+c}, b}\right) & =\sum_{p \in P_{n}} b_{\mu, \nu, \lambda}^{\prime}(p) \hat{p}
\end{aligned}
$$

for some $b_{\mu, \nu, \lambda}(p), b_{\mu, \nu, \lambda}^{\prime}(p) \in \mathbb{C}(q)$, and

$$
\sum_{\nu} c_{\mu, \nu, \lambda} b_{\mu, \nu, \lambda}(p)=0, \forall p \in P_{n}
$$

Take $S_{0}=\iota\left(T_{a+b} \times T_{b, c} \times T_{a+c}\right)$. Note that


On the other hand, $\tilde{y}_{1^{b+c}, b}=\sum_{k} c_{j} \hat{p}_{1, j} \otimes \hat{p}_{2, j}$ for some $c_{j} \in \mathbb{C}(q), p_{1, j} \in T_{b}$ and $p_{2, j} \in T_{c}$ as defined in Equation (18). So


Note that for any $p \in S_{0}$, if we express $\tilde{y}_{1^{b+c}}$ in terms of the basis $\mathcal{B}_{b+c}$, then only the components in $T_{b} \times T_{c}$ contribute non-zero coefficients of $p$. Recall that $\tilde{y}_{1^{b+c}, b}$ is the sum of such components of $\tilde{y}_{1^{b+c}}$ in $T_{b} \times T_{c}$, so

$$
\begin{array}{ll}
b_{\mu, \nu, \lambda}^{\prime}(p)=b_{\mu, \nu, \lambda}(p), & \forall p \in S_{0} \\
b_{\mu, \nu, \lambda}^{\prime}(p)=0, & \\
\forall p \in S \backslash S_{0}
\end{array}
$$

Then

$$
\sum_{p \in S_{0}} \sum_{\nu} c_{\mu, \nu, \lambda} b_{\mu, \nu, \lambda}^{\prime}(p) \hat{p}=\sum_{p \in S_{0}} \sum_{\nu} c_{\mu, \nu, \lambda} b_{\mu, \nu, \lambda}(p) \hat{p}=0
$$

Note that if $b_{\mu, \nu, \lambda}^{\prime}(p) \neq 0, p$ has no pair in $S_{1}$ or $S_{2}$, then $p$ has no pair between the first $b$ points and the last $c$ points in $S_{3}$. So

$$
b_{\mu, \nu, \lambda}^{\prime}(p) \tilde{y}_{\lambda} \hat{p}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \neq 0, \text { only when } p \in S_{0}
$$

By Lemma 2.12, $\tilde{y}_{1^{b+c}, b}\left(\tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right)=c_{0} \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}$ for some $c_{0} \neq 0$ in $\mathbb{C}(q)$. So

$$
\begin{aligned}
& c_{0} \sum_{\nu} c_{\mu, \nu, \lambda} \rho^{\prime}(\mu, \tilde{\nu}, \lambda) \\
= & c_{0} \sum_{\nu} c_{\mu, \nu, \lambda} \tilde{y}_{\lambda} \rho^{\prime}(\mu, \tilde{\nu}, \lambda)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \\
= & \sum_{\nu} c_{\mu, \nu, \lambda} \tilde{y}_{\lambda} \rho^{\prime}(\mu, \tilde{\nu}, \lambda)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b+c}, b}\right)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \\
= & \sum_{\nu} c_{\mu, \nu, \lambda} \tilde{y}_{\lambda}\left(\sum_{p \in P_{n}} b_{\mu, \nu, \lambda}^{\prime}(p) \hat{p}\right)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \\
= & \sum_{\nu} c_{\mu, \nu, \lambda} \sum_{p \in S_{0}} b_{\mu, \nu, \lambda}^{\prime}(p) \tilde{y}_{\lambda} \hat{p}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \\
= & \tilde{y}_{\lambda}\left(\sum_{p \in S_{0}} \sum_{\nu} c_{\mu, \nu, \lambda} b_{\mu, \nu, \lambda}^{\prime}(p) \hat{p}\right)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right) \\
= & 0 .
\end{aligned}
$$

By Lemma 3.8, we have that $c_{\mu, \nu, \lambda}=0$ for all $\nu$, Therefore, the elements $\left\{\rho_{\mu, \nu, \lambda}^{\prime}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ are linear independent in $\mathscr{C}_{n}$. In particular, $\rho_{\mu, \nu, \lambda}^{\prime} \neq 0$, whenever $\nu \in \mu-b, \nu \in \lambda-1^{c}$.

We consider $\hat{p}$ as a morphism in $\operatorname{hom}_{\mathscr{C}}\left(X^{n+b}, X^{a+c}\right)$. Then

$$
\tilde{y}_{\lambda}\left(\sum_{p \in S} \sum_{\nu} c_{\mu, \nu, \lambda} b_{\mu, \nu, \lambda}(p) \hat{p}\right)\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{b}} \otimes \tilde{y}_{1^{c}}\right)=0 .
$$

Note that the set $S$ of pairings are the same as the pairings implemented by

$$
=\sum_{\nu} c_{\mu, \nu, \lambda} c_{r, i} \rho_{\nu, \lambda}\left(\tilde{y}_{\nu} \otimes m_{k} \otimes \tilde{y}_{1^{n-k}}\right)\left(\rho_{\mu, \nu} \otimes \tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{r-i}}\right) .
$$

Note that $\rho_{\nu, \lambda}\left(\tilde{y}_{\nu} \otimes m_{k} \otimes \tilde{y}_{1^{n-k}}\right)\left(\rho_{\mu, \nu} \otimes \tilde{y}_{1^{i}} \otimes \tilde{y}_{1^{r-i}}\right) \neq 0$ and they are linearly independent for different $\tau$. Therefore, $c_{\mu, \nu, \lambda} c_{r, i}=0$. Recall that $c_{r, i} \neq 0$, so $c_{\mu, \nu, \lambda}=0$. Therefore, the morphisms $\left\{\rho_{\mu, \nu, \lambda}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ are linearly independent.

