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Geometric Reductivity at Archimedean Places

Shouwu Zhang

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Introduction

Let $G \to GL(n, \mathbb{C})$ be a representation of a complex reductive group. A theorem of Hilbert says that the algebra $\mathbb{C}[x_1, \ldots, x_n]^G$ of invariant polynomials is finitely generated. Let Y be the projective variety defined by this graded algebra, and then we have a rational morphism $\pi : C^n \cdots \to Y(\mathbb{C})$. A theorem on geometric reductivity of Mumford says that a point $x \in \mathbb{C}^n$ is regular for the map π if and only if the closure of the orbit Gx does not contain the origin 0. Such results have been generalized to more general bases by Haboush and Seshadri, et al., and have been used in constructing moduli spaces of various geometric objects.

We would like to have some analogous results in the Arakelov theory, and apply them to the arithmetic problem. In other words, we want to consider a representation for a reductive group over $\mathbb{Z}$, and invariants with length induced from the standard hermitian structure of $\mathbb{C}^n$. The first paper to appear on this aspect was Burnol's [B], in which he proved a p-adic analogue of a result of Kempf and Ness on stability and length function.

In this paper, we want to prove some analogues of Hilbert's theorem and Mumford's theorem in Arakelov theory. More precisely, we will formulate and prove the geometric reductivity of a reductive group over archimedean places, and give a Hilbert-Samuel formula for the volumes of integral invariants. More details are explained in what follows.

In §1, following a paper of Burnol [B], we will analyze the geometric reductivity of Mumford-Seshadri over a discrete valuation ring in terms of valuations. This will lead to a notion of geometric reductivity at archimedean places. Then we will explain that

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the geometric reductivity in some sense is the ampleness of the quotient metrized line bundle on the quotient variety.

In §2, we will prove the ampleness of the quotient metric for a linear action of a complex reductive group on an arithmetic polarized variety. The proof will use the Mumford theorem [MF] on the geometric reductivity, a theorem in [Z1] about the lifting of sections with small norms, and integration over the maximal compact subgroup. This proof is luckily much simpler than the Seshadri proof [S] for nonarchimedean places.

In §3, we will give a Hilbert-Samuel formula for integral invariants as a direct application of the archimedean geometric reductivity.

I hope that the results in this paper can be used to study the Arakelov theory of moduli spaces.

1 Geometric reductivity at archimedean places

In the first half of this section, following Burnol [B] with some modifications, we will express the geometric reductivity of Seshadri [S] in terms of p-adic norms. Let K be a finite extension of p-adic numbers $\mathbb{Q}_p$, R the valuation ring of K, and G_R a reductive group scheme over $\operatorname{Spec} R$ with a linear action on an n-dimensional affine space $\mathbb{A}^n_R$. A section E of $\mathbb{A}^n_R$ over $\operatorname{Spec} R$ is called semistable if the Zariski closure $\overline{o(E)}$ of the orbit of E is disjoint with the 0-section. The geometric reductivity conjectured by Mumford, proved by Haboush [H], and generalized by Seshadri [S], is the following statement.

Statement 1.1. For any semistable section E, there is an invariant homogeneous polynomial f of positive degree with coefficients in R such that $f(E) \in R^*$.

We say that a K point x of $\mathbb{A}^n$ is residually semistable if there is an element $\alpha \in K^*$ such that αx can be extended to a semistable section of $\mathbb{A}^n_R$ over $\operatorname{Spec} R$. By base change we may define the residual semistability for each $\overline{\mathbb{Q}}_p$ point of $\mathbb{A}^n$.

Let $\|\cdot\|$ be the norm on $\overline{\mathbb{Q}}_p^n$ defined as follows:

$$\|(x_1, \ldots, x_n)\| = \max(|x_1|, \ldots, |x_n|).$$

Then it is clear that (i) the set of residually semistable $\overline{\mathbb{Q}}_p$ points is the set of minimal $\overline{\mathbb{Q}}_p$ points x:

$$\|gx\| \geq \|x\|$$

for all $g \in G_R(\overline{\mathbb{Q}}_p)$; and (ii) each residually semistable point is semistable under the action of $G_{\overline{\mathbb{Q}}_p}$.

