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Positive line bundles on arithmetic surfaces

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Introduction

In [1] Arakelov introduced an intersection theory on an arithmetic surface with the purpose of applying techniques from algebraic geometry to arithmetic problems. Then in [3] Faltings proved a Riemann–Roch theorem and an index theorem as analogues of some properties of an algebraic surface. Furthermore in [7] and [8] Szpiro studied in detail the relation of numerical properties of relative dualizing sheaves to some effective versions of the Mordell conjecture.

In this paper we obtain several results concerning Szpiro's work in [7] and [8].

From Sections 1 to 5 we prove a Nakai–Moishezon theorem on an arithmetic surface: A hermitian line bundle is ample if and only if it is numerically positive. This was conjectured by Szpiro. The statement of this result will be given as Theorem 1.3. In the proof of this theorem we will use a result of Faltings in Section 2 on the existence of effective sections and a result of Tian on Fubini–Study metrics in Section 3. The proof of the theorem will be given in Sections 4 and 5. From Section 6 to Section 9 we prove an analogue of Bogomolov's conjecture about the discreteness of algebraic points on an algebraic curve. This analogue gives discreteness with respect to the distance function induced from an embedding of the curve into a multiplicative group. The precise statement of the result will be given as Theorem 6.2. The main known result, Faltings's index theorem for arithmetic surfaces, will be quoted in Section 7. The theorems in Section 6 will be proved in the last two sections.

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1. Ampleness and numerical positivity

By an arithmetic variety X of dimension $d+1$ we mean an integral scheme X, whose structure morphism $f : X \to \operatorname{spec} \mathbf{Z}$ is projective, flat and of pure relative dimension d, and whose generic fiber $X_{\mathbf{Q}}$ is regular. If $d = 0$, we say that X is an arithmetic curve. If $d = 1$, we say that X is an arithmetic surface. The arithmetic varieties we deal with in this paper always have $d \leq 1$.

A hermitian line bundle $\bar{L}$ on an arithmetic variety is presented by a couple $(\bar{L}, \| \ \|)$, where L is an invertible sheaf on X and $\| \ \|$ is a continuous hermitian metric on $L_{\mathbf{C}}$, which is invariant under the complex conjugation of $X_{\mathbf{C}}$. If X is an arithmetic surface and l is a nonzero meromorphic section of $\bar{L}_{\mathbf{C}}$, then we have a linear function

$$\omega(\bar{L}) = \frac{d'd''}{\pi i} \log \|l\| + \delta_{\operatorname{div}(l)}$$

on the space $\mathcal{C}^{\infty}(X_{\mathbf{C}})$ of smooth functions on $X_{\mathbf{C}}$, where d' and d'' are distributions associated to $\partial, \bar{\partial}$. It is easy to check that $\omega(\bar{L})$ does not depend on the choice of l. We call $\omega(\bar{L})$ the curvature form of $\bar{L}$. In this paper we always assume that $\omega(\bar{L})$ can be extended to a continuous function on the space $\mathcal{C}(X_{\mathbf{C}})$ of continuous functions on X with the supremum norm.

For a smooth hermitian line bundle $\bar{L}$ on a complex manifold M, the curvature $\omega(\bar{L}) = \partial\bar{\partial}/\pi i \log \|l\|$ is said to be positive (resp. semipositive) if for any nonzero tangent vector v of type (1,0) at any point p of M we have $\omega(-iv \wedge \bar{v}) = -v\bar{v}/\pi \log \|l\| > 0$ (resp. ≥ 0), where l is a local section of $\bar{L}$ near p such that $l(p) \neq 0$. For a hermitian line bundle $\bar{L} = (L, \| \ \|)$ (with a continuous metric) on an arithmetic surface X we say that the curvature $\omega(\bar{L})$ is semipositive if there is a sequence of metrics $\| \ \|_n$ with semipositive curvature on $L_{\mathbf{C}}$ such that

(1) $\lim_n \| \ \|_n / \| \ \| = 1$ uniformly on $X_{\mathbf{C}}$, and

(2) $\lim_n \int f\omega(L, \| \ \|_n) = \int f\omega(\bar{L})$ for any continuous function f on $X_{\mathbf{C}}$.

The divisor in this paper always means the Weil divisor on the scheme X. A divisor is called vertical if all of its generic points are contained in the special fibers of X, and a divisor is called horizontal if all of its generic points are contained in the generic fiber of X.

Definition 1.1. Let X be an arithmetic surface and let $\bar{L}$ be a hermitian line bundle on X. A nonzero section l of $\bar{L}$ on X is effective if $\|l\|(x) \leq 1$ for any $x \in X_{\mathbf{C}}$, and it is strictly effective if $\|l\|(x) < 1$ for any $x \in X_{\mathbf{C}}$.

We say that $\bar{L}$ is effective (resp. strictly effective) if there is a section of $\bar{L}$ which is effective (resp. strictly effective).

We say that $\bar{L}$ is ample if L is ample, $\omega(\bar{L})$ is semipositive and there is a basis of $\Gamma(L^{\otimes n})$ over $\mathbf{Z}$ consisting of strictly effective sections for all sufficiently large n.

Let X be an arithmetic curve and $\bar{L}$ be a hermitian line bundle on X. The degree of $\bar{L}$ can be defined as follows: Let l be a section of $\bar{L}$; then

$$\deg(\bar{L}) = \log \frac{|\Gamma(L)/\Gamma(l \cdot O_X)|}{\prod_{x \in X_{\mathbf{C}}} \|l\|(x)} = \deg[\operatorname{div} l] - \sum_{x \in X_{\mathbf{C}}} \log \|l\|(x).$$

Let X be an arithmetic surface and $\bar{L}$ and $\bar{M}$ be two hermitian line bundles. Deligne [2] defined the intersection number $\bar{L} \cdot \bar{M}$ as follows: Choose sections $\bar{l}$ and $\bar{m}$ of $\bar{L}$ and $\bar{M}$, respectively, such that $[\operatorname{div} l_{\mathbf{Q}}]$ and $[\operatorname{div} m_{\mathbf{Q}}]$ are disjoint. Then define

$$\bar{L} \cdot \bar{M} = [\operatorname{div} l] \cdot [\operatorname{div} m] - \langle \|l\|, \|m\| \rangle_{\mathbf{C}},$$

where

$$\langle \|l\|, \|m\| \rangle_{\mathbf{C}} = \int_X \frac{d'd''}{\pi i} \log \|l\| \cdot \log \|m\| + \log \|l\| [\operatorname{div} m] + \log \|m\| [\operatorname{div} l].$$

We need to show that $\langle \|l\|, \|m\| \rangle_{\mathbf{C}}$ makes sense here. Choose a smooth metric $\| \ \|'$ on M. It follows that

$$\langle \|l\|, \|m\| \rangle_{\mathbf{C}} = \int \log \|l\| \omega(M, \| \ \|') + \int \log \frac{\| \ \|}{\| \ \|'} \omega(\bar{L}) + \log \|m\|' [\operatorname{div} l].$$

The first term of the right-hand side makes sense, since $\omega(M, \| \ \|')$ is smooth. The second term makes sense by the hypothesis on $\bar{L}$, since $\log \| \ \| / \| \ \|'$ is a continuous function.

Definition 1.2. Let X be an arithmetic surface and let $\bar{L}$ be a hermitian bundle on X. We say that $\bar{L}$ is positive (resp. semipositive) if $\omega(\bar{L})$ is semipositive, $\bar{L} \cdot \bar{L}$ is positive (resp. nonnegative) and $\deg \bar{L}|_D$ is positive (resp. nonnegative) for any integral divisor D.

We say that $\bar{L}$ is relatively positive (resp. relatively semipositive) if $\deg(L|_F) > 0$ (resp. ≥ 0) for any irreducible vertical divisor F and $\omega(\bar{L})$ is semipositive.

