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A proof of Demailly's strong openness conjecture

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Abstract

In this article, we solve the strong openness conjecture on the multiplier ideal sheaf associated to any plurisubharmonic function, which was posed by Demailly.

1. Introduction

The multiplier ideal sheaf associated to a plurisubharmonic function, which is an invariant of the singularities of the psh function, plays an important role in several complex variables and complex geometry. Various properties about the multiplier ideal sheaves associated to plurisubharmonic functions have been discussed (e.g., see [25], [7], [30], [31]). Demailly's strong openness conjecture means that the strong openness property about the multiplier ideal sheaf holds. In the present article, we establish such a useful strong openness property on the multiplier ideal sheaf associated to any plurisubharmonic function; i.e., Demailly's strong openness conjecture is true.

1.1. *Organization of the paper.* The paper is organized as follows. In the present section, we recall the statement of the strong openness conjecture posed by Demailly and present the main result of the present paper: a solution of the strong openness conjecture. In Section 2, we recall or give some preliminary lemmas used in the proof of the main result. In Section 3, we give the proof of the strong openness conjecture and present some consequences by combining the conjecture with some well-known results.

1.2. *Statement of the main results.* Let X be a complex manifold with dimension n and φ be a plurisubharmonic function (e.g., see [18], [27]) on X. Following Nadel [25], one can define the multiplier ideal sheaf to be the ideal subsheaf $\mathcal{I}(\varphi) \subseteq \mathcal{O}_X$ of germs of holomorphic functions $f \in \mathcal{O}_x$ such that

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$|f|^2e^{-\varphi}$ is locally integrable near $x \in X$. It is well known that $\mathcal{I}(\varphi)$ is a coherent analytic sheaf.

Denote by $\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi)$. In [2], Berndtsson made important progress by giving a proof of the openness conjecture of Demailly and Kollár in [10]:

Openness Conjecture. *Let φ be a plurisubharmonic function on X. Assume that $\mathcal{I}(\varphi) = \mathcal{O}_X$. Then*

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

The dimension two case of the openness conjecture was proved by Favre and Jonsson in [12] (see also [13]).

In the present article, we discuss a more general conjecture — the strong openness conjecture about multiplier ideal sheaves for plurisubharmonic functions, which was posed by Demailly in [7] and [8] (see also [11]):

Strong Openness Conjecture. *Let φ be a plurisubharmonic function on X. Then*

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

For $\dim X \leq 2$, the strong openness conjecture was proved in [16] by studying the asymptotic jumping numbers for the graded sequences of ideals. It is not hard to see that the truth of the strong openness conjecture is equivalent to the following theorem, which will be proved in the present paper:

Theorem 1.1 ([14]). *Let φ be a negative plurisubharmonic function on the unit polydisc $\Delta^n \subset \mathbb{C}^n$. Suppose F is a holomorphic function on Δ^n, which satisfies*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty,$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. Then for some $r \in (0,1)$, there exists a number $p > 1$ such that

$$\int_{\Delta^n_r} |F|^2 e^{-p\varphi} d\lambda_n < +\infty.$$

2. Lemmas used in the proof of the strong openness conjecture

In this section, we will show or recall some results used in the proof of Theorem 1.1 by mathematical induction on dimensions.

2.1. *Lower bound of L^2 norms of some holomorphic functions on singular analytic curves.* The following lemma will be used in the proof of induction for the dimension one case, and also implies Lemma 2.2 for analytic curves located in high dimension domains that will be used in the proof of induction for the general dimension.

LEMMA 2.1. *Let $f \not\equiv 0$ be a holomorphic function on the disc Δ_r of radius r containing the origin o in $\mathbb{C}$. Let h_a be a holomorphic function on Δ_r, which satisfies $h_a(o) = 0$ and $h_a(b) = 1$ for any $b^k = a$ (k is a positive integer), where $a \in \Delta_r$ whose norm is small enough. Then we have*

$$\int_{\Delta_r} |f|^2 |h_a|^2 d\lambda_1 > C_1 |a|^{-2},$$

where C_1 is a positive constant independent of a and h_a.

Proof. As $f \not\equiv 0$, we may write $f = z^m f_1$ near o, where $f_1|_o \neq 0$. Then there exists $r' < r$ such that $|f_1||_{\Delta_{r'}} \geq C_0 > 0$. Therefore it suffices to consider the case that $f = z^m$ on $\Delta_{r'}$.

Using Taylor expansion at o, we have $h_a(z) = \sum_{j=1}^{\infty} c_j z^j$.

