

ANALYSIS OF THE CONVERGENCE RATE FOR THE CYCLIC PROJECTION ALGORITHM APPLIED TO BASIC SEMIALGEBRAIC CONVEX SETS*

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Abstract. In this paper, we study the rate of convergence of the cyclic projection algorithm applied to finitely many basic semialgebraic convex sets. We establish an explicit convergence rate estimate which relies on the maximum degree of the polynomials that generate the basic semialgebraic convex sets and the dimension of the underlying space. We achieve our results by exploiting the algebraic structure of the basic semialgebraic convex sets.

Key words. cyclic projection algorithm, convex polynomial, distance function, Fejér monotone sequence, Hölderian regularity, Lojasiewicz’s inequality, projector operator, basic semialgebraic convex set, von Neumann alternating projection method

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1. Introduction. A very common problem in diverse areas of mathematics and engineering consists of trying to find a point in the intersection of closed convex sets C_i , $i = 1, \dots, m$. This problem is often referred to as the convex feasibility problem. One popular method for solving the convex feasibility problem is the so-called cyclic projection algorithm. Mathematically, the cyclic projection algorithm is formulated as follows. Given finitely many closed convex sets C_1, C_2, \dots, C_m in \mathbb{R}^n with $\bigcap_{i=1}^m C_i \neq \emptyset$, let $x_0 \in \mathbb{R}^n$ and $P_i := P_{C_i}$, $i = 1, 2, \dots, m$, where P_{C_i} denotes the Euclidean projection to the set C_i . The sequence of *cyclic projections*, $(x_k)_{k \in \mathbb{N}}$, is defined by

$$(1.1) \quad x_1 := P_1 x_0, x_2 := P_2 x_1, \dots, x_m := P_m x_{m-1}, x_{m+1} := P_1 x_m \dots$$

When $m = 2$, the cyclic projection method reduces to the well-known von Neumann alternating projection method. (See [41] and also [8, 12, 13, 27] for some recent developments.) The cyclic projection method has attracted much recent interest due to its simplicity and to numerous applications to diverse areas such as engineering and the physical sciences; see [7, 6, 10, 11, 20, 21] and the references therein.

The convergence properties of cyclic projection methods have been examined by many researchers. In particular, Bregman [19] showed that the sequence $(x_k)_{k \in \mathbb{N}}$ generated by the cyclic projection algorithm always converges to a point in C . Moreover, linear convergence of the cyclic projection algorithm onto convex sets with regular intersections was shown in [23]. On the other hand, for convex sets with irregular intersections (for example, when the intersection is a singleton), the cyclic projection algorithm may not exhibit linear convergence even for simple two-dimensional cases

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as observed by [9, Example 5.3]. (See section 5 for more examples.) This then raises the following basic question:

Can we estimate the convergence rate of the cyclic projection algorithm for convex sets with possibly irregular intersections?

In this paper, we provide an answer for the above question by focusing on the case where each set C_i is a *basic semialgebraic convex set* in \mathbb{R}^n in the sense that there exist $\gamma_i \in \mathbb{N}$ and convex polynomial functions, $g_{ij}, j = 1, \dots, \gamma_i$ such that

$$C_i = \{x \in \mathbb{R}^n \mid g_{ij}(x) \leq 0, j = 1, \dots, \gamma_i\}.$$

The main motivation for examining basic semialgebraic convex sets lies with the following two facts. First, as recently established in [2, 3], optimization problems involving semialgebraic structure have a number of remarkable properties (such as the celebrated Kurdyka–Łojasiewicz inequality) which enables us to obtain useful qualitative information of the problem. Second, the class of basic semialgebraic convex sets is a broad class of convex sets which covers polyhedra and convex sets described by convex quadratic functions. Additionally, the structure can often be relatively easily identified [24].

By exploiting the precise algebraic structure, we are able to provide an explicit rate for the cyclic projection algorithm applied to finitely many basic semialgebraic convex sets *without any regularity conditions*. More precisely, let C_i be basic semialgebraic convex sets generated by polynomials in \mathbb{R}^n with degree at most $d \in \mathbb{N}$. We show that the sequence of cyclic projections $(x_k)_{k \in \mathbb{N}}$ (1.1) converges (at least) at the rate of $\frac{1}{k^\rho}$ when $d > 1$, where $\rho := \frac{1}{\min\{(2d-1)^n-1, 2\beta(n-1)d^{n-2}\}}$ and $\beta(s)$ denotes the *central binomial coefficient* with respect to s —which is given by $\binom{s}{[s/2]}$.¹ When $d = 1$, the sequence of cyclic projections converges linearly.

The remainder of this paper is organized as follows. In section 2, we collect notation and auxiliary results for future use and for the reader’s convenience. In section 3, we give a Hölderian regularity result for finitely many basic semialgebraic convex sets. The proof of our main result (Theorem 4.2) forms the bulk of section 4. In section 5, we explore various concrete examples. Finally, we end the paper with some conclusions and open questions.

2. Preliminaries and auxiliary results. We assume throughout that \mathbb{R}^n is a Euclidean space with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, where $n \in \mathbb{N} := \{1, 2, 3, \dots\}$. We reserve $d \in \mathbb{N}$. We denote by $\mathbb{B}(x, \varepsilon) := \{y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon\}$. We adopt standard notation used in these books [12, 14, 18, 37, 39, 42].

Given a subset C of \mathbb{R}^n , $\text{int } C$ is the *interior* of C , $\text{bd } C$ is the *boundary* of C , $\text{aff } C$ is the *affine hull* of C , and \overline{C} is the *norm closure* of C . The *orthogonal set* $C^\perp := \{x^* \in \mathbb{R}^n \mid (\text{for all } c \in C) \langle x^*, c \rangle = 0\}$. The *distance function* to the set C , written $\text{dist}(\cdot, C)$, is defined by $x \mapsto \inf_{c \in C} \|x - c\|$. The *projector operator* to the set C , denoted by P_C , is defined by

$$P_C(x) := \{c \in C \mid \|y - c\| = \text{dist}(x, C)\} \quad \forall x \in \mathbb{R}^n.$$

Let $D \subseteq \mathbb{R}^n$. The distance of two sets: C and D , is $\text{dist}(C, D) := \inf_{c \in C, d \in D} \|c - d\|$. Given $f: X \rightarrow]-\infty, +\infty]$, we set $\text{dom } f := f^{-1}(\mathbb{R})$. We say f is *proper* if $\text{dom } f \neq \emptyset$. Let f be a proper function on \mathbb{R}^n . Its associated *recession function* f^∞ is defined, for

¹Here, $[a]$ denotes the integer part of a .

any $v \in \mathbb{R}^n$, by

$$f^\infty(v) := \liminf_{t \rightarrow \infty, v' \rightarrow v} \frac{f(tv')}{t}.$$

If f is further assumed to be lower semicontinuous and convex, one has (see [4, Proposition 2.5.2])

$$(2.1) \quad f^\infty(v) = \lim_{t \rightarrow \infty} \frac{f(x+tv) - f(x)}{t} = \sup_{t > 0} \frac{f(x+tv) - f(x)}{t} \quad \forall x \in \text{dom} f.$$

2.1. Notation and facts on polynomials. Recall that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a *polynomial* if there exists a number $r \in \mathbb{N}$ such that

$$f(x) := \sum_{0 \leq |\alpha| \leq r} \lambda_\alpha x^\alpha,$$

where $\lambda_\alpha \in \mathbb{R}$, $x = (x_1, \dots, x_n)$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha_i \in \mathbb{N} \cup \{0\}$, and $|\alpha| := \sum_{j=1}^n \alpha_j$. The corresponding constant r is called the *degree* of f .

Next, let us recall a useful property of polynomial functions.

FACT 2.1 (see [5, Remark 4]). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be polynomial, and $\{x_1, x_2\} \subseteq \mathbb{R}^n$. If f is constant on $D := [x_1, x_2]$, then f is constant on $\text{aff} D$.*

We now summarize some basic properties of convex polynomials that will be used later. The first property is a Frank–Wolfe type result for convex polynomial optimization problems while the second one is a directional-constancy property for a convex polynomial.

FACT 2.2 (Belousov [15, Theorem 13, section 4, Chapter II] or [16, Theorem 3] and [35]). *Let f be a convex polynomial on \mathbb{R}^n . Consider a set $D := \{x \mid g_i(x) \leq 0, i = 1, \dots, m\}$, where each $g_i, i = 1, \dots, m$, is a convex polynomial on \mathbb{R}^n . Suppose that $\inf_{x \in D} f(x) > -\infty$. Then f attains its minimum on D .*

FACT 2.3 (see [4, Proposition 3.2.1]). *Let f be a convex polynomial on \mathbb{R}^n and $v \in \mathbb{R}^n$. Assume that $f^\infty(v) = 0$. Then $f(x+tv) = f(x)$ for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$.*

2.2. Notation and facts on semialgebraic sets/functions. Following [17], a set $D \subseteq \mathbb{R}^n$ is said to be semialgebraic if

$$D := \bigcup_{i=1}^l \bigcap_{j=1}^s \{x \in \mathbb{R}^n \mid f_{ij}(x) = 0, h_{ij}(x) \leq 0\}$$

for some integers l, s and some polynomial functions f_{ij}, h_{ij} on \mathbb{R}^n ($1 \leq i \leq l, 1 \leq j \leq s$). Moreover, a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be semialgebraic if its *graph* $\text{gph} f := \{(x, f(x)) \mid x \in \mathbb{R}^n\}$ is semialgebraic.

We now summarize below some basic properties of semialgebraic sets and semialgebraic functions. These properties will be useful for our later work.

FACT 2.4. *The following statements hold (the properties (P1) and (P4) are direct from the definitions):*

- (P1) *Any polynomial is a semialgebraic function.*
- (P2) (see [17, Proposition 2.2.8].) *Let D be a semialgebraic set. Then $\text{dist}(\cdot, D)$ is a semialgebraic function.*
- (P3) (see [17, Proposition 2.2.6].) *If f, g are semialgebraic functions on \mathbb{R}^n and $\lambda \in \mathbb{R}$, then $f + g, \lambda f, \max\{f, g\}, fg$ are semialgebraic.*

(P4) If f_i are polynomials, $i = 1, \dots, m$, and $\lambda \in \mathbb{R}$, then the sets $\{x \mid f_i(x) = \lambda, i = 1, \dots, m\}$, $\{x \mid f_i(x) \leq \lambda, i = 1, \dots, m\}$ are semialgebraic.

(P5) (Lojasiewicz’s inequality [17, Corollary 2.6.7].) If ϕ, ψ are two continuous semialgebraic functions on compact semialgebraic set $K \subseteq \mathbb{R}^n$ such that $\emptyset \neq \phi^{-1}(0) \subseteq \psi^{-1}(0)$, then there exist constants $c > 0$ and $\tau \in (0, 1]$ such that

$$|\psi(x)| \leq c|\phi(x)|^\tau \quad \forall x \in K.$$

Remark 2.5. As pointed out by [34], the corresponding exponent τ in Lojasiewicz’s inequality (P5) is hard to determine and is typically not known. \square

Remark 2.6. Let $g_i, i = 1, \dots, m$ be polynomials on \mathbb{R}^n and set $S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0\}$. Let $\bar{x} \in S$. Then, (P2) and (P4) imply that $\phi = \max_{1 \leq i \leq m} [g_i]_+$ with $[g_i]_+ := \max\{g_i(\cdot), 0\}$ and $\psi = \text{dist}(\cdot, S)$ are semialgebraic functions. Applying (P5), it follows that there exist $c, \varepsilon > 0$ and $\tau \in (0, 1]$ such that

$$(2.2) \quad \text{dist}(x, S) \leq c \max_{1 \leq i \leq m} [g_i(x)]_+^\tau \quad \forall x \in \mathbb{B}(\bar{x}, \varepsilon). \quad \square$$

As explained in Remark 2.5, the exponent τ in (2.2) is hard to determine and is typically unknown. However, there are some special cases where we can provide some effective estimates on the exponent τ : To formulate these results, we introduce the following notation. Define

$$(2.3) \quad \kappa(n, d) := (d - 1)^n + 1.$$

We now present various results which show that the exponent τ in (2.2) can be effectively estimated when g_i have some appropriate extra structure.

FACT 2.7 (Gwoździewicz [22, Theorem 3]). *Let g be a polynomial on \mathbb{R}^n with degree no larger than d . Suppose that $g(0) = 0$ and there exists $\varepsilon_0 > 0$ such that $g(x) > 0$ for all $x \in \mathbb{B}(0, \varepsilon_0) \setminus \{0\}$. Then there exist constants $c, \varepsilon > 0$ such that*

$$(2.4) \quad \|x\| \leq c g(x)^{\frac{1}{\kappa(n, d)}} \quad \forall x \in \mathbb{B}(0, \varepsilon).$$

We denote by $\beta(s)$ the central binomial coefficient with respect to an integer s : $\binom{s}{\lfloor s/2 \rfloor}$ (with $\binom{0}{0} = 1$) [26].