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For a homogeneous polynomial

$$f(x_1, \dots, x_n) = \sum_{i_1+\dots+i_n=d} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

define $\|f\|(x) = |f(x)|/\|x\|^d$. Then one can prove that

$$\max(|a_{i_1,\dots,i_n}| : i_1 + \dots + i_n = d) = \sup_{x\overline{\mathbb{Q}}_p^n} \|f\|(x),$$

and that Statement 1.1 follows from the next statement.

Statement 1.2. If $x \in \overline{\mathbb{Q}}_p^n$ is a semistable point, then there is a nonzero homogeneous invariant polynomial f of positive degree such that

$$\sup_{y\in o(x)} \|f\|(y) = \sup_{y\in\overline{\mathbb{Q}}_p^n} \|f\|(y),$$

where $o(x)$ denotes the orbit $G(\bar{\mathbb{Q}})x$.

Let us prove that Statement 1.2 implies Statement 1.1 as follows. Assume Statement 1.2, and fix a semistable section E of $\mathbb{A}_R^n$ over Spec R. Denote by x the corresponding geometric point at the generic fiber, and let f be as in Statement 1.2 for x. Replacing f by a multiple, we may assume that $\sup_{y\in\overline{\mathbb{Q}}_p^n} \|f\|(y) = 1$. It follows that f has integral coefficients. Since x is minimal and $\|x\| = 1$, the left-hand side of the equation in Statement 1.2 is $|f(x)|$ while the right-hand side is 1. It follows that $f(E) \in R^*$.

Conversely, let us prove as follows that Statement 1.2 is implied by Statement 1.1 and the following semistable reduction theorem: if x is a semistable point of $\mathbb{A}^n$ with respect to action of $G_{\overline{\mathbb{Q}}_p}$, then the Zariski closure $\overline{o(x)}$ has a residually semistable point. Assume Statement 1.1 and the semistable reduction theorem, and fix a semistable point x of $\mathbb{A}^n$. Let x_0 be a residually semistable point (i.e., a minimal point) in the Zariski closure of $o(x)$, and let f be an integral polynomial as in Statement 1.1 for the section E extending $x_0/\|x_0\|$. Now the right-hand side of the equation in Statement 1.2 is 1, while the left-hand side is $|f(x_0)|/\|x_0\|^{\deg f}$. Since $f(E) \in \overline{\mathbb{Z}_p}^*$, $|f(y)|/\|y\|^{\deg f} = 1$. Statement 1.2 follows.

It is obvious that Statement 1.2 implies the following.

Statement 1.3. If $x \in \mathbb{A}^n(\overline{\mathbb{Q}}_p)$ is a semistable point and ϵ is a positive number, then there is a nonzero homogeneous invariant polynomial f of positive degree such that

$$\sup_{y\in o(x)} \|f\|(y) \geq e^{-\deg f\epsilon} \sup_{y\in\overline{\mathbb{Q}}_p^n} \|f\|(y).$$

Conversely, let us prove as follows that Statement 1.2 is implied by Statement 1.3 and the finiteness of invariants: The algebra $R[x_1, x_2, \dots, x_n]^G$ of invariants is of finite type over R. Assume Statement 1.3 and the finiteness of invariants, and fix a semistable point

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x of $\mathbb{A}^n$ and an $\epsilon > 0$. Let $f_1, \ldots, f_m$ be generators of the ring of invariants, and let f be as in Statement 1.3. Replacing f by a multiple, we may assume that f has supremum norm 1. It follows that f has integral coefficients and therefore f is a polynomial of $f_1, \ldots, f_m$ with integral coefficients. Now, for any y, one has

$$|f(y)|^{1/\deg f} \leq \max\{|f_1(y)|^{1/\deg f_1}, \ldots, |f_m(y)|^{1/\deg f_m}\}.$$

The inequality in Statement 1.3 implies that

$$\max\left\{\sup_{y\in o(x)} \|f_1\|(y)^{1/\deg f_1}, \ldots, \sup_{y\in o(x)} \|f_m\|(y)^{1/\deg f_m}\right\} \geq e^{-\epsilon}.$$

Letting $\epsilon \to 0$, the equality in Statement 1.2 holds for one of these f_i's.