Theorem 3.11. Suppose $a, b, c \in \mathbb{N}$ and $r=b+c$. For any Young diagrams $\mu$ and $\lambda,|\mu|=a+b$, $|\lambda|=a+c$, the elements $\left\{\rho_{\mu, \nu, \lambda}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ form a basis of hom $\mathscr{C}^{( }\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\lambda}\right)$. In particular, we obtain the fusion for $X_{1^{r}}$ in a closed form:

$$
X_{\left(1^{r}\right)} X_{\mu}=\sum_{i=0}^{r} \sum_{\nu \in \mu-i} \sum_{\lambda \in \nu+1^{r-i}} X_{\lambda} .
$$

Proof. By Lemmas 3.9 3.10, $\left\{\rho_{\mu, \nu, \lambda}: \nu \in \mu-b, \nu \in \lambda-1^{c}\right\}$ form a basis of hom $\mathscr{C}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{1^{n}}, \tilde{y}_{\lambda}\right)$.
We remove $i$ cells from $\mu$ (no two in the same column), and then we add $r-i$ cells (no two in the same row).

Corollary 3.12. Applying the automorphism $\Omega$, we obtain the fusion with $X_{r}$ in a closed form:

$$
X_{(r)} X_{\mu}=\sum_{i=0}^{r} \sum_{\nu \in \mu-1^{i}} \sum_{\lambda \in \nu+(r-i)} X_{\lambda} .
$$

We remove $i$ cells from $\lambda$ (no two in the same row), and then we add $n-i$ cells (no two in the same column).

Remark 3.13. The morphisms can be constructed explicitly following the construction in 3. They are essentially used in the proof of Theorem 3.11. We are going to compute the characters and the generating functions in the next section using Theorem 3.11.

## 4. Characters, Generating Functions and Fusion Rules for the Generic Case

We begin by introducing the tools we will need from the theory of symmetric functions. All the material we use can be found in the first chapter of [4].
4.1. Symmetric Functions. Recall that the ring of symmetric functions, $\Lambda$, is defined in the following way.

Definition 4.1. Let $n$ be a natural number, and $R_{n}=\mathbb{Z}\left[x_{1}, x_{2}, \ldots, x_{n}\right]^{S_{n}}$ be the ring of symmetric polynomial in $n$ variables. We write $R_{n}^{k}$ for the degree $k$ component of $R_{n}$. For each $k$, we have
maps $\rho_{n}: R_{n}^{k} \rightarrow R_{n-1}^{k}$ defined by setting $x_{n}=0$; these form an inverse system, so we may take the inverse limit $\lim R_{n}^{k}$. Then, as an abelian group, we define:

$$
\Lambda=\bigoplus_{k \geq 0} \lim _{\leftarrow} R_{n}^{k}
$$

The multiplication on $\Lambda$ is inherited from the multiplication $R_{n}^{k_{1}} \otimes R_{n}^{k_{2}} \rightarrow R_{n}^{k_{1}+k_{2}}$. We may complete $\Lambda$ with respect to the grading. In this case we obtain

$$
\hat{\Lambda}=\prod_{k \geq 0} \lim _{\leftarrow} R_{n}^{k}
$$

We introduce some important elements of the ring of symmetric functions.
Proposition 4.2. We have the following facts about $\Lambda$ :
(1) The polynomials $\sum_{i_{1}<i_{2}<\cdots<i_{r}} x_{i_{1}} x_{i_{2}} \cdots x_{i_{r}} \in R_{n}^{r}$ define an element $e_{r} \in \lim _{\leftarrow} R_{n}^{k}$ called the $r$-th elementary symmetric function. These $e_{r}$ freely generate $\Lambda$ as a polynomial ring: $\Lambda=\mathbb{Z}\left[e_{1}, e_{2}, \ldots\right]$. We have the generating function $E(t)=\sum_{r} e_{r} t^{r}=\prod_{i}\left(1+x_{i} t\right)$.
(2) Similarly, the polynomials $\sum_{i_{1} \leq i_{2} \leq \cdots \leq i_{r}} x_{i_{1}} x_{i_{2}} \cdots x_{i_{r}} \in R_{n}^{r}$ define an element $h_{r} \in \lim _{\leftarrow} R_{n}^{k}$ called the $r$-th complete symmetric function. These $h_{r}$ also freely generate $\Lambda$ as a polynomial ring: $\Lambda=\mathbb{Z}\left[h_{1}, h_{2}, \ldots\right]$. We have the generating function $H(t)=\sum_{r} h_{r} t^{r}=\prod_{i}\left(1-x_{i} t\right)^{-1}$.
(3) The polynomials $\sum_{i} x_{i}^{r} \in R_{n}^{r}$ define an element $p_{r} \in \lim _{\leftarrow} R_{n}^{k}$ called the $r$-th power-sum symmetric function. They freely generate $\mathbb{Q} \otimes \Lambda$ as a polynomial ring over $\mathbb{Q}$ (but they do not generate $\Lambda$ over $\mathbb{Z}$ ). We have the generating function $P(t)=\sum_{r} p_{r+1} t^{r}=\sum_{i} \frac{x_{i}}{1-x_{i} t}$.
(4) The generating functions $E(t)$ and $H(t)$ satisfy the relation $H(t) E(-t)=1$, and this equation encodes how to express the elementary symmetric functions in terms of the complete symmetric functions and vice versa. Similarly, we have $H^{\prime}(t) / H(t)=P(t)$, and $E^{\prime}(t) / E(t)=P(-t)$. In particular, we have the equations

$$
\begin{aligned}
& \sum_{r \geq 0} h_{r} t^{r}=\exp \left(\sum_{i \geq 1} \frac{p_{i}}{i} t^{i}\right) \\
& \sum_{r \geq 0} e_{r} t^{r}=\exp \left(\sum_{i \geq 1} \frac{(-1)^{i-1} p_{i}}{i} t^{i}\right)
\end{aligned}
$$

(5) Elements of $\Lambda \otimes \Lambda$ may be viewed as polynomials in two sets of variables, say $x_{i}$ and $y_{j}$, symmetric in each separately. To indicate which variable set is being considered, we write $f(x)$ or $f(y)$. Given $f \in \Lambda$, we write $f(x, y)$ for the element of $\Lambda \otimes \Lambda$ defined by the symmetric function $f$ in the variable set $\left\{x_{i}\right\} \cup\left\{y_{j}\right\}$. (This operation defines a comultiplication $\Lambda \rightarrow \Lambda \otimes \Lambda$.)
(6) Fix a Young diagram $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots\right)$, adding trailing zeros if needed, so that $\lambda$ has $n$ parts (usually we do not distinguish between Young diagrams that differ by trailing zeros). The
polynomials $\operatorname{det}\left(x_{i}^{\lambda_{j}+n-j}\right) / \operatorname{det}\left(x_{i}^{n-j}\right) \in R_{n}^{|\lambda|}$ define an element $s_{\lambda} \in \lim _{\leftarrow} R_{n}^{|\lambda|}$, called the Schur function associated to $\lambda$.
(7) We have that $s_{1^{r}}=e_{r}$ and $s_{r}=h_{r}$.

