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Equidistribution of CM-Points on Quaternion Shimura Varieties

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1 Introduction

The aim of this paper is to show some equidistribution statements of Galois orbits of CM-points for quaternion Shimura varieties. These equidistribution statements will imply the Zariski densities of CM-points as predicted by André-Oort conjecture (see Section 2). Our main result (Corollary 3.7) says that the Galois orbits of CM-points with the maximal Mumford-Tate groups are equidistributed provided that some subconvexity bounds on Rankin-Selberg L-series and on torsions of the class groups. A proof of the subconvexity bound for L-series has been announced by Michel and Venkatesh.

Combining with some work of Cogdell, Michel, Piatetski-Shapiro, Sarnak, and Venkatesh, we obtain the following unconditional results about the equidistribution of CM-points in the following cases.

(1) *Full* CM-*orbits on quaternion Shimura varieties* (Theorem 3.1). This is a generalization of the work of Duke [11] for modular curves, Michel [20], and Harcos and Michel [16] for Shimura curves over $\mathbb{Q}$.

(2) *Galois orbits of* CM-*points with a fixed maximal Hodge-Tate group* (Corollary 3.8). Under our setting, this strengthens a result of Edixhoven and Yafaev [15] about the finiteness of CM-points on a curve with fixed $\mathbb{Q}$-Hodge structure.

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The maximality condition of the Hodge-Tate group automatically holds in dimension-one case (Proposition 7.2), and can be classified in dimension-2 case (Proposition 7.3). In higher-dimension case, we will give many examples of Shimura varieties where maximality condition holds (Propositions 7.4, 7.5, and 7.7).

The proofs of these results have two parts. In the first part (Sections 4-5), we will give an estimate on probability measures on CM-suborbits (Theorem 3.2) which follows from the central value formulas proved in our previous paper and Waldspurger's paper, the study of Hecke orbits of CM-points, and analysis of the spectral decomposition. In the second part (Sections 6-7), we will study the Mumford-Tate group of CM-points and estimate the size of Galois orbits in terms of discriminants of the torsion in class groups.

2 Conjectures

In this section, we will introduce the André-Oort conjecture and the equidistribution conjecture. For background on Shimura varieties, we refer to Deligne [9, 10]. For the André-Oort conjecture, we refer to Moonen [23] and Edixhoven [12]. Notice that questions about the equidistribution of CM-points have previously been addressed by Clozel and Ullmo in [5]. See also Noot [24] for a detailed survey of recent progress.

Let M be a connected Shimura variety defined over a number field E in $\mathbb{C}$. Then for any Shimura subvariety Z, the set of CM-points on Z is Zariski dense. The André-Oort conjecture says that the converse is true.

Conjecture 2.1 (André-Oort conjecture, see [1, 12, 23, 25])**.** Let Z be a connected subvariety of M which contains a Zariski dense subset of CM-points. Then Z is a Shimura subvariety. □

Let us recall the description of Shimura subvarieties. Assume that M is a connected component of a Shimura variety M_U of the form

$$M_U(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\widehat{\mathbb{Q}})/U, \quad M = \Gamma\backslash X, \tag{2.1}$$

where

(i) G is an algebraic group over $\mathbb{Q}$ of adjoint type,

(ii) X is a $G(\mathbb{R})$-conjugacy class of embeddings

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \longrightarrow G_{\mathbb{R}} \tag{2.2}$$

of algebraic groups over $\mathbb{R}$,

(iii) U is an open and compact subgroup of $G(\widehat{\mathbb{Q}})$, $\Gamma = G(\mathbb{Q}) \cap U$.

For each point $x \in M_U(\mathbb{C})$, the minimal (connected) Shimura subvariety containing x can be defined as follows. Let $(h, g) \in X \times G(\widehat{\mathbb{Q}})$ represent x. Let H denote the Zariski closure of $h(U(\mathbb{C}))$ in G as an algebraic subgroup over $\mathbb{Q}$ which is called the Mumford-Tate group of x. The Hodge closure $M_{U,x}$ of x in M_U is defined to be the subvariety of M represented by $H(\mathbb{R})\widetilde{x} \times H(\widehat{\mathbb{Q}})g$. The minimal Shimura subvariety is the connected component of $M_{U,x}$ containing x which has the form

$$M_x := \Gamma' \backslash X', \quad \Gamma' := H(\mathbb{Q}) \cap gUg^{-1},\ X' := H(\mathbb{R})h. \tag{2.3}$$

A point $x \in M_U(\mathbb{C})$ is a CM-point if and only if M_x is 0-dimensional or, equivalently, H is a torus.

Remarks. (1) This conjecture remains open, although many special cases have been treated by Moonen [21, 22, 23], Edixhoven [12, 13, 14], Edixhoven and Yafaev [15], and Yafaev [31, 32]. In particular, the conjecture is true when Z is a curve under one of the following assumptions:

(i) M is a product of two modular curves (see André [2]);

(ii) CM-points on Z are in a single Hecke orbit (see Edixhoven and Yafaev [15]);

(iii) GRH for CM-fields (see Yafaev [32]).

(2) This conjecture is analogous to the Manin-Mumford conjecture proved by Raynaud [26] about torsion points in abelian variety. The Manin-Mumford conjecture is also a consequence of the equidistribution conjecture proved using Arakelov theory (see Szpiro et al. [27], Ullmo [28], Zhang [33]).

In this paper, we want to study the distribution property of CM-points. We want to propose the following conjecture about distributions of CM-points.

