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EXACT SECOND-ORDER CONE PROGRAMMING RELAXATIONS FOR SOME NONCONVEX MINIMAX QUADRATIC OPTIMIZATION PROBLEMS*

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Abstract. In this paper, we study, for the first time, nonconvex minimax separable quadratic optimization problems with multiple separable quadratic constraints and their second-order cone programming (SOCP) relaxations. Under suitable conditions, we establish exact SOCP relaxation for minimax nonconvex separable quadratic programs. We show that various important classes of specially structured minimax quadratic optimization problems admit exact SOCP relaxations under easily verifiable conditions. These classes include some minimax extended trust-region problems, minimax uniform quadratic optimization problems, max dispersion problems, and some robust quadratic optimization problems under bounded data uncertainty. The present work shows that nonconvex minimax separable quadratic problems with quadratic constraints, which contain a hidden closed and convex epigraphical set, exhibit exact SOCP relaxations.

Key words. nonconvex quadratic programs, robust optimization, second-order cone programming, minimax quadratic programs

AMS subject classifications. 90C26, 90C47, 90C22

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1. Introduction.

Nonconvex quadratic optimization problems involving multiple quadratic constraints are a class of important and computationally hard global optimization problems that arise in many practical applications. They have been extensively studied in the literature. Especially, in recent years, a great deal of attention has been focused on studying them using their semidefinite programming (SDP) relaxation problems and second-order cone programming (SOCP) relaxation problems [2, 3, 5, 10, 21, 23, 28, 27, 30].

In this paper, we consider the following class of nonconvex minimax separable quadratic optimization problems of the form

$$(P)\ \inf_{x\in\mathbb{R}^n}\max_{1\le i\le p}\left\{\frac{1}{2}x^T\left(U\Sigma_iU^T\right)x+a_i^Tx+\alpha_i\right\}$$

$$\text{s.t. }\ \frac{1}{2}x^T\left(U\Lambda_jU^T\right)x+b_j^Tx+\beta_j\le 0, j=1,\ldots,q,$$

where U is an orthogonal matrix; Σ_i, $i=1,\ldots,p$, and Λ_j, $j=1,\ldots,q$, are diagonal matrices with diagonal elements given by $\sigma_i^1,\ldots,\sigma_i^n$ and $\mu_j^1,\ldots,\mu_j^n$, respectively, that is, $\Sigma_i=\operatorname{diag}(\sigma_i^1,\ldots,\sigma_i^n)$ and $\Lambda_j=\operatorname{diag}(\mu_j^1,\ldots,\mu_j^n)$; a_i, $i=1,\ldots,p$, and b_j, $j=1,\ldots,q$, are n-dimensional vectors; α_i, $i=1,\ldots,p$, and β_j, $j=1,\ldots,q$, are real numbers.

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The problem (P) is termed the minimax separable quadratic optimization problem because the problem (P) can equivalently be reformulated as a separable quadratic optimization problem by making a linear change of variables. Indeed, letting x by Ux we can reformulate (P) as

$$\inf_{x\in\mathbb{R}^n} \max_{1\leq i\leq p} \left\{ \frac{1}{2}\sum_{l=1}^{n} \sigma_i^l x_l^2 + d_i^T x + \alpha_i \right\}$$

$$\text{s.t. } \frac{1}{2}\sum_{l=1}^{n} \mu_j^l x_l^2 + \gamma_j^T x + \beta_j \leq 0, j = 1, \ldots, q,$$

where $d_i = U^T a_i$, $\gamma_j = U^T b_j$, $U\Sigma_i U^T = \mathrm{diag}(\sigma_i^1, \ldots, \sigma_i^n)$, and $U\Lambda_j U^T = \mathrm{diag}(\mu_j^1, \ldots, \mu_j^n)$.

The separable model problem (P) covers various important classes of specially structured minimax quadratic optimization problems (see sections 3, 4, and 5) such as some minimax extended trust-region problems, minimax uniform quadratic optimization problems, max dispersion problems, and some robust quadratic optimization problems under bounded data uncertainty. These minimax seprable nonconvex quadratic problems (P) are, in general, NP hard problems as the max dispersion problem is known to be an NP-hard problem [18]. In the very special case of problem (P), where $p = 1$ and $q = 2$, the separable model problem (P) has recently been studied in [10], whereas an algorithmic procedure for converting classes of quadratic programs into the separable form (P) was given in [25].

As an illustration for the model problem (P), consider a simple uncertain quadratic optimization problem

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \left\{ a_1 x_1^2 + a_2 x_2^2 : x_1^2 + x_2^2 \leq 1 \right\},$$

where the data $(a_1, a_2) \in \mathbb{R}^2$ is uncertain and it belongs to the interval uncertainty set

$$\mathcal{U} = \{\lambda(1, -1) + (1 - \lambda)(-1, 1) : \lambda \in [0, 1]\}.$$

Then, its robust counterpart [8, 9, 22], which finds the worst-case solution of the uncertain problem, can be formulated as

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \left\{ \max_{(a_1,a_2)\in\mathcal{U}} \left\{ a_1 x_1^2 + a_2 x_2^2 \right\} : x_1^2 + x_2^2 \leq 1 \right\}.$$

This problem is equivalent to the problem

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \left\{ \max\left\{ x_1^2 - x_2^2, x_2^2 - x_1^2 \right\} : x_1^2 + x_2^2 \leq 1 \right\}$$

which is a special case of (P). For a class of robust quadratic optimization problems of this form, we refer the reader to section 5 later in the paper.

The SOCP problem associated with (P) is described as follows:

$$(SOCP) \quad \sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_p,\lambda_j\geq 0\\ v\in\mathbb{R}^n, s_l\in\mathbb{R}}} \mu$$

$$\text{s.t.} \quad \sum_{i=1}^{p}\delta_i\sigma_i^l + \sum_{j=1}^{q}\lambda_j\mu_j^l \geq 0, l = 1,\ldots,n,$$

$$v = U^T\left(\sum_{i=1}^{p}\delta_i a_i + \sum_{j=1}^{q}\lambda_j b_j\right),$$

$$2\left(\sum_{i=1}^{p}\delta_i\alpha_i + \sum_{j=1}^{q}\lambda_j\beta_j - \mu\right) - \sum_{l=1}^{n} s_l \geq 0, s_l \geq 0, \tag{1.1}$$

$$\left\|\left(2v_l, s_l - \sum_{i=1}^{p}\delta_i\sigma_i^l - \sum_{j=1}^{q}\lambda_j\mu_j^l\right)\right\| \tag{1.2}$$

$$\leq s_l + \sum_{i=1}^{p}\delta_i\sigma_i^l + \sum_{j=1}^{q}\lambda_j\mu_j^l, \; l = 1,\ldots,n,$$

where $\sigma_i^1,\ldots,\sigma_i^n$ and $\mu_j^1,\ldots,\mu_j^n$ are given as in problem (P) and Δ_p is the simplex in $\mathbb{R}^p$ given by $\Delta_p = \{x \in \mathbb{R}^p : \sum_{i=1}^{p} x_i = 1, x_i \geq 0\}$. As we show in Lemma 2.1, the value of (SOCP), denoted by $\sup(SOCP)$, provides a lower bound for the optimal value of (P), $\inf(P)$, as $\inf(P) \geq \sup(SOCP)$ always holds, by construction. So, (SOCP) serves as a SOCP relaxation problem for (P).

Unfortunately, the equality between $\inf(P)$ and $\sup(SOCP)$, in general, fails for minimax nonconvex separable quadratic optimization problems (see also Examples 2.2 and 2.3). We say that *exact SOCP relaxation holds* for (P) whenever $\inf(P) = \sup(SOCP)$.

While exact SDP relaxation results have been known (see [5, 15, 16, 21, 23, 14, 28, 27, 30]) for the standard quadratic optimization problems with simple quadratic inequality constraints in the special case where $p = 1$ in (P), an exact SOCP relaxation has only recently been established for standard quadratic problem with one or two constraints [10]. Previous studies suggest that exploiting hidden convexity of (P) in the form of convexity of an epigraphical set often permits the development of such results (see [6, 17, 19, 20, 21, 24, 29]). However, to the best knowledge of the authors, the study of an exact SOCP relaxation for minimax quadratic optimization problems (P) appears to be new in the literature of quadratic optimization.

In this paper, we first establish the role of an epigraphical set (see section 2) associated with (P) in the study of exact SOCP relaxation between (P) and (SOCP) by exploiting hidden convexity of (P). We also provide examples for illustrating the importance of the hidden convexity assumption for the exact relaxation of (P). By providing easily verifiable conditions for the existence of a closed convex epigraphical set, we show that various classes of specially structured minimax quadratic optimization problems enjoy exact SOCP relaxation under these verifiable conditions. They include some minimax extended trust-region problems, uniform minimax quadratic optimization problems, max dispersion problems, and some classes of robust quadratic optimization problems.

The organization of this paper is as follows. In section 2, we provide sufficient conditions for exact SOCP relaxation of (P) in terms of epigraphical convexity. In section 3, we present easily verifiable conditions under which minimax uniform quadratic optimization problems admit exact SOCP relaxations. In section 4, we show that under suitable conditions, minimax extended trust-region problems and max dispersion problems exhibit exact SOCP relaxations. In section 5, as an immediate application of our results, we obtain exact SOCP relaxation for some simple robust quadratic optimization problems.

2. SOCP relaxations and epigraphical sets. In this section, we provide a general sufficient condition for exact SOCP relaxation for problem (P). We begin by fixing the notation and definitions that will be used later in the paper. The real line is denoted by $\mathbb{R}$ and the n-dimensional real Euclidean space is denoted by $\mathbb{R}^n$. The set of all nonnegative vectors of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. The space of all $(n\times n)$ symmetric real matrices is denoted by S^n. The $(n\times n)$ identity matrix is denoted by I_n. The notation $A \succeq B$ means that the matrix $A-B$ is positive semidefinite. The cone which consists of all positive semidefinite matrices is denoted by S_n^+. For a matrix $A \in S^n$, $\mathrm{Ker}(A) := \{d \in \mathbb{R}^n : Ad = 0\}$. For a set M, the convex hull (resp., closure) of the set M is denoted by $\mathrm{co}\, M$ (resp., $\mathrm{cl}(M)$). For a subspace L, we use $\dim L$ to denote the dimension of L. Moreover, we use $\mathrm{span}C$ to denote the span of the set C which is defined by $\mathrm{span}C = \{\sum_{i=1}^k \lambda_i c_i, \lambda_i \in \mathbb{R}, c_i \in C, k \in \mathbb{N}\}$. We use Δ_p to denote the simplex in $\mathbb{R}^p$, that is, $\Delta_p = \{x \in \mathbb{R}^p : \sum_{i=1}^p x_i = 1, x_i \geq 0\}$.

Lemma 2.1 (SOCP relaxation). *For (P), let the feasible set be nonempty. Then,* $\inf(P) \geq \sup(SOCP)$.

Proof. Let x be feasible for (P) and let $\mu \in \mathbb{R}, \delta \in \Delta_p, \lambda_j \geq 0,\ j = 1,\dots,q$, $v \in \mathbb{R}^n, s_l \in \mathbb{R}$, $l = 1,\dots,n$, be feasible for (SOCP). Then,

$$
\left\{
\begin{array}{l}
\sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l \geq 0, l = 1,\dots,n,\\
v = U^T(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j),\\
2(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu) - \sum_{l=1}^n s_l \geq 0,\\
s_l \geq 0, l = 1,\dots,n,\\
\|(2v_l, s_l - \sum_{i=1}^p \delta_i \sigma_i^l - \sum_{j=1}^q \lambda_j \mu_j^l)\|\\
\quad \leq s_l + \sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l,\ l = 1,\dots,n.
\end{array}
\right. \tag{2.1}
$$

The last relation of (2.1) together with $t^2 \leq \alpha\beta,\ \alpha,\beta \geq 0 \ \Leftrightarrow\ \|(2t, \alpha-\beta)\| \leq \alpha+\beta$, implies that

$$
v_l^2 \leq s_l \left(\sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l \right),\ l = 1,\dots,n. \tag{2.2}
$$

In particular,

$$
\sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l = 0 \ \Rightarrow\ v_l = 0. \tag{2.3}
$$

Define $L = \{l \in \{1, \ldots, n\} : \sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l > 0\}$. Note from the third relation of (2.1) that

$$0 \leq 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) - \sum_{l=1}^n s_l$$

$$\leq 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) - \sum_{l \in L} s_l$$

$$\leq 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) - \sum_{l \in L} \frac{v_l^2}{\sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l}\,, \tag{2.4}$$

where the second inequality follows from $s_l \geq 0$, $l = 1, \ldots, n$, and the last inequality follows from (2.2). Denote $M = \mathrm{diag}(\sum_{i=1}^p \delta_i \sigma_i^1 + \sum_{j=1}^q \lambda_j \mu_j^1, \ldots, \sum_{i=1}^p \delta_i \sigma_i^n + \sum_{j=1}^q \lambda_j \mu_j^n)$. For the matrix M and the index L defined as above, let $M_L = (M_{ij})_{i,j \in L}$ and $v_L = (v_l)_{l \in L}$. Note that $M_L \succ 0$. Schur's complement together with (2.4) implies that

$$\begin{pmatrix} M_L & v_L \\ v_L^T & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) \end{pmatrix} \succeq 0. \tag{2.5}$$

