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Admissible pairing on a curve

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Introduction

In this paper, we construct an admissible pairing of divisors on a curve defined over a non-archimedean field, as an analogue of Arakelov's pairing on a Riemann surface.

For an algebraic curve C with a given symmetric metric $|\cdot|_\Delta$ on $O(\Delta)$ on $C \times C$, one can define a pairing $(,)$ on $\mathrm{Div}(C)$ such that $(x, y) = -\log|1|_\Delta(x, y)$. In [A], for a Riemann surface of positive genus, Arakelov constructed a norm $|\cdot|_\Delta$ such that the induced pairing extends the Néron local pairing on $\mathrm{Div}^0(C)$, and satisfies certain adjunction formula and certain normalization condition by integration. For a curve defined over a discrete valuation field K, we will construct a similar metric $|\cdot|_\Delta$ on $O(\Delta)$. From semistable models, one has a canonical norm $|\cdot|_\Delta$ on $O(\Delta)$ such that $i(x, y) = -\log|1|_\Delta(x, y)$ is the normalized intersection number of sections extending x, y on some semistable model of C. If C has potentially good reduction, then $|\cdot|_\Delta$ will satisfies all requirements. In general we need to multiply a term $\exp(-g_\mu(x, y))$, where μ is certain metric on the reduction graph $R(C)$ of C and g_μ is the associated green's function. In §1 we define $i(x, y)$. In §2 we study intersection theory via $R(C)$. In §3 we find the admissible metrics on a metrized graph. In §4 we define admissible pairings and prove all required properties.

In §5 we give some applications to curves defined over global fields. For a curve C defined over a global field K, the local admissible pairings gives a global admissible pairing for divisors on C. We have a relative dualizing sheaf ω_a, a Riemann–Roch formula, an adjunction formula, and an index theorem. Let ω_{Ar} denote the Arakelov dualizing sheaf, then we have the estimate:

$$(\omega_{Ar}, \omega_{Ar}) \geqq (\omega_a, \omega_a) \geqq 0.$$

The first equality holds if and only if C is an elliptic curve, or a curve which has potentially good reduction at all non-archimedean places. The second equality

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holds if and only if there is a sequence $\{x_n\}$ of distinct algebraic points such that the Neron–Tate heights of $(2g-2)x_n - \omega$ converges to 0. We also prove the Bogomolov conjecture for the embeddings $j_D: C \to J(C)$ which takes x to $x - D$, where D is a divisor of C of degree 1 such that $(2g-2)D - \omega$ is not a torsion divisor. When C has potentially good reduction, partial results are obtained by Szpiro in [S], and by author in [Z].

1 Projective systems of semistable models

(1.1) We start from a complete discrete valuation field K with an algebraically closed residue field k. Let R be the valuation ring of K, let π be a uniformizer of R, and let $\bar{K}$ be an algebraic closure of K. Then there is a unique norm $|\cdot|$ on $\bar{K}$ such that $|\pi| = 1/e$. For any extension E of K in $\bar{K}$, let R_E denote the valuation ring of E.

For any projective variety X on spec $\bar{K}$, let MPic(X) denote the category of metrized line bundles on X. An object L consists of a line bundle $L_{\bar{K}}$ on X, and a $\bar{K}$ norm $|\cdot|_L$ on $L_{\bar{K}}$. A morphism between two metrized line bundles is an isomorphism between line bundles which is isometric. We usually write $\overline{\mathrm{Pic}}(\bar{K})$ for MPic(spec $\bar{K}$). We have the following two constructions:

(1.1.1) For an extension E of K in $\bar{K}$, and any free R_E module V of dimension 1, we define a metric on $V_{\bar{K}}$ as follows: for any $v \in V_{\bar{K}}$

$$|v| = \inf_{x \in \bar{K} - \{0\}} \{|x|^{-1} : xv \in V \otimes_{R_E} R_{\bar{K}}\}.$$

This defines a functor from Pic(R_E) to $\overline{\mathrm{Pic}}(\bar{K})$. Let $P(\bar{K})$ denote the full subcategory generated by all images of Pic(R_E)'s.

(1.1.2) For each $r \in \mathbb{R}$, the object $O(r)$ is defined to be $(\bar{K}, |\cdot|_r)$, where $|1|_r = \exp(-r)$. Conversely, for any L and any nonzero section l in L, we have a unique morphism from L to $O(-\log |l|_L)$ which takes section l to section 1. For this reason, we sometimes write $\overline{\mathrm{Div}}(\bar{K})$ for $\mathbb{R}$, and div $l = -\log|l|$. One can easily show that $O(r)$ is in $P(\bar{K})$ if and only if $r \in \mathbb{Q}$. So we sometimes write $D(\bar{K})$ for $\mathbb{Q}$.

(1.2) Let K be a local field defined as in (1.1). Let C be a proper, regular curve of positive genus defined over K. We write $\bar{C}$ for $C_{\mathrm{Spec}\,\bar{K}}$. By the semi-stable reduction theorem, the set Γ of finite extensions of K in $\bar{K}$ over which C has semistable reductions is not empty. For E in Γ, C has a unique projective model X_E on spec R_E which has the following properties: X_E is regular and the special fiber X_{Ek} is a semistable curve. If E in Γ and F is a finite extension of E in $\bar{K}$, then $F \in \Gamma$. There is a unique R_E morphism from X_F to X_E which induces the identity morphism on the generic fibers. So we have a projective system $\{X_E : E \in \Gamma\}$ of schemes. For each X_E, let D_E denote a subgroup of Div$(X_E) \otimes_{\mathbb{Z}} \mathbb{Q}$ expand by integral horizontal divisors and rational vertical divisors. Then $\{D_E : E \in \Gamma\}$ form a direct limit system. Let $D(\bar{C})$ denote its limit. There is an intersection pairing on $D(\bar{C})$: for $D_1, D_2 \in D_E$ such that $|D_{1E}| \cap |D_{2E}| = \emptyset$ then the geometric intersection number $i_E(D_1, D_2)$ on X_E is defined. If we modify pairing as follows

$$i(D_1, D_2) = i_E(D_1, D_2)/[E:K],$$

then one can prove that these pairings compatible with the direct limit system. So we obtain a pairing $i(,)$ on $D(\bar{C})$. There is a canonical map from $D(\bar{C})$ to $\mathrm{Div}(\bar{C})$. We want to define a section of this map. Let $p \in C(\bar{K})$ be any point. There is a E in Γ such that $p \in C(E)$. So there is a section s of X_E over spec R_E extending p. It is easy to see that the image of s in $D(\bar{C})$ does not depend on the choice of E. This gives a map from $C(\bar{K})$ to $D(\bar{C})$. By linear combination, we obtain a homomorphism from $\mathrm{Div}(\bar{C})$ to $D(\bar{C})$. It is easy to see that this is a section of the canonical morphism from $D(\bar{C})$ to $\mathrm{Div}(\bar{C})$. We have the following decomposition

$$D(\bar{C}) = \mathrm{Div}(\bar{C}) \oplus V(\bar{C}),$$

where $V(\bar{C})$ is the direct limit of $\{\mathrm{Ver}(X_E) \otimes_{\mathbb{Z}} \mathbb{Q} : E \in \Gamma\}$, the groups of rational vertical divisors of X_E's. The pull back morphism give an injective morphism from $D(\bar{K})$ to $D(\bar{C})$.

(1.3) We have a similar description for metrized line bundles. For each E in Γ, and each line bundle L on X_E, we can define a metric $|\cdot|$ on $L_{\bar{K}}$ as follows: for any point $p \in C(\bar{K})$ we may find a finite extension F of E such that $p \in C(F)$. Then p can be extended to a unique section s of X_F. Combining s with the morphism from X_F to X_E we obtain a morphism s_1 from spec R_F to X_E. By (1.1.1) we obtain an induced metric on L_p by line bundle $s_1^*(L)$. It is easy to see that this metric does not depend on the choice of F. So we have a metric on the line bundle $L_{\bar{K}}$ induced by L. It is also easy to see that if F is a finite extension of E in $\bar{K}$, then the pull back of L on X_F induces same metric as L does. Let P_E denote a full subcategory of $\mathrm{MPic}(\bar{C})$ described as follows: a metrized line bundle L is in P_E if there is positive integer n such that $L^{\otimes n}$ is isometric to a metrized line bundle induced by a line bundle on X_E. Let $P(\bar{C})$ denote the full subcategory of $\mathrm{MPic}(\bar{C})$ generated by all P_E's.

As usual for each $\bar{D}$ in $D(\bar{C})$, we can define an object $O(\bar{D})$ as follows: there is a E in Γ so that $\bar{D} = D + (v/n)$, $D \in \mathrm{Div}(C_E)$, $v \in \mathrm{Ver}(X_E)$, and n is a positive integer. Then $O(n\bar{D})$ is a line bundle on X_E which induces a metric $|\cdot|_{n\bar{D}}$ on the line bundle $O(nD)$. Let $O(\bar{D})$ be the metrized line bundle which consists of the line bundle $O(D)$ and metric $|\cdot|_{n\bar{D}}^{\frac{1}{n}}$. One can show that $O(\bar{D})$ does not depend on the choice of E, n. Conversely, for each object L in $P(\bar{C})$ and each nonzero rational section l of $L_{\bar{K}}$, one can define a divisor $\mathrm{div}\, l$ in $D(\bar{C})$, and a morphism in $P(\bar{C})$ from L to $O(\mathrm{div}\, l)$ which takes section l to section 1 in $O(\mathrm{div}\, l)$. The pull back morphism gives an embedding from $P(\bar{K})$ to $P(\bar{C})$.

(1.4) We recall the following formulas from [D]. Let $f: C \to S$ be a family of semistable curves. For any line bundle L, M of C, let $\langle L, M\rangle$ denote the Deligne's pairing of L, M. Let $\det \mathrm{R}^* f_*(L)$ denote the determinant of the derived direct image of L. It is simply denoted by $\det \mathrm{H}^*(L)$ if S is the spectrum of a field. Let $\omega_{C/S}$ denote the relative dualizing sheaf of C/S. One has the following canonical isomorphisms.

