The intrinsic topology of linear maps

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Abstract

Based on many experts' former work in the Jacobian conjecture and an essential analysis of intrinsic topology of linear maps, I completely prove the Jacobian conjecture by demonstrating the injectivity of real Keller map of any n-dimensions.

1 Origin and Motivation

Notation:
$\mathbb{C}$: the field of complex numbers; $\mathbb{R}$: the field of real numbers; $\mathbb{Q}$: the field of rational numbers;
C^1: the set of differentiable functions with continuous derivation. $\det A$: the determinant of matrix A;
$JF(x)$: The Jacobian matrix of $F(x)$; $F'(x)$: the derivation of a map F;
$||\cdot||$: the Euclidean norm of $\mathbb{R}^n$;
$JC(R,n)$: n-dimensional Jacobian conjecture over a commutative ring R of characteristic 0.

In 1939, Ott-Heinrich Keller proposed the following question in [Kel39]: Given polynomials $f_1, \cdots, f_n \in \mathbb{Z}[x_1, \cdots, x_n]$ such that $\det J(F) = 1$, where $J(F)$ denotes the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})_{(i,j)}$, can every x_i be expressed as a polynomial in $f_1, \cdots, f_n$ with coefficients in $\mathbb{Z}$? Keller's original question now is known as famous Jacobian Conjecture:

Conjecture 1.1. (Jacobian Conjecture) *Let F be $k^n \to k^n$ a polynomial map, where k is a field of characteristic 0. If the determinant for its Jacobian of the polynomial map is a non-zero constant, i.e.,* $\det JF(x) \equiv c \in k^*$, $\forall x \in k^n$. *Then $F(x)$ is invertible and has a polynomial inverse map.*

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The condition $\det J(F) \in k^*$ is called the **Jacobian condition** and a polynomial map $F : k^n \to k^n$ satisfying the Jacobian condition is called a **Keller map**. The generalized Jacobian conjecture is to replace k by a commutative ring R contained in $\mathbb{Q}$-algebra and consider $F : R[x_1, \cdots, x_n]^n \to R[x_1, \cdots, x_n]^n$, i.e. the every component of F is a polynomial in n-variables with coefficients in R. Obviously, Keller's original problem and Jacobian conjecture are the special cases by taking R as $\mathbb{Z}$, a field of characteristic 0, respectively. In fact, by Lefschetz Principle, the Jacobian conjecture for field $\mathbb{C}$ of complex number implies the generalized Jacobian conjecture (see. page 23, [Ess]). Hence, It is sufficient to consider Jacobian conjecture for the field $k = \mathbb{C}$ of complex numbers. However, for the case real field $\mathbb{R}$, its statement is a little different.

Conjecture 1.2. (Real Jacobian Conjecture) *(RJC) If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map,* $\det J(F)(x)$ *is not zero in $\mathbb{R}^n$, then F is a injective map.*

The real Jacobian conjecture with weaker Jacobian condition has weaker conclusion: injectivity instead of bijection. It is a pity that that is false and Pinchuk [Pin94] constructed a counterexample to (RJC) for $n = 2$.

If we only see the formulation of Jacobian conjecture, it seems as if it is not very interesting or not important . Let us point out that the conjecture is a generalization of several fundamental questions in analysis and algebra.

The first question is Rolle's theorem in analysis. Rolle's theorem state: let F be a continuous function in a real closed interval $[a, b]$ and differentiable in real open interval (a, b). if the function F satisfies $F(a) = F(b)$, then exists a constant $c \in (a, b)$ such that $F'(c) = 0$. A natural generalization is to ask if it happen in higher dimension, i.e. let $F : \mathbb{R}^n \to \mathbb{R}^n$, if $F'(x) \neq 0$, F is a injective map? Or in more general case for $\mathbb{C}^n$. For n-dimensional case, the non-zero determinant of Jacobian matrix of F is naturally the generalization of $F'(x) \neq 0$. The Jacobian conjecture is just the higher dimensional generalization of Real Rolle's theorem for polynomial maps.