Combining this with (2.3), we have $\begin{pmatrix} M & v \\ v^T & 2(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu) \end{pmatrix} \succeq 0$. Define, for simplicity,

$$A_i = U \Sigma_i U^T \text{ and } B_j = U \Lambda_j U^T. \tag{2.6}$$

Then, one has

$$\begin{pmatrix} \sum_{i=1}^p \delta_i A_i + \sum_{j=1}^q \lambda_j B_j & \sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j \\ \left(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j\right)^T & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) \end{pmatrix}$$

$$= \begin{pmatrix} UMU^T & Uv \\ (Uv)^T & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) \end{pmatrix}$$

$$= \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M & v \\ v^T & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}^T \succeq 0.$$

Now, as x is feasible for (P) and $\delta \in \Delta_p$, it follows that

$$\begin{aligned}
&\max_{1\le i\le p}\left\{\frac{1}{2}x^TA_ix + a_i^Tx + \alpha_i\right\} - \mu\\
&\ge \sum_{i=1}^p \delta_i\left(\frac{1}{2}x^TA_ix + a_i^Tx + \alpha_i\right) + \sum_{j=1}^q \lambda_j\left(\frac{1}{2}x^TB_jx + b_j^Tx + \beta_j\right) - \mu\\
&= \frac{1}{2}\begin{pmatrix} x\\ 1\end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^p \delta_iA_i + \sum_{j=1}^q \lambda_jB_j & \sum_{i=1}^p \delta_ia_i + \sum_{j=1}^q \lambda_jb_j\\ \left(\sum_{i=1}^p \delta_ia_i + \sum_{j=1}^q \lambda_jb_j\right)^T & 2\left(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu\right)\end{pmatrix}\begin{pmatrix} x\\ 1\end{pmatrix} \ge 0.
\end{aligned}$$

This shows that $\inf(P) \ge \sup(SOCP)$. ∎

Next, we examine when SOCP relaxation is exact in the sense that $\inf(P) = \sup(SOCP)$. To see this, we first introduce the notion of an epigraphical set. Let $e_p = (1, 1, \dots, 1)^T \in \mathbb{R}^p$, $f_i(x) = \frac{1}{2}x^TA_ix + a_i^Tx + \alpha_i$ for $i = 1, 2, \dots, p$, and let $g_j(x) = \frac{1}{2}x^TB_jx + b_j^Tx + \beta_j$, $j = 1, \dots, q$, where

$$A_i = U\Sigma_iU^T \text{ and } B_j = U\Lambda_jU^T. \tag{2.7}$$

Definition 2.1 (epigraphical set). *For (P), the epigraphical set is given by*

$$\begin{aligned}
&E(f_1, \dots, f_p, g_1, \dots, g_q)\\
&\quad = \{(y, z) \in \mathbb{R}^p \times \mathbb{R}^q : \exists x \in \mathbb{R}^n \textit{ such that } f_i(x) \le y_i,\ i = 1, \dots, p,\\
&\qquad\qquad \textit{and } g_j(x) \le z_j,\ j = 1, \dots, q\}.
\end{aligned}$$

This set has a close connection with the epigraph of the value function of the underlying minimax optimization problem. Indeed, for each $z \in \mathbb{R}^q$, consider the following parameterized minimax optimization problem

$$\begin{aligned}
(P_z)\ &\inf_{x\in\mathbb{R}^n}\max_{1\le i\le p}\left\{\frac{1}{2}x^TA_ix + a_i^Tx + \alpha_i\right\}\\
&\text{s.t. } \frac{1}{2}x^TB_jx + b_j^Tx + \beta_j \le z_j, j = 1, \dots, q.
\end{aligned}$$

Let $v : \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\}$ be defined by $v(z) = \mathrm{val}(P_z)$ and let $\mathrm{epi}\, v$ be the epigraph of the function v. Denote $C = E(f_1, \dots, f_p, g_1, \dots, g_q) \cap (\Lambda \times \mathbb{R}^q)$, where $\Lambda = \{(y_1, \dots, y_p) \in \mathbb{R}^p : y_1 = \cdots = y_p\}$. Then, we see that $C \subseteq L(\mathrm{epi}\, v) \subseteq \mathrm{cl}\big(C\big)$, where $L : \mathbb{R}^{q+1} \to \mathbb{R}^{p+q}$ is a linear map given by

$$L(z_1, \dots, z_q, t) = (te_p, z) = (t, \dots, t, z_1, \dots, z_q).$$

We also note that when $p = 1$, the epigraphical set can be written as the sum of the nonnegative orthant and the so-called joint-range set given by

$$\begin{aligned}
&R(f, g_1, \dots, g_q)\\
&\quad = \{(y, z) \in \mathbb{R} \times \mathbb{R}^q : \exists x \in \mathbb{R}^n \text{ such that } f(x) = y, \text{ and } g_j(x) = z_j,\ j = 1, \dots, q\}.
\end{aligned}$$

The convexity of the joint range set has been studied extensively in the literature (see for example [6, 1, 12, 29] and the reference therein). Clearly, if the joint range set is convex, then the epigraphical set is also convex when $p = 1$. On the other hand, when f, g_j are all convex, the epigraphical set must be convex while the joint range set can be nonconvex.

Note that, if $\inf(P) = -\infty$ then, $\inf(P) = \sup(SOCP) = -\infty$ as $\inf(P) \geq \sup(SOCP)$, by construction. So, *to avoid triviality, we assume that* $\inf(P) > -\infty$ *throughout this paper.*

THEOREM 2.1 (exact SOCP relaxation). *Let* $f_i(x) = \frac{1}{2}x^T A_i x + a_i^T x + \alpha_i$ *for* $i = 1, 2, \dots, p$ *and let* $g_j(x) = \frac{1}{2}x^T B_j x + b_j^T x + \beta_j$, $j = 1, \dots, q$, *where* A_i *and* B_j *are defined as in* (2.7). *For* (P), *suppose that the set* $E(f_1, \dots, f_p, g_1, \dots, g_q)$ *is closed and convex. Then,* $\min(P) = \sup(SOCP)$.

Proof. Step 1: *Guaranteeing exact SDP relaxation by the epigraphical condition.* Let $\epsilon > 0$. As $E(f_1, \dots, f_p, g_1, \dots, g_q)$ is a closed and convex set, by the definition of $\inf(P)$, we have

$$((\inf(P) - \epsilon)\, e_p, 0_q) \notin E(f_1, \dots, f_p, g_1, \dots, g_q).$$

Then, by the strong separation theorem, there exists $(\mu_1, \dots, \mu_{p+q}) \in \mathbb{R}^{p+q} \backslash \{0\}$ such that for all $(y, z) \in E(f_1, \dots, f_p, g_1, \dots, g_q) \subseteq \mathbb{R}^p \times \mathbb{R}^q$,

$$(\inf(P) - \epsilon) \sum_{i=1}^{p} \mu_i < \sum_{i=1}^{p} \mu_i y_i + \sum_{j=1}^{q} \mu_{p+j} z_j. \tag{2.8}$$

Note that if $(y_1, \dots, y_p, z_1, \dots, z_q) \in E(f_1, \dots, f_p, g_1, \dots, g_q)$, then

$$(y_1 + r_1, \dots, y_p + r_p, z + s_1, \dots, z_q + s_q) \in E(f_1, \dots, f_p, g_1, \dots, g_q)$$

for any $r_1, \dots, r_p \geq 0$ and $s_1, \dots, s_q \geq 0$. The standard argument shows that $\mu_i \geq 0$, $i = 1, \dots, p+q$. We now see that $\sum_{i=1}^{p} \mu_i > 0$. Otherwise, $\sum_{i=1}^{p} \mu_i = 0$ and so (2.8) implies that

$$\sum_{j=1}^{q} \mu_{p+j} z_j > 0 \text{ for all } (y, z) \in E(f_1, \dots, f_p, g_1, \dots, g_q). \tag{2.9}$$

As $\inf(P) > -\infty$, the feasible set of (P) is nonempty and so there exists $x_0 \in \mathbb{R}^n$ such that

$$\frac{1}{2} x_0^T B_j x_0 + b_j^T x_0 + \beta_j \leq 0, j = 1, \dots, q.$$

Then, $(c, 0_q) \in E(f_1, \dots, f_p, g_1, \dots, g_q)$ with $c = (c_1, \dots, c_p) \in \mathbb{R}^p$ and $c_i = \frac{1}{2}x_0^T A_i x_0 + a_i^T x_0 + \alpha_i$ This contradicts (2.9). Thus, $\sum_{i=1}^{p} \mu_i > 0$. Let $\delta_i = \mu_i / \sum_{i=1}^{p} \mu_i$, $i = 1, \dots, p$, and $\lambda_j = \mu_{p+j} / \sum_{i=1}^{p} \mu_i$, $j = 1, \dots, q$. Then, $\delta \in \Delta_p$. Dividing by $\sum_{i=1}^{p} \mu_i$ on both sides of (2.8), we have, for each $\epsilon > 0$, that there exist $\delta \in \Delta_p$ and $\lambda \in \mathbb{R}^q_+$ such that for all $x \in \mathbb{R}^n$

$$\begin{aligned}
&\sum_{i=1}^{p} \delta_i \left(\frac{1}{2} x^T A_i x + a_i^T x + \alpha_i \right) + \sum_{j=1}^{q} \lambda_j \left(\frac{1}{2} x^T B_j x + b_j^T x + \beta_j \right) \\
&\quad = \sum_{i=1}^{p} \delta_i f_i(x) + \sum_{j=1}^{q} \lambda_j g_j(x) \\
&\quad \geq \inf(P) - \epsilon.
\end{aligned}$$

Note that

$$\begin{pmatrix} W & w \\ w^T & 2\gamma \end{pmatrix} \succeq 0 \;\Leftrightarrow\; \frac{1}{2} x^T W x + w^T x + \gamma \geq 0 \text{ for all } x \in \mathbb{R}^n. \tag{2.10}$$

This shows that, for each $\epsilon > 0$, there exist $\delta \in \Delta_p$ and $\lambda \in \mathbb{R}^q_+$ such that

$$\begin{pmatrix} \sum_{i=1}^p \delta_i A_i + \sum_{j=1}^q \lambda_j B_j & \sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j \\ \left(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j\right)^T & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - (\inf(P) - \epsilon)\right) \end{pmatrix} \succeq 0. \tag{2.11}$$

It also implies that $\inf(P) = \sup(SDP)$, where (SDP) is an SDP relaxation of (P) given by

$$(SDP) \ \sup \left\{ \mu : \begin{pmatrix} \sum_{i=1}^p \delta_i A_i + \sum_{j=1}^q \lambda_j B_j & \sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j \\ \left(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j\right)^T & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - \mu\right) \end{pmatrix} \succeq 0 \right\}.$$

Step 2: Exact SOCP relaxation from SDP relaxation. Recall from (2.7) that

$$A_i = U\Sigma_i U^T \text{ and } B_j = U\Lambda_j U^T,$$

where Σ_i, Λ_j are diagonal matrices given by $\Sigma_i = \operatorname{diag}(\sigma_i^1, \dots, \sigma_i^n)$ and $\Lambda_j = \operatorname{diag}(\mu_j^1, \dots, \mu_j^n)$. Let $M = \sum_{i=1}^p \delta_i A_i + \sum_{j=1}^q \lambda_j B_j$, $u = \sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j$, and $\delta = 2(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - (\inf(P) - \epsilon))$. Note that

$$\begin{pmatrix} M & u \\ u^T & \delta \end{pmatrix} \succeq 0 \Leftrightarrow \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} M & u \\ u^T & \delta \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} U^T M U & U^T u \\ u^T U & \delta \end{pmatrix} \succeq 0.$$

It follows from (2.11) that

$$\begin{pmatrix} \operatorname{diag}\left(\sum_{i=1}^p \delta_i \sigma_i^1 + \sum_{j=1}^q \lambda_j \mu_j^1, \dots, \sum_{i=1}^p \delta_i \sigma_i^n + \sum_{j=1}^q \lambda_j \mu_j^n\right) & U^T\left(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j\right) \\ \left(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j\right)^T U & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - (\inf(P) - \epsilon)\right) \end{pmatrix} \succeq 0. \tag{2.12}$$