FACT 2.8 (Kollár see [26, Theorem 3(i)]). *Let g_i be polynomials on \mathbb{R}^n with degree $\leq d$ for every $i = 1, \dots, m$. Let $g(x) := \max_{1 \leq i \leq m} g_i(x)$. Suppose that there exists $\varepsilon_0 > 0$ such that $g(x) > 0$ for all $x \in \mathbb{B}(0, \varepsilon_0) \setminus \{0\}$. Then there exist constants $c, \varepsilon > 0$ such that*

$$\|x\| \leq c g(x)^{\frac{1}{\beta(n-1)d^n}} \quad \forall x \in \mathbb{B}(0, \varepsilon).$$

FACT 2.9 (see [29, Theorem 4.2]). *Let g be a convex polynomial on \mathbb{R}^n with degree at most d . Let $S := \{x \mid g(x) \leq 0\}$ and $\bar{x} \in S$. Then, g has a Hölder type local error bound with exponent $\kappa(n, d)^{-1}$, i.e., there exist constants $c, \varepsilon > 0$ such that*

$$\text{dist}(x, S) \leq c [g(x)]_+^{\frac{1}{\kappa(n, d)}} \quad \forall x \in \mathbb{B}(\bar{x}, \varepsilon).$$

FACT 2.10 (see [29, Theorem 4.1]). *Let g be a convex polynomial on \mathbb{R}^n . Let $S := \{x \mid g(x) \leq 0\}$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) < 0$. Then, g has a Lipschitz type global error bound, i.e., there exists a constant $c > 0$ such that*

$$\text{dist}(x, S) \leq c [g(x)]_+ \quad \forall x \in \mathbb{R}^n.$$

FACT 2.11 (see [38, Theorem 2]). Let g be a continuous convex function \mathbb{R}^n . Let $S := \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) < 0$. Then, for every compact subset K of \mathbb{R}^n , there exists $c > 0$ such that

$$\text{dist}(x, S) \leq c[g(x)]_+ \quad \forall x \in K.$$

The following example shows that the conclusion of Fact 2.11 can fail if we allow K to be noncompact.

Example 2.12 (Shironin, see [29, Example 4.1] or [40]). Let $g_1, g_2 : \mathbb{R}^4 \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} g_1(x_1, x_2, x_3, x_4) &:= x_1, \\ g_2(x_1, x_2, x_3, x_4) &:= x_1^{16} + x_2^8 + x_3^6 + x_1x_2^3x_3^3 + x_1^2x_2^4x_3^2 + x_2^2x_3^4 + x_1^4x_3^4 \\ &\quad + x_1^4x_2^6 + x_1^2x_2^6 + x_1^2 + x_2^2 + x_3^2 - x_4. \end{aligned}$$

Then g_1, g_2 are convex polynomials and

$$g_1(-k, 0, 0, k^{16} + k^2 + k) = g_2(-k, 0, 0, k^{16} + k^2 + k) = -k < 0 \quad \forall k \in \mathbb{N}.$$

Let $g(x) := \max\{g_1(x), g_2(x)\}$ for all $x \in \mathbb{R}^4$. Then, g is a continuous convex function and $g(-k, 0, 0, k^{16} + k^2 + k) < 0$ for all $k \in \mathbb{N}$. On the other hand, as shown in [40], there exists a sequence $(x_k)_{k \in \mathbb{N}}$ in \mathbb{R}^4 such that

$$[g(x_k)]_+ \leq 1 \quad \forall k \quad \text{but} \quad \text{dist}(x_k, S) \rightarrow +\infty,$$

where $S := \{x \in \mathbb{R}^4 \mid g_1(x) \leq 0, g_2(x) \leq 0\}$. Let K be a noncompact set such that $\{x_k\} \subseteq K$. Then, the conclusion of Fact 2.11, “ $\text{dist}(x, S) \leq c[g(x)]_+$ for all $x \in K$,” must fail in this case. \square

Recall that a set $C \subseteq \mathbb{R}^n$ is a basic semialgebraic convex set if there exist $\gamma \in \mathbb{N}$ and convex polynomial functions, $g_j, j = 1, \dots, \gamma$ such that $C = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j = 1, \dots, \gamma\}$. Clearly, any basic semialgebraic convex set is convex and semialgebraic. However, the following example shows that a convex and semialgebraic set does not need to be a basic semialgebraic convex set.

Example 2.13. Consider the set $A := \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - x_1x_2 \leq 0, -x_1 \leq 0, -x_2 \leq 0\}$. Clearly, A is convex and semialgebraic while the polynomial $(x_1, x_2) \mapsto 1 - x_1x_2$ is not convex. We now show that A is not a basic semialgebraic convex set, i.e., it cannot be written as $\{x : g_i(x) \leq 0, i = 1, \dots, l\}$ for some convex polynomials $g_i, i = 1, \dots, l, l \in \mathbb{N}$. To see this, we proceed by the method of contradiction. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x_1, x_2) := x_1$. Clearly $\inf_{x=(x_1, x_2) \in A} f(x) = 0$. Then, by Fact 2.2, f should attain its minimum on A . This leads to a contradiction, and so justifies the claim. \square

2.3. Notation and facts for projection methods. From now on, we assume that

$$\begin{aligned} m &\in \mathbb{N}, \gamma_i \in \mathbb{N}, i = 1, \dots, m, \\ g_{i,1}, g_{i,2}, \dots, g_{i,\gamma_i} &\text{ are convex polynomials on } \mathbb{R}^n, i = 1, \dots, m, \\ C_i &:= \left\{ x \in \mathbb{R}^n \mid g_{i,1}(x) \leq 0, g_{i,2}(x) \leq 0, \dots, g_{i,\gamma_i}(x) \leq 0 \right\}, i = 1, \dots, m, \\ P_i &:= P_{C_i} \quad \forall i = 1, 2, \dots, m, \\ C &:= \bigcap_{i=1}^m C_i \neq \emptyset. \end{aligned}$$

FACT 2.14 (Bregman [19]). *Let $x_0 \in \mathbb{R}^n$. The sequence of cyclic projections, $(x_k)_{k \in \mathbb{N}}$, defined by*

$$(2.5) \quad x_1 := P_1 x_0, x_2 := P_2 x_1, \dots, x_m := P_m x_{m-1}, x_{m+1} := P_1 x_m \dots$$

converges to a point in C .

FACT 2.15 (Bauschke and Borwein [9, Lemma 2.2 and Theorem 4.8] or [8, Facts 1.1(iii) and 1.2(ii)]). *Let A, B be nonempty convex subsets of \mathbb{R}^n such that $A - B$ is closed. Let $b_0 \in X$ and $(a_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}}$ be defined as below:*

$$\begin{cases} a_{k+1} := P_A b_k, \\ b_{k+1} := P_B a_{k+1}. \end{cases}$$

Let $v := P_{A-B} 0$. Then, we have

- (i) $\|v\| = \text{dist}(A, B)$ and $a_k \rightarrow a, b_k \rightarrow a + v$,
- (ii) $P_B x = P_{B \cap (A+v)} x = x + v$ for all $x \in A \cap (B - v)$ and $P_A y = P_{A \cap (B-v)} y = y - v$ for all $y \in B \cap (A + v)$

DEFINITION 2.16. *Let A be a nonempty convex subset of \mathbb{R}^n . We say the sequence $(x_k)_{k \in \mathbb{N}}$ in \mathbb{R}^n is Fejér monotone with respect to A if*

$$\|x_{k+1} - a\| \leq \|x_k - a\| \quad \forall k \in \mathbb{N}, a \in A.$$

FACT 2.17 (Bauschke and Borwein [8, Theorem 3.3(iv)]). *Let A be a nonempty closed convex subset of \mathbb{R}^n and let $(x_k)_{k \in \mathbb{N}}$ be Fejér monotone with respect to A , and $x_k \rightarrow x \in A$. Then $\|x_n - x\| \leq 2 \text{dist}(x_n, A)$.*

3. Hölderian regularity for basic semialgebraic convex sets. In this section, we will establish Hölderian regularity for basic semialgebraic convex sets and shall provide an effective estimate of the exponent in the regularity results. This result plays an important role in our following estimation of the convergence speed of the cyclic projection methods.

To do this, we first establish an error bound result which estimates the distance of a point to a basic semialgebraic convex set S in terms of the polynomials which define S . More explicitly, we obtain an explicit exponent $\tau > 0$ such that there exist $c, \varepsilon > 0$,

$$\text{dist}(x, S) \leq c \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^\tau \quad \text{whenever} \quad \|x - \bar{x}\| \leq \varepsilon,$$

where $S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, m\}$.

We note that this error bound property plays an important role in convergence analysis of many algorithms for optimization problems [36] as well as for the variational inequality problem [30], and that the exponent τ in the error bound property has a close relationship with the convergence rate of the algorithm [25, 28, 31, 33]. However, existing results such as the powerful Łojasiewicz’s inequality do not provide any insight on how to explicitly estimate the exponent τ .

Before we proceed, let us use a simple example to illustrate that the exponent τ can be related to the maximum degree of the polynomials that defined the basic semi-algebraic convex set and the dimension of the underlying space. This example is partially inspired by [26, Example 1].

Example 3.1. Let d be an even number. Consider convex polynomials $g_i, i = 1, \dots, n$ on \mathbb{R}^n given by $g_1(x) := x_1^d$ and $g_i, i = 2, \dots, n$ given by $g_i(x) :=$

$x_i^d - x_{i-1}$, $i = 2, \dots, n$. Then, direct verification gives us that $S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, n\} = \{0\}$, and so, $\text{dist}(x, S) = \|x\|$. In this case, consider $x(t) = (t^{d^{n-1}}, t^{d^{n-2}}, \dots, t) \in \mathbb{R}^n$, $t \in (0, 1)$. Then $\text{dist}(x(t), S) = O(t)$ and $\max_{1 \leq i \leq m} [g_i(x(t))]_+ = t^{d^n}$. Therefore, we see that if there exist $c, \epsilon > 0$ and $\tau > 0$ such that

$$\text{dist}(x, S) \leq c \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^\tau \quad \text{whenever} \quad \|x\| \leq \epsilon,$$

then $\tau \leq \frac{1}{d^n}$. Thus, we see that the exponent τ is related to the maximum degree of the polynomials defined the basic semialgebraic convex set and the dimension of the underlying space. \square

We now introduce a decomposition of the index set.

DEFINITION 3.2. For convex polynomials g_1, \dots, g_m on \mathbb{R}^n with $S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, m\}$, the index set $\{1, \dots, m\}$ can be decomposed as $J_0 \cup J_1$ with $J_0 \cap J_1 = \emptyset$, where

$$(3.1) \quad J_0 := \{i \in \{1, \dots, m\} \mid g_i(x) = 0 \ \forall x \in S\} \quad \text{and} \quad J_1 := \{1, \dots, m\} \setminus J_0.$$

Now we come to our key technical result, which provides a local error bound for convex polynomial systems. The main idea of the proof is to use the extreme rigidity of convex polynomial functions (see Facts 2.1 and 2.3) to reduce the problem to the one of comparing a (power of a) nonnegative convex polynomial vanishing only at 0 with the norm so that the estimate of the Lojasiewicz inequality that we introduced in Facts 2.7 and 2.8 can be applied.

THEOREM 3.3 (local error bounds for convex polynomial systems). Let g_i be convex polynomials on \mathbb{R}^n with degree at most d for every $i = 1, \dots, m$. Let $S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, m\}$ and $\bar{x} \in S$. Then there exist $c, \epsilon > 0$ such that

$$\text{dist}(x, S) \leq c \left(\max_{i \in J_1} [g_i(x)]_+ + \left(\max_{i \in J_0} [g_i(x)]_+ \right)^\tau \right) \quad \text{whenever} \quad \|x - \bar{x}\| \leq \epsilon,$$

where $[a]_+ := \max\{a, 0\}$, $\tau := \max\{\frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n}\}$, $\kappa(n, 2d) := (2d - 1)^n + 1$, $\beta(n - 1)$ is the central binomial coefficient with respect to $n - 1$ which is given by $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$, and J_0, J_1 are defined as in (3.1).

Proof. We prove the desired conclusion by induction on the number of the polynomials m .

Trivial case. Suppose that $m = 1$. Then $J_0 = \{1\}$ or $J_0 = \emptyset$. If $J_0 = \{1\}$, then the conclusion follows by Fact 2.9 since $\max\{\frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n}\} \leq \frac{1}{\kappa(n, d)}$. If $J_0 = \emptyset$, then there exists x_0 such that $g_1(x_0) < 0$. In this case, the conclusion follows by Fact 2.10.

Reduction to the active cases. Let us suppose that the conclusion is true for $m \leq p - 1$, $p \in \mathbb{N}$, and look at the case for $m = p$.