The second generalization is from Cramer's Rule in linear algebras. Cramer's Rule says: Given a group of linear equations over a field k (such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), denote the group of n linear equations in n-variables by the matrix form: $AX = \alpha$ with the matrix A of coefficients and $\alpha \in k^n$. Then the group of linear equations has a unique solution in k^n if and only if $\det A \neq 0$. A natural question is that a group of n-equations $\{f_i = \alpha_i\}_{i=1}^n$in n-variables (not necessarily linear) over a field k , what is the conditions that the group of equations has a solution or a unique solution for any $\alpha = (\alpha_1, \cdots, \alpha_n)$ in k^n? The question is just the Jacobian conjecture and the determinant of Jacobian matrix of n-equations $(f_1, \cdots, f_n)$ is the generalization of the matrix of coefficients in linear case.

The third generalization is from complex analysis. The fundamental theorem of algebra says that a non-constant polynomial map $F : \mathbb{C} \to \mathbb{C}$ is a surjective map. Observe that a non-constant polynomial map means $F'(x) \not\equiv 0$. Furthermore, if $F'(x)$ is a non-zero constant, then F is a linear, injective map and hence, surjective . Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. When is the map F surjective or injective for higher dimension? The question is still the Jacobian conjecture answering the properties of F by considering the Jacobian condition instead of derivation of F in one dimension..

By the three basic starting-points, we can sense that the conjecture is fundamentally important.

2 Known results and Jacobian conjecture in other subjects

There are too many experts who already made contributions to the Jacobian conjecture. In [Ess], A. Van dan den Essen already introduced the conjecture from many aspects and pointed out amount of connections with other fields in mathematic. Also, a lot of good references are listed, which is very wonderful resource to researchers. Still, I will continue to mention several topics related to the conjecture and some recent progress for integrity of logic.

(**2**.**1**)The Jacobian conjecture is famous in algebraic geometry because of Abhyankar's work on the formal inversion formula (see[Abh]). He can constructed a formal inversion, i.e. an formal power series by using differential operators for a polynomial map. The inversion formula was first discovered by Guajar (unpublished). Their formula now is called Abhyankar-Gurjar formula which is simplified by Bass , Connell and Wright (see [BCW82]). Since the formula is from utilization of differential operators, so the method is related to D-modules (see page 263,[Ess]).

(**2**.**2**) By Bass, Connell, and Wright in [BCW82], they proved the following theorem: If the $JC(\mathbb{C}, n)$(or $JC(\mathbb{R}, n)$, not confusing with the Real Jacobian conjecture) for all $n \geqslant 2$ and all polynomial map of the form $I + H$ where I is the identity and H is a cubic homogenous, then $JC(\mathbb{C}, n)$ (or $JC(\mathbb{R}, n)$) holds. Furthermore, Drużkowski in [DRT] proved that it is sufficient to prove an$JC(\mathbb{C}, n)$ (or $JC(\mathbb{R}, n)$) for all $n \geqslant 2$ and all special polynomial maps of form $F = I + H = (x_1 + H_1, \cdots, x_n + H_n)$ with $H_i = (\sum a_{ij}x_j)^3, i = 1, \cdots, n$.

(**2**.**3**) From the topological point of view, Gutierrez and Maquera in [GuMa] proved that if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a polynomial map with $\det J(F) \neq 0$ everywhere in $\mathbb{R}^3$ such that $\mathrm{Spec}(F) \bigcap [0, \epsilon) = \emptyset$, for some $\epsilon > 0$, and $\mathrm{codim}(S_F \geqslant 2)$ where $\mathrm{Spec}(F)$ is a set of eigenvalue of the Jacobian of F and S_F is the set of points on which F is not proper, then F is bijective. In [FMV], they have obtained somehow general results by using the semi-algebraic maps instead of polynomial maps.

(**2**.**4**) The equivalence of the Jacobian conjecture, the Diximier conjecture and Poisson conjecture.