We say that $\bar{L}$ is horizontally positive (resp. horizontally semipositive) if $\deg(\bar{L}|_D) > 0$ (resp. ≥ 0) for any irreducible horizontal divisor D.

The Nakai–Moishezon arithmetic theorem, which we will prove in this section, is given as follows:

THEOREM 1.3. *Let X be an arithmetic surface and let $\bar{L}$ be a hermitian line bundle on X. Then $\bar{L}$ is ample if and only if $\bar{L}$ is positive.*

Remark 1.4. This theorem was conjectured by Szpiro in [8], §2.2, and in a letter to the author. The original form of his conjecture is slightly different than Theorem 1.3 and can be stated as follows: If $\bar{L}$ is relatively semipositive and horizontally positive, then, for any irreducible divisor D, one can find an effective section l in some positive power $\bar{L}^{\otimes n}$ such that $l|_D \neq 0$. This original conjecture and Theorem 1.3 are both induced by the following result:

THEOREM 1.5. *Let X be an arithmetic surface and let $\bar{L}$ be a hermitian line bundle on X which is relatively semipositive and horizontally positive. Assume that $L_{\mathbf{Q}}$ is positive and $\bar{L}^2 > 0$. Then there is a positive integer N such that $\mathrm{H}^0(X, L^{\otimes N})$ has a basis consisting of strictly effective sections.*

Proof of Theorems 1.5 $\Rightarrow$ 1.3. The "only if" part of Theorem 1.3 follows directly from our definitions. We have to prove the "if" part. Applying Theorem 1.5, we need only show that L is ample provided that $\bar{L}$ is positive. It is enough to prove that for any line bundle M, for n sufficiently large, $L^{\otimes n} \otimes M$ has no base point.

We claim first that, for any effective divisor D, the bundle $L^{\otimes n} \otimes M|_D$ is ample for some positive n. Since the normalization $\pi : D' \to D$ is finite, it is enough to prove that the pullback L' on D' of $L^{\otimes n} \otimes M|_D$ is ample. By the same reasoning, we reduce the proof of ampleness to its restriction L'' on D'_{red}. We know that D'_{red} is a disjoint sum of smooth curves C_i. The degrees of L'' on each C_i equal those of L' on their images C_i, respectively, which are positive. This implies that L'' is ample on D'_{red}.

Secondly we claim that $\mathrm{H}^1(L^{\otimes n} \otimes M) = 0$ for $n \gg 0$. For this we choose an n sufficiently large that $L^{\otimes n} \otimes M$ has no H^1 on X_p for any p. From the exact sequence

$$0 \to L^{\otimes n} \otimes M \to L^{\otimes n} \otimes M \to L^{\otimes n} \otimes M|_{X_p} \to 0,$$

the morphism

$$p : \mathrm{H}^1(X, L^{\otimes n} \otimes M) \to \mathrm{H}^1(X, L^{\otimes n} \otimes M)$$

is surjective. Notice that $H = \mathrm{H}^1(X, L^{\otimes n} \otimes M)$ is a finite-type module of $\mathbf{Z}$. This implies that p is invertible in H for each p. Hence we must have $H = 0$.

Now we are ready to prove that $L^{\otimes n} \otimes M$ does not have a base point for $n \gg 0$. Since $\mathrm{H}^1(L^{\otimes n} \otimes M) = 0$ for $n \gg 0$, the restriction map

$$\Gamma(X, L^{\otimes n} \otimes M) \to \Gamma(X_p, L^{\otimes n} \otimes M)$$

is surjective; especially we have a nontrivial section l for $L^{\otimes n} \otimes M$. Also the restriction map

$$\Gamma(X, L^{\otimes n} \otimes M) \to \Gamma(\operatorname{div} l, L^{\otimes n} \otimes M)$$

is surjective for n sufficiently large. Choose n so that $\Gamma(\operatorname{div} l, L^{\otimes n} \otimes M)$ has no base point in $\operatorname{div} l$. Then $L^{\otimes n} \otimes M$ does not have any base point. □

For a lattice Γ in a normed real vector space we define the following two numbers $\mu(\Gamma)$ and $\lambda(\Gamma)$: The number $\mu(\Gamma)$ (resp. $\lambda(\Gamma)$) is the smallest number r such that the ball $B(r)$ of radius r contains a basis of Γ (resp. a basis of a sublattice of Γ of full rank). For a hermitian line bundle $\bar{L}$ on an arithmetic variety, let $\| \|_{sup}$ be the supremum norm on $\Gamma(\bar{L}_{\mathbf{R}})$ on $X_{\mathbf{C}}$, where $\Gamma(L_{\mathbf{R}})$ is considered as the invariant subspace of $\Gamma(L_{\mathbf{C}})$ under the complex conjugation of $X_{\mathbf{C}}$. Then the numbers $\mu(\Gamma(L))$ and $\lambda(\Gamma(L))$ are defined.

To conclude this section we will prove this lemma for ampleness:

LEMMA 1.6. *Let $\bar{L}$ be a hermitian line bundle over an arithmetic variety such that $L_{\mathbf{Q}}$ is ample. The following conditions are equivalent:*
(1) $\mu(\Gamma(L^{\otimes n})) < 1$ for $n \gg 0$;
(2) $\lambda(\Gamma(L^{\otimes n})) < 1$ for $n \gg 0$.

We must prove the following lemma first:

LEMMA 1.7. *Let $s_1, \dots, s_n$ be linear independent elements of full rank in a free abelian group G. Then a basis $e_1, \dots, e_n$ of G can be found such that, in $G \otimes \mathbf{Q}$, $e_i = \sum a_{ij} s_j$ with $|a_{ij}| \leq 1$.*

Proof. We use induction on the $n = \operatorname{rank} G$. It is trivial for $n = 0$. Assume that $n > 0$. Let n_1 be the largest integer such that $e_1 = s_1/n_1$ is in G. Then $G' = G/\mathbf{Z}e_1$ is a free group and the images $s'_2, \dots, s'_n$ of $s_2, \dots, s_n$ are of independent elements of G' with full rank. By induction we can find a basis $e'_2, \dots, e'_n$ of G with the form $e'_i = \sum_{j>1} a'_{ij} s'_j$ in $G' \otimes \mathbf{Q}$ such that $|a'_{ij}| \leq 1$. Let f_i $(1 < i \leq n)$ be elements in Γ that have images e'_i. Then in $G \otimes \mathbf{Q}$ the elements $\sum_{j>1} a'_{ij} s_j - f_i$ are contained in $\mathbf{Q}e_1$, namely equal to $c_i e_1$. It is easy to check that $\{e_1, e_i = f_i + [c_i]e_1 = \sum_{j>1} a'_{ij} s'_j + ([c_i] - c_i)e_1 : i > 1\}$ is a required basis for G. □

By Lemma 1.7, for the normed lattice Γ we have the following inequality:

$$\lambda(\Gamma) \leq \mu(\Gamma) \leq \operatorname{rank}_{\mathbf{Z}}(\Gamma)\lambda(\Gamma).$$

Proof of Lemma 1.6. The proof for conditions (1) $\to$ (2) is trivial. Assume condition (2). Since $L_{\mathbf{Q}}$ is ample, it follows that $\lambda(\Gamma(\bar{L}^{\otimes n})) < \exp(-n\epsilon)$ for some $\epsilon > 0$ when $n \gg 0$. Consequently, for $n \gg 0$,

$$\mu(\Gamma(\bar{L}^{\otimes n})) \leq \operatorname{rank}\Gamma(L^{\otimes n})\exp(-\epsilon n) < 1$$

by the Riemann–Roch theorem. □

2. Faltings's theorem on the existence of effective sections

Faltings [3] proved that if $\bar{L}$ is an admissible hermitian line bundle, in Arakelov's sense, and $\bar{L}\cdot\bar{L} > 0$, then there is a section with an L^2 norm less than 1. Recently Gillet–Soulé improved Faltings's result using an $L^2 - L^\infty$ comparison inequality. The final result is quoted as follows:

THEOREM 2.1 (Faltings, Gillet–Soulé). *Let X be an arithmetic surface and $\bar{L}$ be a hermitian line bundle on X such that $L_{\mathbf{Q}}$ is positive, $\bar{L}^2$ is positive and $\omega(\bar{L})$ is semipositive. Then $\bar{L}^{\otimes n}$ is strictly effective for $n \gg 0$.*

Proof. When $\omega(\bar{L})$ is smooth and positive, this is just the result given in [4]. We can eliminate the positive condition for $\omega(\bar{L})$ as follows: Let $\|\ \|_n$ be a sequence of smooth metrics on L, which have semipositive curvatures, such that $\|\ \|_n$ is convergent to the metric $\|\ \|$ of $\bar{L}$, and $\omega(L, \|\ \|_n)$ is convergent to $\omega(L, \|\ \|)$ as a distribution on $C(X_{\mathbf{C}})$. Let $\|\ \|'$ be a smooth positive metric on L. Replacing $\|\ \|_n$ by $\|\ \|^{1-1/n}\|\ \|'^{1/n}$, we may assume that $\omega(L, \|\ \|_n)$ is positive. Let ϵ be any positive number and let $\bar{L}_{n,\epsilon}$ be the hermitian line bundle $(L, \|\ \|_n e^{\epsilon})$. Then the sequence $\bar{L}_{n,\epsilon}^2$ has the limit $\bar{L}^2 - 2\epsilon\deg(L_{\mathbf{Q}})$. Choose ϵ sufficiently small; we may assume that this limit is positive. Choose an n_0 sufficiently large that $\|\ \| \leq \|\ \|_{n_0} e^{\epsilon}$ and $\bar{L}_{n_0,\epsilon}^2 > 0$. Then, for sufficiently large d, the power $\bar{L}_{n_0,\epsilon}^{\otimes d}$ has a strictly effective section s. It is easy to check that s is also a strictly effective section of $\bar{L}^{\otimes d}$. □

Using this theorem, one might prove that if $\bar{L}$ is relatively and horizontally semipositive, then $\bar{L}^2 \geq 0$; i.e., $\bar{L}$ is semipositive. See Section 8 for the details.

To conclude this section we would like to prove an analogue of Theorem 2.1 for an arithmetic curve.

THEOREM 2.2. *Let X be an arithmetic curve and let $\bar{L}$ be a hermitian line bundle on X. Then $\bar{L}$ is ample (i.e., $\mu(\bar{L}^{\otimes n}) < 1$ for $n \gg 0$) if and only if $\deg\bar{L}$ is positive.*

We need the Riemann–Roch formula for an arithmetic curve before giving the proof of Theorem 2.2. Let us define the Euler characteristic $\chi(\bar{L})$ of $\bar{L}$ as follows:

$$\chi(\bar{L}) = -\log\operatorname{vol}(L) = -\log\operatorname{vol}\big(\Gamma(\bar{L})\otimes_{\mathbf{Z}}\mathbf{R}/\Gamma(\bar{L})\big).$$

Then we have the Riemann–Roch formula:

LEMMA 2.3. $\chi(\bar{L}) = \deg \bar{L} + \chi(O_X)$.

Proof. Let l be a section of L. This gives a morphism $l : \Gamma(O_X) \to \Gamma(L)$ and therefore also gives a morphism of two metrized spaces $l_{\mathbf{R}} : \Gamma(O_X) \otimes \mathbf{R} \to \Gamma(L) \otimes \mathbf{R}$. The ratio of volume forms is $\prod_{x \in X_{\mathbf{C}}} l(x)$. It follows that

$$\mathrm{vol}(L) = \frac{\mathrm{vol}(\Gamma(\bar{L}) \otimes \mathbf{R}/\Gamma(l \cdot O_X))}{|\Gamma(L)/\Gamma(l \cdot O_X)|} = \mathrm{vol}(O_X) \frac{\prod_{x \in X_{\mathbf{C}}} \|l\|(x)}{|\Gamma(L)/\Gamma(l \cdot O_X)|}.$$

This gives the Riemann–Roch formula by the application of $-\log$ on both sides. □

Proof of Theorem 2.2. If $\bar{L}$ is ample, then for some integer $n > 0$, $\bar{L}^{\otimes n}$ has a strictly effective section l. This implies that

$$\deg \bar{L} = \frac{1}{n} \deg \bar{L}^{\otimes n} = \frac{1}{n} \left[\deg \operatorname{div} l - \sum_{x \in X_{\mathbf{C}}} \|l\|(x) \right] > 0.$$

Conversely, if $\deg \bar{L}$ is positive, then, by Lemma 2.3, $\chi(\bar{L}) \to \infty$ as $n \to \infty$. So by Minkowski's convex-body theorem, $\bar{L}^{\otimes n}$ has a strictly effective section l for a positive integer n. Letting $\{l_1, l_2, \ldots, l_N\}$ be a basis for $\Gamma(\bar{L})$ over $\mathbf{Z}$, we can choose a positive integer m such that $l^m l_i$, $i = 1, 2, \ldots, N$, are all strictly effective sections of $\bar{L}^{\otimes(mn+1)}$. Since $\Gamma(\bar{L}^n)$ has the same rank for all n, these sections already form a basis of a sublattice of $\Gamma(\bar{L}^{\otimes(mn+1)})$ with full rank. It follows that $\bar{L}$ is ample by Lemma 1.6. □

3. Tian's theorem on Fubini–Study metrics

Let X be a compact complex manifold and let $\bar{L}$ be a hermitian line bundle with smooth metric and positive curvature. Denote by $\mathrm{d}x$ the measure induced from $\omega(\bar{L})$ on X. We take a norm $\|\ \|_n$ on $\Gamma(\bar{L}^{\otimes n})$ as follows:

$$\|\ \|_n = \int_X \|l\|^2(x) \mathrm{d}x,$$

where $\|\ \|$ is the metric on $\bar{L}^{\otimes n}$.

Let $\{l_1, l_2, \ldots, l_N\}$ be an orthonormal basis for $\Gamma(\bar{L}^{\otimes n})$. For $n \gg 0$, by the Kodaira embedding theorem, $L^{\otimes n}$ is very ample. Especially we have $\sum \|l_i\|_n^2(x) \neq 0$. We define the Fubini–Study metric $\|\ \|_n'$ on $\bar{L}^{\otimes n}$ by the formula

$$\|l\|_n'(x) = \frac{n^{\dim(X)/2} \|l\|_n(x)}{\sqrt{\sum \|l_i\|_n^2(x)}}.$$

One easily checks that the metric here does not depend on the choice of the orthonormal basis. The following theorem is Lemma 3.2(i) of [9].