(1.4.1) Adjunction formula: for any section e of C/S, there is a canonical isomorphism

$$\langle \omega_{C/S}(e), O_C(e)\rangle \simeq O_S.$$

(1.4.2) For any section e of C/S and any line bundle L of C, there is a canonical isomorphism

$$\det \mathrm{R}^* f_*(L(e)) \simeq \det \mathrm{R}^* f_*(L) \otimes e^* L(e).$$

(1.4.3) Serre duality: for any line bundle L of C, there is a canonical isomorphism

$$\det \mathrm{R}^* f_*(L^{\otimes -1} \otimes \omega_{C/S}) \simeq \det \mathrm{R}^* f_*(L).$$

(1.4.4) Riemann–Roch formula: for any line bundle L of C, there is a canonical isomorphism

$$\det \mathrm{R}^* f_*(L)^{\otimes 2} \simeq \det \mathrm{R}^* f_*(O_C)^{\otimes 2} \otimes \langle L, L \otimes \omega_{C/S}^{\otimes -1} \rangle.$$

(1.5) For any objects L, M in $P(\bar{C})$, we can find a E in Γ and a positive integer n, such that $L^{\otimes n}$ and $M^{\otimes n}$ are in $\mathrm{Pic}(X_E)$. Let $\langle L^{\otimes n}, M^{\otimes n} \rangle_E$ denote the Deligne pairing on $\mathrm{Pic}(X_E)$ which is a line bundle on $\mathrm{spec}\, R_E$, so induces a metric on $\langle L_{\bar{K}}^{\otimes n}, M_{\bar{K}}^{\otimes n} \rangle$ by (1.1.1). Its n^2-th root gives a metric on $\langle L_{\bar{K}}, M_{\bar{K}} \rangle$. One can prove that this metric does not depend on E, n. So we have a well defined Deligne pairing from $P(\bar{C})$ to $P(\bar{K})$. This pairing is compatible with the pairing of divisors. For any D_1, D_2 in $D(\bar{C})$ such that $|D_{1\bar{K}}| \cap |D_{2\bar{K}}| = \emptyset$ then we have a canonical isometry

$$\langle O(D_1), O(D_2) \rangle = O(i(D_1, D_2)).$$

For any objects L, M in $P(\bar{C})$ and any rational sections l, m of them respectively such that $|\mathrm{div}\, l_{\bar{K}}| \cap |\mathrm{div}\, m_{\bar{K}}| = \emptyset$, then

$$|\langle l, m \rangle| = \exp(-i(\mathrm{div}\, l, \mathrm{div}\, m)).$$

There is a canonical object $\bar{\omega}$ which is induced by relative dualizing sheaves ω_E on X_E for any $E \in \Gamma$. The canonical $\bar{K}$-isomorphism in (1.4.1) is isometric: for any point p in $\bar{C}$,

$$\langle \bar{\omega}(p), O(p) \rangle \simeq O.$$

2 Intersection pairing via reduction graphs

(2.1) A metrized graph G is by definition a finite connected graph with a uniform metric dx on each of its sides. For x in G, let $v(x)$ denote the valence of x in G, that is the number of directions go away from x. Let V_0 be the set of all points of G with valences bigger than 2. Then V_0 is a finite subset of G and $G - V_0$ is a disjoint union of line segments. Let $\mathrm{Div}(G)$ denote the group of divisors, that is $\oplus_{x \in G} \mathbb{Z}$. We may define the degree of a divisor by summing its coefficients. Let $F(G)$ be the set of piecewise smooth functions on G, a continuous function f on G is piecewise smooth, if there is a finite subset V containing V_0 such that f is smooth outside V with respect to the metric dx. We denote by $\overline{\mathrm{Div}}(G)$ the group $\mathrm{Div}(G) \oplus F(G)$ of compactified divisors on G. An intersection pairing on $\overline{\mathrm{Div}}(G)$ is defined as follows:

$$(D_1 + g_1, D_2 + g_2) = g_2(D_1) + g_1(D_2) - \int g_1 \Delta g_2 \, dx,$$

where $D_i \in \mathrm{Div}(G)$, $g_i \in F(G)$, and Δ is the laplacian operator on $F(G)$ as in the appendix. For $D + g \in \overline{\mathrm{Div}}(G)$, we call $h_{D+g} = \delta_D - \Delta g$ the curvature of $D + g$. We call $K = \sum_x (v(x) - 2)x$ the canonical divisor on G.

Now we try to define the reduction graph for a curve C as in (1.2). Let $E \in \Gamma$. Then $R(C)$ is just the dual graph of the special fiber X_{Ek} with length $1/[E:K]$ on each of its sides. More precisely, $R(C)$ has a finite subset V_E containing V_0 indexed by the set of irreducible components in X_{Ek}, the set S_E of sides in $R(C) - V_E$ indexed by the set of double points in X_{Ek}, they satisfy the following rule of connection. Two points in V_E is connected by a side in S_E, if and only if both corresponding components in X_{Ek} contain the corresponding double point. The length of each side in S_E is $1/[E:K]$. One can prove that $R(C)$ does not depend on the choice of E, by the theory of semistable curve. If F is a finite extension of E in $\bar{K}$, then V_E is contained in V_F. We also have a divisor K_C on $R(C)$ induced from C:

$$K_C = \sum_x (2q(x) - 2 + v(x))x = 2q + K,$$

where $q(x)$ is the genus function on G which vanishes out of V_E and coincides with the genus function from X_{Ek}.

(2.2) Let $\overline{\mathrm{Div}}(\bar{C})$ denote the group $\mathrm{Div}(\bar{C}) \oplus F(R(C))$. Then there are two homomorphisms:

$$\gamma: D(\bar{C}) \to \overline{\mathrm{Div}}(\bar{C}),$$

$$R: \overline{\mathrm{Div}}(\bar{C}) \to \overline{\mathrm{Div}}(R(C)).$$

Here, $\gamma(D + v) = D + \gamma(v)$, and $\gamma(v)$ is defined as follows. Let E be in Γ such that v is in $\mathrm{Ver}(X_E)_{\mathbb{Q}}$, v can be considered as a function $\phi_{v,E}$ on V_E. Then $\gamma(v)$ is a continuous function on $R(C)$ such that $\gamma(v)$ is linear on $G - V_E$ and its restriction on V_E is $\phi_{E,v}/[E:K]$. One can prove that $\gamma(v)$ is independent of E. Similarly, $R(D + g) = R(D) + g$ and $R(D)$ is defined as follows. By linearity we may assume D is a point on $C(\bar{K})$. Choose E in Γ such that $D \in C(E)$, then D can be extended to a section s of X_E. Then $R(D)$ is a point in V_E whose corresponding irreducible components in X_{Ek} meets s.

There is an intersection pairing on $\overline{\mathrm{Div}}(\bar{C})$ defined as follows:

(2.2.1) $$(D_1 + g_1, D_2 + g_2) = i(D_1, D_2) + (R(D_1) + g_1, R(D_2) + g_2).$$

Theorem 2.3 *The map γ preserves pairings, this means for any D_1, D_2 in $D(\bar{C})$, $(\gamma(D_1), \gamma(D_2)) = i(D_1, D_2)$.*

Proof. By linearity we need only prove the proposition in the following cases:

(1) Both D_1, D_2 are points of C. The assertion follows from definition (2.2.1), since the second term of the right hand side of (2.2.1) vanishes.

(2) D_1 is point of $C(E)$, and D_2 is an irreducible vertical component of X_E for some E in Γ. Then $(\gamma(D_1), \gamma(D_2)) = \gamma(D_2)(R(D_1))$. It is $1/[E:K]$ if the section extending D_1 meets D_2, otherwise it is zero. The assertion follows from the definition of $i(,)$.

(3) D_1 and D_2 are different irreducible vertical components of X_E. Then

$$(\gamma(D_1), \gamma(D_2)) = -\sum_e \gamma(D_1)'_e \gamma(D_2)'_e/[E:K],$$

where e runs over sides in S_E, and derivatives are defined for a given orientation of e's. By definition, on each e, if e corresponds to intersection point of D_1 and D_2, one of $\gamma(D_i)$ takes values 1 and the other one takes -1, otherwise one of $\gamma(D_i)'$ takes value 0. The assertion follows.

(4) D_1 is an irreducible vertical component of X_E, and D_2 is the special fiber of X_E. Then $i(D_1, D_2) = 0$. But also $(\gamma(D_1), \gamma(D_2)) = 0$, since $\gamma(D_2)$ is a constant.

Theorem 2.4 *The image $\gamma(D(\bar{C}))$ of γ is dense in $\overline{\mathrm{Div}}(\bar{C})$ in the following sense. For any $D + g$ in $\overline{\mathrm{Div}}(\bar{C})$, there is a sequence $D + g_n$ in $\gamma(D(\bar{C}))$ such that g_n converges to g with supremum norm on $F(R(C))$, and h_{D+g_n} converges to h_{D+g} as distributions on $F(R(C))$. Moreover if h_{D+g} is non-negative, then we can choose g_n such that h_{D+g_n} are all non-negative.*

Proof. Since $\mu = \Delta g$ is a measure on $R(G)$ with volume 0, and $\cup_E V_E$ is dense in $R(C)$, we may find E_n and a sequence of pointed measures $\mu_n = \sum_{x \in V_{E_n}} a_x \delta_x$ with $a_x \in \mathbb{Q}$ and $\sum_x a_x = 0$, such that μ_n converges to μ. Let g_n^0 be a function on $R(C)$ such that $\Delta g_n^0 = \mu_n$. Notice that

$$g_n^0 - \int g_n^0 = \int \Delta g_n^0(y) g(x, y)\, dy,$$

where $g(x, y)$ is the Green's function for dx, constructed in the appendix. So $g_n^0 - \int g_n^0$ converges to

$$\int \Delta g(y) g(x, y)\, dy = g - \int g.$$

It follows that

$$g_n = g_n^0 - \int g_n^0 + \int g$$

converges to g. Notice that g_n is linear on $R(C) - V_{E_n}$, and the equation $\Delta g_n = \mu_n$ is equivalent to a system of linear equations of $g_n | V_{E_n}$ with coefficients in $\mathbb{Q}$. Modifying g_n by adding a small number we may assume $g_n | V_{E_n}$ has values in $\mathbb{Q}$. Then $D + g_n$'s are in the image of γ. This proves the first assertion of the proposition. If $h_{D+g} = \delta_D - \Delta g$ is non-negative then we may choose μ_n such that $\delta_D - \mu_n$ are all non-negative. The second assertion follows.