As $h_a(b) = 1$, then

$$\sum_{j=1}^{\infty} c_{kj} a^j = \frac{1}{k} \sum_{1 \leq l \leq k} \sum_{j=1}^{\infty} c_j b_l^j = 1,$$

where $b_l^k = a$, and $\sum_{1 \leq l \leq k} b_l^j = 0$ when $0 < j < k$.

It is clear that

$$\int_{\Delta_{r'}} |f|^2 |h_a|^2 d\lambda_1 = \int_{\Delta_{r'}} |z^m|^2 |h_a|^2 d\lambda_1 = 2\pi \sum_{j=1}^{\infty} |c_j|^2 \frac{{r'}^{2j+2m+2}}{2j+2m+2}.$$

According to the Schwartz Lemma, it follows that

(2.1)
$$\begin{aligned}
&\left(\sum_{j=1}^{\infty} |c_j|^2 \frac{{r'}^{2j+2m+2}}{2j+2m+2}\right)\left(\sum_{j=1}^{\infty} \frac{2kj+2m+2}{{r'}^{2kj+2m+2}} |a|^{2j}\right) \\
&\geq \left(\sum_{j=1}^{\infty} |c_{kj}|^2 \frac{{r'}^{2kj+2m+2}}{2kj+2m+2}\right)\left(\sum_{j=1}^{\infty} \frac{2kj+2m+2}{{r'}^{2kj+2m+2}} |a|^{2j}\right) \geq \left|\sum_{j=1}^{\infty} c_{kj} a^j\right|^2 = 1.
\end{aligned}$$

Note that

$$\sum_{j=1}^{\infty} \frac{2kj+2m+2}{{r'}^{2kj+2m+2}} |a|^{2j} = |\frac{a}{{r'}^k}|^2 \left((2m+2)\frac{{r'}^{-2m-2}}{1-|\frac{a}{{r'}^k}|^2} + 2k\frac{{r'}^{-2m-2}}{(1-|\frac{a}{{r'}^k}|^2)^2}\right)$$

and that $\left((2m+2)\frac{{r'}^{-2m-2}}{1-|\frac{a}{{r'}^k}|^2} + 2k\frac{{r'}^{-2m-2}}{(1-|\frac{a}{{r'}^k}|^2)^2}\right)$ has uniform upper bound independent of a when $|a| < \frac{{r'}^k}{2}$. The lemma thus follows. □

Let π be a projection from $\Delta' \times \Delta'' \subset \mathbb{C} \times \mathbb{C}^{n-1}$ (with coordinate (z', z'')) to Δ' that satisfies $\pi(z', z'') = z'$. Let $\gamma : \Delta \to \Delta' \times \Delta''$ be an analytic curve that satisfies $\gamma \subset \{(z', z'')||z''| \leq |z'|\}$, $\gamma(z) = (z^k, g_2(z), \ldots, g_n(z))$ and $g_2(o) = \cdots = g_n(o) = 0$, where $k \geq 1$ is the degree of the unbranched covering $\pi|_{\gamma \setminus \{o\}}$.

Let f be a holomorphic function on γ satisfying $f \not\equiv 0$. Let h_a be a holomorphic function on γ that satisfies $h_a(o) = 0$ and $h_a(\pi^{-1}(a) \cap \gamma) = 1$.

Using Lemma 2.1, one can obtain the Lemma 2.1 on γ:

$$\int_{(\gamma \backslash o) \cap \pi^{-1}(\Delta'_{1/2})} |f|^2 |h_a|^2 (\pi|_{\gamma \backslash o})^* (d\lambda_{\Delta'}) > C_3 |a|^{-2}, \tag{2.2}$$

when $a \in \Delta'$ and $|a|$ is small enough, where C_3 is a positive constant independent of a and h_a, $\Delta'_{1/2} \subset \Delta'$ is the disc with radius 1/2.

Let F_a be a holomorphic function on $\Delta' \times \Delta''$. Using the submean value property of plurisubharmonic functions and the Fubini Theorem, one can obtain

$$\int_{\Delta' \times \Delta''} |F_a|^2 d\lambda_n \geq C_4 \int_{(\gamma \backslash o) \cap \pi^{-1}(\Delta'_{1/2})} |F_a|_{\gamma \backslash o}|^2 (\pi|_{\gamma \backslash o})^* (d\lambda_{\Delta'}), \tag{2.3}$$

where C_4 is a positive constant independent of F_a.