Example 4.3. We have:

$$
\frac{p_{1}^{2}+p_{2}}{2}=\frac{1}{2}\left(\sum_{i \neq j} x_{i} x_{j}+2 \sum_{i} x_{i}^{2}\right)=\sum_{i \leq j} x_{i} x_{j}=h_{2}
$$

Remark 4.4. Schur functions may be viewed as the characters of irreducible representations of $G L_{n}(\mathbb{C})$ in the following sense. If $M \in G L_{n}(\mathbb{C})$ has eigenvalues $x_{i}$, then the trace of the action of $M$ on the irreducible representation of $G L_{n}(\mathbb{C})$ corresponding to the Young diagram $\lambda$ is $s_{\lambda}\left(x_{i}\right)$ : the Schur function corresponding to $\lambda$ evaluated at the eigenvalues $x_{i}$. Note that this quantity is zero unless the Young diagram $\lambda$ has at most $n$ nonzero parts. This means we have a homomorphism $\Lambda \rightarrow R_{n}$ whose kernel has basis $s_{\mu}$ for Young diagrams $\mu$ with more than $n$ nonzero parts.

We now discuss a bilinear form on $\Lambda$.
Proposition 4.5. The ring $\Lambda$ satisfies the following properties:
(1) The Schur functions form a $\mathbb{Z}$-basis of $\Lambda: \Lambda=\mathbb{Z}\left\{s_{\lambda} \mid \lambda\right.$ a Young diagram $\}$. In particular, there is a bilinear form $\langle-,-\rangle: \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ for which the Schur functions are orthonormal.
(2) The adjoint to multiplication by $p_{i}$ is $i \frac{\partial}{\partial p_{i}}$ (where elements of $\Lambda$ are viewed as polynomials in the $p_{i}$ with possibly rational coefficients).
(3) The adjoint to multiplication by $s_{\mu}$ (with respect to $\langle-,-\rangle$ ) is denoted $s_{\mu}^{\perp}$. The symmetric function $s_{\mu}^{\perp}\left(s_{\lambda}\right)$ is called a skew-Schur function, and denoted $s_{\lambda / \mu}$. It is nonzero if and only if $\mu_{i} \leq \lambda_{i}$ for all $i$.
(4) Schur functions satisfy the following multiplication rule $s_{\mu} s_{\nu}=\sum_{\mu, \nu} c_{\mu, \nu}^{\lambda} s_{\lambda}$, where $c_{\mu, \nu}^{\lambda}$ are the Littlewood-Richardson coefficients (which are zero unless $|\mu|+|\nu|=|\lambda|$ ). They also satisfy $s_{\lambda}(x, y)=\sum_{\mu, \nu} c_{\mu, \nu}^{\lambda} s_{\mu}(x) s_{\nu}(y)$.
(5) The identity $e_{r}(x, y)=\sum_{i=0}^{r} e_{i}(x) e_{r-i}(y)$ shows that $c_{\mu, \nu}^{1^{r}}$ is zero unless $\mu=1^{i}$ and $\nu=1^{r-i}$ for some $0 \leq i \leq r$, in which case it is equal to 1 .
(6) The Littlewood-Richardson coefficient $c_{\mu, r}^{\lambda}$ is zero unless the diagram of $\lambda$ can be obtained by adding $r$ cells to the diagram of $\mu$, with no two cells in the same column; this is the Pieri rule. Similarly, $c_{\mu, 1^{r}}^{\lambda}$ is zero unless the diagram of $\lambda$ can be obtained by adding $r$ cells to the diagram of $\mu$, with no two cells in the same row; this is the dual Pieri rule.
There are two identities that will be important to us, which we now state.
Proposition 4.6. We have the following equations:
(1) The following equality of series holds in a completion of $R_{n} \otimes R_{n}$ for each $n$, and therefore in a completion of $\Lambda$ (note that the homogeneous components of the right-hand side define elements of the inverse limits used to define the ring of symmetric functions):

$$
\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)=\prod_{i, j} \frac{1}{1-x_{i} y_{j}}
$$

This is called the Cauchy Identity.
(2) Similarly, we have the Dual Cauchy Identity:

$$
\sum_{\lambda} s_{\lambda}(x) s_{\lambda^{\prime}}(y)=\prod_{i, j}\left(1+x_{i} y_{j}\right)
$$

Here, $\lambda^{\prime}$ is the Young diagram dual to $\lambda$.
There is another operation on symmetric functions called plethysm.
Definition 4.7. Given symmetric functions $f$ and $g$ which are sums of monomials in the variables $x_{i}$ with coefficients in $\mathbb{Z}_{\geq 0}$, the plethysm of $g$ with $f$ is a symmetric function denoted $g[f]$. It may be calculated in the following way. Express $f\left(x_{1}, x_{2}, \ldots\right)$ as a sum of monomials (repeated according to their multiplicity) $f=\sum_{i} x_{1}^{\alpha_{1}^{(i)}} x_{2}^{\alpha_{2}^{(i)}} \ldots$. Then $g[f]$ is the symmetric function obtained by evaluating $g$ on the variable set given by the monomials $x_{1}^{\alpha_{1}^{(i)}} x_{2}^{\alpha_{2}^{(i)}} \ldots$. It immediately follows that the map $\Lambda \rightarrow \Lambda$ defined by $g \mapsto g[f]$ is an algebra homomorphism (but this is not true for $f \mapsto g[f]$ ).

Remark 4.8. There is a way of generalising the above definition to $f$ and $g$ for which are not necessarily a positive (or even integral, if one is prepared to base change $\Lambda$ ) sum of monomials. The most general definition is the one given in Chapter 1, Section 8 of [4].
Remark 4.9. Let $f=\sum_{\mu} m_{\mu} s_{\mu}$ be the character of a representation $V$ of $G L_{n}(\mathbb{C})$ (where $n$ is taken to be sufficiently large), so $m_{\mu} \in \mathbb{Z}_{\geq 0}$, and all but finitely many $m_{\mu}$ are zero. Thus, $f$ encodes a homomorphism $\varphi_{f}: G L_{n}(\mathbb{C}) \rightarrow G L(V)$. Similarly, fix $g=\sum_{\nu} n_{\nu} s_{\nu}$ (with the same conditions on $n_{\nu}$ as on $m_{\mu}$ ), which uniquely defines a representation $W$ of $G L(V)=G L_{\operatorname{dim}(V)}(\mathbb{C})$, encoding a homomorphism $\varphi_{g}: G L(V) \rightarrow G L(W)$. Then, $W$ is a representation of $G L_{n}(\mathbb{C})$ via the composition $\varphi_{g} \circ \varphi_{f}$ :

$$
G L_{n}(\mathbb{C}) \xrightarrow{\varphi_{f}} G L(V) \xrightarrow{\varphi_{g}} G L(W) .
$$

The character of this representation is the plethysm $g[f]$. The value of $n$ used in this construction does not affect $g[f]$, provided it is large enough (e.g. $n=\operatorname{deg}(f) \operatorname{deg}(g)$ will suffice).

