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Heights of Heegner points on Shimura curves

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Introduction

The purpose of this paper is to generalize some results of Gross-Zagier [20] and Kolyvagin [28] to totally real fields. The main result and the plan of its proof are described as follows.

Main results. Let F be a totally real number field and N a nonzero ideal of $\mathcal{O}_F$. Let f be a newform on $\mathrm{GL}_2(\mathbb{A}_F)$, of (parallel) weight 2, level $K_0(N)$, and with trivial central character, where $K_0(N)$ denotes the subgroup of $\mathrm{GL}_2(\widehat{F})$:

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \middle| c \in \widehat{N} \right\},$$

where for an abelian group M, $\widehat{M}$ denotes $M \otimes \prod_p \mathbb{Z}_p$. Let $\mathcal{O}_f$ denote the subalgebra of $\mathbb{C}$ over $\mathbb{Z}$ generated by eigenvalues $a(f, m)$ of f under the Hecke operators. For each embedding $\sigma : \mathcal{O}_f \to \mathbb{C}$, let f^σ denote the newform with the eigenvalues $a(f^\sigma, m) = a(f, m)^\sigma$. Assume that either $[F : \mathbb{Q}]$ is odd or $\mathrm{ord}_v(N) = 1$ for at least one finite place v of F. Then there is an abelian variety A over F of dimension $[\mathcal{O}_f : \mathbb{Z}]$ such that $L(s, A)$ equals $\prod_{\sigma:\mathcal{O}_f\to\mathbb{C}} L(s, f^\sigma)$ modulo the factors at places dividing N. Our main result is the following:

Theorem A. *Assume the L-function $L(s, f)$ has order ≤ 1 at $s = 1$. Then for any A as above*:

1. *The Mordell-Weil group $A(F)$ has rank given by*

$$\mathrm{rank}\, A(F) = [\mathcal{O}_f : \mathbb{Z}]\, \mathrm{ord}_{s=1} L(s, f);$$

2. *The Shafarevich-Tate group $\text{Ш}(A)$ is finite.*

The theorem would hold with a weaker condition that either $[F : \mathbb{Q}]$ is odd or $\mathrm{ord}_v(N)$ is odd for at least one finite place, provided we could overcome one technical difficulty. That is, it would be enough to prove Lemma 5.2.3 (below) without the assumption that $\mathrm{ord}_\wp(N) \le 1$.

Shimura curves. As in the case $F = \mathbb{Q}$ treated by Gross-Zagier and Kolyvagin, we will prove the theorem by studying Heegner points over some imaginary quadratic extension. Let E be a totally imaginary quadratic extension of F which is unramified over places dividing N. Assume further $\varepsilon(N) = (-1)^{g-1}$ where $g = [F : \mathbb{Q}]$ and

$$\varepsilon = \otimes_v \varepsilon_v : F^\times \backslash \widehat{F}^\times \to \{\pm 1\}$$

is the character on $\mathbb{A}_F^\times / F^\times$ associated to the extension E/F. Let τ be a fixed archimedean place, and let B be a quaternion algebra over F which is nonsplit exactly at all archimedean places other than τ, and finite places v such that $\varepsilon_v(N) = -1$. Fix an embedding $\rho : E \to B$ over F. Let R be an order of B of type (N, E), that is an order of B of discriminant N which contains $\rho(\mathcal{O}_E)$. Fix an isomorphism $B_\tau \otimes \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ such that $\rho(E) \otimes \mathbb{R}$ is sent to the subalgebra of $\mathrm{M}_2(\mathbb{R})$ of elements $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then the group B_+ of the elements in $B^\times$ with totally positive reduced norm acts on the Poincaré half-plane $\mathcal{H}$. Thus we obtain a Shimura curve

$$X_\tau(\mathbb{C}) = B_+ \backslash \mathcal{H} \times \widehat{B}^\times / \widehat{F}^\times \widehat{R}^\times \cup \{\text{cusps}\}$$

where $\{\text{cusps}\}$ is not empty only if $F = \mathbb{Q}$ and $\varepsilon_v(N) = 1$ for any $v|N$. By Shimura's theory [35], $X_\tau(\mathbb{C})$ has a canonical model X defined over F.

The curve X over F is connected but not geometrically connected. Let $\mathrm{Jac}(X)$ denote the connected component subgroup of $\mathrm{Pic}(X/F)$. Then,

$$\mathrm{Jac}(X) = \mathrm{Res}_{\widetilde{F}/F} \mathrm{Pic}^0(X/\widetilde{F}),$$

where $\widetilde{F}$ denotes the abelian Galois extension of F corresponding to the subgroup $F_+ \cdot (\widehat{F}^\times)^2 \cdot \widehat{\mathcal{O}}_F^\times$ via class field theory.

Theorem B. *There is a unique abelian subvariety A of* $\mathrm{Jac}(X)$ *defined over F of dimension $[\mathcal{O}_f : \mathbb{Z}]$ such that $L(s, A)$ is equal to $\prod_{\sigma : \mathcal{O}_f \to \mathbb{C}} L(s, f^\sigma)$ modulo the factors at places dividing N.*

We will prove this theorem in Section 3, by combining the Eichler-Shimura theory and a newform theory for X obtained by using Jacquet-Langlands theory [24]. The key to the newform theory on X is Proposition 3.3.1. I am indebted to H. Jacquet for showing me the proof in the supercuspidal case using results of Waldspurger. (After the paper was submitted, I learned from Gross and the referees that some related results have been obtained by Tunnell [38] and Gross [19].)

Heegner points. Let x denote the image on $X_\tau(\mathbb{C})$ of $\{\sqrt{-1}\} \times \{1\} \in \mathcal{H} \times \widehat{B}^\times$. By Shimura's theory [35], x is defined over the Hilbert class field H of E. We call x a Heegner point on X.

In order to construct a point in the Jacobian $\mathrm{Jac}(X)$ from x, we need to define a map from X to $\mathrm{Jac}(X)$. Write $X_\tau(\mathbb{C})$ as a union $\cup X_i$ of connected compact Riemann surfaces of the form

$$X_i = \Gamma_i \backslash \mathcal{H} \cup \{\text{cusps}\}$$

with $\Gamma_i \subset B_+/F^\times \subset \mathrm{PSL}_2(\mathbb{R})$. Then one has $\mathrm{Jac}(X)(\mathbb{C}) = \prod \mathrm{Jac}(X_i)$. We define a canonical divisor class of degree 1 in $\mathrm{Pic}(X_i) \otimes \mathbb{Q}$ by the formula

$$\xi_i := \left\{ [\Omega^1_{X_i}] + \sum_{p \in X_i} (1 - \frac{1}{u_p})[p] + [\text{cusps}] \right\} \Big/ \int_{X_i} \frac{dxdy}{2\pi y^2},$$

where for any noncuspidal point $p \in X_i$, u_p denotes the cardinality of the group of stabilizers of $\widetilde{p}$ in Γ, where $\widetilde{p}$ is a point in $\mathcal{H}$ projecting to p. Now we define a map $\phi : X \to \mathrm{Jac}(X) \otimes \mathbb{Q}$ which sends a point $p \in X_i$ to the class of $p - \xi_i$. It is easy to see that some positive multiple of ϕ is actually defined over F.

Let z denote the class

$$u_x^{-1} \sum_{\sigma \in \mathrm{Gal}(H/E)} \phi(x^\sigma)$$

in $\mathrm{Jac}(X)(E) \otimes \mathbb{Q}$. Let z_f be the component of z in $A \otimes \mathbb{Q}$.

Gross-Zagier formula. Now we assume that a prime $\wp$ is split in E if either $\wp$ divides 2 or $\mathrm{ord}_\wp(N) > 1$.

THEOREM C. *Let $L_E(s, f)$ denote the product $L(s, f)L(s, \varepsilon, f)$, where $L(s, \varepsilon, f)$ is the L-function of f twisted by ε. Then $L_E(f, 1) = 0$ and*

$$L'_E(f, 1) = \frac{(8\pi^2)^g}{d_F^2 \sqrt{d_E}} [K_0(1) : K_0(N)](f, f)\langle z_f, z_f \rangle,$$

where

1. *$\langle z_f, z_f \rangle$ is the Néron-Tate height of z_f;*
2. *d_F is the discriminant of F, and d_E is the norm of the relative discriminant of E/F;*
3. *(f, f) is the inner product with respect to the standard measure on $Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)$.*

If $F = \mathbb{Q}$ and every prime factor of N is split in E, this is due to Gross-Zagier [21]. Again, the extra condition that $\wp$ is split in E when $\mathrm{ord}_\wp(N) > 1$ can be eliminated if we know how to compute local intersections at $\wp$ when the integral model of Shimura curves has some mild singularities over $\wp$.

For the proof of the second part of Theorem A, we assume that $L(s, f)$ has order less than or equal to 1. By some results in [3] and [40] (the theorem in [3] is stated for $\mathbb{Q}$, but its proof can be easily generalized to any number field), there is an E such that $L_E(f, s)$ has order equal to 1 at $s = 1$. It follows from Theorem C that z_f has infinite order. Now the second part of Theorem A follows from Kolyvagin's method [17], [28], [29], [30], which applies directly to our case without any new difficulty. The only thing we need is to give a correct system of CM-points which we will do at the end of this paper.

Plan of proof. Now we sketch the proof of Theorem C. Let Ψ and Φ be two cusp forms on $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2 and level $K_0(N)$ characterized by the following properties:

- The Fourier coefficients of Ψ are given by

$$a(\Psi, m) = \langle z, \mathrm{T}(m)z \rangle$$

  for all m.

- The form Φ satisfies the equality

$$L'_E(f, 1) = c(f, \Phi)$$

  for any newform f on $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2, level $K_0(N)$, and with trivial central character, where c is some constant, and $(\cdot, \cdot)$ denotes the Weil-Petersson product.

Then the equality in Theorem C is equivalent to $\Phi \equiv \mathrm{const} \cdot \Psi$ modulo old forms, and the proof of Theorem C is reduced to the computations of Fourier coefficients of Ψ and Φ respectively. We will do this by using Arakelov theory and the Rankin-Selberg method respectively. (In a separate paper [14], we will provide a more simple and direct proof for the Fourier coefficients of Φ when $F = \mathbb{Q}$.)

The absence of a cuspidal divisor representative for ξ_i and the absence of Dedekind's η-function in the general case cause some essential difficulties in our height computation. Fortunately, these difficulties can be overcome by using Arakelov theory and the strong multiplicity-one argument. See Section 4 for a detailed explanation of our method. Even in the case $X = X_0(N)$, our method simplifies the computation of Gross and Zagier.

Acknowledgment. Modulo the construction of the map from Shimura curves to their Jacobians, the formula in Theorem C was first conjectured by Gross in [18]. In some cases of $F = \mathbb{Q}$, K. Keating [26] and D. Roberts [33] have made some computations of local intersection numbers of some CM-points based on desingularizations. See also S. Kudla's paper [31].

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Notation.

- $\mathbb{N}_F$: the multiplicative monoid of nonzero ideals of $\mathcal{O}_F$.
- For any ideal m of $\mathcal{O}_F$, we define $\varepsilon(m)$ such that ε is multiplicative on $\mathbb{N}_F$ and such that $\varepsilon(\wp) = \varepsilon_\wp(\pi)$ if $\wp$ is unramified in E and π is a uniformizer of $\wp$ in $\mathcal{O}_\wp$; otherwise, $\varepsilon(\wp) = 0$.
- Let D_F denote the inverse different ideal of F,
$$D_F^{-1} = \{x \in F : \quad \mathrm{tr}_{F/\mathbb{Q}}(x\mathcal{O}_F) \subset \mathbb{Z}\}$$
and let D_E denote the relative discriminant of E over F.
- Let d_F, d_E, d_N denote the absolute norms of N, D_F, and D_E.
- For a quaternion algebra we let det (resp. tr) denote the reduced norm map (resp. reduced trace map). For an order in a quaternion algebra, we call *the reduced discriminant* simply a *discriminant.*

1. Shimura curves

In this section we introduce some of the theory of Shimura curves which will be used in later sections. We start from the construction of the integral model for general Shimura curves through a moduli interpretation in subsections 1.1. and 1.2. Then we give a description of the set of special fibers in 1.3. In 1.4, we study Hecke operators and their reductions. After some modular interpretations, we prove the Eichler-Shimura congruence relation in a special case. Finally in 1.5, we move to the special Shimura curve X constructed in the introduction and define the order R and its corresponding level structure.

1.1. *Modular interpretation.*

1.1.1. *General properties of Shimura curves.* Let F be a totally real field of degree g. This means that all Archimedean places of F are real. Fix a real place τ which allows us to consider F as a subfield of $\mathbb{R}$ by the embedding which we still denote by τ. Let B be a quaternion algebra over F which is unramified at τ but not at the other infinite places. Then we can fix an isomorphism

$$B \otimes \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R}) \oplus \mathbb{H}^{g-1} \tag{1.1.1}$$

where the first factor corresponds to τ, and $\mathbb{H}$ is the quaternion division algebra over $\mathbb{R}$. See [39] for basic properties of quaternion algebras. Let $\mathcal{H}^{\pm}$ denote the Poincaré double-half plane equal to $\mathbb{C}-\mathbb{R}$ equipped with the usual action by $\mathrm{GL}_2(\mathbb{R})$. Thus the first projection in (1.1.1) gives an action of B on $\mathcal{H}^{\pm}$.

For each open subgroup K of $\widehat{B}^{\times}$ which is compact modulo $\widehat{F}^{\times}$, we have a Shimura curve

$$M_K(\mathbb{C}) = B^{\times}\backslash\mathcal{H}^{\pm}\times\widehat{B}^{\times}/K, \tag{1.1.2}$$

where for any abelian group M, $\widehat{M}$ denote the completion $M\otimes\prod_p\mathbb{Z}_p$. For any $g\in\widehat{B}^{\times}$, and open subgroups K_1, K_2 such that $gK_1g^{-1}\subset K_2$, the right multiplication on $(B\otimes\widehat{F})^{\times}$ by g^{-1} induces a morphism $g: M_{K_1}(\mathbb{C})\to M_{K_2}(\mathbb{C})$. By Shimura's theory (see [6]), the curve $M_K(\mathbb{C})$ has a canonical model M_K defined over F and the morphism $g: M_{K_1}\to M_{K_2}$ is also defined over F with respect to these models.

By work of Drinfeld and Carayol [1], [4], [7], one can even define an integral model $\mathcal{M}_K$ over $\mathrm{Spec}\mathcal{O}_F$ such that $\mathcal{M}_K$ is regular if K is sufficiently small. This is what we need for the computation of heights in Sections 4 and 5.

If $F=\mathbb{Q}$, then $M_K(\mathbb{C})$ parametrizes elliptic curves or abelian surfaces. The canonical models and integral models can be obtained by extending the corresponding modular problems to integers. See [25] and [1] for details.

If $F\neq\mathbb{Q}$, $M_K(\mathbb{C})$ does not parametrize abelian varieties in a convenient way. But $M_K(\mathbb{C})$ has a finite map to another Shimura curve $M_{K'}(\mathbb{C})$ which apparently parametrizes abelian varieties. Thus extending the moduli problem to integers gives the integral models. In the following we will describe the curve $\mathcal{M}_{K'}$ and its moduli interpretation.

Let us fix a quadratic extension $F'=F(\sqrt{\lambda})$ of F, where λ is a negative integer. Consider $\sqrt{\lambda}$ as an element in $\mathbb{C}$. Then τ can be extended to a complex place for F':

$$\tau(x+y\sqrt{\lambda}) = \tau(x)+\tau(y)\sqrt{\lambda}. \tag{1.1.3}$$

Let B' denote $B\otimes F'$, let J be a compact open subgroup of $\widehat{F}'^{\times}$, and let K' denote the subgroup $K\cdot J$ of $\widehat{B'}^{\times}$. Then we have a Shimura curve

$$\begin{aligned} M_{K'}(\mathbb{C}) &= F'^{\times}B^{\times}\backslash\mathcal{H}^{\pm}\times B^{\times}\widehat{F}'^{\times}/K' \\ &= M_K(\mathbb{C})\times_{\widehat{F}^{\times}}\left[(F')^{\times}\backslash\widehat{F'}^{\times}/J\right]. \end{aligned} \tag{1.1.4}$$

Again by Shimura's theory, this curve has a canonical model $M_{K'}$ over F' and the morphism

$$M_K(\mathbb{C})\to M_{K'}(\mathbb{C}) \tag{1.1.5}$$

is defined over some extension of F'. For example, we have the abelian extension corresponding to J via class field theory. The image of the morphism in

(1.1.5) is another Shimura curve $M_{\widetilde{K}}$ where

$$\widetilde{K} = K \cdot \left[\widehat{F}^\times \cap \left(F'^\times \cdot J\right)\right].$$

1.1.2. *A moduli problem over F'.* In the following we will explain how the curve $M_{K'}(\mathbb{C})$ parametrizes certain abelian varieties over F' and, therefore, has a model M'_K defined over F'.

For this we write V for B' as a left B'-module and write $V_\mathbb{R}$ for $V \otimes \mathbb{R}$. Then there is a decomposition:

$$V_\mathbb{R} = (B \otimes \mathbb{R}) \otimes_\mathbb{R} F' \otimes \mathbb{R} = (\mathrm{M}_2(\mathbb{R}) \otimes \mathbb{C}) \oplus (\mathbb{H} \otimes \mathbb{C})^{g-1},$$

where we use (1.1.1)) and (1.1.3) with τ replaced by all places of F. Now we define a complex structure on $V_\mathbb{R}$ such that $\sqrt{-1}$ acts on $V_\mathbb{R}$ by right multiplication of the following element $j \in B \otimes \mathbb{C}$:

$$j = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \otimes \sqrt{-1}, \cdots, 1 \otimes \sqrt{-1}\right).$$

Then the space $\mathcal{H}^\pm$ can be identified with the $(B \otimes \mathbb{R})^\times \cdot (F' \otimes \mathbb{R})^\times$-conjugacy classes of j: each $z = x + yi \in \mathcal{H}^\pm$ corresponds to an element given by

$$j_z = \left(\alpha_z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha_z^{-1}, 1 \otimes \sqrt{-1}, \cdots, 1 \otimes \sqrt{-1}\right) \tag{1.1.6}$$

where α_z is an element of $\mathrm{GL}_2(BR)$ such that its action on $\mathcal{H}^\pm$ gives $\alpha_z(\sqrt{-1}) = z$.

Thus $V_\mathbb{R}$ is a $\mathbb{C}$ vector space with an action by B'. The traces of elements ℓ of B' acting on the $\mathbb{C}$-space $V_\mathbb{R}$ are given by the following formula:

$$\mathrm{tr}(\ell, V_\mathbb{R}/\mathbb{C}) = t(\ell)$$

where t is a map $t : B' \to F'$ given by

$$t(\ell) = 2\mathrm{tr}_{F/\mathbb{Q}}(x) + 2\left(\mathrm{tr}_{F/\mathbb{Q}}(y) - y\right)\sqrt{\lambda} \tag{1.1.7}$$

if $\mathrm{tr}_{B'/F'}(\ell) = x + y\sqrt{\lambda}$. The function t characterizes $(V_\mathbb{R}, j)$ uniquely in the sense that a complex B'-module W is B'-linearly isomorphic to $(V_\mathbb{R}, j)$ if and only if $\mathrm{tr}(\ell, W) = t(\ell)$ for every $\ell \in B'$.

Let $v \to \bar{v}$ denote the product of the involutions on both factors of $B' = B \otimes_F F'$, and let δ be a symmetric ($\bar{\delta} = \delta$) and invertible element in B'. Let $\ell \to \ell^*$ be an anticonvolution on V defined by

$$\ell^* = \delta^{-1}\bar{\ell}\delta.$$

Notice that every anticonvolution of B' which extends the convolution on E can be obtained in this manner.

Let ψ_F be a pairing on V with values in F given by

$$\psi_F(u,v) = \mathrm{tr}_{B'/F}\left(\sqrt{\lambda}u\bar{v}\delta\right).$$

Then for any $\ell \in B'$,

$$\psi_F(\ell u, v) = \psi_F(u, \ell^* v).$$

One can show that the similitudes of ψ_F consist of right multiplication on $V = B'$ of elements of $B^\times \cdot F^\times$.

Choose a δ such that $\psi_F(v, vj) \in F \otimes \mathbb{R}$ is totally positive for $v \in V_\mathbb{R}$.

PROPOSITION 1.1.3. *The curve M'_K is the coarse moduli space of the following moduli functor $\mathcal{F}^0_{K'}$ over $\mathbb{C}$: For an F'-scheme S, $\mathcal{F}^0_{K'}(S)$ is the set of the isomorphism classes of objects $[\bar{A}, \iota, \bar{\theta}, \bar{\kappa}]$ where*

1. *$\bar{A}$ is an abelian scheme over S up to isogeny with an action $\iota : B' \to \mathrm{End}_S(\bar{A})$ such that for any $\ell \in B'$ there is the equality*

$$\mathrm{tr}(\iota(\ell), \mathrm{Lie}\bar{A}) = t(\ell).$$

2. *$\bar{\theta}$ is an $F^\times$-class of polarizations $\theta : A \to A^\vee$ for $A \in \bar{A}$ such that for any $\ell \in B'$, the associated Rosati involution takes $\iota(\ell)$ to $\iota(\ell^*)$.*

3. *$\bar{\kappa}$ is a K'-class of B'-linear isomorphisms $\kappa : \widehat{V} \to \widehat{V}(\bar{A})$ which are $\widehat{F}$-symplectic similitudes, where $\widehat{V}(\bar{A}) = \widehat{T}(\bar{A}) \otimes \mathbb{Q}$ with $\widehat{T}(\bar{A}) = \prod T_p(\bar{A})$. This means that each $\kappa \in \bar{\kappa}$ is symplectic between the form ψ_A induced by a polarization $\theta \in \bar{\theta}$, and the form $\mathrm{tr}_{F/\mathbb{Q}}(ua\psi_F)$ for some $u \in \widehat{F}^\times$, $a \in \det K'$.*

Proof. Let x be a point of $M_{K'}(\mathbb{C})$; we want to construct an element $[A, \iota, \bar{\theta}, \bar{\kappa}]$ in $\mathcal{F}^0_{K'}(\mathbb{C})$ as follows. Assume that x is represented by (z, γ).

1. $\bar{A}$ is the abelian variety up to isogeny,

$$\bar{A} = \Lambda \backslash (V_\mathbb{R}, j_z)$$

with Λ any lattice of V, where j_A is the complex structure constructed as in (1.1.6). Thus, $\widehat{V}(A) = \widehat{V}$.

2. $\iota : B' \to \mathrm{End}(\bar{A})$ is induced by left multiplication of B' on V.

3. $\bar{\kappa}$ is the K'-class of the map $\widehat{V} \to \widehat{V}(\bar{A})$ induced by right multiplication of γ.

It follows from the definition that the isomorphic class $[\bar{A}, \rho, \bar{\theta}, \bar{\kappa}]$ is an element of $\mathcal{F}^0_{K'}(\mathbb{C})$.

Conversely, we can construct a point $x \in M_{K'}(\mathbb{C})$ from an element $[A, \iota, \bar{\theta}, \bar{\kappa}]$ of $\mathcal{F}^0_{K'}(\mathbb{C})$ as follows. Let V_A denote $H_1(A, \mathbb{Q})$ and let ψ_A be an alternative form defined by one polarization in $\bar{\theta}$. Then V_A is a B'-algebra which is isomorphic to V at each place of F' by a map in $\bar{\kappa}$. It follows that V_A must be isomorphic to V. We may identify V_A with $V = B'$ by fixing such an isomorphism. By the second condition, the alternative form ψ_A has the form

$$\psi_A(v_1, v_2) = \mathrm{tr}_{F/\mathbb{Q}}\psi_F(v_1 b, v_2)$$

for some $b' \in B^\times$. If $\kappa \in \bar{\kappa}$ then κ is induced by the map $v \to v\gamma$ with $\gamma \in \widehat{B}'^\times$. Condition 3 implies

$$\mathrm{tr}_{F/\mathbb{Q}}(u\psi_F(v_1 x, v_2 x)) = \mathrm{tr}_{F/\mathbb{Q}}(\psi_F(v_1 b, v_2)). \tag{1.1.8}$$

This is equivalent to the equation $ux\bar{x} = b$. By Hasse's principal (see [27, §2.2.3]), this equation must have a solution $x \in B'^\times$. After modifying the isomorphism $\phi : V_A \to V$, we may assume that $b = 1$. Then equation (1.1.8) implies that $\gamma \in \widehat{B}^\times \cdot \widehat{F'}^\times$. Such a γ is uniquely determined modulo right multiplication of K' once ϕ is fixed. We may replace ϕ by $b\phi$ with $b \in B^\times \cdot F'^\times$ which acts on V by left multiplication. Then γ is changed to $b\gamma$.

Let $j_A \in \mathrm{GL}_\mathbb{R}(V_\mathbb{R})$ be multiplication of $\sqrt{-1}$ in the complex structure on $\mathrm{Lie}(A)$. Then j_A commutes with the action of $B'_\mathbb{R}$ so that it is given by right multiplication of an element which we still denote by j_A. Since j_A preserves the alternative form ψ_A or equivalently the form ψ_F, we see that $j_A \in B^\times_\mathbb{R} \otimes \mathbb{R}$. Now,

$$j_A = (\alpha_1 \otimes \beta_1, \cdots, \alpha_g \otimes \beta_g)$$

where for each i, either $\alpha_i = 1, \beta_i = \sqrt{-1}$ or $\alpha_i^2 = -1, \beta_i = 1$. By computing the trace of $B_\mathbb{R}$ over V_A which must satisfy condition 1, we see that j_A must have the form

$$j_A = (\alpha \otimes 1, 1 \otimes \sqrt{-1}, \cdots, 1 \otimes \sqrt{-1})$$

with $\alpha^2 = -1$. Now α must be conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so that j_A must be j_z for some $z \in \mathcal{H}^\pm$. Again this z is unique if $\phi : V \to V_A$ is fixed. If we change ϕ to $b\phi$ then z is changed to $b(z)$. It follows that the image x of (z, γ) in $M_{K'}(\mathbb{C})$ is a well-defined point. □

1.1.4. *Second version.* We may also describe $M_{K'}$ as a coarse moduli space of abelian varieties, rather than abelian varieties up to isogeny. For simplicity, we assume that K is compact. Then there is a maximal order $\mathcal{O}_B$ of B such that K is included in $\widehat{\mathcal{O}}_B^\times$. (Notice that this is not the exact case wanted, as the Shimura curve X defined in the introduction is the compactification of M_K with $K = \widehat{R}^\times \cdot \widehat{F}^\times$.) Let $\mathcal{O}_{B'}$ be the order $\mathcal{O}_B \otimes \mathcal{O}_{F'}$ of B'. Write $V_\mathbb{Z}$ for $\mathcal{O}_{B'}$ as a left $\mathcal{O}_{B'}$-module.

Let $\mathcal{U}$ be a subset of $\widehat{F}^\times$ representing

$$F\backslash\widehat{F}^\times/\det K')$$

such that for each $u \in \mathcal{U}$, the alternating pairing $u\psi_F$ is integral on $V_\mathbb{Z}$. Let $\nu(K')$ denote $\det K' \cap F^\times$ of $\mathcal{O}_F^\times$.

PROPOSITION 1.1.5. *The functor $\mathcal{F}_{K'}^0$ is isomorphic to $\mathcal{F}_{K'}$ defined as follows: For an F'-scheme S, $\mathcal{F}_{K'}(S)$ is the set of isomorphism classes of objects $[A, \iota, \bar\theta, \bar\kappa]$ where:*

1. *A is an abelian scheme over S with an action $\iota : \mathcal{O}_{B'} \to \mathrm{End}_S(A)$ such that for any $\ell \in \mathcal{O}_{B'}$ there is the equality*

$$\mathrm{tr}(\iota(\ell) : \mathrm{Lie}A) = t(\ell).$$

2. *$\bar\theta$ is a $\nu(K')$-class of polarizations $\theta : A \to A^\vee$ such that for any $\ell \in \mathcal{O}_{B'}$, the associated Rosati involution takes $\iota(\ell)$ to $\iota(\ell^*)$.*

3. *$\bar\kappa$ is a K'-class of $\mathcal{O}_{B'}$-linear isomorphisms $\kappa : \widehat{V}_\mathbb{Z} \to \widehat{T}(A)$ which is symplectic with respect to $\psi_{u,a} := \mathrm{tr}_{\widehat{F}/\widehat{\mathbb{Q}}}(ua\psi_F)$ for some $u \in \mathcal{U}$ and $a \in \det K'$.*

Proof. There is an obvious morphism from $\mathcal{F}_{K'}$ to $\mathcal{F}_{K'}^0$. Now we want to define its converse. Let $[\bar A, \rho, \bar\theta, \bar\kappa]$ be an object in $\mathcal{F}_{K'}^0(S)$. Then the lattice $\kappa(\widehat{V}_\mathbb{Z})$ does not depend on the choice of $\kappa \in \bar\kappa$. Let A be the corresponding abelian variety isogenous to $\bar A$. Then A has the action by $\mathcal{O}_{B'}$ such that condition 1 is satisfied. As κ varies in $\bar\kappa = K'\kappa$ and θ varies in $\bar\theta = F^\times\theta$, u in condition 3 in Proposition 1.1.3 varies in a single double-coset of $F^\times\backslash\widehat{F}^\times/\det K'$. Thus we may choose a $\theta_0 \in \bar\theta$ such that $u \in \mathcal{U}$, and the set of such θ's forms a class $\nu(K')\theta_0$. As $\mathrm{tr}_{F/\mathbb{Q}}(ua\psi_F)$ is always integral, ψ_A's corresponding to $\theta_0 \in \nu(K')\theta_0$ take integral values on $T(A)$. It follows that every such θ_0 defines a polarization of A. Condition 2 in this proposition is obviously satisfied. This defines a morphism $\mathcal{F}_{K'}^0 \to \mathcal{F}_{K'}$ which is obviously the inverse of the obvious morphism $\mathcal{F}_{K'} \to \mathcal{F}_{K'}^0$. □

1.1.6. *Remark.* Let $x \in M_{K'}(\mathbb{C})$ be represented by (z, γ). From the proof of Propositions 1.1.3 and 1.1.5, we see that the object $[A, \iota, \bar\theta, \bar\kappa]$ in $\mathcal{F}_{K'}(\mathbb{C})$ parametrized by x has the following form:

1. $$A = V_\mathbb{Z}\gamma^{-1}\backslash(V_\mathbb{R}, j_z).$$

2. ι is induced by left multiplication by $\mathcal{O}_{B'}$ on $V_\mathbb{Z}\gamma^{-1}$.

3. $\bar\theta$ is the unique class induced by alternative forms

$$\left\{\mathrm{tr}_{F/\mathbb{Q}}(t\psi_F): \quad t\in F^\times\cap(\coprod_{u\in\mathcal{U}} u\det\gamma K')\right\}.$$

4. $\bar\kappa$ is the K'-class of the morphism $\widehat{V}_Z\to\widehat{V}_Z\gamma^{-1}$ induced by right multiplication by γ^{-1}.

PROPOSITION 1.1.7. *When K' is sufficiently small, then $\mathcal{F}_{K'}$ (therefore $\mathcal{F}^0_{K'}$) is representable.*

Proof. For each $u\in\mathcal{U}$, let $\mathcal{F}_{K',u}$ denote the subfunctor of $\mathcal{F}_{K'}\otimes F'\widetilde{F}$ with given u in condition 4 of Proposition 1.1.5, where $\widetilde{F}$ is the extension of F corresponding to $F^\times\det K'$ via class field theory. Wishing to show that $\mathcal{F}_{K',u}$ is representable, we need the following:

LEMMA 1.1.8. *There is a positive integer n such that*

$$(1+m\mathcal{O}_F)^\times := (1+m\widehat{\mathcal{O}}_F)^\times\cap F^\times\subset[(1+m\mathcal{O}_F)^\times]^2$$

with some $n\geq 3$.

Proof. We fix an $n\geq 3$ and let S be a finite subset of $(1+n\mathcal{O}_F)^\times$ which contains 1 and represents the quotient

$$(1+n\mathcal{O}_F)^\times/[(1+n\mathcal{O}_F)^\times]^2.$$

For each $s\in S-\{1\}$, let p_s be a prime not dividing $2m$ such that s is not a square in $(\mathcal{O}_F/p_s\mathcal{O}_F)^\times$. Then

$$m := n\prod_{m\in S-\{1\}} p_s$$

will satisfy the requirement. Indeed, the definition of m implies that the morphism

$$\frac{(1+n\mathcal{O}_F)^\times}{[(1+n\mathcal{O}_F)^\times]^2}\to\frac{(\mathcal{O}_F/m\mathcal{O}_F)^\times}{[(\mathcal{O}_F/m\mathcal{O}_F)^\times]^2}$$

is injective. Thus $(1+m\mathcal{O}_F)^\times$ is included in $[(1+n\mathcal{O}_F)^\times]^2$. □

We return to our proof of Proposition 1.1.5. Assume that K' is sufficiently small so that

$$\det K'\subset(1+m\widehat{\mathcal{O}}_B)^\times\cdot(1+m\widehat{\mathcal{O}}_{F'})^\times.$$

Then $\nu(K')\subset(1+n\mathcal{O}_F)^{\times 2}$. Let T denote the set of elements in $(1+n\mathcal{O}_F)^\times$ whose squares are in $\nu(K')$. Let $\widetilde{K'}$ denote $K'\cdot T$. If $t\in T$, then multiplication by t induces an isomorphism

$$[A,\iota,\bar\theta,\bar\kappa]\to[A,\iota,\bar\theta,t\bar\kappa]$$

of objects in $\mathcal{F}_{K',u}(S)$. Here if κ is symplectic with respect to $\psi_{u,a}$, then $t\kappa$ is symplectic with respect to ψ_{u,at^2}. As $T^2 = \nu(K') = \nu(\widetilde{K'})$, it follows that the canonical morphism $\mathcal{F}_{K',u} \to \mathcal{F}_{\widetilde{K'},u}$ is an isomorphism.

Let $\widetilde{\mathcal{F}}_{\widetilde{K'},u}$ denote the functor defined in the same way as $\mathcal{F}_{\widetilde{K'},u}$ but with $\nu(K')$-class $\bar{\theta}$ to replace a single θ. Then multiplication by t induces an isomorphism

$$[A, \iota, \theta, \bar{\kappa}] \to [A, \iota, t^2\theta, \bar{\kappa}]$$

of objects in $\widetilde{\mathcal{F}}_{\widetilde{K},u}(S)$. So the canonical morphism $\widetilde{\mathcal{F}}_{\widetilde{K'},u} \to \mathcal{F}_{\widetilde{K'},u}$ is also an isomorphism.

In this way we have shown that $\mathcal{F}_{K',u}$ is isomorphic to $\widetilde{\mathcal{F}}_{\widetilde{K'},u}$. Now we want to show the representability of $\widetilde{\mathcal{F}}_{\widetilde{K'},u}$. Let d denote the degree of $\psi_{u,1}$. Let $\mathcal{A}$ denote the moduli functor which classifies abelian varieties of dimension $4g$, with a full level n structure and a polarization of degree d. As $n \geq 3$, $\mathcal{A}$ is representable by a scheme $M_{d,n}$ ([32, Prop. 7.9]). The functor $\widetilde{\mathcal{F}}_{\widetilde{K'},u}$ has a finite morphism to $\mathcal{A}$. The conditions in the definition of $\widetilde{\mathcal{F}}_{\widetilde{K'},u}$ defines a finite scheme $\widetilde{M}_{K',u}$ over $M_{d,n} \otimes F'\widetilde{F}$ which represents $\widetilde{\mathcal{F}}_{\widetilde{K'},u}$, also $\mathcal{F}_{K',u}$. Now the union $\widetilde{M}_{K'}$ of $\widetilde{M}_{K',u}$ represents $\mathcal{F}_{K'} \otimes F\widetilde{F}$. Notice that $\widetilde{M}_{K'}$ has an action by

$$\mathrm{Gal}(\widetilde{F}/F) = F^\times \backslash \widehat{F}^\times / \det K'$$

which induces a model $M_{K'}$ of $\widetilde{M}_{K'}$ defined over F'. This model represents $\mathcal{F}_{K'}$. As $M_{K'}(\mathbb{C})$ is a smooth Riemann surface, $M_{K'}$ is a regular scheme. □

1.2. *Integral models.*

1.2.1. *A new version of* $\mathcal{F}_{K'} \otimes F_\wp$. Let $\wp$ be a prime of F of characteristic p. Assume that λ is prime to p and and that $\left(\frac{\lambda}{p}\right)$ is 1; we fix a square root μ_p in $\mathbb{Q}_p$. Then F' can be embedded into $F_\wp$ over F by sending $\sqrt{\lambda}$ to μ_p. We want to extend $M_{K'} \otimes F_\wp$ to a model $\mathcal{M}_{K',\wp}$ over $\mathcal{O}_\wp$, the ring of integers in $F_\wp$. For this we need a new version of the moduli problem $\mathcal{F}_{K'}$ over $F_\wp$-schemes. We start with some notation.

The algebra $\mathcal{O}_{F',p} = \mathcal{O}_{F'} \otimes \mathbb{Z}_p$ is the sum of all completions $\mathcal{O}_{F',q}$ at its places q over p. We have the following decomposition:

$$\mathcal{O}_{F',p} = \mathcal{O}^1_{F',p} + \mathcal{O}^2_{F',p}$$

where $\mathcal{O}^1_{F',p}$ (resp. $\mathcal{O}^2_{F',p}$) denotes the sum of all completions $\mathcal{O}_{F',q}$ such that the map $\mathcal{O}_{F'} \to \mathcal{O}_{F',q}$ takes $\sqrt{\lambda}$ to μ_p (resp. $-\mu_p$). For any $\mathcal{O}_{F',p}$ module M, we let M^1 (resp. $M^1_\wp$, M^2, $M^2_\wp$) denote $\mathcal{O}^1_{F',p}M$ (resp. $\mathcal{O}^2_{F',p}M$). Let $\mathcal{O}^\wp_{F',p}$ denote the sum of components of $\mathcal{O}_{F',p}$ not over $\wp$ and let $M^\wp$ (resp. $M_\wp$) denote $\mathcal{O}^\wp_{F',p}M$ (resp. $\mathcal{O}_{F',\wp}M$).

Choose δ and $\mathcal{U}$ such that for each $u \in \mathcal{U}$, $u\psi_F$ has degree prime to p, and u has component 1 at places dividing p. Then we have the following:

PROPOSITION 1.2.2. *The functor $\mathcal{F}_{K'} \otimes F_\wp$ is equivalent to the following functor $\mathcal{F}_{K',\wp}$: for any $F_\wp$-scheme S, $\mathcal{F}_{K',\wp}(S)$ is the isomorphism classes of objects $[A, \iota, \bar{\theta}, \bar{\kappa}_p, \bar{\kappa}^p]$ where*

1. *A is an abelian scheme over S with an action $\iota : \mathcal{O}_{B'} \to \mathrm{End}(A/S)$ such that the following two condition are satisfied:*

   (a) *$\mathrm{Lie}(A)^2_\wp$ is a locally free $\mathcal{O}_S$ module of rank 2 such that the action of $\iota(F)$ is given by the inclusion $F \to F_\wp \to \mathcal{O}_S$;*

   (b) $\mathrm{Lie}(A)^{2,\wp} = 0$.

   Here $\mathrm{Lie}(A)$ may be viewed as an $\mathcal{O}_{B',p}$ via the action ι.

2. *$\bar{\theta}$ is a $\nu(K')$-class of polarizations on A of degrees prime to p, such that the Rosati involutions take $\iota(\ell)$ to $\iota(\ell^*)$.*

3. *$\bar{\kappa}^2_p$ is a K_p-class of $\mathcal{O}^p_{B'}$-linear isomorphisms:*
$$\kappa^2_p : V^2_{\mathbb{Z},p} \to T_p(A)^2.$$

4. *$\bar{\kappa}^p$ is a ${K'}^p$-class of $\mathcal{O}^p_{B'}$-linear isomorphisms*
$$\kappa^p : \widehat{V}^p_{\mathbb{Z}} \to \widehat{T}(A)^p$$
   which is symplectic with respect to some $\psi^p_{u,a}$. Here, for a $\widehat{\mathcal{O}}_F$-module $M = \prod_q M_q$, let M^p denote the product of components M_q for $q \not| p$.

Proof. Let us first define a morphism from $\mathcal{F}_{K'} \otimes F_\wp$ to $\mathcal{F}_{K',\wp}$. Let $[A, \iota, \bar{\theta}, \bar{\kappa}]$ be an object in $\mathcal{F}_{K'}(S)$ where S is an $F_\wp$-scheme. Then we can decompose any $\bar{\kappa}$ into parts

$$\bar{\kappa} = \bar{\kappa}_p \oplus \bar{\kappa}^p : V_{\mathbb{Z},p} \oplus \widehat{V}^p_{\mathbb{Z}} \to T_p(A) \oplus \widehat{T}(A)^p. \tag{1.2.1}$$

Furthermore we can decompose κ_p into two parts:

$$\kappa^1_p \oplus \kappa^2_p : V^1_{\mathbb{Z},p} \oplus V^2_{\mathbb{Z},p} \to T_p(A)^1 \oplus T^2_p.$$

We claim that the object $[A, \iota, \bar{\theta}, \bar{\kappa}_p, \bar{\kappa}^p]$ is an object of $\mathcal{F}_{K'\wp}(S)$. We need only verify condition 1 in Proposition 1.1.5. Indeed, by a result of Carayol, condition 1 in Proposition 1.1.5, which states that

$$\mathrm{tr}(\iota(\ell), \mathrm{Lie}A) = t(\ell), \qquad \ell \in B',$$

can be replaced by the first condition in Proposition 1.2.2 together with one further condition that

A is an abelian variety of dimension $4g$.

This is a slight generalization of Carayol's proposition in [4, p. 171]. In his case $\wp$ is split in B. His proof can be generalized to our case without any difficulty. In this way, we obtain a morphism $\mathcal{F}_{K'} \otimes F_\wp \to \mathcal{F}_{K',\wp}$.

Now we want to construct the converse of the morphism of functors constructed as above. Since the fact that A has dimension $4g$ is implied by condition 4 in Proposition 1.2.2, we need only show that for a given K_p'-class $\bar\kappa_p^2$ as in Proposition 1.2.2, we can find $\bar\kappa_p$ with a decomposition as in (1.2.1) such that $\bar\kappa_p$ is a K_p'-class of isomorphisms $\kappa_p : V_{\mathbb{Z},p} \to T_p(A)$ which is $\mathcal{O}_{B',p}$-linear and symplectic with respect to $\mathrm{tr} a\psi_F$ and some ψ_A induced by a θ in $\bar\theta$, where a is some element in $\det K_p'$.

Notice that condition 2 in Propositions 1.1.5 and 1.2.2 implies that all these subspaces are null spaces under symplectic forms. So each pair of these spaces forms a complete dual. Now we may take κ_p^1 to be the dual of κ_p^2. □

1.2.3. *Definitions.* Let S be a scheme over $\mathcal{O}_\wp$, $\mathcal{G}$ an $\mathcal{O}_{B,\wp}$-module scheme over S.

1. We say that $\mathcal{G}$ is a *special $\mathcal{O}_{B,\wp}$-module* if the induced action of $\mathcal{O}_{B,\wp}$ on $\mathrm{Lie}(\mathcal{G})$ makes $\mathrm{Lie}(\mathcal{G})$ a locally free module of rank one over $\mathcal{O}_S \otimes_{\mathcal{O}_\wp} \mathcal{O}_{E,\wp}$, where $\mathcal{O}_{E,\wp}$ is any unramified quadratic extension of $\mathcal{O}_\wp$ contained in $\mathcal{O}_{B,\wp}$.
2. Let $n \in \mathbb{N}$ and $x \in \mathcal{G}[n](S)$. We say x is a *Drinfeld base of $\mathcal{G}$ of level n* if, as cycles in $\mathcal{G}$, there exists the identity:
$$[\mathcal{G}[n]] = \sum_{a \in \mathcal{O}_{B,\wp}/\wp^n} [nx].$$

Proposition 1.2.4. *Assume that J is maximal at all places dividing p. Then the functor $\mathcal{F}_{K',\wp}$ can be extended to the following functor over $\mathcal{O}_\wp$ which is still denoted by $\mathcal{F}_{K',\wp}$: For any $\mathcal{O}_\wp$-scheme S, $\mathcal{F}_{K_0',\wp}(S)$ is the set of isomorphism classes of objects $[A, \iota, \theta, \bar x, \bar\kappa^{2,\wp}, \bar\kappa^p]$ where*

1. *A is an abelian scheme over the scheme S with an action $\iota : \mathcal{O}_{B'} \to \mathrm{End}(A/S)$ such that the following two condition are satisfied:*
   (a) *$\mathcal{G} := A[\wp^\infty]^2$ is a special formal $\mathcal{O}_{B,\wp}$-module.*
   (b) *$A[p^\infty]^{2,\wp}$ is an étale $\mathcal{O}_{B',p}^{2,\wp}$-module.*
2. *θ is a $\nu(K')$-class of polarizations on A of degrees prime to p, such that the Rosati involutions take $\iota(\ell)$ to $\iota(\ell^*)$.*
3. *$\bar x$ is a $K_\wp$-class of Drinfeld bases of $\mathcal{G}$ of level n, where n is a positive integer such that $K_\wp$ contains $1 + \wp^n \mathcal{O}_{B,\wp}$.*

4. *$\bar{\kappa}^{2,\wp}$ is a $K_p^{\wp}$-class of $\mathcal{O}_{B',p}^{2,\wp}$-linear isomorphisms:*

$$\kappa^{2,\wp} : V_{\mathbb{Z},p}^{2,\wp} \to T_p(A)^{2,\wp}.$$

5. *$\bar{\kappa}^p$ is a ${K'}^p$-class of $\mathcal{O}_{B'}^p$-linear isomorphisms*

$$\kappa^p : \widehat{V}_{\mathbb{Z}}^p \to \widehat{T}(A)^p$$

which is symplectic with respect to some $\mathrm{tr}(ua\psi_F)$ for some $u \in \mathcal{U}$ and $a \in \det K'$, and some ψ_A induced by some element in $\bar{\theta}$.