THEOREM 3.1 (Tian). *Uniformly on X,*

$$\frac{\|\ \|_n}{\|\ \|'_n} = 1 + O\left(\frac{1}{n}\right).$$

Before using this theorem to give some estimates on restriction maps of a positive hermitian line bundle, we introduce the following notation:

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two normed spaces over $\mathbf{R}$ and let $\alpha : \mathcal{V}_1 \to \mathcal{V}_2$ be a surjective morphism. Let us denote by $\|\ \|_i$ the norm on $\mathcal{V}_i$. We define

$$q(\alpha) = q(\mathcal{V}_1 \to \mathcal{V}_2) = \sup_{x \in \mathcal{V}_2} \inf_{y \in \alpha^{-1}(x)} \log \frac{\|y\|_1}{\|x\|_2}.$$

THEOREM 3.2. *Let X be a compact complex manifold and let Y be a finite set of distinct points that are considered as a subvariety of X. Let $\bar{L}$ and $\bar{M}$ be hermitian bundles on X with continuous metrics. Assume that L is ample and the metric $\|\ \|$ on $\bar{L}$ is a limit of smooth metrics $\|\ \|_n$ with semipositive curvatures; this means that, uniformly on $X_{\mathbf{C}}$, $\|\ \|_n/\|\ \|(x)$ converges to 1. Then, for any $\epsilon > 0$, a positive integer n_0 can be found such that, for any $n > n_0$,*

$$q\big(\Gamma(X, \bar{L}^{\otimes n} \otimes \bar{M})_{\sup} \to \Gamma(Y, \bar{L}^{\otimes n} \otimes \bar{M})_{\sup}\big) \leq n\epsilon.$$

If $X_{\mathbf{R}}$ is a regular algebraic variety defined over $\mathbf{R}$ such that $X = X_{\mathbf{R}} \otimes \mathbf{C}$, $Y_{\mathbf{R}}$ is a reduced subvariety of $X_{\mathbf{R}}$ of dimension 0 such that $Y = Y_{\mathbf{R}} \otimes \mathbf{C}$, and $\bar{L}$ and $\bar{M}$ are invariant under complex conjugation, then the above assertion still holds in the spaces $\Gamma(X_{\mathbf{R}}, \cdot)$ and $\Gamma(X_{\mathbf{R}}, \cdot)$ with subspace norms induced from $\Gamma(X, \cdot)$ and $\Gamma(Y, \cdot)$.

Proof. Let us denote the first assertion in the theorem by $P(\bar{L}, \bar{M})$. Let $\bar{L}_n$ (resp. $\bar{M}_n$) be a sequence of hermitian line bundles with the same line bundle L (resp. M) and let their metrics be convergent to the metric of $\bar{L}$ (resp. $\bar{M}$). Then it is easy to see that $P(\bar{L}_n, \bar{M}_n)$ for all sufficiently large n implies $P(\bar{L}, \bar{M})$. By the hypothesis of the theorem we may assume that the metric on $\bar{L}$ is smooth and its curvature is semipositive and that the metric on M is smooth.

Since L is ample, we can choose a metric $\|\ \|'$ on L such that $\bar{L}' = (L, \|\ \|')$ is positive. Then the sequence of metrics $\|\ \|^{1-1/m}\|\ \|'^{1/m}$ is convergent to the metric of $\bar{L}$. We reduce the problem to the case where $\bar{L}$ is positive.

Suppose now that $\bar{L}$ is positive. Choose a positive integer n_0 such that $L^{\otimes n} \otimes M$ is very ample for $n > n_0$. Put the metrics $\|\ \|_n'^{1/n}$ on L, where $\|\ \|'_n$ is the Fubini–Study metric. Then Tian's theorem claims that $\|\ \|_n'^{1/n}$ is convergent to the metric of $\bar{L}$. We are left with proving $P(\bar{L}_n, M)$ for all $n > n_0$, where $\bar{L}_n = (L, \|\ \|_n'^{1/n})$.

Fix an $n > n_0$. Since the assertions $P(\bar{L}_n^{\otimes n}, \bar{M} \otimes \bar{L}^{\otimes i})$ $(0 \le i \le n-1)$ together imply $P(\bar{L}_n, M)$, we need only prove $P(\bar{L}_n^{\otimes n}, \bar{M})$ for arbitary $\bar{M}$. For each $1 \le i \le p$ we can find a basis $\{l_{i1}, l_{i2}, \dots, \}$ of $\Gamma(X, L^{\otimes n} \otimes M)$, which is orthonormal with respect to the L^2 norm such that $l_{ij}(x_i) = 0$ for $i \ne j$ and $l_{ii}(x_i) \ne 0$, where $Y = \{x_1, \dots, x_p\}$. Let $\| \ \|'$ denote the metric on $L_n^{\otimes n}$; then one easily sees that $\|l_{ii}\|'$ obtains its maximal value at x_i. In fact

$$\|l_{ii}\|'(x) = \frac{n^{\dim(X)/2}\|l_{ii}\|_n(x)}{\sqrt{\sum_j \|l_{ij}\|_n^2(x)}} \le n^{\dim(X)/2} = \|l_{ii}\|'(x_i).$$

For any given $\bar{M}$, choose a positive integer d_0 such that the map

$$\Gamma(X, L^{\otimes d_0 n} \otimes M) \to \Gamma(Y, L^{\otimes d_0 n} \otimes M)$$

is surjective. Hence we can find sections s_i $(1 \le i \le p)$ of $\Gamma(X, L^{\otimes d_0 n} \otimes M)$ such that $s_i(x_j) = 0$ if $i \ne j$ and $s_i(x_i) \ne 0$. It follows that, for $d > d_0$,

$$\Gamma(Y, L^{\otimes nd} \otimes M) = \bigoplus_{i=1}^{p} \mathbf{C} l_{ii}^{d-d_0} s_i(x_i).$$

For any element $\bar{l} = \sum_i a_i l_{ii}^{d-d_0} s_i(x_i)$ in this space, take l to be the section $\sum_i a_i l_{ii} s_i$ in $\Gamma(\bar{L}^{\otimes nd} \otimes \bar{M})$. Then the image of l in $\bigoplus_{i=1}^{p} \bar{L}^{\otimes nd} \otimes \bar{M}|_{x_i}$ is $\bar{l}$. Let $\| \ \|'$ also denote the induced metric on a power of L_n or on a tensor product of a power of $\bar{L}_n$ with $\bar{M}$. Then

$$\begin{aligned} \|l\|'(x) &\le p \max_i |a_i| \|s_i\|' \|l_{ii}\|'^{d-d_0}(x) \\ &\le \mathrm{c} \max_i |a_i| \|s_i\|'(x_i) \|l_{ii}\|'^{d-d_0}(x_i) \\ &= \mathrm{c} \max_i \|\bar{l}\|'(x_i), \end{aligned}$$

where a constant $\mathrm{c} = p \sup_{i,x} \|s_i\|'(x)/\|s_i\|'(x_i)$. Since c does not depend on d and n, it follows that, for any $\epsilon > 0$ and any sufficiently large d,

$$q\big(\Gamma(X, \bar{L}_n^{\otimes nd} \otimes \bar{M})_{\sup} \to \Gamma(Y, \bar{L}_n^{\otimes nd} \otimes \bar{M})_{\sup}\big) \le \mathrm{c} \le d\epsilon.$$

This proves $P(\bar{L}_n^{\otimes n}, \bar{M})$. The first assertion of the theorem follows.

The assertion for real manifolds is obviously true. □

Remark 3.3. The proof of Theorem 3.1 is based on Hörmander's L^2 estimate. Using this method, one can give a direct proof of Theorem 3.2. Ullmo's report [10] has given details for such a proof.

4. On restriction maps of line bundles

Let $X/\operatorname{spec} \mathbf{Z}$ be an arithmetic surface. Let L be a line bundle on X such that L is positive on the generic fiber and semipositive on the special fibers.

Let M be any line bundle on X and D be any irreducible divisor on X. We have the following analogue of Theorem 3.2:

THEOREM 4.1. *Consider the exact sequence*

$$\Gamma(L^{\otimes n} \otimes M) \to \Gamma(D, L^{\otimes n} \otimes M) \to \mathrm{H}^1\big(L^{\otimes n} \otimes M(-D)\big).$$

The image of $\Gamma(D, L^{\otimes n} \otimes M)$ *in* $\mathrm{H}^1(X, L^{\otimes n} \otimes M(-D))$ *has finite order for* n *sufficiently large. Let* h_n *be the order of this image. Then for any* $\epsilon > 0$,

$$h_n \le e^{\epsilon n}$$

for sufficiently large n.

LEMMA 4.2. *Let* $\mathrm{H}(n)$ *denote the group* $\bigoplus_{p\text{:primes}} \mathrm{H}^1(L^{\otimes n} \otimes M(-D)) \otimes \mathbf{Z}/p\mathbf{Z}$. *Then for sufficiently large* n, *the group* $\mathrm{H}(n)$ *is finite and its order is bounded uniformly in* n.