Example 4.10. We show that $e_{1}=\sum_{i} x_{i}$ is a two-sided identity for plethysm. Note that by definition, $e_{1}[f]$ recovers the sum of the monomials of $f$, namely $f$ itself. On the other hand, $f\left[e_{1}\right]$ is the evaluation of $f$ on the variable set $\left\{x_{i}\right\}$ (the monomials of $e_{1}$ ), which again is $f$ itself. This is consistent with the formulation in terms of representations of general linear groups, where $\varphi_{e_{1}}$ represents the identity map $G L_{n}(\mathbb{C}) \rightarrow G L_{n}(\mathbb{C})$ (for any $n$ ).
Remark 4.11. For power-sum symmetric functions $p_{r}$, plethysm has some useful properties. In particular, $p_{r}[f]=f\left[p_{r}\right]$ for arbitrary $f$, because both sides are equal to the symmetric function obtained by multiplying the exponents of all monomials of $f$ by $r$. As a special case, we obtain $p_{r_{1}}\left[p_{r_{2}}\right]=p_{r_{2}}\left[p_{r_{1}}\right]=p_{r_{1} r_{2}}$.

Ultimately, the result we need about plethysm is the following.
Theorem 4.12. We have the following equation:

$$
h_{r}\left[h_{2}\right]=\sum_{|\lambda|=r} s_{2 \lambda},
$$

where, if $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\right)$, then $2 \lambda=\left(2 \lambda_{1}, 2 \lambda_{2}, \ldots, 2 \lambda_{k}\right)$.

Proof. This can be found in Chapter 1, Section 8, Example 6 of [4]. Alternatively, see Example $A 2.9$ of [5].

We now introduce a linear operator which will play an important role in what follows, and prove some properties that it satisfies.

Definition 4.13. Let $L$ be the linear operator on the completion of $\Lambda$ defined by multiplication by $\prod_{i \leq j} \frac{1}{1+x_{i} x_{j}}$.

Proposition 4.14. The adjoint of $L$ with respect to $\langle-,-\rangle$ is:

$$
L^{\dagger}=\sum_{\mu}(-1)^{|\mu|} s_{2 \mu}^{\perp}
$$

Proof. We recognise the product defining $L$ as the generating function of complete symmetric functions evaluated at -1 , with variable set $\left\{x_{i} x_{j}\right\}_{i \leq j}$ (these are the monomials in $h_{2}$ ); the degree $2 r$ component of this sum is precisely what is computed in Theorem4.12. Thus,

$$
\prod_{i \leq j} \frac{1}{1+x_{i} x_{j}}=H(-1)\left[h_{2}\right]=\sum_{r \geq 0}(-1)^{r} h_{r}\left[h_{2}\right]=\sum_{r \geq 0}(-1)^{r} \sum_{|\mu|=r} s_{2 \mu}=\sum_{\mu}(-1)^{|\mu|} s_{2 \mu}
$$

Noting that the adjoint of multiplication by $s_{2 \mu}$ is $s_{2 \mu}^{\perp}$, the proposition follows.

Notation 4.15. Let $\phi_{2}: G L_{n} \rightarrow G L_{n(n+1) / 2}$ be the symmetric square representation of $G L_{n}$ and $\phi_{1^{r}}$ be the $r$-th antisymmetric power representation of $G L_{n(n+1) / 2}$. Then $\phi_{1^{r}} \phi_{2}$ is a representation of $G L_{n}$. The multiplicity of the irreducible representation of $G L_{n}$ with highest weight $\lambda$ in $\phi_{1^{r}} \phi_{2}$ is denoted by $b_{n, r, \lambda}$. We define $b_{r, \lambda}=\lim _{n \rightarrow \infty} b_{n, r, \lambda}$. Then

$$
\begin{align*}
e_{r}\left[h_{2}\right] & =\sum_{\lambda} b_{r, \lambda} s_{\lambda} ;  \tag{25}\\
L^{-1} & =\prod_{i \leq j}\left(1+x_{i} x_{j}\right)=\sum_{r \geq 0} e_{r}\left[h_{2}\right]=\sum_{r \geq 0, \lambda} b_{r, \lambda} s_{\lambda} . \tag{26}
\end{align*}
$$

Lemma 4.16. Let $\theta_{i}=\frac{1+(-1)^{i}}{2}$, so that $\theta_{i}$ is equal to 0 when $i$ is odd, and equal to 1 when $i$ is even. When expressed in terms of power-sum symmetric functions, $L$ has the following form:

$$
L=\exp \left(\sum_{i} \frac{(-1)^{i} p_{i}^{2}+2(-1)^{i / 2} \theta_{i} p_{i}}{2 i}\right)
$$

Proof. We write $L=H(-1)\left[h_{2}\right]$ (as in the proof of Proposition 4.14), where we express $H(-1)$ and $h_{2}$ in terms of power-sum symmetric functions. We use Remark 4.11 to manipulate the plethysm:

$$
\begin{aligned}
L & =\exp \left(\sum_{i} \frac{(-1)^{i} p_{i}}{i}\right)\left[\frac{p_{1}^{2}+p_{2}}{2}\right] \\
& =\exp \left(\sum_{i} \frac{\left.(-1)^{i} p_{i} \frac{p_{1}^{2}+p_{2}}{2}\right]}{i}\right) \\
& =\exp \left(\sum_{i} \frac{(-1)^{i} \frac{p_{1}^{2}+p_{2}}{2}\left[p_{i}\right]}{i}\right) \\
& =\exp \left(\sum_{i} \frac{(-1)^{i} \frac{p_{i}^{2}+p_{2 i}}{2}}{i}\right) \\
& =\exp \left(\sum_{i} \frac{(-1)^{i}\left(p_{i}^{2}+p_{2 i}\right)}{2 i}\right)
\end{aligned}
$$