Moreover, when $K_\wp = (1 + \wp^n \mathcal{O}_{B,\wp})^\times$ and ${K'}^p$ is sufficiently small, the functor $\mathcal{F}_{K',\wp}$ is representable by a regular scheme $\mathcal{M}_{K',\wp}$ over $\mathcal{O}_\wp$. In general, the functor $\mathcal{F}_{K',wp}$ has a coarse moduli space $\mathcal{M}_{K',\wp}$ over $\mathcal{O}_\wp$.

Proof. It is easy to see that the conditions in Proposition 1.2.4 are equivalent to the conditions in Proposition 1.2.2 when S is a $F_\wp$-scheme. The representability can be proved in the same way as in Proposition 1.1.5. Here we need to choose ${K'}^p$ to be sufficiently small and take care of Drinfeld bases. See [4, §5.3 and §7.3]. Also the regularity can be proved using the same argument as in [4, §5.4 and §7.4].

If ${K'}^p$ is not sufficiently small or if $K_\wp$ does not have the form $(1 + \wp^n \mathcal{O}_{B,\wp})^\times$, then $\mathcal{F}_{K',\wp}$ may not be representable. But we may choose a sufficiently small normal subgroup $\widetilde{K'}$ of K' so that the functor $\mathcal{F}_{\widetilde{K'},\wp}$ is representable. The quotient $\mathcal{M}_{\widetilde{K'},\wp}/K'$ does not depend on the choice of $\widetilde{K'}$ and it is actually the coarse moduli space of $\mathcal{F}_{K',\wp}$ □

1.2.5. *Modules $\mathcal{M}_K$ and modules $\mathcal{G}_K$.* Recall that M_K has a finite morphism to $M_{K'}$. We, therefore, obtain a model $\mathcal{M}_{K,\wp}$ for the Shimura curve M_K by taking the normalization of $\mathcal{M}_{K',\wp}$ in M_K. One can show that this model does not depend on the choice of F' and J. By gluing these models, one has a model $\mathcal{M}_K$ over $\mathrm{Spec}\mathcal{O}_F$ for M_K, which is regular when K is sufficiently small.

Assume that K' is sufficiently small so that $\mathcal{F}_{K',\wp}$ is represented by a regular scheme $\mathcal{M}_{K',\wp}$. Then over $\mathcal{M}_{K',\wp}$, we have a divisible $\mathcal{O}_{B,\wp}$-module $\mathcal{G}_{K'} := \mathcal{G}_{\mathcal{A}}$, where $\mathcal{A}$ is the universal abelian variety on $\mathcal{M}_{K'}$. Let $\mathcal{G}_K$ be the pull-back of $\mathcal{G}_{\mathcal{A}}$ on $\mathcal{M}_K$. Then $\mathcal{G}_{K_0}$ does not depend on the choice of J.

Let K_0 denote $\mathcal{O}_{B,\wp}^\times \cdot K^\wp$. Then the scheme $\mathcal{M}_K$ over $\mathcal{M}_{K_0}$ classifies the $K_\wp$-class of Drinfeld bases in $\mathcal{G}_{K_0}[\wp^n]$.

Let x be a geometric point of the special fiber of $\mathcal{M}_{K_0}$. Then over the completion of the strict localization $\widehat{\mathcal{M}_{K_0,x}}$, $\mathcal{G}_{K_0}$ is the universal deformation of $\mathcal{G}_{K_0}|_x$.

1.3. *Reductions of models.* We want to study the set of irreducible components of the special fibers of $\mathcal{M}_K$.

1.3.1. *Split case.* Assuming that B is split at $\wp$, we can fix an isomorphism between $\mathcal{O}_{B,\wp}$ and $\mathrm{M}_2(\mathcal{O}_\wp)$, thus obtaining the decomposition of $\mathcal{O}_\wp$ modules over $\mathcal{M}_{K_0}$:

$$\mathcal{G}_{K_0} = \mathcal{G}^1 \oplus \mathcal{G}^2, \qquad \mathcal{G}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{G}_{K_0}, \quad \mathcal{G}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{G}_{K_0}.$$

The two $\mathcal{O}_\wp$-modules $\mathcal{G}^1, \mathcal{G}^2$ are isomorphic by the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is easy to see that in this setting, $\mathcal{M}_K$ over $\mathcal{M}_{K_0}$ classifies the $K_\wp$-class of morphisms

$$\phi : (\mathcal{O}_\wp/\wp^n)^2 \to \mathcal{G}^1[\wp^n]$$

such that this homomorphism is surjective on cycles.

Let x be a geometric point in the special fiber of $\mathcal{M}_{K_0,\wp}$. Then the $\mathcal{O}_\wp$-module $\mathcal{G}^1_x$ has two possibilities:

1. *Ordinary case*: The group $\mathcal{G}^1_x$ is isomorphic to the product of $(F_\wp/\mathcal{O}_\wp)$ and a formal $\mathcal{O}_\wp$ -module Σ_1 of height 1.

2. *Supersingular case*: The group $\mathcal{G}^1_x$ is isomorphic to a formal $\mathcal{O}_\wp$-module Σ_2 of height 2.

The set of connected geometric components of the special fiber of $\mathcal{M}_{K_0}$ over $\wp$ is the same as that of the generic fiber. Fix a geometrically irreducible component D of the special fiber of $\mathcal{M}_{K_0}$ over $\wp$. Then we have:

PROPOSITION 1.3.2. *Assume that $\wp$ is split in B. Then the set of the irreducible geometric components of $\mathcal{M}_K$ over $\wp$ is indexed by $\mathbb{P}^1(\mathcal{O}_\wp)/K_\wp$. More precisely, for each line $C \subset \mathcal{O}_\wp^2$, the corresponding component of $\mathcal{M}_K$ over $\wp$ will classify the morphism*

$$\phi : (\mathcal{O}_\wp/\wp)^2 \to \mathcal{G}^1[\wp^n]$$

such that $\ker\phi$ *contains* $C \pmod{\wp}$.

1.3.3. *Nonsplit case.* It remains to study the reduction of $\mathcal{M}_K$ in the case that B is not split at $\wp$. In this case, one can show that $\mathcal{G}_K$ is a formal group. It follows that the map

$$\mathcal{M}_K \to \mathcal{M}_{K_0}$$

is purely inseparable at the fiber over $\wp$. So the set of irreducible components in the special fiber of $\mathcal{M}_K$ over $\wp$ is the same as that of $\mathcal{M}_{K_0}$.

To study the irreducible components of $\mathcal{M}_{K_0}$ over $\wp$ we can use the uniformization theorem of Cerednik – Drinfeld [1], but we need some notation. Let $\widehat{\mathcal{M}}_{K_0}$ denote the formal completion of $\mathcal{M}_{K_0}$ along its special fiber over $\wp$.

Let $B(\wp)$ denote the quaternion algebra over F obtained by switching the invariants of B at τ and $\wp$. Fix an isomorphism:

$$\widehat{B(\wp)} \simeq \mathrm{M}_2(F_\wp) \cdot \widehat{B}^\wp$$

where the superscript $\wp$ means that the component at the place $\wp$ is removed. Let $\widehat{\Omega}$ denote Deligne's formal scheme over $\mathcal{O}_\wp$ obtained by blowing-up $\mathbb{P}^1$ along its rational points in the special fiber over the residue field k of $\mathcal{O}_\wp$ successively. The generic fiber Ω of $\widehat{\Omega}$ is a rigid analytic space over $F_\wp$ whose $\bar{F}_\wp$ points are given by $\mathbb{P}^1(\bar{F}_\wp) - \mathbb{P}^1(F_\wp)$. The group $\mathrm{GL}_2(F_\wp)$ has a natural action on $\widehat{\Omega}$. The theorem of Cerednik-Drinfeld gives a natural isomorphism

$$\widehat{\mathcal{M}}_{K_0} \simeq B(\wp)^\times \backslash \widehat{\Omega} \widehat{\otimes} \widehat{\mathcal{O}}_\wp^{nr} \times \widehat{B}^{\times,\wp}/K^\wp$$

where $\widehat{\mathcal{O}}_\wp^{nr}$ denote the completion of the maximal unramified extension of $\mathcal{O}_\wp$.

To obtain a description of the special fiber of $\widehat{\mathcal{M}}_{K_0}$, we notice that the irreducible components of special fibers of $\widehat{\Omega}$ correspond one-to-one to the classes modulo $F^\times$ of $\mathcal{O}_\wp$ lattices in $F_\wp^2$. Consequently, we have the following:

Proposition 1.3.4. *Assume that $\wp$ is not split in B. Then the set of irreducible geometric components of $\widehat{\mathcal{M}}_{K_0}$ over $\wp$ is indexed by the set*

$$\begin{aligned} &B(\wp)^\times \backslash \mathrm{GL}_2(F_\wp)/F_\wp^\times \mathrm{GL}_2(\mathcal{O}_\wp) \times \widehat{B}^{\times,\wp}/K^\wp \\ &\qquad \simeq B(\wp)^\times \backslash \widehat{B(\wp)}^\times / F_\wp^\times \mathrm{GL}_2(\mathcal{O}_\wp) K^\wp. \end{aligned}$$

1.4. *Hecke correspondences.*

1.4.1. *Definition.* Let M_K be a Shimura curve with a compact K contained in $\widehat{\mathcal{O}}_B^\times$. Let m be an ideal of $\mathcal{O}_F$ such that at every prime $\wp$ dividing m, K has maximal components and B is split. Let G_m (resp. G_1) be the set of element g of $\widehat{\mathcal{O}}_B$ which has component 1 at places not dividing m, and such that $\det(g)$ generates m (resp. is invertible) at each place dividing m. Then we may consider G_1 as a subgroup of K. The Hecke operator $\mathrm{T}(m)$ on M_K is defined by the formula

$$\mathrm{T}(m)x = \sum_{\gamma \in G_m/G_1} [(z, g\gamma)], \tag{1.4.1}$$

where (z, g) is a representative of x in $\mathcal{H} \times \widehat{B}^\times$, and $[(z, g\gamma)]$ is the projection of $(z, g\gamma)$ on X. It is easy to see that the correspondence has the degree

$$\deg \mathrm{T}(m) = \sigma_1(m) = \sum_{a|m} \mathrm{N}(a).$$

To see that $\mathrm{T}(m)$ is a correspondence given by algebraic cycles, decompose G_m into a union of double cosets:

$$G_m = \coprod G_1 g_i G_1.$$

For each i, let K_i denote the group $g_iKg_i^{-1} \cap K$. Then we obtain two morphisms p_1, p_2 from M_{K_i} to M_K, induced by right multiplication on $\widehat{B}^\times$ by 1 and g_i respectively. The image of M_{K_i} in $M_K \times M_K$ by (p_1, p_2), as an algebraic cycle, defines a correspondence T_i. Then $\mathrm{T}(m)$ is defined to be the sum of T_i.

If $\mathcal{M}_K$ is the integral model constructed as before then the Hecke correspondences $\mathrm{T}(m)$ can be extended to $\mathcal{M}_K$ by taking Zariski closure of cycles in $\mathcal{M}_K \times \mathcal{M}_K$. See moduli interpretation in the next section.

1.4.2. *Moduli interpretation.* Let $F' = F(\sqrt{\lambda})$ be a quadratic extension as in 1.1.1. Let J be a compact subgroup of $\widehat{F}'^\times$ which has maximal components for places dividing m. Let $K' = K \cdot J$. Then we can use the same formula (1.4.1) to define Hecke correspondence $\mathrm{T}(m)$ on $M_{K'}$. In the following we want to describe a moduli interpretation for $\mathrm{T}(m)$.

1.4.3. *Definition.* Let $[A, \rho, \bar{\theta}, \bar{\kappa}]$ be an object in $\mathcal{F}_{K'}(S)$ as in Proposition 1.1.5, let m be an ideal of $\mathcal{O}_F$, and let D be an $\mathcal{O}_{B'}$-submodule of $A[m]$. We say that D is an *admissible submodule of level* m if the following conditions are satisfied:

1. D is its own annihilator under a Weil pairing

$$(\cdot, \cdot) : A[m] \times A[m] \to \oplus_{\ell | m} \mathcal{O}_\ell / m\mathcal{O}_\ell \tag{1.4.2}$$

   induced by a polarization in $\bar{\theta}$.

2. D^1 and D^2 have the same order.

PROPOSITION 1.4.4. *Assume that each prime factor ℓ of m is split in both B and F'. Let $[A, \rho, \bar{\theta}, \bar{\kappa}]$ be an object in $\mathcal{F}_{K'}(S)$.*

1. *Let D be an admissible submodule of A of level m, let A_D denote the abelian variety A/D, and let ρ_D denote the action of $\mathcal{O}_{B'}$ on A_D induced from that on A. Then there are a unique $\nu(K)$-class $\bar{\theta}_D$ of polarizations on A_D inside of $F^\times \bar{\theta}$, and a unique K'-class $\bar{\kappa}_D$ of level structure which have the same components as $\bar{\kappa}$ out side of ℓ such that $[A_D, \rho_D, \bar{\theta}_D, \bar{\kappa}_D]$ defines an element in $\mathcal{F}_{K'}(S)$.*

2. *The Hecke operator as a correspondence acting on $M_{K'}$ is given by the following formula:*

$$\mathrm{T}(m)[A, \rho, \bar{\theta}, \bar{\kappa}] = \sum_D [A_D, \rho_D, \bar{\theta}_D, \bar{\kappa}_D]$$

   where D runs over all admissible submodule of A of level m.

Proof. Choose a root μ_ℓ of λ in F_ℓ for each ℓ dividing m. Then we have an isomorphism

$$\mathcal{O}_{F',\ell} \to \mathcal{O}_\ell \oplus \mathcal{O}_\ell, \qquad \sqrt{\lambda} \to (\mu_\ell, -\mu_\ell).$$

Any $\oplus_{\ell|m}\mathcal{O}_{F',\ell}$-module M has a corresponding decomposition $M = M^1 + M^2$.

Since $\mathrm{T}(m)$ is multiplicative for coprime m's, we may assume that m is a power of a prime ideal ℓ.

1.4.5. *Models for $\mathcal{O}_{B',\ell}$ and $V_{\mathbb{Z},\ell}$.* In the decomposition

$$\mathcal{O}_{B',\ell} = \mathcal{O}^1_{B',\ell} \oplus \mathcal{O}^2_{B',\ell},$$

the Rosatti convolution switches two factors. Now, we can fix an isomorphism

$$\mathcal{O}_{B',\ell} = \mathcal{O}_{B,\ell} \oplus \mathcal{O}_{B,\ell}, \tag{1.4.3}$$

such that the following conditions are satisfied:

- The second projection is the projection onto $\mathcal{O}^2_{B',\ell}$ composing with the canonical isomorphism $\mathcal{O}^2_{B',\ell} \simeq \mathcal{O}_{B,\ell}$.
- The Rosatti operator is given by
$$(a, b)^* = (\bar{b}, \bar{a}).$$

Similarly, we fix a model for $V_{\mathbb{Z},\ell}$ as follows. First of all, since

$$\psi_F(ax, y) = \psi_F(x, a^* y), \tag{1.4.4}$$

it follows that in the decomposition

$$V_{\mathbb{Z},\ell} = V^1_{\mathbb{Z},\ell} \oplus V^2_{\mathbb{Z},\ell},$$

ψ_F has the form

$$\psi_F(x^1 + x^2, y^1 + y^2) = \psi_F(x^1, y^2) - \psi_F(y^1, x^2)$$

for $x^i, y^i \in V^i_{\mathbb{Z},\ell}$. It follows that ψ_F gives a perfect pairing between $V^i_{\mathbb{Z},\ell}$'s. So we have an isomorphism

$$V_{\mathbb{Z},\ell} \simeq \mathcal{O}_{B,\ell} \oplus \mathcal{O}_{B,\ell} \tag{1.4.5}$$

such that:

- The second projection is the projection onto $V^2_{\mathbb{Z},\ell}$ composing with the canonical isomorphism
$$V^2_{\mathbb{Z},\ell} \simeq \mathcal{O}_{B,\ell}.$$
(Recall that $V_{\mathbb{Z}} = \mathcal{O}_{B'}$ in its definition.)
- The pairing ψ_F is given by

$$\psi_F((x^1,x^2),(y^1,y^2)) = \mathrm{tr}_{B/F}(\bar{x}^1y^2) - \mathrm{tr}_{B/F}(\bar{y}^1x^2) \tag{1.4.6}$$

for $x^i, y^i \in \mathcal{O}_{B,\ell}$.

With respect to the decompositions (1.4.3) and (1.4.5), the action of the second factor of $\mathcal{O}_{B',\ell}$ on $V_{\mathbb{Z},\ell}$ is given by left multiplication on the second factor of $V_{\mathbb{Z},\ell}$. It follows from (1.4.4), that the same is true for the first factor.

Now we want to find a formula for another action $\mathcal{O}_{B,\ell}$ on $V_{B,\ell}$ which is originally given by right multiplication in its definition. Let us denote this action by r. Let $a \in \mathcal{O}_{B,\ell}$. Since $r(a)$ is $\mathcal{O}_{B',\ell}$-linear, $r(a)$ must be given by right multiplication of some element (a_1, a_2) of $\mathcal{O}_{B',\ell}$ with respect to the decomposition (1.4.3). From the definitions of the decomposition, $a_2 = a$. Recall that $r(a)$ is a similitude of ψ_F:

$$\psi_F(r(a)x, r(a)y) = \det(a)\psi_F(x,y).$$

Combining this with (1.4.6), we must have $a_1 = a$. So the action r is still given by right multiplication.

1.4.6. *First statement.* Let κ be one element in $\bar{\kappa}$. Then tensoring with $\mathbb{Q}$ we obtain an isomorphism

$$\kappa : \widehat{V} \to T(A) \otimes \mathbb{Q}.$$

Notice that the natural map $A \to A_D$ induces inclusions

$$T(A) \subset T(A_D) \subset T(A) \otimes \mathbb{Q}.$$

We want to find $\gamma \in G_m$ such that $\widehat{V}_{\mathbb{Z}}\gamma^{-1} = \kappa^{-1}(T(A_D))$. Notice that such a γ is unique modulo G_1 if it exists. We need only work at the place ℓ.

Let W denote $\kappa^{-1}(T_\ell(A_D))$. Then D is isomorphic to $W/V_{\mathbb{Z},\ell}$. Notice that the Weil pairing on $A[m]$ is induced up to an invertible factor by the pairing

$$\alpha^2\psi_F : m^{-1}V_{\mathbb{Z},\ell} \times m^{-1}V_{\mathbb{Z},\ell} \to \mathcal{O}_\ell, \tag{1.4.7}$$

where $\alpha \in \widehat{F}^\times$ is a generator of m. Since D is its own annihilator, it follows that the pairing

$$\alpha\psi_F : W \times W \to \mathcal{O}_\ell$$

is perfect.

With respect to the decomposition (1.4.5), W must have the form

$$W = \mathcal{O}_{B,\ell}\gamma_1^{-1} \oplus \mathcal{O}_{B,\ell}\gamma_2^{-1}$$

with $\gamma_i \in \mathcal{O}_{B,\ell}$. Notice that D_i is isomorphic to $\mathcal{O}_{B,\ell}\gamma_i^{-1}/\mathcal{O}_{B,\ell}$ respectively. So D_i has order $(\det(\gamma_i))$. Since D^1 and D^2 have the same order, it follows that both $\det\gamma_1$ and $\det\gamma_2$ generate m. Now as $\alpha\psi_F$ is perfect on W, γ_2 must be equal to γ_1 times a unit. So $W = V_{\mathbb{Z},\ell}\gamma_1^{-1}$.

Let γ be an element of G_m which has the component γ_1 at the place ℓ; then we have $\kappa^{-1}(T(A_D)) = \widehat{V}_{\mathbb{Z}}\gamma^{-1}$. Now we can define $\bar{\kappa}_D$ as the class of the composition

$$\kappa \circ \gamma^{-1} : \widehat{V}_{\mathbb{Z}} \to \widehat{V}_{\mathbb{Z}}\gamma^{-1} = \kappa^{-1}(T(A_D)) \to T(A_D).$$

As in the proof of Proposition 1.1.5, there will be a unique class θ_D inside $F^\times\theta$ such that $[A_D, \rho_D, \bar{\theta}_D, \bar{\kappa}_D]$ is an object in $\mathcal{F}_{K'}(S)$.

1.4.7. *Second statement.* Let γ be an element in G_m. Recall that in the proof of Proposition 1.1.3, if $[(z, g)]$ represents an object $[\bar{A}, \rho, \bar{\theta}_{\bar{A}}, \bar{\kappa}]$ in $\mathcal{F}^0_{K'}(\mathbb{C})$ then $[z, g\gamma]$ represents the object $[\bar{A}, \rho, \bar{\theta}_{\bar{A}}, \bar{\kappa}\gamma^{-1}]$. Also recall that in the proof of Proposition 1.1.5 of the equivalence $\mathcal{F}^0_{K'}$ and $\mathcal{F}_{K'}$, these two objects are equivalent to

$$[A, \rho, \bar{\theta}, \bar{\kappa}] \quad \text{and} \quad [A', \rho, \bar{\theta}', \bar{\kappa}\gamma^{-1}]$$

where A (resp. A') is the abelian variety isogenous to $\bar{A}$ such that

$$\kappa(\widehat{V}_{\mathbb{Z}}) = T(A) \quad \Big(\text{resp.} \quad \kappa \circ \gamma^{-1}(\widehat{V}_{\mathbb{Z}})) = T(A')\Big).$$

Here $\bar{\theta}$ (resp. $\bar{\theta}'$) is the unique $\nu(K')$-class inside $\bar{\theta}_{\bar{A}}$ to make this an object in $\mathcal{F}_{K'}(\mathbb{C})$.

The inclusion $\widehat{V}_{\mathbb{Z}} \subset \widehat{V}_{\mathbb{Z}}\gamma^{-1}$ induces an isogeny $A \to A'$ with kernel D isomorphic to

$$\widehat{V}_{\mathbb{Z}}\gamma^{-1}/\widehat{V}_{\mathbb{Z}}.$$

We want to show that D is admissible of level m. As the Weil pairing on $A[m]$ up to an invertible scale is induced by a pairing as in (1.4.7), it follows easily that D is its own annihilator. Also D_1 and D_2 have the same cardinality as both of them are isomorphic to $\mathcal{O}_{B,\ell}\gamma^{-1}/\mathcal{O}_{B,\ell}$. Thus D is admissible.

By the first statement, $[A', \rho, \bar{\theta}', \bar{\kappa}\gamma^{-1}]$ is equal to $[A_D, \rho, \bar{\theta}_D, \bar{\kappa}_D]$. From the above arguments, one sees that the correspondence between admissible submodules of level m and γ's in G_m/G_1 is bijective. The second statement of the proposition thus follows. □

1.4.8. *Remarks.* First of all, we may extend Definition 1.4.3 and Proposition 1.4.4 to $\mathcal{M}_{K',\wp}$ where $\wp$ is a prime of F. Indeed, everything is exactly the same as above except when $\ell = \wp$. In this case, we need the following assumptions:

1. Assume further that λ is split in $F_\wp$ and choose a square root $\mu_\wp$ in $F_\wp$.

2. Assume the Weil pairing in (1.4.2) has the values in

$$\Sigma_1[m] \oplus \oplus_{\ell \nmid m} m^{-1}\mathcal{O}_\ell/m\mathcal{O}_\ell$$

where Σ_1 is the formal $\mathcal{O}_\ell$-module of height 1.

Secondly, D is uniquely determined by D^2 as D^1 is the annihilator of D^2 in $A[m]^1$. Actually, the correspondence $D \longmapsto D^2$ gives a bijection between admissible submodules of $A[m]$ and submodules of $A[m]^2$ of order m^2.

Moreover, if each $\ell|m$ is split in B then we may give a further decomposition for M^2. For this we fix an isomorphism $\mathcal{O}_{B,\ell} \simeq \mathrm{M}_2(\mathcal{O}_\ell)$. Then M^2 has a decomposition:

$$M^2 = M^{2,1} + M^{2,2}$$

where

$$M^{2,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^2, \qquad M^{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M^2.$$

The element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ switches $M^{2,1}$ and $M^{2,2}$.

If D is an admissible submodule of A of level m, then $D^{2,1}$ is an $\mathcal{O}_\wp$-submodule of $A[m]^{2,1}$ of order $\mathrm{N}(m)$. The map $M \longmapsto M^{2,1}$ is bijective between the set of admissible submodules of A of level m, and the set of submodules of $A[m]^{2,1}$ of order $\mathrm{N}(m)$. Indeed, for a given $\mathcal{O}_F$-submodule D_1 of $A[m]^{2,1}$ of order m, we can obtain a module $D_2 = D_1 + wD_1$ as an $\mathcal{O}_{B,m}$ module of $A[m]^2$. Let D_3 be the annihilator of D_2 in $A[m]$; then $D = D_2 + D_3$ is an admissible submodule of level m.

1.4.9. *The Eichler-Shimura congruence relation.* Let $\wp$ be a prime in $\mathcal{O}_F$ over which K has the maximal component and B is split. Let $\mathrm{Frob}(\wp)$ be the Frobenius correspondence on $\mathcal{M}_{K,k}$ where k is the residue field of $\mathcal{O}_{F,\wp}$. Then we have the following Eichler-Shimura congruence relation:

PROPOSITION 1.4.10. *Let* $\mathrm{Frob}(\wp)^*$ *denote the dual correspondence of* $\mathrm{Frob}(\wp)$. *Then*

$$\mathrm{T}(\wp) = \mathrm{Frob}(\wp) + \mathrm{Frob}(\wp)^*.$$

Proof. Let $F' = F(\sqrt{\lambda})$ be as before, such that $\wp$ is split in F'. We will only give a proof for the special case where M_K can be embedded into $M_{K'}$ for some K' which is sufficient to apply to the curve X defined in the introduction. (The proof of the general case can be found in Carayol's paper [4, §10.3] where he uses a slightly different definition of $M_{K'}$ so that every M_K can be embedded into his $M_{K'}$.)

It is obvious that we need only prove the same identity for $M_{K'}$. Furthermore, it is true if the identity is true for one K' then it is true for a smaller one, as the identity in the proposition is stable under pushforward of cycles. So we may assume that K' is compact. Now we need only verify the identity for points in $\mathcal{M}_{K',\wp}(\bar{k})$. We may only restrict ourselves to the dense subset of smooth and ordinary points. These points are reductions of points in $\mathcal{M}_{K',\wp}(W)$ where W is the completion of the maximal unramified extension of $F_\wp$.

Let $[A,\rho,\bar\theta,\bar\kappa]$ be one object in $\mathcal{F}_{K'}(W)$. Then

$$\mathrm{T}(\wp)[A,\rho,\bar\theta,\bar\kappa]=\sum_D[A_D,\rho_D,\bar\theta_D,\bar\kappa_D]$$

where D runs through the set of admissible submodules of level m. We want to study the reduction of this identity module $\wp$.

As explained in 1.4.8, D is completely determined by a submodule $D^{2,1}$ of $A[\wp]^{2,1}$ of order $\wp$. Since our object is ordinary, $A[\wp^\infty]^{1,2}$ is isomorphic to

$$\Sigma:=\Sigma_1\oplus F_\wp/\mathcal{O}_\wp$$

where Σ_1 is a formal $\mathcal{O}_\wp$-module on W of height 1. The generic fiber of Σ isomorphic to $F_\wp/\mathcal{O}_\wp\oplus F_\wp/\mathcal{O}_\wp$. For any $t\in\mathcal{O}_\wp/\wp$, let Σ^t denote the submodule of Σ whose generic fiber is a group of points (tx,x). The submodules of Σ order $\wp$ are exactly those of Σ^t and Σ_1.

As the universal deformation space of $[A,\rho,\bar\theta,\bar\kappa]$ is isomorphic to that of $A[\wp^\infty]^{2,1}$, it is easy to see that the isogeny $A\to A_D$ is purely inseparable if $D^{2,1}$ corresponds to Σ_1, and is étale if $D^{2,1}$ does not correspond to Σ_1. Thus in the first case,

$$[A_D,\rho_D,\bar\theta_D,\bar\kappa_D]=\mathrm{Frob}(\wp)[A,\rho,\bar\theta,\bar\kappa]\pmod{\wp}$$

and in the second case

$$[A,\rho,\bar\theta,\bar\kappa]=\mathrm{Frob}(\wp)[A_D,\rho_D,\bar\theta_D,\bar\kappa_D]\pmod{\wp}.$$

Now the congruence relation in the proposition follows. □

1.5. *Order R and its level structure.*

1.5.1. *Construction of R and X.* Let N be a nonzero ideal of $\mathcal{O}_F$ and let E be a totally imaginary quadratic extension of F whose relative discriminant is prime to N. Assume that $\varepsilon(N)=(-1)^{g-1}$, where

$$\varepsilon: F^\times\backslash\widehat{F}^\times\to\{\pm1\}$$

is the character associated to the extension E/F. Then up to isomorphisms, there is a unique quaternion algebra B such that B is ramified exactly at the place τ and finite places $\wp$ where $\varepsilon_\wp(N)=-1$, as this ramification set has even cardinality by our assumption. Also by construction, every ramification place of B is not split in E. So we may fix an embedding $\rho: E\to B$ over F. This allows us to consider E as a subalgebra of B.

In the following we want to construct an order R of B of type (N,E); this means that R contains $\mathcal{O}_E$ and has discriminant N. For each prime $\wp$ dividing N, let $\wp_K$ be a prime of $\mathcal{O}_E$ dividing $\wp$. Let N_E be an ideal of $\mathcal{O}_E$ which is a product of powers of $\wp_E$ and which has relative norm N/N_B. The existence of such N_E follows easily from our assumptions. Indeed, if we write

$$\widetilde{N} = \prod_{\varepsilon(\wp)=1} \wp_E^{\mathrm{ord}_\wp(N)} \cdot \prod_{\varepsilon(\wp)=-1} \wp_E^{[\mathrm{ord}_\wp(N)/2]}$$

then

$$N_E = \prod_{\wp} \wp_E^{\mathrm{ord}_\wp(\widetilde{N})}.$$

Let $\mathcal{O}_B$ be a maximal order of B containing $\mathcal{O}_E$. Then we obtain an order of B by the following formula:

$$R = \mathcal{O}_E + N_E\mathcal{O}_B.$$

Conversely, any order of type (N, E) of B has the above form with some choice of the maximal order $\mathcal{O}_B$.

As in the introduction, our primary curve of study is the compactification X of the Shimura curve associated to the noncompact group $\widehat{F}^\times \cdot \widehat{R}^\times$.

1.5.2. *Cyclic submodule structures.* Let K be an open subgroup of $\widehat{R}$ which has the same components as $\widehat{R}$ over places dividing N. Let J be some compact open subgroup of $\widehat{F}'^\times$ which has maximal components at places dividing N. Let K_0 denote the subgroup of $\widehat{\mathcal{O}}_B^\times$ which is obtained by replacing components of K over places diving N with maximal ones. Let K' denote $K \cdot J$ and K_0' denote $K_0 \cdot J$. Then we have a morphism of functors

$$\mathcal{F}_{K'} \to \mathcal{F}_{K_0'}.$$

In the following we want to show that the fiber of this morphism is given by so-called cyclic submodule structures.

For every prime p which is divided by at least one prime factor $\wp$ in N, we assume that $\left(\frac{\lambda}{p}\right) = 1$, and fix a square root μ_p of $\sqrt{\lambda}$ in $\mathbb{Q}_p$. In this way any $\mathcal{O}_{F'}/N$ module M has decomposition $M = M^1 \oplus M^2$ in the same fashion as before.

1.5.3. *Definition.* Let A be an object of $\mathcal{F}_{K_0}(S)$. By a cyclic submodule structure on A of level N_E, we mean an $\mathcal{O}_E/N_E$-submodule C of $A[N_E]^2$ such that locally there is an element $x \in A[N_E]$ with the following properties:

1. The element x is a Drinfeld base for C. This means that as cycles one has:
$$[C] = \sum_{a \in \mathcal{O}_E/N_E} [ax].$$
2. If $\wp$ is a prime of F over which B is not split, then x is also a Drinfeld base for $\mathcal{O}_B/\widetilde{N}$-module $A[\widetilde{N}]^2$.

Notice that the second condition here is equivalent to the fact that x is not divisible by uniformizers of B in $A[\widetilde{N}]$.

PROPOSITION 1.5.4. *The functor $\mathcal{F}_{K'}$ is equivalent to the functor which sends an F'-scheme S to the set of objects $[A, C]$, where A is an object in $\mathcal{F}_{K_0'}(S)$, and where C is a cyclic submodule structure of level $N_{E'}$ on A.*

Before the proof of this proposition, we need the following crucial lemma. Let $E' = E \otimes F'$. Then every prime factor $\wp_E$ of $\mathcal{O}_E$ can be lifted to a prime $\wp_{E'}$ which is the preimage of $\wp_E$ via the map

$$\mathcal{O}_{E'} \to \mathcal{O}_{E,\wp_E}, \qquad \sqrt{\lambda} \to -\mu_p.$$

Let $N_{E'}$ be the lifting of N_E to the ideal in $\mathcal{O}_{E'}$. Now we have the formulas

$$N_{F'} = \prod_{\wp} \wp_{F'}^{\mathrm{ord}_\wp(\widetilde{N})}.$$

LEMMA 1.5.5. *The following identities hold in $\widehat{B}$:*

$$\begin{aligned}
\widehat{\mathcal{O}}_B^\times \cdot \widehat{\mathcal{O}}_{F'}^\times &= \left\{ g \in \widehat{B}^\times \cdot \widehat{F}'^\times : \widehat{\mathcal{O}}_{B'} g = \widehat{\mathcal{O}}_{B'} \right\}, \\
\widehat{R}^\times &= \left\{ g \in \widehat{\mathcal{O}}_B^\times : \widehat{N}_E^{-1} g = \widehat{N}_E^{-1} \pmod{\widehat{\mathcal{O}}_B} \right\}, \\
\widehat{R}^\times \cdot \widehat{\mathcal{O}}_{F'}^\times &= \left\{ g \in \widehat{\mathcal{O}}_B^\times \cdot \widehat{\mathcal{O}}_{F'}^\times : \widehat{N}_{E'}^{-1} g = \widehat{N}_{E'}^{-1} \pmod{\widehat{\mathcal{O}}_{B'}} \right\}.
\end{aligned}$$

Proof. 1.5.6. *First identity.* We need only prove the inclusion "$\supset$" for each place $\wp$. Let $a \in B_\wp^\times$ and $b \in F_\wp'^\times$ such that $c = ab \in \mathcal{O}_{B',\wp}^\times$.

If $F'_\wp$ is a field unramified over $F_\wp$, then $b = db'$ with $d \in F_\wp^\times$ and $b' \in \mathcal{O}_{F',\wp}^\times$. So we may write $c = a'b'$ with

$$a' = ad = cb'^{-1} \in B_\wp \cap \mathcal{O}_{B',\wp}^\times = \mathcal{O}_{B,\wp}^\times.$$

If $F'_\wp$ is split, then we have a decomposition

$$F'_\wp = F_\wp \oplus F_\wp, \quad B'_\wp = B_\wp \oplus B_\wp.$$

Write $b = (b_1, b_2)$ with respect to these decompositions. Then ab_1 and ab_2 are both in $\mathcal{O}_{B,\wp}^\times$. It follows that $b_1 b_2^{-1} \in \mathcal{O}_{F,\wp}^\times$. So we may write $c = a'b'$ with

$$b' = (b_1 b_2^{-1}, 1) \in \mathcal{O}_{F',\wp}^\times \quad \text{and} \quad a' = ab_1 \in \mathcal{O}_{B,\wp}^\times.$$

Finally let us assume that $F'_\wp$ is a ramified quadratic extension of $F_\wp$. Let π' be a uniformizer for $F'_\wp$. By replacing b with a multiple of elements in $F_\wp^\times \cdot \mathcal{O}_{F'}^\times$, we may assume that b is either 1 or π'. In the first case, a must be a unit and we are done. Now we assume that $b = \pi'$. Since $\wp$ is ramified in F', $\wp$ must be split in B. By writing a as a 2 by 2 matrix over $F_\wp$, we see that the integrality of $a\pi'$ implies that of a. But this implies that $c = a\pi'$ cannot be a unit.

1.5.7. *Second identity.* This identity follows from the definition because

$$\begin{aligned}\widehat{N}_E^{-1} g = \widehat{N}_E^{-1} \pmod{\widehat{\mathcal{O}}_B} &\iff \widehat{N}_E^{-1} g = \widehat{N}_E^{-1} + \widehat{\mathcal{O}}_B \\ &\iff g \in \widehat{\mathcal{O}}_E + N_E \widehat{\mathcal{O}}_B = \widehat{R}.\end{aligned}$$

1.5.8. *Third identity.* For this, we need only show the following:

$$\widehat{\mathcal{O}}_B^{\times} \cap \left(\widehat{\mathcal{O}}_{E'} + \widehat{N}_{E'} \widehat{\mathcal{O}}_{B'}\right) \subset \widehat{R}^{\times}, \tag{1.5.1}$$

since by similar reasoning to that above,

$$\widehat{N}_{E'}^{-1} g = \widehat{N}_{E'}^{-1} \pmod{\widehat{\mathcal{O}}_{B'}} \iff g \in \widehat{\mathcal{O}}_{E'} + \widehat{N}_{E'} \widehat{\mathcal{O}}_{B'}.$$

We need only check (1.5.1) for each place $\wp$ of F. This is clear if $\wp$ does not divide N. But if $\wp$ divides N, then it is split in F' and we have decompositions:

$$B'_\wp = B_\wp \oplus B_\wp, \quad \sqrt{\lambda} \to (\mu_p, -\mu_p),$$

$$N_{E',\wp} = \mathcal{O}_{E,\wp} \oplus N_{E,\wp}.$$

It follows that

$$\mathcal{O}_{E',\wp} + N_{E'} \mathcal{O}_{B',\wp} = \mathcal{O}_{B,\wp} \oplus R_\wp.$$

Thus (1.5.1) is proved. □

1.5.9. *Proof of Proposition* 1.5.4. Let S be an F'-scheme, and $[A, \bar{\kappa}]$ an element of $\mathcal{F}_{K'}(S)$ with $A \in \mathcal{F}_{K'_0}(S)$ and $\bar{\kappa}$ a class modulo K' isomorphisms $\kappa : \widehat{V}_{\mathbb{Z}} \to \widehat{T}(A)$. This κ will induce an isomorphism $\kappa : \widehat{V} \to \widehat{V}(A)$ and a map

$$\widetilde{\kappa} : \widehat{V} \to \widehat{V}(A)/\widehat{T}(A) = A_{\mathrm{tor}}.$$

Thus, we have an $\mathcal{O}_{E'}$-submodule $C_\kappa := \widetilde{\kappa}(N_{E'}^{-1}/\mathcal{O}_{E'})$ of $A[N_{E'}]$. By the above lemma, C_κ does not depend on the choice of κ in the class $\bar{\kappa}$. Since $N_{E'}^{-1}/\mathcal{O}_{E'}$ is a free module of rank 1 over $\mathcal{O}_{F'}/N_{F'}$ and generates $\mathcal{O}_{B'}/N_{F'}$-module $N_{F'}^{-1}\mathcal{O}_{B'}/\mathcal{O}_{B'}$, it follows that C_κ is generated by a Drinfeld base x of the order $N_{F'}$.

Conversely, for any $\mathcal{O}_{E'}$-submodule C of $A[N_{E'}]$ which is generated by a Drinfeld base of order $N_{F'}$, and any level structure κ_0 for the compact subgroup K'_0, we have a unique level structure κ so that $\kappa_0 = \kappa \pmod{K'_0}$ and $C = C_\kappa$.

In a similar manner, we have the following:

Proposition 1.5.10. *Let $\wp$ be a prime of F of characteristic p. Assume that J is maximal at places over p. Then the functor $\mathcal{F}_{K',\wp}$ is equivalent to the functor which sends the W-scheme S to the set of isomorphism classes of objects $[A, C]$ where A is an object in $\mathcal{F}_{K'_0,\wp}(S)$ and C is a cyclic submodule structure on A of order $N_{F'}$.*

2. Heegner points

In this section we study Heegner points. We start in Section 2.1 with the general definition of CM-points and Heegner points as complex points, and their modular interpretations. Then we move to the study of their reductions which are so-called distinguished points, first the structure of formal group in Section 2.2 and then the structure of endomorphism rings in Section 2.3 using Honda-Tate theory. Finally in Section 2.4, we study the lifting of distinguished points by Serre-Tate theory and Gross's theory. In this section we assume that every prime factor of 2 is split in E.

2.1. CM-*points.*

2.1.1. *Definitions and general properties.* Our primary object of study in this paper is the class of Heegner points on the curve X defined in 1.5.1 by the noncompact group $\widehat{F}^\times \widehat{R}^\times$. From the modular point of view, it is more natural to study Heegner points on the Shimura curve Y defined by the compact group $\widehat{R}^\times$:

$$Y = B^\times \backslash \mathcal{H}^\pm \times \widehat{B}^\times / \widehat{R}^\times.$$

The curve X is then a quotient of Y by the action of $\widehat{F}^\times$. As in the introduction, we fix a splitting

$$B \otimes_\tau \mathbb{R} = \mathrm{M}_2(\mathbb{R})$$

such that $\rho(E) \otimes \mathbb{R}$ is sent to the subalgebra of $\mathrm{M}_2(\mathbb{R})$ of elements $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. We then extend $\tau : F \to \mathbb{R}$ to $\tau : E \to \mathbb{C}$ such that

$$\tau(x) = a + bi \Longleftrightarrow \rho(x) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We say a point z in Y is a CM-*point* (*by* E), if z is represented by an element of $\mathcal{H}^\pm \times \widehat{B}^\times$ of the form $(\sqrt{-1}, g)$.

For a CM-point z, let ϕ_z denote the morphism

$$g^{-1} \rho g : E \to \widehat{B}.$$

Then up to conjugation by $\widehat{R}^\times$, ϕ_z does not depend on the choice of g. The order $\mathrm{End}(z) := \phi_z^{-1}(\widehat{R})$ in E, which does not depend on the choice of g, is called *the endomorphism ring of* z. The ideal c of $\mathcal{O}_F$, such that

$$\mathrm{End}(z) = \mathcal{O}_c := \mathcal{O}_F + c\mathcal{O}_E,$$

is called *the conductor of* z.