We now show that there exist $s_l \in \mathbb{R}$, $l = 1, \dots, n$, such that

$$\begin{cases} \sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l \ge 0, l = 1, \dots, n, \\ v = U^T(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j), \\ 2\big(\sum_{i=1}^p \delta_i \alpha_i + \sum_{j=1}^q \lambda_j \beta_j - (\inf(P) - \epsilon)\big) - \sum_{l=1}^n s_l \ge 0, \\ s_l \ge 0, l = 1, \dots, n, \\ \|(2v_l, s_l - \sum_{i=1}^p \delta_i \sigma_i^l - \sum_{j=1}^q \lambda_j \mu_j^l)\| \\ \quad \le s_l + \sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l, \ l = 1, \dots, n. \end{cases} \tag{2.13}$$

To see this, we first note that (2.12) implies that $\sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l \ge 0, l = 1, \dots, n$. Let $v = U^T(\sum_{i=1}^p \delta_i a_i + \sum_{j=1}^q \lambda_j b_j)$. Moreover, (2.12) also gives us that

$$\sum_{i=1}^p \delta_i \sigma_i^l + \sum_{j=1}^q \lambda_j \mu_j^l = 0 \ \Rightarrow \ v_l = 0. \tag{2.14}$$

Define $L = \{l \in \{1,\dots,n\} : \sum_{i=1}^p \delta_i\sigma_i^l + \sum_{j=1}^q \lambda_j\mu_j^l > 0\}$ and

$$s_l = \begin{cases} \frac{v_l^2}{\sum_{i=1}^p \delta_i\sigma_i^l + \sum_{j=1}^q \lambda_j\mu_j^l} & \text{if } l \in L, \\ 0 & \text{otherwise.} \end{cases}$$

The construction of s_l implies that $s_l \geq 0$ and

$$v_l^2 \leq s_l \left(\sum_{i=1}^p \delta_i\sigma_i^l + \sum_{j=1}^q \lambda_j\mu_j^l \right), \ l = 1,\dots,n.$$

Using the following relationship

$$t^2 \leq \alpha\beta,\ \alpha,\beta \geq 0 \ \Leftrightarrow\ \|(2t, \alpha-\beta)\| \leq \alpha+\beta, \tag{2.15}$$

it follows that

$$\left\| \left(2v_l, s_l - \sum_{i=1}^p \delta_i\sigma_i^l - \sum_{j=1}^q \lambda_j\mu_j^l \right) \right\| \leq s_l + \sum_{i=1}^p \delta_i\sigma_i^l + \sum_{j=1}^q \lambda_j\mu_j^l,\ l = 1,\dots,n.$$

Denote $M = \mathrm{diag}(\sum_{i=1}^p \delta_i\sigma_i^1 + \sum_{j=1}^q \lambda_j\mu_j^1, \dots, \sum_{i=1}^p \delta_i\sigma_i^n + \sum_{j=1}^q \lambda_j\mu_j^n)$. For the matrix M and the index L defined as above, let $M_L = (M_{ij})_{i,j\in L}$ and $v_L = (v_l)_{l\in L}$. Then, (2.12) implies that

$$\begin{pmatrix} M_L & v_L \\ v_L^T & 2(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu) \end{pmatrix} \succeq 0. \tag{2.16}$$

Note that $M_L \succ 0$. Schur's complement shows that $2(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu) - v_L^T M_L^{-1} v_L \geq 0$. This implies that

$$\begin{aligned} & 2\left(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu \right) - \sum_{l=1}^n s_l \\ &= 2\left(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu \right) - \sum_{l\in L} s_l \\ &= 2\left(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu \right) - \sum_{l\in L} \frac{v_l^2}{\sum_{i=1}^p \delta_i\sigma_i^l + \sum_{j=1}^q \lambda_j\mu_j^l} \\ &= 2\left(\sum_{i=1}^p \delta_i\alpha_i + \sum_{j=1}^q \lambda_j\beta_j - \mu \right) - v_L^T M_L^{-1} v_L \geq 0. \end{aligned}$$

Thus, (2.13) holds. So, we see that $\sup(SOCP) \geq \inf(P) - \epsilon$. As $\epsilon > 0$ is arbitrary, $\sup(SOCP) \geq \inf(P)$. This together with the fact that $\sup(SOCP) \leq \inf(P)$ gives us that $\sup(SOCP) = \inf(P)$.

Step 3: Guaranteeing the attainment of the minimum of (P) To see the attainment of $\inf(P)$, let $\{x_k\} \subseteq \mathbb{R}^n$ be a sequence satisfying

$$\frac{1}{2}x_k^T B_j x_k + b_j^T x_k + \beta_j \leq 0, j = 1,\dots,q, \text{ and } \max_{1\leq i\leq p} \left\{ \frac{1}{2}x_k^T A_i x_k + a_i^T x_k + \alpha_i \right\} \to \inf(P).$$

Then, there exists $\epsilon_k \to 0$ such that

$$((\inf(P) + \epsilon_k)e_p, 0_q) \in E(f_1, \dots, f_p, g_1, \dots, g_q).$$

Letting $k \to \infty$ and noting that $E(f_1, \dots, f_p, g_1, \dots, g_q)$ is closed, we see that $(\inf(P)e_p, 0_q) \in E(f_1, \dots, f_p, g_1, \dots, g_q)$. Then, there exists $\bar{x} \in \mathbb{R}^n$ such that

$$\frac{1}{2}\bar{x}^T B_j \bar{x} + b_j^T \bar{x} + \beta_j \leq 0, j = 1, \dots, q, \text{ and } \frac{1}{2}\bar{x}^T A_i \bar{x} + a_i^T \bar{x} + \alpha_i \leq \inf(P),\ i = 1, \dots, p.$$

This together with the definition of $\inf(P)$ implies that $\bar{x}$ is a minimizer for (P). ∎

Remark 2.1. Let $f_i(x) = \frac{1}{2}x^T A_i x + a_i^T x + \alpha_i$ for $i = 1, 2, \dots, p$ and let $g_j(x) = \frac{1}{2}x^T B_j x + b_j^T x + \beta_j$, $j = 1, \dots, q$, where A_i and B_j are defined as in (2.7). An alternative approach for studying the exact SOCP relaxation for the minimax quadratic optimization problem is to reformulate the problem (P) as a standard quadratic optimization problem in a higher dimensional space as follows:

$$\begin{aligned} \inf_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} \quad & t \\ \text{s.t.} \quad & \frac{1}{2}x^T B_j x + b_j^T x + \beta_j \leq 0, j = 1, \dots, q, \\ & \frac{1}{2}x^T A_i x + a_i^T x + \alpha_i - t \leq 0, i = 1, \dots, p. \end{aligned}$$

Then, using similar arguments as in the proof of Theorem 2.1, an exact SOCP relaxation result can also be achieved by imposing the closedness and convexity of the following set in a higher-dimensional space:

$$\begin{aligned} &\overline{U}(f_1, \dots, f_p, g_1, \dots, g_q) \\ &= \{(r, y, z) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^q : \exists (x, t) \in \mathbb{R}^n \times \mathbb{R} \text{ such that } t \leq r, f_i(x) \leq y_i,\ i = 1, \dots, p, \\ &\qquad \text{and } g_j(x) - t \leq z_j,\ j = 1, \dots, q\}. \end{aligned}$$

We note that the supremum in the relaxation problem (RP) is in general not attained even when the epigraphical set $E(f_1, \dots, f_p, g_1, \dots, g_q)$ is closed and convex.

Example 2.1 (nonattainment of the SOCP relaxation problem). Consider the one-dimensional problem $\min_{x\in\mathbb{R}}\{\max\{x, 2x\} : x^2 \leq 0\}$. This is of the form (P), where $f_1(x) = x$, $f_2(x) = 2x$, and $g_1(x) = x^2$ with $U = 1$, $\Sigma_1 = \Sigma_2 = 0$, and $\Lambda_1 = 2$. Direct verification shows that

$$\begin{aligned} E(f_1, f_2, g_1) &= \{(y_1, y_2, z_1) : \exists\, x \in \mathbb{R},\ f_1(x) \leq y_1, f_2(x) \leq y_2 \text{ and } g_1(x) \leq z_1\} \\ &= \{(y_1, y_2, z_1) : \exists\, x \in \mathbb{R},\ x \leq y_1, 2x \leq y_2 \text{ and } x^2 \leq z_1\} \end{aligned}$$

is a closed and convex set.

Clearly, $\inf(P) = 0$. Its SOCP relaxation problem is

$$\begin{aligned} \sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_2,\lambda_1\geq 0 \\ v\in\mathbb{R}, s_1\in\mathbb{R}}} \quad & \mu \\ \text{s.t.} \quad & v = \delta_1 + 2\delta_2, \\ & -2\mu - s_1 \geq 0, s_1 \geq 0, \\ & \|(2v, s_1 - 2\lambda_1)\| \leq s_1 + 2\lambda_1. \end{aligned} \tag{2.17}$$

For each $k \in \mathbb{N}$, letting $(\delta_1, \delta_2) = (1, 0)$, $\lambda_1 = k$, $v = 1$, $s_1 = \frac{1}{2k}$, and $\mu = -\frac{1}{k}$, we see that the constraints in (2.17) are satisfied. So, $\sup(SOCP) \geq -\frac{1}{k}$ for all k. This together with $\inf(P) \geq \sup(SOCP)$ shows that $\sup(SOCP) = 0$. We now see that

$\sup(SOCP)$ is not attained. (Otherwise, there exist $\delta \in \Delta_2, \lambda_1 \geq 0,\ v_1 \in \mathbb{R}, s_1 \in \mathbb{R}$ such that

$$\begin{cases} v = \delta_1 + 2\delta_2, \\ s_1 = 0, \\ \|(2v, s_1 - 2\lambda_1)\| \leq s_1 + 2\lambda_1. \end{cases}$$

This implies that $\delta_1 + 2\delta_2 = v = 0$ which contradicts $(\delta_1, \delta_2) \in \Delta_2$.)

We have seen that the closedness and convexity of the epigraphical set $E(f_1, \dots, f_p, g_1, \dots, g_q)$ guarantees the exact SOCP relaxation for (P). Now, we provide two examples which show that exact SOCP relaxation can fail if the epigraphical set is a nonconvex or a nonclosed set.

Example 2.2 (failure of exact SOCP relaxation for nonconvex epigraphical set). Let $f_1(x) = -x^2 + x$, $f_2(x) = -x^2 - x$, and $g_1(x) = x^2 - \frac{1}{4}$. Then, $f_i(x) = \frac{1}{2}x^T A_i x + a_i^T x + \alpha_i$, $i = 1, 2$, with $A_1 = A_2 = -2$, $a_1 = 1$, $a_2 = -1$, $\alpha_1 = \alpha_2 = 0$, and $g_1(x) = \frac{1}{2}x^T B_1 x + b_1^T x + \beta_1$ with $B_1 = 2$, $b_1 = 0$, and $\beta_1 = -\frac{1}{4}$.

Now we see that $E(f_1, f_2, g_1)$ is not convex. Indeed, it is easy to check that

$$\left(0, 0, -\frac{1}{4}\right) = (f_1(0), f_2(0), g_1(0)) \in E(f_1, f_2, g_1)$$

and

$$\left(-2, -2, \frac{15}{4}\right) = (f_1(2), f_2(2) + 4, g_1(2)) \in E(f_1, f_2, g_1).$$

But their midpoint $(-1, -1, \frac{7}{4}) \notin E(f_1, f_2, g_1)$ as the following inequality system

$$-x^2 + x \leq -1, -x^2 - x \leq -1, \text{ and } x^2 - \frac{1}{4} \leq \frac{7}{4},$$

has no solution.

Consider the following minimax quadratic optimization problem

$$\min\{\max\{f_1(x), f_2(x)\} : g_1(x) \leq 0\}.$$

It is of the form (P), where $U = 1$, $\Sigma_1 = \Sigma_2 = -2$, and $\Lambda_1 = 2$. In this case, $\inf(P) = 0$. The corresponding SOCP relaxation becomes

$$\begin{aligned} \sup_{\substack{\mu \in \mathbb{R}, \delta \in \Delta_2, \lambda_1 \geq 0 \\ v_1 \in \mathbb{R}, s_1 \in \mathbb{R}}} \quad & \mu \\ \text{s.t.} \quad & -2(\delta_1 + \delta_2) + 2\lambda_1 \geq 0, \\ & v = \delta_1 - \delta_2, \\ & 2\left(-\frac{1}{4}\lambda_1 - \mu\right) - s_1 \geq 0, s_1 \geq 0, \\ & \|(2v, s_1 + 2(\delta_1 + \delta_2 - 2\lambda_1))\| \leq s_1 - 2(\delta_1 + \delta_2 - 2\lambda_1). \end{aligned} \tag{2.18}$$

Note that $\delta_1 + \delta_2 = 1$. So, any feasible point of the relaxation satisfies $-1 + \lambda_1 \geq 0$ and $-\frac{\lambda_1}{4} - \mu \geq 0$ and, hence, $\mu \leq -\frac{1}{4}$. As $(\lambda, \delta_1, \delta_2, \mu, v, s_1) = (1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, 0, 0)$ is a feasible point, the optimal value of the relaxation problem is $-\frac{1}{4}$. Thus, exact SOCP relaxation fails.