Proof. Apply the Riemann–Roch theorem to fibers of $X/\mathbf{Z}$. □

LEMMA 4.3. *There are a positive integer* n_0 *and an effective vertical divisor* V_0 *of* X *such that* $L^{\otimes n_0}(V_0)$ *is positive on each special fiber of* X.

Proof. Consider the set S of pairs (n, V), a positive number n and an effective vertical divisor V such that $L^{\otimes n}(V)$ is semipositive on each fiber of X. Let Z be a function on S defined as follows: $Z(n, V)$ is the number of irreducible vertical divisors over which $L^{\otimes n}(V)$ has degree 0. Let $(n_0, V_0) \in S$ be such a pair from which Z gets its minimal value. We need to prove that $Z(n_0, V_0) = 0$.

Let $R = \Gamma(O_X)$. Then all special fibers of $X/\operatorname{spec} R$ are connected. If $Z(n_0, V_0)$ is not 0, then we can find two vertical divisors F, G such that $\deg_F(L^{\otimes n_0}(V_0)) = 0$, $\deg_G(L^{\otimes n_0}(V_0)) > 0$, and $F \cdot G > 0$. For sufficiently large n we have

$$Z(nn_0, V_0 n + G) \le Z(n_0, V_0) - 1.$$

This gives a contradiction. □

Proof of Theorem 4.1. For any $n > 0$ let N_n be the smallest positive integer N such that $N\Gamma(D, L^{\otimes n} \otimes M)$ is contained in the image of $\Gamma(X, L^{\otimes n} \otimes M)$. By Lemma 4.2 it suffices to prove that $N_n \le e^{n\epsilon}$ for any given $\epsilon > 0$ and large n.

Choose $L^{\otimes n_0}(V_0)$ as in Lemma 4.2. Since $L^{\otimes n_0}(V_0)$ has the same restriction to the generic fiber as $L^{\otimes n_0}$, we have a positive integer N such that $NL^{\otimes n_0}(V_0)$ is contained in $L^{\otimes n_0}$. For any $n_1 > 0$ choose d sufficiently large that

$$\mathrm{H}^1\big(X, L^{\otimes d(n_0+n_1)}(dV_0) \otimes M(-D)\big) = 0.$$

Now $N^d\Gamma(D, L^{\otimes d(n_0+n_1)} \otimes M)$ is contained in $N^d\Gamma(D, L^{\otimes d(n_0+n_1)}(dV_0) \otimes M)$, which is the image of $N^d\Gamma(X, L^{\otimes d(n_0+n_1)}(dV_0)\otimes M)$ and, in turn, is contained in the image of $\Gamma(D, L^{\otimes d(n_0+n_1)}\otimes M)$. This proves that $N_{(n_0+n_1)d} \leq N^d$. Replacing M by $M \otimes L^{\otimes i}$, $0 \leq i < n_0 + n_1$, we obtain $N_n \leq N^{n/n_1+n_0}$ for $n \gg 0$. Since n_1 can be arbitrarily large, this shows that $N_n \leq e^{\epsilon n}$ for any given $\epsilon > 0$ and any sufficiently large n. $\square$

5. The proof of Theorem 1.5

We need a lemma on a normed lattice before we can give the completed proof of Theorem 1.5. First of all let us fix the following notation.

Let Γ be a finitely generated abelian group. Assume that there is a norm on $V = \Gamma \otimes_{\mathbf{Z}} \mathbf{R}$. Let $\bar{\Gamma}$ be the image of Γ in V. We denote by $\lambda(\Gamma)$ the smallest number λ such that there is a basis $\{l_1, l_2, \dots, l_n\}$ for a sublattice of $\bar{\Gamma}$ of full rank such that $\max_i \|l_i\| \leq \lambda$.

Let $0 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n \subset \Gamma_{n+1} = \Gamma$ be a filtration for the lattice Γ. Assume that $\Gamma_i \otimes \mathbf{R}$ has a norm $\| \ \|_i$; then $\Gamma_i/\Gamma_{i-1} \otimes \mathbf{R}$ has the quotient norm induced by that of $\Gamma_i \otimes \mathbf{R}$. Notice that the metrics on $\Gamma_{n+1} \otimes \mathbf{R}$ and $\Gamma \otimes \mathbf{R}$ may not be the same. We let $\rho(\Gamma_i)$ denote the norm of the morphism $\Gamma_i \otimes \mathbf{R} \to \Gamma \otimes \mathbf{R}$.

LEMMA 5.1.

$$\lambda(\Gamma) \leq \rho(\Gamma_{n+1})\lambda(\Gamma_{n+1}/\Gamma_n) + \sum_{i<n} \rho(\Gamma_{i+1})\lambda(\Gamma_{i+1}/\Gamma_i)\mathrm{rank}_{\mathbf{Z}}(\Gamma_{i+1}/\Gamma_i).$$

Proof. We prove the following special case first: Assume that for each i the norm on $\Gamma_i \otimes \mathbf{R}$ is induced by $\Gamma \otimes \mathbf{R}$. In this case we have $\rho(\Gamma_i) = 1$. By induction on n, we need only prove the inequality $\lambda(\Gamma) \leq \lambda(\Gamma/\Gamma_1) + \mathrm{rank}\,\Gamma_1 \lambda(\Gamma_1)$. Let $V_1 = \Gamma_1 \otimes_{\mathbf{Z}} \mathbf{R}$, $\mathrm{rank}\,\Gamma_1 = d_1$ and $\mathrm{rank}\,\Gamma = d_1 + d_2$. Choose $l'_1, l'_2, \dots, l'_{d_2}$ in $\bar{\Gamma}$ such that their images $m_1, m_2, \dots, m_{d_2}$ form a basis for $\overline{\Gamma/\Gamma_1}$, and $\max_i \|m_i\| = \lambda(\Gamma/\Gamma_1)$. Choose $l_1, l_2, \dots, l_{d_1}$ in $\bar{\Gamma}_1$ such that they form a basis for $\bar{\Gamma}_1$, and $\max_i \|l_i\| = \lambda(\Gamma_1)$. Choose $l''_1, l''_2, \dots, l''_{d_2}$ in V such that, for each i, l''_i has an image m_i and $\|l''_i\| = \|m_i\|$. Since $l''_i - l'_i$ is in V_1, we have real numbers α_{ij}, $1 \leq i \leq d_2$ and $1 \leq j \leq d_1$, such that

$$l''_i - l'_i = \sum_j \alpha_{ij} l_j.$$

Set $l_{d_1+i} = l'_i + \sum_j [\alpha_{ij}] l_j \in \Gamma$ for $1 \leq i \leq d_2$. Now $\{l_1, l_2, \dots, l_{d_1+d_2}\}$ is a basis for a sublattice of full rank in $\bar{\Gamma}$ and

$$\|l_{d_1+i}\| = \left\| l''_i - \sum_j (\alpha_{ij} - [\alpha_{ij}]) l_j \right\| \leq \|l''_i\| + \sum_j \|l_j\|.$$

This implies that $\lambda(\Gamma) \leq \lambda(\Gamma/\Gamma_1) + \operatorname{rank}\Gamma_1\lambda(\Gamma_1)$.