Using inequalities (2.2) and (2.3), one can obtain the following result, which will be used in the induction proof for the dimension n case:

LEMMA 2.2. *Let F be a holomorphic function on $\Delta' \times \Delta''$ satisfying $F|_\gamma \not\equiv 0$. Denote by hyperplane $H_a := \{z' = a\}$ near o. Let F_a be the holomorphic extension of $F|_{H_a}$ on $\Delta' \times \Delta''$ such that there exists a holomorphic function h_a on γ satisfying*
(1) $F_a|_\gamma = F|_\gamma h_a$,
(2) $h_a(o) = 0$.
Then we have

$$\int_{\Delta' \times \Delta''} |F_a|^2 d\lambda_n \geq \frac{C_2}{|a|^2},$$

where C_2 is a positive constant independent of a, H_a and F_a.

2.2. *Curve selection lemma and Noetherian property of coherent sheaves.* Following the proof of the curve selection lemma (see (11.20) in [8]; see also [22]), one can obtain the following lemma:

LEMMA 2.3 (see [8]; see also [22]). *Let $F \in \mathcal{O}_n$ and $g_1, \ldots, g_s \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at the origin $o \in \mathbb{C}^n$. Assume that for any given neighborhood of o, $|F| \leq C|(g_1, \ldots, g_s)|$ does not hold for any constant C. Then there exists a germ of an analytic curve γ through o, satisfying $\gamma \cap \{F = 0\} \subseteq \{o\}$, such that $\frac{g_i}{F}|_\gamma$ is holomorphic on $\gamma \setminus o$ with $\widetilde{\frac{g_i}{F}}|_\gamma(o) = 0$ for any $i \in \{1, \ldots, s\}$, where $\widetilde{\frac{g_i}{F}}$ is the holomorphic extension of $\frac{g_i}{F}$ from $\gamma \setminus o$ to γ.*

Let φ be a negative plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and let $\{\psi_j\}_{j=1,2,\ldots}$ be a sequence of plurisubharmonic functions on Δ^n that is increasingly convergent to φ on Δ^n when $j \to \infty$.

The strong Noetherian property of coherent sheaves (see (3.22) in Chapter II of [5]) implies that $\cup_{j=1}^{\infty}\mathcal{I}(\psi_j)$ is a coherent subsheaf of $\mathcal{I}(\varphi)$; actually for any open $V \subset\subset M$, there exists $j_1 \in \{1,2,\ldots\}$ such that $\cup_{j=1}^{\infty}\mathcal{I}(\psi_j) = \mathcal{I}(\psi_{j_1})$ on V.

Using Lemma 2.3, and the fact that $\mathcal{I}(\psi_j)$ are integrable closed (see [8]), one can derive the following lemma about the generators of the coherent sheaf $\cup_{j=1}^{\infty}\mathcal{I}(\psi_j)$:

LEMMA 2.4. *Assume that $F \in \mathcal{O}_n$ is a holomorphic function on some neighborhood V of o that is not a germ of $\mathcal{I}(\psi_{j_1})_o(= (\cup_{j=1}^{\infty}\mathcal{I}(\psi_j))_o)$. Then there exists a germ of an analytic curve (γ, o) such that $\widetilde{\frac{g\circ\gamma}{F\circ\gamma}}|_o = 0$ holds for any germ (g,o) of $\mathcal{I}(\psi_{j_1})_o$, where $\widetilde{\frac{g\circ\gamma}{F\circ\gamma}}$ is the holomorphic extension of $\frac{g\circ\gamma}{F\circ\gamma}$ from $\gamma \setminus o$ to γ.*

We would like to thank the referee for kindly pointing out that as observed by Lempert in [23], Lemma 2.4 can be replaced by a result of Lejeune-Jalabert and Teissier in [22]. Actually they are equivalent.

3. Proof of the strong openness conjecture

We will prove Theorem 1.1 by the method of induction, and confirm every step of induction by contradiction and by using movably the Ohsawa-Takegoshi L^2 extension theorem ([26]; see also [28], [1], [6], [29], etc.).

3.1. *Step* 1: *Theorem* 1.1 *for the dimension one case.* We first prove Theorem 1.1 for the dimension one case, which is elementary but instructive. Our proof for the general dimension case is quite similar. Actually the proofs for both cases are parallel.