We first consider the case where $J_0 \neq \{1, \dots, m\}$. Then $\{1, \dots, m\} \setminus J_0 \neq \emptyset$. Let $i_0 \notin J_0$. Then there exists $x_0 \in S$ such that $g_{i_0}(x_0) < 0$. Set $J := \{i \in \{1, 2, \dots, m\} \mid g_i(x_0) < 0\}$. Then $i_0 \in J \subseteq J_1$ and $J \cap J_0 = \emptyset$. Let A, B be defined by

$$\begin{aligned} A &:= \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \quad \forall i \in J\}, \\ B &:= \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \quad \forall i \in \tilde{J}\}, \end{aligned}$$

where $\tilde{J} := \{1, 2, \dots, m\} \setminus J$. Thus, $J_0 \subseteq \tilde{J}$. Then we have $x_0 \in \text{int} A \cap B$ and $S = A \cap B$. Since $S \subseteq B$, we have

$$(3.2) \quad \tilde{J}_0 := \{i \in \{1, 2, \dots, m\} \setminus J \mid g_i(B) \equiv 0\} \subseteq J_0.$$

Since $\text{int } A \cap B \neq \emptyset$, [8, Corollary 4.5] implies that for every compact set K there exist $\gamma, \delta > 0$ such that

$$(3.3) \quad \text{dist}(x, S) = \text{dist}(x, A \cap B) \leq \gamma \max\{\text{dist}(x, A), \text{dist}(x, B)\} \quad \forall x \in K.$$

Now applying Fact 2.11 with $f(x) = \max_{i \in J} [g_i(x)]_+$, there exists $c_1 > 0$ such that

$$(3.4) \quad \text{dist}(x, A) \leq c_1 \max_{i \in J} [g_i(x)]_+ \quad \forall x \in K.$$

From the induction hypothesis and (3.2), we see that there exist $\varepsilon > 0$ and $c_2 > 0$ such that for every $\|x - \bar{x}\| \leq \varepsilon$, $\max_{1 \leq i \leq m} [g_i(x)]_+ \leq 1$ such that

$$\begin{aligned} & \text{dist}(x, B) \\ & \leq c_2 \left(\max_{i \in \tilde{J} \setminus \tilde{J}_0} [g_i(x)]_+ + \left(\max_{i \in \tilde{J}_0} [g_i(x)]_+ \right)^\tau \right) \\ & = c_2 \left(\max \left\{ \max_{i \in \tilde{J} \setminus \tilde{J}_0} [g_i(x)]_+, \max_{i \in \tilde{J}_0 \setminus \tilde{J}_0} [g_i(x)]_+ \right\} + \left(\max_{i \in \tilde{J}_0} [g_i(x)]_+ \right)^\tau \right) \\ & \hspace{15em} (\text{since } \tilde{\tilde{J}}_0 \subseteq \tilde{J}_0 \subseteq \tilde{J}) \\ & \leq c_2 \left(\max_{i \in \tilde{J} \setminus \tilde{J}_0} [g_i(x)]_+ + \max_{i \in \tilde{J}_0 \setminus \tilde{J}_0} [g_i(x)]_+ + \left(\max_{i \in \tilde{J}_0} [g_i(x)]_+ \right)^\tau \right) \\ & \leq c_2 \left(\max_{i \in \tilde{J} \setminus \tilde{J}_0} [g_i(x)]_+ + \left(\max_{i \in \tilde{J}_0 \setminus \tilde{J}_0} [g_i(x)]_+ \right)^\tau + \left(\max_{i \in \tilde{J}_0} [g_i(x)]_+ \right)^\tau \right) \\ & \leq c_2 \left(\max_{i \in \tilde{J} \setminus \tilde{J}_0} [g_i(x)]_+ + 2 \left(\max_{i \in \tilde{J}_0} [g_i(x)]_+ \right)^\tau \right) \\ & \leq 2c_2 \left(\max_{i \in \tilde{J} \setminus \tilde{J}_0} [g_i(x)]_+ + \left(\max_{i \in \tilde{J}_0} [g_i(x)]_+ \right)^\tau \right). \end{aligned}$$

Thus, combining (3.3) and (3.4) and noting that $(\tilde{J} \setminus \tilde{J}_0) \cup J \subseteq J_1$, we see that the conclusion follows when $J_0 \neq \{1, \dots, m\}$.

From now on, we may assume that $J_0 = \{1, \dots, m\}$. That is to say,

$$\{x \mid g_i(x) \leq 0, i = 1, \dots, m\} = \{x \mid g_i(x) = 0, i = 1, \dots, m\}.$$

This implies that $\inf_{x \in \mathbb{R}^n} \max_{1 \leq i \leq m} \{g_i(x)\} = 0$. Then,

$$0_{\mathbb{R}^m} \notin \{(g_1(x), \dots, g_m(x)) \mid x \in \mathbb{R}^n\} + \text{int } \mathbb{R}_+^m.$$

Hence the convex separation theorem ensures that there exist $\alpha_i \geq 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\sum_{i=1}^m \alpha_i g_i(x) \geq 0$ for all $x \in \mathbb{R}^n$. Denote $I := \{i \mid \alpha_i > 0\} \neq \emptyset$. Then, we have $\sum_{i \in I} \alpha_i = 1$ and

$$(3.5) \quad \sum_{i \in I} \alpha_i g_i(x) \geq 0 \quad \forall x \in \mathbb{R}^n.$$

Decompose the underlying space into sum of two subspaces M and M^\perp . Consider $D := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\}$. Clearly, D is a convex set and $\bar{x} \in D$. Moreover, for any $v \in D$, (3.5) implies that

$$(3.6) \quad g_i(v) = 0 \quad \forall i \in I.$$

In other words, g_i takes constant value 0 on D . Then, Fact 2.1 implies that D is either a singleton or an affine set with dimension larger than one.

Let $M := D - \bar{x}$. Then M is a subspace. We may decompose $\mathbb{R}^n = M + M^\perp$. Denote $\dim M = k$ ($k \leq n$).

We now see that

$$(3.7) \quad \sum_{i \in I} g_i^2(x) > 0 \quad \forall x - \bar{x} \in M^\perp \setminus \{0\}.$$

Otherwise, there exists $x_0 \in \mathbb{R}^n$ such that $x_0 - \bar{x} \in M^\perp \setminus \{0\}$ and $g_i(x_0) = 0$ for all $i \in I$. This shows that $x_0 \in D$. Thus $x_0 - \bar{x} \in M$ and hence $x_0 - \bar{x} \in M \cap M^\perp$. This contradicts the fact that $x_0 - \bar{x} \neq 0$.

Similarly, we have

$$(3.8) \quad \max_{i \in I} g_i(x) > 0 \quad \forall x - \bar{x} \in M^\perp \setminus \{0\}.$$

Distance estimation on M^\perp . We first show that there exist $\varepsilon_0, \gamma_0 > 0$ such that

$$(3.9) \quad \|x - \bar{x}\| \leq \gamma_0 \left(\sum_{i \in I} g_i^2(x) \right)^{\frac{1}{\kappa(n-k, 2d)}} \quad \forall x - \bar{x} \in M^\perp \cap \mathbb{B}(0, \varepsilon_0).$$

Since $\dim M^\perp = n - k$, there exists an $n \times (n - k)$ matrix Q_0 with the rank $n - k$ such that $Q_0(\mathbb{R}^{n-k}) = M^\perp$. Then Q_0 is a bijective operator from \mathbb{R}^{n-k} to M^\perp . Then (3.7) shows that

$$(3.10) \quad \sum_{i \in I} g_i^2(\bar{x} + Q_0 b) > 0 \quad \forall b \in \mathbb{R}^{n-k} \setminus \{0\}.$$

Define $h : \mathbb{R}^{n-k} \rightarrow \mathbb{R}$ by $h(b) := \sum_{i \in I} g_i^2(\bar{x} + Q_0 b)$. Then $h(0) = \sum_{i \in I} g_i^2(\bar{x}) = 0$ by (3.6). Thus by (3.10) and Fact 2.7, there exist $\varepsilon_1, \gamma_1 > 0$ such that for all

$$\|b\| \leq \gamma_1 h(b)^{\frac{1}{\kappa(n-k, 2d)}} = \gamma_1 \left(\sum_{i \in I} g_i^2(\bar{x} + Q_0 b) \right)^{\frac{1}{\kappa(n-k, 2d)}} \quad \forall \|b\| \leq \varepsilon_1.$$

Setting $x := \bar{x} + Q_0 b$, it follows that

$$\begin{aligned} \|x - \bar{x}\| &= \|Q_0(Q_0^{-1}(x - \bar{x}))\| \leq \|Q_0\| \cdot \|Q_0^{-1}(x - \bar{x})\| \\ &\leq \|Q_0\| \gamma_1 \left(\sum_{i \in I} g_i^2(x) \right)^{\frac{1}{\kappa(n-k, 2d)}} \quad \forall \|Q_0^{-1}(x - \bar{x})\| \leq \varepsilon_1. \end{aligned}$$

Hence, there exist $\varepsilon_1, \gamma_1 > 0$ such that

$$\|x - \bar{x}\| \leq \gamma_0 \left(\sum_{i \in I} g_i^2(x) \right)^{\frac{1}{\kappa(n-k, 2d)}} \quad \forall x - \bar{x} \in M^\perp \cap \mathbb{B}(0, \varepsilon_0).$$

Thence (3.9) holds.

By Fact 2.8 and (3.8), there exist $\tilde{\varepsilon}_0, \tilde{\gamma}_0 > 0$ such that $\tilde{\varepsilon}_0 \leq \varepsilon_0$ and, for all x with $x - \bar{x} \in M^\perp \cap \mathbb{B}(0, \tilde{\varepsilon}_0)$,

$$\max_{i \in I} g_i(x) \leq 1$$

and

$$\begin{aligned}
 \|x - \bar{x}\| &\leq \tilde{\gamma}_0 \left(\max_{i \in I} g_i(x) \right)^{\frac{1}{\beta(n-k-1)d^{n-k}}} \\
 (3.11) \qquad &= \tilde{\gamma}_0 \left(\max_{i \in I} [g_i(x)]_+ \right)^{\frac{1}{\beta(n-k-1)d^{n-k}}}.
 \end{aligned}$$

Distance estimation on M . Set

$$r := \begin{cases} \max\left\{ \frac{\sum_{j \in I \setminus \{i\}} \alpha_j}{\alpha_i} \mid i \in I \right\} > 0 & \text{if } |I| \geq 2; \\ 1 & \text{otherwise.} \end{cases}$$

Thus $r \geq 1$. Note that $\sum_{i \in I} \alpha_i g_i(x) \geq 0$, and we have for each $i \in I$

$$\max_{i \in I} [g_i(x)]_+ \geq g_i(x) \geq -\frac{\sum_{j \in I \setminus \{i\}} \alpha_j g_j(x)}{\alpha_i} \geq -r \max_{i \in I} [g_i(x)]_+.$$

Hence we have $|g_i(x)| \leq r \max_{i \in I} [g_i(x)]_+$. This together with (3.9) implies that

$$\|x - \bar{x}\| \leq \gamma_0 r^2 |I| \left(\max_{i \in I} [g_i(x)]_+ \right)^{\frac{2}{\kappa(n-k, 2d)}} \quad \forall x - \bar{x} \in M^\perp \cap \mathbb{B}(0, \varepsilon_0).$$

Combining this with (3.11), we see that, for every $x - \bar{x} \in M^\perp \cap \mathbb{B}(0, \tilde{\varepsilon}_0)$,

$$(3.12) \quad \|x - \bar{x}\| \leq (\gamma_0 r^2 |I| + \tilde{\gamma}_0) \left(\max_{i \in I} [g_i(x)]_+ \right)^{\max\left\{ \frac{2}{\kappa(n-k, 2d)}, \frac{1}{\beta(n-k-1)d^{n-k}} \right\}}.$$

We now consider two cases.

Case 1. $\dim M = \{0\}$.

We have $D = S = \{\bar{x}\}$. Thus $M = 0$ and $M^\perp = \mathbb{R}^n$. We can assume that $\max_{i \in I} [g_i(x)]_+ \leq 1$ for all $x - \bar{x} \in \mathbb{B}(0, \tilde{\varepsilon}_0)$. Then by (3.12), we have

$$\begin{aligned}
 \text{dist}(x, S) &= \|x - \bar{x}\| \\
 &\leq (\gamma_0 r^2 |I| + \tilde{\gamma}_0) \left(\max_{i \in I} [g_i(x)]_+ \right)^{\max\left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \quad \forall \|x - \bar{x}\| \leq \tilde{\varepsilon}_0.
 \end{aligned}$$

Case 2. $k = \dim M \geq 1$.

Since $\dim M = k$, there exists a full rank matrix $Q \in \mathbb{R}^{n \times k}$ such that $Q(\mathbb{R}^k) = M$. For each $u \in M$ and $i \in I$, (3.6) implies that

$$g_i^\infty(u) = \lim_{t \rightarrow \infty} \frac{g_i(\bar{x} + tu) - g_i(\bar{x})}{t} = 0.$$

Then, Fact 2.3 implies that

$$(3.13) \quad g_i(x + u) = g_i(x) \quad \forall x \in \mathbb{R}^n, u \in M, i \in I.$$

Since $S \subseteq D = \bar{x} + M$, it follows that

$$S = \{x \in \bar{x} + M \in \mathbb{R}^n \mid g_i(x) \leq 0, i \notin I\} = \bar{x} + Q(\widehat{S}),$$

where $\widehat{S} := \{a \in \mathbb{R}^k \mid g_i(\bar{x} + Qa) \leq 0, i \notin I\}$.