It is well known that the semistable reduction theorem and the finiteness of invariants are both true; see papers of Burnol [B] and Seshadri [S]. The three Statements 1.1, 1.2, and 1.3 are therefore equivalent.

In the second half of this section we want to formulate the notion of geometric reductivity at archimedean places. Let G be a complex reductive group with a linear action on $\mathbb{C}^n$ such that a maximal compact subgroup U of G fixes the standard hermitian norm $\|(z_1, \ldots, z_n)\|^2 = \sum |z_i|^2$ on $\mathbb{C}^n$. We define $\|f\|(x) = |f(x)|/\|x\|^{\deg f}$ as before for homogeneous polynomial f. The corresponding Statements 1.1, 1.2, and 1.3 at archimedean places are the following.

Statement 1.4. If x is a nonzero element in $\mathbb{C}^n$ such that $\|x\| = 1 \leq \|gx\|$ for any $g \in G(\mathbb{C})$, then there is an invariant nonzero homogeneous polynomial f of positive degree such that f has supremum norm ≤ 1 and $|f(x)| = 1$.

Statement 1.5. If $x \in \mathbb{C}^n$ is a semistable point, then there is a nonzero homogeneous invariant polynomial f of positive degree such that

$$\sup_{y\in o(x)} \|f\|(y) = \sup_{y\in\mathbb{C}^n} \|f\|(y).$$

Statement 1.6. If $x \in \mathbb{C}^n$ is a semistable point and ϵ is a positive number, then there is a nonzero homogeneous invariant polynomial f of positive degree such that

$$\sup_{y\in o(x)} \|f\|(y) \geq e^{-\deg f \epsilon} \sup_{y\in\mathbb{C}^n} \|f\|(y).$$

As in the first half of this section, Statements 1.4 and 1.5 are equivalent and imply Statement 1.6 since the semistable reduction theorem is trivial at archimedean places. However, I do not know if Statement 1.6 implies 1.5.

In §2 we will prove Statement 1.6, which is sufficient for proving a Hilbert-Samuel formula for integral invariants in §3.

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2 Ampleness of quotient-metrized line bundles

For an ample line bundle $\mathcal{L}$ on a projective complex variety X with a continuous metric $\|\cdot\|$, we call $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ an ample metrized line bundle if, for any $x \in X$ and any $\epsilon > 0$, there is a nonzero section $l \in \Gamma(\mathcal{L}^n)$ for some $n > 0$ such that

$$\|l\|_{\sup} \le \|l\|(x)e^{\epsilon n},$$

where $\|l\|_{\sup} = \sup_{x\in X(\mathbb{C})} \|l\|(x)$.

If X is regular and the metric on $\mathcal{L}$ is smooth, this condition is equivalent to that of the curvature $c'(\bar{\mathcal{L}})$ being semipositive everywhere. In general, $\|\cdot\|$ is ample if and only if there is a sequence of embeddings

$$i_n : X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$$

such that

(a) $i^*\mathcal{O}(1) = \mathcal{L}^{e_n}$ with $e_n > 0$, and

(b) if $\|\cdot\|_n$ denotes $(i_n^*\|\cdot\|_{\mathcal{O}(1)})^{1/e_n}$ on $\mathcal{L}$, then $\log\|\cdot\|_n$ converges uniformly to $\log\|\cdot\|$, where $\|\cdot\|_{\mathcal{O}(1)}$ denotes the Fubini-Study metric on $\mathcal{O}(1)$.

One important property of ample metric is the following lemma.

Lemma 2.1. Let X be a complex variety, $\bar{\mathcal{L}}$ an ample metrized line bundle on X, Z a closed subvariety of X, and $l \in \Gamma(Z, \mathcal{L}|_Z)$ a section. Then, for any $\epsilon > 0$, there is a section $l' \in \Gamma(X, \mathcal{L}^n)$ with $n > 0$ such that $l'|_Z = l^n$ and

$$\sup_{x\in X} \|l\|(x) \le \sup_{x\in Z} \|l'\|(x)e^{n\epsilon}.$$

Proof. See §3 in [Z1].