We choose r_0 small enough such that $\int_{\Delta_{r_0}} |F|^2 e^{-\varphi} d\lambda_1 < +\infty$. Then there exist complex numbers $a_j \to 0$ $(j \to +\infty)$ such that

$$|F(a_j)|^2 e^{-\varphi(a_j)} = o\left(\frac{1}{|a_j|^2}\right).$$

As $|F(a_j)|^2 e^{-\varphi(a_j)} < +\infty$, then one can find $p_j > 1$ small enough such that

$$|F(a_j)|^2 e^{-p_j\varphi(a_j)} \le 2|F(a_j)|^2 e^{-\varphi(a_j)} = o\left(\frac{1}{|a_j|^2}\right). \tag{3.1}$$

Using movably (respect to a_j) the Ohsawa-Takegoshi L^2 extension theorem on Δ, we obtain holomorphic functions F_j on Δ such that $F_j|_{a_j} = F(a_j)$ and

$$\int_{\Delta} |F_j|^2 e^{-p_j\varphi} d\lambda_1 \le \mathbf{C}|F(a_j)|^2 e^{-p_j\varphi(a_j)}, \tag{3.2}$$

where $\mathbf{C}$ is a universal constant.

By inequality (3.1) and negativeness of φ, we obtain that

$$\int_{\Delta} |F_j|^2 d\lambda_1 = o\left(\frac{1}{|a_j|^2}\right). \tag{3.3}$$

By contradiction, assume that Theorem 1.1 does not hold for $n = 1$; that is to say, $\int_{\Delta_r} |F|^2 e^{-p_j\varphi} d\lambda = +\infty$ for any $r > 0$ and any $j \in \{1, 2, \ldots\}$.

Assertion. Since $\{F = 0\} \cap \Delta_{r_0} \subseteq \{o\}$, then it follows from inequality (3.2) that one can derive that F/F_j is unbounded. Otherwise, the boundedness would imply the finiteness of the integral of $|F|^2 e^{-p_j\varphi}$, according to inequality (3.2). This contradicts the assumption. Then there exists a holomorphic function h_j on Δ_{r_0} satisfying

(1) $F_j|_{\Delta_{r_0}} = F|_{\Delta_{r_0}} h_j$,
(2) $h_j(o) = 0$,
(3) $h_j(a_j) = 1$.

According to Lemma 2.1, it follows that

$$\frac{C_1}{|a_j|^2} \leq \int_{\Delta_{r_0}} |F_j|^2 d\lambda_1,$$

which contradicts equality (3.3), where $C_1 > 0$ is independent of j.

We have thus proved Theorem 1.1 for $n = 1$.

3.2. *Step* 2: *Theorem* 1.1 *for* n. By induction on the dimension n, one may assume that Theorem 1.1 holds for $n - 1$.

We prove Theorem 1.1 for the general dimension n by contradiction. Assume that Theorem 1.1 for n is not true, therefore, for some negative psh function φ, there exists a holomorphic function F such that

$$\int_{\Delta^n_{r_0}} |F|^2 e^{-\varphi} d\lambda_n < +\infty, \tag{3.4}$$

for some $r_0 > 0$, and

$$\int_{\Delta^n_r} |F|^2 e^{-p\varphi} d\lambda_n = +\infty, \tag{3.5}$$

for any $r \in (0, r_0)$ and $p > 1$. That is to say, the germ of the holomorphic function F is in $\mathcal{I}(\varphi)_o$ but not in $\mathcal{I}_+(\varphi)_o$.

By Lemma 2.4 and equality (3.5), it follows that there exists a germ of an analytic curve γ through o satisfying $\{F|_\gamma = 0\} \subseteq \{o\}$ such that for any germ of holomorphic function g in $\mathcal{I}_+(\varphi)_o$, there exists a holomorphic function h_g on γ satisfying

$$h_g(o) = 0 \quad \text{and} \quad g|_\gamma = F|_\gamma h_g. \tag{3.6}$$

This plays a similar role with the assertion in the proof for dimension 1.

Using the local parametrization of γ (see [5]), without loss of generality, one may assume that γ and $\Delta' \times \Delta''$ are as in Section 2.1.