Example 2.3 (failure of exact SOCP relaxation for nonclosed epigraphical set). Let $f_1(x) = x_1 x_2$, $f_2(x) = x_2$, and $g_1(x) = x_1$. Then, $f_i(x) = \frac{1}{2}x^T A_i x + a_i^T x + \alpha_i$,

$i=1,2$, with $A_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $A_2=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, $a_1=(0,0)^T$, $a_2=(0,1)^T$, $\alpha_1=\alpha_2=0$, and $g_1(x)=\frac{1}{2}x^TB_1x+b_1^Tx+\beta_1$ with $B_1=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, $b_1=(1,0)^T$, and $\beta_1=0$.

Now we see that $E(f_1,f_2,g_1)$ is not closed. Note that, for all $k\in\mathbb{N}$,

$$\left(-1,-1,\frac{1}{k}\right)=\left(f_1\left(\frac{1}{k},-k\right),f_2\left(\frac{1}{k},-k\right)+k-1,g_1\left(\frac{1}{k},-k\right)\right)\in E(f_1,f_2,g_1).$$

However, its limit $(-1,-1,0)\notin E(f_1,f_2,g_1)$ as the following inequality system has no solution:

$$x_1x_2\le -1, x_2\le -1, \text{ and } x_1\le 0.$$

Now, consider the following minimax quadratic optimization problem

$$\min\{\max\{f_1(x),f_2(x)\}:g_1(x)\le 0\}.$$

It is of the form (P), where $U=\left(\begin{smallmatrix}-\sqrt{2}/2&\sqrt{2}/2\\ \sqrt{2}/2&\sqrt{2}/2\end{smallmatrix}\right)$, $\Sigma_1=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$, and $\Sigma_2=\Lambda_1=0_{2\times 2}$. In this case, $\inf(P)=0$. Its SOCP relaxation becomes

$$\begin{aligned}
\sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_2,\lambda_1\ge 0\\ v\in\mathbb{R}^2,s_1,s_2\in\mathbb{R}}} \quad & \mu\\
\text{s.t.}\quad & \delta_1=0,\\
& \begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}-\sqrt{2}/2&\sqrt{2}/2\\ \sqrt{2}/2&\sqrt{2}/2\end{pmatrix}\left(\delta_2\begin{pmatrix}0\\1\end{pmatrix}+\lambda_1\begin{pmatrix}1\\0\end{pmatrix}\right),\\
& -2\mu-s_1-s_2\ge 0, s_1,s_2\ge 0,\\
& \|(2v_1,s_1-\delta_1)\|\le s_1+\delta_1,\\
& \|(2v_2,s_2+\delta_1)\|\le s_2-\delta_1.
\end{aligned}$$

So, any feasible point of the relaxation satisfies $\delta_1=0$ (and so, $\delta_2=1$), $v_2=\frac{\sqrt{2}}{2}\delta_2=\frac{\sqrt{2}}{2}$, and

$$\|(\sqrt{2},s_2)\|=\|(2v_2,s_2+\delta_1)\|\le s_2-\delta_1=s_2,$$

which is impossible. So, the optimal value of the SOCP relaxation is $-\infty$, and the exact SOCP relaxation fails in this case.

We now show that, in addition to the convexity of the epigraphical set, if a strict feasibility condition is also satisfied for (P), then the exact SOCP relaxation holds with attainment of the relaxation problem. We follow the standard method of proof using separation hyperplane arguments. For a similar method of proof for a convex optimization problem, see, for example, [4, Theorem 12.7].

PROPOSITION 2.1 (exact SOCP relaxation with attainment of sup(SOCP)). *For* (P), *suppose that the set* $E(f_1,\dots,f_p,g_1,\dots,g_q)$ *is convex, and there exists* $x_0\in\mathbb{R}^n$ *such that* $g_j(x_0)<0$, $j=1,\dots,q$. *Then,* $\inf(P)=\max(SOCP)$. *Moreover, if* $E(f_1,\dots,f_p,g_1,\dots,g_q)$ *is further assumed to be closed, then* $\min(P)=\max(SOCP)$.

Proof. Without loss of generality, we assume that $\inf(P)$ is finite. Let $C=\{(y,z)\in\mathbb{R}^p\times\mathbb{R}^q: y_i\le\inf(P), i=1,\dots,p, z_j\le 0, j=1,\dots,q\}$. It is clear that C is a convex set with nonempty interior. By the definition of $\inf(P)$, we have

$$\text{int}C\cap E(f_1,\dots,f_p,g_1,\dots,g_q)=\emptyset.$$

Then, by the convex separation theorem [31, Theorem 1.1.3], there exists $(\mu_1,\dots,\mu_{p+q})\in\mathbb{R}^{p+q}\backslash\{0\}$ such that for all $(y,z)\in E(f_1,\dots,f_p,g_1,\dots,g_q)\subseteq\mathbb{R}^p\times\mathbb{R}^q$, $\inf(P)\sum_{i=1}^p\mu_i\le\sum_{i=1}^p\mu_iy_i+\sum_{j=1}^q\mu_{p+j}z_j$. Clearly, $\mu_i\ge 0$, $i=1,2,\dots,p+q$.

As $(f_i(x), \ldots, f_p(x), g_1(x), \ldots, g_q(x)) \in E(f_1, \ldots, f_p, g_1, \ldots, g_q)$, it follows that, for all $x \in \mathbb{R}^n$

$$\inf(P) \sum_{i=1}^{p} \mu_i \leq \sum_{i=1}^{p} \mu_i f_i(x) + \sum_{j=1}^{q} \mu_{p+j} g_j(x). \tag{2.19}$$

We now observe that $\sum_{i=1}^{p} \mu_i > 0$ (otherwise, $\sum_{i=1}^{p} \mu_i = 0$ and so $\sum_{j=1}^{q} \mu_{p+j} g_j(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $(\mu_{p+1}, \ldots, \mu_{p+q}) \neq 0_q$. This contradicts the strict feasibility assumption). Dividing both sides of (2.19) by $\sum_{i=1}^{p} \mu_i > 0$, we see that

$$\sum_{i=1}^{p} \delta_i f_i(x) + \sum_{j=1}^{q} \lambda_j g_j(x) - \inf(P) \geq 0,$$

where $\delta_i = \frac{\mu_i}{\sum_{i=1}^{p} \mu_i} \geq 0$ with $\sum_{i=1}^{m} \delta_i = 1$ and $\lambda_j = \frac{\mu_{p+j}}{\sum_{i=1}^{p} \mu_i} \geq 0$. By (4.2), we obtain that

(2.20)

$$\begin{pmatrix} \operatorname{diag}\left(\sum_{i=1}^{p} \delta_i \sigma_i^1 + \sum_{j=1}^{q} \lambda_j \mu_j^1, \ldots, \sum_{i=1}^{p} \delta_i \sigma_i^n + \sum_{j=1}^{q} \lambda_j \mu_j^n\right) & U^T\left(\sum_{i=1}^{p} \delta_i a_i + \sum_{j=1}^{q} \lambda_j b_j\right) \\ \left(\sum_{i=1}^{p} \delta_i a_i + \sum_{j=1}^{q} \lambda_j b_j\right)^T U & 2\left(\sum_{i=1}^{p} \delta_i \alpha_i + \sum_{j=1}^{q} \lambda_j \beta_j - \inf(P)\right) \end{pmatrix} \succeq 0.$$

Now, using a similar argument as in Theorem 2.1, we see that $\inf(P) = \sup(SOCP)$ and the optimal value of (SOCP) is attained, that is, $\inf(P) = \max(SOCP)$.

In addition, if $E(f_1, \ldots, f_p, g_1, \ldots, g_q)$ is further assumed to be closed, then, Theorem 2.1 shows that the optimal value of (P) is also attained and, hence, $\min(P) = \max(SOCP)$. ◻

As we have seen in Theorem 2.1, the epigraphical condition plays an important role in our derivation of exact SOCP relaxations. On the other hand, we note that verifying the epigraphical condition can be, in general, difficult for general minimax quadratic problems. In the next three sections, we provide several important classes of minimax quadratic optimization problems for which the epigraphical condition is easily satisfied.

3. Minimax uniform quadratic optimization problems.

Consider the following minimax uniform quadratic optimization problem:

$$\begin{aligned} (UP) \inf_{x \in \mathbb{R}^n} \max_{1 \leq i \leq p} & \left\{ \frac{1}{2} x^T A x + a_i^T x + \alpha_i \right\} \\ \text{s.t. } & \frac{\rho_j}{2} x^T A x + b_j^T x + \beta_j \leq 0, j = 1, \ldots, q, \end{aligned}$$

where $A = U_A \Sigma U_A^T$ is a symmetric $(n \times n)$ matrix and $\rho_j \in \mathbb{R}$, $j = 1, \ldots, q$, U_A is an orthogonal matrix, $\Sigma = \operatorname{diag}(\sigma^1, \ldots, \sigma^n)$ is a diagonal matrix, and the diagonal elements are the eigenvalues of A. So, the problem (UP) is a special case of the model problem (P) with $U = U_A$, $\Sigma_i = \Sigma$, $i = 1, \ldots, p$, and $\Lambda_j = \rho_j \Sigma$, $j = 1, \ldots, q$.

To avoid triviality, we assume that $(\rho_1, \ldots, \rho_q) \neq 0_q$. We note that, for the minimax uniform quadratic optimization problem, the Hessian of the quadratic functions of the objective function and the constraint functions only differ from each other by a constant multiple. In the special case where $p = 1$, the exact SDP relaxation properties have been examined in [7] and they found applications in solving the smallest enclosing ball problem.

The SOCP relaxation of (UP) can be stated as follows:

$$
\begin{aligned}
(SOCP_{UP}) \quad & \sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_p,\lambda_j\geq 0\\ v\in\mathbb{R}^n,s_l\in\mathbb{R}}} \mu \\
\text{s.t.} \quad & \left(1+\sum_{j=1}^{q}\lambda_j\rho_j\right)\sigma^l \geq 0,\ l=1,\dots,n,\\
& v = U_A^T\left(\sum_{i=1}^{p}\delta_i a_i + \sum_{j=1}^{q}\lambda_j b_j\right),\\
& 2\left(\sum_{i=1}^{p}\delta_i\alpha_i + \sum_{j=1}^{q}\lambda_j\beta_j - \mu\right) - \sum_{l=1}^{n} s_l \geq 0,\ s_l\geq 0,\\
& \left\|\left(2v_l, s_l - \left(1+\sum_{j=1}^{q}\lambda_j\rho_j\right)\sigma^l\right)\right\|\\
& \quad \leq s_l + \left(1+\sum_{j=1}^{q}\lambda_j\rho_j\right)\sigma^l,\ l=1,\dots,n,
\end{aligned} \tag{3.1}
$$

where $A = U_A\Sigma U_A^T$.

THEOREM 3.1. *For problem (UP), suppose that A is negative definite,*

$$\{d\in\mathbb{R}^n : (a_i - a_1)^T d = 0, i=2,\dots,p,\ (b_j-\rho_j a_1)^T d = 0, j=1,\dots,q\} \neq \{0\},$$

and there exists $x_0\in\mathbb{R}^n$ such that $\frac{\rho_j}{2}x_0^TAx_0 + b_j^Tx_0 + \beta_j < 0, j=1,\dots,q,$ then $\inf(UP) = \max(SOCP_{UP})$. *Moreover, if we further assume that there exists $j_0\in\{1,\dots,q\}$ with $\rho_{j_0} < 0$, then* $\min(UP) = \max(SOCP_{UP})$.