(2.5) Now we turn to do the theory of metrized line bundles. Let $D + g$ be a compactified divisor on C. Then we have a line bundle $O(D + g) = O(D) \otimes O(g)$, where $O(D)$ is defined in (1.3) and is an object in $P(\bar{C})$, and $O(g)$ is a metrized line bundle whose generic fiber is $O_{C_{\bar{K}}}$, the metric $|\cdot|_g$ is defined so that $|1|_g = \exp(-R^*(g))$ with R defined in (2.2). Let $\overline{\mathrm{Pic}}(\bar{C})$ denote the full subcategory of $\mathrm{MPic}(\bar{C})$ consisting of objects which are isometric to some $O(D + g)$ defined as above. $P(\bar{C})$ can be considered as a full subcategory of $\overline{\mathrm{Pic}}(\bar{C})$. For any object L in $\overline{\mathrm{Pic}}(\bar{C})$ and any non-zero rational section l of L, we can define a divisor $\mathrm{div}\, l$ in $\overline{\mathrm{Div}}(\bar{C})$, such that L is isometric to $O(\mathrm{div}\, l)$ which takes section l to section 1. One can verify that these correspondences between divisors and metrized line bundles give known ones when we restrict them on $D(\bar{C})$ and $P(\bar{C})$.

For each L in $\overline{\mathrm{Pic}}(\bar{C})$, we define the curvature $c_1(L)$ of L to be $h_{\mathrm{div}\, l}$. We need to prove that this definition does not depend on the choice of l. Equivalently, we need to prove that for any nonzero rational function f on X_E for some E in Γ, $h_{\mathrm{div}\, f} = 0$ as distribution on $F(R(C))$. Let g be an element in $F(R(C))$, we have to prove that

$\int g h_{\mathrm{div} f} = 0$. By (2.4) we may assume that $g = \gamma(v)$ for some v in $D(\bar{C})$. By (2.3) we have

$$\int \gamma(v) h_{\mathrm{div} f} = (\gamma(v), \mathrm{div} f) = i(v, \mathrm{div} f) = 0.$$

Our claim follows. It is obvious that $\int c_1(L) = \deg(L_{\bar{K}})$. Conversely, for any measure μ on $R(G)$ with volume $\deg L_{\bar{K}}$, we can find a unique metric on L up to a constant multiple with curvature μ. To prove this, let l be any non-zero rational section on L, then $\delta_{\mathrm{div}\, l_{\bar{K}}} - \mu$ is a measure with volume 0, so there is a function g on $R(C)$ such that $\Delta g = \delta_{\mathrm{div}\, l_{\bar{K}}} - \mu$. Now $O(\mathrm{div}\, l_{\bar{K}} + g)$ is isomorphic to L and has curvature μ. If L has two metrics with curvature μ, then the quotient of these metrics gives a metric on O with curvature 0, or $\Delta \log |1| = 0$. The function $|1|$ must be constant.

We are ready to define Deligne's pairing for objects in $\overline{\mathrm{Pic}}(\bar{C})$. Let L, M be two objects in $\overline{\mathrm{Pic}}(\bar{C})$. Let l, m be two rational sections of L, M respectively such that $|\mathrm{div}\, l_{\bar{K}}| \cap |\mathrm{div}\, m_{\bar{K}}| = \emptyset$, then we define $\langle l, m\rangle = \exp(-(\mathrm{div}\, l, \mathrm{div}\, m))$. We claim this gives a metric on $\langle L_{\bar{K}}, M_{\bar{K}}\rangle$. In other word we need to show that $|\langle fl, m\rangle| = |f(\mathrm{div}\, m)| |\langle l, m\rangle|$. This follows from the following fact

$$(\mathrm{div}\, l, D + g) = -\log|l|(D) + \int g c_1(L).$$

One can prove that the restriction of this pairing on $P(\bar{C})$ gives the known one.

We have the following interpretation for (2.4). For any $L = (L_{\bar{K}}, |\cdot|)$ in $\overline{\mathrm{Pic}}(\bar{C})$, we can find a sequence of metrics $|\cdot|_n$ such that $L_n = (L_{\bar{K}}, |\cdot|_n)$ are in $P(\bar{C})$, $|\cdot|_n$ converges to $|\cdot|$ and $c_1(L_n)$ converges to $c_1(L)$. Moreover If $c_1(L)$ is non-negative we may choose L_n such that each $c_1(L_n)$ is nonnegative. Notice that for a line bundle M in $P(\bar{C})$ whose n-th power ($n > 0$) is induced by a line bundle M' on X_E for E in Γ, the non-negativity of $c_1(M)$ is equivalent to the fact that the restriction of M' on any irreducible vertical component has non-negative degree.

Theorem 2.6 *For a given metric on* $\det \mathrm{H}^*(O)$, *there is a unique functor* $\overline{\det \mathrm{H}^*}$ *from* $\overline{\mathrm{Pic}}(\bar{C})$ *to* $\overline{\mathrm{Pic}}(\bar{K})$ *which is compatible with the functor* $\det \mathrm{H}^*$ *from* $\mathrm{Pic}(\bar{C})$ *to* $\mathrm{Pic}(\mathrm{spec}\, \bar{K})$ *such that the following conditions are verified*:

(1) $\overline{\det \mathrm{H}^*}(O)$ *induces the given metric on det* $\mathrm{H}^*(O)$.

(2) *For any p in* $C(\bar{K})$, *the following canonical* $\bar{K}$*-isomorphism is isometric*:

$$\overline{\det \mathrm{H}^*} L(p) \simeq \overline{\det \mathrm{H}^*}(L) \otimes p^* L(p).$$

(3) *For any function g in* $F(R(C))$, *the following canonical* $\bar{K}$*-isomorphism is isometric*:

$$\overline{\det \mathrm{H}^*} L(g) \simeq \overline{\det \mathrm{H}^*} L \otimes \langle O(g/2), L^{\otimes 2} \otimes \bar{\omega}^{\otimes -1}(g)\rangle.$$

Moreover we have the following properties for $\overline{\det \mathrm{H}^*}$:

(4) *The Serre duality, the following* $\bar{K}$*-isomorphism is isometric*:

$$\overline{\det \mathrm{H}^*}(L^{\otimes -1} \otimes \bar{\omega}) \simeq \overline{\det \mathrm{H}^*}(L).$$

(5) *The Riemann-Roch formula, the following canonical* $\bar{K}$*-isomorphism is isometric*:

$$\overline{\det \mathrm{H}^*}(L)^{\otimes 2} \simeq \overline{\det \mathrm{H}^*}(O)^{\otimes 2} \langle L, L \otimes \bar{\omega}^{\otimes -1}\rangle.$$

Proof. Tfhe uniqueness of $\overline{\det \mathrm{H}^*}$ is obvious. For each L in $\overline{\mathrm{Pic}}(\bar{C})$, we define $\overline{\det \mathrm{H}^*}(L)$ to be the unique element such that (1) and (5) hold. Then all other isomorphisms are isometric, since they are isometric when we square them.

(2.7) We have a similar theory for a regular complete curve C defined over $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $|\cdot|_\Delta$ be a fixed symmetric smooth metric on $O(\Delta)$ on $\bar{C} \times \bar{C}$, where as before $\bar{C}$ denotes $C_\mathbb{C}$ and Δ is the diagonal of $\bar{C} \times \bar{C}$. Let $i(x, y)$ be $-\log |1|_\Delta$, where 1 is the canonical section of $O(\Delta)$. For any D_1, D_2 in $\mathrm{Div}(\bar{C})$ such that $|D_1| \cap |D_2| = \emptyset$, the number $i(D_1, D_2)$ can be defined by linear combination. We denote by $\overline{\mathrm{Div}}(\bar{C})$ the group $\mathrm{Div}(\bar{C}) \oplus C^\infty(\bar{C})$, where $C^\infty(\bar{C})$ is the set of real smooth functions on $\bar{C}$. We define a intersection pairing $(,)$ on $\overline{\mathrm{Div}}(\bar{C})$ as follows: for any $D_1 + g_1, D_2 + g_2$ in $\overline{\mathrm{Div}}(\bar{C})$ such that D_1 and D_2 have disjoint supports,

$$(D_1 + g_1, D_2 + g_2) = i(D_1, D_2) + g_1^*(D_2) + g_2^*(D_1) - \int g_1 \frac{d'd''}{\pi i} g_2,$$

where d', d'' are distributions associated to ∂, $\bar{\partial}$, and

$$g_1^*(D_2) = g_1(D_2) - \int g_1(x) \frac{d'd''}{\pi i} i(D_2, x).$$

Let $\overline{\mathrm{Pic}}(\bar{C})$ be the category of smoothly metrized line bundles on $\bar{C}$. For each $D + g$ we define a metrized line bundle $O(D + g) = (O(D), |\cdot|_g)$ such that the canonical section 1 of $O(D)$ has metric $\exp(-i(D, x) - g)$. Any nonzero rational section l of a metrized line bundle L defines a divisor $\mathrm{div}\, l = \mathrm{div}\, l_\mathbb{C} + (-\log|l| + i(D, \cdot))$. The pairing $\langle , \rangle$ on $\overline{\mathrm{Pic}}(\bar{C})$ is defined such that $|\langle l, m\rangle| = \exp(-(\mathrm{div}\, l, \mathrm{div}\, m))$. One can prove that this pairing coincides with the pairing defined by Deligne in [D]. Let $\bar{\omega} = (\omega, \Delta^*(|\cdot|_\Delta^{-1}))$ via the canonical isomorphism $\omega \simeq \Delta^* O(-\Delta)$. Then we have an adjunction formula:

$$\langle O(p), \bar{\omega}(p)\rangle \simeq O,$$

and Theorem 2.6 holds by the same proof.

3 Admissible metrics on a graph

(3.1) This section is devoted to construct admissible metrics on a metrized graph. Let G be a metrized graph with a uniform metric dx. Let μ be a measure on G of volume 1. Then there is a Green's function $g_\mu(x, y)$ of $G \times G$ with respect to μ. g_μ is continuous and symmetry, and satisfies the following conditions:

$$\Delta_y g_\mu(x, y) = \delta_x - \mu,$$

$$\int g_\mu(x, y)\mu(y) = 0.$$

Actually, let $g(x, y)$ be the Green's function associated to dx constructed in the appendix then

$$g_\mu(x, y) = g(x, y) - \int g(x, y)\mu(y) - \int g(x, y)\mu(x) + \iint g(x, y)\mu(x)\mu(y).$$

For any divisor D on G, let us denote by $g_\mu(D, x)$ or $g_{\mu D}(x)$ the function $\int g_\mu(x, y)\delta_D(y)$.