We firstly make some introduction for these conjecture. Let R be a commutative ring with identity 1 and n a positive integer. The polynomial ring over R in n-variables $x_1, \cdots, x_n$ is denoted by $R[x_1, \cdots, x_n]$. THe n-th $Weyl\ alegbra$ over R, denote by $A_n(R)$, is the associative R-algebra with generators $y_1, \cdots, y_{2n}$ and relations $[y_i, y_{i+n}] = 1$ for all $i \leqslant i \leqslant n$ and $[y_i, y_j] = 0$ otherwise. Dixmier Conjecture claims: For every n, every endomorphism of $A_n(\mathbb{C})$ is an automorphism.

The n-th Poisson algebra $P_n(R)$ over R is the polynomial ring $R[x_1, \cdots, x_{2n}]$ endowed with the canoical Poisson bracket $\{,\}$ defined by

$$\{f, g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_{i+n}} - \frac{\partial f}{\partial x_{i+n}}\frac{\partial g}{\partial x_i})$$

A R-endomorphism φ of $R[x_1, \cdots, x_{2n}]$ is called an endomorphism of $P_n(R)$ if φ preserves Poisson bracket $\{,\}$, i.e. φ with $\{\varphi(f), \varphi(g)\} = \{f, g\}$ for all f, g in $R[x_1, \cdots, x_{2n}]$. Poisson Conjecture claims: For every n, every endomorphism of $P_n(\mathbb{C})$ is an automorphism.

We know that Dixmier conjecture implies the Jacobian conjecture over $\mathbb{C}$ (see [Ess] and [BCW82]). Also, Tsuchimoto (see[Tsu]) have proved that conversely, $JC(\mathbb{C}, 2n)$ implies Dixmier conjecture of n-dimension by p-curvature method. Independently, Belov and Kontsevich proved that the Jacobian conjecture is stably equivalent to the Dixmier conjecture (see [BeKon]), whose proof is completely written by the language of algebraic geometry with talented ideas. Inspired by the work of [BeKon], Essen by using Poisson conjecture prove all these three conjecture are equivalent, whose proof is purely algebraic method.

(**2.5**) Jacobian conjecture in dynamics. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field with $F(0) = 0$. Consider the system of $\begin{cases} \dot{x}_1(t) = F_1(x_1(t), \cdots, x_n(t)), \\ \qquad \vdots \\ \dot{x}_n(t) = F_n(x_1(t), \cdots, x_n(t)), \end{cases}$

ordinary differential equations in brief (2.5.1) $\qquad \dot{x}(t) = F(x(t))$.

Observe that $x(t) = 0$ is a solution of (2.5.1), which we call an equilibrium of (2.5.1). We say that 0 is a global attractor of (2.5.1) if every solution of this system tends to 0 if t tends to infinity.

Markus – Yamabe Conjecture (**MYC**). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field with $F(0) = 0$ satisfying the so-called Markus-Yamabe Assumption, i.e. (MYA): for all $x \in \mathbb{R}^n$ the real parts of all eigenvalues of $J(F)(x)$ are negative then 0 is a global attractor of (2.5.1) .

A result given by Fournier and Martelli (see Page 178, [Ess]): If the MYC is true for all polynomial vector fields of $\mathbb{R}^n$ with degree $\leqslant 3$ and for all $n \geqslant 2$, then the Jacobian conjecture is true. Later, by Meisters and Olech (see page 180, [Ess]), they proved that the two dimension MYC is true for polynomial vector fields. However, by Essen and Hubbers (collaboration with Cima, Gasull and Manñosas), they gave polynomial counter-examples to MYC for all $n \geqslant 3$ (see section 8.3 in [Ess]). In spite of the fact, there are still a lot meaningful work to do in this topic.

(**2.6**) Jacobian conjecture in Quantum Field Theory. Quantum Field Theory (for short, QFT) is a important branch in mathematical physics. QFT is so gigantic and is important to state it in several words, but it is very closed related complex Jacobian conjecture. In [Abdes], A.Abdesselam tried to attacked the Complex Jacobian Conjecture by formal inverse method of perturbative Quantum field theory. In [Tan], A Tanasa recalled that standard QFT method and treat how to apply QFT method to a combinatorial QFT reformulation of the Jacobian conjecture on the invertibility of polynomial systems. From their papers, it seems that the Jacobian conjecture play a important role in QFT.