Now we want to prove the lemma for the general case. Let Γ_i' be the image of Γ_i in Γ with the subspace norm on $\Gamma_i' \otimes \mathbf{R}$ induced by that of $\Gamma \otimes \mathbf{R}$. Then, by the lemma in the above special case,

$$\lambda(\Gamma) \leq \sum_{i<n} \lambda(\Gamma_{i+1}'/\Gamma_i') \operatorname*{rank}_{\mathbf{Z}}(\Gamma_{i+1}/\Gamma_i) + \lambda(\Gamma_{n+1}'/\Gamma_n').$$

We need only prove that

$$\lambda(\Gamma_{i+1}'/\Gamma_i') \leq \lambda(\Gamma_{i+1}/\Gamma_i)\rho(\Gamma_{i+1}).$$

Fix one i. Let $\{\bar{l}_1, \dots, \bar{l}_n\}$ be a basis for the image of Γ_{i+1}/Γ_i in $\Gamma_{i+1}/\Gamma_i \otimes \mathbf{R}$ with norms bounded by $\lambda(\Gamma_{i+1}/\Gamma_i)$. Then we can choose elements $\{l_1, \dots, l_n\}$ in $\Gamma_i \otimes \mathbf{R}$ with images $\{\bar{l}_1, \dots, \bar{l}_n\}$, and their norms are bounded by $\lambda(\Gamma_{i+1}/\Gamma_i)$. Let $\{l_1', \dots, l_n'\}$ be the images of $\{l_1, \dots, l_n\}$ in $\Gamma_{i+1}' \otimes \mathbf{R}$. Then the norms of $\{l_1', \dots, l_n'\}$ are bounded by $\lambda(\Gamma_{i+1}/\Gamma_i)\rho(\Gamma_{i+1})$. Since the images of $\{l_1', \dots, l_n\}$ in $\Gamma_{i+1}'/\Gamma_i' \otimes \mathbf{R}$ are integral and form a basis for the image of Γ_{i+1}'/Γ_i', this implies that

$$\lambda(\Gamma_{i+1}'/\Gamma_i') \leq \lambda(\Gamma_{i+1}/\Gamma_i)\rho(\Gamma_{i+1}). \qquad \square$$

Proof of Theorem 1.5 (completed). Replacing $\bar{L}$ by $\bar{L}^{\otimes n}$ for some positive integer n if necessary, we may assume that there exists a strictly effective section l of $\bar{L}$. Let $[\operatorname{div} l] = D_1 + D_2 + \cdots + D_m$, where the D_i's are irreducible divisors on X. For each i put a hermitian metric on $O(D_i)$ such that the product of these bundles is $\bar{L}$. For each nonnegative integer N let $\bar{L}_N$ denote the bundle $O(D_1 + D_2 + \cdots + D_N)$ with the product metric introduced above, where $D_{i+m} = D_i$. Denote by Γ_N the lattice $\Gamma(X, L_N)$ in the normed space $\Gamma(X_{\mathbf{R}}, L_N)$ with the supremum norm on $X_{\mathbf{C}}$. By Lemma 1.6 we need only prove that $\lambda(\Gamma_N) < 1$ for $N = nm \gg 0$.

We want to apply Lemma 5.1 to $0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_N$ in order to estimate $\lambda(\Gamma_N)$. Notice that if $N' = n'm + r$, then the morphism $\Gamma_{N'} \to \Gamma_N$ is the multiplication by $l^{n-n'}l'$, where l' is the canonical section in $O(D_{r+1} + D_{r+2} + \cdots + D_m)$. So the norm $\rho(\Gamma_{N'})$ of the induced map from $\Gamma_{N'} \otimes \mathbf{R}$ to $\Gamma_N \otimes \mathbf{R}$ is bounded by $\exp((N' - N)c_1 + c_2)$, where c_1 and c_2 are positive numbers.

It remains to estimate $\lambda(\Gamma_{N'+1}/\Gamma_{N'})$. From the exact sequence

$$0 \to L_{N'} \to L_{N'+1} \to L_{N'+1}|_{D_{N'+1}} \to 0$$

we have the embedding

$$0 \to \Gamma_{N'+1}/\Gamma_{N'} \to \Gamma(D_{N'+1}, L_{N'+1}).$$

Since $\deg L_{\mathbf{Q}} > 0$, the cokernel of this map is a finite group. Let $h_{N'+1}$ denote its order. Then we have the following filtration:

$$0 \subset h_{N'+1}(\Gamma(D_{N'+1}, L_{N'+1})) \subset \Gamma_{N'+1}/\Gamma_{N'}.$$

Applying Lemma 5.1 to this filtration, we obtain

$$\lambda(\Gamma_{N'+1}/\Gamma_{N'}) \leq \rho_{N'+1} h_{N'+1} \lambda(\Gamma(D_{N'+1}, L_{N'+1})),$$

where $\rho_{N'+1}$ is the norm of the map from $\Gamma(D_{N'+1}, L_{N'+1}) \otimes \mathbf{R}$ with the supremum norm to $\Gamma_{N'+1}/\Gamma_{N'} \otimes \mathbf{R}$ with the quotient t norm induced from the supremum norm on $\Gamma_{N'+1} \otimes \mathbf{R}$.

We estimate the numbers on the right-hand side of the above inequality as follows:

(a) By Theorem 4.1, $h_{N'+1}$ is bounded by $\mathrm{const} \cdot \exp(N\epsilon_1)$, where ϵ_1 is any positive number.

(b) By Theorem 2.2, $\lambda(\Gamma(D_{N'+1}, L_{N'+1}))$ is bounded by $\mathrm{const} \cdot \exp(-N' \cdot$ positive constant$)$.

(c) By Theorem 3.2, $\rho_{N'+1}$ is bounded by $\mathrm{const} \cdot \exp(N\epsilon_2)$, where ϵ_2 is any positive number.

Combining these estimates with small ϵ_i shows that $\lambda(\Gamma_{N'+1}/\Gamma_{N'})$ is bounded by $\exp(-c_3 N' + c_4)$, where c_3 and c_4 are positive numbers.

Now applying Lemma 5.1, we obtain

$$\begin{aligned}\lambda(\Gamma_N) &\leq \sum_{N' \leq N} \exp((N' - N)c_1 + c_2) \exp(-N' c_3 + c_4) \operatorname*{rank}_{\mathbf{Z}}(\Gamma_{N'}/\Gamma_{N'-1}) \\ &\leq \exp(-c_5 N + c_6) \operatorname*{rank}_{\mathbf{Z}} \Gamma_N,\end{aligned}$$

where c_5 and c_6 are positive numbers. From the Riemann–Roch theorem on algebraic curves we know that $\operatorname{rank}_{\mathbf{Z}} \Gamma(X, L_N)$ is linear in N. This implies finally that

$$\lambda(\Gamma(X, \bar{L}^{\otimes n} 0)) < 1$$

for $n \gg 0$, completing the proof of the theorem. □

6. Discreteness and positivity

Let A be an abelian variety defined over a number field K and let $C \subset A$ be an embedding of a smooth, projective, algebraic curve of genus ≥ 2. Let h_{NT} be the Néron–Tate height on $A(\bar{K})$ defined by an ample line bundle on A, where $\bar{K}$ is an algebraic closure of K. Let us define a function d_{NT} on $A(\bar{K}) \times A(\bar{K})$ as follows:

$$d_{\mathrm{NT}}(x, y) = \sqrt{h_{\mathrm{NT}}(x - y)}.$$

We can easily check that d_{NT} is a semipositive distance function on $A(\bar{K})$. This means that, for any x, y, z on $A(\bar{K})$,

(1) $d_{\mathrm{NT}}(x,y) \geq 0$; actually $d_{\mathrm{NT}}(x-y)=0$ if and only if $x-y$ is a torsion point,

(2) $d_{\mathrm{NT}}(x,y)+d_{\mathrm{NT}}(y,z) \geq d_{\mathrm{NT}}(x,z)$,

(3) $d_{\mathrm{NT}}(x,y)=d_{\mathrm{NT}}(y,x)$.

Conjecture 6.1 (Bogomolov). $C(\bar{K})$ is discrete under d_{NT}. This means that, for any $p \in C(\bar{K})$, there is a positive number ϵ such that the set of points $q \in C(\bar{K})$, with the property that $d_{\mathrm{NT}}(x,y) \leq \epsilon$, is finite.

Clearly Conjecture 6.1 implies a result of Raynaud [6] that, for any point $p \in C(\bar{K})$, there are only finitely many points $q \in C(\bar{K})$ such that $p-q$ is torsion.