Note that $0 \in \widehat{S}$. The induction hypothesis implies that there exist $\widetilde{\varepsilon}_1, \widetilde{\gamma}_1 > 0$ such that $\max_{i \notin I} [g_i(\bar{x} + Qa)]_+ \leq 1$ and

$$\text{dist}(a, \widehat{S}) \leq \widetilde{\gamma}_1 \left(\max_{i \notin I} [g_i(\bar{x} + Qa)]_+ \right)^{\max \left\{ \frac{2}{\kappa(k, 2d)}, \frac{1}{\beta(k-1)d^k} \right\}} \quad \forall \|a\| \leq \widetilde{\varepsilon}_1.$$

This implies that there exist $\varepsilon_2, \gamma_2 > 0$ such that

$$(3.14) \quad \text{dist}(x, S) \leq \gamma_2 \left(\max_{i \notin I} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(k, 2d)}, \frac{1}{\beta(k-1)d^k} \right\}} \quad \forall x - \bar{x} \in M \cap \mathbb{B}(0, \varepsilon_2).$$

Combining the estimation and simplification. Now let $\varepsilon \leq \min\{\widetilde{\varepsilon}_0, \varepsilon_2\}$ be such that $\max_{1 \leq i \leq m} [g_i(x)]_+ \leq 1$ for all $x \in \mathbb{B}(\bar{x}, \varepsilon)$. Let K be a compact set containing $\mathbb{B}(\bar{x}, \varepsilon) \cup \mathbb{B}(0, \varepsilon)$. Denote the Lipschitz constant of g_i over K by L_i , i.e., $|g_i(x_1) - g_i(x_2)| \leq L_i \|x_1 - x_2\|$ for all $x_1, x_2 \in K$. Set $L := \max_{1 \leq i \leq m} L_i$ and $\gamma := \max\{\gamma_0 r^2 |I| + \widetilde{\gamma}_0, \gamma_2\}$.

To arrive at the conclusion, we only need to show that for any $x \in \mathbb{B}(\bar{x}, \varepsilon)$,

$$\text{dist}(x, S) \leq c \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}},$$

where $c := 2\gamma + L\gamma^2$. To see this, let us fix an arbitrary $x \in \mathbb{B}(\bar{x}, \varepsilon)$. Note that $\mathbb{R}^n = M + M^\perp$. Then, one can decompose $x - \bar{x} = u + v$ for some $u \in M \cap \mathbb{B}(0, \varepsilon)$ and $v \in M^\perp \cap \mathbb{B}(0, \varepsilon)$. This, together with (3.14) and (3.12), implies that

$$(3.15) \quad \text{dist}(u + \bar{x}, S) \leq \gamma \left(\max_{i \notin I} [g_i(u + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(k, 2d)}, \frac{1}{\beta(k-1)d^k} \right\}},$$

$$(3.16) \quad \|v\| \leq \gamma \left(\max_{i \in I} [g_i(v + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(n-k, 2d)}, \frac{1}{\beta(n-k-1)d^{n-k}} \right\}}.$$

Therefore,

$$\begin{aligned} \text{dist}(x, S) &\leq \text{dist}(u + \bar{x}, S) + \|x - (u + \bar{x})\| \\ &= \text{dist}(u + \bar{x}, S) + \|v\| \\ &\leq \gamma \left(\max_{i \notin I} [g_i(u + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(k, 2d)}, \frac{1}{\beta(k-1)d^k} \right\}} \\ &\quad + \gamma \left(\max_{i \in I} [g_i(v + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(n-k, 2d)}, \frac{1}{\beta(n-k-1)d^{n-k}} \right\}} \\ &= \gamma \left(\max_{i \notin I} [g_i(u + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(k, 2d)}, \frac{1}{\beta(k-1)d^k} \right\}} \\ &\quad + \gamma \left(\max_{i \in I} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\ &\leq \gamma \left(\max_{1 \leq i \leq m} [g_i(u + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(k, 2d)}, \frac{1}{\beta(k-1)d^k} \right\}} \\ &\quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}}, \end{aligned}$$

where the second equality follows by (3.13) and $v + \bar{x} + u = x$. Note that

$$\begin{aligned}
 (3.17) \quad & \left| \max_{1 \leq i \leq m} [g_i(u + \bar{x})]_+ - \max_{1 \leq i \leq m} [g_i(x)]_+ \right| \\
 & \leq \max_{1 \leq i \leq m} |g_i(u + \bar{x}) - g_i(x)| \\
 & \leq L \|u + \bar{x} - x\| = L \|v\| \\
 & \leq L \gamma \left(\max_{i \in I} [g_i(v + \bar{x})]_+ \right)^{\max \left\{ \frac{2}{\kappa(n-k, 2d)}, \frac{1}{\beta(n-k-1)d^{n-k}} \right\}} \quad (\text{by (3.16)}) \\
 & = L \gamma \left(\max_{i \in I} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n-k, 2d)}, \frac{1}{\beta(n-k-1)d^{n-k}} \right\}} \quad (\text{by 3.13) and } v + \bar{x} + u = x).
 \end{aligned}$$

As $\max_{1 \leq i \leq m} [g_i(x)]_+ \leq 1$ for all $x \in \mathbb{B}(\bar{x}, \varepsilon)$, it follows that

$$\begin{aligned}
 (3.18) \quad & \text{dist}(x, S) \\
 & \leq \gamma \left(\max_{1 \leq i \leq m} [g_i(u + \bar{x})]_+ \right)^{\frac{2}{\kappa(k, 2d)}} + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
 & \leq \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ + L \gamma \left(\max_{i \in I} [g_i(x)]_+ \right)^{\frac{2}{\kappa(n-k, 2d)}} \right)^{\frac{2}{\kappa(k, 2d)}} \quad (\text{by (3.17)}) \\
 & \quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
 & \leq \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{2}{\kappa(n-k, 2d)}} + L \gamma \left(\max_{i \in I} [g_i(x)]_+^{\frac{2}{\kappa(n-k, 2d)}} \right) \right)^{\frac{2}{\kappa(k, 2d)}} \\
 & \quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
 & \leq (\gamma + L \gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{4}{\kappa(n-k, 2d) \cdot \kappa(k, 2d)}} + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
 & = (\gamma + L \gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{4}{(2d-1)^{n-k+1} \cdot (2d-1)^{k+1}}} \\
 & \quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
 & \leq (\gamma + L \gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{2}{(2d-1)^{n+1}}} + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}}.
 \end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\text{dist}(x, S) &\leq \gamma \left(\max_{1 \leq i \leq m} [g_i(u + \bar{x})]_+ \right)^{\frac{1}{\beta(k-1)d^k}} + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
&\leq \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ + L\gamma \left(\max_{i \in I} [g_i(x)]_+ \right)^{\frac{1}{\beta(n-k-1)d^{n-k}}} \right)^{\frac{1}{\beta(k-1)d^k}} \quad (\text{by (3.17)}) \\
&\quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
&\leq \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{1}{\beta(n-k-1)d^{n-k}}} + L\gamma \left(\max_{i \in I} [g_i(x)]_+^{\frac{1}{\beta(n-k-1)d^{n-k}}} \right) \right)^{\frac{1}{\beta(k-1)d^k}} \\
&\quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
&= (\gamma + L\gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{1}{(\beta(n-k-1)d^{n-k}) \cdot (\beta(k-1)d^k)}} \\
&\quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
&\leq (\gamma + L\gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\frac{1}{\beta(n-1)d^n}} + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}},
\end{aligned}$$

where the last inequality was obtained by the Chu–Vandermonde identity.

In combination with (3.18), we obtain

$$\begin{aligned}
\text{dist}(x, S) &\leq (\gamma + L\gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\max \left\{ \frac{2}{(2d-1)^n+1}, \frac{1}{\beta(n-1)d^n} \right\}} \\
&\quad + \gamma \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\max \left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\}} \\
&= (2\gamma + L\gamma^2) \max_{1 \leq i \leq m} [g_i(x)]_+^{\max \left\{ \frac{2}{(2d-1)^n+1}, \frac{1}{\beta(n-1)d^n} \right\}}.
\end{aligned}$$

This completes the proof. \square

As a corollary, we obtain a local error bound result which is independent of the partition of the index set.

COROLLARY 3.4. *Let g_i be convex polynomials on \mathbb{R}^n with degree at most d for every $i = 1, \dots, m$. Let $S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, m\}$ and $\bar{x} \in S$. Then there exist $c, \varepsilon > 0$ such that*

$$\text{dist}(x, S) \leq c \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^\tau \quad \text{whenever} \quad \|x - \bar{x}\| \leq \varepsilon,$$

where $[a]_+ := \max\{a, 0\}$, $\tau := \max\left\{ \frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n} \right\} = \frac{1}{\min\left\{ \frac{(2d-1)^n+1}{2}, \beta(n-1)d^n \right\}}$ and $\beta(n-1)$ is the central binomial coefficient with respect to $n-1$.

Proof. Choose ε small enough so that $\max_{1 \leq i \leq m} [g_i(x)]_+ \leq 1$. Then, the conclusion follows immediately from the preceding Theorem 3.3 by noting that $[g_i(x)]_+ \leq (\max_{1 \leq i \leq m} [g_i(x)]_+)^{\frac{2}{\kappa(n, 2d)}}$ for each $i = 1, \dots, m$. \square

Remark 3.5 (discussion of the exponent). Let g_i be convex polynomials on \mathbb{R}^n with degree at most d for every $i = 1, \dots, m$. Let $S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i =$

$1, \dots, m\}$ and $\bar{x} \in S$. We now make some discussion on the exponent in our local error bound results.

- (1) Theorem 3.3 shows that in the case when $d = 1$ or if there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0, i = 1, \dots, m$, we indeed obtain a Lipschitz type local error bound. That is to say, in these cases, there exist $c, \varepsilon > 0$ such that

$$\text{dist}(x, S) \leq c \max_{1 \leq i \leq m} [g_i(x)]_+ \quad \text{whenever} \quad \|x - \bar{x}\| \leq \varepsilon,$$

where $[a]_+ := \max\{a, 0\}$. To see this, if $d = 1$, then $\frac{2}{\kappa(n, 2d)} = 1$ and $\frac{1}{\beta(n-1)d^n} \geq 1$. So, $\tau = \max\{\frac{2}{\kappa(n, 2d)}, \frac{1}{\beta(n-1)d^n}\} = 1$ and the conclusion follows immediately from Theorem 3.3. On the other hand, if there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0, i = 1, \dots, m$, then $J_0 = \emptyset$. So, the conclusion follows immediately from the same theorem.

- (2) In the case when $n = 1$, we see that $\frac{2}{\kappa(n, 2d)} = \frac{1}{\beta(n-1)d^n} = \frac{1}{d}$. So, when each g_i is a univariate convex polynomial, there exist $c, \varepsilon > 0$ such that

$$\text{dist}(x, S) \leq c \left(\max_{1 \leq i \leq m} [g_i(x)]_+ \right)^{\frac{1}{d}} \quad \text{whenever} \quad \|x - \bar{x}\| \leq \varepsilon.$$

Note that, for the naive simple example $g_1(x) := x^d$, local error bound holds at 0 with exponent $\frac{1}{d}$. This suggests that our result matches what one might expect in the univariate case.

- (3) On the other hand, in general, our estimation on the exponent will not be optimal.

For example, if the inequality system consists of one single convex polynomial, Fact 2.9 shows that the exponent can be set as $\frac{1}{(d-1)^{n+1}}$ while our results produce a weaker exponent $\max\{\frac{2}{(2d-1)^{n+1}}, \frac{1}{\beta(n-1)d^n}\}$. An interesting feature of the exponent $\frac{1}{(d-1)^{n+1}}$ in Fact 2.9 is that, in the convex quadratic case, it collapses to $\frac{1}{2}$, which is independent of the dimension of the underlying space and which agrees with the known result presented in [32]. By contrast, our estimate $\max\{\frac{2}{3^{n+1}}, \frac{1}{\beta(n-1)2^n}\}$ depends heavily on the dimension n .

Moreover, as indicated in Example 3.1, the best possible exponent might be $\frac{1}{d^n}$. (See [26] for some relevant discussion regarding the best possible exponent for a general nonconvex polynomial system.) It would be interesting to find how to could improve our estimate here.

Making better sense of these estimates will be one of our future research topics. □

Given $D \subseteq \mathbb{R}^n$, we set $\text{dist}^r(\cdot, D) := (\text{dist}(\cdot, D))^r$ for every $r \in \mathbb{R}$.