Consider a projective complex variety X, an ample metrized line bundle $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on X, a complex reductive group G, a maximal compact subgroup U of G, and a linear action

$$\sigma : G \times X \to X, \qquad \phi : \sigma^*\mathcal{L} \simeq p_2^*\mathcal{L}$$

of G on $(X, \mathcal{L})$. We assume that this action is hermitian with respect to U. This means that $\phi|_{U\times X}$ is an isometry of hermitian line bundles. Let gx denote $\sigma(g, x)$, and let $l(gx)$ denote $\phi(\sigma^* l(g, x))$.

Denote by $\pi : X^{ss} \to Y = X^{ss}//G$ the uniform categorical quotient of the set X^{ss} of semistable points of X by G. As schemes, X is $\mathrm{Proj}(\oplus_{n\ge 0}\Gamma(X, \mathcal{L}^n))$ and Y is $\mathrm{Proj}(\oplus_{n\ge 0}\Gamma(X, \mathcal{L}^n)^G)$. Denote by $\mathcal{M}$ the line bundle on Y induced by the graded algebra $\oplus_{n\ge 0}\Gamma(X, \mathcal{L}^n)^G$, and then $\pi^*\mathcal{M}$ is naturally identified with $\mathcal{L}_{X^{ss}}$. We define a metric on $\mathcal{M}$ as follows: for any $y \in Y$ and any $m \in \mathcal{M}(y)$,

$$\|m\|(y) = \sup_{x\in\pi^{-1}(y)} \|\pi^* m\|(x).$$

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It is clear that $\|m\|_{\sup} = \|\pi^* m\|_{\sup}$ for any section $m \in \Gamma(Y, \mathcal{M}^n)$ whenever $\pi^* m$ can be extended to a global section of $\mathcal{L}$ on X.

Theorem 2.2. The metric defined as above on $\mathcal{M}$ is ample.

Proof. Fix a positive number ϵ and a point y of Y. Denote by Z the Zariski closure of $\pi^{-1}(y)$ in X. Since $\mathcal{M}$ is ample, there is a section m_1 of a positive power $\mathcal{M}^{n_1}$ on Y such that $m_1(y) \neq 0$. Replacing n_1 by a multiple, we may assume that $\pi^* m_1$ can be extended to an invariant section l_1 of $\mathcal{L}^{n_1}$ on X. By Lemma 2.1, there is a section $l_2 \in \Gamma(X, \mathcal{L}^{n_1 n_2})$ with $n_2 > 0$ such that $l_2|_Z = l_1|_Z^{n_2}$ and

$$\|l_2\|_{\sup} \le \|l_1|_Z^{n_2}\|_{\sup} e^{\epsilon n_1 n_2}.$$

Consider the section

$$l(x) = \int_U l_2(ux)\, du$$

of $\Gamma(X, \mathcal{L}^{n_1 n_2})$, where du is the invariant measure of U with volume 1. Since G is the complexification of the real Lie group U and l is invariant under U, it follows that U is Zariski dense in G and l is G invariant. From $l|_Z = l_1|_Z^{n_2}$ one has

$$\begin{aligned}\|l\|_{\sup} &\le \int_U \sup_{x \in X(\mathbb{C})} \|l_2(ux)\|\, du = \|l_2\|_{\sup} \\ &\le \|l_1|_Z^{n_2}\|_{\sup} e^{\epsilon n_2} = \|l|_Z\|_{\sup} e^{\epsilon n_2}.\end{aligned}$$

There is a section $m \in \Gamma(Y, \mathcal{M}^n)$ with $n = n_1 n_2$ such that $l = \pi^* m$. One sees that $m \neq 0$, and

$$\|m\|_{\sup} \le \|m\|(y) e^{\epsilon n}.$$

This completes the proof of the theorem.