We rearrange the sum so that all instances of $p_{i}$ occur in the $i$-th summand. This means moving the term $(-1)^{i} p_{2 i} / 2 i$ from the $i$-th summand to the $2 i$-th summand. Upon noting that only even index summands obtain a contribution in this way, we obtain the stated formula.
Proposition 4.17. Consider $\Lambda \otimes \Lambda$ as the set of symmetric functions in two sets of variables $\left\{x_{i}^{(1)}\right\}$ and $\left\{x_{i}^{(2)}\right\}$. Suppose that the symmetric function $f$ satisfies $f\left(x^{(1)}, x^{(2)}\right)=\sum_{i} g_{i}\left(x^{(1)}\right) h_{i}\left(x^{(2)}\right)$. Then, we have the following equation:

$$
\left(\sum_{\lambda} s_{\lambda}\left(x^{(1)}\right) s_{\lambda^{\prime}}\left(x^{(2)}\right)\right) L(f)\left(x^{(1)}, x^{(2)}\right)=L\left(g_{i}\right)\left(x^{(1)}\right) L\left(h_{i}\right)\left(x^{(2)}\right)
$$

Proof. In the definition of $L$ (considered to have variable set $\left\{x_{i}^{(1)}\right\} \cup\left\{x_{i}^{(2)}\right\}$ ), products of pairs of variables take one of three forms: either both variables come from $\left\{x_{i}^{(1)}\right\}$, or both variables come from $\left\{x_{i}^{(2)}\right\}$, or one variable comes from each. Giving $L$ a subscript to show its variable set, we obtain:

$$
L_{\left\{x_{i}^{(1)}\right\} \cup\left\{x_{i}^{(2)}\right\}}=\prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i_{1}}^{(1)} x_{i_{2}}^{(1)}} \prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i_{1}}^{(2)} x_{i_{2}}^{(2)}} \prod_{i_{1}, i_{2}} \frac{1}{1+x_{i_{1}}^{(1)} x_{i_{2}}^{(2)}}
$$

Moving the last factor to the left-hand side, and using the Dual Cauchy Identity,

$$
\left(\sum_{\lambda} s_{\lambda}\left(x^{(1)}\right) s_{\lambda^{\prime}}\left(x^{(2)}\right)\right) L_{\left\{x_{i}^{(1)}\right\} \cup\left\{x_{i}^{(2)}\right\}}=L_{\left\{x_{i}^{(1)}\right\}} L_{\left\{x_{i}^{(2)}\right\}} .
$$

This is equivalent to the statement of the proposition.
4.2. Characters, Generating Functions and Fusion Rules. In this section, we recall some properties of the Grothendieck ring $\mathcal{G}$, and then study its structure using symmetric functions. Recall that $\mathcal{G}$ has basis $\left\{Y_{\lambda}\right\}$ indexed by Young diagrams.

Notation 4.18. By Schur-Weyl duality, we obtain a ring isomorphism $\Phi: \mathcal{G} \rightarrow \Lambda$, such that $\Phi\left(Y_{\lambda}\right)=s_{\lambda}$. Moreover,

$$
Y_{\mu} Y_{\nu}=\sum_{\lambda} c_{\mu, \nu}^{\lambda} Y_{\lambda}
$$

where $c_{\mu, \nu}^{\lambda}$ are the Littlewood-Richardson coefficients.
Recall that $s_{\lambda / 2 \mu}$ is a skew-Schur function, and $2 \mu$ is the Young diagram obtained by doubling each part of $\mu$.

Definition 4.19. By Theorem 2.8, we define a ring isomorphism $\Phi: \mathcal{G} \rightarrow \Lambda$, such that

$$
\Phi\left(Y_{\lambda}\right)=s_{\lambda}, \forall \lambda
$$

In particular,

$$
\Phi\left(X_{1^{r}}\right)=\Phi\left(Y_{1^{r}}\right)=s_{1^{r}}, \forall r \geq 0
$$

Theorem 4.20. For any Young diagram $\lambda$, we call $\Phi\left(X_{\lambda}\right)$ the character of $X_{\lambda}$ in $\mathcal{G}$. Then

$$
\begin{align*}
\Phi\left(X_{\lambda}\right) & =L^{\dagger} s_{\lambda}=\sum_{\mu}(-1)^{|\mu|} s_{\lambda / 2 \mu}  \tag{27}\\
X_{\lambda} & =\sum_{\substack{\mu, \nu \\
2|\mu|+|\nu|=|\lambda|}}(-1)^{|\mu|} c_{2 \mu, \nu}^{\lambda} Y_{\nu} \tag{28}
\end{align*}
$$

Proof. To prove the first statement, it suffices (by induction on $\lambda$ ) to show that the claimed expressions for the $\Phi\left(X_{\lambda}\right)$ multiply according to the rule defined by Theorem 3.11. When we encode the operations of removing and adding cells via the Pieri rule and dual Pieri rule, what we must prove becomes

$$
e_{r} L^{\dagger}\left(s_{\lambda}\right)=\sum_{i=0}^{r} L^{\dagger}\left(e_{r-i} h_{i}^{\perp} s_{\lambda}\right)
$$

This is precisely the assertion of the following equality of operators: $e_{r} L^{\dagger}=\sum_{i=0}^{r} L^{\dagger} e_{r-i} h_{i}^{\perp}$. We prove the adjoint of this equality, namely $L e_{r}^{\perp}=\sum_{i=0}^{r} h_{i} e_{r-i}^{\perp} L$. To prove this statement for all $r$ simultaneously, we multiply by $t^{r}$ and sum over $r \geq 0$; it is equivalent to prove the following identity of (operator-valued) generating functions:

$$
L E(t)^{\perp}=H(t) E(t)^{\perp} L
$$

We rewrite all quantities in terms of power-sum symmetric functions. We have:

$$
\begin{aligned}
E(t)^{\perp} & =\exp \left(\sum_{i} \frac{(-1)^{i-1} p_{i}^{\perp}}{i} t^{i}\right)=\exp \left(\sum_{i}(-1)^{i-1} \frac{\partial}{\partial p_{i}} t^{i}\right) \\
H(t) & =\exp \left(\sum_{i} \frac{p_{i}}{i} t^{i}\right) \\
L & =\exp \left(\sum_{i} \frac{(-1)^{i} p_{i}^{2}+2(-1)^{i / 2} \theta_{i} p_{i}}{2 i}\right)
\end{aligned}
$$