By inequality (3.4), it follows that there exist hyperplanes $H_j := H_{a_j} = \{z' = a_j\}$ that satisfy $a_j \to 0$ ($j \to \infty$) and

$$\int_{H_j} |F|^2 e^{-\varphi} d\lambda_{n-1} = o\left(\frac{1}{|a_j|^2}\right).$$

According to the induction assumption and the Lebesgue dominated convergence theorem, it follows that for any given j, there exists a small enough $p_j > 1$ such that

$$\int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1} \leq 2 \int_{H_j} |F|^2 e^{-\varphi} d\lambda_{n-1} = o(\frac{1}{|a_j|^2}). \tag{3.7}$$

Using movably (respect to $j \to \infty$) the Ohsawa-Takegoshi L^2 extension theorem on $\Delta' \times \Delta''$, we obtain a holomorphic function $F_j := F_{a_j}$ on $\Delta' \times \Delta''$ for each j such that $F_j|_{H_j} = F|_{H_j}$, and

$$\int_{\Delta' \times \Delta''} |F_j|^2 e^{-p_j\varphi} d\lambda_n \leq \mathbf{C} \int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1},$$

where $\mathbf{C}$ is a universal constant.

By inequality (3.7) and negativeness of φ, it follows that

$$\int_{\Delta' \times \Delta''} |F_j|^2 d\lambda_n = o\left(\frac{1}{|a_j|^2}\right). \tag{3.8}$$

Note that $F_j|_{H_j} = F|_{H_j}$ and $(F_j, o) \in \mathcal{I}_+(\varphi)_o$ but $(F, o) \notin \mathcal{I}_+(\varphi)_o$. According to equality (3.6) and Lemma 2.2, it follows that

$$\int_{\Delta' \times \Delta''} |F_j|^2 d\lambda_n \geq \frac{C_2}{|a_j|^2},$$

where $C_2 > 0$ is independent of j, which contradicts equality (3.8).

We have thus proved Theorem 1.1 for n. The proof of Theorem 1.1 is now complete.

3.3. *Some remarks on Theorem* 1.1. Let φ be a negative plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and let $\{\psi_j\}_{j=1,2,\ldots}$ be a sequence of plurisubharmonic functions on Δ^n that is increasingly convergent to φ on Δ^n when $j \to \infty$.

Without loss of generality, one can assume that $\psi_1 \not\equiv -\infty$.

By just replacing $p_j\varphi$ by ψ_{k_j} in the proof of Theorem 1.1, the same proof gives the following in [15] (Prof. L. Lempert also observed this after reading [14]):

Let F be a holomorphic function on Δ^n satisfying

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty.$$

Then there exists a number $j_0 \geq 1$ such that

$$\int_{\Delta_r^n} |F|^2 e^{-\psi_{j_0}} d\lambda_n < +\infty$$

for some $r \in (0, 1)$.

That is to say,

$$\cup_{j=1}^{\infty} \mathcal{I}(\psi_j) = \mathcal{I}(\varphi). \tag{3.9}$$

In particular, let $\psi_j = \varphi + \frac{1}{j}\varphi_0$ in equality (3.9). Then we get the following modified version of the strong openness conjecture, which was conjectured in [17]:

Let φ be a negative plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and let $\varphi_0 \not\equiv -\infty$ be a negative plurisubharmonic function on Δ^n. Then

$$\cup_{\varepsilon>0} \mathcal{I}(\varphi + \varepsilon\varphi_0) = \mathcal{I}(\varphi).$$

3.4. *Some consequences of the strong openness conjecture.* In the present subsection, combining our Theorem 1.1 (the truth of the strong openness conjecture) with some known results, one can obtain some direct conclusions that solve some problems.

3.4.1. *Singular metric with minimal singularities.* Let L be a line bundle on a smooth projective complex variety X, whose Kodaira-Iitaka dimension $\kappa(X, L) \geq 0$ (see [9], [19], [20], [21]). Then the asymptotic multiplier ideal $\mathcal{J}(||L||)$ can be defined as the maximal member of the family of ideals $\{\mathcal{J}(\frac{1}{k} \cdot |kL|)\}$ (k large). (See Definition 1.7 in [9]; see also [19], [20].)

Demailly has shown that if L is any pseudo-effective divisor, then up to equivalence of singularities, $\mathcal{O}_X(L)$ has a unique singular metric $h_{\min}$ with minimal singularities having nonnegative curvature current. (See [8]; see also [19], [20].)

Let $\mathcal{J}(h_{\min})$ be the associated multiplier ideal sheaf of $h_{\min}$ (see [8]). In [9], [20], the authors conjectured the following: for a big line bundle L, the equality

$$\mathcal{J}(||mL||) = \mathcal{J}(h_{\min}^m) \tag{3.10}$$

holds for every $m > 0$.