For a place $\wp$ prime to c, the homomorphism ϕ_z defines an *orientation* in

$$U_\wp = \mathrm{Hom}(\mathcal{O}_{E,\wp}, R_\wp)/R_\wp^\times.$$

This set has only one element if $\wp$ does not divide N; otherwise it has two elements: the image of ρ which we called *the positive orientation*, and the image of $\bar{\rho}$ which we called *the negative orientation*. We say two CM-points have *the same orientation*, if they define the same elements in $U_\wp$ for $\wp|N$. If we write

$$\mathcal{O}_{E,\wp} = \mathcal{O}_{F,\wp} + \mathcal{O}_{F,\wp}e$$

with $e^2 \in F$, then two embeddings

$$\phi_1, \phi_2 : \quad \mathcal{O}_{E,\wp} \to R_\wp$$

define the same element in $U_\wp$ if and only if

$$(\phi_1(e) - \phi_2(e))^2 \equiv 0 \pmod{N}. \tag{2.1.1}$$

Indeed, write $R_\wp = \mathcal{O}_{E,\wp} + t\mathcal{O}_{E,\wp}$ with $t \in R_\wp$ such that $\det(t)$ generates $N_\wp$. Then if e_1 and e_2 have the same orientation, it follows that $e_1 - e_2 \in tR_\wp$. This implies that

$$(e_1 - e_2)^2 = -\det(e_1 - e_2) = 0 \pmod{N_\wp}.$$

If e_1 and e_2 do not have the same orientation, then $e_1 - e_2 = 2e_1 \bmod tR_\wp$. Thus

$$(e_1 - e_2)^2 = 4e_1^2 \neq 0 \pmod{N_\wp}.$$

The curve Y admits an action by the group

$$\mathcal{W} = \{b \in \widehat{B}^\times : b^{-1}\widehat{R}^\times b = \widehat{R}^\times\}/\widehat{R}^\times.$$

This group has 2^s elements, where s is the number of prime factors of N. The action of $\mathcal{W}$ on CM-points does not change the conductors, and the induced action on $\prod_{\wp|N} U_\wp$ is free and transitive.

Let Y_c denote the subscheme of the positively oriented CM-points of the conductor c. Then Y_c is defined over E and every point in $Y_c(\bar{E}) = Y_c(\mathbb{C})$ is defined over the ring class field H_c of $\mathcal{O}_c$. Indeed, if $(\sqrt{-1}, g)$ is a CM-point of Y with positive orientation and conductor c, then Y_c is identified with the set of points represented by $(\sqrt{-1}, \widehat{E}^\times g)$. The correspondence which sends x to the class of $(\sqrt{-1}, xg)$, therefore, defines a bijection

$$E^\times \backslash \widehat{E}^\times / \widehat{\mathcal{O}}_c^\times \simeq Y_c.$$

The Galois action of $\mathrm{Gal}(H_c/E)$ on Y_c is given by the inverse of the map,

$$\mathrm{Gal}(H_c/E) \simeq E^\times \backslash \widehat{E}^\times / \widehat{\mathcal{O}}_c^\times,$$

via class field theory.

A CM-point z by E is called a *Heegner point* if its conductor is the trivial ideal $\mathcal{O}_F$. Obviously, the point $(\sqrt{-1}, 1)$ is a Heegner point. In this paper we only consider CM-points with conductors prime to ND_E and with positive orientation, where D_E is the relative discriminant ideal in $\mathcal{O}_F$ for the extension E/F.

Notice that the property of a point to be a CM-point of conductor c is invariant under the action by $\widehat{F}^\times$. Thus, all the above discussion is valid for X or any Shimura curves between X and Y.

2.1.2. *Modular interpretation.* We fix F' as in Section 1.5. In the following we want to give a modular interpretation of Heegner points over $E' = F' \cdot E$. We let Y' denote the Shimura curve $M_{K'}$ with

$$K' = \widehat{R}^\times \cdot \widehat{\mathcal{O}}_{F'}^\times.$$

Then Y has a finite morphism to Y'.

Let $\mathcal{F}$ denote the functor $\mathcal{F}_{K'}$ and let $\mathcal{F}_0$ denote $\mathcal{F}_{K_0'}$ where

$$K_0' = \widehat{\mathcal{O}}_B^\times \cdot \widehat{\mathcal{O}}_{F'}^\times.$$

Then every point x in $Y(\mathbb{C})$ represents an object $[A, C]$, where A stands for an object $[A, \rho_A, \bar{\theta}_A, \bar{\kappa}_A]$ of $\mathcal{F}_0(\mathbb{C})$ and C is a cyclic $\mathcal{O}_E$-submodule structure of A of level N_E. We need some notation to state our result:

- For $[A, C]$ in $\mathcal{F}(S)$,
  - let $\mathrm{End}_{\mathcal{F}_0}(A)$ denote the $\mathcal{O}_{F'}$-subalgebra of $\mathrm{End}_{\mathcal{O}_{B'}}(A)$ generated by elements $\phi : A \to A$ such that $\phi\phi^* \in F^\times$, where $\phi \to \phi^*$ is a Rosati involution induced by a polarization in $\bar{\theta}_A$.
  - let $\mathrm{End}_{\mathcal{F}}(A, C)$ denote the subalgebra of $\mathrm{End}_{\mathcal{F}_0}(A)$ of elements ϕ such that $\phi(C) \subset C$.
- Let $t' : E' \to E'$ be a map defined by
$$t'(a + b\sqrt{\lambda}) = \mathrm{tr}_{E/\mathbb{Q}}(a) + a - \bar{a} + \left(\mathrm{tr}_{E/\mathbb{Q}}(b) - b - \bar{b}\right)\sqrt{\lambda}$$
  for any $a, b \in E$.

PROPOSITION 2.1.3. *Let x be a point on $Y(\mathbb{C})$, and let $[A, , C]$ be an object represented by x. Then the point x is a* CM*-point by E if and only if $\mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q} \simeq E'$. Moreover, if x is a* CM*-point by E, then*:

1. *There is a unique isomorphism*
$$\alpha : E' \simeq \mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q}$$
   over F' such that for any $a \in E'$,
$$\mathrm{tr}(\alpha(a) : \mathrm{Lie}A) = 2s(a).$$

2. *With α as above,*

$$\mathrm{End}(x) = \{a \in E : \quad \alpha(a) \in \mathrm{End}_{\mathcal{F}}(A, C)\}.$$

Proof. Let x be represented by (z, γ). With notation as in the proof of Proposition 1.1.5, the endomorphism ring $\mathrm{End}_{\mathcal{F}_0}(A)$ can be identified with the subring of B generated by elements $b \in B^\times \cdot F^\times$ such that

1. $V_\gamma b \subset V_\gamma$, or equivalently, $b \in \gamma \widehat{\mathcal{O}}_{B'} \gamma^{-1}$;
2. $bj_z = j_z b$, or equivalently, $\tau(b) \in a\rho(E)a^{-1} \otimes_\tau \mathbb{R}$, where $a \in \mathrm{GL}_2(\mathbb{R})$ such that $a(\sqrt{-1}) = z$.

It follows that $\mathrm{End}_{\mathcal{F}_0}(A_x) \otimes \mathbb{Q}$ is an F'-subalgebra of B' generated by elements $b \in B^\times$ satisfying the second condition.

2.1.4. *Equivalence.* If $\mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q} \simeq E'$, then there is an embedding $\beta : E \to B$ over F such that in B',

$$\mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q} = \beta(E) \otimes F'.$$

As all embeddings of E into B are conjugate, it follows that $\beta = b\rho b^{-1}$ where $b \in B^\times$ is uniquely determined by β modulo $\rho(E)^\times$. Now condition 2 implies that in $B \otimes_\tau \mathbb{R}$,

$$b\rho(E)b^{-1} \otimes_\tau \mathbb{R} = a\rho(E)a^{-1} \otimes_\tau \mathbb{R}.$$

It follows that $b = ak$ with some $k \in \rho(E) \otimes_\tau \mathbb{R}$. As $a(\sqrt{-1}) = z$, $k(\sqrt{-1}) = \sqrt{-1}$, one must have $z = \beta(\sqrt{-1})$. Thus x can be represented by

$$\beta^{-1}(z, \gamma) = (\sqrt{-1}, \beta^{-1}\gamma).$$

So x is a CM-point.

Conversely, if x is a CM-point and is represented by $(\sqrt{-1}, g)$, then in the above description of $\mathrm{End}_{\mathcal{F}_0}(A)$, we may take $a = 1$ in condition 2. Now,

$$\mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q} = \rho(E) \otimes F'.$$

This is isomorphic to E' by the following map:

$$\alpha : E' = E \otimes F' \to \mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q},$$

$$\alpha(x \otimes y) = \rho(x) \otimes y.$$

2.1.5. *First property.* It remains to show that α satisfies both properties in the proposition. If $a \in E$ then a acts on $V_\mathbb{R}$ via right multiplication by $\rho(a)$. Write $\rho(a) = (a_1, \cdots, a_g)$ with respect to the decomposition

$$B \otimes \mathbb{R} = \mathrm{M}_2(\mathbb{R}) \oplus (\mathbb{H})^{g-1}.$$

Then by the definition of complex structure in Section 1.1 on

$$V_{\mathbb{R}} = \mathrm{M}_2(\mathbb{R}) \otimes \mathbb{C} \oplus (\mathbb{H} \otimes \mathbb{C})^{g-1},$$

one has

$$\mathrm{tr}(a + b\sqrt{\lambda}) = 4a + 2\sum_{i\geq 2}\left(\mathrm{tr}_{\mathbb{H}/BR}(a_i) + \mathrm{tr}_{\mathbb{H}/BR}(b_i)\sqrt{\lambda}\right) = 2t'(a).$$

The only other isomorphism between $\mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q}$ and E' is $\bar{\alpha}$ defined by

$$\bar{\alpha}(x \otimes y) = \alpha(\bar{x} \otimes y)$$

which does not satisfies property 1 as $t'(\bar{a}) \neq t'(a)$ for $a \in E - F$.

2.1.6. *Second property.* Finally we want to prove the second property in the proposition. By the proof of Proposition 1.1.5 and 1.5.4, C is isomorphic to $N_{E'}^{-1}\gamma^{-1}$ modulo $V_\gamma = \widehat{\mathcal{O}}_{B'}\gamma^{-1}$. It follows that

$$\begin{aligned}
&\{a \in E, \quad \alpha(a) \in \mathrm{End}_{\mathcal{F}}(A, C)\} \\
&\qquad = \left\{a \in E, \quad \begin{matrix} \rho(a) \in \gamma\widehat{\mathcal{O}}_{B'}\gamma^{-!}, \\ N_{E'}^{-1}\gamma^{-1}\rho(a) \subset N_{E'}^{-1}\gamma^{-1} \pmod{\widehat{\mathcal{O}}_{B'}\gamma^{-1}} \end{matrix}\right\} \\
&\qquad = \left\{a \in E, \quad \gamma^{-1}\rho(a)\gamma \in \widehat{\mathcal{O}}_{E'} + N_{E'}\widehat{\mathcal{O}}_{B'}\right\}.
\end{aligned}$$

Similarly, as in 1.5.9, it is easy to see that

$$\widehat{B} \cap \left(\widehat{\mathcal{O}}_{E'} + N_{E'}\widehat{\mathcal{O}}_{B'}\right) = \widehat{R}.$$

Thus we have

$$\begin{aligned}
\{a \in E, \quad \rho(a) \in \mathrm{End}_{\mathcal{F}}(A, C)\} &= \left\{a \in E, \quad \rho(a) \in \gamma\widehat{R}\gamma^{-1}\right\} \\
&= \mathrm{End}(x).
\end{aligned}$$

□

PROPOSITION 2.1.7. *Let x and y be two* CM-*points with conductors prime to N, and representing the objects $[A, C]$ and $[A', C']$. Then x and y have the same orientation if and only if there is an $(\iota(\mathcal{O}_{B'}) \otimes \alpha(\mathcal{O}_E))_N$-linear symplectic similitude from $T(A)_N$ to $T(A')_N$ which takes C to C'. Here for an $\mathcal{O}_F$-module M, $M_N = M \otimes \oplus_{\ell|N}\mathbb{Z}_\ell$.*

Proof. We may assume that x is represented by $(\sqrt{-1}, 1)$ and prove only the local statement for each $\wp$ dividing N. Let y be represented by $(\sqrt{-1}, \gamma)$. Then we have isomorphisms of $\iota(\mathcal{O}_{B'}) \otimes \alpha(\mathcal{O}_{E,\wp})$-modules

$$T_\wp(A) \simeq \mathcal{O}_{B',\wp} \quad \text{and} \quad T_\wp(A') \simeq \mathcal{O}_{B',\wp}\gamma_\wp^{-1}$$

where $\iota(\mathcal{O}_{B'})$ acts by left multiplication and $\alpha(\mathcal{O}_E)$ acts by right multiplication of $\rho(\mathcal{O}_E)$, and isomorphisms of $\alpha(\mathcal{O}_{E'})$-submodules

$$C_\wp \simeq N_{E',\wp}^{-1} \pmod{\mathcal{O}_{B',\wp}} \quad \text{and} \quad C'_\wp \simeq N_{E'}^{-1}\gamma^{-1} \pmod{\mathcal{O}_{B',\wp}\gamma^{-1}}.$$

As any $B'_\wp$-linear endomorphism of $B'_\wp$ is given by right multiplication by an element of $B'_\wp$, the "if" part of the proposition is, therefore, equivalent to the existence of $a \in B'_\wp$ such that the following conditions are verified:

1. $\gamma_\wp^{-1} a \in \mathcal{O}_{B',\wp}^\times$;
2. a commutes with $\rho(E)$;
3. $a \in B_\wp^\times \cdot F_\wp'^\times$;
4. $N_{E'}\gamma_\wp^{-1} a \subset N_{E'} \pmod{\mathcal{O}_{B',\wp}}$.

By the first identity of Lemma 1.5.5, conditions 1 and 3 here are equivalent to the fact that a has the form $\gamma_\wp^{-1} a = bc$ where $b \in \mathcal{O}_{B,\wp}^\times$ and $c \in \mathcal{O}_{F',\wp}^\times$. Replacing a by ac^{-1}, we may assume that $c = 1$ and then $a \in B_\wp$.

Now condition 2 is equivalent to $a \in \rho(E)$, and condition 4 is equivalent to $\gamma_\wp^{-1} a \in R_\wp$ by a similar argument to 1.5.8. It follows that the "if" part of the proposition is equivalent to $\gamma_\wp \in \rho(E)^\times \cdot R_\wp^\times$, or equivalently to the fact that the map

$$\gamma_\wp^{-1}\rho\gamma_\wp : E_\wp \to B_\wp$$

has positive orientation. □

2.2. *Formal groups.* Let q be a finite place of E and let E_q^{ur} be the completion of the maximal unramified extension of E_q with ring of integers $\mathcal{O}_q^{\mathrm{ur}}$, and residue field k. Let y be a CM-point of Y with conductor c prime to ND_E and q. Then y is defined over E_q^{ur}. Let $\bar{y}$ denote the Zariski closure of y in $\mathcal{Y} \otimes \mathcal{O}_q^{\mathrm{ur}}$, where $\mathcal{Y}$ is the integral model of Y over $\mathcal{O}_F$ constructed in Section 1.2. We want to study the reduction y_k of $\bar{y}$ in $\mathcal{Y}_k := \mathcal{Y} \otimes k$.

Let p denote the characteristic of k and let $\wp$ denote the prime of $\mathcal{O}_F$ under q. As usual, we will choose an auxiliary negative integer λ as in Section 1.1 and work on $F' = F(\sqrt{\lambda})$. We assume that $\left(\frac{\lambda}{p}\right) = 1$ and choose a square root μ_p of λ in $\mathbb{Q}_p$. Then there is the usual decomposition $M = M^1 \oplus M^2$ for F'_p modules M. Let i denote the embedding

$$i : E' = E(\sqrt{\lambda}) \to E_q^{\mathrm{ur}}$$

which takes $\sqrt{\lambda}$ to μ_p.

Let $[\bar{A}, \bar{C}]$ be the abelian variety represented by $\bar{y}$. Then the action of $\mathrm{End}(y) \otimes \mathcal{O}_{F'}$ on $A = \bar{A} \otimes E_q^{\mathrm{ur}}$ extends to an action on $\bar{A}$. Let $\mathcal{G}$ denote the divisible $\mathcal{O}_{E'}$-module $\bar{A}[\wp^\infty]^2$.

PROPOSITION 2.2.1. *The action of $\mathcal{O}_E$ on the $\mathcal{O}_q^{\mathrm{ur}}$-module $\mathrm{Lie}(\mathcal{G})$ induced by the action α on $\bar{A}$ is given by the canonical embedding $\mathcal{O}_E \to \mathcal{O}_q^{\mathrm{ur}}$.*

Proof. We want to prove the proposition by computing the trace of the action of $\alpha(\mathcal{O}_{E'})$. Recall that the action $\alpha : E' \to \mathrm{End}(A) \otimes \mathbb{Q}$ induces an action of $E'_p = E' \otimes \mathbb{Q}_p$ on $\mathrm{Lie}(A)$, therefore an action of $E'_\wp = E' \otimes F_\wp$ on $\mathrm{Lie}(A[\wp^\infty])$. This last module has a projection

$$\mathrm{Lie}(A[\wp^\infty]) \to \mathrm{Lie}(\mathcal{G}), \quad \sqrt{\lambda} \to -\mu_p.$$

We denote all of these actions by α. By Proposition 2.1.3, the action α of E' on $\mathrm{Lie}(A)$ has the trace map $i \circ 2t' : E' \to E_q^{\mathrm{ur}}$. It is easy to see that the action α of $E'_\wp$ on $\mathrm{Lie}(A[\wp]^\infty)$ will have the trace $2t'_\wp$, where $t'_\wp : E'_\wp \to E'_\wp$ has the same formula as t' but with $\mathrm{tr}_{E/\mathbb{Q}}$ replaced by $\mathrm{tr}_{E_\wp/\mathbb{Q}}$. The trace $2t''$ of E on $\mathrm{Lie}(\mathcal{G})$ is given by composing $2t'_\wp$ with the embedding

$$E \to E'_\wp, \quad x \to \frac{x}{2} - \frac{x}{2}\frac{\sqrt{\lambda}}{\mu_p}$$

and the projection

$$E'_\wp \to E_q^{\mathrm{ur}}, \quad a + b\sqrt{\lambda} \to a + b\mu_p.$$

So for $x \in E$,

$$\begin{aligned} t''(x) &= \mathrm{tr}_{E/\mathbb{Q}}\left(\frac{x}{2}\right) + \frac{x}{2} - \frac{\bar{x}}{2} - \left(\mathrm{tr}_{E_\wp/\mathbb{Q}_p}\left(\frac{x}{2\mu_p}\right) - \frac{x}{2\mu_p} - \frac{\bar{x}}{2\mu_p}\right)\mu_p \\ &= x. \end{aligned}$$

Now, the action α of E on $\mathrm{Lie}(\mathcal{G})$ has the trace $2x$. Thus $\mathrm{Lie}(\mathcal{G})$ is a two dimensional space of E_q^{ur} and the action α of E is given by the usual scalar multiplication of $E \subset E_q^{\mathrm{ur}}$. □

2.2.2. *The structure of* $\mathcal{G}$. Let $\mathcal{C}$ be the component of C in $\mathcal{G}$. In the following we want to identify the structure of $[\mathcal{G}, \mathcal{C}]$ as an $\mathcal{O}_{B,\wp} - \mathcal{O}_{E,\wp}$-module. First of all let us construct a special object $[\mathcal{G}^0, \mathcal{C}^0]$.

Let Σ denote the following $\mathcal{O}_\wp$-module:

$$\Sigma = \begin{cases} \Sigma_2 & \text{if } \wp \text{ is not split in } E, \\ \Sigma_1 \oplus F_\wp/\mathcal{O}_\wp & \text{if } \wp \text{ is split in } E. \end{cases}$$

Here for any positive integer h, let Σ_h denote a formal $\mathcal{O}_\wp$-module of height h over $\mathcal{O}_q^{\mathrm{ur}}$ which is special in the sense that the induced action on the tangent space is given by scalar multiplication, which exists uniquely up to isomorphism. Let

$$\mathcal{O}_{E,\wp} \times \Sigma \to \Sigma, \quad (a, x) \to ax$$

be a faithful $\mathcal{O}_\wp$-linear action such that the induced action of $\mathcal{O}_{E,\wp}$ on $\mathrm{Lie}(\Sigma)$ is given by the reduction $\mathcal{O}_{E,\wp} \to \mathcal{O}_q \to k$.

Let us define a special $\mathcal{O}_{B,\wp} - \mathcal{O}_{E,\wp}$-module $[\mathcal{G}^0, \mathcal{C}^0]$ such that

1. $\mathcal{G}^0 = \Sigma \oplus \Sigma$;

2. The action α^0 of $\mathcal{O}_{E,\wp}$ is given by the multiplication:

$$\alpha^0(x)(u,v) = (xu, xv);$$

3. $\mathcal{C}^0 = \Sigma[N_E] \oplus 0$ as $\mathcal{O}_{E,\wp}$-modules;

4. The action of $\mathcal{O}_{B,\wp}$ is given as follows:

   (a) If $\wp$ is ramified in E we fix an isomorphism $\mathcal{O}_{B,\wp} \simeq \mathrm{M}_2(\mathcal{O}_\wp)$. Define the action $\iota^0 : \mathcal{O}_{B,\wp} \to \mathrm{End}_{\mathcal{O}_\wp}(\mathcal{G}^0)$ by matrix multiplications.

   (b) If $\wp$ is not ramified in E, then $\mathcal{O}_{B,\wp}$ is generated by $\mathcal{O}_{E,\wp}$ and an element ϖ such that $\varpi x = \bar{x}\varpi$ and such that $\pi := \varpi^2$ is a uniformizer of $\mathcal{O}_\wp$ if $\wp$ is ramified in B, and 1 if $\wp$ is split in B. Then we define the action of $\mathcal{O}_{B,\wp}$ on Σ^2 by the following formula:

$$\iota^0(x)(u,v) = (xu, \bar{x}v), \qquad \iota^0(\varpi)(u,v) = (\pi v, u).$$

PROPOSITION 2.2.3. *The object $[\mathcal{G},\mathcal{C}]$ is isomorphic to $[\mathcal{G}^0,\mathcal{C}]$. In other words, there is an isomorphism $\phi : \mathcal{G} \to \mathcal{G}^0$ such that*

1. *ϕ is $\mathcal{O}_{E,\wp}$-linear with respect to the actions α, α^0,*

2. *ϕ is $\mathcal{O}_{B,\wp}$-linear with respect to the actions ι, ι^0,*

3. *$\phi(\mathcal{C}) = \mathcal{C}^0$.*

Proof.

2.2.4. *Case* 1: *$\wp$ is ramified in E.* In this case $\mathcal{C} = \mathcal{C}^0 = 0$. Define

$$\mathcal{G}_1 = \iota\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\mathcal{G}, \qquad \mathcal{G}_2 = \iota\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathcal{G}.$$

Then $\mathcal{G}$ is isomorphic to $\mathcal{G}_1 \oplus \mathcal{G}_2$ and $\iota\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ switches two factors. Since the $\mathcal{G}_i$'s are stable under the action of $\mathcal{O}_q$, they are isomorphic to Σ_2.

2.2.5. *Case* 2: *$\wp$ is not ramified in E.* Let $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) be the maximal $\alpha(\mathcal{O}_{E,\wp})$-submodule over which $\iota(x) = \alpha(x)$ (resp. $\iota(x) = \alpha(\bar{x})$) for any $x \in \mathcal{O}_{E,\wp}$; then the $\mathcal{G}_i$'s are $\mathcal{O}_{E,\wp}$-modules (via α) of height 1 and $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. The action of $\iota(\varpi)$ gives two $\mathcal{O}_{E,\wp}$-linear morphisms $u : \mathcal{G}_1 \to \mathcal{G}_2$ and $v : \mathcal{G}_2 \to \mathcal{G}_1$ such that $uv = vu = \pi$. The object $[\mathcal{G}, \iota, \alpha]$ is completely determined by $[\mathcal{G}_1, \mathcal{G}_2, u, v]$. As up to isomorphism there is only one special formal $\mathcal{O}_{E,\wp}$-module of height 1, so $\mathcal{G}_i$ is isomorphic to Σ and one of u and v is an isomorphism. In other words, up to isomorphism, $[\mathcal{G}_1, \mathcal{G}_2, u, v]$ is isomorphic to $[\Sigma, \Sigma, 1, \pi]$ or $[\Sigma, \Sigma, \pi, 1]$.

By Proposition 2.1.7, the generic fiber of the $(\mathcal{O}_{B,\wp}, \mathcal{O}_{E,\wp})$-module $(\mathcal{G}, \mathcal{C})$ is isomorphic to that corresponding to $(\sqrt{-1}, 1)$; that is,

$$(\mathcal{G}, \mathcal{C})_{E_q^{\mathrm{ur}}} \simeq (B_\wp/\mathcal{O}_{B,\wp}, N_E^{-1}\mathcal{O}_{E,\wp}/\mathcal{O}_{E,\wp})$$

with the action ι by multiplication from the left and the action of α by multiplication from the right. It follows that $[\mathcal{G}_1, \mathcal{G}_2, u, v]$ is isomorphic to $[\Sigma, \Sigma, 1, \pi]$ and $\mathcal{C}$ is isomorphic to $\Sigma[N_E] \oplus 0$. □

2.3. *Endomorphisms.* Now we assume that y is a Heegner point. We want to study $\mathrm{End}_{\mathcal{F}}(A_k, C_k)$, where A_k is the reduction of A over $\mathrm{Spec}(k)$. Let $\mathbb{F}$ be a finite subfield of k over which $[A_k, C_k]$ and α are defined. In other words, $[A_k, C_k]$ is the base change of some object $[A_{\mathbb{F}}, C_{\mathbb{F}}]$ with an action of $\mathcal{O}_E$. Let σ be the Frobenius over $\mathbb{F}$ which acts on $A_{\mathbb{F}}$. By the Honda-Tate theorem and the Tate theorem ([37] and [42]), $\mathrm{End}(A_{\mathbb{F}})$ is a semisimple algebra with center $\mathbb{Q}(\sigma)$, and for any prime ℓ,

$$\mathrm{End}(A_{\mathbb{F}})_\ell \simeq \mathrm{End}(A_{\mathbb{F}}[\ell^\infty]) \simeq \mathrm{End}^\sigma(A_k[\ell^\infty])$$

where $\mathrm{End}^\sigma(\cdot)$ means the commutator of σ in $\mathrm{End}(\cdot)$. It follows that $\mathrm{End}_{\mathcal{F}}(A_{\mathbb{F}}, C_{\mathbb{F}})$ is also a semisimple algebra with center containing $\mathcal{O}_{F'(\sigma)}$, and such that for any place ℓ' of F',

$$\begin{aligned} \mathrm{End}_{\mathcal{F}}(A_{\mathbb{F}}, C_{\mathbb{F}})_{\ell'} &\simeq \mathrm{End}_{\mathcal{F}}\left(A_{\mathbb{F}}[\ell'^\infty], C_{\mathbb{F}}[\ell'^\infty]\right) \\ &\simeq \mathrm{End}_{\mathcal{F}}^\sigma\left(A_k[\ell'^\infty], C_k[\ell'^\infty]\right). \end{aligned}$$

Here two $\mathrm{End}_{\mathcal{F}}$'s on the right are defined in the same way as in 2.1.1.

Fix $\mathcal{O}_{B'}$-linear isomorphisms from the level structure of $[A_{\mathbb{F}}, C_{\mathbb{F}}]$:

$$\text{(2.3.1)} \qquad \begin{cases} \kappa_\wp^2 : & [\mathcal{G}^0, \mathcal{C}^0] \to [\mathcal{G}, \mathcal{C}], \\ \kappa_p^{2,\wp} : & [V_{\mathbb{Z},p}^{2,\wp}, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)_p^{2,\wp}] \to [T(A)_p^{2,\wp}, C_p^{2,\wp}], \\ \kappa^p : & [\widehat{V}_{\mathbb{Z}}^p, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)^p] \to [\widehat{T}(A)^p, C^p]. \end{cases}$$

Then we obtain isomorphisms:
(2.3.2)

$$\mathrm{End}_{\mathcal{F}}(A_{\mathbb{F}}, C_{\mathbb{F}})_{\ell'} = \begin{cases} \mathrm{End}_{\mathcal{O}_B}^{\widetilde{\sigma}}(\mathcal{G}^0, \mathcal{C}^0) & \text{if } \ell' \mid \wp, \\ \mathrm{End}_{\mathcal{O}_B}^{\widetilde{\sigma}}\left(V_{\mathbb{Z},p}^{2,\wp}, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)_p^{2,\wp}\right)_{\ell'} & \text{if } \ell \mid p \text{ and } \ell \not| \wp \\ \mathrm{End}_{\mathcal{F}}^{\widetilde{\sigma}}\left(\widehat{V}_{\mathbb{Z}}^p, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)^p\right)_{\ell'} & \text{otherwise,} \end{cases}$$

where $\widetilde{\sigma}$ denotes the endomorphism induced from σ through the isomorphism κ's. It follows that $\mathrm{End}_{\mathcal{O}_{B'}}(A_{\mathbb{F}}, C_{\mathbb{F}})$ is the commutator of σ in a quaternion algebra over $\mathcal{O}_{F'}$.

PROPOSITION 2.3.1. *If $\wp$ is split over E, then*

$$\mathrm{End}_{\mathcal{F}}(A_k, C_k) = \mathcal{O}_{E'}.$$

Proof. From the definition of $\mathcal{G}^0$, one sees that

$$\mathrm{End}_{\mathcal{O}_B}(\mathcal{G}^0) \simeq \mathrm{End}_{\mathcal{O}_F}(\Sigma) \simeq \mathcal{O}_\wp \oplus \mathcal{O}_\wp.$$

It follows that for $\ell'|\wp$, $\mathrm{End}_{\mathcal{F}}(A_k, C_k)'_\ell$ can only be an algebra over $\mathcal{O}_\wp$ of degree at most 2. Thus $\mathrm{End}_{\mathcal{F}}(A_k, C_k)$ is an algebra over $\mathcal{O}_F$ of degree at most 2.

Obviously, the right side is isomorphic to $\mathrm{End}_{\mathcal{F}}(\bar{A}, \bar{C})$; therefore, it is included in the left-hand side. As $\mathcal{O}_{E'}$ is a maximal order, we must have an equality. ☐

PROPOSITION 2.3.2. *Assume that $\wp$ is not split in E. Let $B(\wp)$ be the quaternion algebra obtained by changing invariants at τ and $\wp$. Then there is an order $R(\wp)$ of $B(\wp)$ of type $(N(\wp), E)$ such that*

$$\mathrm{End}_{\mathcal{F}}(A_k, C_k) \simeq R(\wp) \otimes \mathcal{O}_{F'},$$

where $N(\wp) = N\wp^{1-2\mathrm{ord}_q(N_E)}$.

Proof. We need only prove this identity locally at each place ℓ' of F' using (2.3.2). In this case, one can show that for $\mathbb{F}$ sufficiently large, $\sigma \in F'$. (See [4, §§11.4.4, 11.4.5] for a proof.)

It is easy to check that if ℓ' does not divide $\wp$ then $\mathrm{End}_{\mathcal{F}}(A, C)_{\ell'}$ is isomorphic to $R \otimes \mathcal{O}_{F', \ell'}$. It remains to show that $\mathrm{End}_{\mathcal{O}_B}(\mathcal{G}^0, \mathcal{C}^0)$ is isomorphic to $R(\wp)_\wp$. Notice that $\mathcal{C}^0$ has only the geometric point 0, thus does not play any role in the computation.

Let D denote $\mathrm{End}_{\mathcal{O}_\wp}(\Sigma)$ which is the maximal order of a quaternion division algebra over $F_\wp$. The action of $\mathcal{O}_{E,\wp}$ defines an embedding of $\mathcal{O}_{E,\wp}$ into D. By a direct computation, we have the following description of $\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}^0, \mathcal{C}^0)$:

$$\mathrm{End}_{\mathcal{O}_\wp}(\mathcal{G}^0) = \mathrm{End}_{\mathcal{O}_\wp}(\Sigma \oplus \Sigma) = \mathrm{M}_2(D):$$

1. If $\wp$ is ramified in E, then

$$\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}^0) = D \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. If $\wp$ is ramified in B, then

$$\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}^0) \simeq \mathcal{O}_{E,\wp} + \mathcal{O}_{E,\wp}\wp \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{pmatrix},$$

where ϖ is an element in D such that $\varpi^2 = \wp$ and such that $\varpi x = \bar{x}\varpi$ for $x \in E_\wp$.

3. If $\wp$ is unramified in both B and E, then

$$\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}^0, \mathcal{C}^0) \simeq \mathcal{O}_{E,\wp} + \mathcal{O}_{E,\wp}\varpi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ ☐

2.3.3. *Some remarks and definitions.* Let y_k be point in $\mathcal{Y}(k)$. We call y_k a *distinguished point* in $Y(k)$ if it is the reduction of Heegner points in $Y(E_q^{\mathrm{ur}})$. We can define a similar concept for any curve between Y and X.

Assume that y_k is a CM-point representing (A_k, C_k). If $\wp$ is split in E then

$$\mathrm{End}_{\mathcal{F}}(A_k, C_k) \simeq \mathcal{O}_{E'}.$$

We write $\mathrm{End}(y_k)$ or $\mathrm{End}^{\mathrm{a}}(A_k, C_k)$ for the unique subring of $\mathrm{End}_{\mathcal{F}}(A_k, C_k)$ corresponding to $\mathcal{O}_E$ (the superscript a stands for the "admissible endomorphisms").

If $\wp$ is not split in E then

$$\mathrm{End}_{\mathcal{F}}(A_k, C_k) \simeq R(\wp) \otimes \mathcal{O}_{F'}$$

with $R(\wp)$ an order of $B(\wp)$ of type $(N^{\wp}, E)$. One may fix the isomorphism so that the involution $\mathrm{End}_{\mathcal{F}_0}(A_k)_{\mathbb{Q}}$ induced by the polarization corresponds to the product of the involutions on $B(\wp)$ and F' respectively. In this way the image of $R(\wp)$ does not depend on the choice of the isomorphism. Denote this image by $\mathrm{End}(y_k)$ or $\mathrm{End}^{\mathrm{a}}(A_k, C_k)$. Notice that two orders in $B(\wp)$ of type $(N(\wp), E)$ are isomorphic if and only if they are conjugate under $B(\wp)^\times$.

For a fixed point z of type $(N\wp, E)$ with $\mathrm{End}(z) = R(\wp)$, the reduction thus defines a map from the set of CM-points reducing to z, with conductor c prime to $N\wp$, to the set

$$\prod_{v | N\wp} R(\wp)_v^\times \backslash \mathrm{Hom}(\mathcal{O}_{E,v}, R(\wp)_v)$$

of orientations. This set has $2^{s(\wp)}$ elements, where $s(\wp)$ is the number of prime factors of $N\wp$ which do not divide D_E. Two CM-points x and y reducing to z have the same orientation with respect to R if and only if they have the same orientation with respect to $R(\wp)$. We call the *orientation* defined by the reduction of the point $(\sqrt{-1}, 1)$ *the positive orientation.*

PROPOSITION 2.3.4. *Assume that $\wp$ is not split in E. Then the map $x \to \mathrm{End}(x)$ gives a bijection between the set of distinguished points in $\mathcal{X}(k)$ and the set of conjugacy classes of orders of $B(\wp)$ of type $(N\wp, E)$.*

Proof. The set of distinguished points on $\mathcal{X}(k)$ is the set of $\widehat{F}^\times$-orbits of distinguished points on $\mathcal{Y}(k)$ or any curves between X and Y.

Notice that the set of Heegner points in $Y(\mathbb{C})$ is represented by $(\sqrt{-1}, \widehat{E})$. Thus the corresponding objects are

$$[A_\gamma, C_\gamma] = \left[\widehat{V}_{\mathbb{Z}}\gamma^{-1}\backslash(V_{\mathbb{R}}, j), \quad \left(N_{E'}^{-1}/\mathcal{O}_E\right)\gamma^{-1}\right],$$

where $\gamma \in \widehat{E}^\times$. Let y_γ be the point in $Y'(\mathbb{C})$ representing the object $[A_\gamma, C_\gamma]$. Then y_γ depends only on the class of γ in $E^\times\backslash\widehat{E}^\times/\widehat{\mathcal{O}}_E^\times$. Thus we may only consider y_γ with γ integral and having components 1 at places over $N\wp$. Then we have isogenies ϕ_γ from $[A_1, C_1]$ to $[A_\gamma, C_\gamma]$ given by right multiplication of γ^{-1} on $V_{\mathbb{Z}}$. Let $y_{\gamma,k}$ be the reduction of y_γ and let $\phi_{\gamma,k}$ denote the reduction of ϕ_γ. Then we can choose isomorphisms in (2.3.1) such that for places not dividing $\wp$ they are induced by multiplication of γ^{-1}, and that at place $\wp$, $\phi_{\gamma,k}$ induces the identity on $[\mathcal{G}^0, \mathcal{C}^0]$.

Using the Honda-Tate theorem, we show easily that

$$\phi_{\gamma,k}^{-1} \circ \mathrm{End}(y_k) \circ \phi_{\gamma,k} = \gamma \widehat{R}(\wp)\gamma^{-1} \cap B(\wp)$$

where $R(\wp)$ (resp. $B(\wp)$) denotes $\mathrm{End}(y_{1,k})$ (resp. $\mathrm{End}(y_{1,k}) \otimes \mathbb{Q}$), and we identify both sides as subrings in

$$\widehat{B}^{\times,\wp} \simeq \widehat{B}(\wp)^{\times,\wp}.$$

As every order of $B(\wp)$ of type $(N\wp, E)$ is conjugate to one of $\gamma R(\wp)\gamma^{-1}$, the map in the proposition is surjective.

Let y_{γ_1} and y_{γ_2} be two Heegner points. By (2.3.1), it is easy to see that the injective map

$$\mathrm{Isom}_{\mathcal{F}}\left([A_{\gamma_1,k}, C_{\gamma_1,k}], [A_{\gamma_2,k}, C_{\gamma_2,k}]\right) \to \mathrm{End}_{\mathcal{F}_0}(A_{1,k}) \otimes \mathbb{Q},$$
$$\alpha \to \phi_{\gamma_2,k}^{-1} \alpha \phi_{\gamma_1,k}$$

has the image consisting of elements b such that

$$\begin{cases} [\mathcal{G}^0, \mathcal{C}^0] \cdot b = [\mathcal{G}^0, \mathcal{C}^0], \\ [V_{\mathbb{Z},p}^{2,\wp}, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)_p^{2,\wp}] \cdot \gamma_1^{-1} b = [V_{\mathbb{Z},p}^{2,\wp}, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)_p^{2,\wp}] \cdot \gamma_2^{-1}, \\ [\widehat{V}_{\mathbb{Z}}^{p}, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)^{p}] \cdot \gamma_1^{-1} b = [\widehat{V}_{\mathbb{Z}}^{p}, \left(N_{E'}^{-1}/\mathcal{O}_{E'}\right)^{p}] \cdot \gamma_2^{-1}. \end{cases}$$

This is equivalent to

$$\gamma_1^{-1} b \gamma_2 \in \widehat{R(\wp)}^{\times} \mathcal{O}_{F'}^{\times}.$$

Thus $y_{\gamma_1,k}$ and $y_{\gamma_2,k}$ are in the same orbit under $\widehat{F}^\times$ if and only if

$$\gamma_2 \in B(\wp)^\times \cdot \gamma_1 \cdot \widehat{R(\wp)}^{\times} \cdot \widehat{F}^\times.$$

This in turn is equivalent to the fact that $\mathrm{End}(y_{1,k})$ and $\mathrm{End}(y_{\gamma_2,k})$ are conjugate in $B(\wp)$. $\square$

2.4. *Liftings of distinguished points.*

2.4.1. *A deformation problem.* Let y_k be a point of $Y(k)$ which represents an object $[A_k, C_k]$ of $\mathcal{F}(k)$. Let $\mathcal{G}_k$ denote $A[\wp^\infty]^2$ and let $\mathcal{C}_k$ denote the component of C in $\mathcal{G}_k$. Let $\alpha_k : E \to \mathrm{End}_{\mathcal{F}_0}(A) \otimes \mathbb{Q}$ be a homomorphism with order

$$\mathcal{O}_c := \{x \in E : \alpha(x) \in \mathrm{End}_{\mathcal{F}}([A_k, C_k])\}.$$

Assume:

1. The order $\mathcal{O}_c$ has conductor prime to $N\wp$, and the restriction of α_k on this order has the positive orientation.

2. The action of $\mathcal{O}_c$ on $\mathrm{Lie}(\mathcal{G})$ is given by the map
$$i : \mathcal{O}_{\alpha_k} \to \mathcal{O}_{\alpha_k}/q \to k.$$

3. The object $[\mathcal{G}_k, \mathcal{C}_k]$ is isomorphic to the reduction of $[\mathcal{G}_k^0, \mathcal{C}_k^0]$ with respect to both the actions of $\iota(\mathcal{O}_B)$ and $\alpha_k(\mathcal{O}_c)$.

Let us consider the deformation functor $\mathcal{F}_\alpha$ over $\mathcal{O}_q^{\mathrm{ur}}$-algebra with residue field k which sends an algebra W to the set of isomorphism classes of objects $[A, C, \alpha]$. Here $[A, C]$ is an object in $\mathcal{F}(W)$, and $\alpha : \mathcal{O}_c \to \mathrm{End}_{\mathcal{F}}[A, C]$ is a homomorphism such that the following conditions are satisfied:

- The reduction of $[A, C, \alpha]$ at k is isomorphic to $[A_k, C_k, \alpha_k]$.
- The Rosati involution induced by $\bar{\theta}_A$ takes $\alpha(x)$ to $\alpha(\bar{x})$ for any $x \in \mathcal{O}_c$.
- The action of $\alpha(\mathcal{O}_c)$ on $\mathrm{Lie}(A)_\wp^2$ is given by the composition:
$$\mathcal{O}_c \to \mathcal{O}_q^{\mathrm{ur}} \to \mathcal{O}_S.$$

Proposition 2.4.2. *The functor $\mathcal{F}_\alpha$ is representable by $\mathcal{O}_q^{\mathrm{ur}}$.*

Proof. The deformation space of $[A_k, C_k, \alpha_k]$ is the same as that of $[\mathcal{G}_k, \mathcal{C}_k, \alpha_k]$. This is isomorphic to $[\mathcal{G}_k^0, \mathcal{C}_k^0, \alpha_k^0]$ by Proposition 2.2.3. Notice that $\mathcal{C}_k^0 = 0$. Now the conclusion of Proposition 2.4.2 follows from the fact that the formal E_q-module Σ has universal deformation space E_q^{ur}. □

Corollary 2.4.3. *Let y_k be a point of $Y(k)$ which represents an object $[A_k, C_k]$ of $\mathcal{F}(k)$. Then y_k is a distinguished point if and only if there is a homomorphism*
$$\alpha_k : \mathcal{O}_E \to \mathrm{End}_{\mathcal{F}}([A, C])$$
such that the above conditions 1–3 *are satisfied.*

2.4.4. *Canonical liftings.* The universal object over $\mathcal{O}_q^{\mathrm{ur}}$ is called the *canonical lifting* of $[A_k, C_k, \alpha_k]$. In this way, if $\wp$ is not split in E then for a fixed distinguished point $y_k \in Y(k)$, the set of positively oriented CM-points with conductor c prime to $N\wp$, which reduce to y_k modulo $\wp$, is bijective to the set of positively oriented homomorphisms $E \to B(\wp)$ with conductor c.

If $\wp$ is split over F, then α is an isomorphism, and the canonical lifting of y_k is a Heegner point y (of characteristic 0).

PROPOSITION 2.4.5. *Assume* $\wp$ *is not split in* E *and* $\mathrm{ord}_\wp(N) \le 1$. *Let* $y_m = [A_m, C_m]$ *be the deformation of* $y_k = [A_k, C_k]$ *to* $\mathcal{O}_q^{\mathrm{ur}}/q^m$ *with respect to* α_k. *Then* $\mathrm{End}(y_m)$ *has the same localization as* $\mathrm{End}(y_k)$ *at places different from* $\wp$, *and*

$$\mathrm{End}(y_m)_\wp = \mathcal{O}_{E,\wp} + q^{m-1}\mathrm{End}(y_k)_\wp.$$

In other words, $\mathrm{End}(y_m)$ *is the unique sub-order of* $\mathrm{End}(y_k)$ *of discriminant* $\wp^{b_m}N$ *where*

$$b_m = \begin{cases} m & \text{if } \wp \text{ is ramified in } E \\ 2m-1 & \text{if } \wp \text{ is unramified in } E. \end{cases}$$

Moreover the action on the formal module $\mathcal{G}_m = A_m[\wp^\infty]^2$ *is given by the following composition of canonical homomorphisms*:

$$\mathcal{O}_{E,\wp} + q^{m-1}\mathrm{End}(y_k)_\wp \to \mathcal{O}_{E,\wp}/q^m \to \mathcal{O}_q^{\mathrm{ur}}/q^m.$$

Proof. By a fundamental theorem of Serre and Tate [7], one can show that

$$\mathrm{End}(y_m) = \mathrm{End}(y_k) \cap \mathrm{End}([\mathcal{G}_m, \mathcal{C}_m])$$

where $\mathcal{C}_m$ is the component of C_m in $\mathcal{G}_m$. It follows that $\mathrm{End}(y_m)$ has the same localization as $\mathrm{End}(y_k)$ at places different from $\wp$, and

$$\mathrm{End}(y_m)_\wp = \mathrm{End}([\mathcal{G}_m, \mathcal{C}_m]) \simeq \mathrm{End}([\mathcal{G}_m^0, \mathcal{C}_m^0])$$

where $[\mathcal{G}_m^0, \mathcal{C}_m^0]$ is the restriction of $[\mathcal{G}^0, \mathcal{C}^0]$ on $\mathcal{O}_q^{\mathrm{ur}}/q^m$. We want to use the description in the proof of Proposition 2.3.2 and Gross' result [15] to describe $\mathrm{End}([\mathcal{G}_m^0, \mathcal{C}_m^0])$.