The main result in this section is the following theorem:

Theorem 3.2 *Let D be a divisor on G with $\deg(D) \neq -2$. Then there is a unique metric μ on G of volume 1, and a unique constant c such that the following equality holds for any point x in G:*

$$c + g_\mu(D, x) + g_\mu(x, x) = 0.$$

Moreover μ is positive if $D - K$ is effective, where K is the canonical divisor defined in (2.1).

We call μ the admissible metric on G with respect to D. When $D = 0$, this is a theorem of Rumely and Chinberg [C-R]. Our proof is based on their method. For a point p in G, let $g_p(x, y)$ or $g_p(G; p, q)$ denote $g_{\delta_p}(x, y)$. For points p, q in G, let $r(p, q)$ or $r(G; p, q)$ denote $g_p(q, q)$. This is the resistance between p and q if we consider G as an electric circuit. One can prove the following well known properties from physics.

Proposition 3.3 (1) *$r(p, q)$ is a symmetric, nonnegative function. $r(p, q) = 0$ if and only if $p = q$.*

(2) *Let e be a line segment of G with length l, and with endpoints p, q. Let H, I be two metrized subgraph of G. Assume that $G = H \cup I \cup e$, $\{p\} = H \cap e$, $\{q\} = I \cap e$, and $H \cap I = \emptyset$. Then $r(p, q) = l$.*

(3) *Let p, q, s are three points of G, and let H, I are two metrized subgraph of G. Assume that $G = H \cup I$, $\{s\} = H \cap I$, $p \in H$, and $q \in I$. Then*

$$r(G; p, q) = r(H; p, s) + r(I; s, q).$$

(4) *Let p, q are two points of G, and let H, I are metrized subgraph of G such that $G = H \cup I$ and $\{p, q\} = H \cap I$. Then*

$$r(G; p, q)^{-1} = r(H; p, q)^{-1} + r(I; p, q)^{-1}.$$

Proof. Since $\Delta_x g_p(x, q) = \delta_q - \delta_p$, so $g_p(x, q)$ obtains its maximal value $r(p, q)$ at $x = q$, and its minimal value 0 at $x = p$, this proves the nonnegativity of $r(p, q)$. If $r(p, q) = 0$, then $g_p(x, q) = 0$ for all x, so we must have $p = q$. By definition, one has $\Delta_x(g_p(x, q) + g_q(x, p)) = 0$, so $g_p(x, q) + g_q(x, p)$ is a constant function of x. This shows $r(p, q)$ is symmetric by setting $x = p$ and $x = q$. This proves (1).

For (2), let x be the coordinate on e such that $x(p) = 0$, then assertion follows from the following easily verified equality

$$g_p(s, q) = \begin{cases} 0, & s \in H \\ x(s), & s \in e \\ l, & s \in I. \end{cases}$$

Assertion (3) follows from the following easily verified equality

$$g_p(x, q) = \begin{cases} g_p(H; x, s), & x \in H \\ r(H; p, s) + g_s(I; x, q), & x \in I. \end{cases}$$

Assertion (4) follows from the following easily verified equality

$$g_p(x, q) = \begin{cases} \dfrac{r(I, p, q)}{r(H; p, q) + r(I; p, q)} g_p(H; x, q), & x \in H \\ \dfrac{r(H, p, q)}{r(H; p, q) + r(I; p, q)} g_p(I; x, q), & x \in I. \end{cases}$$

Corollary 3.4 *Let e be a closed line segment of G with length l_e and with endpoints p, p'. Define r_e to be $r(G - e^0; p, p')$ if $G - e^0$ is connected, and to be ∞ if $G - e^0$ is not connected, where $e^0 = e - \{p, p'\}$. Let x be a coordinate on e such that $x(p) = 0$. Then for $q \in e$*

$$r(p, q) = x(q) - (l_e + r_e)^{-1} x(q)^2.$$

Proof. Fix a q in e. Let e_1, e_2 be two closed subsegments of e which connect p, q and p', q respectively. If $G - e^0$ is not connected the assertion follows from (2) of the proposition. So we may assume $G - e^0$ is connected.

$$\begin{aligned} r(p, q) &= \{r(e_1; p, q)^{-1} + r(e_2 \cup (G - e^0); p, q)^{-1}\}^{-1} && \text{by (4)} \\ &= \{r(e_1; p, q)^{-1} + [r(e_2; p', q) + r(G - e^0; p, p')]^{-1}\}^{-1} && \text{by (3)} \\ &= \{x(q)^{-1} + [l - x(q) + r_e]^{-1}\}^{-1} && \text{by (2)} \\ &= x(q) - (l_e + r_e)^{-1} x(q)^2. \end{aligned}$$

(3.5) Now we are ready to prove Theorem 3.2. We will use (3.4) and the following equation

(3.5.1) $$r(p, q) = g_\mu(q, q) - 2g_\mu(q, p) + g_\mu(p, p),$$

where μ is any measure on G with volume 1. Notice that (3.5.1) follows from the following easily verified equality

$$g_p(x, y) = g_\mu(x, y) - g_\mu(x, p) - g_\mu(p, y) + g_\mu(p, p).$$

Let $V_D = V_0 \cup \operatorname{supp}(D)$, and let S_D denote sides in $G - V_D$.

Lemma 3.6 *Let M_D denote the vector space of all measures on G with form*

$$\sum_{v \in V_D} a_v \delta_v + \sum_{e \in S_D} a_e \delta_e,$$

where for each v and e, a_v and a_e are real numbers, and δ_e is the uniform measure on e induced by dx. Then there is a measure in M_D such that the equality in (3.2) *holds.*

Proof. Let Q_D denote the vector space of all continuous functions on G which are quadratic on $G - V_D$. Then it is easy to see that both M_D and Q_D have dimension $\# V_D + \# S_D$. For any measure μ, let g_μ^0 denote the function

$$g(x, y)\mu(G) - \int g(x, y)\mu(y) - \int g(x, y)\mu(x).$$

We claim that for any $\mu \in M_D$, the function

$$\alpha(\mu)(q) = g_\mu^0(q, q) + g_\mu^0(D, q)$$

is in Q_D. Since M_D is generated by measures of volume 1, we may assume μ has volume 1. Now $g_\mu - g_\mu^0$ is a constant function. Let e be a side of S_D, we need to prove that $\alpha(\mu)(x)$ is quadratic on e. Let p be an endpoint of e, then we still have (3.5.1) for g_μ^0, and therefore $\alpha(\mu)$ is a linear combination of $r(p, q)$, $g_\mu^0(p, q)$, and $g_\mu^0(D, q)$. Since $r(p, q)$ is quadratic in q by (3.4), and since for any $v \in V_D$, $\Delta_q g_\mu^0(v, q) = a_e dx(q)$, it follows that $g_\mu^0(p, q)$ and $g_\mu^0(D, q)$ are quadratic on e, our claim follows.

Now $\alpha: M_D \to Q_D$ is well defined. Since M_D and Q_D have the same dimension, $\alpha^{-1}(\mathbb{R})$ has positive dimension, where $\mathbb{R}$ is considered as constant functions. The lemma follows if one can prove there is an element μ in $\alpha^{-1}(\mathbb{R})$ with volume 1. If not, then every element in $\alpha^{-1}(\mathbb{R})$ has volume 0. Fix an element μ with volume 0. Let $f(x)$ denote $\int -g(x, y)\mu(y)$, then $g_\mu^0(x, y) = f(x) + f(y)$ and $\alpha(\mu)(x) = (\deg D + 2) f(x) + f(D)$. Since $\deg D \neq -2$, $\alpha(\mu)$ is constant only if $\mu = 0$. This proves our lemma.

The remain parts of theorem 3.2 follow from the following lemma

Lemma 3.7 *If μ is a measure such that the equality in* (3.2) *holds, then*

$$\mu = (\deg D + 2)^{-1}\left(\delta_D - \delta_K + 2 \sum_{e \in E_D} (l_e + r_e)^{-1} \delta_e\right).$$

Proof. We apply Δ_q to both sides of (3.5.1). Notice that $g_\mu(q, q) + g_\mu(D, q)$ is constant, so the Δ_q of the right hand side of (3.5.1) is $(\deg D + 2)\mu - 2\delta_p - \delta_D$. By (3.4), the Δ_q of the left hand side has value $-v(p)\delta_p$ at $q = p$, and has $2(l_e + r_e)^{-1}\delta_e$ on e. The lemma follows.

4 Admissible pairing

(4.1) Let C be a complete and regular curve of positive genus defind over a local field K as in (1.2). Let μ be the metric on $R(C)$ defined in (3.2) with respect to the divisor K_C defined in (2.1). We call μ the admissible metric on $R(C)$, and call $g_\mu(x, y)$ the admissible Green's function on $R(C)$. Let $\mathrm{Div}_a(\bar{C})$ denote the group $\mathrm{Div}(\bar{C}) \otimes \mathbb{R}$ of admissible divisors. We have an injective map $-_a$ from $\mathrm{Div}_a(\bar{C})$ to $\overline{\mathrm{Div}}(\bar{C})$ which takes $D + r$ to $D + g_{\mu D} + r$. The image of this map consists of all divisors whose curvatures are multiples of μ. The pairing $(,)$ on $\overline{\mathrm{Div}}(\bar{C})$ gives a pairing $(,)_a$ on $\mathrm{Div}_a(C)$. Let $\mathrm{Pic}_a(\bar{C})$ denote the full subcategory of $\overline{\mathrm{Pic}}(\bar{C})$ consisting of objects whose curvatures are multiples of μ. We call such objects are admissible line bundles. There are some correspondences between admissible divisors and admissible line bundles via $-_a$: for admissible divisor $D + r$, $O(D_a + r)$ is admissible, conversely for any nonzero rational section l of an admissible line bundle L, there is a unique admissible divisor $\mathrm{div}_a l$ such that $(\mathrm{div}_a l)_a = \mathrm{div}\, l$. Let c be the constant defined as in (3.2). Let ω_a denote the admissible line bundle $\bar{\omega}(c + g_{\mu K_C})$, where c is defined in (3.2) for divisor K_C.

Theorem 4.2 *For any p in $C(\bar{K})$, the following canonical $\bar{K}$-isomorphism on $\bar{C}$ is isometric*:

$$\langle O(p_a), \omega_a(p_a)\rangle \simeq O.$$

Proof. The left hand side of the above equation is

$$\langle O(p)(g_{\mu p}), \bar{\omega}(p)(g_{\mu p} + c + g_{\mu K_C})\rangle,$$

which is canonically isometric to

$$\langle O(p), \bar{\omega}(p)\rangle \otimes O((R(p) + g_{\mu p}, R(p) + K_C + g_{\mu p} + c + g_{\mu K_C})).$$

The first factor is canonically trivial by adjunction formula in (1.5). The second factor is canonically isometric to $O(c + g_\mu(D, x) + g_\mu(x, x))$ by definition, which is canonically trivial by (3.2).