Note that mL is big and $\kappa(X, mL) \geq 0$ for any m. Let $\tilde{h}_{\min}^m$ be the singular metric with minimal singularities on L^m. By the uniqueness of the singular metric with minimal singularities, it follows that $\frac{\tilde{h}_{\min}^m}{h_{\min}^m}$ is a function with uniformly positive upper and lower bound on X. Therefore $\mathcal{J}(\tilde{h}_{\min}^m) = \mathcal{J}(h_{\min}^m)$. Then it suffices to consider the case $m = 1$.

In [21], the author conjectured the following analogue of the above conjecture:

Let X be a smooth projective complex variety and L be a pseudo-effective $\mathbb{R}$-divisor on X. Then

$$\mathcal{J}(T_{\min}) \subseteq \mathcal{J}_\sigma(L), \tag{3.11}$$

where $T_{\min}$ is a current of minimal singularities in the numerical class of L and $\mathcal{J}_\sigma(L)$ is the diminished ideal in [21].

By the arguments after Theorem 1.2 in [21], the above conjecture can be proved by the strong openness conjecture.

In [21], it was shown that

$$\mathcal{J}_\sigma(L) = \mathcal{J}(||L||)$$

when L is a big line bundle (see [21, Cor. 6.12]). Note that $\mathcal{J}(h_{\min})$ in [9] is just $\mathcal{J}(T_{\min})$ in [21]. Using inequality (3.11), one has

$$\mathcal{J}(h_{\min}) \subseteq \mathcal{J}(||L||).$$

In [9], it was shown that

$$\mathcal{J}(||L||) \subseteq \mathcal{J}(h_{\min}).$$

Then equality (3.10) holds.

3.4.2. *Kawamata-Viehweg-Nadel type vanishing theorem.* Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n, and let $\mathrm{nd}(L, \varphi)$ be the numerical dimension of (L, φ) as defined in [4].

In [4], Cao obtained a Kawamata-Viehweg-Nadel type vanishing theorem for $\mathcal{I}_+(\varphi)$ on any compact Kähler manifold:

$$H^p(X, K_X \otimes L \otimes \mathcal{I}_+(\varphi)) = 0$$

holds for any $p \geq n - \mathrm{nd}(L, \varphi) + 1$. In the same paper, Cao asked whether his result holds for $\mathcal{I}(\varphi)$.

Combining Theorem 1.1 with the above result of Cao, one can obtain the following:

Corollary 3.1. *Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n. Then*

$$H^p(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$$

for any $p \geq n - \mathrm{nd}(L, \varphi) + 1$.

A similar result when X is a projective manifold can be found in [24].

3.4.3. *Multiplier ideal sheaves with analytic singularities.* It is known that $\mathcal{I}_+(\varphi)$ is essentially with analytic singularities (see [4]) using Demailly's approximation of plurisubharmonic functions (see [8]). Then it follows from Theorem 1.1 that $\mathcal{I}(\varphi)$ is essentially with analytic singularities; that is to say,

Corollary 3.2. *There is a plurisubharmonic function φ_A with analytic singularities such that $\mathcal{I}(\varphi) = \mathcal{I}(\varphi_A)$.*

3.4.4. *Proper modifications, multiplier ideal sheaves and Lelong numbers.* Let $u, v : (\mathbb{C}^n, 0) \to \mathbb{R} \cup \{-\infty\}$ be two plurisubharmonic germs.

By Theorem 1.1, one has the following result.

COROLLARY 3.3. *Statements* (1) *through* (3) *are equivalent and imply statement* (4):

(1) *For any proper modification* $\pi := X_\pi \to \mathbb{C}^n$ *above* 0 *and all points* $p \in \pi^{-1}(0)$, *we have the Lelong number* $\mu(u \circ \pi, p) = \mu(v \circ \pi, p)$.
(2) *For all* $t > 0$, *we have* $\mathcal{I}(tu) = \mathcal{I}(tv)$.
(3) *For any tame maximal plurisubharmonic weight* φ (*see* [3]), *the relative types* $\sigma(u, \varphi) := \sup\{c > 0, u \le c\varphi + O(1)\}$ *and* $\sigma(v, \varphi)$ *are equal.*
(4) *For any tame maximal plurisubharmonic weight* φ, *the generalized Lelong numbers* $\mu_\varphi(u) := dd^c u \wedge (dd^c \varphi)^{n-1}\{0\}$ *and* $\mu_\varphi(v)$ *are equal.*

The result was conjectured in [3], where the authors show that $\mathcal{I}_+(tu) = \mathcal{I}_+(tv)$ for all $t > 0$ is equivalent to (1) and (3) and implies (4).

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