As in the proof of Proposition 2.3.2, let D denote $\mathrm{End}_{\mathcal{O}_\wp}(\Sigma_k)$ and let D_m denote the suborder $\mathrm{End}_{\mathcal{O}_\wp}(\Sigma_{2,m})$. Then by Gross' result [15, Prop. 3.3],

$$D_m = \mathcal{O}_q + q^{m-1}D.$$

Now in

$$\begin{aligned} \mathrm{End}_{\mathcal{O}_\wp}(\mathcal{G}_k^0) &= \mathrm{End}_{\mathcal{O}_\wp}(\Sigma_k \oplus \Sigma_k) = \mathrm{M}_2(D), \\ \mathrm{End}_{\mathcal{O}_\wp}([\mathcal{G}_m^0, \mathcal{C}_m^0]) &= \mathrm{End}\mathcal{O}_\wp([\mathcal{G}_k^0, \mathcal{C}_k^0]) \cap \mathrm{M}_2(D_m). \end{aligned}$$

Using the description in the proof of Proposition 2.3.2, we have the following:

1. If $\wp$ is ramified in E, then $\mathcal{C} = 0$ and

$$\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}_m^0) = D_m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. If $\wp$ is ramified in B, then

$$\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}_m^0, \mathcal{C}_m^0) \simeq \mathcal{O}_{E,\wp} + \mathcal{O}_{E,\wp} q^m \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{pmatrix},$$

where ϖ is an element in D such that $\varpi^2 = \wp$ and such that $\varpi x = \bar{x}\varpi$ for $x \in E_\wp$.

3. If $\wp$ is unramified in both B and E, then

$$\mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{G}_m^0, \mathcal{C}_m^0) \simeq \mathcal{O}_{E,\wp} + \mathcal{O}_{E,\wp} \wp^{m-1} \varpi \begin{pmatrix} 0 & 1 \\ 1 & o \end{pmatrix}.$$ □

2.4.6. *Quasi-canonical liftings.* We need also to consider the quasi-canonical lifting. Let y be a Heegner point representing $[A, C]$ in $\mathcal{F}(E_q^{\mathrm{ur}})$. Let D be an admissible submodule of A of order $m = \wp^n$ $(n \neq 0)$ prime to N and let $[A_D, C_D]$ be the quotient constructed in Proposition 1.4.4. Assume that D is connected (this is automatically satisfied if $\wp$ is not split in E). Then $[A_D, C_D]$ is an object of $\mathcal{F}(W)$ where W is a finite extension of $\mathcal{O}_q^{\mathrm{ur}}$. Then $[A, C]$ and $[A_D, C_D]$ have the same reduction modulo q. Notice that $[A_D, C_D]$ is not a canonical lifting of the reduction of $[A_k, C_k]$. We call $[A_D, C_D]$ a *quasi-canonical lifting* of $[A_k, C_k]$.

Proposition 2.4.7. *The objects $[A, C]$ and $[A_D, C_D]$ are not isomorphic modulo q^2.*

Proof. By the Honda-Tate theorem we need only check whether or not the associated divisible groups are isomorphic. It suffices to consider the groups $A[\wp^\infty]^2$ and $A_D[\wp^\infty]^2$. Then the conclusion follows from our precise description for $A[\wp^\infty]^2$ and corresponding results of Gross ([15, Prop. 5.3]) on formal groups of dimension 1. □

3. Modular forms and L-functions

In this section we will collect various facts about Hilbert modular forms and associated L-functions. In Section 3.1, we will recall definitions of modular forms and Atkin-Lehner's theory on newforms. In Sections 3.2 and 3.3, we will give a newform theory for X using Jacquet-Langlands correspondence and some work of Waldspurger. In Section 3.4, we will first recall Hecke's theory of L-functions and then prove Theorem B in the introduction. In Section 3.5, we will study some standard Eisenstein series and theta series attached to quadratic characters.

3.1. *Modular forms.*

3.1.1. *Some definitions.* Let k be a positive integer, N an ideal of $\mathcal{O}_F$, and $\omega = \prod \omega_v$ a finite character of $\mathbb{A}_F^\times/F^\times$ with conductor dividing N such that $\omega_v(-1) = (-1)^k$ for $v|\infty$. We want to define the space of modular forms of (parallel) weight k and level N. See [2], [10] for general background and references.

Let $K_0(N)$ denote the following subgroup of $\mathrm{GL}_2(\widehat{F})$:

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{F}) : c \equiv 0 \pmod{\widehat{N}} \right\}.$$

Let K^∞ denote the compact subgroup $\prod_{v|\infty} \mathrm{GL}_2(F_v)$ of matrices of the form

$$r(\theta) = (r(\theta_v), \quad v|\infty) \in \mathrm{GL}_2(F \otimes \mathbb{R})$$

where for $\theta = (\theta_v, v|\infty) \in \mathbb{R}^g$,

$$r(\theta_v) = \begin{pmatrix} \cos 2\pi\theta_v & \sin 2\pi\theta_v \\ -\sin 2\pi\theta_v & \cos 2\pi\theta_v \end{pmatrix}.$$

Let Z denote the center of GL_2. Extend ω to a character on $Z(\mathbb{A}_F)K_0(N)K^\infty$ by the formula

$$\omega\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} r(\theta) \right) = \omega(z) \cdot \prod_{\mathrm{ord}_v(N)>0} \omega_v(a_v) \cdot \prod_{v|\infty} e^{2\pi i k\theta_v}.$$

Now by a *modular form over F of weight k, level N, character ω* we mean a function ϕ on $\mathrm{GL}_2(\mathbb{A}_F)$ satisfying the following conditions:

1. $\phi(\gamma g) = \phi(g)$ for γ in $\mathrm{GL}_2(\mathbb{Q})$;
2. $\phi(gk) = \phi(g)\omega(k)$ for k in $Z(\mathbb{A}_F)K_0(N)K^\infty$;
3. ϕ is slowly increasing: for every $c > 0$, and any compact subset Ω of $\mathrm{GL}_2(\mathbb{A}_F)$, there are a constant C and N such that

$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \le C|a|^N$$

for all $g \in \Omega$, and $a \in \mathbb{A}^\times$ with $|a| \ge c$.

Let ψ be a character on $F\backslash\mathbb{A}_F$ defined by

$$\psi(x) = \exp[2\pi i(\mathrm{tr}_{F/\mathbb{Q}}(x_\infty) - \mathrm{tr}_{F/\mathbb{Q}}(x_f)].$$

Then every character on $F\backslash\mathbb{A}_F$ has the form

$$x \to \psi(\alpha x)$$

with some $\alpha \in F$.

Let dx be a Haar measure on $\mathbb{A}_F$ which is a product of local Haar measures dx_v such that if v is archimedean, dx_v is the usual Haar measure on $\mathbb{R}$, and that if v is nonarchimedean, the volume of $\mathcal{O}_v$ is 1. In this way, the volume of $\mathbb{A}_F/F$ is $d_F^{-1/2}$ where d_F denotes $\mathrm{N}(D_F)$.

For a modular form ϕ as above, let $W_\phi(g)$ denote the corresponding Whittaker function on $\mathrm{GL}_2(\mathbb{A})$:

$$W_\phi(g) = d_F^{-1/2} \int_{F\backslash\mathbb{A}_F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x)dx \tag{3.1.1}$$

where $W_\phi(g)$ satisfies the same condition 2 above as ϕ, and in addition the following property:

$$W_\phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W_\phi(g). \tag{3.1.2}$$

Now ϕ has the following Fourier expansion

$$\phi(g) = C_\phi(g) + \sum_{\alpha\in F^\times} W_\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right). \tag{3.1.3}$$

where

$$C_\phi(g) = d_F^{-1/2} \int_{F\backslash\mathbb{A}_F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx. \tag{3.1.4}$$

We say a form ϕ is *cuspidal*, if for almost every g,

$$C_\phi(g) = 0.$$

Thus cuspidal forms are determined by their Whittaker functions.

Notice that any double coset in

$$Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/K_0(N)K_\infty$$

can be represented by an element of the form $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ with $y \in \mathbb{A}_F^\times$, $y_\infty > 0$, and $x \in \mathbb{A}_F$. We say a form ϕ is *holomorphic* if

$$\omega^{-1}(y)|y|^{-k/2}\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$$

is holomorphic in

$$x_\infty + iy_\infty \in \mathcal{H}^g.$$

PROPOSITION 3.1.2. *Let ϕ be a holomorphic form. Then*

1. $W_\phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0$ *only if* $y_\infty > 0$.

2. *There is a function* a *on the set of fractional ideals which vanishes on nonintegral ideals, such that*

$$C_\phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \omega(y)|y|^{k/2}a(0),$$
$$W_\phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \omega(y)|y|^{k/2}a(y_f D_F)\psi(iy_\infty),$$

where for $y \in \mathbb{A}_F^\times$ *with* $y_\infty > 0$, $x \in \mathbb{A}_F$, *and* D_F *the inverse of the different ideal of* F:

$$D_F{}^{-1} = \{x \in F : \mathrm{tr}(x\mathcal{O}_F) \subset \mathbb{Z}\}.$$

Proof. From (3.1.2), one sees that

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \omega(y)|y|^{k/2}\sum_{\alpha\in F} c(\alpha y)\psi(\alpha x)$$

where $c(y)$ is defined by

$$C_\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \omega(y)|y|^{k/2}c(0),$$
$$W_\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \omega(y)|y|^{k/2}c(y)\psi(x).$$

As ϕ is holomorphic in $x_\infty + iy_\infty$, it follows that $c(y) \neq 0$ only if $y_\infty > 0$. Moreover if $y_\infty > 0$, then $c(y)$ has the decomposition $c(y) = c(y_f)\psi(iy_\infty)$.

For any $\alpha \in \widehat{\mathcal{O}}_{F,+}^\times$, $\beta \in \widehat{\mathcal{O}}_F$, since

$$W_\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}\right) = W_\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\omega(\alpha),$$

one has

$$c(\alpha y_f)\psi(\beta y_f) = c(y_f).$$

It follows that $c(y_f) \neq 0$ only if $y_f D_F$ is integral, and that $c(y_f)$ only depends on the ideal $y_f D_F$. In other words, there is a function $a(m)$ on the ideals m of $\mathcal{O}_F$ which vanishes on nonintegral ideals and such that

$$c(y_f) = a(y_f D_F). \qquad \square$$

3.1.3. *Hecke operators.* Now a holomorphic form is uniquely determined by $a(m)$. We call $a(m)$ *the* m*-th coefficient* of ϕ and denote it as $a_\phi(m)$ when ϕ is referred.

Now let m be a nonzero ideal of $\mathcal{O}_F$. We want to define *the Hecke operator* $\mathrm{T}(m)$ on the space of cusp forms. Let $H(m)$ denote the following subset of $\mathrm{GL}_2(\widehat{F})$:

$$H(m)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{M}_2(\widehat{\mathcal{O}}_F):\quad (d,N)=1, c\in\widehat{N}, (ad-bc)\widehat{\mathcal{O}}_F=\widehat{m}\right\}.$$

We define $\mathrm{T}(m)$ by the formula:

$$(\mathrm{T}(m)\phi)(g)=\mathrm{N}(m)^{k/2-1}\int_{H(m)}\phi(gh)dh$$

where dh is a Haar measure on $\mathrm{GL}_2(\widehat{F})$ such that $K_0(N)$ has volume 1.

Proposition 3.1.4. *The Fourier coefficients of* $\mathrm{T}(m)\phi$ *are given by the following formula*:

$$a_{\mathrm{T}(m)\phi}(\ell)=\sum_{a|m+\ell}\mathrm{N}(a)^{k-1}a_\phi(m\ell/a^2).$$

Proof. We need only prove the corresponding statement for W_ϕ. The set $H(m)$ is stable under right multiplication by $K_0(N)$ and has disjoint decomposition:

$$H(m)=\coprod_{a,b,d}\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}K_0(N)$$

where (a,d) are representatives in the class

$$\widehat{\mathcal{O}}_F\cap(\widehat{F})^\times/\widehat{\mathcal{O}}_F^\times$$

such that ad generates m, and for fixed (a,d), b are representatives in $\widehat{\mathcal{O}}_F/a\widehat{\mathcal{O}}_F$. Now we have

$$\begin{aligned}\mathrm{T}(m)W_\phi\left(\begin{pmatrix} y_f & 0 \\ 0 & 1\end{pmatrix}\right) &= \mathrm{N}(m)^{k/2-1}\sum_{a,b,d}W_\phi\left(\begin{pmatrix} y_fa/d & y_fb/d \\ 0 & 1\end{pmatrix}\right)\\ &= \mathrm{N}(m)^{k/2-1}\sum_{a,b,d}W_\phi\left(\begin{pmatrix} y_fa/d & 0 \\ 0 & 1\end{pmatrix}\right)\psi(y_fb/d)\\ &= \mathrm{N}(m)^{k/2-1}\sum_{a}W_\phi\left(\begin{pmatrix} y_fa/d & 0 \\ 0 & 1\end{pmatrix}\right)\sum_{b,d}\psi(y_fb/d).\end{aligned}$$

For fixed a,d, if $a_\phi(\alpha y_f a/dD_F)\neq 0$ then $\alpha y_f a/dD_F$ is an integral ideal. In this case $b\to\psi(\alpha y_f b/d)$ is a character on $\widehat{\mathcal{O}}_F/a\widehat{\mathcal{O}}_F$. So the last sum over b is $|a|^{-1}$ if this character is trivial; otherwise it is 0. Notice that this character is trivial if and only if $\alpha y_f d^{-1}D_F$ is an integral ideal. In terms of Fourier coefficients, we obtain

$$a_{\mathrm{T}(m)\phi}(\alpha y_f D_F)=\mathrm{N}(m)^{k/2-1}\sum_{\substack{a,d\\ d|\alpha y_f D_F}}|a/d|^{k/2}|a|^{-1}a_\phi(\alpha y_f a/dD_F).$$

For any given nonzero ideal ℓ of $\mathcal{O}_F$, we always can find α, y such that $\alpha y_f D_F = \ell$. So the above formula gives

$$a_{\mathrm{T}(m)\phi}(\ell) = \mathrm{N}(m)^{k/2-1} \sum_{\substack{a,d \\ d|\ell}} |a/d|^{k/2} |a|^{-1} a_\phi(\ell a/d).$$

Let $a = d\mathcal{O}_F$; then $\ell a/d = m\ell/a^2$, and $|a|^{-1} = \mathrm{N}(m/a)$ and $|d|^{-1} = \mathrm{N}(a)$. The above formula, therefore, gives the proposition. □

Setting $\ell = 1$ in the formula, we obtain

$$a_{\mathrm{T}(m)\phi}(1) = a_\phi(m).$$

COROLLARY 3.1.5. *If ϕ is a nonzero eigenform for all $\mathrm{T}(m)$ then $a_\phi(1) \neq 0$ and*

$$a_\phi(1)\mathrm{T}(m)\phi = a_\phi(m)\phi.$$

3.1.6. *Newforms and multiplicity one.* The Hecke operators are generated by $\mathrm{T}(\wp)$ with prime $\wp$ and satisfy the formal identity

$$\sum \frac{\mathrm{T}(m)}{m^s} = \sum_{\wp|N} (1 - \mathrm{T}(\wp)\wp^{-s})^{-1} \prod_{\wp\nmid N} (1 - \mathrm{T}(\wp)\wp^{-s} + m^{1-2s})^{-1}.$$

It follows that if two eigenforms ϕ_1 and ϕ_2 have the same eigenvalues under all $\mathrm{T}(\wp)$, then ϕ_1 and ϕ_2 are proportional. This will not be true if we only consider Hecke operators $\mathrm{T}(m)$ with m prime to some given ideal m'. Let N' be a factor of N, let $d \in \mathrm{GL}_2(\widehat{F})$ be such that

$$d^{-1}K_0(N)d \subset K_0(N'),$$

and let ϕ' be a form for $K_0(N')$. Then the function $\phi'_d(g) = \phi'(gd)$ is a form for $K_0(N)$. The subspace of $S_k(K_0(N))$ generated by these ϕ'_d with $N' \neq N$ is called the space of *old forms*.

We say a form ϕ for $K_0(N)$ is *new*, if it is perpendicular to the space of old forms. The space $S_k^{\mathrm{new}}(K_0(N))$ of new forms is generated by *newforms*: eigenforms for $\mathrm{T}(m)$ $((m, N) = 1)$ whose first coefficients are 1. Then we have the strong multiplicity one theorem:

THEOREM 3.1.7. *Let ϕ_i, $(i = 1, 2)$, be two newforms of weight k of levels N_1, N_2 respectively, such that $a_{\phi_1}(\wp) = a_{\phi_2}(\wp)$ for all but finitely many $\wp$. Then $N_1 = N_2$ and $\phi_1 = \phi_2$.*

Proof. See [2, Th. 1.4.4 and 3.3.6] and [5]. □

In particular if ϕ is a newform of level N then $w_N(\phi) = \pm\phi$ since $w_N(\phi)$ is also a newform and shares the same eigenvalues as ϕ, where

$$w_N(\phi)(g) = \phi\left(g\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\right)$$

with t a generator of $\widehat{N}$.

One application of this is the rank of the Hecke algebra. Let $\mathbb{T} = \mathbb{T}_k(K_0(N))$ denote the subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(K_0(N))$ generated by $\mathrm{T}(m)$ with $(m, N) = 1$. Then $\mathbb{T}$ acts faithfully on

$$S_N := \oplus_{N'|N} S_k^{\mathrm{new}}(K_0(N))$$

and there is a nondegenerate bilinear form

$$S_N \otimes_{\mathbb{C}} \mathbb{T} \xrightarrow{(\cdot,\cdot)} \mathbb{C}$$

such that

$$(\phi, \mathrm{T}(m)) = a_{\mathrm{T}(m)\phi}(1) = a_\phi(m).$$

Now, we have:

Corollary 3.1.8. *For any linear map $\alpha : \mathbb{T} \to \mathbb{C}$, there is a unique form ϕ such that*

$$a_\phi(m) = \alpha(\mathrm{T}(m))$$

whenever $(m, N) = 1$.

3.2. *Newforms on X.* As in the modular curve case, one may define the notion of modular form on the curve X defined in the introduction:

$$X = B_+\backslash\mathcal{H} \times \widehat{B}^\times/\widehat{F}^\times\widehat{R}^\times \cup \{\text{cusps}\}.$$

Here we are only interested in forms of weight 2, which are functions f on $\mathcal{H} \times \widehat{B}^\times$ such that $f(z)dz$ gives a differential form on X. For m prime to N, we define the action by the Hecke operator $\mathrm{T}(m)$ by the following formula:

$$\mathrm{T}(m)\alpha = \sum_{\gamma \in G_m/G_1} \gamma^*\alpha,$$

where $\alpha \in \Gamma(X, \Omega^1)$, and G_m and G_1 are defined as in Section 1.4. Let $\mathbb{T}'$ denote the subalgebra of $\mathrm{End}(\Gamma(X, \Omega^1_X))$ generated by images of $\mathrm{T}(m)$ $((m, N) = 1)$. For every newform ϕ of level dividing N, let α_ϕ be a character of $\mathbb{T}$ defined by ϕ as in Corollary 3.1.8.

The following theorem translates newforms for $K_0(N)$ into newforms for $R^\times$:

Theorem 3.2.1. 1. *The algebra $\mathbb{T}'$ is a quotient algebra of the Hecke algebra $\mathbb{T}$ defined in* 3.1.7.

2. *If f is a newform of weight 2 for $K_0(N)$ with trivial character, then the eigen subspace of $\Gamma(X, \Omega_X^1)$ of $\mathbb{T}$ with character α_f has dimension* 1.

Proof. Indeed, as in modular curve case, one can show that T' is diagonalizable and every character $\alpha : \mathbb{T}' \to \mathbb{C}$ of $\mathbb{T}'$ corresponds to an irreducible automorphic representation of $(B \otimes \mathbb{A})^\times$. By Jacquet-Langlands theory [24], this representation corresponds to a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Thus there is a character $\beta : \mathbb{T} \to \mathbb{C}$ such that $\alpha(\mathrm{T}(m)) = \beta(\mathrm{T}(m))$. So $\mathbb{T}'$ is a quotient of $\mathbb{T}$. This proves the first part.

For the second part, let π be the cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to f. Then for each place $\wp$ with $\mathrm{ord}_\wp(N)$ odd, the local component $\pi_\wp$ of π is special or supercuspidal. (Otherwise $\pi_\wp$ is principal with trivial central character.) So $\pi_\wp = \pi(\mu, \mu^{-1})$ and the conductor of $\pi_\wp$ is the square of the conductor of μ. This implies that $\mathrm{ord}_\wp(N)$ is even. (See [10, p. 73] for a discussion of conductors.) By Jacquet-Langlands' theory [24], π corresponds to a unique admissible representation π' of $B^\times(\mathbb{A}_F)$. Let V' be the space of the representation of π'. Then the proposition is equivalent to the following: *The space of invariant vectors under $\widehat{R}^\times$ has dimension* 1. This is a local problem. In other words, we may check the above problem for each finite place $\wp$. Thus the proof is reduced to the following theorem. ☐

THEOREM 3.2.2. *Let F be a nonarchimedean local field, B a quaternion algebra over F, E an unramified quadratic extension of F embedded in B. Let $\mathcal{O}_B$ be a maximal order of B containing $\mathcal{O}_E$. Let (ι, V) be an admissible representation of $B^\times$ with trivial central characters. Assume that the conductor of ι is $2n$ if B is split, and $2n+1$ if B is not split. Then the subspace of V of vectors invariant under the action by $\Gamma = (\mathcal{O}_E + \wp^n \mathcal{O}_B)^\times$ is one dimensional, where $\wp$ is a uniformizer of F.*

Proof. Case 1. *E is split.* The theorem in this case is a special case of a result of Casselman [5]. Indeed, in this case we may assume that $\mathcal{O}_B$ is the matrix algebra $\mathrm{M}_2(\mathcal{O}_F)$ and $\mathcal{O}_E$ is the algebra of diagonal matrices. Let $w = \begin{pmatrix} 0 & 1 \\ \wp & 0 \end{pmatrix}$. Then $w^n \Gamma w^{-n} = \Gamma_0(\wp^{2n})$. In the following we assume that $\mathcal{O}_E$ is not split and $n > 0$.

Case 2. *B is split, and ι is a principal series with conductor $\wp^{2n}$.* Then $\iota = \pi(\mu, \mu^{-1})$ with μ a quasicharacter of conductor $\wp^n$, and $\mu^2(x) \neq |x|^{\pm 1}$. Recall that $\pi(\mu, \mu^{-1})$ acts by right translation on the space $\mathcal{B}(\mu, \mu^{-1})$ of locally constant functions f on $\mathrm{GL}_2(F)$ such that

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \mu(a/b)|a/b|^{1/2} f(g).$$

The restriction on $\mathrm{GL}_2(\mathcal{O}_F)$ gives an isomorphism from $\mathcal{B}(\mu,\mu^{-1})$ to the space of functions f on $\mathrm{GL}_2(\mathcal{O}_F)$ such that

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \mu(a/b)f(g).$$

The subspace of invariant vectors f for Γ are functions f on $\mathrm{GL}_2(\mathcal{O}_F)$ such that

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \mu(a/b)f(g)$$

for all $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in Γ.

Since the embedding of $\mathcal{O}_E$ into $\mathrm{M}_2(\mathcal{O}_F)$ is unique up to conjugation, the assertion of the theorem does not depend on the choice of the embedding. Now write $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\varepsilon$ with $\varepsilon^2 \in \mathcal{O}_F^\times \backslash (\mathcal{O}_F^\times)^2$. Define an action of $\mathrm{M}_2(\mathcal{O}_F)$ on $\mathcal{O}_E$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x+y\varepsilon) = (dx+cy) + (bx+ay)\varepsilon.$$

Then action of $\mathcal{O}_E$ on $\mathcal{O}_E$ given by multiplication induces an embedding α from $\mathcal{O}_E$ into $\mathrm{M}_2(\mathcal{O}_F)$. Let $g \in \mathrm{GL}_2(\mathcal{O}_F) = \mathrm{Aut}_{\mathcal{O}_F}(\mathcal{O}_E)$. Let $s = \varepsilon \cdot g^{-1}(\varepsilon)^{-1}$ and $g' = g \cdot \alpha(s)^{-1}$. Then g' will fix ε, so it has the form

$$g' = \beta(a,x) := \begin{pmatrix} 1 & x \\ 0 & a \end{pmatrix}.$$

The decomposition

$$g = \beta(a,x)\alpha(s)$$

of this form is obviously unique. The element g is in Γ if and only if

$$\mathrm{ord}(a-1) \geq n.$$

Now it is easy to see that the space of functions invariant under Γ is the one dimensional space generated by

$$f_0(\beta(a,x)\alpha(s)) = \mu(a)^{-1}. \tag{3.2.1}$$

Case 3. *B is split, and ι is a special representation.* Now ι is the quotient representation of $\mathcal{B}(\mu|\cdot|^{-1/2}, \mu|\cdot|^{1/2})$ with $\mu^2 = 1$, modulo the one dimensional representation $\mu\circ\det(g)$. The restriction of this one dimensional representation on Γ has the form

$$(\mu\cdot\det)(\beta(a,x)\alpha(s)) = \mu(\mathrm{N}_{E/F}s)).$$

Since ι has a conductor of even order, so μ has a conductor of positive order. As $\mathrm{N}_{E/F}\mathcal{O}_E^\times = \mathcal{O}_F^\times$, this one-dimensional representation $\mu\cdot\det$ is nontrivial on Γ. It follows that the image of f_0 defined in (3.2.1) on the space of ι gives a nonzero generator of the space of invariant vectors for Γ.

Case 4. *B is nonsplit or B is split but ι is supercuspidal.* The proof was shown to me by H. Jacquet and will be given in the next subsection. □

3.3. *The supercuspidal case.* We prove the theorem in the supersingular case in two steps. First we prove that $V^{\mathcal{O}_E^\times}$ is one dimensional and then we show that this space is also invariant under Γ.

PROPOSITION 3.3.1. *The subspace $V^{\mathcal{O}_E^\times}$ of $\mathcal{O}_E^\times$-invariants in V has dimension* 1.

Proof. Let π be a representation of $\mathrm{GL}_2(\mathcal{O}_F)$ such that $\pi=\iota$ if B is split, and π is the Jacquet-Langlands correspondence of ι if B is nonsplit. Let m be the conductor of π, so that $m=2n$ if B is split and $m=2n+1$ if B is not split.

Since ι has trivial central character and $\mathcal{O}_E/\mathcal{O}_F$ is unramified, $\mathcal{O}_E^\times$ invariants are simply $E^\times$ invariants. According to Waldspurger, ([41, Th. 2]), V has a nonzero vector invariant under $E^\times$ if and only if

$$\varepsilon\left(\frac{1}{2},\pi\otimes\varepsilon_E\right)=\varepsilon\left(\frac{1}{2},\pi\right)$$

if B is split, and

$$\varepsilon\left(\frac{1}{2},\pi\otimes\varepsilon_E\right)=-\varepsilon\left(\frac{1}{2},\pi\right)$$

if B is nonsplit, where ε_E is the quadratic character of $F^\times$ attached to the extension E/F. Moreover by Proposition 1 in [41], the space of $E^\times$-invariants has dimension 1 if these conditions are verified.

Now the proposition follows from the identity:

$$\varepsilon\left(\frac{1}{2},\pi\otimes\varepsilon_E\right)=(-1)^m\varepsilon\left(\frac{1}{2},\pi\right).$$

Let ψ be a nontrivial character of F and let W be a vector in the Whittaker model $\mathcal{W}(\pi,\psi)$. As the L-function of π is 1, one has the functional equation:

$$\varepsilon(s,\pi,\psi)\int W\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right]|a|^{s-1/2}d^\times a=\int \widetilde{W}\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right]|a|^{1/2-s}d^\times a$$

where

$$\widetilde{W}(g)=W\left[\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}{}^tg^{-1}\right].$$

Now assume that W is the essential vector. This means that

$$W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{cases} 1 & \text{if } |a| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that

$$\varepsilon(s, \pi, \psi) = \int \widetilde{W}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] |a|^{1/2-s} d^\times a.$$

Recall that

$$\varepsilon(s, \pi, \psi) = q^{(1/2-s)m} \varepsilon\left(\frac{1}{2}, \pi\right)$$

where q is the cardinality of the residue field of F. Thus

$$\widetilde{W}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] \neq 0$$

implies $|a| = q^m$. Consequently,

$$\varepsilon\left(\frac{1}{2}, \pi\right) = \int_{|a|=q^m} \widetilde{W}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] d^\times a.$$

Replacing π by $\pi \otimes \varepsilon_E$ and $W(g)$ by $W(g)\varepsilon_E(\det g)$, we obtain the required equality,

$$\varepsilon\left(\frac{1}{2}, \pi \otimes \varepsilon_E\right) = (-1)^m \varepsilon\left(\frac{1}{2}, \pi\right). \qquad \square$$

Let $v \in V$ be a nonzero vector invariant under $E^\times$. Let ϖ be a square root of $\wp$ in $\mathcal{O}_B$. Then v is invariant under $K_r = 1 + \varpi^r \mathcal{O}_B$ for some sufficiently large r.

Proposition 3.3.2. *With the notation and assumption as in Proposition* 3.3.1, *the smallest r such that v is invariant under K_r is $r = m$ if B is split, and $r = m - 1$ if B is not split.*

Proof. We will prove the case that B is not split. The case that B is split is similar. Let $f(g) = (\pi(g)v, v)$ be the coefficient function attached to v. Let Φ be the characteristic function of K_r. Its Fourier transform is (apart from a positive factor) the function $\psi(-\mathrm{tr}(g))\Psi$, where Ψ is the the characteristic function of the set $\varpi^{1+r}\mathcal{O}_B$. The Godement-Jacquet equation [13] reads, apart from a nonzero constant factor,

$$\varepsilon(s, \pi, \psi) = \int f(g^{-1})\Psi(g)\psi(-\mathrm{tr}(g))|\det g|^{1/2-s} d^\times g.$$

Since $\varepsilon(s, \pi, \psi) = q^{m(1/2-s)}\varepsilon(1/2, \pi)$, we see that the integral does not change if we restrict the domain of the integral to the set $\mathcal{O}_B^\times \varpi^{-m}$. Thus Ψ must be nonzero on this set, which implies that $r \geq m-1$. Moreover the nonvanishing of the above integral implies that for at least one $g \in \mathcal{O}_B^\times \varpi^{-m}$ the following integral is nonzero:

$$\int_{K_{m-1}} f(k^{-1}g^{-1})\psi(-\mathrm{tr} gk)dk.$$

Since $\psi(-\mathrm{tr} gk)$ does not depend on k,

$$\int_{K_{m-1}} f(k^{-1}g^{-1})dk \neq 0.$$

This implies that

$$v' = \int_{K_{m-1}} \pi(k)v dk \neq 0.$$

As K_m is a normal subgroup of $\mathcal{O}_B^\times$ and v is invariant under $\mathcal{O}_E^\times$, v' is invariant under the action of $\mathcal{O}_E^\times$. So v is a multiple of v' by the previous proposition and v is invariant under K_{m-1}. □

3.4. *L-functions associated to newforms.*

3.4.1. *Definitions.* Let ϕ be a newform for $K_0(N)$ of weight 2 with trivial central character. Let $a_\phi(m)$ be the Fourier coefficients of ϕ. Then the L-function for ϕ is defined to be

$$\mathrm{L}(s,\phi) = \sum_{m\in\mathbb{N}_F} \frac{a_\phi(m)}{\mathrm{N}(m)^s} = \prod_{\wp|N} \frac{1}{1-a_\phi(\wp)\mathrm{N}(\wp)^{-s}} \prod_{\wp\nmid N} \frac{1}{1-a_\phi(\wp)\mathrm{N}(\wp)^{1-2s}}$$

which is absolutely convergent for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. Recall that

$$w_N(\phi)(g) := \phi\left(g\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\right) = \gamma\phi$$

with $\gamma = \pm 1$, where t is an element of $\mathbb{A}_F^\times$ such that

- at archimedean places, t has component -1;
- at finite place, t generates $\widehat{N}$.

Proposition 3.4.2. *The function $L(s,\phi)$ is holomorphic in s and satisfies a functional equation*:

$$\begin{aligned} L^*(s,\phi) : &= d_N^{s/2} d_F^s \left[\frac{\Gamma(s)}{(2\pi)^s}\right]^g L(s,\phi) \\ &= \gamma L^*(s,\phi). \end{aligned}$$

Proof. Let $d^\times x$ be a Haar measure on $\mathbb{A}_F^\times$ which is a product of local Haar measures $dx_v^\times$ on $F_v^\times$ such that $d^\times x_v = dx/x$ if v is archimedean, and such that the volume of $\mathcal{O}_v^\times$ equals 1 if v is nonarchimedean. Let $\Lambda(s,\phi)$ denote the function

$$\Lambda(s,\phi) = \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|^{s-1/2} d^\times y$$

where F_+^* (resp. $\mathbb{A}_{F,+}$) denotes the subgroup of $F^\times$ (resp. $\mathbb{A}_F^\times$) of elements which are totally positive at archimedean places. Then $\Lambda(s,\phi)$ is absolutely convergent for all $s\in\mathbb{C}$ and defines an entire function on $\mathbb{C}$. Using the Fourier expansion of ϕ and Proposition 3.1.2, we have

$$\begin{aligned}\Lambda(s,\phi) &= \int_{\widehat{F}^\times} a_\phi(yD_F)|y|^{s+1/2}d^\times y \cdot \int_{F_\infty^\times} |y|^{s+1/2}\psi(iy_\infty)d^\times y \\ &= d_F^{s+1/2}\mathrm{L}(s+1/2,\chi,\phi)\cdot\left(\frac{\Gamma(s+1/2)}{(2\pi)^{s+1/2}}\right)^g.\end{aligned}$$

Thus we need only prove the corresponding functional equation for $\Lambda(s,\phi)$. By definition of $w_N(\phi)$,

$$\begin{aligned}\phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= \gamma\phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\right) \\ &= \gamma\phi\left(\frac{-1}{y}\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\right) \\ &= \gamma\phi\left(\begin{pmatrix} -t/y & 0 \\ 0 & 1 \end{pmatrix}\right).\end{aligned}$$

Bringing this to our definition of $\Lambda(s,\chi,\phi)$, we obtain

$$\begin{aligned}\Lambda(s,\phi) &= \gamma\int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} -t/y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|^{s-1/2} d^\times y \\ &= \gamma\cdot\mathrm{N}(N)^{1/2-s}\cdot\Lambda(1-s,\phi).\end{aligned}$$

□

3.4.3. *Remarks.* Let ε be the character associated to the imaginary quadratic extension E/F. Let $L(s,\varepsilon,\phi)$ be the twisted L-series:

$$L(s,\varepsilon,f) = \sum_m \frac{\varepsilon(m)a_\phi(m)}{\mathrm{N}(m)^s}.$$

Then this series is essentially the L-series associated to a new form in the space of the representation $\pi\otimes\varepsilon$ if π is the representation associated to ϕ. Thus it has a functional equation.

The base change of $L(s,\phi)$ is defined to be the product:

$$L_E(s,\phi) := L(s,\phi)L(s,\varepsilon,\phi).$$

In Section 6, using the Rankin-Selberg convolution method, we will prove that $L_E(s,\phi)$ has a functional equation with sign $\varepsilon(N)(-1)^g$.

3.4.4. *Proof of Theorem* B. Let J_X denote the Jacobian variety of X. Let $\mathcal{T}$ be the $\mathbb{Z}$-subalgebra in $\mathrm{End}_{\mathbb{Z}}(J_X)$ generated by $\mathrm{T}(m)$ $((m,N)=1)$. Then $\mathcal{T}\otimes\mathbb{C}=\mathbb{T}'$. For every newform ϕ of level dividing N, let α_ϕ be a character of $\mathbb{T}$ defined by ϕ as in Corollary 3.1.8, let $\mathcal{O}_\phi$ be the subalgebra of $\mathbb{C}$ generated by Fourier coefficients $a_\phi(m)$ $((m,N)=1)$, and let J_ϕ be the maximal abelian subvariety of J killed by $\ker(\alpha_\phi)$. We say two forms ϕ_1 and ϕ_2 are *conjugate* if $\ker(\alpha_{\phi_1})=\ker(\alpha_{\phi_2})$, or equivalently, there is an automorphism σ of $\mathbb{C}$ such that $a_{\phi_1}^\sigma(m)=a_{\phi_2}^2$ for all m prime to N.

LEMMA 3.4.5. 1. *J_X is isogenous to $\oplus_{[\phi]}J_\phi$ where $[\phi]$ runs through the conjugacy classes of newforms ϕ in S_N.*

2. *If J_ϕ is nonzero, then $\mathcal{O}_\phi$ is totally real with finite rank over $\mathbb{Z}$.*

3. *If ϕ is a newform of level N, then* $\mathrm{Lie}(J_\phi)$ *is a free module of rank* 1 *over* $\mathcal{O}_\phi\otimes\mathbb{C}$.

Proof. Parts (1) and (3) are reformulations of parts (1) and (2) of Theorem 3.2.1. As $\mathcal{T}$ acts faithfully on $H^1(J,\mathbb{Z})$, the characteristic polynomial of $\mathrm{T}(m)$ is monic and integral. It follows that the $a(m)$ are algebraic integers, and that the subalgebra $\mathcal{O}_\phi$ generated by $a_\phi(m)$ over $\mathbb{Z}$ has finite rank. Also as $\mathbb{T}'$ is self-adjoint, the characteristic polynomial of $\mathrm{T}(m)$ has only real roots. So $\mathcal{O}_\phi$ is an order in a totally real number field. □

Now fix a newform f of weight 2 for $K_0(N)$ with trivial character. Let A denote J_f. Fix a place $\wp$ not dividing N. Then A has good reduction at $\wp$. Let $\ell\neq\wp$ be a prime. Then $A\otimes k(\wp)$ has the same ℓ-adic Tate module as A, that is $H^1(A,\mathbb{Q}_l)$. The local zeta function of A at $\wp$ is

$$Z_\wp(t)=\det(1-t\mathrm{Frob}(\wp)|H^1(A,\mathbb{Q}_\ell)).$$

As $\mathrm{Frob}(\wp)^*$ has the same characteristic polynomial as $\mathrm{Frob}(\wp)$,

$$\begin{aligned} Z_\wp(t)^2 &= \det\left[(1-t\mathrm{Frob}(\wp))(1-t\mathrm{Frob}(\wp)^*)\right]\\ &= \det(1-t(\mathrm{Frob}(\wp)+\mathrm{Frob}(\wp)^*)+t^2\mathrm{Frob}(\wp)\mathrm{Frob}(\wp)^*). \end{aligned}$$

As $\mathrm{Frob}(\wp)$ has degree $\mathrm{N}(\wp)$, we see that $\mathrm{Frob}(\wp)\mathrm{Frob}(\wp)^*=\mathrm{N}(\wp)$. Now the congruence relation

$$a(\wp)=\mathrm{Frob}(\wp)+\mathrm{Frob}(\wp)^*$$

implies

$$Z_\wp(t)^2=\det(1-a(\wp)t+\mathrm{N}(\wp)t^2).$$

As $\dim H^1(A, \mathbb{Q}) = 2[\mathcal{O}_f : \mathbb{Z}]$,

$$Z_\wp(t) = \mathrm{N}_{\mathcal{O}_f/\mathbb{Z}}(1 - a(\wp)t + \mathrm{N}(\wp)t^2).$$

Thus Theorem B follows, as the L-function of A is defined as

$$L^{(N)}(s, A) = \prod_{\wp \nmid N} Z_\wp(\mathrm{N}(\wp)^{-s})^{-1}.$$

3.5. *Eisenstein series and theta series.*

3.5.1. *Some definitions.* Let k be a positive integer. Let χ be a quadratic character on $\mathbb{A}_F^\times/F^\times$ with a square-free conductor D_χ such that $\chi_v(-1) = (-1)^k$, and that D_χ is prime to D_F. We extend χ to $K_0(D_\chi)$ as in §3.1. For s a complex number, we define a function H_s on $\mathrm{GL}_2(\mathbb{A}_F)$ by

$$H_s(g) = \begin{cases} \left|\frac{a}{d}\right|^s \chi(aur(\theta)) & \text{if } u \in K_0(D_\chi) \\ 0 & \text{otherwise,} \end{cases}$$

where every element $g \in \mathrm{GL}_2(\mathbb{A}_F)$ has the form

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ur(\theta)$$

with $ur(\theta) \in K_0(1)K^\infty$, the standard maximal subgroup of $\mathrm{GL}_2(\mathbb{A}_F)$. Let B denote the Borel subgroup (the group of upper triangular matrices), then $H_s(g)$ is left invariant under $B(F)$.

For $\mathrm{Re}(s) > 1$, the Eisenstein series

$$E_s(g) = L(2s, \chi) \sum_{\gamma \in B(F)\backslash \mathrm{GL}_2(F)} H_s(\gamma g)$$

is absolutely convergent and defines a (nonholomorphic and noncuspidal) form for $K_0(D_\chi)$ of (parallel) weight k, and character χ.

PROPOSITION 3.5.2. 1. *The constant term of E_s at $\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ is given by the following formula*:

$$C_{E_s}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} L(2s, \chi)\chi(y)|y|^s & \text{if } \chi \neq 1 \\ \zeta_F(2s)|y|^s + d_F^{-1/2}\zeta_F(2s-1)V_s(0)^g|y|^{1-s} & \text{if } \chi = 1. \end{cases}$$

2. *The Whittaker function at $\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ of E_s is 0 if yD_F is not integral*; *otherwise it is given by the following formula*:

$$W_{E_s}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \frac{1}{\sqrt{d_F d_\chi}}\sigma_s(y)|y|^{1-s} \cdot \prod_{v|D_\chi} |y_v \pi_v|^{2s-1}\varepsilon(y_v)\kappa(v)$$

with

$$\sigma_s(y) = \prod_{\substack{v \nmid D_\chi \\ v \nmid \infty}} \frac{1 - \chi(y_v \delta_v \pi_v)|y_v \delta_v \pi_v|^{2s-1}}{1 - \chi(\pi_v)|\pi_v|^{2s-1}} \cdot \prod_{v|\infty} V_s(y_v),$$

where

- π_v *is a uniformizer of* F_v *such that* $\varepsilon(\pi_v) = 1$ *if* $\pi_v \mid D_\chi$.
- $\kappa(v)$ *is a square root of* $(-1)^k$ *defined by*

$$\kappa(v) = |\pi_v|^{1/2} \sum_{a \in (\mathcal{O}_v/\pi_v)^\times} \chi(a/\pi_v)\psi_v(-a/\pi_v).$$

- $\delta \in \widehat{F}^\times$ *is a generator of* D_F.
- $$V_s(y) = \int_{-\infty}^{\infty} \frac{e^{2\pi i y_v x}}{(x^2+1)^{s-k/2}(x+i)^k} dx.$$

Proof. For $\alpha = 0$ or 1, let

$$c_s(\alpha, y) = d_F^{-1/2} \int_{\mathbb{A}_F/F} E_s\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \psi(-\alpha x) dx.$$

Then

$$W_{E_s}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = c_s(1, y), \qquad C_{E_s}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = c_s(0, y).$$

The group $\mathrm{GL}_2(F)$ has the Bruhat decomposition

$$\mathrm{GL}_2(F) = B(F) \coprod \coprod_{u \in F} B(F) w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

where

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned} c_s(\alpha, y) \quad &= \quad L(2s, \chi) d_F^{-1/2} \int_{\mathbb{A}_F/F} H_s\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \psi(-\alpha x) dx \\ &\quad + L(2s, \chi) d_F^{-1/2} \int_{\mathbb{A}_F} H_s\left(w \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \psi(-\alpha x) dx. \end{aligned}$$

By definition the first term is equal to

$$L(2s, \chi) d_F^{-1/2} \int_{\mathbb{A}_F/F} \chi(y)|y|^s \psi(-\alpha x) dx$$

which is

$$L(2s,\chi)\chi(y)|y|^s$$

if $\alpha=0$; otherwise it is zero.