3 The proof and some byproducts

First of all, we are back to the title of this paper-"The intrinsic topology of linear maps". We will obtain the heuristic proof for the injectivity of Keller maps from linear maps in $\mathbb{R}^1$ by considering the intrinsic topology of linear maps.

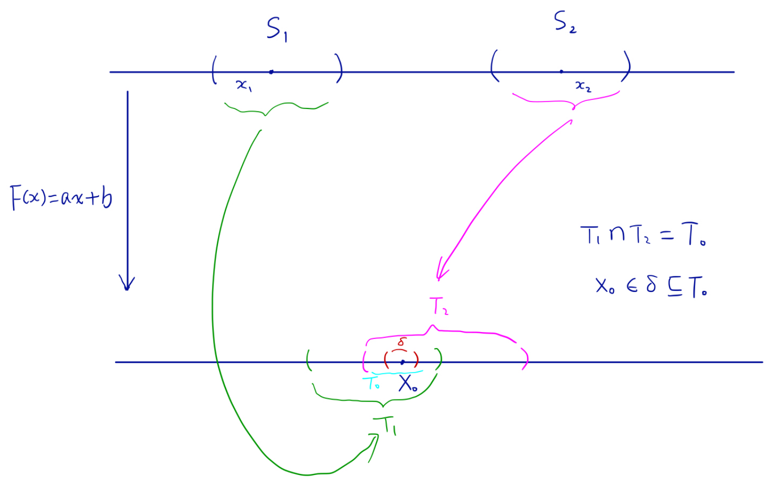
"**Linear**" is originally a terminology from the algebra while "**Injective**" is a terminology more from the topology. By the property of linearity to realize the injectivity, actually, it is a translation of definitions between different branches in math, i.e. from the algebra to the topology. In $\mathbb{R}^1$, this procedure is very simple. However, in higher dimension, it seems that it is completely non-trivial if we want from the algebraic information of a Keller map directly to get topological injective. So, by analyzing the simplest case, i.e. linear maps in $\mathbb{R}^1$, we try to find out the topological information of linear maps which can induce injectivity and also would be valid for Keller maps in any dimension.

In brief, how to prove the injectivity of linear maps $\mathbb{R}^1$ by adopting the topological approach (the algebraic way is very trivial)? If we find such a topological way, can we apply it to Keller maps? The answer is affirmative.

A linear map have topological information including length-preserving map (see Def.3.1 on length-preserving map), open map, closed map, C^1-map, which already can induces injectivity. Actually, a map which is length-preserving map, open map, and C^1-map already ensures the map is injective. For Keller maps, we also try to prove that it is volume-preserving map, open map, and C^1-map so that we can obtain the injectivity of the map.

Let us firstly treat linear case in $\mathbb{R}^1$. Let $F = ax + b : \mathbb{R} \to \mathbb{R}$ be a linear map with $a \neq 0$ a non-zero constant and b any constant. Now, we will use topological property of linear maps to prove F is injective. The geometric picture is as below so as to have a better understanding for the proof.

If there exist $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = X_0$. Then there exist two



open neighbourhoods S_1 of x_1 and S_2 of x_2 such that $S_1 \bigcap S_2 = \emptyset$. Denote $F(S_i) = T_i$ for $i = 1, 2$ and $X_0 \in T_1 \bigcap T_2$ (denote their intersection by T_0). Therefore, $T_1 \bigcap T_2 = T_0$ is a non-empty open subset of $\mathbb{R}^1$ since the linear map F is a open map. Denote the length of the open neighbourhood (in the sense of Lebesegue measure) by $L()$. So there is a open neighbourhood δ of X_0 with $L(\delta) > 0$ by the property of open set.Then $L(T_1)+L(T_2) = |a|(L(S_1)+L(S_2)) = |a|L(S_1 \bigcup S_2)$ because of $S_1 \bigcap S_2 = \emptyset$. Moreover, $|a|L(S_1 \bigcup S_2) = L(T_1 \bigcup T_2)$, since a linear map is a length-preserving map. Then