Theorem 4.3 *Choose a metric on* $\det \mathrm{H}^*(O)$. *Then there is a unique functor* $\det \mathrm{H}_a^*$ *from* $\mathrm{Pic}_a(\bar{C})$ *to* $\overline{\mathrm{Pic}}(\bar{K})$ *which is compatible with the functor* $\det \mathrm{H}^*$ *from* $\mathrm{Pic}(\bar{C})$ *to* $\mathrm{Pic}(\mathrm{spec}\,\bar{K})$, *such that the following two conditions are verified:*

(1) $\det \mathrm{H}_a^*(O)$ *induces the given metric on* $\det \mathrm{H}^*(O)$.

(2) *For any* p *in* $C(\bar{K})$ *and any admissible line bundle* L *on* $\bar{C}$, *the following canonical* $\bar{K}$*-isomorphism is isometric*

$$\det \mathrm{H}_a^*(L(p_a)) \simeq \det \mathrm{H}_a^*(L) \otimes p^* L(p_a).$$

We have the following properties for $\det \mathrm{H}_a^*$:

(3) *Let* $\overline{\det \mathrm{H}^*}$ *denote the functor defined in* (2.6) *with given metric on* $\det \mathrm{H}^*(O)$, *and* c *is the constant defined in* (3.2) *then*

$$\det \mathrm{H}_a^*(L) \simeq \overline{\det \mathrm{H}^*}(L) \otimes O\left(-\frac{c}{2} \deg L_{\bar{K}}\right).$$

(4) *The Serre duality, the following canonical* $\bar{K}$*-isomorphism is isometric:*

$$\det \mathrm{H}_a^*(L^{\otimes -1} \otimes \omega_a) \simeq \det \mathrm{H}_a^*(L).$$

(5) *The Riemann–Roch formula, the following canonical* $\bar{K}$*-isomorphism is isometric:*

$$\det \mathrm{H}_a^*(L)^{\otimes 2} \simeq \det \mathrm{H}_a^*(O)^{\otimes 2} \otimes \langle L, L \otimes \omega_a^{\otimes -1}\rangle.$$

Proof. The uniqueness of $\det \mathrm{H}_a^*$ is obvious. For each admissible line bundle L, we define $\det \mathrm{H}_a^*(L)$ to be the unique element such that (1) and (5) holds. Then isomorphisms in (1), (2), and (4) are isometric as their square are isometric by Riemann-Roch. It remains to prove (3). By (5), and the Riemann-Roch for $\overline{\det \mathrm{H}^*}$,

$$\begin{aligned}
\det \mathrm{H}_a^*(L)^{\otimes 2} &\simeq \overline{\det \mathrm{H}^*}(L)^{\otimes 2} \otimes \langle L, O(-c - g_{\mu K_c})\rangle \\
&\simeq \overline{\det \mathrm{H}^*}(L)^{\otimes 2} \otimes O(\textstyle\int(-c - g_{\mu K_c}) c_1(L)) \\
&\simeq \overline{\det \mathrm{H}^*}(L)^{\otimes 2} \otimes O(\textstyle\int(-c - g_{\mu K_c}) \deg L \mu) \\
&\simeq \overline{\det \mathrm{H}^*}(L)^{\otimes 2} \otimes O(-c \deg L).
\end{aligned}$$

Theorem 4.4 *The identity over* $\mathrm{spec}\,\bar{K}$ *induces the following isometry:*

$$\langle \omega_a, \omega_a\rangle = \langle \bar{\omega}, \bar{\omega}\rangle \otimes O(r),$$

where

$$r = -\int g_\mu(x, x)((2g - 2)\mu + \delta_{K_c}).$$

Moreover r is nonpositive, and $r = 0$ if and only if $g = 1$ or $R(C)$ is a point, equivalently C is an elliptic curve or has a potentially good reduction.

Proof. By definitions, we have the following computation

$$\begin{aligned}\langle \omega_a, \omega_a \rangle &= \langle \bar{\omega} + c + g_{\mu K_c}, \bar{\omega} + c + g_{\mu K_c} \rangle \\ &= \langle \bar{\omega}, \bar{\omega} \rangle \otimes O(K_C + c + g_{\mu K_c}, K_C + c + g_{\mu K_c}).\end{aligned}$$

So we obtain that

$$\begin{aligned}r &= (K_C + c + g_{\mu K_c}, K_C + c + g_{\mu K_c}) \\ &= 2(c + g_{\mu K_c})(K_C) - \int (c + g_{\mu K_c}) \Delta (c + g_{\mu K_c}).\end{aligned}$$

From $c + g_{\mu K_c}(x) + g(x, x) = 0$, we obtain $c = -\int g(x, x)\mu$, $c + g_{\mu K_c} = -g(x, x)$. Combining these equalities, and $\Delta g_{\mu K_c} = \delta_{K_c} - (2g - 2)\mu$, we obtain the required formula for r. Since $g_\mu(x, x)$ is the maximal value of $g_{\mu x}(y)$, and since $\int g_\mu(x, y)\mu(y) = 0$, it follows that $r \leqq 0$. If $g = 1$, or $R(C)$ is a point then $r = 0$. If $g > 1$ and $r = 0$, then all $g_\mu(x, y)$ are 0, so $R(C)$ must be a point.

(4.5) Now let C be a complete curve defined over $\mathbb{R}$ or $\mathbb{C}$. Let us recall the following Arakelov calculus. See [A, F] for details. Let μ be the Arakelov metric on $\bar{C}$ with volume 1. Let $g_{Ar}(x, y)$ be the Arakelov-Green's function on $\bar{C}$. That is a smooth symmetric function on $\bar{C} \times \bar{C} - \Delta$ such that for all x, y,

$$\frac{d'_y d''_y}{\pi i} g_{Ar}(x, y) = \delta_x - \mu,$$

$$\int g_{Ar}(x, y)\mu(y) = 0,$$

where d', d'' are distributions associated to ∂ and $\bar{\partial}$. Let $\mathrm{Div}_a(\bar{C})$ denote the group $\mathrm{Div}(\bar{C}) \oplus \mathbb{R}$ of admissible divisors as before. Let $(,)_a$ be a pairing on $\mathrm{Div}_a(\bar{C})$: for all $D_1 + r_1$, $D_2 + r_2$ such that $|D_1| \cap |D_2| = \emptyset$, the number

$$(D_1 + r_1, D_2 + r_2)_a = (D_1, D_2)_a + r_1 \deg D_2 + r_2 \deg D_1$$

is defined and bilinear, such that $(p, q)_a = g_a(p, q)$ if $p \neq q$ are points. Similarly we let $\mathrm{Pic}_a(\bar{C})$ denote the full subcategory of smooth metrized line bundles consists of all objects whose curvature are multiples of μ. We also call such objects admissible line bundles. There are some correspondences between admissible divisors and admissible line bundles: Let $D + r$ be an admissible divisor then $O(D_a) = (O(D), |\cdot|_r)$ is an admissible line bundle such that $\int \log |1|_r = -r$. Conversely, let l be any nonzero rational section of an admissible line bundle L, there is a unique admissible divisor $\mathrm{div}_a l$, such that the isomorphism from $O((\mathrm{div}_a l)_a)$ to L which takes 1 to l is isometric. We define a pairing $\langle , \rangle$ on $\overline{\mathrm{Pic}}(\bar{C})$ as follows. Let L, M be two admissible line bundles, and let l, m be rational sections of them respectively such that $|\mathrm{div}\, l| \cap |\mathrm{div}\, m| = \emptyset$. Then

$$|\langle l, m \rangle| = \exp(-(\mathrm{div}_a l, \mathrm{div}_a m)_a).$$

In [A], Arakelov proved that there is a unique admissible line bundle $\omega_a = (\omega, |\cdot|_{\omega_a})$ such that (4.2) holds, where ω is the canonical line bundle on $\bar{C}$. The Theorem 4.3 in this case is known as Faltings theorem. We want to give another description for $(,)_a$.

Theorem 4.6 *Let C be a regular curver defined over a local field as in* (1.2) *or* (4.5). *Then the restriction of* $(\,,)_a$ *on* $\mathrm{Div}(\bar{C})$ *satisfies the following conditions*:

(1) *The* $(\,,)_a$ *is symmetric, bilinear, and defined for all D_1, D_2 such that* $|D_1| \cap |D_2| = \emptyset$.

(2) *For any divisor D of $\bar{C}$, $(D, x)_a$ is a Weil function associated to D. This means that if locally on a Zariski open subset U of $\bar{C}$ over which D is defined by a rational function f, then $(D, x)_a + \log|f(x)|$ is a bounded continuous function on U.*

(3) *For any nonzero rational function f of $\bar{C}$, there is a constant c_f such that for any x*

$$(\operatorname{div} f, x)_a = -\log|f(x)| + c_f.$$

(4) *Let α be any nonzero rational* 1*-form on $\bar{C}$, there is a constant c_α such that for any point x on $\bar{C}$ which is not in the support of* $\operatorname{div}\alpha$, *and for any rational function f of $\bar{C}$ which has a simple pole at x, the following equality holds*:

$$c_\alpha + \lim_{y \to x}((K + x, y)_a + \log|f\alpha/df|(y)) = 0.$$

Moreover if $[\,,]$ *is any pairing on* $\mathrm{Div}(\bar{C})$ *which satisfies the above conditions, then* $[\,,] = (\,,)_a + $ *a constant.*

Proof. The assertion (1) is obvious. For assertion (2) we may assume $D = p$ is a point by (1). If C defined over an archimedean field this follows from the definition of Arakelov-Green's function. If C is defined over a non-archimedean field, then

$$(p, x)_a = i(p, x) + g_\mu(Rp, Rx).$$

Since g_μ is continuous and bounded we need only prove that $i(p, x)$ is a Weil function associated to p. Choose some E in Γ such that p is in $C(E)$. (see notations in (1.2).) There is a section s on X_E extending x. Assume E as a Cartier divisor is defined by $\{U_i, f_i\}$. Then $\{U_{i\bar{K}}, f_i\}$ defines p. One can prove that on each U_i, $i(p, x) = -\log|f_i(x)|$. This proves (2). For assertion (3), we notice that $(\operatorname{div} f, x)_a = -\log|1|(x)$, where 1 is the canonical section of the admissible line bundle $O((\operatorname{div} f)_a)$. Since $O(\operatorname{div} f)$ is isomorphic to the trivial bundle O on $\bar{C}$, the metric on $O((\operatorname{div} f)_a)$ must have curvature 0, and must be the pull-back of a constant metric. The assertion (3) follows. The assertion (4) should follow from (2.5) as follows. Let s be the canonical section of $O(p)$. Then $\alpha s/f$ gives a local section of $\omega_a(p)$ at p. By (2.5) we have

$$|\alpha s/f|(p) = |\mathrm{Res}_p(\alpha/f)|. \tag{4.6.1}$$

Write

$$\operatorname{div}_a(\alpha) = \operatorname{div}(\alpha_{\bar{K}}) + c_\alpha \tag{4.6.2}$$

where c_α is a constant. Since s/f is an invertible regular section of $O(p)$ near p, it follows that

$$-\log|s/f|(p) = \lim_{q \to p}((p, q)_a + \log|f|(q)). \tag{4.6.3}$$

The assertion (4) follows from (4.6.1)–(4.6.3), and the fact that $\mathrm{Res}_p(\omega/f) = (\omega/df)(p)$. This proves the first part of the theorem.