Proof. Let $f_i(x) = \frac{1}{2}x^TAx + a_i^Tx + \alpha_i$, $i=1,\dots,p$, and $g_j(x) = \frac{\rho_j}{2}x^TAx + b_j^Tx + \beta_j$, $j=1,\dots,q$. We first establish that the epigraphical set $E(f_1,\dots,f_p,g_1,\dots,g_q)$ is convex. To see this, let

$$R(f_1,\dots,f_p,g_1,\dots,g_q) = \{(f_1(x),\dots,f_p(x),g_1(x),\dots,g_q(x))\in\mathbb{R}^{p+q} : x\in\mathbb{R}^n\}.$$

Then, $E(f_1,\dots,f_p,g_1,\dots,g_q) = R(f_1,\dots,f_p,g_1,\dots,g_q) + \mathbb{R}_+^{p+q}$. Next, let

$$
\begin{aligned}
C := \Bigg\{ & \left(\frac{1}{2}x^TAx + a_1^Tx + \alpha_1, \overline{a}_2^Tx + \overline{\alpha}_2, \dots, \overline{a}_p^Tx + \overline{\alpha}_p, \overline{b}_1^Tx + \overline{\beta}_1, \dots, \overline{b}_q^Tx + \overline{\beta}_q\right)\\
& \in \mathbb{R}^{p+q} : x\in\mathbb{R}^n\Bigg\},
\end{aligned}
$$

where $\overline{a}_i = a_i - a_1\in\mathbb{R}^n$, $\overline{\alpha}_i = \alpha_i - \alpha_1\in\mathbb{R}$, $i=2,\dots,p$, $\overline{b}_j = b_j - \rho_j a_1\in\mathbb{R}^n$ and $\overline{\beta}_j = \beta_j - \rho_j\alpha_1\in\mathbb{R}$, $j=1,\dots,q$. We now verify that C is convex. To see this, let $u=(u_1,\dots,u_{p+q})\in C$, $v=(v_1,\dots,v_{p+q})\in C$, and $\lambda\in[0,1]$. Then, there exist $y,z\in\mathbb{R}^n$ such that

$$u_1 = \frac{1}{2}y^TAy + a_1^Ty + \alpha_1,\ u_i = \overline{a}_i^Ty + \overline{\alpha}_i,\ i=2,\dots,p,\ u_{p+j} = \overline{b}_j^Ty + \overline{\beta}_j,\ j=1,\dots,q,$$

and

$$v_1 = \frac{1}{2}z^T Az + a_1^T z + \alpha_1,\ v_i = \overline{a}_i^T z + \overline{\alpha}_i,\ i = 2, \dots, p,\ v_{p+j} = \overline{b}_j^T z + \overline{\beta}_j,\ j = 1, \dots, q.$$

Let $x = \lambda y + (1 - \lambda)z$. Then,

$$\overline{a}_i^T x + \overline{\alpha}_i = \lambda u_i + (1-\lambda)v_i,\ i = 2, \dots, p, \text{ and } \overline{b}_j^T x + \overline{\beta}_j = \lambda u_{p+j} + (1-\lambda)v_{p+j},\ j = 1, \dots, q.$$

Moreover, by the negative definiteness of A

$$\frac{1}{2}x^T Ax + a_1^T x + \alpha_1 \geq \lambda u_1 + (1 - \lambda)v_1.$$

Now, let $d \neq 0_n$ be such that

$$\overline{a}_i^T d = (a_i - a_1)^T d = 0, i = 2, \dots, p, \text{ and } \overline{b}_j^T d = (b_j - \rho_j a_1)^T d = 0, j = 1, \dots, q.$$

Consider $x(s) := x + sd$ with $s \geq 0$. It then follows that, for all $s \geq 0$,

$$\overline{a}_i^T x(s) + \overline{\alpha}_i = \lambda u_i + (1 - \lambda)v_i,\ i = 2, \dots, p, \text{ and}$$
$$\overline{b}_j^T x(s) + \overline{\beta}_j = \lambda u_{p+j} + (1 - \lambda)v_{p+j},\ j = 1, \dots, q.$$

Moreover, as $d^T Ad < 0$ (thanks to the negative definiteness assumption of A), one has

$$\lim_{s \to +\infty} \left\{ \frac{1}{2}x(s)^T Ax(s) + a_1^T x(s) + \alpha_1 \right\} = -\infty.$$

It then follows from the intermediate value theorem that there exists $s_0 \geq 0$ such that, for $x(s_0) := x + s_0 d$,

$$\frac{1}{2}x(s_0)^T Ax(s_0) + a_1^T x(s_0) + \alpha_1 = \lambda u_1 + (1 - \lambda)v_1.$$

This together with the fact that

$$\overline{a}_i^T x(s_0) + \overline{\alpha}_i = \lambda u_i + (1 - \lambda)v_i,\ i = 2, \dots, p, \text{ and}$$
$$\overline{b}_j^T x(s_0) + \overline{\beta}_j = \lambda u_{p+j} + (1 - \lambda)v_{p+j},\ j = 1, \dots, q,$$

implies that $\lambda u + (1 - \lambda)v \in C$. So, C is convex. Now, note that

$$R(f_1, \dots, f_p, g_1, \dots, g_q) = L(C),$$

where L is a linear mapping given by

$$\begin{aligned} &L(x_1, x_2, \dots, x_p, x_{p+1}, \dots, x_{p+q}) \\ &\quad = (x_1, x_2 + x_1, \dots, x_p + x_1, x_{p+1} + \rho_1 x_1, \dots, x_{p+q} + \rho_q x_1) \end{aligned}$$

Thus, $R(f_1, \dots, f_p, g_1, \dots, g_q)$ is convex and, hence, $E(f_1, \dots, f_p, g_1, \dots, g_q)$ is also convex. Clearly, the strict feasibility assumption is satisfied. It then follows from Proposition 2.1 that $\inf(UP) = \max(SOCP_{UP})$.

Moreover, if there exists $j_0 \in \{1, \dots, q\}$ with $\rho_{j_0} < 0$, then $\rho_{j_0} A \succ 0$. So, the feasible set of (UP) is bounded. Thus, the optimal value of (P) is attained and, hence, $\min(UP) = \max(SOCP_{UP})$.[1] ∎

[1] This can also be obtained from Proposition 2.1 by verifying $E(f_1, \dots, f_p, g_1, \dots, g_q)$ is closed in this case.

Remark 3.1. Theorem 3.1 provides an improvement to the exact SDP relaxation result of the uniform quadratic optimization problem of [7] where an exact SDP relaxation is obtained in the case $p = 1$ under the condition that A is negative definite and $q \leq n - 1$.

Here, we obtained an exact SOCP relaxation by assuming a slightly weaker assumption. To see this, note that, if $q \leq n - 1$, then

$$\left\{d \in \mathbb{R}^n : (b_j - \rho_j a_1)^T d = 0, j = 1, \dots, q\right\} \neq \{0\},$$

and so, the assumption in Theorem 3.1 is satisfied with $p = 1$.

4. Minimax QPs with generalized trust-region constraints. In this section, we consider the following minimax quadratic programming problem with the generalized trust-region constraint

$$\begin{aligned}(\tilde{P})\ \min_{x \in \mathbb{R}^n} \max_{1 \leq i \leq p} & \left\{\frac{1}{2} x^T A x + a_i^T x + \alpha_i\right\} \\ \text{s.t. } & \|x - x_0\|^2 \leq \rho, \\ & b_j^T x + \beta_j \leq 0, j = 1, \dots, r,\end{aligned}$$

where the matrix A is a given symmetric $(n \times n)$ matrix, $x_0 \in \mathbb{R}^n$, $\rho > 0$, $a_i, b_j \in \mathbb{R}^n$, and $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, \dots, p$, $j = 1, \dots, r$. Here, we note that, the solution is always attained (due to the norm constraint), and all the quadratic functions $f_i(x) = \frac{1}{2} x^T A x + a_i^T x + \alpha_i$, $i = 1, \dots, p$, have the *same Hessian matrix*.

In the special case where $p = 1$, the model problem $(\tilde{P})$ reduces to the nonconvex quadratic optimization problem with extended trust-region problem discussed in [2, 21]. Moreover, this model problem $(\tilde{P})$ covers the max dispersion problem [13, 18] which will be examined later in this section .

We first see $(\tilde{P})$ is indeed a special case of our model problem (P). To do this, write $A = U_A \Sigma U_A^T$, where U_A is an orthogonal matrix and $\Sigma = \text{diag}(\sigma^1, \dots, \sigma^n)$ is a diagonal matrix whose diagonal element σ^l, $l = 1, \dots, n$, are eigenvalues of A. Then, $(\tilde{P})$ is a special case of our model problem (P) with $U = U_A$, $\Sigma_i = \Sigma$, $i = 1, \dots, p$, and $\Lambda_j = 2I_n$, $j = 1$, and $\Lambda_j = 0_{n \times n}$ for $j \geq 2$. Its corresponding SOCP relaxation problem (1.1) reduces to

$$\begin{aligned}(\widetilde{SOCP}) \quad & \sup_{\substack{\mu \in \mathbb{R}, \delta \in \Delta_p, \lambda_j \geq 0 \\ v \in \mathbb{R}^n, s_l \in \mathbb{R}}} \mu \\ \text{s.t.} \quad & \sigma^l + 2\lambda_1 \geq 0, l = 1, \dots, n, \\ & v = U_A^T \left(\sum_{i=1}^p \delta_i a_i - 2\lambda_1 x_0 + \sum_{j=1}^r \lambda_{j+1} b_j\right), \\ & 2\left(\sum_{i=1}^p \delta_i \alpha_i + \lambda_1(\|x_0\|^2 - \rho) + \sum_{j=1}^r \lambda_{j+1} \beta_j - \mu\right) \\ & - \sum_{l=1}^n s_l \geq 0, s_l \geq 0, \\ & \|(2v_l, s_l - \sigma^l - 2\lambda_1)\| \leq s_l + \sigma^l + 2\lambda_1,\ l = 1, \dots, n,\end{aligned}$$

where $A = U_A \Sigma U_A^T$. We now show that, for the specially structured model $(\tilde{P})$, a relaxed assumption of Theorem 2.1 leads to an exact SOCP relaxation result in terms of the subepigraphical set, $E(f_1,\dots,f_p,g_1,\dots,g_{r+1})\cap(\Lambda\times\mathbb{R}^{r+1})$. To do this, let $f_i(x)=\frac{1}{2}x^TAx+a_i^Tx+\alpha_i$, $i=1,\dots,p$, $g_1(x)=\|x-x_0\|^2-\rho$, and $g_{j+1}(x)=b_j^Tx+\beta_j$, $j=1,\dots,r$. We also denote $\Lambda=\{(y_1,\dots,y_p)\in\mathbb{R}^p : y_1=\cdots=y_p\}$.

LEMMA 4.1 (exact relaxation under a subepigraphical condition). *For problem $(\tilde{P})$, suppose that the subepigraphical set $E(f_1,\dots,f_p,g_1,\dots,g_{r+1})\cap(\Lambda\times\mathbb{R}^{r+1})$ is convex. Then,* $\min(\tilde{P})=\sup(\widetilde{SOCP})$.

Proof. Let $C=E(f_1,\dots,f_p,g_1,\dots,g_{r+1})\cap(\Lambda\times\mathbb{R}^{r+1})$. We first show by construction that C is a closed set. To see this, let $(y_1^k,\dots,y_p^k,z_1^k,\dots,z_{r+1}^k)\in C$ and

$$(y_1^k,\dots,y_p^k,z_1^k,\dots,z_{r+1}^k)\to(y_1,\dots,y_p,z_1,\dots,z_{r+1}).$$

Then, there exists $\{x_k\}\subseteq\mathbb{R}^n$ such that, for all $k\in\mathbb{N}$, $y_1^k=\cdots=y_p^k$, $f_i(x_k)\le y_i^k$, $i=1,\dots,p$, and $g_j(x_k)\le z_j^k$, $j=1,\dots,r+1$. As $g_1(x_k)=\|x_k-x_0\|^2-\rho\le z_1^k\to z_1$, we see that $\{x_k\}$ is a bounded sequence. By passing to subsequences, we may assume that $x_k\to\bar{x}$. Letting $k\to\infty$, we have, $y_1=\cdots=y_p$, $f_i(\bar{x})\le y_i$, $i=1,\dots,p$, and $g_j(\bar{x})\le z_j$, $j=1,\dots,r+1$. So, C is closed.