THEOREM 3.6 (Hölderian regularity). *Let $\gamma_i \in \mathbb{N}, i = 1, \dots, m$, and $g_{i,j}$ be convex polynomials on \mathbb{R}^n with degree $d \in \mathbb{N}, j = 1, \dots, \gamma_i, i = 1, \dots, m$. Recall that*

$$C_i = \left\{ x \in \mathbb{R}^n \mid g_{i,j}(x) \leq 0, j = 1, \dots, \gamma_i \right\} \text{ and } C = \bigcap_{i=1}^m C_i.$$

Let $\theta > 0$ and $K \subseteq \mathbb{R}^n$ be a compact set. Then there exists $c > 0$ such that

$$\text{dist}^\theta(x, C) \leq c \left(\sum_{i=1}^m \text{dist}^\theta(x, C_i) \right)^\tau \quad \forall x \in K,$$

where $\tau := \frac{1}{\min\{\frac{(2d-1)^{n+1}}{2}, \beta(n-1)d^n\}}$ and $\beta(n-1)$ is the central binomial coefficient with respect to $n-1$ which is given by $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

Proof. To see the conclusion, we only need to show that for each $\bar{x} \in \mathbb{R}^n$, there exist $c, \varepsilon > 0$ such that

$$(3.19) \quad \text{dist}^\theta(x, C) \leq c \left(\sum_{i=1}^m \text{dist}^\theta(x, C_i) \right)^\tau \quad \forall \|x - \bar{x}\| \leq \varepsilon.$$

Indeed, granting this and fixing a compact set K , for any $\bar{x} \in K$ there exist $c_{\bar{x}}, \varepsilon_{\bar{x}} > 0$ such that

$$\text{dist}^\theta(x, C) \leq c_x \left(\sum_{i=1}^m \text{dist}^\theta(x, C_i) \right)^\tau \quad \forall \|x - \bar{x}\| \leq \varepsilon_x.$$

As K is compact and $\bigcup_{\bar{x} \in K} \mathbb{B}(\bar{x}; \varepsilon_{\bar{x}}) \supseteq K$, we can find finitely many points $\bar{x}_1, \dots, \bar{x}_s \in K$, $s \in \mathbb{N}$, such that $\bigcup_{i=1}^s \mathbb{B}(\bar{x}_i; \varepsilon_{\bar{x}_i}) \supseteq K$. Let $c := \max\{c_{\bar{x}_1}, \dots, c_{\bar{x}_s}\}$. Then, for any $x \in K$, there exists $i_0 \in \{1, \dots, s\}$ such that $x \in \mathbb{B}(\bar{x}_{i_0}; \varepsilon_{\bar{x}_{i_0}})$, and hence

$$\text{dist}^\theta(x, C) \leq c_{\bar{x}_{i_0}} \left(\sum_{i=1}^m \text{dist}^\theta(x, C_i) \right)^\tau \leq c \left(\sum_{i=1}^m \text{dist}^\theta(x, C_i) \right)^\tau.$$

We now show (3.19) holds. Fix $\bar{x} \in \mathbb{R}^n$. We consider two cases.

Case 1. $\bar{x} \notin C$.

Then there exist $\varepsilon_1, \eta, M > 0$ such that

$$\sum_{i=1}^m \text{dist}^\theta(x, C_i) \geq \eta \text{ and } \text{dist}^\theta(x, C) \leq M \quad \forall \|x - \bar{x}\| \leq \varepsilon_1.$$

Therefore, $\text{dist}^\theta(x, C) \leq M = \frac{M}{\eta^\tau} \eta^\tau \leq \frac{M}{\eta^\tau} (\sum_{i=1}^m \text{dist}^\theta(x, C_i))^\tau$ for all $\|x - \bar{x}\| \leq \varepsilon_1$, and hence (3.19) holds.

Case 2. $\bar{x} \in C$.

We have

$$C = \left\{ x \in \mathbb{R}^n \mid g_{i,1}(x) \leq 0, g_{i,2}(x) \leq 0, \dots, g_{i,\gamma_i}(x) \leq 0, \quad i = 1, \dots, m \right\}.$$

By Corollary 3.4, there exist positive constants c_0 and δ such that

$$\text{dist}(x, C) \leq c_0^{\frac{1}{\theta}} \left(\max_{1 \leq i \leq m} \{ [g_{i,1}(x)]_+, \dots, [g_{i,\gamma_i}(x)]_+ \} \right)^\tau \quad \forall \|x - \bar{x}\| \leq \delta.$$

Hence

$$(3.20) \quad \text{dist}^\theta(x, C) \leq c_0 \left(\max_{1 \leq i \leq m} \{ [g_{i,1}(x)]_+, \dots, [g_{i,\gamma_i}(x)]_+ \} \right)^{\theta\tau} \quad \forall \|x - \bar{x}\| \leq \delta.$$

Now we claim that there exists $\beta > 0$ such that

$$(3.21) \quad \left(\max_{1 \leq i \leq m} \{ [g_{i,1}(x)]_+, \dots, [g_{i,\gamma_i}(x)]_+ \} \right)^\theta \leq \beta \sum_{i=1}^m \text{dist}^\theta(x, C_i) \quad \forall \|x - \bar{x}\| \leq \delta.$$

Suppose to the contrary that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ in $\mathbb{B}(\bar{x}, \delta)$ such that

$$(3.22) \quad \left(\max_{1 \leq i \leq m} \{ [g_{i,1}(x_k)]_+, \dots, [g_{i,\gamma_i}(x_k)]_+ \} \right)^\theta > k \sum_{i=1}^m \text{dist}^\theta(x_k, C_i) \quad \forall k \in \mathbb{N}.$$

Without loss of generality, we can assume that $g_{i,1}, g_{i,2}, \dots, g_{i,\gamma_i}$ have the Lipschitz constant $L > 0$ on $\mathbb{B}(\bar{x}, \delta)$ for every $i = 1, 2, \dots, m$. Then, there exists a subsequence $(x_{k_l})_{l \in \mathbb{N}}$ of $(x_k)_{k \in \mathbb{N}}$, $1 \leq i_0 \leq m$ and $1 \leq j_0 \leq \gamma_{i_0}$, such that

$$\max_{1 \leq i \leq m} \{ [g_{i,1}(x_{k_l})]_+, \dots, [g_{i,\gamma_i}(x_{k_l})]_+ \} = [g_{i_0,j_0}(x_{k_l})]_+ \quad \forall l \in \mathbb{N}.$$

It follows from (3.22) that

$$(3.23) \quad ([g_{i_0,j_0}(x_{k_l})]_+)^{\theta} > k_l \sum_{i=1}^m \text{dist}^{\theta}(x_{k_l}, C_i) \quad \forall l \in \mathbb{N}.$$

Then $[g_{i_0,j_0}(x_{k_l})]_+ = g_{i_0,j_0}(x_{k_l})$, and hence for every $l \in \mathbb{N}$,

$$(3.24) \quad (g_{i_0,j_0}(x_{k_l}))^{\theta} > k_l \sum_{i=1}^m \text{dist}^{\theta}(x_{k_l}, C_i) = k_l \sum_{i=1}^m \|x_{k_l} - P_i(x_{k_l})\|^{\theta} \geq k_l \|x_{k_l} - P_{i_0}(x_{k_l})\|^{\theta}.$$

Since $P_{i_0}(x_{k_l}) \in C_{i_0}$, we have $g_{i_0,j_0}(P_{i_0}(x_{k_l})) \leq 0$ and $\|x_{k_l} - P_{i_0}(x_{k_l})\| \leq \|x_{k_l} - \bar{x}\| < \delta$ by $\bar{x} \in C_{i_0}$. Combining this with (3.24), we have

$$\begin{aligned} L^{\theta} \|x_{k_l} - P_{i_0}(x_{k_l})\|^{\theta} &\geq \left(g_{i_0,j_0}(x_{k_l}) - g_{i_0,j_0}(P_{i_0}(x_{k_l})) \right)^{\theta} \\ &> k_l \|x_{k_l} - P_{i_0}(x_{k_l})\|^{\theta} \quad \forall l \in \mathbb{N}. \end{aligned}$$

Hence we have $L^{\theta} > k_l$ for every $l \in \mathbb{N}$; this contradicts the fact that $k_l \rightarrow +\infty$. Thus, (3.21) holds.

Combining (3.21) and (3.20), we see that

$$\text{dist}^{\theta}(x, C) \leq c_0 \beta^{\tau} \left(\sum_{i=1}^m \text{dist}^{\theta}(x, C_i) \right)^{\tau} \quad \forall \|x - \bar{x}\| \leq \delta,$$

and so the conclusion follows. \square

4. Convergence rate for the cyclic projection algorithm. In this section, we derive an explicit convergence rate of the cyclic projection algorithm applied to finite intersections of basic semialgebraic convex sets.

Before we come to our main result, we need the following useful lemma, Lemma 4.1, which is a special case of Alber and Reich’s result in [1]. For the reader’s convenience, we provide a direct and self-contained proof.

LEMMA 4.1 (recurrence relationships). *Let $p > 0$, and let $\{\delta_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers satisfying the conditions*

$$\beta_{k+1} \leq \beta_k (1 - \delta_k \beta_k^p) \quad \text{as } k = 0, 1, \dots$$

Then, we have

$$(4.1) \quad \beta_k \leq \left(\beta_0^{-p} + p \sum_{i=0}^{k-1} \delta_i \right)^{-\frac{1}{p}} \quad \forall k \in \mathbb{N}.$$

We use the convention that $\frac{1}{0} = +\infty$. In particular, we have $\lim_{k \rightarrow \infty} \beta_k = 0$ whenever $\sum_{k=0}^{\infty} \delta_k = \infty$.

Proof. It follows from our assumption that

$$0 \leq \beta_{i+1} \leq \beta_i \leq \dots \leq \beta_0 \quad \text{and} \quad \delta_i \beta_i^{p+1} \leq \beta_i - \beta_{i+1} \quad \text{as } i \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. We consider two cases.

Case 1. $\beta_k = 0$.

Clearly, (4.1) holds.

Case 2. $\beta_k \neq 0$.

Thus $\beta_k > 0$ and hence $\beta_i > 0$ for every $i \leq k$. Define the nonincreasing function $h : \mathbb{R}_{++} \rightarrow]-\infty, +\infty]$ by $h(x) := x^{-(p+1)}$. As $\delta_i h(\beta_i)^{-1} = \delta_i \beta_i^{p+1} \leq \beta_i - \beta_{i+1}$, we get

$$\delta_i \leq (\beta_i - \beta_{i+1})h(\beta_i) \leq \int_{\beta_{i+1}}^{\beta_i} h(x)dx = \frac{\beta_{i+1}^{-p} - \beta_i^{-p}}{p}.$$

This implies that

$$(4.2) \quad \beta_{i+1}^{-p} - \beta_i^{-p} \geq p\delta_i \quad \forall i \in \mathbb{N} \cup \{0\}.$$

Now fix any $k \in \mathbb{N}$ and, summing (4.2) from $i = 0$ to $i = k - 1$, we get

$$\beta_k^{-p} - \beta_0^{-p} \geq p \sum_{i=0}^{k-1} \delta_i,$$

which implies the conclusion in (4.1). \square

We also need the following technical result. The proof of it follows in part that of [23, Lemmas 3 and 4]; one may also consult [10].

PROPOSITION 4.2 (cyclic convergence rate). *Let $D_i \subseteq \mathbb{R}^n$ be a closed convex set, for all $i = 1, \dots, m$, and $D := \bigcap_{i=1}^m D_i \neq \emptyset$. Let $x_0 \in \mathbb{R}^n$ and the sequence of cyclic projections, $(x_k)_{k \in \mathbb{N}}$, be defined by*

$$x_1 := P_1 x_0, x_2 := P_2 x_1, \dots, x_m := P_m x_{m-1}, x_{m+1} := P_1 x_m \dots,$$

where we set $P_i := P_{D_i}$ for the convenience. Suppose that Hölderian regularity with exponent τ ($0 < \tau \leq 1$) holds: for any compact set $K \subseteq \mathbb{R}^n$ and $\theta > 0$, there exists $c_0 > 0$ such that

$$\text{dist}^\theta(x, D) \leq c_0 \left(\sum_{i=1}^m \text{dist}^\theta(x, D_i) \right)^\tau \quad \forall x \in K.$$

Then x_k converges to $x_\infty \in D$. Moreover, there exist $M > 0$ and $r_0 \in]0, 1[$ such that

$$\|x_k - x_\infty\| \leq \begin{cases} M k^{-\frac{1}{2\tau-1-2}} & \text{if } \tau \in]0, 1[\\ M r_0^k & \text{if } \tau = 1. \end{cases} \quad \forall k \in \mathbb{N},$$

Proof. We denote by $\alpha_i := (i \bmod m) + 1$ for all $i \in \mathbb{N}$. Thus $x_{k+1} = P_{D_{\alpha_k}} x_k$. By Fact 2.14, there exists $x_\infty \in D$ such that $x_k \rightarrow x_\infty$.

We first follow closely the proofs of [23, Lemmas 3 and 4] to get that

$$(4.3) \quad \text{dist}^2(x_k, D) - \text{dist}^2(x_{k+1}, D) \geq \text{dist}^2(x_k, D_{\alpha_k}) \quad \forall k \in \mathbb{N}.$$

Indeed, using the definition of projection operator, we have

$$\begin{aligned}
 \text{dist}^2(x_k, D) - \text{dist}^2(x_{k+1}, D) &\geq \|x_k - P_D x_k\|^2 - \|x_{k+1} - P_D x_k\|^2 \\
 &= \|x_k - P_D x_k\|^2 - \|P_{\alpha_k} x_k - P_D x_k\|^2 \\
 &= \|x_k - P_D x_k\|^2 - \|P_{\alpha_k} x_k - x_k + x_k - P_D x_k\|^2 \\
 &= \|x_k - P_D x_k\|^2 - \|P_{\alpha_k} x_k - x_k\|^2 - \|x_k - P_D x_k\|^2 + 2\langle x_k - P_{\alpha_k} x_k, x_k - P_D x_k \rangle \\
 &\geq -\|x_k - P_{\alpha_k} x_k\|^2 + 2\langle x_k - P_{\alpha_k} x_k, x_k - P_{\alpha_k} x_k + P_{\alpha_k} x_k - P_D x_k \rangle \\
 &= -\|x_k - P_{\alpha_k} x_k\|^2 + 2\|x_k - P_{\alpha_k} x_k\|^2 + 2\langle x_k - P_{\alpha_k} x_k, P_{\alpha_k} x_k - P_D x_k \rangle \\
 &\geq \|x_k - P_{\alpha_k} x_k\|^2 = \text{dist}^2(x_k, D_{\alpha_k}) \quad \forall k \in \mathbb{N}.
 \end{aligned}$$

Hence (4.3) holds.