$$\begin{aligned}
L(T_1) + L(T_2) &= |a|L(S_1 \bigcup S_2) \\
&= L(T_1 \bigcup T_2) \\
&= L((T_1 \backslash T_0) \bigcup (T_2 \backslash T_0) \bigcup T_0) \\
&= L(T_1 \backslash T_0) + L(T_2 \backslash T_0) + L(T_0) \\
&< L(T_1 \backslash T_0) + L(T_2 \backslash T_0) + L(T_0) + L(\delta) \\
&\leqslant L(T_1 \backslash T_0) + L(T_2 \backslash T_0) + 2L(T_0) \\
&= L(T_1) + L(T_2)
\end{aligned}$$

i.e. $L(T_1) + L(T_2) < L(T_1) + L(T_2)$, this is a contradiction. This prove that a linear map is injective.

Before proceeding to our proof, we need to recall the property of n-dimensional Lebesgue measure under some special maps.

Definition 3.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. F is called a volume-preserving map if for any non-empty measurable set $\Omega \in \mathbb{R}^n$, the image $F(\Omega)$ has same measure with Ω up to a fixed non-zero constant c, i.e. $m(F(\Omega)) = cm(\Omega)$, where $m()$ represents the Lebesgue measure of a measurable set.

For example, F is the usual area-preserving map for $n = 2$ and the length-preserving map for $n = 1$. There are many references about measure theory , e.g. "Measure and integration" by Dietmar A. Salamon. If F is the Keller map for $\mathbb{R}^n$, the obvious conclusion is that non-zero constant $\det J(F)$ is equivalent to the volume-preserving map F. In order to make clarity, we prove the following proposition.

Proposition 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. Then F is a volume-preserving map if and only if* $\det J(F)$*, the determinant of Jacobian is a non-zero constant. In particular, the equivalence is true for the Keller map.*

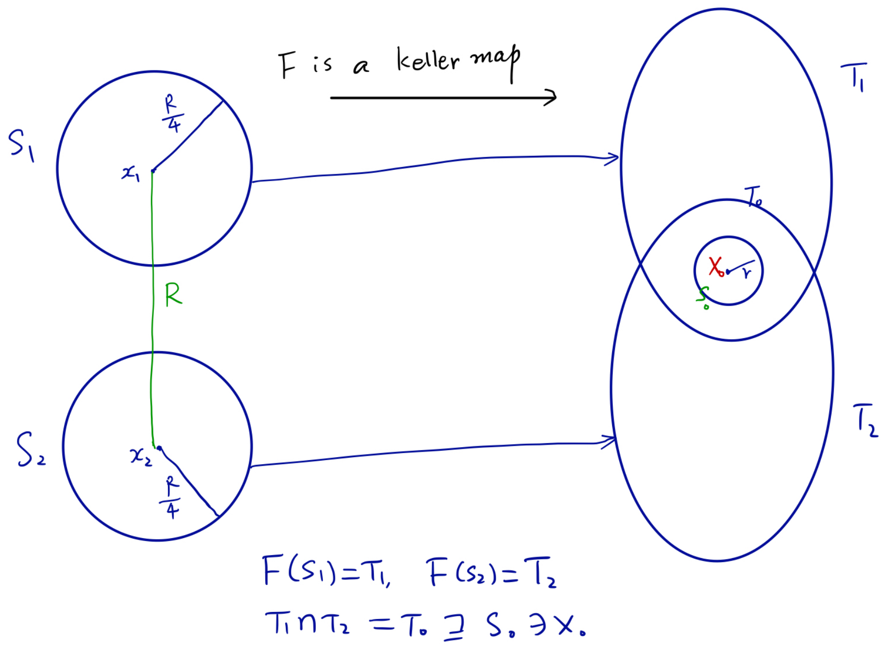
Proof. Sufficiency is almost trivial. Let $\Omega \subseteq \mathbb{R}^n$ be any non-empty measurable set and $V = F(\Omega) = \{F(x)|x \in \Omega \subseteq \mathbb{R}^n\}$. Then $v_1 = m(V) = \int_V dv = \int_\Omega dF(\Omega) = \int_\Omega \det J(F)dx$ where dv (resp. dx) is the corresponding element of volume and the volume of V (resp. Ω) is denoted by v_1 (resp. u_1). But $\det J(F) = c$ for a non-zero constant c implies $v_1 = cu_1$.