Proof of Statement 1.6. Denote by $p : \mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}(\mathbb{C})$ the canonical map. Then

$$\mathbb{C}[x_1, \ldots, x_n] = \oplus_{d \ge 0} \Gamma(\mathbb{P}^{n-1}, \mathcal{O}(d)),$$

and $\|f\|(x)$ is the induced metric on $\mathcal{O}(\deg f)$ by the standard Fubini-Study metric on $\mathcal{O}(1)$. Denote by

$$\pi : (\mathbb{P}^{n-1})^{ss} \to Y = (\mathbb{P}^{n-1})^{ss} // G$$

the uniform categorical quotient, and by $\mathcal{M}$ the quotient line bundle; then

$$\mathbb{C}[x_1, \ldots, x_n]_d^G = \Gamma(Y, \mathcal{M}^d)$$

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for $d \gg 0$, where $\mathbb{C}[x_1, \dots, x_n]_d^G$ is the set of invariant homogeneous polynomials of degree d. Since $\sup_{y\in Y} \|m\|(y) = \sup_{x\in\mathbb{P}^{n-1}(\mathbb{C})} \|\pi^* m\|(x)$, the reductivity Statement 1.6 is reduced to the ampleness of the metric on $\mathcal{M}$: for any $\epsilon > 0$ and any $y \in Y$, there is a nonzero section $m \in \Gamma(Y, \mathcal{M}^d)$ for some $d > 0$ such that $\|m\|(y) \geq e^{-\epsilon d} \sup_{z\in Y} \|m\|(z)$. The reductivity Statement 1.6 therefore follows from Theorem 2.2.

3 Hilbert-Samuel formula for integral invariants

Consider a number field K, a projective variety X over Spec $\mathcal{O}_K$, an ample line bundle $\mathcal{L}$ on X with an ample metric $\|\cdot\|$ on $\mathcal{L}_{\mathbb{C}}$ that is invariant under the complex conjugation, and a reductive group scheme G over Spec $\mathcal{O}_K$ with a linear action on $(X, \mathcal{L})$. Assume that, for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, this linear action induces a hermitian action of $G_\sigma = G \times_{\mathcal{O}_K,\sigma} \mathbb{C}$ on $(X_\sigma, \mathcal{L}_\sigma) = (X \times_\sigma \mathbb{C}, \mathcal{L} \otimes_\sigma \mathbb{C})$ with respect to a maximal compact subgroup U_σ of G_σ.

For each $n \geq 0$, let $\|\cdot\|_{\sup}$ denote the norm on $\Gamma(X, \mathcal{L}^n) \otimes_{\mathbb{Z}} \mathbb{R}$ as usual:

$$\|l\|_{\sup} = \sup_{x\in X(\mathbb{C})} \|l\|(x).$$

There is a unique invariant measure μ of $\Gamma(X, \mathcal{L}^n)_{\mathbb{R}}^G$ such that $\mu(B_n^G) = 1$, where $B_n^G = \{l \in \Gamma(X, \mathcal{L}^n)_{\mathbb{R}}^G : \|l\|_{\sup} \leq 1\}$. Define the Euler characteristic of $\Gamma(X, \mathcal{L}^n)$ as follows:

$$\chi\left(\Gamma(X, \bar{\mathcal{L}}^n)^G\right) = -\log \mathrm{vol}\left(\Gamma(X, \bar{\mathcal{L}}^n)_{\mathbb{R}}^G / \Gamma(X, \mathcal{L}^n)^G\right).$$

Theorem 3.1. The following limit exists:

$$\lim_{n\to\infty} \frac{\chi\left(\Gamma(X, \bar{\mathcal{L}}^n)^G\right)}{n \dim \Gamma(X_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q}}^n)^G}.$$