To evaluate the second integral, we notice that

$$w\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & xy^{-1} \end{pmatrix}.$$

Replacing x by xy, we see that the second integral becomes

$$\int_{\mathbb{A}_F} H_s\left(w\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\psi(-\alpha x)dx = |y|^{1-s}\prod_v V_s(\alpha_v y_v)$$

where for $y\in F_v$,

$$V_s(y) = \int_{F_v} H_s\left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\right)\psi(-xy)dx.$$

The case where v is archimedean. If v is archimedean, we have the decomposition

$$\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{x^2+1}} & \frac{-x}{\sqrt{x^2+1}} \\ 0 & \sqrt{x^2+1} \end{pmatrix}\begin{pmatrix} \frac{x}{\sqrt{x^2+1}} & \frac{-1}{\sqrt{x^2+1}} \\ \frac{1}{\sqrt{x^2+1}} & \frac{x}{\sqrt{x^2+1}} \end{pmatrix}.$$

It follows that

$$\begin{aligned} V_s(y) &= \int_{\mathbb{R}} \frac{1}{(x^2+1)^s}\left(\frac{x-i}{\sqrt{x^2+1}}\right)^k e^{-2\pi i yx}dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-2\pi i yx}}{(x^2+1)^{s-k/2}(x+i)^k}dx. \end{aligned} \tag{3.5.1}$$

The case where v is nonarchimedean. If v is nonarchimedean, then

$$\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \in K_v$$

if $x\in\mathcal{O}_v$; otherwise we have the decomposition

$$\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Thus,

$$H_v\left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\right) = \begin{cases} \chi_v(x)|x|^{-2s} & \text{if } x\notin\mathcal{O}_v; \\ 1 & \text{if } x\in\mathcal{O}_v,\ v\nmid D_\chi \\ 0 & \text{if } x\in\mathcal{O}_v,\ v\mid D_\chi. \end{cases}$$

It follows that

$$\begin{aligned} V_s(y) &= \sum_{n\geq 1}\int_{\mathcal{O}_v^\times}\chi(x\pi_v^{-n})|x\pi_v^{-n}|^{-2s}\psi(-xy\pi^{-n})d(\pi^{-n}x) \\ &\quad + \begin{cases} \int_{\mathcal{O}_v}\psi(-yx)dx & \text{if } v\nmid D_\chi, \\ 0 & \text{if } v\mid D_\chi \end{cases} \\ &= \sum_{n\geq 1}\chi(\pi_v)^n|\pi_v|^{2ns-n}\int_{\mathcal{O}_v^\times}\chi(x)\psi(-xy\pi^{-n})dx \\ &\quad + \begin{cases} 1 & \text{if } v\nmid D_\chi,\ y\in {D_F}^{-1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The case where $v \mid D_\chi$. If v divides D_χ, then $\int_{\mathcal{O}_v^\times}\chi(x)\psi(-xy\pi^{-n})dx$ is nonzero only if $y\neq 0$ and $\mathrm{ord}_v(y)=n-1$. In this case it equals $\chi(y\pi_v^n)\kappa(v)|\pi_v|^{1/2}$. So if v divides D_χ, we obtain the following formula for $V_s(y)$:

$$V_s(y)=\begin{cases} |y|^{2s-1}\chi(y)\kappa(v)|\pi_v|^{2s-1/2} & \text{if } y\neq 0 \text{ and } \mathrm{ord}_v(y)\geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.2}$$

Consequently, if χ is nontrivial, $V_s(0)=0$ and the 0-th Fourier coefficient of $E_s(g)$ is

$$C_s(y)=L(2s,\chi)\chi(y)|y|^s.$$

The case where $v\nmid D_\chi$. In this case

$$\int_{\mathcal{O}_v^\times}\psi(-xy\pi^{-n})dx=\begin{cases} 1-|\pi_v| & \text{if } \mathrm{ord}_v(yD_F)\geq n; \\ -|\pi_v| & \text{if } \mathrm{ord}_v(yD_F)=n-1; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $V_s(y)$ is nonzero only if $\mathrm{ord}_v(yD_F)\geq 0$ and in this case

$$\begin{aligned} V_s(y) &= \sum_{1\leq n\leq \mathrm{ord}_v(yD_F)}\chi(\pi_v)^n|\pi_v|^{2ns-n}(1-|\pi_v|) \\ &\quad + 1-(\chi(\pi_v)|\pi_v|^{2s-1})^{\mathrm{ord}_v(yD_F)+1}|\pi_v| \\ &= (1-\chi_v(\pi_v)|\pi_v|^{2s})\sum_{n=0}^{\mathrm{ord}_v(y_vD_{F\,v})}\chi_v(\pi_v)^n|\pi_v|^{2ns-n}. \end{aligned} \tag{3.5.3}$$

□

COROLLARY 3.5.3. *If* $(F,k,\chi)\neq(\mathbb{Q},2,1)$ *then there is a unique holomorphic form* $E_{\chi,k}$ *of weight* k *and central character* χ *for* $K_0(D_\chi)$ *such that the* m^{th} *Fourier coefficient of* $E_{\chi,k}$ *is given by*

$$\sigma_{\chi,k-1}(m)=\sum_{n\mid m}\chi(n)\mathrm{N}(n)^{k-1}.$$

Proof. The function $V_s(y)$ can be analytically extended to a function for all $\mathrm{Re}(s) > 0$ and has exponential decay with respect to y. When $s = k/2$,

$$V_{k/2}(y) = \begin{cases} (-2\pi i)^k y^{k-1} e^{-2\pi y} & \text{if } y > 0 \\ 0 & \text{if } y < 0. \end{cases}$$

Now, $E_s(g)$ can be analytically continued to a form for $\mathrm{Re}(s) > 0$ and $E_{k/2}(g)$ is a holomorphic form whose m^{th} Fourier coefficients are given by

$$\begin{cases} L(k,\chi) & \text{if } m = 0; \\ A_{\chi,k}\sigma_{\chi,k-1}(m) & \text{if } m \neq 0 \end{cases}$$

where

$$A_{\chi,k} = \frac{(-2\pi i)^{kg}}{(d_F d_\chi)^{k-1/2}} \chi(D_F) \prod_{v|D_\chi} \kappa(v). \qquad \square$$

3.5.4. *Remarks.* 1. When $k = 1$ and χ is the character attached to an imaginary quadratic extension E/F, the form $E_{\chi,k}$ is called the theta series associated to the extension E/F and is denoted as $\theta_{E/F}$ or simply θ; thus

$$E_{1/2} = A_{\varepsilon,1}\theta.$$

Notice that in this case $\sigma_{\chi,k-1}(m)$ is the number of integral ideals in E with norm m', where m' is the maximal factor of m prime to D_E. We denote this number simply by $r(m)$.

2. When $(F, k, \chi) = (\mathbb{Q}, 2, 1)$, $E_1(g)$ is holomorphic except for the constant term.

4. Global intersections

In this section we will study the Néron-Tate height pairing $\langle z, \mathrm{T}(m)z\rangle$ of the Heegner points and the CM-points. More precisely, we will first show that $\langle z, \mathrm{T}(m)z\rangle$ is the coefficient of a modular form Ψ, and then express the heights as the arithmetic intersections using the arithmetic Hodge index theorem [8]. Finally we decompose this number as a sum of local intersections. Compared with the case $F = \mathbb{Q}$, there are two major difficulties: one is the absence of cusps which were used to map the modular curves to their Jacobians; another is the absence of the Dedekind η-function which was used to compute the self-intersection. Therefore we can only obtain an expression of $\langle z, \mathrm{T}(m)z\rangle$ as a sum of the local intersections of CM-points which meet properly at special fibers, modulo some multiple of the coefficients in the Dirichlet series $\zeta_E(s)$ and $\zeta_F(s)\zeta_F(s-1)$. At the end of this section, we will use the multiplicity-one theorem to show that the modular form Ψ actually is uniquely determined by our expression. This section contains most of the new ideas of this paper.

4.1. *Height pairing.*

4.1.1. *Height pairings as Fourier coefficients.* In the introduction, we defined a Shimura curve X and a Heegner point z in the Jacobian $J(E)\otimes\mathbb{Q}$ of X. In Section 1.4 we defined the Hecke operator $\mathrm{T}(m)$ as a correspondence for m prime to N. As in the modular curve case we want to show that

$$\langle z, \mathrm{T}_m z\rangle, \qquad m\in\mathbb{N}_F, \quad (m,N)=1$$

are Fourier coefficients of a holomorphic cusp form for $K_0(N)$, where $\langle\cdot,\cdot\rangle$ is the Néron-Tate height pairing on $J(\bar{F})\otimes\mathbb{Q}$. Actually this is a general fact:

LEMMA 4.1.2. *Let S_N denote the sum of $S_2^{\mathrm{new}}(K_0(N'))$ for all $N'|N$. For any $x\in\mathrm{Jac}(X)(\bar{F})$, there is a unique element f_x in S_N such that $\langle x, \mathrm{T}(m)x\rangle$ is the m^{th} coefficient in the Fourier expansion of f at ∞ for all $m\in\mathbb{N}_F$ prime to N.*

Proof. Now $\mathbb{T}'$ also acts on $J(\bar{F})\otimes\mathbb{C}$. So $\mathrm{T}\to\langle x,\mathrm{T}x\rangle$ gives a linear function on $\mathbb{T}'$ and, therefore, on $\mathbb{T}$. Now the conclusion follows from Corollary 3.1.8 and Lemma 3.4.5. □

4.1.3. *Height pairings as intersection pairings.* Let Ψ denote the form f_z defined in the lemma. The purpose of this section is to show that Ψ is determined by the local arithmetic intersections of some CM-divisors.

We have constructed an integral model $\mathcal{X}$ for X over $\mathcal{O}_F$. However this model is not fine enough for the computation of intersection numbers. Instead of X we will consider $\widetilde{X}$ which is the Shimura curve corresponding to a smaller group $\widetilde{K}$ such that the corresponding curve has a regular model. For example, we may take $\widetilde{K}:=(1+N_E\widehat{\mathcal{O}}_{B,\wp})^\times\cap U$ where U is an open compact subgroup of $G(\mathbb{A}_f)$ which is maximal at places dividing ND_E. When U is sufficiently small, $\widetilde{X}$ has a regular model over $\widetilde{\mathcal{X}}$ over $\mathcal{O}_F$. As U is maximal at places dividing D_E, $\widetilde{\mathcal{X}}\times\mathrm{Spec}\mathcal{O}_E$ is also regular. Let $\pi:\widetilde{X}_E\to X_E$ be the projection induced by the inclusion $\widetilde{K}\to K$, and let $\widetilde{z}$ be the pullback of z on $\widetilde{X}_E$. Then $\widetilde{z}$ has degree 0 on each irreducible component of $\widetilde{X}_E$. The projection formula for heights gives

$$\langle z, \mathrm{T}(m)z\rangle = \langle\widetilde{z}, \mathrm{T}(m)\widetilde{z}\rangle/\deg\pi.$$

Here the pairing on the right-hand side is the Néron-Tate pairing on the Jacobian of $\widetilde{X}\otimes E$, which by definition is the product of Jacobians of irreducible components.

We may write $\langle\widetilde{z},\mathrm{T}(m)\widetilde{z}\rangle$ as an intersection of arithmetic divisors on $\widetilde{\mathcal{X}}\otimes\mathcal{O}_E$ ([9], [11], [12]). More precisely, let $\widehat{z}$ be the arithmetic divisor on $\widetilde{\mathcal{X}}\otimes\mathcal{O}_E$ which has curvature 0 on the Riemann surface $\widetilde{X}(\mathbb{C})$ and has zero degree on each irreducible component C of the special fibers of $\widetilde{\mathcal{X}} \otimes \mathcal{O}_E$; then the Hodge index theorem gives

$$\langle \widetilde{z}, \mathrm{T}(m)\widetilde{z}\rangle = -(\widehat{z}, \mathrm{T}(m)\widehat{z}).$$

Here the right-hand side is the arithmetic intersection.

4.1.4. *A formula for $\widehat{z}$.* We write a formula for $\widehat{z}$ and let η be the divisor

$$\eta = u^{-1}\sum_x [x]$$

where $u = [\mathcal{O}_E^\times : \mathcal{O}_F^\times]$ and x runs through the set of positively oriented Heegner points on X. Let $\widetilde{\eta}$ be the pull-back of η on $\widetilde{X} \otimes E$. Let $\bar{\eta}$ denote the Zariski closure of $\widetilde{\eta}$ on $\widetilde{\mathcal{X}} \otimes \mathcal{O}_E$. For each infinite place τ of F, $X_\tau(\mathbb{C})$ is a Riemann surface compactified from a quotient of $\mathcal{H}$. Let $d\mu$ be a volume form on $\widetilde{X}_E(\mathbb{C})$ such that on each irreducible component X_i of $\widetilde{X}_E(\mathbb{C})$, $d\mu$ has volume 1, and the pull-back of $d\mu$ on $\mathcal{H}$ is proportional to the Poincaré metric $dxdy/y^2$ for $x+yi \in \mathcal{H}$. Let g denote Green's function on $X(\mathbb{C})$ with respect to the Poincaré volume form $d\mu$:

$$\frac{\partial\bar{\partial}}{\pi i} g = \delta_{\eta_i} - \deg(\eta_i)d\mu, \quad \text{where} \quad \eta_i = \widetilde{\eta}|_{X_i}.$$

Let $\widehat{\eta}$ denote the arithmetic divisor $(\bar{\eta}, g)$.

Let ξ be the class in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ which has component ξ_i on each geometrically connected component X_i defined as in the introduction. Then z is the class of $\eta - h\xi$ where h is a number such that z has degree 0 on each irreducible component of X. Let $\widetilde{\xi}$ be the pull-back of ξ on $\widetilde{X}_E$. Then $\widetilde{\xi}$ is the class of the bundle $\Omega^1_{\widetilde{X}}[\text{cusps}]$ divided by its degree. We will find an extension of $\widetilde{\xi}$ to an arithmetic class $\widehat{\xi}$ whose curvature is a multiple of $d\mu$ on each component X_i. We need only do this locally at each place v of $\mathcal{O}_F$.

Choose F' as before. Let $\widetilde{X}'$ be the Shimura curve defined over F' associated to the open compact subgroup $K' = \widetilde{K} \cdot J$ of $\widehat{B}'^\times$, where J is an open compact subgroup of $\widehat{\mathcal{O}}_{F'}^\times$ which is maximal at places dividing N. Choose U and J sufficiently small so that $\widetilde{\mathcal{F}} := \mathcal{F}_{K'}$ is representable. Let $\mathcal{A}$ be the universal abelian variety over $\widetilde{X}'$ and let $\mathcal{L}_{F'}$ denote $\det(\mathrm{Lie}\mathcal{A})^\vee$. Then by the Kodaira-Spencer map, $\mathcal{L}_{F'}$ equals the canonical bundle $\Omega^1[\text{cusps}]$ on $\widetilde{X}'$.

If v is an infinite place τ of F, then $\widetilde{X}_\tau$ can be embedded into $\widetilde{X}'_\tau$. The bundle $\mathcal{L}_v$ has a Peterson-Weil metric $\|\cdot\|$: for a point $x \in X_\tau(\mathbb{C})$ representing an abelian variety A, and for an element $\alpha \in \mathcal{L}_v = \Gamma(A, \Omega_A^{4g})$,

$$\|\alpha\|^2 = (-i)^{g^2} \int_{A(\mathbb{C})} \alpha \wedge \bar{\alpha}.$$

So we obtain a metric on ξ; this is nothing else but the standard hyperbolic metric up to a constant multiple.

If v is a finite place $\wp$, we assume that F' is split at $\wp$ and that J is maximal at places dividing $\wp$. We assume that $\mathcal{F}_{K',\wp}$ is representable by a regular scheme $\widetilde{\mathcal{X}}'$ over $\mathcal{O}_\wp$. Then we can define a bundle $\mathcal{L}_v$ on $\widetilde{\mathcal{X}}'$ the same way. Let $\mathcal{O}_\wp^{\mathrm{ur}}$ be the completion of the maximal unramified extension of $\mathcal{O}_\wp$; then $\widetilde{\mathcal{X}}_{\mathcal{O}_\wp^{\mathrm{ur}}}$ can be embedded into $\widetilde{\mathcal{X}}'_{\mathcal{O}_\wp^{\mathrm{ur}}}$. Now the restriction of $\mathcal{L}_v$ on $\widetilde{\mathcal{X}}_{\mathcal{O}_\wp^{\mathrm{ur}}}$ defines an extension of $\Omega^1[\text{cusps}]$. If $\wp$ does not divide N, then this integral structure is the same as that induced by Ω^1 on $\widetilde{\mathcal{X}}$ at v.

Let $\mathcal{L}$ be the extension of $\Omega^1[\text{cusps}]$ on $\widetilde{\mathcal{X}}$ such that $\mathcal{L}_\wp = \mathcal{L}\otimes\mathcal{O}_\wp^{\mathrm{ur}}$ for every $\wp$. Let $\widehat{\xi}$ be the arithmetic divisor class of the hermitian line bundle $(\mathcal{L}, \|\cdot\|)$ dividing by its degree. Then $\widehat{\eta} - h\widehat{\xi}$ has curvature zero.

For simplicity of notation and computation, we will assume that E/F is not unramified. In this case η will have the same degree on each geometrically connected component of X, and so will $\widetilde{\eta}$. Now we can write

$$\widehat{z} := \widehat{\eta} - h\widehat{\xi} + Z$$

where h is a number such that $\widehat{z}$ has degree 0 on each geometrically connected component of the generic fiber, and Z is a vertical divisor of $\widetilde{\mathcal{X}} \otimes \mathcal{O}_E$ such that $\widehat{z}$ has the degree 0 on any irreducible component of the special fibers of $\widetilde{\mathcal{X}} \otimes \mathcal{O}_E$. In the following subsections we will compute $\mathrm{T}(m)\eta$, $\mathrm{T}(m)\widehat{\xi}$, and $\mathrm{T}(m)Z$ respectively.

4.2. *Computing* $\mathrm{T}(m)\eta$.

PROPOSITION 4.2.1. *For c prime to N, let*

$$\eta_c = u_c^{-1}\sum_x x,$$

where u_c is the cardinality of $\mathcal{O}_c^\times/\mathcal{O}_F^\times$, and where the sum runs through the set of positively oriented CM-*points of conductor c. Then for m prime to N,*

$$\mathrm{T}(m)\eta_1 = \sum_{\substack{c\in\mathbb{N}_F\\ c|m}} r(m/c)\eta_c$$

where $r(m)$ denotes the number of integral ideals in $\mathcal{O}_E$ with norm m.

Proof. The map $(\sqrt{-1}, g) \to g$ identifies the set of CM-points with the set

$$E^\times\backslash\widehat{B}^\times/\widehat{R}^\times.$$

For any $\mathcal{O}_F$-module M, write $M^\flat$ for $M\otimes\mathcal{O}^\flat$, where $\mathcal{O}^\flat$ is the product $\prod_{\wp\nmid N}\mathcal{O}_\wp$. Also write $E^\sharp$ for the group of elements in $E^\times$ which is a unit at $\wp$ for any place $\wp$ dividing N. Then the set of positively oriented CM-points is identified with

$$E^\times\backslash\prod_{\wp|N} E_\wp^\times R_\wp^\times\cdot B^{\flat,\times}/\widehat{R}^\times = E^\sharp\backslash B^{\flat,\times}/R^{\flat,\times}.$$

As B is unramified off N, there is an isomorphism $\mathcal{O}_B^\flat \simeq \mathrm{End}_{\mathcal{O}_F}(\mathcal{O}_E^\flat)$ of the left $\mathcal{O}_E^\flat$-algebras. Now the correspondence $g \to g\mathcal{O}_E^\flat$ gives a bijection between the set of positively oriented CM-points and the set of classes of $\mathcal{O}^\flat$-lattices in $E^\flat$:

$$E^\sharp \backslash \{\mathcal{O}^\flat - \text{ lattices in } E^\flat\}$$

where $E^\sharp$ acts on the lattices by left multiplication. It is not difficult to show that if a CM-point has order $\mathcal{O}_c$ then the corresponding lattice class has the form $g\mathcal{O}_c^\flat$ with g an element in $E^\flat$. This shows that the set of CM-points of conductor c is bijective to

$$E^\sharp \backslash E^{\flat,\times} / \mathcal{O}_c^{\flat,\times}.$$

More precisely, let S_c denote a subset of $E^\flat$ representing the above set; then η_c has the expression

$$\eta_c = u_c^{-1} \sum_{\gamma \in S_c} [\sqrt{-1}, \gamma]$$

where S_c is considered as a subset of $\widehat{B}^\times$ by setting components 1 at places dividing N.

The action of $\mathrm{T}(m)$ on CM-points can be described as follows. If x is represented by a lattice L in $E^\flat$ then $\mathrm{T}(m)x$ is the sum of classes of all sublattices M of norm m (this means that the product of elementary factors of $\mathcal{O}_F$-module L/M is m).

Let $[g\mathcal{O}_c^\flat]$ be a lattice class with $g \in E^{\flat,\times}$. Then the multiplicity of $[g\mathcal{O}_c^\flat]$ in $\mathrm{T}(m)\eta_1$ is equal to u_1^{-1} times the number of pairs

$$(\gamma, k) \in S_1 \times E^\sharp / \mathcal{O}_c^\times$$

such that $kg\mathcal{O}_c^b$ is a sublattice of $\gamma\mathcal{O}_E^b$ of norm m, or equivalently

$$\gamma^{-1} g k \in \widehat{\mathcal{O}}_E^b, \qquad \mathrm{N}(\gamma^{-1} g k) = m/c.$$

Now the surjective map

$$S_1 \times E^\sharp / \mathcal{O}_c^\times \to (E^\flat)^\times / \mathcal{O}_E^{\flat,\times}, \quad (g, k) \to \gamma^{-1} g k \pmod{\mathcal{O}_E^{\flat,\times}}$$

is $[\mathcal{O}_E^\times : \mathcal{O}_c^\times]$ to 1. Thus the multiplicity is equal to

$$u_1^{-1} [\mathcal{O}_E^\times : \mathcal{O}_c^\times] \# \{\gamma \in \widehat{\mathcal{O}}_E^\flat / \mathcal{O}_E^{\flat,\times} : \quad \mathrm{N}(\gamma) = m/c\} = u_c^{-1} r(m/c). \qquad \square$$

Let η_c^0 denote the sum of η_a for all $a | c$ and $a \neq \mathcal{O}_F$, and define

$$\mathrm{T}^0(m)\eta = \sum_{c|m} \varepsilon(c) \eta_{m/c}^0. \tag{4.2.1}$$

Then $\mathrm{T}^0(m)\eta$ is disjoint to η. As $r(m) = \sum_{n|m} \varepsilon(n)$, we obtain:

Corollary 4.2.2. *If m is prime to ND_E, then*

$$\mathrm{T}(m)\eta = \mathrm{T}^0(m)\eta + r(m)\eta.$$

4.3. *Computing* $\mathrm{T}(m)\widehat{\xi}$.

4.3.1. *Some definitions.* Let $\pi : U \to V$ be a finite flat morphism of integral schemes. Let $\mathbf{Pic}(U)$, $\mathbf{Pic}(V)$ be categories of line bundles on U and V respectively. Then we can define the pull-back functor $\pi^* : \mathbf{Pic}(V) \to \mathbf{Pic}(U)$ as usual, and the norm functor $\mathrm{N}_\pi : \mathbf{Pic}(U) \to \mathbf{Pic}(V)$ as follows. If L is a line bundle on U then $\mathrm{N}_\pi(L)$ is a line bundle on V which is locally generated by $\mathrm{N}_\pi(\ell)$ with ℓ a section of $\mathcal{L}$ such that

$$\mathrm{N}_\pi(f\ell) = \mathrm{Norm}(f)\mathrm{N}_\pi(\ell)$$

where Norm is the norm map $f_*\mathcal{O}_U \to \mathcal{O}_V$ for the algebra extension $\mathcal{O}_V \to f_*\mathcal{O}_U$. It follows from the definition that if $L = \mathcal{O}_U(D)$ for a divisor D on U, then $N_\pi(L)$ is canonically isomorphic to $\mathcal{O}_V(\pi_* D)$.

If W is an integral subscheme of $U \times V$ such that the projection from W to U is finite and flat then we can define a functor $W : \mathbf{Pic}(V) \to \mathbf{Pic}(U)$ as $W(L) = \mathrm{N}_{\pi_U}\pi_V^*(L)$ where π_U, π_V are projections from W to U and V respectively. We may extend this definition linearly to any correspondence W of $U \times V$ such that W has all irreducible components finite and flat over U.

It is easy to see that at the generic fiber

$$\mathrm{T}(m)\xi = \sigma_1(m)\xi.$$

The following proposition gives the corresponding formula for $\mathrm{T}(m)\widehat{\xi}$.

Proposition 4.3.2. *There is a morphism*

$$\psi_m : \mathrm{T}(m)\mathcal{L} \to \mathcal{L}^{\sigma_1(m)}$$

such that the following conditions are verified:

1. *Let $c \in \mathbb{N}_F$ be such that*

   $$\psi_m(\mathrm{T}(m)\mathcal{L}) = c\mathcal{L}^{\sigma_1(m)}.$$

   Then for each finite place $\wp$,

   $$\mathrm{ord}_\wp(c) = 2\sigma_1(m\wp^{-\mathrm{ord}_\wp(m)})\sum_{i=0}^{n} i\mathrm{N}(\wp^{n-i}).$$

2. *Let ψ be the following function on $\widetilde{X}(\mathbb{C})$*

   $$\|\psi\|_m(x) := \frac{\|\psi_m\beta\|}{\|\beta\|},$$

   where β is a nonzero element in $\mathrm{T}(m)(\mathcal{L})(x)$. Then

   $$\|\psi\|_m(x) = \mathrm{N}(m)^{2\sigma_1(m)}.$$

Proof. We need only prove the corresponding statement on $\widetilde{\mathcal{X}}'$. For this we extend $\mathrm{T}(m)$ to $\widetilde{\mathcal{X}}'$ by the formula (1.4.1). By Proposition 1.4.2, we have the following modular interpretation for $\mathrm{T}(m)$: For any object $[A, C]$ of $\widetilde{\mathcal{F}}(S)$,

$$\mathrm{T}(m)[A, C] = \sum_{D} [A_D, C_D]$$

where D runs through the set of admissible submodules of A of order m, $A = A/D$, $C_D = C + D/D$. Let $\mathcal{X}_m$ be the subscheme of $\mathcal{X}' \times \mathcal{X}'$ which represents the isogenies $A_1 \to A_2$ with admissible kernel of order m; then $\mathrm{T}(m)$ is induced by $\mathcal{X}_m$.

Let $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ be the universal isogeny over $\mathcal{X}_m$, and let p_1, p_2 be the projection of $\mathcal{X}_m$ to $\mathcal{X}'$. Then $p_i^*\mathcal{L} = \det \mathrm{Lie}(\mathcal{A}_i)^\vee$. The morphism $\pi^* : \mathrm{Lie}(\mathcal{A}_2) \to \mathrm{Lie}(\mathcal{A}_1)$ therefore induces a morphism of line bundles on $\mathcal{X}'$:

$$\mathrm{N}_{p_1}(p_2^*\mathcal{L}) \to \mathrm{N}_{p_1}(p_1^*\mathcal{L}).$$

Notice that by definition $\mathrm{T}(m)\mathcal{L} = \mathrm{N}_{p_1} p_2^*(\mathcal{L})$, and $\mathrm{N}_{p_1} p_1^*\mathcal{L} = \mathcal{L}^{\sigma_1(m)}$. Therefore, we obtain a morphism of line bundles:

$$\psi_m : \mathrm{T}(m)\mathcal{L} \to \mathcal{L}^{\sigma_1(m)}.$$

To prove (1), we need only check the proposition locally at each finite place $\wp$ prime to N. Write $m = m'\wp^n$ with $(m', \wp) = 1$. Then ψ_m is factored as a composition of $\psi_{m'}$ and $\psi_{\wp^n}$:

$$\mathrm{T}(m)(\mathcal{L}) = \mathrm{T}(\wp^n)\mathrm{T}(m')\mathcal{L} \xrightarrow{\mathrm{T}(\wp^n)\psi_{m'}} \mathrm{T}(\wp^n)\mathcal{L}^{\sigma_1(m')} \xrightarrow{\psi_{\wp^n}^{\otimes \sigma_1(m')}} \mathcal{L}^{\sigma_1(m)}.$$

As $\mathrm{T}(m')$ is étale at $\wp$, it follows that if $\psi_{\wp^n}$ has order t at $\wp$, then ψ_m has order $\sigma_1(m')t$.

Let $x : \mathrm{Spec} W \to \mathcal{X}_\wp$ be a strictly henselian point represented by an abelian variety A with ordinary reduction. Then

$$\mathrm{T}(\wp^n)(\mathcal{L}_x) = \otimes_D \det \mathrm{Lie}(A/D)^\vee$$

where D runs through the set of admissible submodules of A of order m. Fix an isomorphism $\mathcal{O}_{B,\wp} \simeq \mathrm{M}_2(\mathcal{O}_\wp)$. Let G denote the $\mathcal{O}_\wp$-module $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A[\wp^\infty]^2$ of dimension 1; then $\det \mathrm{Lie}(A) = \mathrm{Lie}(G)^{\otimes 2}$. It follows that

$$\mathrm{T}(\wp^n)\mathcal{L} = \prod_H (\mathrm{Lie} G/H)^{\otimes -2}$$

where H runs through the set of submodules of G of order $\wp^n$, and that the morphism $\psi : \mathrm{T}(\wp^n)\mathcal{L} \to \mathcal{L}^{\sigma_1(\wp^n)}$ is induced by morphisms $\pi^* : \mathrm{Lie}(G/H)^\vee \to \mathrm{Lie}(G)^\vee$. Let

$$0 \to H' \to H \to H'' \to 0$$

be the formal-étale decomposition. Then

$$\mathrm{Lie}(G)^\vee \Big/ \pi^*\mathrm{Lie}(G/H)^\vee \simeq 0^*(\Omega_H) \simeq 0^*(\Omega_{H'})$$

where 0 is the 0-section of G.

Now G has a decomposition $G = \Sigma_1 \oplus F_\wp/\mathcal{O}_\wp$ where Σ_1 is a formal $\mathcal{O}_\wp$-module of height 1. It follows that H has the form $H = \Sigma_1[\wp^i] \otimes G_{n-i,\lambda}$ where $0 \le i \le n$, $\lambda \in \wp^{i-n}\mathcal{O}_\wp/\mathcal{O}_\wp$, and $G_{n-i,\lambda}$ is the subgroup with the generic fiber $\{(\lambda x, x) : x \in \wp^{i-n}\mathcal{O}_\wp/\mathcal{O}_\wp\}$. Thus

$$0^*(\Omega_H) \simeq 0^*(\Omega_{\Sigma_1[\wp^i]}) \simeq \mathrm{Lie}(\Sigma_1)^\vee/\wp^i\mathrm{Lie}(\Sigma_1)^\vee \simeq \mathcal{O}_\wp/\wp^i.$$

It follows that the quotient of ψ has the order

$$\sum_{i=0}^{n} 2i\mathrm{N}(\wp^{n-i}).$$

It remains to prove (1). Recall that $\mathrm{T}(m)\mathcal{L}(x)$ is equal to

$$\otimes_D \det \mathrm{Lie}(A/D)^\vee$$

where D runs through the set of admissible submodules of order m. As ψ is induced by the maps

$$\pi_D^* : \Omega^1_{A/D} \to \Omega^1_A,$$

the norm of ψ is the product of the norms of

$$\det \pi_D^* : \det \Gamma(\Omega^1_{A/D}) \to \det \Gamma(\Omega^1_A)$$

which is $(\deg \pi_D)^{1/2} = \mathrm{N}(m)^2$. Then for any infinite place τ,

$$\|\psi\|_\tau(x) = \mathrm{N}(m)^{2\sigma_1(m)}. \qquad \square$$

Corollary 4.3.3. *Let ϕ be a function on the set of elements of $\mathbb{N}_F$ prime to* N *with values in the group of arithmetic divisors on $\mathcal{O}_F$ defined by the formula*

$$\mathrm{T}(m)\widehat{\xi} = \sigma_1(m)(\widehat{\xi} + \phi(m)).$$

Then ϕ is quasi-additive: for any m' and m'' such that $(m', m'') = 1$ then

$$\phi(m'm'') = \phi(m') + \phi(m'').$$

Proof. We decompose $\phi(m) = \sum \phi(m)_v[v]$ where v runs through all places of F. Then by the proposition,

$$\phi(m)_v = \begin{cases} c\sigma(\wp^{\mathrm{ord}_\wp(m)})^{-1} \sum_{i=0}^n i\mathrm{N}(\wp^{n-i}) & \text{if } v = \wp \text{ is finite,} \\ c \log \mathrm{N}(m) & \text{if } v \text{ is infinite} \end{cases}$$

where c is some fixed constant. Thus ϕ is additive for coprime m's. $\square$

4.4. *Computing* $\mathrm{T}(m)Z$.

4.4.1. *Decompositions.* For each finite place $\wp$ of F, let $V_\wp$ denote the group of $\mathbb{Q}$-divisors of $\widetilde{\mathcal{X}}$ supported in the fiber over $\wp$ modulo the subgroup of $\mathbb{Q}$-divisors of connected components. Then we have the decomposition

$$Z = \sum_\wp Z_\wp$$

where $Z_\wp$ are elements in $V_\wp$. We want to study $\mathrm{T}(m)Z_\wp$ for m prime to ND_E. If we choose different models $\widetilde{\mathcal{X}}$, then the decomposition is preserved by the pull-back maps. So we assume that $\widetilde{\mathcal{X}}$ has the same level structure as $\mathcal{X}$ at the place $\wp$.

PROPOSITION 4.4.2. *Assume that $\wp$ is split in B. Then*

$$\mathrm{T}(m)Z_\wp = \sigma_1(m)Z_\wp.$$

Proof. By definition $\mathrm{T}(m)Z$ is a unique solution to the equations

$$(\mathrm{T}(m)\hat{\eta} - h\mathrm{T}(m)\widehat{\xi} + \mathrm{T}(m)Z, P) = 0$$

for any irreducible vertical divisor P on $\widetilde{\mathcal{X}} \otimes \mathcal{O}_E$. As $\widetilde{\mathcal{X}} \otimes E$ is smooth at the places not dividing N, we need only check that the differences

$$Z_1 = \mathrm{T}(m)\hat{\eta} - \sigma_1(m)\hat{\eta} \quad \text{and} \quad Z_2 = \mathrm{T}(m)\widehat{\xi} - \sigma_1(m)\widehat{\xi}$$

both have degree 0 on each irreducible component of $\widetilde{\mathcal{X}}$ over $\wp$ dividing N. For Z_2 this follows from Proposition 4.3.2. It remains to study Z_1.

Case 1. *$\wp$ does not divide N.* In this case, each geometrically connected component of $\widetilde{\mathcal{X}}_\wp$ has only one irreducible component. Thus $Z_\wp = 0$.

Case 2. *$\wp$ split in E.* Let $\widetilde{K_0}$ denote the level structure obtained by replacing the level structure K_p by the maximal one $\mathcal{O}_{B,\wp}^\times$. Let $\widetilde{\mathcal{X}}_0$ denote the corresponding Shimura curve. Then the natural map $\widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}_0$ induces a bijection on the set of connected components. Over $\widetilde{\mathcal{X}}_0$ we have the divisible $\mathcal{O}_\wp$-module $\mathcal{G}^1$ of height 2 and $\widetilde{\mathcal{X}}$ classifies the "cyclic" submodules C of $\mathcal{G}^1$ of order $\wp^{\mathrm{ord}_\wp(N)}$. For a fixed irreducible component D of the special fiber of $\widetilde{\mathcal{X}}_0$ over $\wp$, by Proposition 1.3.2, the set of irreducible components of $\widetilde{\mathcal{X}}$ over D is indexed by the types of the subgroups over the ordinary points over D. By Proposition 2.2.3, all divisors η_c will have ordinary reduction at $\wp$ and the corresponding subgroups are of the same type: either all étale or all formal. It follows that all CM-divisors η_c with positive orientation will reduce to the same irreducible component of $\widetilde{\mathcal{X}}_\wp$ over D. This implies that Z_1 has degree 0 on each irreducible component of $\widetilde{\mathcal{X}}_\wp$.

Case 3. *$\wp$ is split in B and inert in E.* We claim that each connected component of $\widetilde{\mathcal{X}}$ over $\wp$ has only one irreducible component. With the notation as Section 1.3.1 and Proposition 1.3.2, the set of irreducible components of $\widetilde{\mathcal{X}}_\wp$ over D is indexed by $\mathbb{P}^1(F_\wp)/K^\wp$ where $K^\wp = F_\wp^\times R_\wp^\times$. As R contains $\mathcal{O}_{E,\wp}$, and $F_\wp^\times \mathcal{O}_{E,\wp}^\times = E_\wp^\times$, it suffices to show that $\mathbb{P}^1(F_\wp)$ has only one orbit under the action of $E_\wp^\times$ for any embedding $E_\wp \to \mathrm{M}_2(F_\wp)$. Up to a conjugation, we may identify $\mathbb{P}^1(F_\wp)$ as the set of surjective $F_\wp$-homomorphisms, from $E_\wp$ to $F_\wp$ and the action of $E_\wp^\times$ is given by multiplication on $E_\wp$. It follows that $\mathbb{P}^1(F_\wp)$ has one element $\mathrm{tr} : E_\wp \to F_\wp$. As the pairing

$$E_\wp \times E_\wp \to F_\wp, \qquad (x, y) \to \mathrm{tr}(xy)$$

is nondegenerate, any other surjective $F_\wp$-homomorphism $\phi : E_\wp \to F_\wp$ will have the form

$$\phi(x) = \mathrm{tr}(ax)$$

where a is a nonzero element of $E_\wp$. In other words, $\phi = a(\mathrm{tr})$, or the action of $E_\wp$ on $\mathbb{P}^1(F_\wp)$ is transitive. Consequently, each connected component of $\widetilde{\mathcal{X}}_\wp$ has only one irreducible component. As in case 1, we have $Z_\wp = 0$. $\square$

It remains to consider the case where $\wp$ is not split in B. The conclusion of the previous proposition will definitely not be true. But we have the following:

PROPOSITION 4.4.3. *Assume that $\wp$ is not split in B. Then for any element $D \in V_\wp$, there is a holomorphic $V_\wp$-valued cusp form f of weight 2 and level $K_0(N^\wp)$ such that for all but finitely many m, the m-coefficient of f is given by $\mathrm{T}(m)D$, where $N^\wp$ denotes $N\wp^{-\mathrm{ord}_\wp(N)}$.*

Proof. Actually by Proposition 1.3.4, the set of irreducible components of $\widetilde{\mathcal{X}}$ is identified with

$$S_{\widetilde{K}_0} = B(\wp)^\times \backslash \widehat{B(\wp)}^\times / \widehat{F}^\times \mathrm{GL}_2(\mathcal{O}_\wp)\widetilde{K}^\wp.$$

The group $V_\wp$ is therefore identified with a subgroup of the space $\widetilde{V}$ of complex functions on $S_{\widetilde{K}_0}$. By Jacquet-Langlands theory [24], the action of the Hecke correspondences is factored through the action of the Hecke algebra of holomorphic cusp forms of weight 2 and level $\widehat{F}^\times \mathrm{GL}_2(\mathcal{O}_\wp)\widetilde{K}^\wp$, so the proposition is true with the level structure $K_0(N)$ replaced by $K_0(N^\wp)_N \widetilde{K}^N$.

Using the pull-back of divisors, we notice that the minimal level of the forms which have $\mathrm{T}(m)D$ as Fourier coefficients does exist and does not depend on the choice of $\widetilde{K}$. Thus this minimal level must be $K_0(N^\wp)$. $\square$

4.4.4. *Some definitions.* Let $\mathcal{S}$ denote the vector space of complex-valued functions on $\mathbb{N}_F$ modulo an equivalence relation so that two functions a and b are equivalent if and only if there is some element M such that $a(\ell) = b(\ell)$ for any ℓ prime to M. The strong multiplicity theorem 3.1.7 implies that the map

$$f \longrightarrow \widetilde{f} : n \to a_f(n)$$

is an embedding from S_N into $\mathcal{S}$. We say a function h in $\mathcal{S}$ is *quasi-multiplicative* if there is an $M \in \mathbb{N}_F$ such that

$$f(mn) = f(m)f(n)$$

for all $m, n \in \mathbb{N}_F$ such that

$$(m, n) = (mn, M) = 1.$$

For a quasi-multiplicative function f, a function h is called *an f-derivative* if

$$h(mn) = f(m)h(n) + f(n)h(m)$$

for all (m, n) as above.

Let σ_1 and r denote the elements in $\mathcal{S}$ defined by: $m \to \sigma_1(m)$ and $m \to r(m)$ respectively, and let $\mathcal{D}_N$ be the subspace of $\mathcal{S}$ generated by σ_1, r, σ_1-derivatives, and r-derivatives, and the Fourier coefficients corresponding to the old cusp forms of weight 2. Then we have:

Proposition 4.4.5. *Let $\widehat{\Psi}$ denote the image of Ψ in $\mathcal{S}$. Then in $\mathcal{S}$,*

$$\widehat{\Psi}(m) = -\left(\widehat{\eta}, \mathrm{T}^0(m)\widehat{\eta}\right) / \deg \pi \pmod{\mathcal{D}_N}.$$

Proof. By discussions in 4.1.3 and 4.1.4, for m prime to ND_E,

$$\widehat{\Psi}(m) = -\left(\widehat{\eta} - h\widehat{\xi} + Z, \mathrm{T}(m)(\widehat{\eta} - h\widehat{\xi} + Z)\right) / \deg \pi.$$

Now we have shown:

- $\mathrm{T}(m)Z_\wp = \sigma_1(m)Z_\wp$ if $\wp$ is split in B, and $m \to \mathrm{T}(m)Z_\wp$ is given by an old cusp form of weight 2 if $\wp$ is not split in B;
- $\mathrm{T}(m)\widehat{\xi} = \sigma_1(m)(\xi + \psi(m))$ with ψ quasi-additive;
- $\mathrm{T}(m)\widehat{\eta} = r(m)\widehat{\eta} + \mathrm{T}^0(m)\widehat{\eta}$.

It follows that

$$\widehat{\Psi}(m) = -\left(\widehat{\eta}, \mathrm{T}^0(m)\widehat{\eta}\right) / \deg \pi \pmod{\mathcal{D}_N}. \qquad \square$$

4.5. *A uniqueness theorem.* Now we are going to prove that the relation in Proposition 4.4.5 determines a newform projection of Ψ uniquely:

PROPOSITION 4.5.1. *Let f be an element in the space S_N such that in $\mathcal{S}$,*

$$\widehat{f} \equiv 0 \pmod{\mathcal{D}_N}.$$

Then f is an old cusp form of weight 2.

Proof. We start from the following:

LEMMA 4.5.2. *Let $\alpha_1, \cdots, \alpha_\ell$ be distinct nonzero quasi-multiplicative elements in $\mathcal{S}$. Then the equation*

$$(c_1\alpha_1 + h_1) + \cdots + (c_\ell\alpha_\ell + h_\ell) = 0$$

in $\mathcal{S}$ does not have a nonzero solution

$$x = (c_1, h_1, \cdots, c_\ell, h_\ell),$$

where for each i, c_i is a constant and h_i is an α_i-derivative.