Next we claim that for every $i \in \mathbb{N}$

$$(4.4) \quad \text{dist}(x_k, D_{\alpha_i}) \leq \text{dist}(x_k, D_{\alpha_k}) + \text{dist}(x_{k+1}, D_{\alpha_{k+1}}) + \dots + \text{dist}(x_{k+m-1}, D_{\alpha_{k+m-1}}).$$

To see this, note that there exists $i_0 \leq m - 1$ such that $\alpha_{i_0+k} = \alpha_i$. Then, we have

$$\begin{aligned}
 \text{dist}(x_k, D_{\alpha_i}) &= \text{dist}(x_k, D_{\alpha_{i_0+k}}) \leq \|x_k - x_{i_0+k}\| + \|x_{i_0+k} - P_{\alpha_{i_0+k}} x_{i_0+k}\| \\
 &\leq \|x_k - x_{k+1}\| + \dots + \|x_{i_0+k-1} - x_{i_0+k}\| + \|x_{i_0+k} - P_{\alpha_{i_0+k}} x_{i_0+k}\| \\
 &= \|x_k - P_{\alpha_k} x_k\| + \dots + \|x_{i_0+k-1} - P_{\alpha_{i_0+k-1}} x_{i_0+k-1}\| + \|x_{i_0+k} - P_{\alpha_{i_0+k}} x_{i_0+k}\| \\
 &= \text{dist}(x_k, D_{\alpha_k}) + \text{dist}(x_{k+1}, D_{\alpha_{k+1}}) + \dots + \text{dist}(x_{i_0+k}, D_{\alpha_{i_0+k}}) \\
 &\leq \text{dist}(x_k, D_{\alpha_k}) + \text{dist}(x_{k+1}, D_{\alpha_{k+1}}) + \dots + \text{dist}(x_{k+m-1}, D_{\alpha_{k+m-1}}) \\
 &\hspace{15em} (\text{by } i_0 \leq m - 1).
 \end{aligned}$$

Hence (4.4) holds.

Thus by (4.4),

$$\begin{aligned}
 \text{dist}^2(x_k, D_{\alpha_i}) &\leq \left(m \max_{k \leq i \leq k+m-1} \text{dist}(x_i, D_{\alpha_i}) \right)^2 \\
 &\leq m^2 \left(\text{dist}^2(x_k, D_{\alpha_k}) + \text{dist}^2(x_{k+1}, D_{\alpha_{k+1}}) \right. \\
 (4.5) \quad &\quad \left. + \dots + \text{dist}^2(x_{k+m-1}, D_{\alpha_{k+m-1}}) \right).
 \end{aligned}$$

By the assumption, there exists $c_0 > 0$ such that

$$\text{dist}^2(x_k, D) \leq c_0 \left(\sum_{i=1}^m \text{dist}^2(x_k, D_i) \right)^\tau \quad \forall k \in \mathbb{N}.$$

By enlarging c_0 if necessary, we may assume that $c_0 > 1$.

Let $r := \tau^{-1}$. Then by (4.5), for every $k \in \mathbb{N}$,

$$\begin{aligned}
 & \frac{1}{c_0^r} \text{dist}^{2r}(x_k, D) \\
 & \leq \sum_{i=1}^m \text{dist}^2(x_k, D_i) \leq m \max_{1 \leq i \leq m} \text{dist}^2(x_k, D_i) \\
 & \leq m^3 \left(\text{dist}^2(x_k, D_{\alpha_k}) + \text{dist}^2(x_{k+1}, D_{\alpha_{k+1}}) + \cdots + \text{dist}^2(x_{k+m-1}, D_{\alpha_{k+m-1}}) \right) \\
 & \leq m^3 \sum_{i=k}^{k+m-1} \text{dist}^2(x_i, D) - \text{dist}^2(x_{i+1}, D) \quad (\text{by (4.3)}) \\
 (4.6) \quad & = m^3 \left(\text{dist}^2(x_k, D) - \text{dist}^2(x_{k+m}, D) \right).
 \end{aligned}$$

Thus we have

$$(4.7) \quad \text{dist}^2(x_{k+m}, D) \leq \text{dist}^2(x_k, D) - \frac{1}{m^3 c_0^r} \text{dist}^{2r}(x_k, D).$$

Now we consider two cases.

Case 1. $\tau \in]0, 1[$.

Thus $r > 1$. Fix $k_0 \in \mathbb{N}$. Let $\beta_i := \text{dist}^2(x_{k_0+im}, D)$ for all $i \in \mathbb{N} \cup \{0\}$. Then (4.7) shows that

$$(4.8) \quad \beta_{i+1} \leq \beta_i - \frac{1}{m^3 c_0^r} \beta_i^r = \beta_i \left(1 - \frac{1}{m^3 c_0^r} \beta_i^{r-1} \right).$$

By Lemma 4.1,

$$\text{dist}^2(x_{k_0+im}, D) = \beta_i \leq \left(\beta_0^{1-r} + (r-1)i \frac{1}{m^3 c_0^r} \right)^{-\frac{1}{r-1}} \quad \forall i \in \mathbb{N}.$$

Thus, there exists $M_0 > 0$ such that

$$\text{dist}(x_{k_0+im}, D) \leq M_0 \frac{1}{2^{(r-1)\sqrt{i}}} \quad \forall i \in \mathbb{N}.$$

Hence there exists $M_1 > 0$ such that

$$(4.9) \quad \text{dist}(x_k, D) \leq M_1 \frac{1}{2^{(r-1)\sqrt{k}}} = M_1 k^{-\frac{1}{2\tau-1-2}} \quad \forall k \in \mathbb{N}.$$

So, we have

$$(4.10) \quad \|x_k - x_\infty\| \leq \|x_k - P_D(x_k)\| + \|P_D(x_k) - x_\infty\| = \text{dist}(x_k, D) + \|P_D(x_k) - x_\infty\|.$$

By [23, Lemma 3],

$$\|x_{k+l} - P_D(x_k)\| \leq \|x_k - P_D(x_k)\| = \text{dist}(x_k, D) \quad \forall l \in \mathbb{N}.$$

Letting $l \rightarrow \infty$ in the above inequality, we obtain that

$$(4.11) \quad \|x_\infty - P_D(x_k)\| \leq \text{dist}(x_k, D).$$

Combining (4.10), (4.11), and (4.9),

$$(4.12) \quad \|x_k - x_\infty\| \leq 2 \operatorname{dist}(x_k, D) \leq 2M_1 k^{-\frac{1}{2\tau-1-2}} \quad \forall k \in \mathbb{N}.$$

Thus, the conclusion follows by letting $M := 2M_1$.

Case 2. $\tau = 1$.

Then we have $r = \tau^{-1} = 1$, and so (4.7) implies that for all $k \in \mathbb{N}$

$$\operatorname{dist}(x_{k+m}, D) \leq \sqrt{1 - \frac{1}{m^3 c_0^r}} \operatorname{dist}(x_k, D).$$

Hence there exist $M'_1 > 0$ and $r_0 \in]0, 1[$ such that $\operatorname{dist}(x_k, D) \leq M'_1 r_0^k$ for all $k \in \mathbb{N}$. Then, using a similar method of proof as in Case 1, we obtain that

$$(4.13) \quad \|x_k - x_\infty\| \leq 2 \operatorname{dist}(x_k, D) \leq 2M'_1 r_0^k \quad \forall k \in \mathbb{N}.$$

Thus, the conclusion follows by letting $M := 2M'_1$. \square

Remark 4.3 (connection to the existing result on linear convergence). In the case where there exists $i_0 \in \{1, \dots, m\}$ such that $D_{i_0} \cap \operatorname{int}(\bigcap_{i \neq i_0} D_i) \neq \emptyset$ (in this case, we say the intersection is regular), the Hölderian regularity result holds with exponent $\tau = 1$. So, the preceding proposition implies that the cyclic projection algorithm converges linearly in the regular intersection case. Thus, this recovers the linear convergence result for cyclic projection algorithm established in [23]. \square

We are now ready for one of our main results.

THEOREM 4.4 (estimate of the cyclic convergence rate). *Let $x_0 \in \mathbb{R}^n$ and the sequence of cyclic projections, $(x_k)_{k \in \mathbb{N}}$, be defined by*

$$x_1 := P_1 x_0, x_2 := P_2 x_1, \dots, x_m := P_m x_{m-1}, x_{m+1} := P_1 x_m \dots$$

Then x_k converges to $x_\infty \in C$, and there exist $M > 0$ and $r_0 \in]0, 1[$ such that

$$\|x_k - x_\infty\| \leq \begin{cases} M \frac{1}{k^\rho} & \text{if } d > 1 \\ M r_0^k & \text{if } d = 1 \end{cases} \quad \forall k \in \mathbb{N},$$

where $\rho := \frac{1}{\min\{(2d-1)^{n-1}, 2\beta(n-1)d^{n-2}\}}$ and $\beta(n-1)$ is the central binomial coefficient with respect to $n-1$ which is given by $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

Proof. Combining Theorem 3.6 and Proposition 4.2, we directly obtain $\tau := \frac{1}{\min\{\frac{(2d-1)^{n+1}}{2}, \beta(n-1)d^n\}}$. Note that $2\tau^{-1} - 2 = \min\{(2d-1)^n - 1, 2\beta(n-1)d^n - 2\}$. Thus the conclusion follows from the preceding proposition. \square

Remark 4.5 (discussion on our estimation of the convergence rate). Although our estimate of the convergence rate works for cyclic projection algorithm with finitely many basic semialgebraic convex sets without any regularity condition, the estimated convergence rate is quite poor when the dimension n of the underlying space and the maximal degree d are large. This is mainly due to the fact that the estimated convergence rate is derived by using the local error bound result for general convex polynomial systems. It would be interesting to see how one could improve the estimation of the convergence rate by either adopting other approaches or by further exploiting the structure of the underlying convex sets. For example, one possibility would be to examine problems involving some suitable additional curvature or uniform convexity assumptions. This will be another of our future research topics.

4.1. Alternating projection algorithm. In this subsection, we discuss the convergence rate of the alternating projection algorithm. We assume throughout this subsection that

$$\begin{aligned} &g_i, h_j \text{ are convex polynomials with degree at most } d \forall i = 1, 2, \dots, m, j = 1, 2, \dots, l, \\ &A := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, m\}, \\ &B := \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, j = 1, \dots, l\}, \\ &b_0 \in \mathbb{R}^n, \quad a_{k+1} := P_A b_k, \quad b_{k+1} := P_B a_{k+1}. \end{aligned}$$

As an immediate corollary of Theorem 4.4, we first obtain the following estimate on the convergence rate of the alternating projection algorithm in the case where the two sets have nonempty intersection.

The case of two sets with nonempty intersection.

COROLLARY 4.6 (alternating convergence rate). *Suppose that $A \cap B \neq \emptyset$. Let the sequence $\{(a_k, b_k)\}$ be generated by the alternating projection algorithm. Then, $a_k, b_k \rightarrow c \in A \cap B$. Moreover, there exist $M > 0$ and $r_0 \in]0, 1[$ such that for every $k \in \mathbb{N}$*

$$\|a_k - c\| \leq \begin{cases} M \frac{1}{k^\rho} & \text{if } d > 1, \\ M r_0^k & \text{if } d = 1 \end{cases} \quad \text{and} \quad \|b_k - c\| \leq \begin{cases} M \frac{1}{k^\rho} & \text{if } d > 1, \\ M r_0^k & \text{if } d = 1, \end{cases}$$

where $\rho := \frac{1}{\min\{(2d-1)^n - 1, 2\beta(n-1)d^{n-2}\}}$ and $\beta(n-1)$ is the central binomial coefficient with respect to $n-1$ which is given by $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

Recently, [2] established a local convergence rate analysis for proximal alternating projection methods for very general nonconvex problems, and where the corresponding convergence rate involves the exponent of the Kurdyka–Lojasiewicz inequality. The *proximal alternating projection method* is a variant of the alternating projection algorithm we discussed here. On the other hand, as we discussed before, in general, the actual exponent of the Kurdyka–Lojasiewicz inequality is typically unknown and hard to estimate. Corollary 4.6 above complements the result of [2] in the case of basic convex semialgebraic cases by providing an explicit estimate of the convergence rate.

The case of two sets with empty intersection. In this part, we consider the general case where the intersection of these two sets is (possibly) empty. We first need the following lemma.