Let $[\,,]$ be another pairing on $\mathrm{Div}(\bar{C})$ which satisfies the conditions in theorem. Let $h(,) = [\,,]-(,)_a$ then we have the following properties for h:

(4.6.4) h is bilinear, symmetric, and defined over all $\mathrm{Div}(\bar{C}) \times \mathrm{Div}(\bar{C})$.
(4.6.5) For each D in $\mathrm{Div}(\bar{C})$, $h(D, x)$ is a bounded continuous function of x.
(4.6.6) For any nonzero rational function f of $\bar{C}$, $h(\mathrm{div}\, f, x)$ is a constant function of x.
(4.6.7) For any canonical divisor K there is a constant c_K such that for all x, $c_K = h(x, x) + h(K, x)$.

We need to prove that h is a constant function from these properties. Let x, y, z be any three points of $\bar{C}$, and let m be any positive integer. Then by Riemann–Roch theorem, the bundle $O(gz + m(y - z))$ has a nontrivial section, where g is the genus of C. There are g points $y_1, \ldots, y_g$ and a nonzero function f such that

$$m(y - z) = \sum_{i=1}^{g} (y_i - z) + \mathrm{div}\, f.$$

It follows that

$$h(x - z, y - z) = \frac{1}{m}\left(\sum_{i=1}^{g} h(x - z, y_i) - \sum_{i=1}^{g} h(x - z, z) \right)$$

by (4.6.4), (4.6.6). This must be 0 as m tends to ∞, by (4.6.5). Let $j(x) = h(x, z) - \frac{1}{2} h(z, z)$ then $h(x, y) = j(x) + j(y)$ for any x, y in $\bar{C}$. With function j, (4.6.7) can be read as

$$-2gj(x) = c_K + j(K).$$

Since $g > 0$, it follows that j is a constant function. This proves the second part of the theorem.

(4.7) We have the following applications for (4.6):

By (4.6), the restriction on the group $\mathrm{Div}^0(\bar{C})$ of degree 0 gives a pairing satisfies (1)–(3). This is just the axioms for the minus Neron local pairing. This can be used to prove the positivities of certain line bundles in the next section. We refer [L] for Neron pairing.

We can give another construction of admissible metrics using θ divisor. Let a be any point of $\bar{C}$, then we have an embedding $j_a : \bar{C} \to J_0 = J$ by sending x to the class of $x - a$. Let θ be the canonical line bundle on J_{g-1} and let θ^a denote $T^*_{(g-1)a}\theta$ on J. Then for any line bundle L on $\bar{C}$, we can find a positive integer n and a line bundle M in J such that the class of M in $NS(J)$ is a multiple of θ^a, and such that $j_a^* M = L^{\otimes n}$. Notice that there is a $\bar{K}$ metric on M such that $-\log|m|$ is the Neron function associated to $\mathrm{div}\, m$ for any non zero rational section of M, we call such a metric on M an admissible metric. The pullback of this admissible metric on M gives a metric on L. One can prove that this metric on L modulo a positive constant factor does not depend on the choice of a, M, and the chosen metric on M. We temporarily call L with such a metric a θ-admissible mertrized line bundle. Let $\phi : \bar{C} \times \bar{C} \to J$ be a morphism by sending (x, y) to the class of $x - y$. Then one can prove that there are two line bundles L_1, L_2 on $\bar{C}$ such that

$$O(\Delta) = \phi^* \theta^a \otimes p_1^* L_1 \otimes p_2^* L_2.$$

One obtains a metric $|\cdot|_\Delta$ on $O(\Delta)$ by choosing an admissible metric on θ^a and θ-admissible metric on L_i's. The pairing $(x, y) = -\log|1|_\Delta(x, y)$ can be extended to a pairing on $\mathrm{Div}(\bar{C})$. One can prove that all conditions in (4.6) are verified. So this pairing is $(,)_a$ up to a constant, and θ-admissible line bundles are just admissible line bundles on $\bar{C}$.

One can use this construction to compute the determinant of cohomology of a family of line bundles. Actually, let $\mathscr{U}_n^a$ be the universal line bundle on $\bar{C} \times J_n$ with a trivialization on $a \times J_n$, then there is a unique metric on $\mathscr{U}_n^a$ compatible with the trivialization, such that for any j in J_n, its restriction to $\bar{C} \times \{j\}$ is admissible. Then one can prove that $\det \mathrm{H}_a^*(\mathscr{U}_n^a)$ is admissible.

We refer [M-B] for all details. When $\bar{C}$ is defined over $\mathbb{R}$ or $\mathbb{C}$, Faltings [F] proved this by comparing curvatures. Moret-Baily [M-B] algebraized Faltings' proof.

5 Curves over global fields

(5.1) Now we turn to global fields. Let K be a global field. This means that K is a function field of an algebraic curve, or a number field. Let S denote the set of places of K. For each σ in S, if it is non-archimedean, let K_σ be a strictly henselian closure of K with respect to σ, that is an algebraic extension of K which is unramified over σ and whose residue field is algebraically closed. If it is archimedean, K_σ denotes the completion of K with respect to σ, so which is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. We fix the norms in S as follows. For each σ in S, the number N_σ is defined to be the cardinality of the residue field of σ if K is a number field and σ is a finite place, to be e^2 if K is a number field and K_σ is $\mathbb{C}$, otherwise N_σ is e. The norms in S are defined such that $|\pi_\sigma| = N_\sigma^{-1}$, where π_σ is a uniformizer of K_σ if σ is non-archimedean, and is e^{-1} if it is archimedean. We have the product formula for K with norms defined as above.

For any projective variety X on $\operatorname{spec} K$ let $\mathrm{MPic}(X)$ denote the category of metrized line bundles on X which is defined as follows.

An object L consists of a line bundle L_K on X, and a set $|\cdot|_L = \{|\cdot|_{L,\sigma} : \sigma \in S\}$ of metrics, where $|\cdot|_{L\sigma}$ is the $\bar{K}_\sigma$ norm on $L_{\bar{K}_\sigma}$, $|\cdot|_L$ is assumed to verify the following property. There is a finite subset S_∞ of S containing all archimedean places, so $S_f = S - S_\infty$ can be considered as the set of closed points of an integral scheme which we still denote by S_f, and there is a line bundle L_f on a projective model X_f of X on S_f extending L_K, such that for any $\sigma \in S_f$ the norm $|\cdot|_\sigma$ is induced by line bundle $L_f \otimes_{S_f} \operatorname{spec} R_\sigma$ as in (1.1.1). Here R_σ is the ring of integers of K_σ.

A morphism from L_1 to L_2 is an isomorphism from base line bundle of L_1 to that of L_2 which induces isometrics over all $\bar{K}_\sigma$ norms.

Let $\overline{\mathrm{Pic}}(K)$ denote $\mathrm{MPic}(\operatorname{spec} K)$, and let $\overline{\mathrm{Div}}(K)$ denote the group $\sum_{\sigma\in S} \mathbb{R}$ of compactified divisors. For each compactified divisor $D = \sum r_\sigma \sigma$, we denote by $\deg(D)$ the number $\sum r_\sigma \log N_\sigma$. For each D we can define an object $O(D) = (O, \{|\cdot|_{D\sigma}\})$, where $|1|_{D,\sigma} = N_\sigma^{-r_\sigma}$. If S_∞ is a finite subset of S containing all archimedean places and all σ such that $r_\sigma \neq 0$, then $|\cdot|_\sigma$ is induced by the trivial line bundle O_f on $S_f = S - S_\infty$. Conversely, let $L = (L_K, |\cdot|)$ be an object of $\overline{\mathrm{Pic}}(K)$, and let l be nonzero section of L_K, then $\operatorname{div} l = \sum \log_{N_\sigma} |l|_\sigma \sigma$ is a finite sum. This is because on an open subset S_f of S, $\{|\cdot|_\sigma\}$ are induced by a line bundle L_f on S_f. One can prove that the morphism from $O(\operatorname{div} l)$ to L by l is an isometric map.

Moreover one can verify that deg(div l) does not depend on the choice of l, we call it the degree of L. If E is an extension of K, then the pull back of L gives an object in $\overline{\mathrm{Pic}}(E)$ which has degree $[E:K]\deg(L)$.