Proof. Assume that the lemma is not true, then we will have one solution $x_0 = (c_1, h_1, \cdots, c_l, h_l)$. Let M be an element in $\mathbb{N}_F$ such that

$$(c_1\alpha_1(n) + h_1(n)) + \cdots + (c_\ell\alpha_\ell(n) + h_\ell(n)) = 0$$

for any n prime to M. Let m be any ideal prime to M; then for any n prime to mM, we have

$$(c_1\alpha_1(mn) + h_1(mn)) + (c_2\alpha_2(mn) + h_2(mn)) + \cdots = 0.$$

So we have a new solution

$$x_1 = (c_1\alpha_1(m) + h_1(m), \alpha_1(m)h_1, \cdots, c_\ell\alpha_\ell(m) + h_\ell(m), \alpha_\ell(m)h_\ell).$$

If $h_1(m) \neq 0$ then we obtain a solution

$$x' = x_1 - \alpha_1(m)x_0 = (h_1(m), 0, \cdots),$$

in which $h_1 = 0$ and $c_1 \neq 0$. Doing this for each i, we obtain a solution in which every $h_i = 0$ but some c_i will not be 0. We need only to show that $\alpha_1, \cdots, \alpha_\ell$ are linearly independent. This is similar to the proof of the linear independence of the characters of a group. □

Now go back to the proof of our proposition. Decompose f into a sum of newforms of levels dividing N and forms of type $\phi\left(g\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right)$, where $d \neq 1$ is a divisor of N in $\widehat{F}^\times$ and ϕ is a newform of level $d^{-1}N$. Then the above lemma implies the proposition if we can show that σ_1 and r are distinct and not in the image of newforms. For any quasi-multiplicative a in $\mathcal{S}$, we define the Dirichlet series

$$L(s, a) = \sum_n a(n)\mathrm{N}(n)^{-s}$$

which is well-defined modulo finitely many factors. Then it is easy to see up to finitely many factors,

$$L(s,r)=\zeta_E(s),\qquad L(s,\sigma_1)=\zeta_F(s-1)\zeta_F(s).$$

So $L(s,r)$ has a pole at $s=1$ and $L(s,\sigma_1)$ has a pole at $s=2$. If r is multiple of σ_1 in $\mathcal{S}$, then $L(s,r)$ should be equal to a multiple of $L(s,\sigma_1)$ up to finitely many Euler factors. This is impossible as they have different poles. The same argument shows that σ_1 and r should not be equal to $\hat{f}$ for any cusp form f, as $L(s,\widehat{f})=L(s,f)$ is holomorphic at $s=2$ and $s=1$. □

4.5.3. *Remarks.* The number $(\widehat{\eta},\mathrm{T}^0(m)\widehat{\eta})/\deg\pi$ does not depend on the choice of $\pi:\widetilde{X}\to X$. Let us denote it by $(\eta,\mathrm{T}^0(m)\eta)$. As two divisors $\widehat{\eta}$ and $\mathrm{T}^0(m)\widehat{\eta}$ are disjoint at the generic fibers, it has decomposition

$$(\eta,\mathrm{T}^0(m)\eta)=\sum_v(\eta,\mathrm{T}^0(m)\eta)_v$$

where v runs through the set of all places of F, and

$$(\eta,\mathrm{T}^0(m)\eta)_v=\sum_{w|v}(\widehat{\eta},\mathrm{T}^0(m)\widehat{\eta})_w/\deg\pi$$

where w runs through all places of E over v.

Assume that m is prime to ND_E, then by (4.2.1), the computation of Ψ modulo oldforms is reduced to the computation of

$$(\eta,\eta_c^0)_v:=(\widehat{\eta},\widehat{\eta}^0)/\deg\pi.$$

In the following section we will first compute the local intersections then add them together.

5. Local intersections

In this section we are going to compute $(\eta,\mathrm{T}(m)^0\eta)_v$ where v is a place of E. We follow the method of Gross- Kohnen-Zagier [22]. However we are working on CM-points with the discriminants not necessarily coprime. Again, we need to assume that every factor of 2 is split in E.

5.1. *Archimedean intersections.* In this subsection we want to compute the infinite local intersections $(\eta,\eta_c^0)_v$ where $c\in\mathbb{N}_F$ is prime to ND_E and $\pi:\widetilde{X}\to X$ is some covering of X constructed as in Section 4.1. First of all let us assume that v is over τ, the embedding chosen in the introduction.

5.1.1. *Intersections as Green's functions.* Let R_i's be nonconjugate orders of B of type (N,E). Then $X(\mathbb{C})$ is a union of Riemann surfaces

$$X_i=R_{i+}^\times\backslash\mathcal{H}=R_i^\times\backslash\mathcal{H}^\pm.$$

We want to compute the intersections $(\eta_i, \eta^0_{c,i})_v$ separately, where η_i and $\eta^0_{c,i}$ are restrictions of η and η^0_c on X_i. Let $g_i(x,y)$ be Green's function on the compactification of X_i with respect to the Poincaré metric $d\mu$:

$$\frac{\partial_x \bar{\partial}_x}{\pi i} g_i(x,y) = \delta_y - d\mu(x).$$

We can linearly extend g_i to a function on the set of disjoint pairs of divisors.

LEMMA 5.1.2. $(\eta_i, \eta^0_{c,i})_v = g_i(\eta_i, \eta^0_{c,i})$.

Proof. Let $\widetilde{X}_i$ be the part of $\widetilde{X}$ which projects into X_i. Then the Riemann surface $\widetilde{X}_i$ is a union of Riemann surfaces of the form: $\widetilde{X}_{i,j} = \bar{\Gamma}_{i,j}\backslash\mathcal{H}$, where $\bar{\Gamma}_{i,j} \subset \mathrm{PGL}_2(\mathbb{R})^+$ acts freely on $\mathcal{H}$. Let $\eta_{i,j}$ and $\eta^0_{c,i,j}$ be the restrictions of $\widetilde{\eta}_i$ and $\widetilde{\eta}^0_{c,i}$ on $\widetilde{X}_{i,j}$; then by construction of $\widehat{\eta}_i$, as in 4.1.4,

$$(\widehat{\eta}_i, \widehat{\eta}^0_{c,i})_v = \sum_j g_{i,j}(\eta_{i,j}, \eta^0_{c,i,j})/\deg \pi,$$

where $g_{i,j}(x,y)$ are Green's functions on the compactifications of $\widetilde{X}_{i,j}$'s with respect to the Poincaré metric. Now the lemma follows easily from the projection formula

$$g_i(x,y) = \sum_j g_{i,j}(\pi^{-1}(x), \pi^{-1}(y))/\deg \pi$$

for any distinct x and y in $X_i(\mathbb{C})$. □

5.1.3. *Construction of Green's functions.* Now $g_i(x,y)$ can be constructed as follows [16]: for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define the function

$$G_s(z,w) = Q_{s-1}\left(1 + \frac{|z-w|^2}{2\mathrm{Im}z\mathrm{Im}w}\right)$$

where $Q_{s-1}(u)$ is the Legendre function of the second kind:

$$Q_{s-1}(u) = \int_0^\infty (u + \sqrt{u^2-1}\cosh t)^s dt.$$

Then the function on X_i,

$$g_{s,i}(z,w) = \sum_{\gamma \in \Gamma_i} G_s(z, \gamma w)$$

is convergent and has a simple pole at $s = 1$ with residue $1/\chi_i$ where Γ_i is the image of $R_i^\times$ in $\mathrm{PSL}_2(\mathbb{R})$ and χ_i is the Euler characteristic of X_i. Then we have an identity

$$g_i(z,w) = \lim_{s \to 1}\left(g_{s,i}(z,w) - \frac{1}{s(s-1)\chi_i}\right).$$

Let x, y be two points on $X_\tau(\mathbb{C})$ represented by z, w on $\mathcal{H}$. Let u_x and u_y be the orders of stabilizers of x and y in Γ_i respectively. Let P_x and P_y be the sets of points on $\mathcal{H}$ mapping to x and y, respectively. Then,

$$g_i(x/u_x, y/u_y) = \lim_{s\to 1}\left[\sum_{(z,w)\in\Gamma_i\backslash P_x\times P_y} g_s(z,w) - \frac{u_x^{-1}u_y^{-1}}{s(s-1)\chi_i}\right].$$

Applying this to components in η_i and $\eta^0_{c,i}$, we see that Lemma 5.1.2 gives

$$(\eta_i, \eta^0_{c,i})_v = \lim_{s\to 1}\left[\sum_{(z,w)\in\Gamma_i\backslash P_i\times P^0_{c,i}} g_s(z,w) - \frac{\deg\eta\deg\eta^0_c}{s(s-1)\chi_i}\right], \tag{5.1.1}$$

where P_i and $P^0_{c,i}$ are the sets of points in $\mathcal{H}$ mapping to components of η_i and $\eta^0_{c,i}$.

5.1.4. *Descriptions of* CM-*points.* We may identify $\mathcal{H}^\pm$ with

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathrm{M}_2(\mathbb{R}))$$

such that if $z = g(\sqrt{-1}) \in \mathcal{H}^\pm$ with $g \in \mathrm{GL}_2(\mathbb{R})$, then the corresponding element $\phi_z : \mathbb{C} \to \mathrm{M}_2(\mathbb{R})$ takes $a+bi$ to $g\begin{pmatrix} a & b \\ -b & a\end{pmatrix}g^{-1}$. In this way, the CM-points on X_i are those points induced by a homomorphism $\phi : K \to B$ with order given by $\phi^{-1}(R_i)$.

For two points z and w in $\mathcal{H}^\pm$ corresponding to two homomorphisms ϕ_z and ϕ_w in $\mathrm{Hom}(\mathbb{C}, \mathrm{M}_2(\mathbb{R}))$ it is easy to check that

$$1 + \frac{|z-w|^2}{2\mathrm{Im}z\mathrm{Im}w} = -\frac{1}{2}\mathrm{tr}(i_z i_w),$$

where $i_z = \phi_z(i)$ and $i_w = \phi_w(i)$. It follows that z and w are in the same connected component and $z \neq w$ if and only if $-\frac{1}{2}\mathrm{tr}(i_z i_w) > 1$. Let $\mathcal{P}_i$ (resp. $\mathcal{P}^0_{c,i}$) denote the inverse image of η_i and $\eta^0_{c,i}$ on $\mathcal{H}^\pm$, and let $\mathcal{P}_{c,i}$ denote the union of $\mathcal{P}_i$ and $\mathcal{P}^0_{c,i}$. Then we have

$$(\eta_i, \eta^0_{c,i})_v = \lim_{s\to 1}\left\{\sum_{\substack{(z,w)\in\mathcal{P}_i\times\mathcal{P}^0_{c,i}/R_i^\times \\ -\frac{1}{2}\mathrm{tr}(i_z i_w)>1}} Q_{s-1}\left(-\frac{1}{2}\mathrm{tr}(i_z i_w)\right) + \frac{\deg\eta_i\deg\eta^0_{c,i}}{s(s-1)\chi}\right\}.$$

For an element $c \in \mathbb{N}_F$, define

$$u_{v,s}(c,i) = \sum_{\substack{(z,w)\in\mathcal{P}\times\mathcal{P}_{c,i}/R^\times \\ -\frac{1}{2}\mathrm{tr}(i_z i_w)>1}} Q_{s-1}\left(-\frac{1}{2}\mathrm{tr}(i_z i_w)\right). \tag{5.1.2}$$

Then formula (5.1.1) gives

$$(\eta_i, \eta_{c,i}^0)_{\tau_1} = \lim_{s\to 1}\left(u_{v,s}(c,i) - u_{v,s}(1,i) + \frac{\deg \eta_i \deg \eta_{c,i}^0}{s(s-1)\chi}\right). \tag{5.1.3}$$

5.1.5. *Linking numbers.* Each pair $(z,w)\in \mathcal{P}_i\times\mathcal{P}_{c,i}$ of X_i determines two homomorphisms ϕ_z and ϕ_w from K to B such that $\phi_z(\mathcal{O}_K)\subset R_i$ and $\phi_w(\mathcal{O}_c)\subset R_i$ and such that ϕ_z and ϕ_w have the same orientation. Let a, b be two totally positive elements of c and D_E respectively such that both a and b are prime to N and that $\sqrt{-b}$ is in E. Let $e_1=\phi_z(\sqrt{-b})$ and $e_2=\phi_w(c\sqrt{-b})$ in R_i. As ϕ_z and ϕ_w have positive orientation, we have

$$(ae_1-e_2)^2\equiv 0 \pmod{4N}.$$

In other words there is an $n\in Na^{-1}b^{-1}$ such that

$$\mathrm{tr}(e_1e_2) = -2ab + 4nab.$$

It is easy to verify that n is independent of the choice of c and d. So $n\in Nc^{-1}{D_E}^{-1}$. We call n the *linking number* of z and w (or ϕ_z and ϕ_w) and denote it by $n(z,w)$ (or $n(\phi_z,\phi_w)$).

As

$$-\frac{1}{2}\mathrm{tr}(i_z i_w) = 1 - 2\tau_1(n),$$

formula (5.1.2) becomes

$$u_{v,s}(c,i) = \sum_{\substack{n\in Nc^{-1}{D_E}^{-1}\\ \tau(n)<0}} \varrho_v(c,n,i)Q_{s-1}(1-2\tau_1(n)), \tag{5.1.4}$$

where $\varrho_v(c,n,i)$ is the number of conjugacy classes of pairs $(z,w)\in\mathcal{P}_i\times\mathcal{P}_{c,i}$ such that $n(z,w)=n$.

5.1.6. *Summing up.* We need to sum up formula (5.1.3) for all i, but only for $\varrho_v(c,n,i)$'s and residues. Let $P(n)_i$ denote the set of conjugacy classes of pairs $(\phi_1,\phi_2)\in\mathrm{Hom}(E,B)^2$ such that

$$\phi_1(\mathcal{O}_E)\subset R_i,\quad \phi_2(\mathcal{O}_c)\subset R_i,\qquad n(\phi_1,\phi_2)=n.$$

Then any pair (ϕ_1,ϕ_2) defines two CM-points $(z,w)\in\mathcal{H}\times\mathcal{H}$ with conductor 1 and c respectively. These two points are in $\mathcal{P}_i\times\mathcal{P}_{c,i}$ if and only if the morphism ϕ_z defined by z has positive orientation. The orientation group $\mathcal{W}$ acts freely on $\cup_i P(n)_i$ which, therefore, has cardinality $\varrho_\tau(c,n) := 2^s\sum_i \varrho_v(c,n,i)$.

Now we want to treat the residue term in formula (5.1.3).

Lemma 5.1.7. *The numbers* $\deg\eta_i$, $\deg\eta_{c,i}$, χ_i *do not depend on* i, *if they are nonzero.*

Proof. The natural projection from $X_\tau(\mathbb{C})$ onto the set of its connected components is given by the determinant map

$$X(\mathbb{C}) \to \pi_0(X(\mathbb{C}) := F_+^\times \backslash \widehat{F}^\times / \widehat{F}^{\times,2}\widehat{\mathcal{O}}_F^\times,$$
$$(z, g) \in \mathcal{H} \otimes \widehat{B}^\times \to \det g.$$

Now η_c is u_c^{-1} times the sum of CM-points represented by $(\sqrt{-1}, g)$ with g in $E^\times\backslash\widehat{E}^\times/\widehat{F}^\times\widehat{\mathcal{O}}_c^\times$. The determinant map restricted on these CM-points is given by the norm homomorphism

$$\mathrm{N}_{E/F} : E^\times\backslash\widehat{E}^\times/\widehat{F}^\times\widehat{\mathcal{O}}_c^\times \to F_+^\times\backslash\widehat{F}^\times/\widehat{F}^{\times,2}\widehat{\mathcal{O}}_F^\times.$$

Thus the pre-image of every point in $\pi_0(X(\mathbb{C})$ has the same cardinality if it is not empty. This implies that $\deg \eta_{c,i}$, therefore, $\deg \eta_i$ and $\deg \eta_{c,i}^0$ do not depend on i if they are nonzero.

It remains to show that χ_i does not depend on i. Recall that χ_i is the volume of X_i with respect to the measure $dxdy/y^2$ on $\mathcal{H}$ times an absolute constant. We need only show that the volume of X_i does not depend on i. For this we use Hecke's correspondence $\mathrm{T}(m)$. By the definition of $\mathrm{T}(m)$ in Section 1.4, the induced action of $\mathrm{T}(m)$ on $\pi_0(X(\mathbb{C}))$ is given by $[x] \to \sigma_1(m)[mx]$, where $[x]$ denote a point represented by $x \in \widehat{F}^\times$. On the other hand, $\mathrm{T}(m)$ changes volume form $dxdy/y^2$ to $\sigma_1(m)dxdy/y^2$. Thus all connected components of $X_\tau(\mathbb{C})$ must have the same volume. □

5.1.8. *Intersection on other archimedean places.* Now we want to compute the archimedean intersection for places of E over $\tau_2, \cdots, \tau_g$. For this we need to describe the conjugation $X_{\tau_k}(\mathbb{C})$ of $X(\mathbb{C})$ over F. Let $B(\tau_k)$ denote a quaternion algebra obtained from B by switching invariants at τ_1 and τ_k. Fix an order $R(\tau_k)$ of $B(\tau_k)$ of type (N, E); then

$$X_{\tau_k}(\mathbb{C}) \simeq B(\tau_k)^\times\backslash\mathcal{H}^\pm \times \widehat{B}(\tau_k)^\times/\widehat{R}(\tau_k)^\times.$$

So the above formulas (5.1.2)–(5.1.4) and Lemma 5.1.7 for (η, η_c^0) work for each τ_k.

More precisely for each infinite place τ_k, let $\varrho_{\tau_k}(c, n)$ be defined as above for $B(\tau_k)$; then we have the following:

PROPOSITION 5.1.9. *For each infinite place v of E over an infinite place τ_k of F, the local intersection $(\eta, \eta_c^0)_v$ is given by the formula*

$$\lim_{s\to 1}\left(u_{\tau_k,s}(c) - u_{\tau_k,s}(\mathcal{O}_F) + \frac{\deg\eta\deg\eta_c^0}{s(s-1)\chi}\right),$$

where χ is a constant independent of c and τ_k, and

$$u_{\tau_k,s}(c) = \sum_{\substack{n\in Nc^{-1}D_E{}^{-1}\\ \tau_k(n)<0}} 2^{-s}\varrho_{\tau_k}(c, n)Q_{s-1}\left(1 - 2\tau_k(n)\right).$$

5.2. *Nonarchimedean intersections.* In this section we want to compute the intersection of η and η_c^0 at a place v of E over a prime $\wp$ of F.

5.2.1. *Some intersection settings.* Let q denote the prime of $\mathcal{O}_E$ corresponding to v, and let $\mathcal{O}_q^{\mathrm{ur}}$ be the completion of the maximal unramified extension of $\mathcal{O}_q$ with a uniformizer π. Let E_q^{ur} denote its field of fractions. Then $\bar{X} := \widetilde{\mathcal{X}} \otimes \mathcal{O}_q^{\mathrm{ur}}$ can be embedded into $\bar{X}' = \widetilde{\mathcal{X}}' \otimes \mathcal{O}_q^{\mathrm{ur}}$. For any $\mathcal{O}_q^{\mathrm{ur}}$-scheme S, the set $\mathrm{Hom}_{\mathcal{O}_q^{\mathrm{ur}}}(S, \bar{\mathcal{X}}')$ parametrizes isomorphism classes of objects $[A, C, \kappa_0]$ where $[A, C]$ is an object of $\mathcal{F}(S)$, and κ_0 is a level structure defined by the compact subgroup $U \times J$.

Let x and y be two integral components of η and η_c^0, respectively, over E_q^{ur}. Then x is the image of a morphism from E_q^{ur} to X and y is the image of a morphism from $F(W)$ to X where $F(W)$ is the fraction field of a finite extension W of E_q^{ur}. Let us denote

$$(x, y)_q = (\overline{\pi^*(x)}/u_x, \overline{\pi^*(y)}/u_y)/\deg \pi,$$

where $\overline{\pi^*(x)}$ and $\overline{\pi^*(y)}$ are the Zariski closures of $\pi^*(x)$ and $\pi^*(y)$.

Assume that U and J are maximal at places dividing c; then

$$\pi^*(x) = u_x \sum x_i, \qquad \pi^*(y) = u_y \sum y_j$$

where the x_i are points of $\widetilde{X}$ defined over E_q^{ur} and the y_j are points defined over $F(W)$. Now, we have

$$(x, y)_q = \frac{1}{\deg \pi} \sum_{(i,j)} (\bar{x}_i, \bar{y}_j). \tag{5.2.1}$$

5.2.2. *Moduli interpretation.* The schemes $\overline{\pi^*(x)} = u_x \sum \overline{x_i}$, and $\overline{\pi^*(y)} = u_y \sum \overline{y_j}$ represent objects $[A, C, \kappa_i]$ and $[A', C', \kappa_j']$, where $[A, C]$ and $[A', C']$ are objects represented by the Zariski closures $\bar{x}$ and $\bar{y}$ of x and y in $\mathcal{X}$, respectively, and κ_i and κ_j are level structures on them for the group $U \cdot J$.

Now let us study the local intersection $(\eta, \eta_c^0)_q$ in two cases: $\wp \nmid c$ and $\wp | c$.

Case 1. $\wp \nmid c$. Let x and y be integral components of η and η_c^0 over E_q^{ur}. Then all x_i and y_j are sections of $\mathcal{X}$ over $\mathcal{O}_q^{\mathrm{ur}}$. Let $z_1, z_2, \cdots,$ be the inverse images of the reduction z of $\bar{x}$ on $\bar{X}$. Let $[A^0, C^0, \kappa_k^0]$ be the corresponding objects.

If $\wp$ is split in E and $\bar{x}_i$ and $\bar{y}_j$ intersect at some z_k, then both $\bar{x}_i$ and $\bar{y}_j$ are canonical liftings of z_k with the same multiplication by $\mathrm{End}(x)$, so that $x_i = y_j$. This is impossible and now $(x, y)_q = 0$.

If $\wp$ is not split in E and $\bar{x}_i$ and $\bar{y}_j$ intersect at some z_k in the special fiber, then there are two embeddings $\alpha_x : \mathrm{End}(x) \to \mathrm{End}(z)$ and $\alpha_y : \mathrm{End}(y) \to \mathrm{End}(z)$. With respect to α_x and α_y, x and y are canonical liftings.

Fix isomorphisms

$$\text{(5.2.2)} \qquad \begin{cases} \mathrm{End}(x) \simeq \mathcal{O}_E, \\ \mathrm{End}(y) \simeq \mathcal{O}_c, \\ \mathrm{End}(z) \simeq R(\wp), \end{cases}$$

where $R(\wp)$ is an order of type $(N(\wp), E)$ in the quaternion algebra $B(\wp)$. We require that the first two isomorphisms satisfy the conditions of Proposition 2.1.3. Let

$$n = n(\alpha_x, \alpha_y) \in N(\wp)c^{-1}{D_E}^{-1}$$

be the link number defined as in 5.1.5.

LEMMA 5.2.3. *Assume that* $\mathrm{ord}_\wp(N) \le 1$. *Then the intersection of* x *and* y *is given by* $(x, y)_q = m(n)$ *where*

$$m(n) = \begin{cases} \mathrm{ord}_\wp(n\wp) & \text{if } \wp \mid D_E \\ [\mathrm{ord}_\wp(n\wp/N)/2] & \text{if } \wp \nmid D_E. \end{cases}$$

Proof. In this case the component of C at $\wp$ is 0. Thus the formal deformation of the formal group gives a formal neighborhood of $z_i's$ in $\widehat{X}$. By (5.2.1), it is not difficult to show that $(\bar{x}, \bar{y})$ equals the maximal integer m such that

1. $\mathrm{End}(x_m)$ contains the images of α_x and α_y where x_m is the restriction of x on $\mathcal{O}_q^{\mathrm{ur}}/q^m\mathcal{O}_q^{\mathrm{ur}}$;
2. $\alpha_x = \alpha_y \pmod{q^{m-1}\pi}$ in $\mathcal{O}_{B(\wp)}$ as x and y have the same orientation at $\wp$, where

$$\pi = \begin{cases} q & \text{if } \wp \mid ND_F \\ \varpi & \text{otherwise,} \end{cases}$$

   and where ϖ is a uniformizer of $B(\wp)$

By Proposition 2.4.5, $\mathrm{End}(x_m)$ is the unique suborder of $R(\wp)$ of type $(E, \wp^{b_m}N)$ where

$$b_m = \begin{cases} 2m - 1 & \text{if } \wp \nmid D_E \\ m & \text{if } \wp \mid D_E. \end{cases}$$

On the other hand, the algebra $\mathcal{O}_{x,y}$ generated by the images of α_x, α_y has discriminant $D_n := c^2{D_E}^2 n(1 - n)$. Thus $m(n)$ is the largest number such that $\mathrm{ord}_\wp(D_n) \ge b_m$. Now, the first condition is equivalent to

$$\text{(5.2.3)} \qquad m \le \begin{cases} \mathrm{ord}_\wp(n(1-n)\wp^2) & \text{if } \wp \mid D_E \\ \left[\frac{1}{2}\mathrm{ord}_\wp(n(1-n)\wp/N)\right] & \text{otherwise.} \end{cases}$$

For the second condition we let t be an element in $\mathcal{O}_q$ such that

$$\mathcal{O}_q = \mathcal{O}_\wp + \mathcal{O}_\wp t, \qquad t^2 \in \mathcal{O}_\wp.$$

Then the second condition is equivalent to

$$\alpha_x(t) - \alpha_y(t) = 0 \pmod{q^{m-1}\pi}.$$

Let μ be an element in $\mathcal{O}_{B(\wp)}$ such that the following conditions hold:

$$\mathcal{O}_{B(\wp)} = \mathcal{O}_q + \mathcal{O}_q\mu, \qquad \mu^2 \in \mathcal{O}_\wp$$
$$\mu x = \bar{x}\mu \qquad \text{for all } x \in \mathcal{O}_q.$$

Consider t as an element in $R(\wp)$ via α_x then $\alpha_y(t)$ will have the form

$$\alpha_y(t) = t(\alpha + \beta\mu), \qquad \alpha^2 - \beta\bar{\beta}\mu^2 = 1,$$

where $\alpha \in \mathcal{O}_\wp$, $\beta \in \mathcal{O}_q$. Now the second condition is equivalent to the following:

$$(5.2.4) \qquad \alpha - 1 = 0 \pmod{q^{m-1}\pi t^{-1}}, \qquad \beta\mu = 0 \pmod{q^{m-1}\pi t^{-1}}.$$

By the definition of n,

$$\operatorname{tr}(\alpha_x(t)\alpha_y(t)) = 2t^2 - 4t^2 n.$$

Thus

$$\alpha - 1 = -2n, \qquad \beta\bar{\beta}\mu^2 = 4n(n-1)$$

and (5.2.4) is equivalent to

$$n = 0 \pmod{q^{m-1}\pi t^{-1}}, \qquad n(1-n) = 0 \pmod{(q^{m-1}\pi t^{-1})^2},$$

or equivalently,

$$\begin{aligned} m &\leq \operatorname{ord}_q(tq/\pi) + \operatorname{ord}_q(\wp)\min\left\{\operatorname{ord}_\wp(n), \frac{1}{2}\operatorname{ord}_\wp(n(n-1))\right\} \\ &\leq \operatorname{ord}_q(tq/\pi) + \operatorname{ord}_q(\wp)\frac{1}{2}\operatorname{ord}_\wp(n). \end{aligned}$$

Thus the second condition is equivalent to

$$(5.2.5) \qquad m \leq \begin{cases} \operatorname{ord}_\wp(n\wp) & \text{if } \wp \mid D_F \\ \frac{1}{2}\operatorname{ord}_\wp(n) & \text{if } \wp \mid N \\ \frac{1}{2}\operatorname{ord}_\wp(n\wp) & \text{otherwise.} \end{cases}$$

The lemma follows from (5.2.3), (5.2.5), and the fact that $\operatorname{ord}_\wp(n) > 0$ if $\wp$ is unramified in E, as $n \in N(\wp)c^{-1}D_E^{-1}$. □

Conversely if α_1 and α_2 are two homomorphisms from $\mathcal{O}_E$ and $\mathcal{O}_c$ to $\operatorname{End}(z)$, respectively, which have positive orientation, then by Proposition 1.5.1, we can find objects $[A, C]$ and $[A', C']$ which are canonical liftings of $[A^0, C^0]$ with respect to α_1 and α_2. This defines a component x for η and a component y for η_c^0. Now for each z_k, the level structure κ_k^0 can be uniquely extended to level structure on $[A, C]$ and $[A', C']$ so that we obtain some sections x_i and y_j which intersect at z_k. It is easy to see that the number of z_k is

$\deg\pi/c_z$ where $c_z = \#[R(\wp)^\times/\mathcal{O}_F^\times]$. Now the total intersection of $(\eta, \eta_c^0)_q$ at z is given by

$$\sum_n \varrho(z, c, n) m(n),$$

where $\varrho(z, c, n)$ is the number of $R(\wp)$-conjugacy classes of pairs (ϕ_1, ϕ_2) as above with link number n.

Write $\varrho_\wp(c, n)$ as the sum of $\varrho(R(\wp), c, n)$ over all nonconjugate orders $R(\wp)$ of $B(\wp)$ of type $(N(\wp), E)$, where $\varrho(R(\wp), c, n)$ is the number of $R(\wp)$-conjugacy classes of pairs (α_1, α_2) of homomorphisms from $\mathcal{O}_E$ and $\mathcal{O}_c$ to $R(\wp)$ with the same orientation and link number n in $N(\wp)c^{-1}{D_E}^{-1}$. As in the archimedean case,

$$\sum_z \varrho(z, c, n) = 2^{-s(\wp)} \varrho_\wp(c, n)$$

where $s(\wp)$ is the number of prime factors of $N(\wp)$ not dividing D_E. Then we obtain

$$(\eta, \eta_c^0)_v = u_\wp(c) - u_\wp(1), \tag{5.2.6}$$

where

$$u_\wp(c) = 2^{-s(\wp)} \sum_{n \in Nc^{-1}{D_E}^{-1}} \varrho_\wp(c, n) m(n),$$

with $m(n)$ given by formula (5.2.6). Here 2^{-1} appears in the formula because of the symmetry between n and $1 - n$.

Case 2. $\wp | c$. Write $c = c'\wp^s$ with $s = \mathrm{ord}_\wp(c)$. Then η_c^0 can be written as

$$\eta_c^0 = \sum_{x' \in \eta} x'(s) + \sum_{y' \in \eta_{c'}^0} (y' + y'(s))$$

where $x'(s)$ and $y'(s)$ are sums of quasi-canonical liftings of the reductions of x' and y' of levels up to s. If x is a section of η then a component $\bar{x}_i$ of $\overline{\pi^* x}$ has an intersection with a component $\overline{x'(s)}_j$ of $\overline{\pi^*(x'(s))}$ if and only if $x = x'$, and then $(\bar{x}_i, \overline{x'(s)}_j) = s$. Similarly, a component $\bar{x}_i$ of $\overline{\pi^* x}$ has an intersection with a component $\overline{y(s)}_i$ of $\overline{\pi^* y(s)}$ if and only if $\bar{x}_i$ has an intersection with $\bar{y}_j$, and then $(\bar{x}_i, \overline{y(s)}_j)_q = s$. It follows that

$$(\eta, \eta_c^0)_q = s h_1,$$

if $\wp$ is split in E, and that

$$(\eta, \eta_c^0)_q = s h_2 + u_\wp(c) - u_\wp(1),$$

if $\wp$ is inert in E, where h_1, h_2 are constants independent of c, and where

$$u_\wp(c) = 2^{-s(\wp)} \sum_{n' \in Nc^{-1}{D_E}^{-1}} \varrho_\wp(c', n')(s + m(n')).$$

In summary we have proved the following:

PROPOSITION 5.2.4. *Assume that c is prime to ND_E.*

1. *If $\varepsilon(\wp) = 1$, then*
$$(\eta, \eta_c^0)_v = \mathrm{ord}_\wp(c) h_1$$
where h_1 is a constant independent of c.

2. *If $\varepsilon(\wp) = 0$, then*
$$(\eta, \eta_c^0)_v = u_\wp(c) - u_\wp(1),$$
where $u_\wp$ is given by the formula
$$u_\wp = 2^{-s(\wp)} \sum_{n \in Nc^{-1}D_E{}^{-1}} \varrho_\wp(c, n) m(n).$$

3. *If $\varepsilon(\wp) = -1$, then*
$$(\eta, \eta_c^0)_v = \mathrm{ord}_\wp(c) h_2 + u_\wp(c) - u_\wp(\mathcal{O}_F)$$
where h_2 is a constant independent of c, and where $u_\wp$ is given by the formula
$$u_\wp(c) = 2^{-s(\wp)} \sum_{n' \in Nc'^{-1}D_E{}^{-1}} \varrho_\wp(c', n')(\mathrm{ord}_\wp(c) + m(n')),$$
where $c' = c\wp^{-\mathrm{ord}_\wp(c)}$.

5.3. *Clifford algebras.*

5.3.1. *Determining the ramification type.* Let Δ be a quaternion algebra over F with embeddings ϕ_1 and ϕ_2 from E into Δ. Let d be a nonzero element in $F^\times$ such that $\sqrt{-d} \in E$, and denote
$$e_1 = \phi_1(\sqrt{-d}), \qquad e_2 = \phi_2(\sqrt{-d}).$$
Let $m \in F^\times$ be defined by
$$e_1 e_2 + e_2 e_1 = 2dm.$$
Then m does not depend on the choice of d. We want to describe the places at which Δ is ramified in terms of m.

PROPOSITION 5.3.2. *Let v be a place of F. The algebra Δ is ramified at v if and only if $\varepsilon_v(m^2 - 1) = -1$.*

Proof. Let Δ^0 denote the vector space of trace 0 elements in Δ. Then Δ^0 is ramified at a place v of F if and only if $\Delta^0 \otimes F_v$ has no nonzero element with square 0. Now Δ^0 is a vector space over F generated by e_1, e_2 and $e_1e_2 - md$ and it is easy to check that for any x, y, z in F_v,

$$[xe_1 + ye_2 + z(e_1e_2 - md)]^2 = -dx^2 + 2mdxy - dy^2 + (m^2 - 1)d^2z^2.$$

This form is linearly equivalent to

$$-dx^2 + (m^2 - 1)y^2 - z^2.$$

Thus Δ is ramified at v if and only if $(m^2 - 1)$ is not a norm from E_v, or equivalently, $\varepsilon_v(m^2 - 1) = -1$. □

5.3.3. *Counting orders.* Let c be a nonzero ideal of $\mathcal{O}_E$ prime to ND_E . Let S denote the $\mathcal{O}_F$-subalgebra in Δ generated by $\phi_1(\mathcal{O}_E)$ and $\phi_2(\mathcal{O}_c)$. Then S is finite over $\mathcal{O}_F$ if and only if $m + 1 \in 2c^{-1}{D_E}^{-1}$. The discriminant of S is $D_S = (m^2 - 1)c^2{D_E}^2$.

Let ℓ be an ideal of $\mathcal{O}_F$ such that the following conditions are satisfied:

1. $\mathrm{ord}_v(\ell)$ is even if v is split in Δ and inert in E;
2. $\mathrm{ord}_v(\ell)$ is odd if v is ramified in Δ and inert in E;
3. $\mathrm{ord}_v(\ell)$ is 0 if v is split in Δ and ramified in E;
4. $\mathrm{ord}_v(\ell)$ is 1 if v is ramified in both Δ and E.

In the following we want to compute the number of orders in Δ of type (ℓ, E) containing S. The above conditions imply the existence of the orders in Δ of type (ℓ, E). Indeed, conditions 1 and 2 imply that there is an ideal ℓ_E in $\mathcal{O}_E$ with norm ℓ/D_Δ where D_Δ is the product of primes in F over which Δ is ramified. Let $\mathcal{O}_\Delta$ be any maximal order of Δ containing $\phi_1(\mathcal{O}_E)$. Then

$$\phi_1(\mathcal{O}_E) + \phi_1(\ell_E)\mathcal{O}_\Delta$$

is an order in Δ of type (ℓ, E). Let $\varrho(S)$ denote the number of orders in Δ of discriminant ℓ containing S.

Proposition 5.3.4. *Assume $m + 1 \in 2c^{-1}{D_E}^{-1}$. There is an order of discriminant ℓ containing S only if D_S is divisible by ℓ. If $\ell | D_S$, then*

$$\varrho(S) = r(D_S/\ell) \cdot \prod_{\substack{v|(D_S, D_E) \\ \varepsilon_v(m^2-1)=1}} 2.$$

Proof. Since the correspondence $\mathcal{O} \to \widehat{\mathcal{O}}$ gives a bijection between the set of orders of Δ and the orders of $\widehat{\Delta}$, it follows that $\varrho(S)$ equals the product of the numbers $\varrho_v(S)$ of orders on Δ_v of type (ℓ, E) containing S_v for all finite places v of F. Fix a finite place $v = \wp$. We want to compute $\varrho_v(S)$ case by case.

Let W denote the ring $\phi_1(\mathcal{O}_{E,v})$ contained in S. Recall that the discriminant of S is D_S.

If $\varepsilon(v) = -1$, or v is ramified in Δ, then there is a unique order in Δ_v of discriminant $\wp^{\mathrm{ord}_v(\ell)}$ containing W. This order contains S_v if and only if $\mathrm{ord}_v(\ell) \le \mathrm{ord}_v(D_S)$. In other words, $\varrho_v(S) = 1$ if $\mathrm{ord}_v(D_S) \le \mathrm{ord}_v(\ell)$ and $\varrho_v(S) = 0$ if $\mathrm{ord}_v(D_S) < \mathrm{ord}_v(\ell)$.

If $\varepsilon(\wp) = 1$ then $W \simeq \mathcal{O}_\wp^2$ as a ring. It follows that $S_\wp$ is an Eichler order conjugate to an order of the form

$$\left\{ \begin{pmatrix} a & b \\ \wp^{\mathrm{ord}_\wp(D_S)}c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathcal{O}_\wp \right\}.$$

This order is contained in an order of the discriminant $\wp^{\mathrm{ord}_\wp(\ell)}$ if and only if $\mathrm{ord}_\wp(\ell) \le \mathrm{ord}_\wp(D_S)$. If this is the case, then each order of Δ of the type (ℓ, E) containing this order has the form

$$\left\{ \begin{pmatrix} a & \wp^{-k}\ell b \\ \wp^k c & d \end{pmatrix} \,\middle|\, a, b, c, d, \in \mathcal{O}_\wp \right\}$$

with $0 \le k \le \mathrm{ord}_\wp(D_S) - \mathrm{ord}_\wp(\ell)$. Hence $\varrho_\wp(S) = 1 + \mathrm{ord}_\wp(D_S/\ell)$.

If $\varepsilon(\wp) = 0$ and $\wp$ is not ramified on Δ, then S is generated by e_1 and e_2 such that

$$e_1e_2 + e_2e_1 = 2md$$

and W is generated by e_1, where $e_i^2 = d$. The correspondence $I \to \mathrm{End}_{\mathcal{O}_\wp}(I)$ gives a bijection between the set of maximal orders of $\Delta_\wp$ containing W and the set of ideals in W modulo an equivalence relation: $I_1 \sim I_2$ if and only if $I_1 = I_2\alpha$ for an $\alpha \in F^\times$. We have two maximal orders $\mathrm{End}_{\mathcal{O}_\wp}(W)$ and $\mathrm{End}_{\mathcal{O}_\wp}(We_1)$ corresponding to ideals W and We_1. One of them must contain S, say the first one. We want to prove that the second one also contains S. It suffices to show that the second one contains e_2, or in other words $e_2(We_1) \subset We_1$. First of all, as $e_2W \subset W$ and

$$e_1e_2 + e_2e_1 = 2md,$$

we have

$$e_2(We_1) \subset 2mdW + We_1.$$

Secondly, as $\wp|(D_S, D_E)$ with $D_S = (m^2-1)d^2\mathcal{O}_F$, we must have $\mathrm{ord}\wp(m) \ge 0$. So $e_1|md$ at the place $\wp$. It follows that $e_2(We_1) \subset We_1$. So we have proved that $\varrho_\wp(S) = 2$. This completes the proof of the proposition. □

We will use Proposition 5.3.4 to compute the embeddings from S into orders of type (ℓ, E):

Proposition 5.3.5. *Let $\mathcal{O}_1, \dots, \mathcal{O}_h$ be a representing set of all conjugacy classes of orders in Δ of type (ℓ, E). Then*

$$\sum_{i=1}^{h} \#\{\phi : S \to \mathcal{O}_i \pmod{\mathcal{O}_i^*}\} = 2^{t(\ell)} \varrho(S),$$

where $\mathcal{O}_i^\times$ acts on the set of embeddings from S into $\mathcal{O}_i$ by conjugations, and $t(\ell)$ is the number of finite places dividing ℓ.

Proof. The proof is almost exactly like the modular curve case treated by Gross, Kohnen, and Zagier [22]. We omit the details. □

5.4. *The final formula.* For each place w of F, let $(\eta, \eta_c^0)_w$ denote the total intersection over the places over w:

$$(\eta, \eta_c^0)_w = \sum_{v|w} (\eta, \eta_c^0)_v \log \mathrm{N}(v) \tag{5.4.1}$$

where the v are places of E, and $\log \mathrm{N}(v)$ is set to be 2 if v is a complex place. In this section we want to compute the local intersection

$$(\eta, \mathrm{T}(m)^0 \eta)_w = \sum_{c|m} \varepsilon(c) (\eta, \eta_{m/c}^0)_w. \tag{5.4.2}$$

5.4.1. *The archimedean case*: *Computation.* By Proposition 5.1.9, for a place τ_i, $(\eta, \mathrm{T}(m)^0 \eta)_{\tau_i}$ is equal to

$$2 \lim_{s \to 1} \left(U_{\tau_i,s}(m) - r(m) U_{\tau_i,s}(\mathcal{O}_F) + \frac{(\deg \eta)^2 \deg \mathrm{T}^0(m)}{s(s-1)\chi} \right) \tag{5.4.3}$$

where $U_{\tau_i,s}(m)$ is

$$\begin{aligned} & 2^{-s} \sum_{c|m} \varepsilon(c) \sum_{\substack{n \in Ncm^{-1}D_E{}^{-1} \\ \tau_i(n) < 0}} \varrho_{\tau_i}(m/c, n) Q_{s-1}(1 - 2\tau_i(n)) \\ & \quad = 2^{-s} \sum_{\substack{n \in Nm^{-1}D_E{}^{-1} \\ \tau_i(n) < 0}} \sum_{\substack{c|m \\ c|nmD_E N^{-1}}} \varepsilon(c) \varrho_{\tau_i}(m/c, n) Q_{s-1}(1 - 2\tau_i(n)). \end{aligned}$$

Lemma 5.4.2. *Let $n \in Nc^{-1}D_E{}^{-1}$ be such that $\tau_i(n) < 0$. Then the following assertions hold*:

1. *$\varrho_{\tau_i}(c, n) \neq 0$ if and only if the following are satisfied*:
   (a) *$0 < \tau_j(n) < 1$ for $j \neq i$;*
   (b) *$\varepsilon_\wp(n(n-1)) = 1$ for any $\wp | D_E$;*
   (c) *$r(n(n-1)c^2 N^{-1}) \neq 0$.*

2. *Assume conditions* (a) *and* (b). *Then*

$$\varrho_{\tau_i}(c,n) = 2^s r(n(n-1)c^2N^{-1})\delta(n)$$

where $\delta(n) = \prod_{\wp|(D_E,n\wp)} 2$.

Proof. Let Δ and S be a Clifford algebra and an order defined as in 5.3.1 and 5.3.3 with $m = 2n-1$. Then S has discriminant

$$D_S = n(n-1)c^2{D_E}^2.$$

By 5.1.6, Propositions 5.3.5 and 5.3.4, $\varrho_{\tau_i}(c,n) \neq 0$ is equivalent to the following:

- Δ is isomorphic to $B(\tau_i)$, or equivalently by Proposition 5.3.2,

  $$\varepsilon_v(n(n-1)) = -1$$

  if and only if v is ramified in $B(\tau_i)$.

- D_S is divisible by $\ell = N$, or equivalently $n(n-1)c^2{D_E}^2N^{-1}$ is an integer.

Recall that $B(\tau_i)$ is ramified exactly at archimedean place τ_j $(j \neq i)$ and finite places $\wp$ such that $\varepsilon_\wp(N) = -1$. Thus these two conditions are equivalent to the conditions (a), (b), and (c) because of the following:

- For an infinite place τ_j,

  $$\varepsilon_{\tau_j}(n(n-1)) < 0 \Longleftrightarrow \tau_j(n(n-1)) < 0;$$

- $(D_S, N) = 1$ so that $B(\tau_i)$ is unramified at all places dividing D_E;

- $r(n(n-1)c^2N^{-1}) \neq 0$ if and only if $n(n-1)c^2{D_E}^2N^{-1}$ is an integer and

  $$\varepsilon_\wp(n(n-1)c^2{D_E}^2N^{-1}) = 1$$

  for all finite place $\wp \nmid D_E$.

This proves the first assertion in the lemma.