Now, from the definition of C, we observe that $((\mathrm{val}(\tilde{P})-\epsilon)\,e_p,0_{r+1})\notin C$ where $e_p=(1,1,\dots,1)^T\in\mathbb{R}^p$. As C is closed and convex, it follows from the strong separation theorem [31, Theorem 1.1.5] that there exists a vector $(\mu_1,\dots,\mu_{p+r+1})\in\mathbb{R}^{p+r+1}\backslash\{0\}$ such that, for all $(y,z)\in C$,

$$(\mathrm{val}(\tilde{P})-\epsilon)\sum_{i=1}^p\mu_i<\sum_{i=1}^p\mu_iy_i+\sum_{j=1}^{r+1}\mu_{p+j}z_j=\left(\sum_{i=1}^p\mu_i\right)y_1+\sum_{j=1}^{r+1}\mu_{p+j}z_j, \tag{4.1}$$

where the last equality follows from the fact that $y_1=\cdots=y_p$ and $(y_1,\dots,y_p,z_1,\dots,z_{r+1})\in C$. Using the same line of argument as in the proof of Theorem 2.1, we see that $\sum_{i=1}^p\mu_p>0$ and $\mu_{p+1},\dots,\mu_{p+r+1}\ge 0$. Let $\delta_i=\mu_i/\sum_{i=1}^p\mu_i$, $i=1,\dots,p$, and $\lambda_j=\mu_{p+j}/\sum_{i=1}^p\mu_i$, $j=1,\dots,q$. Dividing $\sum_{i=1}^p\mu_p$ on both sides of (4.1) and noting that, for all $x\in\mathbb{R}^n$,

$$\left(\max_{1\le i\le p}\{f_i(x)\},\dots,\max_{1\le i\le p}\{f_i(x)\},g_1(x),\dots,g_{r+1}(x)\right)\in C,$$

we obtain that, for all $x\in\mathbb{R}^n$,

$$\begin{aligned}&\frac{1}{2}x^TAx+\max_{1\le i\le p}\left\{a_i^Tx+\alpha_i\right\}+\lambda_1\left(\|x-x_0\|^2-\rho\right)+\sum_{j=1}^r\lambda_{j+1}\left(b_j^Tx+\beta_j\right)\\&\quad=\max_{1\le i\le p}\{f_i(x)\}+\sum_{j=1}^{r+1}\lambda_jg_j(x)\ge\mathrm{val}(\tilde{P})-\epsilon.\end{aligned}$$

In particular, this shows that $A+2\lambda_1I_n\succeq 0$ (otherwise, there exists $d\in\mathbb{R}^n$ such that $d^T(A+2\lambda_1I_n)d<0$. This implies that, for each fixed $a\in\mathbb{R}^n$,

$$\max_{1\le i\le p}\{f_i(a+td)\}+\sum_{j=1}^{r+1}\lambda_jg_j(a+td)\to-\infty\text{ as }t\to\infty,$$

which is impossible). Define $h:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ by

$$h(x,\delta)=\frac{1}{2}x^TAx+\sum_{i=1}^p\delta_i\left(a_i^Tx+\alpha_i\right)+\lambda_1(\|x-x_0\|^2-\rho)+\sum_{j=1}^r\lambda_{j+1}\left(b_j^Tx+\beta_j\right).$$

Then, $h(x,\cdot)$ is affine for each fixed $x \in \mathbb{R}^n$ and $h(\cdot,\delta)$ is convex for all $\delta \in \mathbb{R}^p$. So, the convex minimax theorem shows that

$$\begin{aligned}
&\max_{\delta\in\Delta_p}\inf_{x\in\mathbb{R}^n} h(x,\delta)\\
&\quad= \inf_{x\in\mathbb{R}^n}\max_{\delta\in\Delta_p} h(x,\delta)\\
&\quad= \inf_{x\in\mathbb{R}^n}\left\{\frac{1}{2}x^TAx + \max_{1\le i\le p}\left\{a_i^Tx+\alpha_i\right\} + \lambda_1\left(\|x-x_0\|^2-\rho\right) + \sum_{j=1}^r \lambda_{j+1}\left(b_j^Tx+\beta_j\right)\right\}\\
&\quad\geq \mathrm{val}(\tilde{P})-\epsilon.
\end{aligned}$$

Hence, there exists $\delta \in \Delta_p$ such that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
\sum_{i=1}^p \delta_i\left(\frac{1}{2}x^TAx + a_i^Tx+\alpha_i\right) + \lambda_1\left(\|x-x_0\|^2-\rho\right) + \sum_{j=1}^r \lambda_{j+1}\left(b_j^Tx+\beta_j\right) &= h(x,\delta)\\
&\geq \mathrm{val}(\tilde{P})-\epsilon,
\end{aligned}$$

where the first equality follows from the assumption that $\delta \in \Delta_p$. Thus, it follows from (2.10) that

(4.2)
$$\begin{pmatrix} A+2\lambda_1 I_n & \sum_{i=1}^p \delta_i a_i - 2\lambda_1 x_0 + \sum_{j=1}^r \lambda_{j+1}b_j \\ \left(\sum_{i=1}^p \delta_i a_i - 2\lambda_1 x_0 + \sum_{j=1}^r \lambda_{j+1}b_j\right)^T & 2\left(\sum_{i=1}^p \delta_i\alpha_i + \lambda_1(\|x_0\|^2-\rho) + \sum_{j=1}^r \lambda_{j+1}\beta_j - (\mathrm{val}(\tilde{P})-\epsilon)\right)\end{pmatrix} \succeq 0.$$

Now, using the same method of proof as in Theorem 2.1, we see that $\min(\tilde{P}) = \sup(\widetilde{SOCP})$. ∎

Next, we provide a lemma to show that problem $(\tilde{P})$ admits an exact SOCP relaxation under an abstract boundary attainment condition. A simple sufficient condition, expressed in terms of the original data, will be given after this abstract result.

LEMMA 4.2 (boundary attainment condition and exact SOCP relaxation). *For problem $(\tilde{P})$ and its relaxation problem $(\widetilde{SOCP})$, let*

$$\hat{f}(x) = \max_{1\le i\le p}\{\frac{1}{2}x^TAx + a_i^Tx + \alpha_i\}$$

and let

$$D = \{z\in\mathbb{R}^{r+1} : \|x-x_0\|^2 \le \rho + z_1, b_j^Tx+\beta_j \le z_{j+1}, j=1,\dots,r, \text{ for some } x\in\mathbb{R}^n\}.$$

Suppose that, for each $z=(z_1,\dots,z_{r+1})\in D$, the convex minimization problem

(4.3)
$$\min_{x\in\mathbb{R}^n}\left\{\hat{f}(x) - \frac{\lambda_{\min}(A)}{2}\|x-x_0\|^2 : \|x-x_0\|^2 \le \rho+z_1, b_j^Tx+\beta_j \le z_{j+1}, j=1,\dots,r\right\}$$

attains its minimum at some $\overline{x}\in\mathbb{R}^n$ with $\|\overline{x}-x_0\|^2 = \rho+z_1$. Then, $\min(\tilde{P}) = \sup(\widetilde{SOCP})$.

Proof. The conclusion will follow from Lemma 4.1 if we show that $E(f_1,\dots,f_p,g_1,\dots,g_{r+1})\cap(\Lambda\times\mathbb{R}^{r+1})$ is a convex set, where $\Lambda = \{(y_1,\dots,y_p)\in\mathbb{R}^p : y_1=\cdots=y_p\}$.

To see the convexity of $E(f_1,\dots,f_p,g_1,\dots,g_{r+1}) \cap (\Lambda \times \mathbb{R}^{r+1})$, we note that, if A is positive semidefinite, then f_i and g_j are convex functions, and so $E(f_1,\dots,f_p,g_1,\dots,g_{r+1}) \cap (\Lambda \times \mathbb{R}^{r+1})$ is always convex. Therefore, we may assume that A is not positive semidefinite and hence $\lambda_{\min}(A) < 0$. Recall that $\hat{f}(x) = \max_{1\le i\le p}\{\frac{1}{2}x^TAx + a_i^Tx + \alpha_i\}$ and

$$D = \{z \in \mathbb{R}^{r+1} : \|x-x_0\|^2 \le \rho + z_1, b_j^Tx + \beta_j \le z_{j+1}, j = 1,\dots,r, \text{ for some } x \in \mathbb{R}^n\}.$$

Clearly, D is a convex set. Define the value function h of $(\tilde{P})$ by

$$h(z) = \begin{cases} \min\{\hat{f}(x) : \|x-x_0\|^2 \le \rho + z_1, b_j^Tx + \beta_j \le z_{j+1}, j = 1,\dots,r\}, & z \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, $E(f_1,\dots,f_p,g_1,\dots,g_{r+1}) \cap (\Lambda \times \mathbb{R}^{r+1}) = \Pi(\text{epi}h)$, where epi$h$ denotes the epigraph of h and $\Pi : \mathbb{R}^{r+1} \times \mathbb{R} \to \mathbb{R}^p \times \mathbb{R}^{r+1}$ is a linear map given by $\Pi(z,t) = (te_p, z) = (t,\dots,t,z)$ for all $(z,t) \in \mathbb{R}^{r+1} \times \mathbb{R}$. As a linear map preserves convexity, D is convex and $h(z) = +\infty$ for all $z \notin D$; to see $E(f_1,\dots,f_p,g_1,\dots,g_{r+1}) \cap (\Lambda \times \mathbb{R}^{r+1})$ is convex, we only need to show that the value function h is convex on D. From our assumption, we have for each $z = (z_1,\dots,z_{r+1}) \in D$,

$$\begin{aligned} h(z) &= \min_{x\in\mathbb{R}^n}\{\hat{f}(x) : \|x-x_0\|^2 \le \rho + z_1, b_j^Tx + \beta_j \le z_{j+1}, j = 1,\dots,r\} \\ &\ge \min_{x\in\mathbb{R}^n}\left\{\hat{f}(x) - \frac{\lambda_{\min}(A)}{2}\|x-x_0\|^2 : \|x-x_0\|^2 \right. \\ &\qquad\qquad \left. \le \rho + z_1, b_j^Tx + \beta_j \le z_{j+1}, j = 1,\dots,r\right\} + \frac{\lambda_{\min}(A)}{2}(\rho + z_1) \\ &= \hat{f}(\bar{x}) - \frac{\lambda_{\min}(A)}{2}(\rho + z_1) + \frac{\lambda_{\min}(A)}{2}(\rho + z_1) = \hat{f}(\bar{x}) \ge h(z). \end{aligned}$$

Hence, for each $z \in D$,

$$\begin{aligned} h(z) = \min_{x\in\mathbb{R}^n}&\left\{\hat{f}(x) - \frac{\lambda_{\min}(A)}{2}\|x-x_0\|^2 : \|x-x_0\|^2 \right. \\ &\qquad \left. \le \rho + z_1, b_j^Tx + \beta_j \le z_{j+1}, j = 1,\dots,r\right\} \\ &+ \frac{\lambda_{\min}(A)}{2}(\rho + z_1). \end{aligned} \tag{4.4}$$

Note that

$$\begin{aligned} &\hat{f}(x) - \frac{\lambda_{\min}(A)}{2}\|x-x_0\|^2 \\ &\quad = \frac{1}{2}x^T(A - \lambda_{\min}(A)I_n)x + \max_{1\le i\le p} a_i^Tx + \lambda_{\min}(A)x_0^Tx - \frac{\lambda_{\min}(A)}{2}x_0^Tx_0 \end{aligned}$$

and $A - \lambda_{\min}(A)I_n$ is positive semidefinite, $\hat{f}(x) - \frac{\lambda_{\min}(A)}{2}\|x-x_0\|^2$ is convex. Thus, for each $z \in D$, the optimization problem (4.3) is a convex optimization problem. It then follows from (4.4), $h(z) = +\infty$ for all $z \notin D$, and the convexity of D that h is a convex function. Hence $E(f_1,\dots,f_p,g_1,\dots,g_{r+1}) = \Pi(\text{epi}h)$ is also a convex set. Now, the conclusion follows from Lemma 4.1. ∎

We show that a simple dimension condition, expressed in terms of the original data of the problem, guarantees exact SOCP relaxation for $(\tilde{P})$.

THEOREM 4.1 (generalized dimension condition for exact SOCP relaxation). *For problem* $(\tilde{P})$ *and its relaxation problem* $(\widetilde{SOCP})$*, define the matrix*

$$Q = (A - \lambda_{\min}(A) I_n, b_1, \ldots, b_r)^T \in \mathbb{R}^{(n+r)\times n}.$$

Suppose that $\dim\big(\mathrm{Ker} Q\big) \geq p$. *Then,* $\min(\tilde{P}) = \sup(\widetilde{SOCP})$.