(5.2) Let K be a global field as in (5.1). Let C be a regular, proper curve of positive genus defined over K. We need the following notation and assumptions. For each σ in S, let C_σ denote $C \times_{\operatorname{spec} K} \bar{K}_\sigma$. For each archimedean place σ, let us fix a symmetric smooth metric on $O(\Delta_\sigma)$ on $C_\sigma \times C_\sigma$, where Δ_σ is the diagonal. For each nonarchimedean σ, $F_\sigma(C)$ will denote $F(R(C_\sigma))$, and for archimedean σ, $F_\sigma(C)$ will denote $C^\infty(C_\sigma)$. Let $F(C)$ denote $\oplus_{\sigma \in S} F_\sigma(C)$, and let $\overline{\mathrm{Div}}(C)$ denote the group $\mathrm{Div}(C) \oplus F(C)$ of compactified divisors. There is a pairing $\langle\,,\rangle$ on $\overline{\mathrm{Div}}(C)$ with values in $\overline{\mathrm{Div}}(K)$ defined to be the sum of local pairings. Let $D_i + \sum g_\sigma \sigma$, $i = 1, 2$, be two compactified divisors on C, such that $|D_1| \cap |D_2| = \emptyset$.

$$\left\langle D_1 + \sum_\sigma g_{1\sigma}\sigma, D_2 + \sum_\sigma g_{2\sigma}\sigma \right\rangle = \sum_\sigma (D_1 + g_{1\sigma}, D_2 + g_{2\sigma})\sigma,$$

where $(\,,)$ on the right hand side is the local pairing defined in (2.2.1) for non-archimdean place, and in (2.7) for archimedean place. The global intersection number $(\,,)$ is defined by

$$(D_1 + g_1, D_2 + g_2) = \deg\langle D_1 + g_1, D_2 + g_2\rangle.$$

The corresponding theory for metrized line bundles goes as follows. Let $\overline{\mathrm{Pic}}(C)$ denote the full subcategory of $\mathrm{MPic}(C)$ consisting of objects $L = (L_K, \{|\cdot|_\sigma\})$ such that for each σ, the metrized line bundle $(L_{\bar{K}_\sigma}, |\cdot|_\sigma)$ is an object of $\overline{\mathrm{Pic}}(C_\sigma)$. The following stuffs are straight forward from local stuffs: the correspondences between $\overline{\mathrm{Div}}(C)$ and $\overline{\mathrm{Pic}}(C)$; the pairing on $\overline{\mathrm{Pic}}(C)$ with values in $\overline{\mathrm{Pic}}(K)$. The intersection number (L, M) of two objects L, M is defined to be $\deg\langle L, M\rangle$. We have a canonical object $\bar{\omega}$ in $\overline{\mathrm{Div}}(C)$, and an adjunction formula

$$\langle \bar{\omega}(p), O(p)\rangle \simeq O.$$

We have a functor $\overline{\det H^*}$ on $\overline{\mathrm{Pic}}(C)$ which satisfies the standard corresponding properties as in (1.8). We leave all details to reader. Here we will generalize a statement for positive line bundles in [Z].

Theorem 5.3 *Let $L = (L_K, \{|\cdot|_\sigma\})$ be an object in $\overline{\mathrm{Pic}}(C)$. Assume $\deg L_K$ is positive and L is relative semipositive, that is for each σ the curvature $c_1(L_{\bar{K}_\sigma}, |\cdot|_\sigma)$ is nonnegative point-wisely. We have the following inequality about heights:*

$$\liminf_{x \in C(\bar{K})} h_L(x) \geqq \frac{(L, L)}{2\deg L_K} \geqq \frac{1}{2}\left(\liminf_{x \in C(\bar{K})} h_L(x) + \inf_{x \in C(\bar{K})} h_L(x)\right).$$

Proof. Let S_∞ be a finite subset of S containing all archimedean places of K, such that X has a smooth projective model X_f over S_f, and that for each $\sigma \in S_f$, $|\cdot|_\sigma$ is induced by a line bundle $\underline{L_f}$ on X_f. If $L_n = (L, \{|\cdot|_{n\sigma}\})$ is a sequence of relative semi-positive objects in $\overline{\mathrm{Pic}}(C)$ such that $|\cdot|_{n\sigma} = |\cdot|_\sigma$ for all σ in S_f, that $|\cdot|_{n\sigma}$ converges to $|\cdot|_\sigma$ for σ in S_∞, and that the theorem are true for L_n, then the theorem is true for L. By (2.4), (2.5) we may assume for each $\sigma \in S_\infty$, some positive power of L_σ is induced by some line bundle on a semistable model over a finite extension of

K_σ. But this can be realized globally after a finite extension with sufficiently large ramification on S_∞. There is a finite Galois extension E of K such that C has a semistable model X on $S - \{\text{archimedean places}\}$, and that a hermitian line bundle M which induces the object $\pi^* L^{\otimes m}$ in $\overline{\text{Pic}}(E)$, where π is the morphism $\text{spec} E \to \text{spec} K$, n is a positive integer. Now the theorem for algebraic line bundle M has been proved in [Z]. The theorem for L follows since the assertion is invariant under base change or after a positive power.

(5.4) Now we are going to do the global admissible intersection theory. We may define the group $\text{Div}_a(C)$ as $\text{Div}(C) \oplus \oplus_{\sigma \in S} \mathbb{R}$, the group of admissible divisors; an intersection pairing on this group with values in $\overline{\text{Div}}(K)$ as the sum of local pairing; the intersection number of two divisors; the category $\text{Pic}_a(C)$ of admissible line bundles; some correspondences between $\text{Div}_a(C)$ and $\text{Pic}_a(C)$; an intersection pairing on $\text{Pic}_a(C)$ with values in $\overline{\text{Div}}(K)$ or in $\mathbb{R}$; a canonical line bundle ω_a which we call the admissible canonical line bundle for which there is an adjunction formula; a determine $\det H_a^*$ on $\text{Pic}_a(C)$ which satisfies the standard properties, etc. They are all straight forward from the local theory. We leave them to reader to check details.

Notice that $\text{Div}_a(C)$ is $f^* \overline{\text{Div}}(K) \oplus \text{Div}(C)$, where f denotes the structure morphism $f: C \to \text{spec} K$, the pairing $(,)_a$ factors through $\mathbb{R} \oplus \text{Div}(C)$ by the degree morphism on $\overline{\text{Div}}(K)$. If E is a finite extension of K, then $(,)_a/[E:\text{K}]$ on $\text{Div}_a(C_E)$ is compatible with the pull-back map of compactified divisors. So we have a pairing on $\mathbb{R} \oplus (\text{Div}(\bar{C}) \otimes \mathbb{Q})$, where $\bar{C}$ denotes $C_{\bar{K}}$. As a consequence of (4.6), (4.7), it follows that the restriction of pairing $(,)_a$ to $\text{Div}^0(C)$ is just $-h_{NT}(,)$, the minus canonical Neron–Tate height pairing.

We are interested in the number (ω_a, ω_a). First of all we want to compare it with Arakelov relative dualizing sheaf $\omega_{Ar} = (\omega, \{|\cdot|_\sigma\})$, where norms $|\cdot|_\sigma$ are chosen such that $(\omega_\sigma, |\cdot|_\sigma) = \bar{\omega}_\sigma$ if σ is non-archimedean, and $(\omega_\sigma, |\cdot|_\sigma) = \omega_{\sigma a}$ if σ is archimedean. So if C has a semistable model X then ω_{Ar} is induced by the ordinary Arakelov dualizing sheaf on X. The local theorem (4.4) tells us

Theorem 5.5 *The following identity holds*:

$$\langle \omega_a, \omega_a \rangle = \langle \omega_{Ar}, \omega_{Ar} \rangle \otimes O\left(\sum_\sigma r_\sigma \sigma\right),$$

where $r_\sigma \leqq 0$ and $r_\sigma = 0$ if and only if C is an elliptic curve, or σ is archimedean, or σ is nonarchimedean and C has a potentially good reduction on σ. In particular,

$$(\omega_{Ar}, \omega_{Ar}) \geqq (\omega_a, \omega_a),$$

the equality holds if and only C is of genus 1, *or C has potentially good reduction on all nonarchimedean places.*

Theorem 5.6 *Let D be a divisor of degree* 1 *in a curve C which is regular, proper and of genus $g > 1$. Let $j_D: C \to J$ be an embedding of C to its jacobian J by sending x to the class of $x - D$. Write*

$$a'(D) = \liminf_{x \in C(\bar{K})} h_{NT}(j_D(x)),$$

$$a(D) = \inf_{x \in C(\bar{K})} h_{NT}(j_D(x)).$$

Then we have the following estimate

$$a'(D) \geqq \frac{(\omega_a, \omega_a)}{4(g-1)} + \left(1 - \frac{1}{g}\right) h_{NT}\left(D - \frac{\omega}{2g-2}\right) \geqq \frac{1}{2}(a(D) + a'(D)).$$

Proof. By (5.4) and adjunction formula it follows that

$$h_{NT}(j_D(x)) = -(x - D, x - D)_a = (O(x_a), \omega_a(2D_a)) - (D, D)_a = h_L(x),$$

where $L = \omega_a(2D_a - (D, D)_a)$. Now by Theorem 5.3, the above theorem follows if we can prove the middle term (I) of the above inequality is equal to the middle term (II) $= \frac{(L, L)}{2\deg L_K}$ in (5.3). Computing (II) directly, and (I) by replacing h_{NT} term by $-()^2_a$, both (I) and (II) are equal to

$$\frac{1}{4g}(\omega_a, \omega_a) + \frac{1}{g}(O(D_a), \omega_a) - \left(1 - \frac{1}{g}\right)(D, D)_a.$$

This completes the proof of the theorem.

Corollary 5.7 (1) *The self-intersection (ω_a, ω_a) is always nonnegative, and is* 0 *if and only if there is a sequence of distinct points $x_1, x_2, \ldots,$ such that $h_{NT}((2g-2)x_n - \omega)$ converges to* 0.

(2) *If $\omega(-(2g-2)D)$ is not a torsion line bundle then $a'(D) > 0$.*

(3) *The self-intersection $(\omega_{Ar}, \omega_{Ar})$ is positive, if there is a non-archimedean place of K over which C does not have potentially good reduction.*

Proof. The assertion (1) follows by setting $D = \frac{\omega}{2g-2}$ in (5.6). The assertion (2) follows from (1) and (5.6). The assertion (3) follows from (5.5) and (1).

(5.8) Remarks. (1) When C has a potentially good reduction on all non-archimedean places of K, the 'if' parts of (1) and (2) are due to Szpiro [S]. It is his assignments to author to generalize his result. Part (3) is proved first time in [Z] by a computation of Green's function. In the case that C has a reducible stable reduction at a place of K, Burnol recently gave a different proof using Weierstrass points.

(2) We actually proved the Bogomolov conjecture for a big class of situations. This conjecture claims that for any embedding of a non-elliptic and nonisotrivial curve C to an abelian variety A over a global field, there are only finitely many small points. Notice for a general curve, its jacobian has Neron–Severi group of rank 1, so any such embedding can be factored to j_D for some D.

(3) As a consequence of Bogomolov's conjecture, (ω_a, ω_a) should always be positive if C is nonisotrivial.