By assertion 1, the equality in assertion 2 follows if $\varrho_{\tau_i}(c,n) = 0$. Otherwise, by Propositions 5.3.5 and 5.3.4, $\varrho_{\tau_i}(c,n) = 2^s\varrho(S)$ and $\varrho(S)$ is given by

$$r(D_S/\ell) \cdot \prod_{\substack{v|(D_S,D_E) \\ \varepsilon_v(m^2-1)=1}} 2.$$

Now for any $\wp \mid D_E$, condition (a) implies $\varepsilon_\wp(m^2-1) = 1$, and $\wp \mid D_S$ is equivalent to $\mathrm{ord}_\wp(n) \geq 0$. Thus we have assertion 2. □

Lemma 5.4.3. *Let a and b be two nonzero ideals. Then*

$$\sum_{c|(a,b)} \varepsilon(c) r(\frac{ab}{c^2}) = r(a)r(b).$$

Proof. It is easy to reduce to the case where $a = \wp^m$ and $b = \wp^n$ both are powers of a prime ideal in $\mathcal{O}_F$. In this case the lemma is obvious. □

Applying Lemmas 5.4.2 and 5.4.3 to formula (5.4.3), we, therefore, obtain the following:

Proposition 5.4.4. *Let τ_i be an infinite place of F. Then in $\mathcal{S}/\mathcal{D}_N$, $(\eta, \mathrm{T}(m)^0\eta)_{\tau_i}$ is given by the limit as $s \to 1$ of*

$$2 \sum_{\substack{n \in Nm^{-1}D_E{}^{-1}, \tau_i(n)<0, \\ 0<\tau_j(n)<1, \forall j \neq i \\ \varepsilon_\wp(n(n-1))=1, \forall \wp | D_E}} \delta(n) r(ncN^{-1}) r((n-1)m) Q_{s-1}\left(1 - 2\tau_i(n)\right).$$

Here in $\mathcal{S}/\mathcal{D}_N$, the limit makes sense, as the term

$$\frac{(\deg \eta)^2 \deg \mathrm{T}^0(m)}{s(s-1)\chi}$$

is an element in $\mathcal{D}_N$ as a function of m.

5.4.5. *The nonarchimedean case.* Now let us treat the nonarchimedean case. Fix a prime $\wp$. To compute $(\eta, \mathrm{T}(m)^0\eta)_\wp$, there are three cases:

Case 1. $\varepsilon(\wp) = 1$. By Proposition 5.2.3,

$$(\eta, \mathrm{T}(m)^0\eta)_\wp = 2h_1 j_\wp(m), \tag{5.4.4}$$

where h_1 is a constant independent of m, and where

$$j_\wp(m) = \sum_{c|m} \varepsilon(c) \mathrm{ord}_\wp(m/c) \log \mathrm{N}(\wp) = \frac{1}{2} r(m) \mathrm{ord}_\wp(m) \log \mathrm{N}(\wp). \tag{5.4.5}$$

Case 2. $\varepsilon(\wp) = 0$. As m is prime to D_E, by 5.2.3, one has

$$(\eta, \mathrm{T}(m)^0\eta)_\wp = (U_\wp(m) - R(m)U_\wp(1)) \log \mathrm{N}(\wp). \tag{5.4.6}$$

where

$$U_\wp(m) = 2^{-s(\wp)} \sum_{n \in N(\wp)m^{-1}D_E{}^{-1}} \sum_{\substack{c|m \\ c|nmD_E N^{-1}}} \varepsilon(c) \varrho_\wp(m/c, n) m(n)$$

where $m(n)$ is as defined by formula (5.2.3). By the proof of Lemma 5.4.2 we have:

LEMMA 5.4.6. *Let $n \in N(\wp)c^{-1}{D_E}^{-1}$. Then the following assertions hold*:

1. $\varrho_\wp(c,n) \neq 0$ *if and only if the following are satisfied*
   (a) *For all infinite places* τ_i, $0 \leq \tau_i(n) \leq 1$;
   (b) $\varepsilon_\wp(n(n-1)) = -1$, *and* $\varepsilon_q(n(n-1)) = 1$ *for all* $q|D_E$, $q \neq \wp$;
   (c) $r(n(n-1)c^2N^{-1}) \neq 0$;
2. *If conditions* (a) *and* (b) *are satisfied then*

$$\varrho_\wp(c,n) = 2^{s(\wp)}\delta(n)r(n(n-1)c^2N^{-1}).$$

Applying Lemmas 5.4.6 and 5.4.3, we see that $U_\wp(m)$ is equal to

$$\sum_{\substack{n\in Nm^{-1}{D_E}^{-1},0<n<1\\ \varepsilon_\wp(n(n-1))=-1\\ \varepsilon_q(n(n-1))=1,\forall\wp\neq q|D_E}} \delta(n)r(nmN^{-1}/\wp)r((n-1)m)m(n) \tag{5.4.7}$$

where the inequality $0 < n < 1$ means $0 < \tau_i(n) < 1$ for all τ_i.

Case 3. $\varepsilon(\wp) = -1$. Again by Proposition 5.2.3,

$$(\eta, \mathrm{T}(m)^0\eta)_\wp = h_2 j_\wp(m) + (U_\wp(m) - R(m)U_\wp(1))\log \mathrm{N}(\wp). \tag{5.4.8}$$

Here h_2 is a constant independent of m, and

$$\begin{aligned} j_\wp(m) &= 2\sum_{c|m}\varepsilon(c)\mathrm{ord}_\wp(m/c)\log \mathrm{N}(\wp) \\ &= r(m)\mathrm{ord}_\wp(m)\log \mathrm{N}(\wp) + \mathrm{ord}_\wp(m\wp)r(m/\wp)\log \mathrm{N}(\wp), \end{aligned} \tag{5.4.9}$$

and $U_\wp(m)$ is equal to

$$2^{1-s(\wp)}\sum_{c|m}\varepsilon(c)\sum_{n\in m'^{-1}c'{D_E}^{-1}N\wp}\varrho_\wp(\frac{m'}{c'},n)\left[\mathrm{ord}_\wp(\frac{m}{c}) + m(n)\right],$$

where $m' = m'\wp^{-\mathrm{ord}_\wp(m)}$ and $c' = c\wp^{-\mathrm{ord}_\wp(c)}$. Changing the order of sums and writing $c = c'\wp^t$, we see that $U_\wp(m)$ is equal to

$$\begin{aligned} 2^{1-s(\wp)} &\sum_{n\in Nm'^{-1}{D_E}^{-1}\wp}\ \sum_{\substack{c|m'\\ c|nm'D_EN^{-1}}}\varepsilon(c')\varrho_\wp(\frac{m'}{c},n) \\ &\cdot\sum_{t=0}^{\mathrm{ord}_\wp(m)}(-1)^t\left[m(n) + \mathrm{ord}_\wp(m) - t\right]. \end{aligned}$$

The last two sums are independent. Let us evaluate them separately.

As in the other two case, we have the following lemma:

LEMMA 5.4.7. *Let c be an integer prime to $\wp$ and let $n \in Nc^{-1}{D_E}^{-1}\wp$. Then the following two assertions hold*:

1. $\varrho_\wp(c,n) \neq 0$, *if and only if the following conditions are satisfied*:
   (a) $0 < n < 1$;
   (b) $\varepsilon_\ell(n(n-1)) = 1$, *for all* $\ell | D_E$;
   (c) $r(n(n-1)c^2N^{-1}\wp^{-1}) \neq 0$.
2. *Moreover if conditions* (a) *and* (b) *are satisfied then*
$$\varrho_\wp(c,n) = 2^{s(\wp)}\delta(n)r(n(n-1)c^2N^{-1}\wp^{-1}).$$

Applying Lemmas 5.4.7 and 5.4.3, we obtain
$$2^{-s(\wp)} \sum_{\substack{c|m' \\ c|nm'D_EN^{-1}}} \varepsilon(c)\varrho_\wp(\frac{m'}{c}, n) = r(nm'N^{-1}/\wp)r((n-1)m').$$

The second sum (in Case 3) can be evaluated directly:
$$\sum_{t=0}^{\mathrm{ord}_\wp(m)} (-1)^t \left[m(n) + \mathrm{ord}_\wp(m) - t\right]$$
$$= \begin{cases} \left[m(n) + \frac{1}{2}\mathrm{ord}_\wp(m)\right] & \text{if } \mathrm{ord}_\wp(m) \text{ is even,} \\ \frac{1}{2}\mathrm{ord}_\wp(m\wp) & \text{if } \mathrm{ord}_\wp(m) \text{ is odd.} \end{cases}$$

Thus $U_\wp(m)$ is equal to

$$(5.4.10) \qquad \sum_{\substack{n \in {m'}^{-1}{D_E}^{-1}N(\wp) \\ 0<n<1 \\ \varepsilon_\ell(n(n-1))=1, \forall \ell | D_E}} r(nm'/N^{-1}\wp)r((n-1)m')\delta(n) \cdot$$
$$\cdot \begin{cases} 2\left[m(n) + \frac{1}{2}\mathrm{ord}_\wp(m)\right] & \text{if } \mathrm{ord}_\wp(m) \text{ is even,} \\ \mathrm{ord}_\wp(m\wp) & \text{if } \mathrm{ord}_\wp(m) \text{ is odd.} \end{cases}$$

In summary we obtain the following:

PROPOSITION 5.4.8. *Assume that* $\varepsilon(\wp) = 1$ *if either* $\mathrm{ord}_\wp(N) > 1$ *or* $\wp \mid 2$. *Then the local intersection* $(\eta, \mathrm{T}(m)^0\eta)_\wp$ *is given by the following formulas*:

1. *If* $\varepsilon(\wp) = 1$ *then* $(\eta, \mathrm{T}(m)^0\eta)_\wp \pmod{\mathcal{D}_N}$ *is equal to*
$$h_1 r(m)\mathrm{ord}_\wp(m)\log \mathrm{N}(\wp)$$
where h_1 *is a constant independent of* m *and* $\wp$.

2. *If* $\varepsilon(\wp) = 0$ *then* $(\eta, \mathrm{T}(m)^0\eta)_\wp$ *is equal to*

$$(U_\wp(m) - R(m)U_\wp(1)) \log \mathrm{N}(\wp)$$

where $U_\wp(m)$ *is equal to*

$$\sum_{\substack{n \in m^{-1}D_E{}^{-1}N(\wp), 0<n<1 \\ \varepsilon_\wp(n(n-1))=-1 \\ \varepsilon_q(n(n-1))=1, \forall \wp \neq q | D_E}} \delta(n) r(nm/N\wp) r((n-1)m) \mathrm{ord}_\wp(n\wp).$$

3. *If* $\varepsilon(\wp) = -1$ *then* $(\eta, \mathrm{T}(m)^0\eta)_\wp$ *is equal to*

$$h_2 r(m) \mathrm{ord}_\wp(m) \log \mathrm{N}(\wp) + h_2 \mathrm{ord}_\wp(m\wp) r(m/\wp) \log \mathrm{N}(\wp) \\ + (U_\wp(m) - U_\wp(1)) \log \mathrm{N}(\wp)$$

where h_2 *is a constant independent of* $m, \wp$, *and* $U_\wp(m)$ *is equal to*

$$\sum_{\substack{n' \in m'^{-1}D_E{}^{-1}N(\wp) \\ 0<n'<1 \\ \varepsilon_\ell(n'(n'-1))=1, \forall \ell | D_E}} r(n'm'N^{-1}/\wp) r((n'-1)m') \delta(n') \cdot \begin{cases} 2\left[\frac{1}{2}\mathrm{ord}_\wp(n'\wp m)\right] & \textit{if } \mathrm{ord}_\wp(m) \textit{ is even,} \\ \mathrm{ord}_\wp(m\wp) & \textit{if } \mathrm{ord}_\wp(m) \textit{ is odd.} \end{cases}$$

Proof. All these statements follow from formulas (5.4.4)–(5.4.10) and Lemma 5.2.3, with the fact that in the case $\varepsilon(\wp) = -1$, the term for an n' in (5.4.10) has nonzero contribution only if $\mathrm{ord}_\wp(n'N(\wp))$ is even. Thus

$$m(n') = \left[\frac{1}{2}\mathrm{ord}_\wp(n'\wp)\right]$$

even when $\mathrm{ord}_\wp(N)$ is odd. □

6. Derivatives of L-series

In this section, we will compute $L'_E(f, s)$ using the method of Gross and Zagier shown in [21]. We will start with a formula which expresses $L_E(f, 1/2 + s)$ as an inner product of f with a nonholomorphic form $\Phi_s(z)$. Then we compute the Fourier expansion for $\Phi_s(z)$ and get a formula for some multiple $\widetilde{\Phi}$ of $\frac{\partial}{\partial s}\Big|_{s=1/2} \widetilde{\Phi}_s(z)$. Finally the holomorphic projection of $\widetilde{\Phi}$ gives a holomorphic form Φ with Fourier coefficients given explicitly.

6.1. *The Rankin-Selberg method.* Let f be a new form for $K_0(N)$ and let E be an imaginary quadratic extension of F as before. Then the base change L-function of f to E is defined to be $L_E(s,f)=L(s,f)L(s,\varepsilon,f)$ where ε is the character attached to E/F. See Section 3.4 for definitions. For any nonzero ideal m let $r(m)$ denote the number of integral ideals in $\mathcal{O}_E$ with the norm m. Using Proposition 3.1.4, one shows that

$$L_E(f,s)=L^N(2s-1,\varepsilon)\sum_{m\in\mathbb{N}_F}a(m)r(m)\mathrm{N}(m)^{-s} \tag{6.1.1}$$

where $L^N(s,\varepsilon)$ denotes the series

$$\sum_{\substack{m\in\mathbb{N}_F\\(m,ND_E)=1}}\varepsilon(m)\mathrm{N}(m)^{-s},$$

where D_E is the conductor of ε. In other words, $L_E(f,s)$ is essentially the Rankin-Selberg convolution of $L(s,f)$ with $\zeta_E(s)$. We want to express this convolution as an inner product of f with a modular form. We will construct such a form using the Eisenstein series defined in Section 3.5.

6.1.1. *$L_E(f,s)$ as an inner product of f with some other form.* We need to define a Haar measure on $Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. Let $dk=\otimes dk_v$ be the Haar measure on $K_0(1)$ with volume 1 on each component. Recall that $dx=\otimes dx_v$ is defined in 3.1.1 to be a measure on $\mathbb{A}_F$ such that dx_v is the usual Euclidean measure if v is infinite, and that $\mathcal{O}_v$ has volume 1 if v is finite. Also recall that $d^\times x=\otimes d^\times x_v$ is defined in the proof of 3.4.2 to be a Haar measure on $\mathbb{A}_F^\times$ such that $d^\times x_v=|dx_v/x_v|$ if v is infinite, and that $\mathcal{O}_v^\times$ has volume 1 if v is finite. Now dg on $G(\mathbb{A}_F)$ is defined by the formula

$$\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}f(g)dg=\int_{\mathbb{A}_F^\times}\int_{\mathbb{A}_F}\int_K f\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}k\right)dkdx\frac{d^\times y}{|y|}.$$

For any two function f and g on $Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$ the integral $f\bar{g}$ (if it is absolutely convergent) is denoted as (f,g).

Let E_s be the Eisenstein series defined in 3.5.1 with $\chi=\varepsilon$ associated to the extension E/F. Let $E_{s,N}$ be an Eisenstein series defined by the formula

$$E_{s,N}(g)=E_s\left(g\begin{pmatrix}1&0\\0&\pi_N\end{pmatrix}\right) \tag{6.1.2}$$

where π_N is an idele with components 1 at places not dividing N and such that π_N generates $\widehat{N}$. Then $E_{s,N}$ is a form of level $K_0(D_EN)$.

PROPOSITION 6.1.2. *Let f be a new cusp form of weight* 2, *for $K_0(N)$ a trivial central character. Then*

$$(f,E_{1/2}E_{s,N})=A(s)L_E(s+1/2,f)$$

where

$$A(s) = \left[\frac{\Gamma(s+1/2)}{2^{2s}\pi^{s-1/2}}\right]^g d_F^{s+1/2} d_N^s d_E^{-1/2} \mu(ND_E)$$

where $\mu(ND_E)$ *is the volume of* $K_0(ND_E)$.

Proof. For each factor e of N, let E_s^e be the Eisenstein series defined in the same way as E_s in 3.5.1 with factor $L(2s,\varepsilon)$ replaced by $L^e(2s,\varepsilon)$ and with H_s replaced by the following H_s^e:

$$H_s^e(g) = \begin{cases} |\frac{a}{d}|^s \varepsilon(akr(\theta)) & \text{if } k \in K_0(D_E e) \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathrm{Re}(s) > 1$, $E_s^e(g)$ is absolutely convergent and defines a (nonholomorphic) form for $K_0(D_E e)$ of (parallel) weight 1 with character ε. If $e = \mathcal{O}_F$, $E_s^e = E_s$.

LEMMA 6.1.3. *Let* f *be a cusp form of weight* 2 *for* $K_0(ND_E)$ *with trivial central character. Let* θ *be the theta series with Fourier coefficients* $r(m)$ *defined as in* 3.4.5. *Then*

$$(f, \theta E_{\bar{s}}^N) = \mu(ND_E) d_F^{s+1} \left[\frac{\Gamma(s+1/2)}{(4\pi)^{s+1/2}}\right]^g L_E(s+1/2, f).$$

Proof. By definition of E_s^N, up to a factor $L(2s,\varepsilon)$, $(f, \theta E_{\bar{s}}^N)$ is given by

$$\int_{Z(\mathbb{A}_F)B(F)\backslash G(\mathbb{A}_F)} f\overline{\theta H_{\bar{s}}^N} dg$$

$$= \int_{\mathbb{A}_{F,+}^\times/F_+} \int_{\mathbb{A}_F/F} \int_K (f\overline{\theta H_{\bar{s}}^N}) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k \right) dk dx \frac{d^\times y}{|y|},$$

where $\mathbb{A}_{F,+}^\times$ denote ideles with positive components at the infinite places. By definition of H_s^N, the inner integral over K is

$$\mu(ND_E)(f\bar{\theta}) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) |y|^s \varepsilon(y).$$

Using Fourier expansions of f and θ in (3.1.3) and Proposition 3.1.2,

$$\int_{\mathbb{A}_F/F} (f\bar{\theta}) \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) dx$$

$$= d_F^{1/2} |y|^{3/2} \varepsilon(y) \sum_{\alpha > 0} a(\alpha y_f D_F) r(\alpha y_f D_F) \psi(2\alpha y_\infty i),$$

where $a(m)$ are the Fourier coefficients of f as defined in Proposition 3.1.2. Combining these, $(f, \theta E_{\bar{s}}^N)$ up to a factor $L(2s,\varepsilon)$, is equal to

$$\mu(ND_E) d_F^{1/2} \int_{\mathbb{A}_{F,+}^\times} |y|^{s+1/2} \sum_{\alpha > 0} a(y_f D_F) r(y_f D_F) \psi(2y_\infty i) d^\times y.$$

This integral is the product of the integrals over infinite ideles $\prod_{v|\infty} F_{v,+}$ and over finite ideles $\widehat{F}^\times$. The integral over infinite ideles gives

$$\left[\frac{\Gamma(s+1/2)}{(4\pi)^{s+1/2}}\right]^g$$

while the integral over the finite ideles gives

$$d_F^{s+1/2}\sum_m \frac{a(m)r(m)}{\mathrm{N}(m)^{s+1/2}}.$$

The lemma follows from (6.1.1). □

The next lemma gives a comparison between E_s^e and $E_{s,n}$.

LEMMA 6.1.4. $E_{s,N} = d_N^s \sum_{a|N} \frac{\varepsilon(a)}{\mathrm{N}(a)^{2s}} E_s^{N/a}$.

Proof. Let $H_{s,N}$ be the function on $G(\mathbb{A}_F)$ defined in the same way as $E_{s,N}$:

$$H_{s,N}(g) = H_s\left(g\begin{pmatrix}1 & 0\\ 0 & \pi_N\end{pmatrix}\right).$$

It suffices to prove the corresponding statement for $H_{s,N}$ on $K_0(1)$:

$$H_{s,N} = d_N^s \sum_{a|N} \frac{\varepsilon(a)}{\mathrm{N}(a)^{2s}} \frac{L^{N/a}(2s,\varepsilon)}{L(2s,\varepsilon)} H_s^{N/a}.$$

We do this by testing their values on elements $k = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ of K_v for finite places v not dividing D_E. Now we have the decomposition:

$$k\begin{pmatrix}1 & 0\\ 0 & \pi_N\end{pmatrix} = \begin{pmatrix}\frac{1}{d}(ad-bc) & b\pi_N\\ 0 & d\pi_N\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{c}{d\pi_N} & 1\end{pmatrix}$$

if $N|c$, and

$$k\begin{pmatrix}1 & 0\\ 0 & \pi_N\end{pmatrix} = \begin{pmatrix}\frac{\pi_N}{c}(ad-bc) & a\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & \frac{d\pi_N}{c}\end{pmatrix}$$

if $c|\frac{\pi_N}{\pi_v}$. It follows that

$$H_{s,N}(k) = \begin{cases} |\pi_N|^{-s} & \text{if } N|c\\ \varepsilon_v\left(\frac{\pi_N}{c}\right)\left|\frac{\pi_N}{c^2}\right|^s & \text{if } c|\frac{\pi_N}{\pi_v}.\end{cases}$$

Let $\wp_v$ be the prime ideal of $\mathcal{O}_F$ corresponding to v. By definition of H_s^e,

$$H_s^e(k) - H_s^{\wp e}(k) = \begin{cases} 1 & \text{if } \mathrm{ord}_v(N) = \mathrm{ord}_v(e)\\ 0 & \text{otherwise.}\end{cases}$$

It follows that

$$\begin{aligned}
&\mathrm{N}(N_v)^s H_{s,N}(k)\\
&\qquad = H_s^N(k) + \sum_{1\le i\le \mathrm{ord}_v(N)} \varepsilon(\wp_v^i)\mathrm{N}(\wp_v^i)^{2s}(H_s^{N/\wp^i}(k) - H_s^{N/\wp^{i-1}}(k))\\
&\qquad = \sum_{0\le i\le \mathrm{ord}_v(N)} \frac{L^{N/\wp^i}(2s,\varepsilon)}{L(2s,\varepsilon)}\varepsilon(\wp^i)\mathrm{N}(\wp^i)^{2s}H_s^{N/\wp^i}(k). \qquad \square
\end{aligned}$$

Now go back to the proof of Proposition 6.1.2. By Proposition 3.5.4,

$$E_{1/2} = \frac{(2\pi)^g}{\sqrt{d_F d_E}}\theta.$$

By the two lemmas above, the proof of the proposition is reduced to showing that

$$(f, E_{1/2}E_s^e) = 0$$

for any factor $e \neq N$ of N.

Let tr_{D_E} be the trace operator from the space of cusp forms of level $K_0(ND_E)$ to $K_0(N)$: for any form ϕ of level $D_E N$,

$$(\mathrm{tr}_{D_E}\phi)(g) = \sum_{\gamma\in K_0(N)/K_0(ND_E)} \phi(g\gamma). \tag{6.1.3}$$

Then

$$(f, E_{1/2}E_s^e) = [K_0(N) : K_0(ND_E)]^{-1}(f, \mathrm{tr}_{D_E}(E_{1/2}E_s^e).$$

As representatives of $K_0(N)/K_0(ND_E)$ will also serve as representatives for $K_0(e)/K_0(eD_E)$ for any $e|N$, $\mathrm{tr}_{D_E}(E_{1/2}E_s^e)$ is a form of level $K_0(e)$. Thus it is orthogonal to f as f is a newform. □

6.1.5 *Definition of* Φ_s. We define

$$\Phi_s(g) = \mathrm{tr}_{D_E}\left(\frac{1}{2^{\#\{v:v|D_E\}}}\sum_{e|D_E}\mathrm{N}(e)^{s-1/2}\Phi_s^e\right). \tag{6.1.4}$$

Here, for e a divisor of D_E,

$$\Phi_s^e(g) = (E_{1/2}E_{s,N})(g\gamma_e) \tag{6.1}$$

where γ_e is an element of $\mathrm{GL}_2(\mathbb{A}_F)$ which has components 1 at places not dividing e and at a place v dividing e it has the component $\begin{pmatrix} 0 & -1 \\ \pi_v & 0 \end{pmatrix}$ where π_v is normalized such that $\varepsilon(\pi_v) = 1$, and where tr_{D_E} is as defined in (6.1.3). It is easy to check that Φ_s is a form of weight 1 for $K_0(N)$ with trivial character.

Corollary 6.1.6. *Let f be a newform of weight 2 for $K_0(N)$ with trivial central character. Then*

$$(f, \Phi_{\bar{s}}) = B(s) L_E(f, 1/2 + s)$$

where

$$B(s) = \left[\frac{\Gamma(s+1/2)}{2(4\pi)^{s-1/2}}\right]^g d_F^{s+1/2} d_N^s d_E^{-1/2} \mu(N).$$

Proof. Fix any factor e of D_E. Write f^e for the form $g \to f(g\gamma_e^{-1})$. Then again f^e is a form of level $K_0(N)$. By definition, $(f, \Phi_{\bar{s}}^e)$ is equal to

$$[K_0(N) : K_0(ND_E)](f^e, E_{1/2} E_{\bar{s},N}).$$

By Proposition 6.1.2, this is

$$A(s)[K_0(N) : K_0(ND_E)] L_E(s + 1/2, f^e).$$

As

$$f^e\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} y/\pi_e & x \\ 0 & 1 \end{pmatrix}\right),$$

if f has the Fourier coefficients $a(m)$ then f^e will have the Fourier coefficients

$$a(f^e, m) = \mathrm{N}(e) a(m/e).$$

It follows that

$$L_E(f^e, s) = \mathrm{N}(e)^{1-s} L_E(f, s).$$

The proposition follows. □

6.2. *Fourier coefficients.*

6.2.1. *Strategy.* In this section we want to compute the Fourier coefficients $c_s(\alpha, y)$ ($\alpha \in F$) for Φ_s defined by (6.1.4), where

$$c_s(\alpha, y) = d_F^{-1/2} \int_{\mathbb{A}_F/F} \Phi_s\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \psi(-\alpha x) dx. \tag{6.2.1}$$

It suffices to compute $c_s(\alpha, y)$ for $\alpha = 0$ or 1, as for $\alpha \in F^\times$,

$$c_s(\alpha, y) = c_s(1, \alpha y).$$

We proceed with the following steps:

1. Compute the Fourier coefficients $c_s^e(\alpha, y)$ for $E_s(g\gamma_e)$. This will give the Fourier coefficients for $\Phi_s^e(g)$ defined by (6.1.5).

2. For a factor g of D_E and an integral adele a which is 0 at places not dividing g, let $\gamma_{g,a}$ denote the element in $\mathrm{GL}_2(\mathbb{A})$ which has the component 1 at places v not dividing g, otherwise it is given by $\begin{pmatrix} a_v & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$\left\{\gamma_{g,a}\middle| \quad g|D_E, \quad a \pmod g\right\}$$

forms a set of representatives for $K_0(N)/K_0(D_E N)$. Compute the Fourier coefficients $c_s^{e,g}(\alpha, y)$ for

$$\Phi_s^{e,g} := \sum_{a \pmod g} \Phi^e(g\gamma_{g,a}). \tag{6.2.2}$$

3. Compute the Fourier coefficients of Φ_s using the following expression:

$$\Phi_s = 2^{-\#\{v:v|D_E\}} \sum_{e,g|D_E} \mathrm{N}(e)^{s-1/2}\Phi_s^{e,g}. \tag{6.2.3}$$

Lemma 6.2.2. *The Fourier coefficient $c_s^e(\alpha, y)$ of $E_s(g\gamma_e)$ is zero if $\alpha y D_F$ is nonintegral. Otherwise, it is given by the following expressions*:

$$c_s^e(0, y) = \begin{cases} \varepsilon(y)L(2s, \varepsilon)|y|^s & \text{if } e = 1 \\ \frac{(-1)^g}{d_F^{1/2} d_E^s} V_s(0)^g L(2s-1, \varepsilon)|y|^{1-s} & \text{if } e = D_F \\ 0 & \text{otherwise,} \end{cases}$$

$$c_s^e(1, y) = \frac{(-1)^g}{\sqrt{d_F d_E}} \mathrm{N}(e)^{1/2-s}\sigma_s(y)|y|^{1-s} \prod_{v|D_E/e} |y_v\pi_v|^{2s-1}\varepsilon(-y_v)\kappa(v),$$

where

$$\sigma_s(y) = \prod_{\substack{v\nmid D_E \\ v\nmid\infty}} \frac{1-\varepsilon(y_v\delta_v\pi_v)|y_v\delta_v\pi_v|^{2s-1}}{1-\varepsilon(\pi_v)|\pi_v|^{2s-1}} \cdot \prod_{v|\infty} V_s(y_v),$$

and for $y \in \mathbb{R}$,

$$V_s(y) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i y x}}{(x^2+1)^{s-1/2}(x+i)} dx.$$

Proof. The case $e = 1$ was covered in Proposition 3.5.2. So we assume $e \neq 1$ in the following. Again by Bruhat decomposition, $c^e(\alpha, y)$ is equal to

$$L(2s, \varepsilon)d_F^{-1/2} \int_{\mathbb{A}_F/F} H_s\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\gamma_e\right)\psi(-\alpha x)dx$$
$$+ L(2s, \varepsilon)d_F^{-1/2} \int_{\mathbb{A}_F} H_s\left(w\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\gamma_e\right)\psi(-\alpha x)dx.$$

Let v be a place dividing e, then $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \gamma_e$ has the component

$$\begin{pmatrix} y_v & x_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \pi_v & 0 \end{pmatrix} = \begin{pmatrix} y_v & x_v\pi_v \\ 0 & \pi_v \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It follows that

$$H_s\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \gamma_e\right) = 0$$

and that $c^e(\alpha, y)$ is given by

$$\begin{aligned} & L(2s,\varepsilon)d_F^{-1/2} \int_{\mathbb{A}_F} H_s\left(w\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \gamma_e\right) \psi(-\alpha x)dx \\ &= L(2s,\varepsilon)d_F^{-1/2}|y|^{1-s} \prod_{v\nmid e} V_s(\alpha_v y_v) \cdot \prod_{v|e} V_s'(\alpha_v y_v), \end{aligned}$$

where V_s is as defined in the proof of Proposition 3.5.2, and V_s' is given by the formula

$$V_s'(y) = \int_{F_v} H_s\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \pi_v & 0 \end{pmatrix}\right) \psi(-xy)dx.$$

Now

$$w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \pi_v & 0 \end{pmatrix} = \begin{pmatrix} -\pi_v & 0 \\ x\pi_v & -1 \end{pmatrix}$$

has the decomposition

$$\begin{pmatrix} -\pi_v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x\pi_v & -1 \end{pmatrix},$$

if $\mathrm{ord}_v(x) \geq 0$, and

$$\begin{pmatrix} -x^{-1} & \pi_v \\ 0 & -x\pi_v \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & x^{-1}\pi_v^{-1} \end{pmatrix},$$

if $\mathrm{ord}_v(x) < 0$. It follows that

$$H_s\left(\begin{pmatrix} -\pi_v & 0 \\ x\pi_v & -1 \end{pmatrix}\right) = \begin{cases} |\pi_v|^s \varepsilon_v(-1) & \text{if } \mathrm{ord}_v(x) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$V_s'(y) = \begin{cases} |\pi_v|^s \varepsilon_v(-1) & \text{if } \mathrm{ord}_v(y) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now the lemma follows easily from this formula and the formulas for V_s derived in the proof of Proposition 3.5.2. □

LEMMA 6.2.3. *The Fourier coefficient $c_s^{e,g}(\alpha, y)$ $(\alpha = 0, 1)$ of $\Phi_s^{e,g}$ is zero if $\alpha y D_F$ is nonintegral. Otherwise, it is given by*

$$c_s^{e,g}(\alpha, y) = \mathrm{N}(g)\varepsilon(N) \sum_{n \in F} c_{1/2}^{e*g}(\alpha - n, \pi_g y) c_s^{e*g}(n, \pi_g y/\pi_N)$$

where $e * g$ denotes $eg/(e,g)^2$.

Proof. By definition,

$$\Phi_s^{e,g}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{a \pmod g} \Phi_s^e\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\gamma_a\right).$$

From the following decomposition at any place v dividing g,

$$\begin{pmatrix} a_v & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\pi_v}\begin{pmatrix} \pi_v & a_v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \pi_v & 0 \end{pmatrix},$$

we see that

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\gamma_a = \frac{1}{\pi_g}\begin{pmatrix} \pi_g y & ay + x \\ 0 & 1 \end{pmatrix}\gamma_g.$$

It follows that

$$\Phi_s^{e,g}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{a \pmod g} \Phi_s^{e*g}\left(\begin{pmatrix} \pi_g y & ay + x \\ 0 & 1 \end{pmatrix}\right).$$

Thus the Fourier coefficients of $\Phi_s^{e,g}$ are given by

$$a_s^{e*g}(\alpha, \pi_g y) \sum_{a \pmod g} \psi(\alpha y a),$$

or in other words, $c_s^{e,g}(\alpha, y)$ is nonzero only if αy is integral at places dividing g. In this case it is given by

$$c_s^{e,g}(\alpha, y) = \mathrm{N}(g) a_s^{e*g}(\alpha, \pi_g y)$$

where $a_s^e(\alpha, y)$ is the Fourier coefficient of Φ_s^e which can be expressed as

$$a_s^e(\alpha, y) = \varepsilon(N) \sum_{n \in F} c_{1/2}^e(\alpha - n, y) c_{s,N}^e(n, y/\pi_N). \qquad \square$$

PROPOSITION 6.2.4. *The Fourier coefficient $c_s(\alpha, y)$ $(\alpha = 0, 1)$ of Φ_s is nonzero only if $\alpha y D_F$ is integral. In this case it is given by*

$$c_s(\alpha, y) = \frac{\varepsilon(N) d_F^{1-s}}{d_E d_F} \sum_{n \in F} a_s^n(\alpha, y)$$

where $a_s^n(\alpha, y)$ is as given by the following formulas if it is nonzero.

1. *If* $n \neq 0$ *and* $n \neq \alpha$, *then* $a_s^n(\alpha, y) \neq 0$ *only if* $nyD_ED_FN^{-1}$ *is integral. In this case* $a_s^n(\alpha, y)$ *is equal to*

$$|y|^{3/2-s}\delta(ny)\prod_{v|D_E}\frac{1+|n_vy_v\pi_v|^{2s-1}\varepsilon_v((n-\alpha)n)}{2} \cdot\sigma_{1/2}((\alpha-n)y)\sigma_s(ny/\pi_N),$$

where for an idele y, $\delta(y) = 2^{\#\{v|D_E,\mathrm{ord}_v(y)\geq 0\}}$.

2. *If* $n = 0$, $\alpha = 1$, *then* $a_s^n(\alpha, y)$ *is equal to*

$$\sigma_{1/2}(y)d_F^{1/2}d_E^{1/2}d_N^{2s-1}\varepsilon(N)i^gL(2s,\varepsilon)|y|^{1/2+s} + \sigma_{1/2}(y)V_s(0)^gL(2s-1,\varepsilon)|y|^{3/2-s}.$$

3. *If* $n = \alpha = 0$, *then* $a_s^n(\alpha, y)$ *is equal to*

$$\varepsilon(N)d_Fd_Ed_N^{2s-1}L(1,\varepsilon)L(2s,\varepsilon)|y|^{1/2+s} + V_{1/2}(0)V_s(0)L(0,\varepsilon)L(2s-1,\varepsilon)|y|^{3/2-s}.$$

4. *If* $n = 1$, $\alpha = 1$, *then* $a_s^n(\alpha, y)$ *is equal to*

$$\left[d_F^{1/2}d_E^{1/2}L(1,\varepsilon)i^g|\pi_{D_E}y_{D_E}|^{2s-1}\varepsilon^{D_E}(y) + L(0,\varepsilon)V_{1/2}(0)^g\right] \cdot\sigma_s(y/\pi_N)|y|^{3/2-s},$$

where $\varepsilon^{D_E}(y)$ *denotes* $\varepsilon(y)\prod_{v|D_E}\varepsilon_v(y)$.

Proof. By formula (6.2.3), $c_s(\alpha, y)$ is equal to

$$\frac{1}{\delta(1)}\sum_{e,g}\mathrm{N}(e)^{s-1/2}c_s^{e,g}(\alpha,y) = \frac{\varepsilon(N)d_N^{1-s}}{d_Ed_F}\sum_{n\in F}a_s^n(\alpha,y)$$

where $a_s^n(\alpha, y)$ is equal to

$$\frac{d_Fd_Ed_N^{s-1}}{\delta(1)}\sum_{e,g}\mathrm{N}(g)\mathrm{N}(e)^{s-1/2}c_{1/2}^{e*g}(\alpha-n,\pi_gy)c_s^{e*g}(n,\pi_gy/\pi_N). \tag{6.2.4}$$

Case 1. $n \neq 0, \alpha$. If $a_s^n(\alpha, y) \neq 0$, one must have

$$\mathrm{ord}_v(ny\pi_v) \geq 0 \quad \text{for each} \quad v \mid D_E.$$

Assume this is the case and let g_0 be the factor of D_E consisting of places v such that $|n_vy_v\pi_v| = 1$. Then

$$c_{1/2}^{e*g}(\alpha-n,\pi_gy)c_s^{e*g}(n,\pi_gy/\pi_N) \neq 0$$

only if $g_0|g$ and in this case by Lemma 6.2.2, it equals

$$\frac{|\pi_N|^{s-1}}{d_E d_F}|\pi_g y|^{3/2-s}\sigma_{1/2}((\alpha-n)y)\sigma_s(ny/\pi_N)\mathrm{N}(g*e)^{1/2-s} \cdot \prod_{v|D_E/(g*e)} |n_v y_v \pi_{g,v}\pi_v|^{2s-1}\varepsilon_v((\alpha-n)n).$$

It follows that $a_s^n(\alpha, y)$ is equal to

$$\frac{|y|^{3/2-s}}{\delta(1)}\sigma_{1/2}((\alpha-n)y)\sigma_s(ny/\pi_N) \cdot \sum_{\substack{g_0|g|D_E\\ e|D_E}} \frac{\mathrm{N}(ge)^{s-1/2}}{\mathrm{N}(g*e)^{s-1/2}} \prod_{v|D_E/(g*e)} |n_v y_v \pi_{g,v}\pi_v|^{2s-1}\varepsilon_v((\alpha-n)n). \tag{6.2.5}$$

Notice that

$$\frac{\mathrm{N}(ge)}{\mathrm{N}(e*g)} \prod_{v|D_E/(g*e)} |\pi_{g,v}|^2 = 1.$$

Now the last sum is

$$\sum_{\substack{g_0|g|D_E\\ e|D_E}} \prod_{v|D_E/(g*e)} |n_v y_v \pi_v|^{2s-1}\varepsilon_v((\alpha-n)n).$$

When e is exchanged for $(D_E/e)*g$, this sum equals

$$\delta(1/g_0)\prod_{v|D_E}\left[1+|n_v y_v \pi_v|^{2s-1}\varepsilon_v((\alpha-n)n)\right].$$

Bringing this to (6.2.4), we obtain the formula for $a_s^n(\alpha, y)$ in the proposition.

Case 2. $n=0$, $\alpha=1$. In this case, $a_s^0(1,y)$ is equal to

$$\frac{d_F d_E d_N^{s-1}}{\delta(1)}\sum_g \mathrm{N}(g)^{1/2+s}c_{1/2}^1(1,\pi_g y)c_s^1(0,\pi_g y/\pi_N) + \frac{d_E d_F d_N^{s-1}}{\delta(1)}\sum_g d_E^{s-1/2}\mathrm{N}(g)^{3/2-s}c_{1/2}^{D_E}(1,\pi_g y)c_s^{D_E}(0,\pi_g y/\pi_N).$$

The formula in the proposition follows, as $c_{1/2}^1(1,\pi_g y)c_s^1(0,\pi_g y/\pi_N)$ is equal to

$$\frac{i^g\varepsilon(N)d_N^s}{d_E^{1/2}d_F^{1/2}}\sigma_{1/2}(y)L(2s,\varepsilon)|y\pi_g|^{1/2+s}\varepsilon^{D_E}(y)$$

and $c_{1/2}^{D_E}(1,\pi_g y)c_s^{D_E}(0,\pi_g y/\pi_N)$ is equal to

$$\frac{d_N^{1-s}}{d_E d_F}d_E^{1/2-s}V_s(0)^g\sigma_{1/2}(y)L(2s-1,\varepsilon)|y\pi_g|^{3/2-s}$$

where $\varepsilon^{D_E}(y)$ denotes $\varepsilon(y)\prod_{v|D_E}\varepsilon_v(y)$ which equals 1 if $\sigma_{1/2}(y)\neq 0$.