Proof. The conclusion will follow from the preceding lemma if we show that the minimum of the auxiliary problem (4.3) is attained on the sphere. We do this by the method of contradiction. Assume to the contrary that any minimizer x^* of the problem

$$\min_{x\in\mathbb{R}^n} \left\{ \hat{f}(x) - \frac{\lambda_{\min}(A)}{2}\|x - x_0\|^2 : \|x-x_0\|^2 \leq \rho + z_1, b_j^T x + \beta_j \leq z_{j+1}, j = 1, \ldots, r \right\}$$

satisfies $\|x^* - x_0\|^2 < \rho + z_1$. Note that

$$\dim\big(\mathrm{Ker}Q\big) \geq p > p-1 \geq \dim \mathrm{span}\{a_1 + \lambda_{\min}(A)x_0, \ldots, a_{p-1} + \lambda_{\min}(A)x_0\}.$$

Then, there exists $\bar{v} \neq 0$ such that $\bar{v} \in \mathrm{Ker}Q$ and $(a_i + \lambda_{\min}(A)x_0)^T \bar{v} = 0$, $i = 1, \ldots, p-1$ (otherwise, $(\mathrm{Ker}Q) \cap L^{\perp} = \{0\}$ where

$$L = \mathrm{span}\{a_1 + \lambda_{\min}(A)x_0, \ldots, a_{p-1} + \lambda_{\min}(A)x_0\}$$

and $L^{\perp}$ is the orthogonal complement of L. This shows that

$$n+1 = p + (n-(p-1)) = \dim\big(\mathrm{Ker}Q\big) + \dim\big(L^{\perp}\big) = \dim\big(\mathrm{Ker}Q + L^{\perp}\big) \leq n,$$

which is impossible.) In particular, we have

$$\big(A - \lambda_{\min}(A)I_n\big)\bar{v} = 0 \text{ and } b_j^T \bar{v} = 0, j = 1, \ldots, r$$

and, hence, $\bar{v}$ is an eigenvector associated with $\lambda_{\min}(A)$. By replacing $\bar{v}$ with $-\bar{v}$ if necessary, we may assume that $(a_p + \lambda_{\min}(A)x_0)^T\bar{v} \leq 0$. It follows that

$$\max_{1\leq i\leq p} a_i^T \bar{v} + \lambda_{\min}(A)x_0^T\bar{v} = \max_{1\leq i\leq p} \big\{(a_i + \lambda_{\min}(A)x_0)^T\bar{v}\big\} \leq 0.$$

So, $\bar{v}$ is a nonzero vector that satisfies $\max_{1\leq i\leq p} a_i^T\bar{v} + \lambda_{\min}(A)x_0^T\bar{v} \leq 0$ and $b_j^T\bar{v} = 0$, $j = 1, \ldots, r$. Note that $\|x^* - x_0\|^2 < \rho + z_1$ and $b_j^T x^* + \beta_j \leq z_{j+1}$, $j = 1, \ldots, r$. Then we can find $t > 0$ such that $\|x^* + t\bar{v} - x_0\|^2 = \rho + z_1$ and $b_j^T(x^* + t\bar{v}) + \beta_j \leq 0$, $j = 1, \ldots, r$. Moreover, we have

(4.5)

$$\begin{aligned}
&\hat{f}(x^* + t\bar{v}) - \frac{\lambda_{\min}(A)}{2}\|x^* + t\bar{v} - x_0\|^2 \\
&= \frac{1}{2}(x^* + t\bar{v})^T A (x^* + t\bar{v}) + \max_{1\leq i\leq p}\big\{a_i^T(x^* + t\bar{v}) + \alpha\big\} - \frac{\lambda_{\min}(A)}{2}\|x^* - x_0\|^2 \\
&\quad - t\lambda_{\min}(A)(x^* - x_0)^T\bar{v} - \frac{t^2}{2}\lambda_{\min}(A)\bar{v}^T\bar{v} \\
&\leq \frac{1}{2}{x^*}^T A x^* + \max_{1\leq i\leq p}\big\{a_i^T x^* + \alpha\big\} - \frac{\lambda_{\min}(A)}{2}\|x^* - x_0\|^2 + t\left({x^*}^T A\bar{v} - \lambda_{\min}(A){x^*}^T\bar{v}\right) \\
&\quad + \frac{t^2}{2}\left(\bar{v}^T A \bar{v} - \lambda_{\min}(A)\bar{v}^T\bar{v}\right) + t\left(\max_{1\leq i\leq p} a_i^T\bar{v} + \lambda_{\min}(A)x_0^T\bar{v}\right).
\end{aligned}$$

Since $A\bar{v} = \lambda_{\min}(A)\bar{v}$ and $\max_{1\le i\le p} a_i^T\bar{v} + \lambda_{\min}(A)x_0^T\bar{v} \le 0$, we have,

$$\hat{f}(x^* + t\bar{v}) - \frac{\lambda_{\min}(A)}{2}\|x^* + t\bar{v} - x_0\|^2 \le \hat{f}(x^*) - \frac{\lambda_{\min}(A)}{2}\|x^* - x_0\|^2.$$

This shows that $x^* + t\bar{v}$ also solves the minimization problem and $\|x^* + t\bar{v} - x_0\|^2 = \rho + z_1$, which contradicts our assumption. ∎

In the special case where $p = 1$, that is, the quadratic optimization with the extended trust-region constraints, problem $(\tilde{P})$ reduces to

$$\begin{aligned}(\tilde{P}_1)\ \min_{x\in\mathbb{R}^n}\ & \frac{1}{2}x^TAx + a_1^Tx + \alpha_1\\ \text{s.t.}\ & \|x - x_0\|^2 \le \rho,\\ & b_j^Tx + \beta_j \le 0, j = 1,\dots,r.\end{aligned}$$

Its SOCP relaxations are given, respectively, by

$$\begin{aligned}(\widetilde{SOCP_1})\quad \sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_p,\lambda_j\ge 0\\ v\in\mathbb{R}^n, s_l\in\mathbb{R}}}\ & \mu\\ \text{s.t.}\quad & \sigma^l + 2\lambda_1 \ge 0, l = 1,\dots,n,\\ & v = U^T\left(a_1 - 2\lambda_1 x_0 + \sum_{j=1}^r \lambda_{j+1}b_j\right),\\ & 2\left(\alpha_1 + \lambda_1(\|x_0\|^2 - \rho) + \sum_{j=1}^r \lambda_{j+1}\beta_j - \mu\right) - \sum_{l=1}^n s_l \ge 0, s_l \ge 0,\\ & \left\|\left(2v_l, s_l - \sigma^l - 2\lambda_1\right)\right\| \le s_l + \sigma^l + 2\lambda_1,\ \ l = 1,\dots,n.\end{aligned}$$

As a corollary, we obtain an exact SOCP relaxation result for $(\tilde{P}_1)$ under the condition that "$\dim(\text{Ker}Q) \ge 1$, where $Q = (A - \lambda_{\min}(A)I_n, b_1,\dots,b_r)^T \in \mathbb{R}^{(n+r)\times n}$." We note that, in the special case where $b_j = 0$ and $r_j = 0$, $j = 1,\dots,r$, this condition is automatically satisfied, and so our result collapses to the exact SOCP relaxation result obtained in [10] for a standard quadratic optimization problem under a single quadratic constraint.

COROLLARY 4.1. *For problem* $(\tilde{P}_1)$ *and* $(\widetilde{SOCP_1})$*, define*

$$Q = (A - \lambda_{\min}(A)I_n, b_1,\dots,b_r)^T \in \mathbb{R}^{(n+r)\times n}.$$

Suppose that $\dim(\text{Ker}Q) \ge 1$*. Then,* $\min(\tilde{P}_1) = \sup(\widetilde{SOCP_1})$.

Proof. The conclusion follows from Theorem 4.1 by letting $p = 1$. ∎

Remark 4.1 (comparison with known dimension condition). Corollary 4.1 improves the recently established exact SDP relaxation result of the quadratic optimization problem with extended trust-region constraints in [21] where an exact SDP relaxation is obtained under a slightly stronger dimension condition:

$$(DC)\quad \dim\text{Ker}(A - \lambda_{\min}(A)I_n) \ge s + 1 \text{ with } s = \dim\text{span}\{b_1,\dots,b_r\}.$$

Here, we make two improvements. Firstly, we obtain exact SOCP relaxation (and so, in particular exact SDP relaxation) under a weaker condition. To see the dimension condition of Corollary 4.1 is weaker than the condition (DC), let $Q = (A - \lambda_{\min}(A)I_n, b_1, \ldots, b_r)^T \in \mathbb{R}^{(n+q-1)\times n}$ with $s = \dim \operatorname{span}\{b_1, \ldots, b_r\}$. From the condition (DC), there exists $z \in \mathbb{R}^n \backslash \{0\}$ such that

$$(A - \lambda_{\min}(A)I_n)z = 0 \text{ and } b_j^T z = 0,\ j = 1, \ldots, r.$$

As A is symmetric, $A - \lambda_{\min}(A)I_n$ is also symmetric. So, we have

$$Qz = \left((A - \lambda_{\min}(A)I_n)^T z, b_1^T z, \ldots, b_r^T z\right) = 0.$$

Thus, $\dim(\operatorname{Ker}Q) \geq 1$. Second, we showed that under the dimension condition of Corollary 4.1, an exact SOCP relaxation holds which is a stronger conclusion compared to the exact SDP relaxation result derived in [21].

Moreover, the following simple example shows that condition (DC) is, in general, strictly stronger than the condition "$\dim(\operatorname{Ker}Q) \geq 1$" used in Corollary 4.1 for a standard quadratic optimization problem.

Example 4.1. Consider $\min_{(x_1,x_2)\in\mathbb{R}^2}\{x_1^2 - x_2^2 : x_1^2 + x_2^2 \leq 1, -x_1 \leq 0\}$. Clearly, the optimal value of this problem equals -1. Moreover, in this case, $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $x_0 = 0$, $\rho = 1$, and $b_1 = (-1, 0)^T$. Then, $\dim \operatorname{Ker}(A - \lambda_{\min}(A)I_n) = \dim \operatorname{Ker}\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = 1$ and $s = \dim \operatorname{span}\{b_1\} = 1$. So, condition (DC) fails. On the other hand, $Q = (A - \lambda_{\min}(A)I_n, b_1)^T = \begin{pmatrix} 4 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$. So, $\dim \operatorname{Ker}(Q) = 1$.

Max dispersion problems. Recall that the max dispersion problem (or max-min location problem) over a polyhedral constraint with Euclidean metric or a ball constraint is given by [13, 18]:

$$\begin{aligned} (\tilde{P}_{MD}) \max_{x\in\mathbb{R}^n} \min_{1\leq i\leq p} \ & \|x - u_i\|^2 \\ \text{s.t.} \quad & \|x - x_0\|^2 \leq \rho, \\ & b_j^T x + \beta_j \leq 0, j = 1, \ldots, r, \end{aligned}$$

where u_i, $i = 1, \ldots, p$, b_j, $j = 1, \ldots, r$, and x_0 are given points in $\mathbb{R}^n$. Its SOCP relaxation problem can be formulated as

$$\begin{aligned} (RP_{MD}) \inf_{\substack{\mu\in\mathbb{R},\delta\in\Delta_p,\lambda_j\geq 0 \\ v\in\mathbb{R}^n, s_l\in\mathbb{R}}} & -\mu \\ \text{s.t.} \qquad & \lambda_1 \geq 1, \\ & v = \sum_{i=1}^{p} 2\delta_i u_i - 2\lambda_1 x_0 + \sum_{j=1}^{r} \lambda_{j+1} b_j, \\ & 2\left(\sum_{i=1}^{p} -\delta_i\|u_i\|^2 + \lambda_1(\|x_0\|^2 - \rho) + \sum_{j=1}^{r} \lambda_{j+1}\beta_j - \mu\right) - \sum_{l=1}^{n} s_l \geq 0, \\ & s_l \geq 0, \|(2v_l, s_l + 2 - 2\lambda_1)\| \leq s_l - 2 + 2\lambda_1,\ l = 1, \ldots, n. \end{aligned}$$

It was recently shown in [18] that the max dispersion problem (or max-min location problem) over a box constraint, is an NP-hard problem. Note that the max dispersion problem over a box constraint is a special case of $(\tilde{P}_{MD})$ with $x_0 = 0$, $\rho = \sqrt{n}$, $r = 2n$,

$b_j = e_j$, $\beta_j = -1$, $j = 1, \dots, n$, and $b_j = -e_j$, $\beta_j = -1$, $j = n+1, \dots, 2n$. So, $(\tilde{P}_{MD})$ is also an NP-hard problem, in general. Below, we provide a simple condition that guarantees exact SOCP relaxation for (P_{MD}).

Corollary 4.2 (max dispersion problem). *For problem $(\tilde{P}_{MD})$ and its relaxation problem (RP_{MD}), let $s = \operatorname{dimspan}\{b_1, \dots, b_r\}$. Suppose that $p + s \le n$. Then,* $\min(P_{MD}) = \sup(RP_{MD})$.

Proof. We note that $\min(P_{MD})$ equals the negative of the optimal value of the following minimax quadratic optimization problem:

$$\begin{aligned} \inf_{x \in \mathbb{R}^n} \max_{1 \le i \le p} \; & -\|x\|^2 + 2u_i^T x - \|u_i\|^2 \\ \text{s.t.} \quad & \|x - x_0\|^2 \le \rho, \\ & b_j^T x + \beta_j \le 0, j = 1, \dots, r. \end{aligned} \tag{4.6}$$

Let $A = -I_n$. As $p + s \le n$ with $s = \operatorname{dimspan}\{b_1, \dots, b_r\}$, we see that

$$Q := (A - \lambda_{\min}(A) I_n, b_1, \dots, b_r)^T = (0_{n\times n}, b_1, \dots, b_r)^T,$$

and $\operatorname{dimKer}(Q) = n - s \ge p$. So, applying Theorem 4.1 with $U = I_n$ implies that the optimal value of (4.6) equals the optimal value of the following SOCP problem

$$\begin{aligned} (RP_{MD}) \quad \sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_p,\lambda_j\ge 0\\ v\in\mathbb{R}^n, s_l\in\mathbb{R}}} \; & -\mu \\ \text{s.t.} \quad & \lambda_1 \ge 1, \\ & v = \sum_{i=1}^p 2\delta_i u_i - 2\lambda_1 x_0 + \sum_{j=1}^r \lambda_{j+1} b_j, \\ & 2\left(\sum_{i=1}^p -\delta_i \|u_i\|^2 + \lambda_1(\|x_0\|^2 - \rho) + \sum_{j=1}^r \lambda_{j+1}\beta_j - \mu\right) - \sum_{l=1}^n s_l \ge 0, \\ & s_l \ge 0, \|(2v_l, s_l + 2 - 2\lambda_1)\| \le s_l - 2 + 2\lambda_1, \; l = 1, \dots, n. \end{aligned}$$

Thus, the conclusion follows by noting that the negative of the optimal value of the above problem is equal to $\sup(RP_{MD})$. ∎

5. Applications to some robust quadratic problems. In this section we provide some robust quadratic optimization problems which admit exact SOCP relaxation under some easily verifiable conditions.