On the contrary, $v_1 = \int_V dv = \int_\Omega \det J(F)dx = c \int_\Omega dx = cm(\Omega)$ for any non-empty measurable set Ω and a fixed constant c. Then $\int_\Omega(\det J(F) - c)dx = 0$ holds for any non-empty measurable set Ω. If $\det J(F) \not\equiv c$ in $\mathbb{R}^n$, i.e. $a = \det J(F)(x_0) - c > 0$ at some x_0 which will not loss generality, then we can take a neighbourhood Ω_1 of x_0 with measure $m(\Omega_1) > 0$ of x_0 such that $\det J(F)(x) - c > a/100$ in the Ω_1 since $\det J(F)(x)$ is continuous in $\mathbb{R}^n$. Furthermore, $\int_{\Omega_1}(\det J(F) - c)dx > a/100\ m(\Omega_1) > 0$ will induces contradiction with the equality $\int_\Omega(\det J(F) - c)dx = 0$ for any non-empty measurable set Ω. This proves its necessity. $\square$

Now, I can prove the injectivity of Keller map by similar method used in linear maps.

Theorem 3.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map with Jacobian determinant* $\det J(F) \equiv c \neq 0$*. If F is an open map under Euclidean topology, then F is an injective map.*

Proof. According to Prop. 3.2 , F is a volume-preserving map. We prove the injectivity of the map by contradiction.The picture nearby will help us have a better geometry instuition. Let a point $X_0 \in \mathbb{R}^n$ in the image of F. If there are two different points x_1, x_2 such that $F(x_1) = F(x_2) = X_0$ with distance $R > 0$, then there exist two open balls $S_i = \{x \in \mathbb{R}^n| \ ||x - x_i|| < R/4\}$ for $i = 1, 2$. Obvious, the two balls have empty intersection, i.e. $S_1 \bigcap S_2 = \emptyset$. Denote the images of S_i under the map F by T_i, i.e. $T_i = F(S_i)$. Their images are open



subsets in $\mathbb{R}^n$, since F is a open map. Observe that $X_0 \in T_1 \bigcap T_2$ and denote their intersection by $T_0 = T_1 \bigcap T_2$. Then T_0 is non-empty open subset.

Because X_0 is in an open subset T_0, we can take an open ball $S_0 = \{x \in T_0 | \; ||x - X_0|| < r\}$ for some $r > 0$ by the definition of open set. In the one hand, $m(T_1) + m(T_2) = m(F(S_1)) + m(F(S_2)) = c\ (m(S_1) + m(S_2)) = c\, m(S_1 \bigcup S_2)$ by the volume-preserving of F and $S_1 \bigcap S_2 = \emptyset$. In another hand, $c\ m(S_1 \bigcup S_2) = m(F(S_1 \bigcup S_2)) = m((T_1 \backslash T_0) \bigcup (T_2 \backslash T_0) \bigcup T_0) = m(T_1 \backslash T_0) + m(T_2 \backslash T_0) + m(T_0) < m(T_1 \backslash T_0) + m(T_2 \backslash T_0) + m(T_0) + m(S_0) \leqslant m(T_1 \backslash T_0) + m(T_2 \backslash T_0) + 2m(T_0) = m(T_1) + m(T_2)$, i.e. $m(T_1) + m(T_2) < m(T_1) + m(T_2)$, which is a contradiction. Hence, F is a injective map. $\square$

For a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^p$, the well-known result (see page 132 in [Nara]) is that F is open if and only if its fibers have pure dimension $n - p$. The result in the real case for F from $\mathbb{R}^n$ to itself is the following theorem proved by J.Gamboa and F. Ronga in [GR96].