(Recall from Lemma 4.16 that $\theta_{i}$ is equal to 0 if $i$ is odd, and equal to 1 if $i$ is even.) We use an operator-theoretic version of Taylor's theorem, namely

$$
\exp \left(a \frac{\partial}{\partial x}\right) f(x)=f(x+a)
$$

Applying this termwise to the composition of operators $E(t)^{\perp} L$, we obtain:

$$
\begin{aligned}
E(t)^{\perp} L= & \exp \left(\sum_{i}(-1)^{i-1} \frac{\partial}{\partial p_{i}} t^{i}\right) \exp \left(\sum_{i} \frac{(-1)^{i} p_{i}^{2}+2(-1)^{i / 2} \theta_{i} p_{i}}{2 i}\right) \\
= & \exp \left(\sum_{i} \frac{(-1)^{i}\left(p_{i}+(-1)^{i-1} t^{i}\right)^{2}+2(-1)^{i / 2} \theta_{i}\left(p_{i}+(-1)^{i-1} t^{i}\right)}{2 i}\right) \exp \left(\sum_{i}(-1)^{i-1} \frac{\partial}{\partial p_{i}} t^{i}\right) \\
= & \exp \left(\sum_{i} \frac{(-1)^{i} p_{i}^{2}-2 t^{i} p_{i}+(-1)^{i} t^{2 i}+2(-1)^{i / 2} \theta_{i} p_{i}+2(-1)^{i / 2} \theta_{i}(-1)^{i-1} t^{i}}{2 i}\right) E(t)^{\perp} \\
= & \exp \left(-\sum_{i} \frac{p_{i} t^{i}}{i}\right) \exp \left(\sum_{i} \frac{(-1)^{i} p_{i}^{2}+2(-1)^{i / 2} \theta_{i} p_{i}}{2 i}\right) \\
& \times \exp \left(\sum_{i} \frac{(-1)^{i} t^{2 i}+2(-1)^{i / 2} \theta_{i}(-1)^{i-1} t^{i}}{2 i}\right) E(t)^{\perp} .
\end{aligned}
$$

We recognise the first term as $H(t)^{-1}$, the second term as $L$, and the third term as 1 (noting that all powers of $t$ cancel out). Thus we have:

$$
H(t)^{-1} L E(t)^{\perp}=E(t)^{\perp} L
$$

which is equivalent to the statement

$$
\Phi\left(X_{\lambda}\right)=L^{\dagger} s_{\lambda}=\sum_{\mu}(-1)^{|\mu|} s_{\lambda / 2 \mu}
$$

Furthermore,

$$
\Phi\left(X_{\lambda}\right)=\sum_{\mu}(-1)^{|\mu|} s_{\lambda / 2 \mu}=\sum_{\substack{\mu, \nu \\ 2|\mu|+|\nu|=|\lambda|}}(-1)^{|\mu|} c_{2 \mu, \nu}^{\lambda} s_{\nu}=\sum_{\substack{\mu, \nu \\ 2|\mu|+|\nu|=|\lambda|}}(-1)^{|\mu|} c_{2 \mu, \nu}^{\lambda} \Phi\left(Y_{\nu}\right)
$$

Recall that $\Phi$ is an isomorphism, so

$$
X_{\lambda}=\sum_{\substack{\mu, \nu \\ 2|\mu|+|\nu|=|\lambda|}}(-1)^{|\mu|} c_{2 \mu, \nu}^{\lambda} Y_{\nu}
$$

Theorem 4.21. For a Young diagram $\lambda$, let us define $\lambda_{<}$to be set of proper sub Young diagrams $\mu$, such that $|\lambda|-|\mu| \in 2 \mathbb{N}^{+}$. Then

$$
\begin{align*}
Y_{\lambda} & =X_{\lambda}+\sum_{\mu \in \lambda_{<}} n_{\lambda, \mu} X_{\mu}  \tag{29}\\
\sum_{\lambda} n_{\lambda, \mu} s_{\lambda} & =L^{-1} s_{\mu}=s_{\mu} \prod_{i \leq j}\left(1+x_{i} x_{j}\right),  \tag{30}\\
n_{\lambda, \mu} & =\sum_{r \geq 0, \nu} b_{r, \nu} c_{\mu, \nu}^{\lambda} \tag{31}
\end{align*}
$$

Proof. We assume that

$$
Y_{\lambda}=\sum_{\mu} n_{\lambda, \mu} X_{\mu}, \text { for some } n_{\lambda, \mu} \in \mathbb{N}
$$

By Theorem 4.20.

$$
s_{\lambda}=\sum_{\mu} n_{\lambda, \mu} L^{\dagger} s_{\mu}
$$

Then

$$
\left\langle L^{-1} s_{\nu}, s_{\lambda}\right\rangle=\sum_{\mu} n_{\lambda, \mu}\left\langle L^{-1} s_{\nu}, L^{\dagger} s_{\mu}\right\rangle=\sum_{\mu} n_{\lambda, \mu}\left\langle s_{\nu}, s_{\mu}\right\rangle=n_{\lambda, \nu}
$$

By Equation 26),

$$
\sum_{\lambda} n_{\lambda, \mu} s_{\lambda}=\sum_{\lambda}\left\langle L^{-1} s_{\mu}, s_{\lambda}\right\rangle s_{\lambda}=L^{-1} s_{\mu}=s_{\mu} \prod_{i \leq j}\left(1+x_{i} x_{j}\right)
$$

Moreover,

$$
n_{\lambda, \mu}=\left\langle L^{-1} s_{\mu}, s_{\lambda}\right\rangle=\left\langle\sum_{r \geq 0, \nu} b_{r, \nu} s_{\nu} s_{\mu}, s_{\lambda}\right\rangle=\sum_{r \geq 0, \nu} b_{r, \nu} c_{\mu, \nu}^{\lambda}
$$

Theorem 4.22. We have the following generating function for $\Phi\left(X_{\lambda}\right)$,

$$
\sum_{\lambda} s_{\lambda}(x) \Phi\left(X_{\lambda}\right)(y)=\prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i} x_{j}} \prod_{i, j} \frac{1}{1-x_{i} y_{j}}
$$

(Here $\Phi\left(X_{\lambda}\right)(y)$ means that the symmetric function $\Phi\left(X_{\lambda}\right)$ has variable set $\left.\left\{y_{j}\right\}.\right)$
Proof. Now we apply Theorem 4.20 to prove this theorem. We consider the first equation of Theorem 4.20 as a having symmetric function variables $\left\{y_{j}\right\}$, and multiply by $s_{\lambda}(x)$. Summing over $\lambda$, we are required to show:

$$
\sum_{\lambda} s_{\lambda}(x) \sum_{\mu}(-1)^{|\mu|} s_{\lambda / 2 \mu}(y)=\prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i} x_{j}} \prod_{i, j} \frac{1}{1-x_{i} y_{j}}
$$