The first class of minimax quadratic programming problems with the generalized trust-region constraints arises when we study the following uncertain trust-region problem via robust optimization [9]:

$$\begin{aligned} (TR) \min_{x\in\mathbb{R}^n} \; & \frac{1}{2} x^T A x + a^T x + \alpha \\ \text{s.t.} \; & \|x - x_0\|^2 \le \rho, \\ & b_j^T x + \beta_j \le 0, j = 1, \dots, r, \end{aligned}$$

where the data $(A, a, \alpha) \in S^n \times \mathbb{R}^n \times \mathbb{R}$ are uncertain, A belongs to the spectral norm uncertainty set $\mathcal{V} = \{\bar{A} + M : \|M\|_{\text{spec}} \le \eta\}$ and (a, α) belongs to the polytope uncertainty set

$$\mathcal{W} = \left\{ (a_0, \alpha_0) + \sum_{i=1}^{p} u_j (a_i, \alpha_i) : (u_1, \ldots, u_p) \in \Delta_p \right\} \text{ for some } p \in \mathbb{N}. \tag{5.1}$$

Here, M_{spec} is the spectral norm given by $\|M\|_{\text{spec}} = \sqrt{\lambda_{\max}(M^T M)}$ and $\lambda_{\max}$ denotes the maximum eigenvalue; $\bar{A} \in S^n$, $\eta > 0$, and $(a_i, \alpha_i) \in \mathbb{R}^n \times \mathbb{R}$, $i = 0, 1, \ldots, p$. The robust counterpart of the problem (TR) can be formulated as

$$\begin{aligned} (TR_r) \min_{x \in \mathbb{R}^n} \max_{A \in \mathcal{V}, (a,\alpha) \in \mathcal{W}} & \left\{ \frac{1}{2} x^T A x + a^T x + \alpha \right\} \\ \text{s.t. } & \|x - x_0\|^2 \le \rho, \\ & b_j^T x + \beta_j \le 0, j = 1, \ldots, r. \end{aligned}$$

Let $\bar{A} = U \bar{\Sigma} U^T$, where U is an orthogonal matrix and $\bar{\Sigma} = \text{diag}(\sigma^1, \ldots, \sigma^n)$ is a diagonal matrix whose diagonal element σ^l, $l = 1, \ldots, n$, are eigenvalues of $\bar{A}$. We associate an SOCP relaxation problem for the above robust counterpart (TP_r) as follows:

$$\begin{aligned} (\widetilde{SOCP}) \sup_{\substack{\mu \in \mathbb{R}, \delta \in \Delta_p, \lambda_j \ge 0 \\ v \in \mathbb{R}^n, s_l \in \mathbb{R}}} & \mu \\ \text{s.t.} \quad & \sigma^l + \eta + 2\lambda_1 \ge 0, l = 1, \ldots, n, \\ & v = U^T \left(a_0 + \sum_{i=1}^{p} \delta_i a_i - 2\lambda_1 x_0 + \sum_{j=1}^{r} \lambda_{j+1} b_j \right), \\ & 2 \left(\alpha_0 + \sum_{i=1}^{p} \delta_i \alpha_i + \lambda_1 (\|x_0\|^2 - \rho) + \sum_{j=1}^{r} \lambda_{j+1} \beta_j - \mu \right) - \sum_{l=1}^{n} s_l \ge 0, \\ & s_l \ge 0, \|(2v_l, s_l - \sigma^l - \eta - 2\lambda_1)\| \le s_l + \sigma^l + \eta + 2\lambda_1, \ l = 1, \ldots, n, \end{aligned}$$

where $\bar{A} = U \bar{\Sigma} U^T$ and $\bar{\Sigma} = \text{diag}(\sigma^1, \ldots, \sigma^n)$.

THEOREM 5.1 (robust trust-region problems). *For problem (TR_r) and its SOCP relaxation problem $(SOCP_r)$, let*

$$Q = \left(\bar{A} - \lambda_{\min}(\bar{A}) I_n, b_1, \ldots, b_r \right)^T \in \mathbb{R}^{(n+r) \times n}.$$

Suppose that $\dim(\text{Ker}Q) \ge p$*, where* p *is defined as in* (5.1)*. Then, we have* $\min(TR_r) = \sup(SOCP_r)$.

Proof. We first note that $\max_{A \in \mathcal{V}} x^T A x = x^T (\bar{A} + \eta I_n) x$ for all $x \in \mathbb{R}^n$. Moreover, as $(A, \alpha) \mapsto x^T A x + \alpha$ is linear and any linear function attains its minimum over a polytope on some extreme points of the polytope, we have

$$\max_{(a,\alpha) \in \mathcal{V}} \left\{ a^T x + \alpha \right\} = \left(a_0^T x + \alpha_0 \right) + \max_{1 \le i \le p} \left\{ a_i^T x + \alpha_i \right\} \text{ for all } x \in \mathbb{R}^n.$$

So, problem (TR_r) can be equivalently rewritten as

$$\min_{x\in\mathbb{R}^n} \max_{1\le i\le p} \left\{ \frac{1}{2}x^T(\bar{A}+\eta I_n)x + (a_0+a_i)^T x + (\alpha_0+\alpha_i) \right\}$$
$$\text{s.t. } \|x-x_0\|^2 \le \rho,$$
$$b_j^T x + \beta_j \le 0, j = 1,\ldots,r.$$

The above problem can be rewritten as a form of $(\tilde{P})$ with A replaced by $\bar{A}+\eta I_n$, a_i replaced by a_0+a_i, and α_i replaced by $\alpha_0+\alpha_i$ $i=1,\ldots,p$. Note that $U(\Sigma+\eta I_n)U^T = \bar{A}+\eta I_n$ and

$$\left(\bar{A}+\eta I_n - \lambda_{\min}(\bar{A}+\eta I_n)I_n, b_1,\ldots,b_r\right)^T = \left(\bar{A} - \lambda_{\min}(\bar{A})I_n, b_1,\ldots,b_r\right)^T.$$

Thus, the conclusion follows from Theorem 4.1. ◻

Finally, to illustrate our example in the introduction, we establish the exact SOCP relaxation for a robust least squares optimization problem under polytope objective uncertainty. Recall that the weighted least squares optimization problem can be stated as

$$(LS) \min_{x\in\mathbb{R}^n} \sum_{l=1}^n w_l x_l^2 + \sum_{l=1}^n u_l x_l + \alpha$$
$$\text{s.t. } \sum_{l=1}^n \gamma_{lj} x_l^2 \le r_j,\ j=1,\ldots,q,$$

where, $w_l, \gamma_{lj}, r_j \in \mathbb{R}$, $l=1,\ldots,n$, $j=1,\ldots,q$, $u\in\mathbb{R}^n$, and $\alpha\in\mathbb{R}$. Let $w = (w_1,\ldots,w_n)^T \in \mathbb{R}^n$. Suppose that the coefficients $(w_1,\ldots,w_n)\in\mathbb{R}^n$ of the objective function are uncertain and they belong to a polytope uncertainty set given by $w \in \mathcal{U}_p := \text{co}\{w^1,\ldots w^p\}$. Then, the robust counterpart of (LS) can be formulated as

$$(RLS) \min_{x\in\mathbb{R}^n} \max_{w\in\mathcal{U}_p} \left\{ \sum_{l=1}^n w_l x_l^2 + \sum_{l=1}^n u_l x_l + \alpha \right\}$$
$$\text{s.t. } \sum_{l=1}^n \gamma_{lj} x_l^2 \le r_j,\ j=1,\ldots,q,$$

which is equivalent to the following minimax quadratic optimization problem

$$(RLS_1) \min_{x\in\mathbb{R}^n} \max_{1\le i\le p} \left\{ \sum_{l=1}^n w_l^i x_l^2 + \sum_{l=1}^n u_l x_l + \alpha \right\}$$
$$\text{s.t. } \sum_{l=1}^n \gamma_{lj} x_l^2 \le r_j,\ j=1,\ldots,q.$$

We note that the example in the introduction is a special case of (RLS) with $u_l = 0$, $l=1,\ldots,n$.

As all the Hessian matrices of the associated quadratic functions in (RLS_1) are diagonal matrices and, hence, (RLS_1) is a special case of problem (P) with $U = I_n$.

Its associated SOCP relaxation is

$$(SOCP_{RLS}) \quad \sup_{\substack{\mu\in\mathbb{R},\delta\in\Delta_p,\lambda_j\geq 0\\ v\in\mathbb{R}^n, s_l\in\mathbb{R}}} \mu$$

$$\text{s.t.} \quad \sum_{i=1}^p \delta_i w_l^i + \sum_{j=1}^q \lambda_j \gamma_{lj} \geq 0, l=1,\ldots,n,$$

$$2\left(\alpha - \sum_{j=1}^q \lambda_j r_j - \mu\right) - \sum_{l=1}^n s_l \geq 0, s_l \geq 0, \tag{5.2}$$

$$\left\|\left(2u_l, s_l - \sum_{i=1}^p \delta_i w_l^i - \sum_{j=1}^q \lambda_j\gamma_{lj}\right)\right\| \leq s_l + \sum_{i=1}^p \delta_i w_l^i + \sum_{j=1}^q \lambda_j \gamma_{lj}, \; l=1,\ldots,n.$$

Corollary 5.1 (robust weighted least squares problems under polytope uncertainty). *For the problem* (RLS) *with* $r_j > 0$, $j=1,\ldots,q$, *and its SOCP relaxation problem* $(SOCP_{RLS})$, *we have* $\inf(RLS) = \max(SOCP_{RLS})$. *In addition, if we further assume that there exists* $j_0 \in \{1,\ldots,q\}$ *such that* $\gamma_{lj_0} > 0$ *for all* $l=1,\ldots,n$, *then* $\min(RLS) = \max(SOCP_{RLS})$.

Proof. Let $f_i(x) = \sum_{l=1}^n w_l^i x_l^2 + \sum_{l=1}^n u_l x_l + \alpha$, $i=1,\ldots,p$, and $g_j(x) = \sum_{l=1}^n \gamma_{lj} x_l^2 - r_j$, $j=1,\ldots,q$. We now verify that $E(f_1,\ldots,f_p,g_1,\ldots,g_q)$ is convex. To see this, using the variable transform $x_l = -\text{sign}(u_l)\sqrt{s_l}$, one can directly verify that

$$\begin{aligned} &E(f_1,\ldots,f_p,g_1,\ldots,g_q)\\ &= \left\{\left(\sum_{l=1}^n w_l^1 s_l + \sum_{l=1}^n (-|u_l|\sqrt{s_l}) + \alpha, \ldots \sum_{l=1}^n w_l^p s_l + \sum_{l=1}^n (-|u_l|\sqrt{s_l}) + \alpha,\right.\right.\\ &\quad \left.\left.\sum_{l=1}^n \gamma_{l1}s_l - r_1 \ldots, \sum_{l=1}^n \gamma_{lq} s_l - r_q : s_l \geq 0\right\}\right. + \mathbb{R}_+^{p+q}. \end{aligned}$$

Note that $t \mapsto -\sqrt{t}$ is convex on $\mathbb{R}_+$. This implies that, for all $i=1,\ldots,p$, $\hat{f}_i(s) = \sum_{l=1}^n w_l^i s_l - \sum_{l=1}^n |u_l|\sqrt{s_l} + \alpha$ are convex on $\mathbb{R}_+^n$ and, for all $j=1,\ldots,q$, $\hat{g}_j(s) = \sum_{l=1}^n \gamma_{lj} s_l - r_j$ are affine. So, $E(f_1,\ldots,f_p,g_1,\ldots,g_q)$ is a convex set. Thus, the first conclusion follows from Proposition 2.1.

If we further assume that there exists $j_0 \in \{1,\ldots,q\}$ such that $\gamma_{lj_0} > 0$ for all $l=1,\ldots,n$, then the feasible set is compact. Thus, the second conclusion follows. ∎

6. Conclusion. In this paper, we have established exact SOCP relaxations for nonconvex minimax separable quadratic optimization problems with multiple separable quadratic constraints under an epigraphical condition. We exploited hidden convexity in the form of a convex epigraphical set to achieve our results. We have also provided various classes of minimax problems for which our results hold under easily verifiable conditions.

In the cases where exact SDP relaxations are not possible, convergent hierarchies of SDP relaxations are known [26, 11] for various classes of hard nonconvex optimization problems. It would be of interest to study hierarchies of SOCP relaxations for minimax quadratic optimization problems with multiple quadratic constraints. This will form an interesting topic for further study.

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