LEMMA 4.7. *The difference $B - A$ of two basic semialgebraic sets A, B is closed.*

Proof. Let $b_k \in B$ and $a_k \in A$ be such that $b_k - a_k \rightarrow c$. We now show that $c \in B - A$. Consider the following convex polynomial optimization problem:

$$\begin{aligned} (P) \quad &\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^n} \|(y - x) - c\|^2 \\ &\text{subject to } g_i(x) \leq 0, i = 1, \dots, m, \\ &h_j(y) \leq 0, j = 1, \dots, l. \end{aligned}$$

Note that (a_k, b_k) are feasible for (P). Hence we see that $\inf(P) = 0$. By Fact 2.2, the optimal solution of (P) exists. Thus there exists $x \in A$ and $y \in B$ such that $c = y - x \in B - A$. Hence the conclusion follows. \square

Remark 4.8. With A and B defined as above, Fact 2.2 implies that $B - A$ is closed convex. Hence $P_{B-A}0 \neq \emptyset$. Let $v := P_{B-A}0$. Then, there exist $a \in A$ and $b \in B$ such that $v = b - a$ and hence $\text{dist}(A, B) = \|v\|$. \square

Remark 4.9. In general, the distance between two convex and semialgebraic sets need not be attained. For instance, consider $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$ and $E := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$. It is clear that D, E are both convex and semialgebraic, while D is not a basic semialgebraic convex set (as explained in Example 2.13). Clearly, $\text{dist}(D, E) = 0$ but $D \cap E = \emptyset$. Thus, the distance is not attained in this case. \square

The proof of Theorem 4.10 partially follows that of [8, Theorem 3.12].

THEOREM 4.10 (convergence rate in the infeasible case). *Let the sequence $\{(a_k, b_k)\}$ be generated by the alternating projection algorithm. Then $a_k \rightarrow \tilde{a} \in A$ and $b_k \rightarrow \tilde{b} \in B$ with $\tilde{b} - \tilde{a} = v$, where $v := P_{B-A}0$. Moreover, there exist $M > 0$ and $r_0 \in]0, 1[$ such that for every $k \in \mathbb{N}$*

$$(4.14) \quad \|a_k - \tilde{a}\| \leq \begin{cases} M \frac{1}{k^\rho} & \text{if } d > 1, \\ M r_0^k & \text{if } d = 1 \end{cases} \quad \text{and} \quad \|b_k - \tilde{b}\| \leq \begin{cases} M \frac{1}{k^\rho} & \text{if } d > 1, \\ M r_0^k & \text{if } d = 1, \end{cases}$$

where $\rho := \frac{1}{\min\{(2d-1)^{n-1}, 2\beta(n-1)d^{n-2}\}}$ and $\beta(n-1)$ is the central binomial coefficient with respect to $n-1$ which is given by $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

Proof. Lemma 4.7 implies that $B - A$ is closed. Then, by Fact 2.15(i), there exist $\tilde{a} \in A, \tilde{b} \in B$ such that $a_k \rightarrow \tilde{a} \in A$ and $b_k \rightarrow \tilde{b} \in B$ with $\tilde{b} - \tilde{a} = v$. By Theorem 3.6, there exists $c_0 > 1$ such that

$$(4.15) \quad \text{dist}(a_k, A \cap (B - v)) \leq c_0 (\text{dist}(a_k, A) + \text{dist}(a_k, B - v))^{\frac{1}{r}} = c_0 \text{dist}^{\frac{1}{r}}(a_k, B - v),$$

where $r := \min\{\frac{(2d-1)^{n+1}}{2}, \beta(n-1)d^n\}$. Fix $x \in A \cap (B - v)$. Note that $v = P_{B-A}0$; we have $P_B(x) = x + v$ by Fact 2.15(ii). This implies that

$$\begin{aligned} \text{dist}^2(a_k, B - v) &\leq \|a_k - (b_k - v)\|^2 = \|(a_k - x) - (b_k - (v + x))\|^2 \\ &= \|(a_k - x) - (P_B a_k - P_B x)\|^2 \\ &\leq \|a_k - x\|^2 - \|P_B a_k - P_B x\|^2 \quad (\text{by [12, Proposition 4.8]}) \\ &= \|a_k - x\|^2 - \|b_k - (x + v)\|^2 \\ &\leq \|a_k - x\|^2 - \|P_A b_k - P_A(x + v)\|^2 \\ &= \|a_k - x\|^2 - \|a_{k+1} - x\|^2 \quad (\text{by Fact 2.15(ii)}). \end{aligned}$$

In particular, choose $x = P_{A \cap (B-v)} a_k$. Then, we have

$$\begin{aligned} \text{dist}^2(a_k, B - v) &\leq \text{dist}^2(a_k, A \cap (B - v)) - \|a_{k+1} - P_{A \cap (B-v)} a_k\|^2 \\ &\leq \text{dist}^2(a_k, A \cap (B - v)) - \text{dist}^2(a_{k+1}, A \cap (B - v)). \end{aligned}$$

Combining with (4.15), we have

$$(4.16) \quad \begin{aligned} \frac{1}{c_0^{2r}} \text{dist}^{2r}(a_k, A \cap (B - v)) &\leq \text{dist}^2(a_k, (B - v)) \\ &\leq \text{dist}^2(a_k, A \cap (B - v)) - \text{dist}^2(a_{k+1}, A \cap (B - v)). \end{aligned}$$

Thus

$$\text{dist}(a_{k+1}, A \cap (B - v))^2 \leq \text{dist}(a_k, A \cap (B - v))^2 - \frac{1}{c_0^{2r}} \text{dist}(a_k, A \cap (B - v))^{2r}.$$

Now, let $\beta_k := \text{dist}(a_k, A \cap (B - v))^2$, $k \in \mathbb{N}$. Then, we have

$$(4.17) \quad \beta_{k+1} \leq \beta_k \left(1 - \frac{1}{c_0^{2r}} \beta_k^{r-1} \right).$$

Now we consider two cases.

Case 1. $d > 1$.

In this case, we have $r > 1$. Applying the preceding Lemma 4.1 with $\delta_k := \frac{1}{c_0^{2r}}$ and $p := r - 1$, by (4.17), we see that

$$\text{dist}^2(a_k, A \cap (B - v)) = \beta_k \leq \left(\beta_0^{1-r} + \frac{(r-1)}{c_0^{2r}} k \right)^{-\frac{1}{r-1}} \quad \forall k \in \mathbb{N}.$$

Thence there exists $M_0 > 0$ such that

$$\text{dist}(a_k, A \cap (B - v)) \leq M_0 \frac{1}{k^\rho} \quad \forall k \in \mathbb{N},$$

where $\rho := \frac{1}{\min\{(2d-1)^n - 1, 2\beta(n-1)d^{n-2}\}}$. Then, [8, Example 3.2] shows that $(a_k)_{k \in \mathbb{N}}$ is Fejér monotone with respect to $A \cap (B - v)$. Thus, by Fact 2.17,

$$\|a_k - \tilde{a}\| \leq 2 \text{dist}(a_k, A \cap (B - v)) \leq 2M_0 \frac{1}{k^\rho} \quad \forall k \in \mathbb{N}.$$

Case 2. $d = 1$.

Thus $r = 1$. Then by (4.17), $\text{dist}(a_{k+1}, A \cap (B - v)) \leq \theta \text{dist}(a_k, A \cap (B - v))$, where $\theta := \sqrt{1 - \frac{1}{c_0^2}}$. Then $\theta < 1$ since $c_0 > 1$. Hence there exists $M_1 > 0$ such that

$$\text{dist}(a_k, A \cap (B - v)) \leq M_1 \theta^k.$$

From [8, Example 3.2], we see that $(a_k)_{k \in \mathbb{N}}$ is Fejér monotone with respect to $A \cap (B - v)$. Thus, by Fact 2.17,

$$\|a_k - \tilde{a}\| \leq 2 \text{dist}(a_k, A \cap (B - v)) \leq 2M_1 \theta^k.$$

Set $M_2 := \max\{2M_0, 2M_1\}$. Combining the above two cases, we have

$$\|a_k - \tilde{a}\| \leq \begin{cases} M_2 \frac{1}{k^\rho} & \text{if } d > 1 \\ M_2 \theta^k & \text{if } d = 1 \end{cases} \quad \forall k \in \mathbb{N}.$$

Similarly, we can show that there exist $L > 0$ and $\eta \in]0, 1[$ such that

$$\|b_k - \tilde{b}\| \leq \begin{cases} L \frac{1}{k^\rho} & \text{if } d > 1 \\ L \eta^k & \text{if } d = 1 \end{cases} \quad \forall k \in \mathbb{N}.$$

Therefore, the conclusion follows by taking $M := \max\{M_2, L\}$ and $r_0 := \max\{\theta, \eta\}$. \square

5. Examples and remarks. In this section, we will provide several examples of the rates of convergence of the cyclic projection algorithm and the von Neumann alternating projection algorithm. We first start with some examples where the basic semialgebraic convex sets are described by convex quadratic functions. Subsequently, we will examine examples where the basic semialgebraic convex sets are described by higher degree convex polynomials.

Basic semialgebraic convex sets described by convex quadratic functions.

Example 5.1. Let

$$\begin{aligned} C_1 &:= \{(x, y) \in \mathbb{R}^2 \mid (x + 1)^2 + y^2 - 1 \leq 0\}, \\ C_2 &:= \{(x, y) \in \mathbb{R}^2 \mid x + y - 1 \leq 0\}, \\ C_3 &:= \{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 - 1 \leq 0\}, \\ C_4 &:= \{(x, y) \in \mathbb{R}^2 \mid x + (y + 2)^2 - 4 \leq 0\}. \end{aligned}$$

Take $x_0 \in \mathbb{R}^2$. Let $(x_k)_{k \in \mathbb{N}}$ be defined by

$$x_1 := P_1 x_0, x_2 := P_2 x_1, x_3 := P_3 x_2, x_4 := P_4 x_3, x_5 := P_1 x_4 \dots$$

Then $\|x_k\| = O(\frac{1}{k^{\frac{1}{6}}})$. \square

Proof. Clearly, $\bigcap_{i=1}^4 C_i = \{0\}$. Then apply $n = 2$ and $d = 2$ to Theorem 4.2. \square

Example 5.2. Let $\alpha \geq 0$ and

$$\begin{aligned} A &:= \{(x, y) \in \mathbb{R}^2 \mid (x + 1)^2 + y^2 - 1 \leq 0\} = (-1, 0) + \overline{\mathbb{B}}(0, 1), \\ B &:= \{(x, y) \in \mathbb{R}^2 \mid -x + \alpha \leq 0\}. \end{aligned}$$

Let $(a_k)_{k \in \mathbb{N}} := (u_k, v_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}} := (s_k, t_k)_{k \in \mathbb{N}}$ be defined by

$$b_0 \in \mathbb{R}^2, \quad a_{k+1} := P_A b_k, \quad b_{k+1} := P_B a_{k+1}.$$

Then for every $k \geq 2$

$$\begin{aligned} b_k &= \left(\alpha, \frac{t_1}{\sqrt{(1 + \alpha)^{2(k-1)} + t_1^2 \sum_{i=0}^{k-2} ((1 + \alpha)^{2i})}} \right), \\ a_{k+1} &= \left(-1 + \frac{\alpha + 1}{\sqrt{(\alpha + 1)^2 + \frac{t_1^2}{(1 + \alpha)^{2(k-1)} + t_1^2 \sum_{i=0}^{k-2} ((1 + \alpha)^{2i})}}}, \right. \\ &\quad \left. \frac{t_1}{\sqrt{(1 + \alpha)^{2k} + t_1^2 \sum_{i=0}^{k-1} ((1 + \alpha)^{2i})}} \right). \end{aligned}$$

Consequently, $a_k \rightarrow 0$ and $b_k \rightarrow (\alpha, 0)$ at the rate of $k^{-\frac{1}{2}}$ when $\alpha = 0$. When $\alpha \neq 0$ (then $A \cap B = \emptyset$), $a_k \rightarrow 0$ and $b_k \rightarrow (\alpha, 0)$ at the rate of $(1 + \alpha)^{-k}$. \square

Proof. We first claim that

$$(5.1) \quad b_{k+1} = (\alpha, t_{k+1}) = \left(\alpha, \frac{t_k}{\sqrt{(1 + \alpha)^2 + t_k^2}} \right) \quad \forall k \geq 1.$$

By [12, Examples 3.17 and 3.21 and Proposition 3.17], we have

$$\begin{aligned} (5.2) \quad P_A(x, y) &= (-1, 0) + \frac{(x + 1, y)}{\max\{1, \|(x + 1, y)\|\}} \quad \forall (x, y) \in \mathbb{R}^2, \\ P_B(x, y) &= (\alpha, y) \quad \forall (x, y) \notin \text{int } B. \end{aligned}$$

Let $k \geq 1$. Since $A \cap B = \{0\}$ or $A \cap B = \emptyset$, $a_k \notin \text{int } B$. Then by (5.2), $b_k = (\alpha, v_k)$ and then

$$a_{k+1} = P_A b_k = (-1, 0) + \frac{(1 + \alpha, v_k)}{\max\{1, \|(1 + \alpha, v_k)\|\}} = (-1, 0) + \frac{(1 + \alpha, v_k)}{\sqrt{(1 + \alpha)^2 + v_k^2}},$$

$$b_{k+1} = P_B(a_{k+1}) = \left(\alpha, \frac{v_k}{\sqrt{(1 + \alpha)^2 + v_k^2}} \right) = \left(\alpha, \frac{t_k}{\sqrt{(1 + \alpha)^2 + t_k^2}} \right).$$

Hence (5.1) holds. Next we show that

$$(5.3) \quad b_k = \left(\alpha, \frac{t_1}{\sqrt{(1 + \alpha)^{2(k-1)} + t_1^2 \sum_{i=0}^{k-2} ((1 + \alpha)^{2i})}} \right) \quad \forall k \geq 2.$$

We prove (5.3) by the induction on k .