We now calculate:

$$
\begin{aligned}
\sum_{\lambda} s_{\lambda}(x) \sum_{\mu}(-1)^{|\mu|} s_{\lambda / 2 \mu}(y) & =\sum_{\lambda} s_{\lambda}(x) L^{\dagger}\left(s_{\lambda}\right)(y) \\
& =\sum_{\lambda} \sum_{\rho}\left\langle s_{\rho}, L^{\dagger}\left(s_{\lambda}\right)\right\rangle s_{\lambda}(x) s_{\rho}(y) \\
& =\sum_{\rho} \sum_{\lambda}\left\langle L\left(s_{\rho}\right), s_{\lambda}\right\rangle s_{\lambda}(x) s_{\rho}(y) \\
& =\sum_{\rho} L\left(s_{\rho}\right)(x) s_{\rho}(y) \\
& =\prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i} x_{j}} \sum_{\rho} s_{\rho}(x) s_{\rho}(y) \\
& =\prod_{i_{1} \leq i_{2}} \frac{1}{1+x_{i} x_{j}} \prod_{i, j} \frac{1}{1-x_{i} y_{j}}
\end{aligned}
$$

Theorem 4.23. We have the following fusion rules:

$$
R_{\mu, \nu}^{\lambda}=\sum_{\alpha, \beta, \gamma} c_{\alpha, \beta}^{\mu} c_{\beta^{\prime}, \gamma}^{\nu} c_{\alpha, \gamma}^{\lambda}
$$

(Here $\beta^{\prime}$ is the Young diagram dual to $\beta$.)

Proof. Now we apply Theorem 4.22 to prove this theorem. To do this, we consider a suitable generating function for the $R_{\mu, \nu}^{\lambda}$, and express it in terms of two instances of the generating function in the second part of the theorem. We work with three variable sets: $\left\{x_{i}^{(1)}\right\},\left\{x_{i}^{(2)}\right\}$, and $\left\{y_{j}\right\}$, and use Proposition 4.17 .

$$
\begin{aligned}
& \sum_{\mu, \nu, \lambda} R_{\mu, \nu}^{\lambda} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right) \Phi\left(X_{\lambda}\right)(y) \\
= & \sum_{\mu, \nu} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right) \Phi\left(X_{\mu}\right)(y) \Phi\left(X_{\nu}\right)(y) \\
= & \sum_{\mu} s_{\mu}\left(x^{(1)}\right) \Phi\left(X_{\mu}\right)(y) \sum_{\nu} s_{\nu}\left(x^{(2)}\right) \Phi\left(X_{\nu}\right)(y) \\
= & \sum_{\mu} L\left(s_{\mu}\right)\left(x^{(1)}\right) s_{\mu}(y) \sum_{\nu} L\left(s_{\nu}\right)\left(x^{(2)}\right) s_{\nu}(y) \\
= & \sum_{\mu, \nu} L\left(s_{\mu}\right)\left(x^{(1)}\right) L\left(s_{\nu}\right)\left(x^{(2)}\right) \sum_{\lambda} c_{\mu, \nu}^{\lambda} s_{\lambda}(y) \\
= & \sum_{\mu, \nu} L_{\left\{x^{(1)}\right\}} L_{\left\{x^{(2)}\right\}} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right) \sum_{\lambda} c_{\mu, \nu}^{\lambda} s_{\lambda}(y) \\
= & \sum_{\mu, \nu}\left(\sum_{\beta} s_{\beta}\left(x^{(1)}\right) s_{\beta^{\prime}}\left(x^{(2)}\right)\right) L_{\left\{x^{(1)}\right\} \cup\left\{x^{(2)}\right\}} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right) \sum_{\lambda} c_{\mu, \nu}^{\lambda} s_{\lambda}(y) \\
= & \left(\sum_{\beta} s_{\beta}\left(x^{(1)}\right) s_{\beta^{\prime}}\left(x^{(2)}\right)\right) L_{\left\{x^{(1)}\right\} \cup\left\{x^{(2)}\right\}} \sum_{\lambda} s_{\lambda}(y) \sum_{\mu, \nu} c_{\mu, \nu}^{\lambda} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right) \\
= & \left(\sum_{\beta} s_{\beta}\left(x^{(1)}\right) s_{\beta^{\prime}}\left(x^{(2)}\right)\right) L_{\left\{x^{(1)}\right\} \cup\left\{x^{(2)}\right\}} \sum_{\lambda} s_{\lambda}(y) s_{\lambda}\left(x^{(1)}, x^{(2)}\right) \\
= & \left(\sum_{\beta} s_{\beta}\left(x^{(1)}\right) s_{\beta^{\prime}}\left(x^{(2)}\right)\right) \sum_{\lambda} s_{\lambda}\left(x^{(1)}, x^{(2)}\right) \Phi\left(X_{\lambda}\right)(y) .
\end{aligned}
$$

At this point, we may take the coefficient of $\Phi\left(X_{\lambda}\right)(y)$ (these form a basis of $\Lambda$ ) to deduce

$$
\begin{aligned}
\sum_{\mu, \nu} R_{\mu, \nu}^{\lambda} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right) & =\left(\sum_{\beta} s_{\beta}\left(x^{(1)}\right) s_{\beta^{\prime}}\left(x^{(2)}\right)\right) s_{\lambda}\left(x^{(1)}, x^{(2)}\right) \\
& =\left(\sum_{\beta} s_{\beta}\left(x^{(1)}\right) s_{\beta^{\prime}}\left(x^{(2)}\right)\right) \sum_{\alpha, \gamma} c_{\alpha, \gamma}^{\lambda} s_{\alpha}\left(x^{(1)}\right) s_{\gamma}\left(x^{(2)}\right) \\
& =\sum_{\mu, \nu} \sum_{\alpha, \beta, \gamma} c_{\alpha, \beta}^{\mu} c_{\beta^{\prime}, \gamma}^{\nu} c_{\alpha, \gamma}^{\lambda} s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right)
\end{aligned}
$$

Taking coefficient of $s_{\mu}\left(x^{(1)}\right) s_{\nu}\left(x^{(2)}\right)$, we recover the formula for $R_{\mu, \nu}^{\lambda}$.
Note that if $|\alpha|=a,|\beta|=b,|\gamma|=c$, and

$$
c_{\alpha, \beta}^{\mu} c_{\beta^{\prime}, \gamma}^{\nu} c_{\alpha, \gamma}^{\lambda} \neq 0
$$

then $|\mu|=a+b,|\nu|=b+c,|\lambda|=a+c$. Conversely, $a, b, c$ are determined by $|\mu|,|\nu|,|\lambda|$. Thus the equation in Theorem 4.23 is a finite sum.