Theorem 3.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map. Denote* $\det J(F) : \mathbb{R}^n \to \mathbb{R}$ *the Jacobian determinant of F. Then F is an open map if and only if the fibers of F are finite and the sign of* $\det J(F)$ *does not change (i.e.* $\det J(F)(x) \geqslant 0$*, for $\forall x \in \mathbb{R}^n$ or* $\det J(F)(x) \leqslant 0$*, for $\forall x \in \mathbb{R}^n$).*

For the Keller map, its Jacobian determinant is a non-zero constant in $\mathbb{R}$, so its Jacobian determinant does not change in $\mathbb{R}^n$. In order to ensure that the Keller map F is an open map, we still need to show the fibers of the Keller map F are finite. On the finiteness of fibers of F, this is already proved by M. Drużkowski and K. Tutaj (see Lemma 3.1, [DRT]). Actually, their results is more than finiteness. We directly cite it as my proposition without proof.

Proposition 3.5. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map such that $\det J(F)(x) \neq 0$ for every $x \in \mathbb{R}^n$. Then for every $a \in \mathbb{R}^n$, the equation $F(x) = a$ has only isolated solutions and*

$$\sharp\{x \in \mathbb{R}^n : F(x) = a\} \leqslant (degf_1) \cdot \cdots \cdot (degf_n),$$

where we denote the degree of f_i in n-varibles by $degf_i$.

Remark 3.6. It is very interesting to observe that A. Fernandes, C. Maquera and J. Venato-Santos, gave a more general result (see Cor. 2.5 [FMV])by introducing semi-algebraic set and semi-algebraic map. Their result is : If $F : \mathbb{R}^n \to \mathbb{R}^n$ is semi-algebraic local homeomorphism, then there exists $k \in \mathbb{N}$ such that the cardinality of the fibers of F: $\sharp F^{-1}(p) \leqslant k$ for all $p \in \mathbb{R}^n$. The semi-algebraic condition is somehow general than polynomial and local homeomorphism condition is corresponding to the Jacobian condition. In the paper (loc. cit.), they consider the Jacobian conjecture from the angle of topology by using foliation and semi-algebraic knowledge, which is worthy to reading.

Therefore, the Keller map is an open map and also injective map by Th.3.3. Finally, we can prove the conjecture:

Theorem 3.7. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a Keller map, then F is invertible and the inverse of F is also a polynomial map.*

Proof. Białynicki-Birula and Rosenlicht in [BBR] proved that if F is injective for Keller maps, then F is invertible. Cynk and Rusek in [CR91] proved if F is invertible then the inverse of F is again a polynomial map. Therefore, our preoccupation is to treat the global injectivity of F.

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a complex polynomial map: $(z_1, \cdots, z_n) \mapsto (f_1, \cdots, f_n)$ and consider the associated real polynomial map $\widetilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which sends $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ to $(Ref_1, Imf_1, \cdots, Ref_n, Imf_n)$, where $z_k = x_k + iy_k, Ref_i$ and Imf_i are the real part and imaginary part of f_i. Then their determinant of Jacobian satisfies $\det J(\widetilde{F}) = |\det J(F)|^2$. Therefore, F is a Keller map if and only if $\widetilde{F}$ is a keller map. Also, F is a injective map if and only if $\widetilde{F}$ is a injective map. Hence, by [BBR] , if we can prove that the real Keller map F of any $2n$-dimension is a injective, then complex Jacobian conjecture of n-dimension is true. By Th.3.3, we can get the injectivity of real Keller maps for any dimension, which induces complex Jacobian conjecture in any dimension. □

The following table is an analogue made between linear maps and Keller maps to let us see how to convert the algebraic definition into topological properties and to get required topological property: injective and an additional topological property :surjective

(Tab.1)

Map in $\mathbb{R}^{(*)}$	Algebraic Description	Possessing Top. Properties	Required Top. property	Additional Top. Property
Linear maps in $\mathbb{R}$	$F'(x)$ is a non-zero const.	C^1-map, open map, closed map, length-preserving	Injective	Surjective
Keller maps in $\mathbb{R}^n$	$\det J(F)$ is a non-zero const.	C^1-map, open map, volume-preserving, closed map(?)	Injective	Surjective

At present. I can not directly prove that the real Keller map is a closed map. The real Keller map is an open map from Theo.3.4, which is a key ingredient to our proof. The Keller map can be viewed as 'geometrical linear map', although I do not define the terminology: geometrical linear map.