By (5.1), (5.3) holds when $k = 2$. Now assume that (5.3) holds when $k = p$, where $p \geq 2$. Now we consider the case of $k = p + 1$. By the assumption, we have

$$(5.4) \quad b_p = \left(\alpha, \frac{t_1}{\sqrt{(1 + \alpha)^{2(p-1)} + t_1^2 \sum_{i=0}^{p-2} ((1 + \alpha)^{2i})}} \right).$$

Then by (5.1), we have

$$b_{p+1} = \left(\alpha, \frac{t_p}{\sqrt{(1 + \alpha)^2 + t_p^2}} \right)$$

$$= \left(\alpha, \frac{\frac{t_1}{\sqrt{(1 + \alpha)^{2(p-1)} + t_1^2 \sum_{i=0}^{p-2} ((1 + \alpha)^{2i})}}}{\sqrt{(1 + \alpha)^2 + \frac{t_1^2}{(1 + \alpha)^{2(p-1)} + t_1^2 \sum_{i=0}^{p-2} ((1 + \alpha)^{2i})}}} \right)$$

$$= \left(\alpha, \frac{t_1}{\sqrt{(1 + \alpha)^{2p} + t_1^2 \sum_{i=0}^{p-1} ((1 + \alpha)^{2i})}} \right).$$

Hence (5.3) holds.

Combining (5.2) and (5.3), we have for every $k \geq 2$

$$a_{k+1} = P_A b_k$$

$$= \left(-1 + \frac{\alpha + 1}{\sqrt{(\alpha + 1)^2 + \frac{t_1^2}{(1 + \alpha)^{2(k-1)} + t_1^2 \sum_{i=0}^{k-2} ((1 + \alpha)^{2i})}}}, \right.$$

$$\left. \frac{t_1}{\sqrt{(1 + \alpha)^{2k} + t_1^2 \sum_{i=0}^{k-1} ((1 + \alpha)^{2i})}} \right).$$

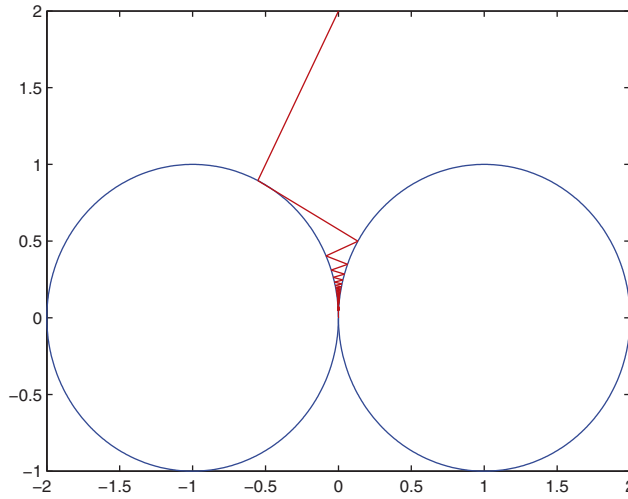


FIG. 1. The iteration commencing at (0, 2).

Hence $a_k \rightarrow 0$ and $b_k \rightarrow (\alpha, 0)$ at the rate of $k^{-\frac{1}{2}}$ when $\alpha = 0$. When $\alpha \neq 0$, $a_k \rightarrow 0$ and $b_k \rightarrow (\alpha, 0)$ at the rate of $(1 + \alpha)^{-k}$. \square

Remark 5.3. According to Theorem 4.10, we can only deduce that $(a_k)_{k \in \mathbb{N}}$ in Example 5.2 converge to $(0, 0)$ and $(b_k)_{k \in \mathbb{N}}$ converge to $(\alpha, 0)$ at the rate of at least of $k^{-\frac{1}{6}}$. \square

Example 5.4. Let

$$A := \{(x, y) \in \mathbb{R}^2 \mid (x + 1)^2 + y^2 - 1 \leq 0\},$$

$$B := \{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 - 1 \leq 0\}.$$

Let $(x_k, y_k)_{k \in \mathbb{N}}$ be defined by

$$(x_0, y_0) \in \mathbb{R}^2, (x_1, y_1) := P_A(x_0, y_0), (x_2, y_2) := P_B(x_1, y_1), (x_3, y_3) := P_A(x_2, y_2), \dots$$

Note that

$$P_A(x, y) = \left(-1 + \frac{x - 1}{\sqrt{(x + 1)^2 + y^2}}, \frac{y}{\sqrt{(x + 1)^2 + y^2}} \right) \quad \forall (x, y) \in \mathbb{R}_+ \times \mathbb{R}_{++},$$

$$P_B(x, y) = \left(1 + \frac{x + 1}{\sqrt{(x - 1)^2 + y^2}}, \frac{y}{\sqrt{(x - 1)^2 + y^2}} \right) \quad \forall (x, y) \in \mathbb{R}_- \times \mathbb{R}_{++},$$

Figure 1 depicts the algorithm’s trajectory with the starting point $(0, 2)$.

Suppose, without loss of generality, that one starts on a point on one of the half-circles nearest the other circle. Then the distance from zero (for every $k \in \mathbb{N}$), $r_k := \sqrt{x_k^2 + y_k^2}$, satisfies $r_k^2 = 2\alpha_k$, where $\alpha_k := |x_k|$, since $(x_k, y_k) \in \text{bd } A \cup \text{bd } B$. Hence

$$1 - \alpha_{k+1} = \frac{1 + \alpha_k}{\sqrt{1 + 4\alpha_k}}.$$

Linearizing, we obtain that $w_k := 4\alpha_k$ approximately satisfies the logistics equation

$$w_{k+1} \approx w_k(1 - w_k).$$

This can be explicitly solved by writing

$$\frac{1}{w_{k+1}} - \frac{1}{w_k} = \frac{1}{1 - w_k}.$$

Summing and dividing by N leads to

$$\begin{aligned} (5.5) \quad \lim_{N \rightarrow \infty} \frac{1}{Nw_N} &= \lim_{N \rightarrow \infty} \left(\frac{1}{Nw_N} - \frac{1}{Nw_0} \right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{1 - w_k} \\ &= \lim_{N \rightarrow \infty} \frac{1}{1 - w_N} = 1, \end{aligned}$$

since Césaro summability is conservative and $w_N \rightarrow 0$. Hence $\alpha_k \sim 1/(4k)$ and so

$$\sqrt{x_k^2 + y_k^2} = r_k \sim \frac{1}{\sqrt{2k}}.$$

For instance, with $\alpha_0 = 1$, $N = 10^6$, we obtain $\alpha_N \approx 0.0000002499992442$. A similar analysis can be performed in the previous example. \square

Remark 5.5. According to Theorem 4.10, we can only deduce that $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ in Example 5.4 converge to $(0, 0)$ at the rate of at least of $k^{-\frac{1}{6}}$. \square

Basic semialgebraic convex sets described by convex polynomials. In general, identifying the exact convergence rate of the cyclic projection method in a direct way can be quite arduous when applied to finitely many basic semialgebraic convex sets. Below, we provide two simple examples to illustrate how the convergence rate depends on the maximum degree of the polynomials that described the basic semialgebraic convex sets and on the dimension of the underlying space.

Example 5.6. Let A, B be defined by

$$\begin{aligned} A &:= \{(x, y) \in \mathbb{R}^2 \mid x \leq 0\}, \\ B &:= \{(x, y) \in \mathbb{R}^2 \mid y^d - x \leq 0\}, \end{aligned}$$

where d is an even number. Let $(a_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}} := (x_k, y_k)_{k \in \mathbb{N}}$ be defined by

$$b_0 := (x_0, y_0) \in \mathbb{R} \times \mathbb{R}_{++} \text{ with } \|b_0\| \leq 1, \quad a_{k+1} := P_A b_k, \quad b_{k+1} := P_B a_{k+1}.$$

Then for every $k \in \mathbb{N}$

$$a_{k+1} = (0, y_k), \quad b_k = (y_k^d, y_k), \quad \text{and} \quad d y_{k+1}^{2d-1} + y_{k+1} = y_k.$$

Consequently, $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ converge to 0 at the rate of at least of $k^{-\frac{1}{2d-2}}$.

Proof. Let $k \in \mathbb{N}$. Since $A \cap B = \{0\}$, $b_k \notin \text{int } A$. Note that $b_k \in \text{bd } B$. Then, $x_k = y_k^d$. Thus $a_{k+1} = P_A b_k = P_A(x_k, y_k) = (0, y_k)$. Then we have $y_k \rightarrow 0$,

$$b_{k+1} = (x_{k+1}, y_{k+1}) = (y_{k+1}^d, y_{k+1}) = P_B a_{k+1} = P_B(0, y_k).$$

Thus y_{k+1} is a minimizer of the function

$$y \mapsto \|(y^d, y) - (0, y_k)\|^2 = \|(y^d, y - y_k)\|^2 = y^{2d} + y^2 - 2yy_k + y_k^2.$$

Thus $(dy_{k+1}^{2d-2} + 1)y_{k+1} - y_k = dy_{k+1}^{2d-1} + y_{k+1} - y_k = 0$. Then we have

$$\frac{1}{(dy_{k+1}^{2d-2} + 1)^{2d-2}} \frac{1}{y_{k+1}^{2d-2}} = \frac{1}{y_k^{2d-2}}.$$

Now, for the function $h(x) := \frac{1}{(x+1)^{2d-2}}$, we have

$$h'(x) = -(2d - 2) \frac{1}{(x + 1)^{2d-1}} \forall x \in \mathbb{R}.$$

Note that $y_k \rightarrow 0$. So, for all large k ,

$$\frac{1}{(dy_{k+1}^{2d-2} + 1)^{2d-2}} = h(dy_{k+1}^{2d-2}) \approx h(0) + h'(0)dy_{k+1}^{2d-2} = 1 - d(2d - 2)y_{k+1}^{2d-2}.$$

This gives us that for all large k ,

$$\frac{1}{y_k^{2d-2}} \approx (1 - d(2d - 2)y_{k+1}^{2d-2}) \frac{1}{y_{k+1}^{2d-2}} = \frac{1}{y_{k+1}^{2d-2}} - d(2d - 2).$$

In other words, there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$

$$(5.6) \quad \frac{1}{y_{l+1}^{2d-2}} - \frac{1}{y_l^{2d-2}} \approx d(2d - 2).$$

Let $k > l_0$. Summing the above relation from $l = l_0$ to k , we have

$$\frac{1}{y_k^{2d-2}} - \frac{1}{y_{l_0}^{2d-2}} \approx d(2d - 2)(k - l_0 + 1),$$

and so

$$y_k \approx \left(\frac{1}{y_{l_0}^{2d-2}} + d(2d - 2)(k - l_0 + 1) \right)^{-\frac{1}{2d-2}} = O(k^{-\frac{1}{2d-2}}).$$

Thence $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ converge to 0 at that rate. \square

Remark 5.7. Similarly, according to Theorem 4.10, we can only deduce that $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ in Example 5.6 converge to $(0, 0)$ at the rate of at least of $k^{-\frac{1}{2d^2-2}}$.

Example 5.8. Let

$$A := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1 + 1)^4 + \sum_{i=2}^n x_i^4 - 1 \leq 0 \right\},$$

$$B := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1 - 2)^4 + \sum_{i=2}^n x_i^4 - 1 \leq 0 \right\}.$$

Let $(a_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}}$ be defined by

$$b_0 \in \mathbb{R}^2, \quad a_{k+1} := P_A b_k, \quad b_{k+1} := P_B a_{k+1}.$$

Then $\|a_k\| = O(\frac{1}{k^{\rho_n}}), \|b_k - (1, 0, \dots, 0)\| = O(\frac{1}{k^{\rho_n}})$ with $\rho_n = \frac{1}{\min\{7^n - 1, 2\beta(n-1)4^n - 2\}}$. \square

Proof. By the assumption, there exist unique points $a_0 \in \text{bd } A, b_0 \in \text{bd } B$ such that $1 = \text{dist}(A, B) = \|a_0 - b_0\|$. Clearly, $a_0 = (0, 0, \dots, 0)$ and $b_0 = (1, 0, \dots, 0)$. Then, the conclusion follows by applying Theorem 4.10 with $d = 4$. \square

6. Conclusion and open questions. In this paper, we studied the rate of convergence of the cyclic projection algorithm applied to finitely many basic semialgebraic convex sets. We established an explicit convergence rate estimate which relies on the maximum degree of the polynomials that generate the basic semialgebraic convex sets and the dimension of the underlying space. We also examined some concrete examples and compared the actual convergence rate with our estimate.

Although our estimate of the convergence rate works for a cyclic projection algorithm with finitely many basic semialgebraic convex sets without any regularity condition, the limitation of our approach is that the estimated convergence rate behaves quite badly when the dimension n of the underlying space and the maximal degree d are large. Our results have suggested the following future research topics and open questions:

- The explicit examples (Examples 5.2, 5.4, and 5.6) show that, in general, our estimate of the convergence rate of the cyclic projection algorithm will not be tight. It would be interesting to see how one can sharpen the estimate obtained in this paper and get a tight estimate for the cyclic projection algorithm. In particular, finding the right exponent when each set is defined by convex quadratic functions would be a good starting point.
- Can we extend the approach here to analyze the convergence rate of the Douglas–Rachford algorithm? Almost nothing is known except for affine sets.

These will be our future research topics and will be examined later on.

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