Definition 4.24. Recall that the simple objects $\tilde{y}_{\gamma}$ and $\tilde{y}_{\gamma^{\prime}}$ are dual to each other in $\mathscr{C}$. We denote $\cup_{\beta}$ to be the evaluation map in the hom space $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\beta} \otimes \tilde{y}_{\beta^{\prime}}, \emptyset\right)$. For Young diagrams $\mu, \nu, \lambda$, $\alpha$, $\beta$, $\gamma$, with $|\mu|=a+b,|\nu|=b+c,|\lambda|=a+c,|\alpha|=a,|\beta|=b,|\gamma|=c$, we define the triangle map $\nabla: \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\alpha} \otimes \tilde{y}_{\beta^{\prime}}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\beta} \otimes \tilde{y}_{\gamma}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\alpha} \otimes \tilde{y}_{\gamma}, \tilde{y}_{\lambda}\right) \rightarrow \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)$ as


Theorem 4.25. For any Young diagrams $\mu, \nu, \lambda$, the triangle map $\nabla$ is an embedding map and

$$
\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)=\bigoplus_{\alpha, \beta, \gamma} \nabla\left(\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\alpha} \otimes \tilde{y}_{\beta^{\prime}}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\beta} \otimes \tilde{y}_{\gamma}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\alpha} \otimes \tilde{y}_{\gamma}, \tilde{y}_{\lambda}\right)\right) .
$$

Proof. Similarly to the proof of Lemma 3.9 , we take

$$
\tilde{x}_{p_{1}, p_{2}, p_{3}}=\tilde{y}_{\lambda} x_{p_{1}, p_{2}, p_{3}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}\right) .
$$

Then $\left\{\tilde{x}_{p_{1}, p_{2}, p_{3}}: p_{1} \in P_{a+b}, p_{2} \in P_{b+c}, p_{3} \in P_{a+c} \cdot\right\}$ is a spanning set of $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)$. Note that the $180^{\circ}$ rotation of $s_{b}$ is $s_{b}$. So

for some Young diagrams $\alpha, \beta, \gamma$, with $|\alpha|=a,|\beta|=b,|\gamma|=c$, and some morphisms $\rho_{1} \otimes \rho_{2} \otimes \rho_{3} \in$ $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\alpha} \otimes \tilde{y}_{\beta^{\prime}}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\beta} \otimes \tilde{y}_{\gamma}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\alpha} \otimes \tilde{y}_{\gamma}, \tilde{y}_{\lambda}\right)$. Therefore

$$
\bigcup_{\alpha, \beta, \gamma} \nabla\left(\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\alpha} \otimes \tilde{y}_{\beta^{\prime}}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\beta} \otimes \tilde{y}_{\gamma}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\alpha} \otimes \tilde{y}_{\gamma}, \tilde{y}_{\lambda}\right)\right)
$$

is a spanning set of $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)$. By Proposition 2.5,

$$
\begin{aligned}
R_{\mu, \nu}^{\lambda} & =\operatorname{dim} \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right) \\
& \leq \sum_{\alpha, \beta, \gamma} \operatorname{dim}_{\operatorname{hom}_{\mathscr{C}}}\left(\tilde{y}_{\mu}, \tilde{y}_{\alpha} \otimes \tilde{y}_{\beta^{\prime}}\right) \times \operatorname{dim}_{\operatorname{hom}_{\mathscr{C}}}\left(\tilde{y}_{\mu}, \tilde{y}_{\beta} \otimes \tilde{y}_{\gamma}\right) \times \operatorname{dim}_{\operatorname{hom}_{\mathscr{C}}}\left(\tilde{y}_{\alpha} \otimes \tilde{y}_{\gamma}, \tilde{y}_{\lambda}\right) \\
& =\sum_{\alpha, \beta, \gamma} \operatorname{dim}_{\operatorname{hom}_{H}}\left(y_{\mu}, y_{\alpha} \otimes y_{\beta^{\prime}}\right) \times \operatorname{dim} \operatorname{hom}_{H}\left(y_{\mu}, y_{\beta} \otimes y_{\gamma}\right) \times \operatorname{dim\operatorname {hom}_{H}(y_{\alpha }\otimes y_{\gamma },y_{\lambda })} \\
& =\sum_{\alpha, \beta, \gamma} c_{\alpha, \beta}^{\mu} c_{\beta^{\prime}, \gamma}^{\nu} c_{\alpha, \gamma}^{\lambda}
\end{aligned}
$$

By Theorem 4.23, the equality holds. So the triangle map $\nabla$ is an embedding map and

$$
\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)=\bigoplus_{\alpha, \beta, \gamma} \nabla\left(\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\alpha} \otimes \tilde{y}_{\beta^{\prime}}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu}, \tilde{y}_{\beta} \otimes \tilde{y}_{\gamma}\right) \otimes \operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\alpha} \otimes \tilde{y}_{\gamma}, \tilde{y}_{\lambda}\right)\right) .
$$

Remark 4.26. Combining Theorem 4.25 and Proposition 2.5, we can construct an explicit basis of $\operatorname{hom}_{\mathscr{C}}\left(\tilde{y}_{\mu} \otimes \tilde{y}_{\nu}, \tilde{y}_{\lambda}\right)$ using $s_{n}$ and the basis of the hom spaces $\operatorname{hom}_{H}\left(\tilde{y}_{\mu}, y_{\alpha} \otimes y_{\beta^{\prime}}\right), \operatorname{hom}_{H}\left(y_{\mu}, y_{\beta}, y_{\gamma}\right)$, $\operatorname{hom}_{H}\left(y_{\alpha} \otimes y_{\gamma}, y_{\lambda}\right)$ in the Hecke algebra H. Applying the evaluation algorithm of the Yang-Baxter relation, we obtain the 6j-symbols of $\mathscr{C}$.

When the Young diagrams are small, the 6j-symbols can be computed by hand or by computer. We do not expect to compute 6j-symbols for large Young diagrams in this way, the complexity of this algorithm grows exponentially w. r. t. the size of the Young diagrams. Even computing the $6 j$-symbols for $\operatorname{Rep}(H(q))$ remains challenging.

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[^0]:    ${ }^{1}$ It was called the Jones projection is the operator algebraic setting.