Remark 3.8. Our proof for Jacobian conjecture is from an analogue of topological properties of linear maps. As it is known, for Jacobian conjecture in two dimension, there are a lot of papers in which researchers try to find inverse of Keller maps with not too large degree. Their results still are valuable.

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References

[Abdes] A. Abdesselam. *The Jacobian Conjecture as a Problem of Perturbative Quantum Field Theory.* Ann. Henri Poincaré 4 (2003) 199-215.

[Abh] S.S. Abhyankar, *Lectures on Expansion Techniques in Algebraic Geometry.* Part Two: The Jacobian Problem, Tata Institute of Fundamental Research, Bombay, 1977.

[BCW82] H. Bass, E. Connell and D. Wright. *The Jacobian Conjecture: reduction of degree and formal expansion of the inverse.* Bull. Amer. Math. Soc. 7 (1982), no. 2, 287-330.

[BeKon] A. Belov and M. Kontsevich,*Jacobian Conjecture is stably equivalent to Dixmier Conjec-ture,* Moscow Mathematical Journal volume 7 (2007), issue 2, 209-218.

[BBR] A. Białyniki-Birula and M.Rosenlicht. *Injective morphisms of real algebraic varieties.* Proc. Amer. Math. Soc. 13(2) (1962), 200–203.

[CR91] Cynk and K. Rusek. *Injective endomorphisms of algebraic and analytic sets.* Ann. Polonici Math. 56, no 1, (1991).

[DRT] L. Drużkowski. *An effective approach to Keller's Jacobian Conjecture.* Math. Ann. 264 (1983),303–313 .

[DRT] L. Drużkowski and H. Tutaj. *Differential conditions to verify the Jacobian Conjecture.* Ann.Polon. Math. 46 (1992), 85–90.

[Ess] A. van den Essen. *Polynomial automorphsims and Jacobian Conjecture* Proc. Amer. Math. Soc. 133 (2005), no. 8, 2201-2205.

[FMV] A. Fernandes, C. Maquera and J. Venato-Santos. *Jacobian conjecture and semi-algebraic maps.* Mathematical Proceedings of the Cambridge Philosophical Society.Vol.157. Issue 02 Sept. 2014. pp 221-229.

[GR96] J. M. Gamboa and F. Ronga. *On open real polynomial maps.* Journal of Pure and Applied Algebra 110 (1996) 297-304.

[GuMa] C. Gutierrez and C. Maquera. *Foliations and polynomial diffeomorphisms of* $\mathbb{R}^3$. Math. Zeits. 262 (2009), 613-626.

[Jel] Z. Jelonek. *Geometry of real polynomial mappings.* Math. Zeitschrift 239 (2002), 321–333.

[Kel39] O. H. Keller. *Ganze Gremona-Transformation.* Monats. Math. Physik 47 (1939), 299-306.

[Nara] R. Narasimhan, *Introduction to the Theory of Analytic Spaces*, Lecture Notes in Mathematics, Vol. 25 (Springer, Berlin).

[Tsu] Y. Tsuchimoto, *Endomorphisms of Weyl algebra and p-curvatures*, Osaka J. Math. 42, (2005), 435-452.

[Pin94] S. Pinchuk. *A counterexamle to the strong real Jacobian conjecture.* Math. Zeitschrift. 217 (1994), 1-4.

[Tan] A. Tanasa *Combinatorial quantum field theory and the Jacobian conjecture*. https://arxiv.org/pdf/2002.07453.pdf.

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