Proof. Denote by Y the uniform categorical quotient of X by G, and by $\overline{\mathcal{M}}$ the quotient line bundle with the quotient metric defined as in §2. Then, by Theorem 2.2, $\overline{\mathcal{M}}$ is an ample metrized line bundle on X. One has a Hilbert-Samuel formula for $\overline{\mathcal{M}}$ as in [Z2]:

$$\chi\left(\Gamma(Y, \overline{\mathcal{M}}^n)\right) = (n h_{\overline{\mathcal{M}}}(Y) + o(n)) \dim_{\mathbb{Q}} \Gamma(Y_{\mathbb{Q}}, \mathcal{M}_{\mathbb{Q}}^n),$$

where $h_{\overline{\mathcal{M}}}(Y)$ is the height of Y with respect to $\overline{\mathcal{M}}$. The theorem follows from the fact that

$$\Gamma(Y, \overline{\mathcal{M}}^n) = \Gamma(X, \bar{\mathcal{L}}^n)^G.$$

We want to give an application to the case that a reductive group acts on an affine space. Let K be a number field, and G a reductive group over Spec $\mathcal{O}_K$ with a linear action on $\mathbb{A}^r_{\mathcal{O}_K}$. Assume that for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, there is a maximal compact subgroup U_σ of $G_\sigma = G \times_{\mathrm{Spec}\,\mathcal{O}_k,\sigma} \mathbb{C}$ that fixes the standard hermitian inner product of $\mathbb{C}^r$. One defines a norm $\|f\|$ of a homogeneous polynomial f as follows: if

$$f(x_1, \dots, x_r) = \sum_{i_1+\cdots+i_r=d} a_{i_1,\dots,i_r} x_1^{i_1} \cdots x_r^{i_r} \in \mathbb{C}[x_1, \dots, x_n]_d,$$

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then

$$\|f\|^2 = \sum_{i_1+\cdots+i_r=d} |a_{i_1,\ldots,i_r}|^2 \frac{i_1!\cdots i_r!}{d!};$$

if $f \in K[x_1,\ldots,x_r]_d \otimes_{\mathbb{Q}} \mathbb{R}$, then

$$\|f\|^2 = \sum_{\sigma K \hookrightarrow \mathbb{C}} \|\sigma f\|^2.$$

Theorem 3.2. There is a constant h such that

$$-\log \operatorname{vol}\left(K[x_1,\ldots,x_r]_d^G \otimes_{\mathbb{Q}} \mathbb{R} / \mathcal{O}_K[x_1,\ldots,x_r]\right) = (dh + o(d)) \dim(\mathbb{R}[x_1,\ldots,x_r]_d^G),$$

where the volume is computed for hermitian norm $\|\cdot\|$ defined as above.

Proof. We claim for any $\epsilon > 0$ that

$$\|\cdot\|_{\sup} e^{-d\epsilon} \le \|\cdot\| \le \|\cdot\|_{\sup} e^{d\epsilon}$$

for $d \gg 0$, where $\|\cdot\|_{\sup}$ is the norm defined as in §1. For a homogeneous polynomial f of degree d, as a section of bundle $\mathcal{O}(d)$ on $\mathbb{P}^{r-1}$, define the L^2-norm of f as follows:

$$\|f\|_{L^2}^2 = \sum_{\sigma:K\to\mathbb{C}} \int_{\mathbb{P}^{r-1}(\mathbb{C})} \frac{|\sigma(f)(x)|^2}{(\sum_{i=1}^r |x_i|^2)^d}\, dx,$$

where dx is the unique measure on $\mathbb{P}^{r-1}(\mathbb{C})$ which is invariant under the action of U and has volume 1. Then

$$\|\cdot\| = \frac{d!(r-1)!}{(d+r-1)!} \|\cdot\|_{L^2}.$$

Our claim follows from this equality and an inequality of Gromov: there is a constant C independent of d such that

$$\|\cdot\|_{L^2} \le \|\cdot\|_{\sup} \le C d^{r-1} \|\cdot\|_{L^2}.$$

This proves our claim. Now, to prove the theorem, it suffices to prove the statement for norm $\|\cdot\|_{\sup}$; this is a special case of Theorem 3.1.

Remark 3.3. Theorems 3.1 and 3.2 give only the existence of h. It should be an interesting problem to find h for some concrete examples, such as representations of SL(2).

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Department of Mathematics, Princeton University, Princeton, New Jersey 08544, USA; szhang@math.princeton.edu

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