The main result we will prove here is an analogue of Conjecture 6.1 for multiplicative groups. Let us explain it as follows: Recall the definition of a canonical height function on the projective line. Then, for p a place of $\mathbf{Q}$, let $\bar{\mathbf{Q}}_p$ denote one fixed algebraic closure of the p-adic numbers $\mathbf{Q}_p$ and let $|\ |_p$ be the canonical norm function on $\bar{\mathbf{Q}}_p$ such that $|p|_p = p^{-1}$ if p is finite and $|a+b\sqrt{-1}|_\infty = \sqrt{a^2+b^2}$ for $a, b \in \mathbf{R}$. For a point (x_0, x_1) in $\mathbf{P}^1(\bar{\mathbf{Q}})$, let K denote the Galois closure of $\mathbf{Q}(x_0, x_1)$. Now define

$$h_\infty(x_0,x_1) = \frac{1}{[K:\mathbf{Q}]} \sum_p \sum_{\sigma:K\to\bar{\mathbf{Q}}_p} \log\max(|\sigma x_0|_p, |\sigma x_1|_p).$$

Let $\mathbf{G}_m$ be the multiplicative group of dimension 1, which we consider as the complement of $\{x_0x_1=0\}$ in $\mathbf{P}^1$. Let us still denote by h_∞ its restriction on $\mathbf{G}_m$ and its product on $\mathbf{G}_m^n$. Then the function d_∞on $(\mathbf{G}_m^n)^2$ is defined as follows:

$$d_\infty(x,y) = h_\infty(xy^{-1}),$$

where if $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$, then $xy^{-1}=(x_1y_1^{-1},\ldots,x_ny_n^{-1})$. It is easy to check that d_∞ is a semipositive distance function on $\mathbf{G}_m^n(\bar{\mathbf{Q}})$. This means that d_∞ has the corresponding properties to (1), (2) and (3) of d_{NT} described before Conjecture 6.1.

THEOREM 6.2. *Let $C \to \mathbf{G}_m^n$ be an embedding of a curve defined over a number field K. Assume that C is not a translate of a subgroup $\mathbf{G}_m^n$. Then $C(\bar{K})$ is discrete under d_∞.*

In the spirit of Szpiro [8], the discreteness of algebraic points follows from the positivity of certain line bundles. Let us explain this in more detail.

Let $X/\mathbf{Z}$ be an arithmetic surface and $\bar{L}$ be a hermitian line bundle on X such that $L_{\mathbf{Q}}$ is positive. Then $\bar{L}$ defines a height function $h_{\bar{L}}$ on $X(\bar{\mathbf{Q}})$ as follows: Let $x \in X(\bar{\mathbf{Q}})$ and let $D(x)$ be the irreducible divisor on X with a generic point x. Then

$$h_{\bar{L}}(x) = \frac{\bar{L} \cdot D(x)}{\deg D(x)_{\mathbf{Q}}}.$$

Let us define

$$e_{\bar{L}}(X_{\mathbf{Q}}) = \inf_{x \in X(\bar{\mathbf{Q}})} h_{\bar{L}}(x)$$

and

$$e'_{\bar{L}}(X_{\mathbf{Q}}) = \lim \inf_{x \in X(\bar{\mathbf{Q}})} h_{\bar{L}}(x).$$

THEOREM 6.3. *If $\bar{L}$ is relatively semipositive and* $\deg L_{\mathbf{Q}} > 0$, *then*

$$2e'_{\bar{L}} \geq \frac{\bar{L}^2}{\deg L_{\mathbf{Q}}} \geq e'_{\bar{L}} + e_{\bar{L}}.$$

Remark 6.4. Here is an example which shows that the equalities in Theorem 6.3 may hold. Let E/O_K be a semistable model of an elliptic curve with a j-invariant integral. Then E/O_K has good reduction everywhere. Let P be any section of E, which we identify with the line bundle $O(P)$ with a canonical Arakelov metric. By a theorem of Szpiro [8], $P^2 = 0$ and, by Theorem 7.1, we can show that $e'_P = e_P = 0$.

For an application to Theorem 6.2 we need to define a metric $|\ |_\infty$ on the bundle $O(1)$ on $\mathbf{P}^1$. Let s_0 and s_1 be two canonical sections of $O(1)$. The metric $|\ |_\infty$ is defined so that

$$|s_0|_\infty(x_0, x_1) = 1/\max\left(1, |\frac{x_1}{x_0}|\right).$$

Its curvature $\omega(|\ |_\infty)$ can be extended to a function on $C(\mathbf{P}^1_{\mathbf{C}})$: for any continous function f on $\mathbf{P}^1$,

$$\int_{\mathbf{P}^1(\mathbf{C})} f\omega(O(1), |\ |_\infty) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\mathrm{d}\theta,$$

where $x_1 = e^{i\theta}x_0$. Also the metric $|\ |_\infty$ (resp. its curvature) is the limit of the metrics $|\ |_l$ (resp. the curvature of the metrics $|\ |_l$), where the $|\ |_l$ denote the metrics defined by

$$|s_0|_l(x_0, x_1) = \left(1 + |\frac{x_1}{x_0}|^l\right)^{-1/l}.$$

This shows that the results in Sections 1–5 are valid for $|\ |_\infty$.

It is easy to see that the height function h_∞ on $\mathbf{P}^1(\bar{\mathbf{Q}})$ is defined by $(O(1), |\ |_\infty)$.

Let C be any curve in $\mathbf{G}_m^n$ defined over K. Let us denote its Zariski closure in $(\mathbf{P}^1_{\mathbf{Z}})^n$ by X. Let $\pi : \tilde{X} \to X$ be its normalization and $\bar{L}_{\infty,C}$ be the pullback of $(O(1), |\ |_\infty)$ on $\tilde{X}$. Then we have the following result:

THEOREM 6.5. *Assume that C is not a translate of a subgroup of $\mathbf{G}_m^n$; then*

$$\bar{L}^2_{\infty,C} > 0.$$

Proof of Theorems 6.5 + 6.3 $\Rightarrow$ 6.2. Let p be an algebraic point on C. We need only prove that

$$\lim \inf_{q \in C(\bar{K})} d_\infty(p, q) > 0.$$

Let $\mathbf{G}_m^n \to \mathbf{G}_m^n$ be a morphism of a variety by the multiplication of p^{-1} and let C' be the image of C under this morphism. Then C' is still not a translate of a subgroup and

$$\lim \inf_{q \in C(\bar{K})} d_\infty(p, q) = \lim \inf_{q \in C'(\bar{K})} d_\infty(q, 1) = \lim \inf_{q \in C'(\bar{K})} h_\infty(q) = e'_{\bar{L}_{\infty,C'}}.$$

This number is positive by Theorems 6.3 and 6.5. $\square$

7. Faltings's results on the index theorem

Let $\pi : X \to \operatorname{spec} O_K$ be a morphism from an arithmetic surface to the spectrum of the ring O_K of integers of the number field K. Assume that the generic fiber of π is regular and of positive genus. Let $\bar{L}$ be a hermitian line bundle on X of degree 0. Then there are a vertical divisor ϕ_0 with coefficients in $\mathbf{Q}$ and a smooth function ϕ_∞ on $X_{\mathbf{C}}$ such that the following conditions are verified:

(1) for any vertical divisor F of X, $(L - \phi_0)F = 0$,

(2) $\omega(\bar{L}) + \partial\bar{\partial}/\pi i \phi_\infty = 0$.

Replacing L by a positive power of L, we may assume that ϕ_0 has integral coefficients.