Appendix: The Green's function on a metrized graph

(a.1) By a metrized graph, we mean a locally metrized and compact topological space G which has the following properties: For any $p \in G$, there is an $\varepsilon > 0$, an

integer $d = v(p) > 0$, and an open neighborhood U_ε for which we have an isometric map

$$\phi: U_\varepsilon \to S_{d,\varepsilon} = \{re^{\frac{2\pi ik}{d}} \in \mathbb{C} : 0 \leqq r < \varepsilon, 0 \leqq k < d\}.$$

Let E_i be the connected components of $U - \{p\}$. We denote by x_i the restrictions of the function $x = r \cdot \phi$ on E_i.

(a.2) We want to do some harmonic analysis on a metrized graph G. For simplicity we restrict our discussion to the space $F(G)$ of continuous and piecewise smooth real functions on G. Let f be a continuous function on G. We say f is piecewise smooth if for any point p, the restriction of f on a neighborhood U_ε of p is smooth in x_i and all derivatives have limits as $x_i \to 0$.

Let $f \in F(G)$. We may define a functional f'' on G in the following way. If p is a point in G satisfies $v(p) = 2$ and $\lim_{x_1 \to 0} f''(x_1) = \lim_{x_2 \to 0} b''(x_2)$, then we denote this limit by $f''(p)$. Notice that f'' is defined at all but finitely many points on G, and is piecewise smooth on G, so it defines a linear function on $L(G)$. For any $g \in F(G)$,

$$\langle f'', g \rangle = \int_G f'' g \, d\mu,$$

where $d\mu$ is defined locally as $|dx|$.

We also define a Dirac-function associated to f: Let p be a point on G. The linear functional $\delta f(p)$ on $F(G)$ is defined so that for any $g \in F(G)$, we have

$$\langle \delta f(p), g \rangle = g(p) \sum_i \lim_{x_i \to 0} f'(x_i).$$

It is easy to see that $\delta f(p)$ is zero at all but finitely many points of G so $\delta f = \sum_p \delta f(p)$ is a well defined linear function on $F(G)$.

Definition a.3 The *Laplacian Δ* is defined to be the following linear map from the space $F(G)$ to the space of linear functions of $F(G)$:

$$\Delta f = -f'' - \delta f,$$

for all f in $F(G)$.

Lemma a.4 (1) *If G is a union of two subspaces G_1 and G_2 so that $G_1 \cap G_2$ is a finite subset, then*

$$\Delta f = \Delta f|_{G_1} + \Delta f|_{G_2}$$

(2) *The Laplacian Δ is self-adjoint. For any two functions f, g in $F(G)$, we have*

$$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle.$$

(3) *The Laplacian Δ is a semi-positive. For any f in $F(G)$, we have $\langle f, \Delta f \rangle \geqq 0$ and $\langle f, \Delta f \rangle = 0$ if and only if f is locally constant.*

Proof. Part (1) follows from definition. By (1) we may reduce (2) and (3) to the case that G a closed line segment $[0, l]$ and f and g are smooth functions on $(0, l)$. By definition we have that

$$\langle \Delta f, g \rangle = \int_0^l f' g' \, d\mu.$$

Parts (2) and (3) follow immediately.

(a.5) As one example, let us compute the Laplacian of a continuous and piecewise linear function on a connected G. We say a subset V of finite points of G is a vertex set if $G-V$ is a disjoint union of open line segments. We say a function f is piecewise linear if there is a vertex set V such that f is linear on $G - V$. It is easy to see that the morphism $f \to f|_V$ gives an isomorphism from the space of piecewise linear functions with vertex V to the space $\mathbb{R}^{|V|}$ of functions on V. We also have an isomorphism ψ_V from $\mathbb{R}^{|V|}$ to the space of Dirac-functions with support in V: $\langle \psi_V(c), g \rangle = \sum c_v g(v)$. Now we can define an endomorphism L_V of $\mathbb{R}^{|V|}$ such that for any piecewise linear function f with vertex V we have that

$$\Delta f = \psi_V \cdot L_V(f|_V).$$

Precisely, L_V is defined as follows:

$$L_V(c)(v) = \sum_{v' \in V} \sum_{e \in E_{vv'}} \frac{c_{v'} - c_v}{l(e)}.$$

Lemma a.6 *Let G be a connected metrized graph. If we denote by H_V the subspace of $\mathbb{R}^{|V|}$ consisting of all elements c with $\sum_v c_v = 0$, then L_V has image H_V. The kernel of L_V consists of all constant functions on V. Especially L_V is invertible over H_V.*

Proof. Let c be in $\mathbb{R}^{|V|}$ such that $L_V(c) = 0$. Then we can find a continuous function f_c on G such that $f_c|_V = c$ and f_c is linear on $G - V$. Then by the argument in (a.5) we have that $\Delta f = 0$. By (a.4) we have that f is locally constant. Since G is connected, this implies that f is constant. This implies that c is constant and the kernel of L_V is one dimensional. The other assertions follow immediately.

The main result of this section is the following result about the Green's function:

Theorem a.7 *Let G be a connected metrized graph. There is a unique function $g(p, q)$ on $G \times G$ which satisfies the following conditions:*

(1) *g is continuous and piecewise smooth in both p and q.*

(2) *For each fixed p, as a function of q, we have that*

$$\Delta g(p, q) = \delta_p - 1/\text{volume}(G).$$

(3) *For each fixed p, we have $\int_G g(p, q)\, d\mu(q) = 0$.*

Proof. The uniqueness follows from (a.4). We need to prove the existence of a $g(p, q)$ which satisfies (2), (3), and the continuity (1) of $g(p, q)$. Without loss of generality, we assume that volume$(G) = 1$. Fix a vertex set V_0 of G such that no element is connected to itself by a line segment of $G - V_0$.

Fix a point p_0 in G. We need only find a function in $F(G)$ which satisfies the condition (2), since $f - \int f$ satisfies both (2) and (3). Let $V = V_0 \cup \{p_0\}$. Since $f'' = 1$ on $G - V$, it follows that f is determined completely by its values on V. We want to use $f|_V$ to compute δf. Let $v \neq v'$ be two elements in V connected by a line segment e in $G - V$. We have a unique isometric map from $\bar{e}$ to $[0, l(e)]$ such that the image of v is 0. The condition that $f'' = 1$ on e gives that

$$f = \tfrac{1}{2} t^2 + at + b.$$

So

$$f'(0) = \frac{f(v') - f(v)}{l(e)} - \frac{1}{2} l(e).$$

This implies that

$$\delta f(v) = L_V(f|_V)(v) - \frac{1}{2}\sum_{e\in E_v} l(e)$$

where E_v is the set of all line segments in G_V which connect v with points $v' \neq v$. The existence of f is equivalent the existence of the solution c of the following equation:

(a.7.1) $$L_V(c) = l$$

where l is in $\mathbb{R}^{|V|}$ and l is given by

$$l_v = \begin{cases} \frac{1}{2}\sum_{e\in E_v} l(e), & v \neq p. \\ \frac{1}{2}\sum_{e\in E_v} l(e) - 1, & v = p. \end{cases}$$

Since the $\sum_v l_v = 0$, it follows from (a.6) that (a.7.1) has at least one solution c in H_V.

Now we need to prove that $g(p, q)$ is continuous and piecewise smooth in p. Notice that if p is not in V_0, the coefficients in equation (a.7.1) are all smooth in p. This implies that the solution c is smooth in p. This gives us a function $f(p, q)$ which is piecewise smooth in p and satisfies condition (2). Now $g(p, q) = f(p, q) - \int f(p, q)\, d\mu(q)$ is smooth in p and satisfies (2) and (3).

It remains to prove that $g(p, q)$ is continuous in p. For a fixed p_0 we want to show that

$$\lim_{p\to p_0} g(p, q) = g(p_0, p)$$

holds uniformly in q. Let $h_p(q) = g(p_0, q) - g(p, q)$. Then we want to use the fact that

(a.7.2) $$\Delta h_p(q) = \delta_{p_0} - \delta_p \quad \text{and} \quad \int h_p = 0$$

to prove our assertion. As $p \to p_0$ we may assume that p and p_0 are in the same line segment e in $G - V$ and $p \notin V$. The point p cuts e in two parts: one part e_1 connects to p_0 and another part e_2 connects $v_0 \in V_0$. Notice that h_p is a piecewise linear function with the vertex set $V \cup \{p\}$, we have

$$\Delta h_p = \psi'_V(L'_V(h|_V)) + \frac{h(v_0) - h(p)}{l(e_2)}\delta_{v_0} + \frac{h(p_0) - h(p)}{l(e_1)}\delta_{p_0}$$

(a.7.3) $$+ \left(\frac{h(p) - h(v_0)}{l(e_2)} + \frac{h(p) - h(p_0)}{l(e_1)}\right)\delta_p,$$

where ψ'_V and L'_V are defined with respect to the subgraph $G' = G - \{e\}$. By (a.7.2) it follows that the coefficient of δ_p is -1. This implies that

$$h(p) = \frac{l(e_1)}{l(e)}h(v_0) + \frac{l(e_2)}{l(e)}h(p_0) - \frac{l(e_1)l(e_2)}{l(e)},$$

and in turn $L_V(h|_V) = l(e_1)d$, where

$$d = \begin{cases} \dfrac{1}{l(e)}, & \text{if } v = v_0 \\ -\dfrac{1}{l(e)}, & \text{if } v = p_0 \\ 0, & \text{otherwise.} \end{cases}$$

By (a.6), we have a unique h^0 in H_V such that $L_V(h^0|_V) = d$. Let h^1 be the unique piecewise linear function with vertex set $V \cup \{p\}$ such that

$$h^1|_V = l(e_1)h^0|_V,$$

and

$$h^1(p) = l(e_1)\left(\frac{l(e_1)}{l(e)}h^0(v_0) + \frac{l(e_2)}{l(e)}h^0(p_0) - \frac{l(e_2)}{l(e)}\right).$$

Then h^1 satisfies the same differential equation (a.7.3) as h. This implies that $h = h^1 - \int h^1$. As $p \to p_0$, we have $l(e_1) \to 0$, so $h^1 \to 0$ and therefore $h \to 0$ uniformly in q. This completes the proof of the theorem.

(a.8) To conclude this section let us give explicitly the Green's function on a circle. Let G be a circle of length l and choose a coordinate t on G with $0 \leqq t < l$. Let p and q be two points on G. Then we have

$$g(p, q) = \frac{1}{2l}(t(p) - t(q))^2 - \frac{1}{2}|t(p) - t(q)| + \frac{l}{12}.$$

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