Case 3. $n = \alpha = 0$. In this case, $a_s^0(1, y)$ is equal to

$$\frac{d_F d_E d_N^{s-1}}{\delta(1)} \sum_g \mathrm{N}(g)^{1/2+s} c_{1/2}^1(0, \pi_g y) c_s^1(0, \pi_g y/\pi_N) + \frac{d_E d_F d_N^{s-1}}{\delta(1)} \sum_g d_E^{s-1/2} \mathrm{N}(g)^{3/2-s} c_{1/2}^{D_E}(0, \pi_g y) c_s^{D_E}(0, \pi_g y/\pi_N).$$

The formula in the proposition follows, as $c_{1/2}^1(0, \pi_g y) c_s^1(0, \pi_g y/\pi_N)$ is equal to

$$\varepsilon(N) d_N^s L(1, \varepsilon) L(2s, \varepsilon) |\pi_g y|^{1/2+s}$$

while $c_{1/2}^{D_E}(0, \pi_g y) c_s^{D_E}(0, \pi_g y/\pi_N)$ is equal to

$$\frac{d_N^{1-s}}{d_F d_E} d_E^{-s} V_{1/2}(0) V_s(0) L(0, \varepsilon) L(2s-1, \varepsilon) |y|^{3/2-s}.$$

Case 4. $n = \alpha = 1$. This case can be treated similarly. We have $a_s^1(1, y)$ equal to

$$\frac{d_F d_E d_N^{s-1}}{\delta(1)} \sum_g \mathrm{N}(g)^{1/2+s} c_{1/2}^1(0, \pi_g y) c_s^1(1, \pi_g y/\pi_N) + \frac{d_F d_E d_N^{s-1}}{\delta(1)} \sum_g d_E^{s-1/2} \mathrm{N}(g)^{3/2-s} c_{1/2}^{D_E}(0, \pi_g g) c_s^{D_E}(1, \pi_g y/\pi_N),$$

where $c_{1/2}^1(0, \pi_g y) c_s^1(1, \pi_g y/\pi_N)$ is equal to

$$\frac{d_N^{1/2-s} i^g L(1, \varepsilon)}{d_F^{1/2} d_E^{1/2}} \sigma_s(y/\pi_N) |y|^{1-s} |\pi_{D_E} y_{D_E}|^{2s-1} \varepsilon^{D_E}(y) |\pi_g|^{1/2+s},$$

and $c_{1/2}^{D_E}(0, \pi_g y) c_s^{D_E}(1, \pi_g y/\pi_N)$ is equal to

$$\frac{L(0, \varepsilon) d_N^{1-s}}{d_E d_F} d_E^{1/2-s} V_{1/2}(0)^g \sigma_s(y/\pi_N) |\pi_g y|^{3/2-s}. \qquad \square$$

6.3. *Functional equations and derivatives.*

PROPOSITION 6.3.1. *The Fourier coefficient $c_s^e(\alpha, y)$ of $E(g\gamma_e)$ has the following functional equation*:

$$\begin{aligned} \widetilde{c}_s^e(\alpha, y) &:= (d_F d_E d_e)^{s-1/2} \left[\Gamma(s+1/2) \pi^{1/2-s}\right]^g c_s^e(\alpha, y) \\ &= i^g \varepsilon(y) \prod_{v | D_E/e} \varepsilon_v(-1) \widetilde{c}_{1-s}^{D_E/e}(\alpha, y). \end{aligned}$$

Proof. If $\alpha = 1$, then by Lemma 6.2.2, up to a factor independent of s and e, $\widetilde{c}_s^e(1, y)$ is given by

$$\prod_{\substack{v \nmid D_E \\ v \nmid \infty}} |y_v \delta_v|^{1/2-s} \frac{1 - \varepsilon(y_v \delta_v \pi_v)|y_v \delta_v \pi_v|^{2s-1}}{1 - \varepsilon(\pi_v)|\pi_v|^{2s-1}}$$
$$\cdot \prod_{v|\infty} \left[\Gamma(s + 1/2)\pi^{1/2-s}\right] |y_v|^{1/2-s} V_s(y_v)$$
$$\cdot \prod_{v|e} |y_v \pi_v|^{1/2-s}$$
$$\cdot \prod_{v|D_E/e} |y_v \pi_v|^{s-1/2} \varepsilon(-y_v)\kappa(v).$$

By Proposition 3.3 in [20, p. 278], with $k = 1$, $V_s(t)$ $(t \neq 0)$ has a functional equation

$$V_s^*(t) := (\pi|t|)^{1/2-s}\Gamma(s + 1/2)V_s(t) = \operatorname{sgn}(t)V_{1-s}^*(t).$$

(Notice that V_s as defined in Lemma 6.2.2 is $V_{s+1/2}$ in [20].) Thus the functional equation in the lemma follows from the local equations and the equality

$$\prod_{v|D_E} \kappa(v) = i^g \varepsilon(D_F).$$

Now we want to treat the case where $\alpha = 0$. By Lemma 6.2.2, we need only consider the case where $e = 1$ or $e = D_E$. Recall that $L(s, \varepsilon)$ has a functional equation:

$$\begin{aligned} L^*(s, \varepsilon) &:= (d_E d_F)^{s/2} \left[\Gamma(s/2 + 1/2)\pi^{1/2-s/2}\right]^g L(s, \varepsilon) \\ &= L^*(1 - s, \varepsilon). \end{aligned}$$

(This can be proved by using functional equations for both ζ_E and ζ_F and the identity $\zeta_E(s) = L(s, \varepsilon)\zeta_F(s)$.) Thus $\widetilde{c}_s^1(0, y)$ is equal to

$$(d_F d_E)^{s-1/2} \left[\Gamma(s + 1/2)\pi^{1/2-s}\right]^g L(2s, \varepsilon)\varepsilon(y)|y|^s \tag{6.3.1}$$
$$= (d_F d_E)^{-1/2} L^*(2s, \varepsilon)\varepsilon(y)|y|^s = (d_F d_E)^{-1/2} L^*(1 - 2s, \varepsilon)\varepsilon(y)|y|^s.$$

On the other hand, by Proposition (3.3) in [20, p. 277],

$$\begin{aligned} V_s(0) &= -\pi i 2^{2-2s}\Gamma(2s - 1)/\Gamma(s - 1/2)\Gamma(s + 1/2) \\ &= -i\pi^{1/2}\Gamma(s)/\Gamma(s + 1/2). \end{aligned}$$

Thus $\widetilde{c}_s^{D_E}(0,y)$ is equal to

$$\begin{aligned}(d_F d_E^2)^{s-1/2}\left[\Gamma(s+1/2)\pi^{1/2-s}\right]^g \frac{(-1)^g}{d_F^{1/2}d_E^s} V_s(0)^g L(2s-1,\varepsilon)|y|^{1-s} \\ = (d_F d_E)^{s-1}\left[i\pi^{1-s}\Gamma(s)\right]^g L(2s-1,\varepsilon)|y|^{1-s} \\ = i^g (d_F d_E)^{-1/2} L^*(2s-1,\varepsilon).\end{aligned}$$

Combining this with (6.3.1), we have shown

$$\widetilde{c}_{1-s}^{D_E}(0,y) = i^g \varepsilon(y)\widetilde{c}_s^1(0,y).$$

Thus, the lemma is proved in this case. □

Corollary 6.3.2. *The function Φ_s satisfies the following functional equation*:

$$\begin{aligned}\Phi_s^* &:= (d_F d_E)^{s-1/2}\left[\Gamma(s+1/2)\pi^{1/2-s}\right]^g \Phi_s \\ &= (-1)^g \varepsilon(N)\Phi_{1-s}^*.\end{aligned}$$

Proof. We need only prove the following functional equation for Φ_s^e defined in (6.1.4):

$$\begin{aligned}\Phi_s^{e,*} &:= (d_F d_E \mathrm{N}(e))^{s-1/2}\left[\Gamma(s+1/2)\pi^{1/2-s}\right]^g \Phi_s^e \\ &= (-1)^g \varepsilon(N)\Phi_{1-s}^{e,*}.\end{aligned}$$

As both sides are modular forms for $K_0(ND_E)$ with trivial character, it suffices to check the functional equation for its Fourier coefficients. But this follows from Lemma 6.3.1, as the Fourier coefficients of Φ_s^e are expressed in the form

$$a_s^e(\alpha,y) = \varepsilon(N)\sum_{n\in F} c_{1/2}^e(\alpha-n,y)c_{s,N}^e(n,y/\pi_N).$$ □

Theorem 6.3.3. *The function $L_E(s,f)$ satisfies the following functional equation*:

$$\begin{aligned}L^*(s,f) &:= (d_F^2 d_E d_N)^{s-1}\left[\Gamma(s)(2\pi)^{1-s}\right]^{2g} L_E(s,f) \\ &= (-1)^g \varepsilon(N) L_E^*(2-s,f).\end{aligned}$$

Proof. This follows from Corollaries 6.3.2 and 6.1.6. □

PROPOSITION 6.3.4. *Assume that* $\varepsilon(N) = (-1)^{g-1}$. *Then* $\Phi_{1/2} = 0$ *and the Fourier coefficient* $c'(\alpha, y)$ $(\alpha = 0, 1)$ *of*

$$\Phi' := \frac{\partial}{\partial s}\Phi_s\Big|_{s=1/2}$$

is nonzero only if $\alpha y D_F$ *is integral. In this case it is given by*

$$c'(\alpha, y) = \frac{\varepsilon(N) d_N^{1/2}}{d_E d_F} \sum_{n \in F} b^n(\alpha, y)$$

where $b^n(\alpha, y)$ *is give by the following formulas if it is nonzero*:

1. *If* $n \neq 0$ *and* $n \neq \alpha$, *then* $b^n(\alpha, y) \neq 0$ *only if* $nyD_E D_F N^{-1}$ *is integral and* $(\alpha - n)y$ *is totally positive. In this case* $b^n(\alpha, y)$ *is equal to*

$$(-4\pi^2)^g |y| \psi(i\alpha y_\infty) \delta(ny) r((\alpha - n)yD_F) \sum_v b_v^n(\alpha, y)$$

where v *runs through all places of* F, $\delta(y) = 2^{\#\{v | D_E, \mathrm{ord}_v(y) \geq 0\}}$, *and* b_v^n *is given by the following formulas*:

(a) *If* v *is an infinite place, then* $b_v^n(\alpha, y)$ *is nonzero only if*

- ny *is negative at place* v *and positive at other infinite places,*
- $\varepsilon_\wp((n - \alpha)n) = 1$ *for every place* $\wp$ *of* D_E.

In this case, $b_v^n(\alpha, y)$ *is equal to*

$$r(nyD_F/N) q(4\pi |ny_v|)$$

where

$$q(t) = \int_1^\infty e^{-xt} \frac{dx}{x}, \qquad (t > 0).$$

(b) *If* v *is a finite place ramified in* E, *then* $b_v^n(\alpha, y)$ *is nonzero only if*

- ny *is totally positive,*
- $\varepsilon_v((n - \alpha)n) = -1$ *but* $\varepsilon_\wp((n - \alpha)n) = 1$ *for every place* $\wp$ *of* D_E.

In this case, $b_v^n(\alpha, y)$ *is equal to*

$$-r(nyD_F/N) \log |n_v y_v \pi_v / \pi_{N,v}|.$$

(c) *If* v *is a finite place inert in* E, *then* $b_v^n(\alpha, y) \neq 0$ *only if*

- ny *is totally positive,*
- $\varepsilon_\wp((n - \alpha)n) = 1$ *for every place* $\wp$ *of* D_E,
- $\mathrm{ord}_v(nyD_F/N)$ *is odd.*

In this case, $b_v^n(\alpha, y)$ *is equal to*

$$-r(nyD_F/(N\wp_v))\log|n_v y_v D_{F,v}\wp_v|$$

where $\wp_v$ *is the prime corresponding to* v.

(d) *If* v *is a finite place split in* E, *then* $b_v^n(\alpha, y) = 0$.

2. *If* $n = 0$, $\alpha = 1$, *then* $b^n(\alpha, y)$ *is nonzero only if* y *is totally positive. In this case, it is equal to*

$$r(yD_F)\psi(iy_\infty)|y|(c_1 + c_2 \log|y|)$$

where c_1 *and* c_2 *are constants.*

3. *If* $n = \alpha = 0$, *then* $b^n(\alpha, y)$ *is equal to*

$$|y|(c_3 + c_4 \log|y|)$$

where c_3 *and* c_4 *are constants.*

4. *If* $n = 1$, $\alpha = 1$, *then* $b^n(\alpha, y)$ *is equal to*

$$|y|\psi(iy_\infty)\left[c_5 r(yD_F/N)\log|\pi_{D_E} y_{D_E}| + c_6 r'(yD_F/N)\right]$$

where c_5, c_6 *are constants, and for a nonzero integral ideal* m,

$$r'(m) = \sum_{n|m} \varepsilon(n)\log \mathrm{N}(n).$$

Proof. The vanishing of Φ_s at $1/2$ follows from Theorem 6.3.3. To compute the Fourier coefficients of Φ' we use formulas in Proposition 6.2.4 with

$$b^n(\alpha, y) = \frac{\partial}{\partial s} a_s^n(\alpha, y)\Big|_{s=1/2}.$$

Notice that a_s^n vanishes at $s = 1/2$. This can be checked from its expression, or from formula (6.2.4) and Proposition 6.3.1.

The case where $n \neq 0$, $n \neq \alpha$. In this case, a_s^n is a product of

$$y^{3/2-s}\delta(ny)\sigma_{1/2}((\alpha - n)y) \cdot \prod_v \sigma_{s,v}^n(\alpha, y/\pi_N)$$

where v runs through the set of all places of F, and

$$\sigma_{s,v}^n(\alpha, y) = \begin{cases} \frac{1+\varepsilon((n-\alpha)n)|n_v y_v \pi_v|^{2s-1}}{2} & \text{if } v|D_E \\ V_s(n_v y_v) & \text{if } v \mid \infty \\ \frac{1-\varepsilon(n_v y_v \delta_v \pi_v)|n_v y_v \delta_v \pi_v|^{2s-1}}{1-\varepsilon(\pi_v)|\pi_v|^{2s-1}} & \text{otherwise.} \end{cases}$$

If $b^n(\alpha, y) \neq 0$, then $\sigma_{1/2}((\alpha - n)y) \neq 0$ and one and only one factor of $\sigma_{s,v}$ vanishes at $s = \frac{1}{2}$. If this is the case, then

$$\sigma_{1/2}((\alpha - n)y) = r((\alpha - n)yD_F) \prod_{v|\infty} V_{1/2}((\alpha_v - n_v)y_v).$$

By Proposition 3.3 in [20, p. 278], we know that $V_{1/2}(t)$ for $t \in \mathbb{R}$, is given by

$$V_{1/2}(t) = \begin{cases} 0 & \text{if } t < 0 \\ -2\pi i e^{-2\pi t} & \text{if } t > 0. \end{cases}$$

Thus, if $b^n(\alpha, y) \neq 0$, then $(\alpha - n)y$ must be totally positive and $r((\alpha - n)y) \neq 0$. In this case b^n has the expression in the proposition with $b^n_v(\alpha, y)$ equal to

$$\psi(-iny_\infty) \frac{\partial}{\partial s} \sigma^n_{v,s}(\alpha, y)\Big|_{s=1/2} \cdot \prod_{w \neq v} \sigma^n_{w,1/2}(\alpha, y).$$

The proposition in this case can be checked case by case. Notice that when v is archimedean, we have used the identity

$$\frac{\partial}{\partial s} V_s(t)\Big|_{s=1/2} = -2\pi i q(t) e^{-2\pi i t} \qquad (t < 0)$$

in Proposition 3.3 in [20].

The cases where $n = 0$ or $n = \alpha$. These cases can be verified from the expressions in Proposition 6.2.4. □

The same proof will also give the following:

PROPOSITION 6.3.5. *Assume that $\varepsilon(N) = (-1)^g$. Then the Fourier coefficient $c_{1/2}(\alpha, y)$ $(\alpha = 0, 1)$ of $\Phi_{1/2}$ is nonzero only if $\alpha y D_F$ is integral. In this case it is given by*

$$c_{1/2}(\alpha, y) = \frac{\varepsilon(N) d_F^{1/2}}{d_E d_F} \sum_{n \in F} a^n_{1/2}(\alpha, y)$$

where $a^n_{1/2}(\alpha, y)$ is given by the following formulas if it is nonzero:

1. *If $n \neq 0$ and $n \neq \alpha$, then $a^n_{1/2}(\alpha, y) \neq 0$ only if the following holds:*

   (a) *$nyD_E D_F N^{-1}$ is integral,*

   (b) *both $(\alpha - n)y$ and ny are totally positive,*

   (c) *$\varepsilon_v(n(n-1)) = 1$ for all $v \mid D_E$.*

   In this case $a^n_{1/2}(\alpha, y)$ is equal to

   $$(-4\pi^2)^g |y| \delta(ny) r((\alpha - n)yD_F) r(nyD_F/N) \psi(i\alpha y_\infty)$$

   where for an idele y, $\delta(y) = 2^{\#\{v|D_E, \mathrm{ord}_v(y) \geq 0\}}$.

2. *If* $n = 0$, $\alpha = 1$, *then* $a^n_{1/2}(\alpha, y)$ *is equal to*

$$c_1 r(yD_F)|y|\psi(iy_\infty)$$

where c_1 *is a constant independent of* y.

3. *If* $n = \alpha = 0$, *then* $a^n_{1/2}(\alpha, y)$ *is equal to*

$$c_2|y|$$

where c_2 *is a constant independent of* y.

4. *If* $n = 1$, $\alpha = 1$, *then* $a^n_{1/2}(\alpha, y)$ *is equal to*

$$c_3 r(yD_F/N)|y|\psi(iy_\infty)$$

where c_3 *is a constant independent of* y.

6.4. *Holomorphic projections.*

6.4.1. *Asymptotic formula for* Φ' *near cusps.* Assume that $\varepsilon(N) = (-1)^{g-1}$. The form Φ' defined in Proposition 6.3.4 is not holomorphic. We want to find a *holomorphic projection* Φ which is a holomorphic cusp form for $K_0(N)$ and has the property that for any newform f, f has the same scalar product with both $\widetilde{\Phi}$ and Φ.

As in the case $F = \mathbb{Q}$ treated by Gross and Zagier [20, Chap. IV, §6], Φ' satisfies the growth condition

$$\Phi'\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g\right) = a_g|y|\log|y| + b_g|y| + O_g(|y|^{1-\varepsilon}) \tag{6.4.1}$$

for each $g \in \mathrm{GL}(\mathbb{A}_F)$. By Proposition 6.3.4, the asymptotic formula (6.4.1) is true for $g = 1$, and

$$\Phi'\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g\right) = c_3|y|\log|y| + c_4|y| + O(|y|^{1-\varepsilon})$$

where c_3 and c_4 are constants independent of g as defined in Proposition 6.3.4.

For any $e|N$, let g_e denote an element in $\mathrm{GL}_2(\mathbb{A}_F)$ which has components 1 at places not dividing N and has components $\begin{pmatrix} 1 & 0 \\ \pi_v^{\mathrm{ord}_v(e)} & 1 \end{pmatrix}$ at each place v dividing N. Using the same method, we may compute the Fourier coefficients of $\Phi'(gg_e)$ and will obtain the formula similar to (6.4.1). As $\mathrm{GL}_2(\mathbb{A}_F)$ is a union of the form $B(\mathbb{A})g_eK_0(N)$, formula (6.4.1) is true for every $g \in \mathrm{GL}_2(\mathbb{A}_F)$.

We have to subtract some Eisenstein series to make $a_g = b_g = 0$ for every $g \in \mathrm{GL}_2(\mathbb{A}_F)$. Let $E_{2,s}(g)$ be the Eisenstein series constructed in 3.5.1 with

$k = 2$, $\chi = 1$, and $N = 1$. Then $E_{2,s}(g)$ is perpendicular to all holomorphic cusp forms. Proposition 3.5.2 implies the following asymptotic formula:

$$E_{2,s}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \zeta_F(2s)|y|^s + c(s)|y|^{1-s} + O(1) \tag{6.4.2}$$

as $y \to \infty$, where $c(s)$ and $O(1)$ are holomorphic in s near $s = 1$. Define by continuation

$$E(g) = E_{2,s}(g)|_{s=1} \quad \text{and} \quad F(g) = \frac{\partial}{\partial s}\Big|_{s=1} E_{2,s}(g).$$

For each $e \mid N$, let h_e be an element of $\mathrm{GL}_2(\mathbb{A}_F)$ which has components 1 at places not dividing N and has components $\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{\mathrm{ord}_v(e)} \end{pmatrix}$ at places v dividing N.

Lemma 6.4.2. *There are some pairs of numbers (α_e, β_e) $(e \mid N)$ such that the form*

$$\widetilde{\Phi}(g) := \Phi'(g) - \sum_e \left[\alpha_e F(gh_e) + \beta_e E(gh_e)\right]$$

has the same holomorphic projection as Φ', and $\widetilde{\Phi}$ satisfies the bound

$$\widetilde{\Phi}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g\right) = O(|y|^{1-\varepsilon})$$

as $y \to \infty$, for every $g \in \mathrm{GL}_2(\mathbb{A}_F)$.

Proof. We need only find (α_e, β_e)'s so that the equation in the lemma holds for $g = g_f$'s, as $\mathrm{GL}_2(\mathbb{A}_F)$ is a union of $B(\mathbb{A}_F)\gamma_e K_0(N)$. Now $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g_f h_e$ has the decomposition at a place v of N:

$$\begin{cases} \begin{pmatrix} y_v & x_v\pi_v^{m_v} \\ 0 & \pi_v^{m_v} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \pi_v^{n_v - m_v} & 1 \end{pmatrix} & \text{if } n_v \geq m_v \\ \begin{pmatrix} y_v\pi_v^{m_v-n_v} & y_v + x_v\pi_v^{n_v} \\ 0 & \pi_v^{n_v} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & \pi_v^{m_v-n_v} \end{pmatrix} & \text{if } m_v > n_v \end{cases}$$

where $m_v = \mathrm{ord}_v(e)$, $n_v = \mathrm{ord}_v(f)$. Thus (6.4.2) implies

$$E_{2,s}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g_f h_e\right) = C_N(e,f)^s \zeta_F(2s)|y|^s + c(s)C_N(e,f)^{1-s}|y|^{1-s} + O(1)$$

where $C_N(e,f) = \mathrm{N}(e,f)^2/\mathrm{N}(e)$. It follows that,

$$E\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g_f h_e\right) = \zeta_F(2)C_N(e,f)|y| + O(1),$$

$$F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} g_f h_e\right) = \zeta_F(2)C_N(e,f)|y|\log|y| + O(\log|y|).$$

Now the asymptotic formula in the lemma is equivalent to

$$\begin{aligned}\sum_{e|N}\alpha_e C_N(e,f) &= \zeta_F(2)^{-1}a_{g_f},\\ \sum_{e|N}\beta_e C_N(e,f) &= \zeta_F(2)^{-1}b_{g_f},\end{aligned}$$

for all $f \mid N$, where a_{g_f} and b_{g_f} are constants in (6.4.1). Thus it suffices to show that the matrix $C_N = (C_N(e,f))_{e,f|N}$ is invertible. It is easy to see that C_N is multiplicative for coprime N's in the sense of tensor products. Thus it suffices to prove that C_N is invertible for $N = \wp^n$ to be a power of a prime. In this case, $C_{\wp^n}$ has determinant $\pm(\mathrm{N}(\wp)^2-1)^n$. This completes the proof of the lemma. □

Lemma 6.4.3. *Let $\widetilde{\phi}$ be a form which has growth $O(y^{1-\varepsilon})$ near each cusp. Let $\widetilde{c}(y)$ denote the Whittaker function at $\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}$ of $\widetilde{\phi}$:*

$$\widetilde{c}(y) = d_F^{-1/2}\int_{\mathbb{A}_F/F}\widetilde{\phi}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)\psi(-x)dx.$$

Then the Fourier coefficient of the holomorphic projection ϕ of ϕ^* is given by

$$a(m) = (4\pi)^g \lim_{s\to 1}\int_{\mathbb{R}_+^g} |t|^{-1}\widetilde{c}(ty_\infty)\psi(iy_\infty)|y_\infty|^{s-2}dy_\infty$$

where t is a generator of mD_F^{-1} in $\widehat{F}^\times$.

Proof. For m a nonzero ideal of $\mathcal{O}_F$, let $P_{m,s}(g)$ be the m^{th} Poincaré series defined by

$$P_{m,s}(g) = \sum_{\gamma\in Z(F)U(F)\backslash \mathrm{GL}_2(F)} H_m(\gamma g)$$

where U denotes the algebraic group of matrices $\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}$ and $H_{m,s}$ denotes a function on $Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ such that for $y\in\mathbb{A}_F^\times, x\in\mathbb{A}_F, r(\theta)k\in K$,

$$H_{m,s}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix} r(\theta)k\right) = |y|^s\psi(2\theta + x + iy_\infty)$$

if $y_\infty > 0$, $k\in K_0(N)$, and $y_f D_F = m$; otherwise, it is zero. Then the Petersson product $(\widetilde{\phi}, P_{m,\bar{s}})$ is equal to

$$\int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \widetilde{\phi}\overline{P_{m,\bar{s}}}dg = \int_{Z(\mathbb{A}_F)U(F)\backslash G(\mathbb{A}_F)} \widetilde{\phi}\overline{H_{m,\bar{s}}}dg$$
$$= \int_{\mathbb{A}_F^\times}\int_{\mathbb{A}_F/F}\int_K (\widetilde{\phi}\overline{H_{m,s}})\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}k\right)dkdx\frac{d^\times y}{|y|}$$
$$= \mu(N)\int_{y_\infty\in\mathbb{R}_+^g}\int_{t\widehat{\mathcal{O}}_F^\times}\int_{\mathbb{A}_F/F}\widetilde{\phi}\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)|y|^{s-1}\psi(-x+iy_\infty)dxd^\times y$$
$$= \mu(N)d_F^{1/2}\int_{\mathbb{R}_+^g}\int_{t\widehat{\mathcal{O}}_F^\times}\widetilde{c}(y)\psi(iy_\infty)|y|^{s-1}d^\times y.$$

Thus we have

$$(\widetilde{\phi}, P_{m,\bar{s}}) = |t|^{s-1}\mu(N)d_F^{1/2}\int_{\mathbb{R}_+^g}\widetilde{c}(ty_\infty)\psi(iy_\infty)|y_\infty|^{s-2}dy. \tag{6.4.3}$$

If we replace $\widetilde{\phi}$ by ϕ with the Whittaker function

$$c(y) = |y|a(y_f D_F)\psi(iy_\infty),$$

then

$$(\phi, P_{m,s}) = |t|^s\mu(N)d_F^{1/2}\left[\frac{\Gamma(s)}{(4\pi)^s}\right]^g a(m). \tag{6.4.4}$$

As $P_m = \lim_{s\to 0} P_{m,s}$ is a holomorphic form,

$$(\phi, P_m) = (\widetilde{\phi}, P_m). \tag{6.4.5}$$

The lemma follows from (6.4.3)–(6.4.5). □

We want to apply this lemma to $\widetilde{\Phi}$.

LEMMA 6.4.4. *Let $a(m)$ be the Fourier coefficient of the holomorphic projection of Φ'. Then for m prime to ND_E,*

$$a(m) \pmod{\mathcal{D}_N} = (4\pi)^g\lim_{s\to 1}\int_{\mathbb{R}_+^g}|t|^{-1}c'(1,ty_\infty)\psi(iy_\infty)|y_\infty|^{s-2}dy_\infty$$

where t is a generator of $\widehat{m}\widehat{D}_F^{-1}$ in $\widehat{F}^\times$.

Proof. Let $a(y)$, $b(y)$, and $\widetilde{c}(y)$ be Fourier coefficients of $E(g)$, $F(g)$, and $\widetilde{\Phi}(g)$; then

$$\widetilde{c}(y) = c'(1,y) - \alpha_1 a(y) - \beta_1 b(y)$$

where $c'(1,y)$ is the Whittaker function of Φ', and α_1, β_1 are constants as in Lemma 6.4.2. By Lemma 6.4.3, $a(m)$ is equal to

$$(4\pi)^g\lim_{s\to 0}\int_{(\mathbb{R}^+)^g}|t|^{-1}c'(1,ty)\psi(iy_\infty)|y|^{s-2}dy - \alpha_1 c_s(m) - \beta_1 b_s(m) \tag{6.4.6}$$

where

$$b_s(m) = (4\pi)^g \int_{(\mathbb{R}^+)^g} |t|^{-1} b(ty)\psi(iy_\infty)|y_\infty|^{s-2} dy_\infty$$

and

$$c_s(m) = (4\pi)^g \int_{(\mathbb{R}^+)^g} |t|^{-1} c(ty_\infty)\psi(iy_\infty)|y_\infty|^{s-2} dy_\infty.$$

Write $\sigma_s(m) = \sum_{a|m} \mathrm{N}(a)^s$ and $\sigma'(m) = \frac{\partial}{\partial s}\Big|_{s=1} \sigma_s(m)$. One can show from the Fourier expansion of $E_{2,s}$ that

$$b_s(m) = k_1\sigma_1(m) + o(s-1),$$

and

$$c_s(m) = k_2\sigma_1(m) + k_3\sigma'(m) + \frac{k_4}{s-1} + k_5 + o(s-1).$$

Here the k_i's are constants independent of m. Thus $c_s(m)$, $b_s(m)$ only contribute elements in $\mathcal{D}_N$ in (6.4.6). The lemma follows. □

Applying the formula for Fourier coefficients of $\widetilde{\Phi}$, we obtain the following:

Proposition 6.4.5. *Let $a(m)$ be the Fourier coefficients for the holomorphic projection Φ of Φ'. Then for m prime to ND_E,*

$$a(m) \pmod{\mathcal{D}_N} = -\frac{(2\pi)^{2g} d_N^{1/2}}{d_E d_F} \sum_v a_v(m)$$

where $\mathrm{N}(v) = 1$ if v is archimedean and $a_v(m)$ is given by the following formulas:

1. *If $v|\infty$, then $a_v(m)$ is equal to the constant term in the Taylor expansion in $s-1$ of*

$$\sum_{\substack{n\in Nm^{-1}D_E{}^{-1}, n_v<0 \\ 0<n_w<1 \forall v\neq w|\infty \\ \varepsilon_w(n(n-1))=1 \forall w|D_E}} \delta(n) r((1-n)m) r(nmD_E/N) p_s(|n_v|)$$

where the sum is over the set of places of F,

$$\delta(n) = 2^{\#\{v|D_E, \mathrm{ord}_v(n)\geq 0\}},$$

and

$$p(s,t) = \int_1^\infty (1+tx)^{-s} \frac{dx}{x}, \qquad (t>0).$$

2. *If* $v = \wp \not| \infty, \varepsilon(v) = 0$, $a_v(m)$ *is equal to*

$$\sum_{\substack{n \in Nm^{-1}D_E{}^{-1} \\ \varepsilon_v((n-1)n)=1 \\ \varepsilon_w((n-1)n)=1 \forall v \neq w | D_E \\ 0<n<1}} \delta(n) r((1-n)m) r(nm/N) \mathrm{ord}_v(nm\wp) \log \mathrm{N}(v).$$

3. *If* $v = \wp \not| \infty, \varepsilon(v) = -1$, $a_v(m)$ *is equal to*

$$\sum_{\substack{n \in Nm^{-1}D_E{}^{-1} \\ \varepsilon_w((n-1)n)=1 \forall v | D_E \\ 0<n<1}} \delta(n) r((1-n)m) r(nm/N\wp) \mathrm{ord}_v(nm\wp/N) \log \mathrm{N}(v).$$

4. *If* $v \not| \infty, \varepsilon(v) = 1$,

$$a_v(m) = 0.$$

Proof. By Proposition 6.3.4 and 6.4.4, the Fourier coefficients $a(m) \pmod{\mathcal{D}_N}$ are given by

$$\frac{\varepsilon(N) d_N^{1/2}}{d_E d_F} (4\pi)^g \lim_{s \to 1} \sum_{n \in F} b_s^n(m) \tag{6.4.7}$$

where

$$b_s^n(m) = \int_{\mathbb{R}_+^g} |t|^{-1} b^n(1, ty_\infty) \psi(iy_\infty) |y_\infty|^{s-2} dy_\infty.$$

From formulas of $b^n(1, ty_\infty)$ one can show that if $n = 0$ or 1, $b_s^n(m)$ is a linear combination of a multiple of $r(m)$ and its derivatives. Thus, modulo $\mathcal{D}_N$, we may assume that $n \neq 0, 1$ in (6.4.7). Moreover $b_s^n(m) \neq 0$ only if $nD_E mN^{-1}$ is integral and $1 - n$ is totally positive. In this case $b_s^n(m)$ is equal to

$$(-4\pi^2)^g \delta(n) r((1-n)m) \sum_v b_{v,s}^n(m) \tag{6.4.8}$$

where v runs through all places of F, and

$$b_{v,s}^n(m) = \int_{\mathbb{R}_+^g} b_v^n(1, ty_\infty) \psi(2iy_\infty) |y_\infty|^{s-1} dy_\infty.$$

Now we compute $b_{v,s}^n(m)$ case by case using Proposition 6.3.4.

Case 1. $v \mid \infty$. In this case, $b_{v,s}^n(m)$ is nonzero only if

- n is negative at place v, but positive at other infinite places,
- $\varepsilon_\wp((n-1)n) = 1$ for every place $\wp$ of D_E.

In this case, $b^n_{v,s}(m)$ is equal to

$$r(nm/N)\int_{\mathbb{R}^g_+} q(4\pi|ny_v|)\psi(2iy_\infty)|y_\infty|^{s-1}dy_\infty = r(nm/N)\left[\frac{\Gamma(s)}{(4\pi)^s}\right]^g p(s,|n_v|). \tag{6.4.9}$$

Case 2. $v\nmid\infty, \varepsilon(v)=0$. In this case, $b^n_{v,s}(m)$ is nonzero only if

- n is totally positive,
- $\varepsilon_v((n-1)n)=-1$ but $\varepsilon_\wp((n-1)n)=1$ for every place $\wp$ of D_E.

In this case, $b^n_{v,s}(m)$ is equal to

$$r(nm/N)\mathrm{ord}_\wp(nm\wp)\log \mathrm{N}(\wp)\left[\frac{\Gamma(s)}{(4\pi)^s}\right]^g. \tag{6.4.10}$$

Case 3. $v\nmid\infty, \varepsilon(v)=-1$. In this case, $b^n_{v,s}(m)\neq 0$ only if

- n is totally positive,
- $\varepsilon_\wp((n-1)n)=1$ for every place $\wp$ of D_E.
- $\mathrm{ord}_v(nm/N)$ is odd.

In this case, $b^n_{v,s}(m)$ is equal to

$$r(nm/(N\wp_v))\mathrm{ord}_\wp(nm\wp/N)\log \mathrm{N}(\wp)\left[\frac{\Gamma(s)}{(4\pi)^s}\right]^g. \tag{6.4.11}$$

Case 4. v is a finite place split in E. In this case,

$$b^n_{v,s}(m)=0. \tag{6.4.12}$$

The proposition follows from (6.4.7)–(6.4.12) with

$$a_v(m)=-(4\pi)^g\lim_{s\to 1}\sum_{\substack{n\in F\\ n\neq 0,1}} b^n_{v,s}(m).$$ □

The same proof will also give the following:

Proposition 6.4.6. *Assume that $\varepsilon(N)=(-1)^g$. Let $b(m)$ denote the Fourier coefficient of the holomorphic projection of $\Phi_{1/2}$. Then for m prime to ND_E,*

$$b(m)\pmod{\mathcal{D}_N}=\frac{(2\pi)^{2g}d_F^{1/2}}{d_Ed_F}\sum_{\substack{n\in ND_E^{-1}m^{-1}\\ 0<n<1\\ \varepsilon_v(n(n-1))=1,\forall v|D_E}}\delta(n)r((1-n)m)r(nm/N).$$

7. Proof of the main theorems

In this section we will finish the proofs of the theorems stated in the introduction. We need only prove Theorem C and A.

7.1. *Proof of Theorem* C. Recall that Φ is the holomorphic form of weight 2 for $K_0(N)$ with trivial character as constructed in 6.4.1, which is the holomorphic projection of $\frac{\partial}{\partial s}\Phi_s\Big|_{s=1/2}$ where Φ_s is a form constructed as in 6.1.5. By Corollary 6.1.6, we thus have

$$(f, \Phi) = B(1/2)L'_E(f, 1) \tag{7.1.1}$$

where

$$B(1/2) = 2^{-g} d_F d_N^{1/2} d_E^{-1/2} \mu(N).$$

Recall also that we have constructed a form Ψ in 4.1.3 whose Fourier coefficients are height pairings of CM-points $\langle z, \mathrm{T}(m)z\rangle$, where z is the class of η in the Jacobian of X and the pairing here is the Neron-Tate height pairing.

The proof of Theorem C will be easily reduced to the following:

Proposition 7.1.1. *With the notation of* 4.4.4,

$$\widetilde{\Phi} = \frac{(2\pi)^{2g} d_N^{1/2}}{d_E d_F} \widetilde{\Psi} \pmod{\mathcal{D}_N}.$$

Proof. We need to show that both sides have the same value for all $m \in \mathbb{N}_F$ prime to ND_E, modulo $\mathcal{D}_N$. By Proposition 4.4.5 and 4.5.3, modulo $\mathcal{D}_N$, $\widetilde{\Psi}(m)$ is equal to the sum of $-(\eta, \mathrm{T}^0(m)\eta)_v$. On the other hand, we have studied the Fourier coefficients $a(m) \pmod{\mathcal{D}_N}$ in Proposition 6.4.5 by decomposing it into a sum of local terms $a_v(m)$. Now we only need to show that

$$\sum_v (\eta, \mathrm{T}^0(m)\eta)_v = \sum_v a_v(m) \pmod{\mathcal{D}_N}.$$

We will prove this by splitting the sum according to types of v. More precisely we want to show

$$\sum_{v\in S} (\eta, \mathrm{T}^0(m)\eta)_v = \sum_{v\in S} a_v(m) \pmod{\mathcal{D}_N}, \tag{7.1.2}$$

where S is one of the following:

1. S_∞: the set of infinite places;
2. S_0: the set of finite places ramified in E;
3. S_1: the set of finite places split in E;
4. S_{-1}: the set of finite places inert in E.

The case of archimedean places. Since S_∞ is finite, we need only show the individual identity

$$(\eta, \mathrm{T}^0(m)\eta)_v = a_v(m) \pmod{\mathcal{D}_N}.$$

In view of Proposition 5.4.4 and 6.5.4, we need only show that the quantity

$$E(s) = \sum_{\substack{n\in Nm^{-1}D_E{}^{-1}, n_v<0\\ 0<n_w<1 \forall v\neq w|\infty \\ \varepsilon_w(n(n-1))=1\forall w|D_E}} \delta(n)r((1-n)m)r(nmD_E/N)\varepsilon_s(|n_v|)$$

has limit 0 as $s \to 1$, where

$$\varepsilon_s(t) = p_s(t) - 2Q_{s-1}(1+2t) \qquad (t>0).$$

One can show that

$$\varepsilon_1(t) = 0, \qquad \varepsilon_s(t) = O(t^{-1-s})$$

as $t \to \infty$. Thus $E(s)$ is absolutely convergent for $\mathrm{Re}(s) > 0$, and has limit 0 as $s \to 1$.

The case of ramified places. Again S_0 is finite, we need only show the individual identity

$$(\eta, \mathrm{T}^0(m)\eta)_v = a_v(m) \pmod{\mathcal{D}_N}.$$

This follows directly from Proposition 5.4.8 and (6.4.5).

The case of split places. In this case S_1 is not finite. But by Proposition 5.4.8, the sum

$$\sum_{v\in S_1}(\eta, \mathrm{T}^0(m)\eta)_v = r(m)\sum_{v\in S}\mathrm{ord}_v(m)\log \mathrm{N}(v)$$

has only finitely many nonzero terms and defines an element in $\mathcal{D}_N$. On the other hand $\sum_{v\in S_1} a_v(m) = 0$.

The case of inert places. Again by Proposition 5.4.8, the left-hand side of (7.1.2) is equal to

$$\sum_{\wp\in S_{-1}}(U_\wp(m) - U_\wp(1)R(m))\log \mathrm{N}(\wp) \pmod{\mathcal{D}_N}. \tag{7.1.3}$$

We need to compare $(U_\wp(m) - U_\wp(1)R(m))\log \mathrm{N}(\wp)$ with $a_\wp(m)$. For $\ell \in \mathbb{N}_F$ prime to $\wp$ define

$$k_\wp(\ell) = \sum_{\substack{n\in \ell^{-1}D_E{}^{-1}N\\ 0<n<1\\ \varepsilon_q(n(n-1))=1, \forall q|D_E}} r(n\ell N^{-1}/\wp)r((n-1)\ell)\delta(n),$$

$$k'(\ell) = \sum_{\substack{n\in\ell^{-1}D_E^{-1}N \\ 0<n<1 \\ \varepsilon_q(n(n-1))=1,\forall q|D_E}} r(n\ell N^{-1})r((n-1)\ell/\wp)\delta(n).$$

LEMMA 7.1.2. *Let* $m' = m\wp^{-\mathrm{ord}_\wp(m)}$.

1. *If* $\mathrm{ord}_\wp(m)$ *is even, then*

$$U_\wp(m)\log \mathrm{N}(\wp) - a_\wp(m) = 0.$$

2. *If* $\mathrm{ord}_\wp(m)$ *is odd, then*

$$\begin{aligned} U_\wp(m) &= \mathrm{ord}_\wp(m\wp)k_\wp(m'), \\ a_\wp(m) &= \mathrm{ord}_\wp(m\wp)\log \mathrm{N}(\wp)k'_\wp(m'). \end{aligned}$$

Proof. If $\mathrm{ord}_\wp(m)$ is even, then the only nonzero terms in $a_\wp(m)$ are for those n which lie in $m'^{-1}D_E{}^{-1}N$, where $m' = m\wp^{-\mathrm{ord}_\wp(m)}$. This is clear if $\wp \nmid m$. Otherwise, $\wp \nmid N$ and then $r(nm/N\wp) \neq 0$ will imply that $\mathrm{ord}_\wp(n)$ is odd. But then $r((n-1)m) \neq 0$ will imply that $\mathrm{ord}_\wp(n)$ is nonnegative. Thus $U_\wp(m)\log \mathrm{N}(\wp) = a_\wp(m)$.

If $\mathrm{ord}_\wp(m)$ is odd, then the only nonzero terms in $a_\wp(m)$ are for those n which have zero order at $\wp$. Indeed, $r(nm/N\wp) \neq 0$ implies $\mathrm{ord}_\wp(n)$ is even, but $r((1-n)m) \neq 0$ implies $\mathrm{ord}_\wp(n) = 0$. Actually, $\mathrm{ord}_\wp(1-n)$ is positive and even. Thus

$$a_\wp(m) = \sum_{\substack{n\in m'^{-1}D_E{}^{-1}N \\ 0<n'<1 \\ \varepsilon_q(n(n-1))=1,\forall q|D_E}} r(nm'N^{-1})r((n-1)m'/\wp)\delta(n)\mathrm{ord}_\wp(m\wp). \quad \square$$

LEMMA 7.1.3. *Let* $\ell \in \mathbb{N}_F$ *be prime to* $\wp$. *Then*

$$k(\ell) - k(1) = k'(\ell) - k'(1).$$

Proof. From the proof of Proposition 5.4.8, it is not difficult to see that $k_\wp(\ell) - k_\wp(1)$ is the local intersection of η and $\mathrm{T}^0(\ell)\eta$ over $\wp$ without counting multiplicities. See formula (5.4.10) with $m = \ell$ and $m(n') = 1$. But this does not give a description for $k'_\wp(\ell) - k'_\wp(\ell)$. So we need to give a description in a different setting.

Let $R(\wp)$ be an order of $B(\wp)$ of type $(E, N(\wp))$ and consider the projection map

$$\pi : C = E^\times\backslash\widehat{B}(\wp)^\times/\widehat{R}(\wp)^\times \to S = B^\times\backslash\widehat{B}(\wp)^\times/\widehat{R}(\wp)^\times.$$

The set C can be considered as the set of CM-points, and the set S as supersingular points, the reduction of CM-points. We may define conductors for elements in C, and orientations for elements in C with conductor prime to $N\wp$ for each place dividing $N\wp$. The group

$$\mathcal{W} = \{b \in \widehat{B}(\wp)^\times : b^{-1}\widehat{R}(\wp)b\}/\widehat{R}(\wp)^\times$$

acting on C does not change reductions and conductors but is free and transitive on orientations. For each place v dividing $N\wp$, we call the orientation defined by 1 the positive orientation.

By a $\mathbb{Q}$-divisor in C we just mean an element in the free abelian group $\mathbb{Q}[C]$. For ℓ prime to $N(\wp)$ we can also define the Hecke operator $\mathrm{T}(\ell)$ on $\mathbb{Z}[C]$. Let $\eta(\wp)$ (resp. $\eta(\wp)'$) be the set of elements in the first set with trivial conductor and positive orientations at all places of $N\wp$ (resp. positive orientations at places dividing N but negative orientation at place $\wp$). Then $\eta(\wp)$ and $\eta(\wp)'$ have the exact same reduction because of the action of $\mathcal{W}$. Now, $k(\ell) - k(1)$ (resp. $k'(\ell) - k'(1)$) is the intersection number of $\eta(\wp)$ (resp. $\eta(\wp)'$) and $\mathrm{T}(\ell)^0(\eta(\wp))$ under the specialization map. Thus they are same since $\eta(\wp)$ and $\eta(\wp)'$ have the same reductions. □

We return to the proof of (7.1.2) for S_{-1}. By (7.1.3), the difference of two sides of (7.1.2) for S_{-1} is equal to

$$\sum_{\varepsilon(\wp)=-1} (U_\wp(m)\log \mathrm{N}(\wp) - a_\wp(m)) \quad - \sum_{\varepsilon(\wp)=-1} (U_\wp(1)\log \mathrm{N}(\wp) - a_\wp(1))r(m)$$
$$- \sum_{\varepsilon(\wp)=-1} a_\wp(1)r(m).$$

The first two terms vanish by the above two lemmas. The last sum is absolutely convergent, and thus defines an element in $\mathcal{D}_N$.

Corollary 7.1.4. *For any newform f for $K_0(N)$,*

$$L'_E(f,1) = \frac{(8\pi^2)^g}{d_F^2\sqrt{d_E}}[K_0(1) : K_0(N)](f,\Psi).$$

Proof. By Propositions 4.5.1 and 7.1.1,

$$\Phi = \frac{(2\pi)^{2g} d_N^{1/2}}{d_E d_F}\Psi + \text{an old form}.$$

Now the conclusion follows from formula (7.1.1). □

7.1.5. *Proof of Theorem* C. The ideal is exactly as in [20, p. 308]. By Lemma 3.4.5, we may decompose z in $\mathrm{Jac}(X)\otimes\mathbb{C}$ into eigenvectors z_ϕ with the same eigenvalues as ϕ under Hecke operators $\mathrm{T}(m)$ with m prime to N:

$$z = \sum_{\phi\in S_N} z_\phi, \qquad \mathrm{T}(m)z_\phi = a_\phi(m)z_\phi.$$

As Hecke operators on $\mathrm{Jac}(\mathbb{C}) \otimes \mathbb{C}$ are self-adjoint with respect to the Neron-Tate height pairing, one has the decomposition

$$\Psi = \sum_{\phi \in S_N} \langle z_\phi, z_\phi \rangle \phi.$$

Now Theorem C follows from this equality and Corollary 7.1.4.

7.2. *Proof of Theorem* A. By Theorems B and C, it suffices to prove the following generalization of a theorem of Kolyvagin:

PROPOSITION 7.2.1. *Assume that the Heegner point y_f in A is not a torsion point. Then*

- *$A(F)$ has rank given by*

$$\mathrm{rank} A(F) = [\mathcal{O}_f : \mathbb{Z}] \mathrm{ord}_{s=1} L(s, f),$$

- *Ш(A) is finite.*

In view of Kolyvagin's method for other cases ([17], [28], [29], [30]), we need only to construct certain Euler system of CM-points. We consider square-free elements $n \in \mathbb{N}_F$ which are square-free and prime to ND_E and such that every prime factor ℓ is inert in K. For every such n, we choose a CM-point x_n of the conductor n such that

$$x_n \text{ is included in } \mathrm{T}(\ell) x_m$$

if $n = m\ell$ with ℓ a prime. Then x_n is defined over E_n, the ring class field of the conductor n over E.

LEMMA 7.2.2. *If $n = m\ell$ as above, then*

$$u_n^{-1} \sum_{\sigma \in \mathrm{Gal}(E_n/E_m)} x_n^\sigma = u_m^{-1} \mathrm{T}(\ell) x_m,$$

where u_n is the cardinality of the group $\mathcal{O}_c^\times / \mathcal{O}_F^\times$.

Proof. By an argument similar to the proof of Proposition 4.2.1, one can show that $P := \frac{u_m}{u_n} \mathrm{T}(\ell) x_m$ is a divisor with integral coefficients. It follows that $P = Q$ because of the following facts:

- P includes the divisor $Q = \sum_{\sigma \in \mathrm{Gal}(E_n/E_m)} x_n^\sigma$;
- $\deg P = \deg Q$;
- Q is irreducible over E_m. □

As in the case $F = \mathbb{Q}$, this lemma implies that the collection of x_n forms an Euler system [28].

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