Let $\bar{L}(-\phi_0 - \phi_\infty)$ denote the line bundle $L(-\phi_0)$ with the metric $\|\ \|_{\bar{L}_{\mathbf{C}}} \exp(\phi_\infty)$ on $L_{\mathbf{C}}$. Then $\bar{L}(-\phi_0 - \phi_\infty)$ has 0 intersections with each vertical divisor and each hermitian line bundle of the form $O(f) = (O, \|\ \| \exp(-f))$, where f is a smooth function on $X_{\mathbf{C}}$ and $\|\ \|$ is the canonical metric on O. We have the following index theorem:

THEOREM 7.1 (Faltings, Hriljac). *Let $h(L_K)$ denote the Néron–Tate height of the class of L_K; then*

$$\bar{L} \cdot \bar{L} = -[K : \mathbf{Q}] h(L_K) + \phi_0^2 + \int_{X_{\mathbf{C}}} \phi_\infty \frac{\partial\bar{\partial}}{\pi i} \phi_\infty.$$

In particular $\bar{L}\cdot\bar{L}=0$ *if and only if some positive power is isometric to the trivial line bundle with some constant metric.*

Proof. If $\phi_0=\phi_\infty=0$, the theorem follows from Theorem 4 in [3]. In general, replacing L by some positive power (the assertion does not change), we may assume that ϕ has integral coefficients. Since $\bar{L}(-\phi_0-\phi_\infty)\cdot O(\phi_0+\phi_\infty)=0$, one has

$$-[K:\mathbf{Q}]h(L)=\bar{L}(-\phi_0-\phi_\infty)^2=\bar{L}^2-O(\phi_0)^2-O(\phi_\infty)^2.$$

We obtain the formula in the theorem, since $O(\phi_\infty)^2=\int_{X_{\mathbf{C}}}\phi_\infty\,\partial\bar\partial/\pi i\,\phi_\infty$. Since the three terms in the right-hand side of the formula in the theorem are all nonpositive, $\bar{L}\cdot\bar{L}=0$ will imply that all of them are 0. This implies that L_K is a torsion line bundle, ϕ_0 is a combination of vertical fibers and ϕ_∞ is a constant function on $X_{\mathbf{C}}$. This is equivalent to the second assertion in the theorem. □

8. The proof of Theorem 6.3

We prove the lower bound for $\bar{L}\cdot\bar{L}$ first.

Let $\bar{L}'$ be the hermitian line bundle $\bar{L}(-e_{\bar{L}})$ on X. We want to prove that $\bar{L}'\cdot\bar{L}'\geq 0$. Suppose that $\bar{L}'\cdot\bar{L}'<0$. Let $\bar{M}$ be an ample hermitian line bundle on X and $p(t)$ be the polynomial

$$\bar{L}'\cdot\bar{L}'+2\bar{L}'\cdot\bar{M}t+\bar{M}\cdot\bar{M}t^2.$$

Then there is a positive number t_0 such that $p(t_0)=0$ and $p(t)>0$ for any $t>t_0$. Let a and b be two positive integers such that $a/b>t_0$ and let $\bar{L}''$ be the hermitian line bundle $\bar{L}'^{\otimes b}\otimes\bar{M}^{\otimes a}$. Then

$$\bar{L}''\cdot\bar{L}''=b^2p\left(\frac{a}{b}\right)>0.$$

By Theorem 2.1, $\bar{L}''^{\otimes n}$ is effective for some positive integer n, and this implies that $\bar{L}'\cdot\bar{L}''^{\otimes n}\geq 0$ or $\bar{L}'\cdot\bar{L}'+\bar{L}'\cdot\bar{M}\,a/b\geq 0$. Since a/b can be any rational number bigger than t_0, and $p(t_0)=0$, we have

$$\bar{L}'\cdot\bar{L}'+\bar{L}'\cdot\bar{M}t_0\geq 0$$

and

$$\bar{L}'\cdot\bar{M}t_0+\bar{M}\cdot\bar{M}t_0^2\leq 0.$$

We get a contradiction, since $\bar{L}'\cdot\bar{M}\geq 0$ and $\bar{M}\cdot\bar{M}>0$. This implies that $\bar{L}'\cdot\bar{L}'\geq 0$.

Now let D_0 be any effective divisor on X and ϵ be any positive number. Then $\bar{L}'(\epsilon)$ is horizontally positive. By Theorem 1.5 there is a section s of a power $\bar{L}'(\epsilon)$ such that $\mathrm{div}(s)_{\mathbf{Q}}$ and $D_{0\mathbf{Q}}$ are disjoint. Write $\mathrm{div}(s)=\sum D_i+\phi$, where the D_i's are all irreducible and horizontal and ϕ is vertical. Then we have

$$\inf_{D_{0\mathbf{Q}}\cap D_{\mathbf{Q}}=\phi} \frac{\bar{L}\cdot D}{\deg(D_{\mathbf{Q}})} \leq \min_i \frac{\bar{L}\cdot D_i}{\deg(D_{i\mathbf{Q}})} \leq \frac{\bar{L}\cdot \bar{L}'(\epsilon)}{\deg L_{\mathbf{Q}}}.$$

Since D_0 and ϵ are arbitrary,

$$\bar{L}\cdot\bar{L} \geq \deg L_{\mathbf{Q}}(e_{\bar{L}} + e'_{\bar{L}}).$$

We can now start to prove the upper bound for $\bar{L}\cdot\bar{L}$. Fix a positive number ϵ. Let us denote by $\bar{L}''$ the bundle $\bar{L}(-\bar{L}^2/(2\deg(L_{\mathbf{Q}})) + \epsilon)$. Then $\bar{L}''\cdot\bar{L}'' > 0$. By Theorem 2.1 there is a strictly effective section l of a power of $\bar{L}''$. This implies that any irreducible horizontal divisor D of X, which is not in the support of $\operatorname{div}(l)$, has a positive intersection with $\bar{L}''$. This implies that $e'_{\bar{L}''}$ is nonnegative or

$$2e'_{\bar{L}} \geq \frac{\bar{L}^2}{\deg L_{\mathbf{Q}}} - \epsilon.$$

Since ϵ is arbitrary, we obtain our inequality in the theorem. □

9. The proof of Theorem 6.5

Before we give the proof of Theorem 6.5 let us recall a theorem of Ihara, Serre and Tate on the torsion points on a curve in a multiplicative group. Note that this is an analogue of Raynaud's theorem [6] for an abelian variety and both of these theorems were conjectured by S. Lang.

THEOREM 9.1 (Ihara, Serre and Tate). *Let C be a curve in a multiplicative group $\mathbf{G}_m^n$, which is not a translate of a subgroup. Then there are only finitely many torsion points of $\mathbf{G}_m^n$ lying on C.*

For the proof see [5].

Proof of Theorem 6.5. Let π_i be the i^{th} component of the morphism $\tilde{X} \to (\mathbf{P}^1)^n$. Let $d_i = \deg \pi_i$ and $\bar{L}_i = \pi_i^*(O(1), |\ |_\infty)$. Then we have $\bar{L}_\infty = \sum \bar{L}_i$. For each i, since the curvature and the height associated to L_i are pull-backs of those associated to $\omega(O(1), \|\ \|_\infty)$, it follows that L_i is both relative semipositive and horizontally semipositive. By Theorem 6.3, or by some direct computation, one has $L_i^2 = 0$. It follows that L_i is semipositive. For each i, j one has $L_i\cdot L_j \geq 0$. Hence $\bar{L}_\infty^2 \geq 0$.

If $\bar{L}_\infty^2 = 0$, then we have $\bar{L}_i\cdot\bar{L}_j = 0$ for any i, j. This implies that

$$(d_j\bar{L}_i - d_i\bar{L}_j)^2 = 0.$$

By Theorem 7.1, the $d_j\bar{L}_i - d_i\bar{L}_j$ are all numerically equivalent to 0.

Let $S \subset C$ consist of all points that have at least one torsion component. This means that, for any point p in S, we can find an i such that

$$\sum_{q \in \pi_i^{-1}(p)} h_{\bar{L}_i}(p) = 0.$$

By the above argument, this equation holds for all i. In other words, S is the set of all torsion points on C. By the above construction, we know that S is an infinite set. The assertion follows from Theorem 9.1. □

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