Conjecture 2.2 (equidistribution conjecture). Let x_n be a sequence of CM-points on M. Assume that for any proper Shimura subvariety Z, there are only finitely many points in x_n contained in Z. Then the Galois orbit $O(x_n)$ of x_n is equidistributed with respect to the canonical measure on M. □

Here, the canonical measure $d\mu$ means the probability measure induced from the invariant measure on the Hermitian symmetric domain X in the definition of Shimura variety. Equidistribution means that for any continuous function f on $M(\mathbb{C})$ with compact support, we have the limit

$$\frac{1}{\#O(x_n)} \sum_{y \in O(x_n)} f(y) \longrightarrow \int_{M(\mathbb{C})} f(x) d\mu(x). \tag{2.4}$$

Remarks. (1) To see how the equidistribution conjecture implies the André-Oort conjecture, we first assume that M does not have a proper Shimura subvariety containing Z. Then we may list all Shimura subvarieties of M in a sequence $M_1, M_2, \ldots, M_n, \ldots$. Now by induction, for each n, we may find a CM-point x_n on Z which is not in the union of the first n M_i's. In this way, $\{x_n\}$ becomes a strict sequence of CM-points in M. The equidistribution conjecture implies that the Galois orbits of the x_n are equidistributed. Since all these Galois orbits are included in $Z(\mathbb{C})$, we must have $Z = M$.

(2) In the simplest case where M is the modular curve $X_0(1)$, the conjecture is a theorem of Duke [11]. In the case where M is defined by an algebraic group with positive $\mathbb{Q}$-rank, the equidistribution of Hecke orbits has been proved by Clozel and Ullmo [5] and by Clozel et al. [4].

(3) The equidistribution conjecture also implies (and is implied by) the equidistribution of Shimura subvarieties in M. When these subvarieties are defined by semisimple subgroups not included in any proper parabolic subgroup, the equidistribution has been proved by Clozel and Ullmo [7] by ergodic method. In [18], Jiang et al. proved an explicit period integral formula for cycles in the middle dimension, and were able to deduce the equidistribution with precise rate of convergence of probability measures.

3 Statements

In this section, we state our main results on the equidistribution of Galois orbits of CM-points on quaternion Shimura varieties. Let us start with some definitions and notations.

Quaternion Shimura varieties. Let F be a totally real number field of degree g, and let B be a quaternion algebra over F. Then for each real embedding σ of F, $B \otimes_\sigma \mathbb{R}$ is either isomorphic to the matrix algebra $M_2(\mathbb{R})$ or to the Hamilton quaternion algebra $\mathbb{H}$. Let G denote the algebraic group $B^\times/F^\times$ over $\mathbb{Q}$. Then

$$G(\mathbb{R}) = \prod_\sigma \left(B \otimes_\sigma \mathbb{R}\right)^\times / \mathbb{R}^\times \simeq \mathrm{PGL}_2(\mathbb{R})^d \times \mathrm{SO}_3^{g-d}, \tag{3.1}$$

where σ runs through the set of real embeddings of F. Via such an isomorphism, $G(\mathbb{R})$ acts on

$$X := (\mathbb{C} \setminus \mathbb{R})^d = \left(\mathcal{H}^\pm\right)^d. \tag{3.2}$$

In terms of Shimura datum, X can be considered as the $G(\mathbb{R})$-conjugacy class of the embedding

$$h_0 : \mathbb{S} \longrightarrow G_{\mathbb{R}}, \tag{3.3}$$

which sends $a + bi \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ to an element whose components are represented by $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ in $PGL_2(\mathbb{R})$ and by 1 in $SO_3(\mathbb{R})$.

For any abelian group A, let $\widehat{A}$ denote $A \otimes \varprojlim \mathbb{Z}/n\mathbb{Z}$. Then for any open and compact subgroup U of $G(\widehat{\mathbb{Q}})$, we have an analytic variety

$$M_U(\mathbb{C}) := G(\mathbb{Q}) \backslash X \times G(\widehat{\mathbb{Q}})/U. \tag{3.4}$$

If $d > 0$, by Shimura's theory, the variety M_U is defined over the following totally real subfield:

$$\widetilde{F} = \mathbb{Q}\left(\sum_{\sigma \in S} \sigma(x),\ \forall x \in F\right), \tag{3.5}$$

where S denotes the real embeddings σ such that $B \otimes_\sigma \mathbb{R} \simeq M_2(\mathbb{R})$. The action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\widetilde{F})$ on the connected components is given by a reciprocity homomorphism:

$$\mathrm{Gal}\left(\bar{\mathbb{Q}}/\widetilde{F}\right) \longrightarrow F_+^\times \backslash \widehat{F}/\nu(U). \tag{3.6}$$

In the following, we will fix a maximal order $\mathcal{O}_B$ of B and will take U to be the subgroup $\widehat{\mathcal{O}}_B^\times / \widehat{\mathcal{O}}_F^\times$ of $G(\widehat{\mathbb{Q}})$. Let N denote the level of M, which is by definition the product of prime ideals $\wp$ over which B does not split. Up to isomorphisms, both B and M_U are determined by the pair (S, N).

CM-points. A point x in $M_U(\mathbb{C})$ is a CM-point if and only if it is represented by a pair $(h, g) \in X \times G(\widehat{\mathbb{Q}})$ such that the stabilizer of x in $G(\mathbb{Q})$ is a torus $T := K^\times/F^\times$, where K is a quadratic CM-extension of F embedded into B. Here are some invariants of CM-points:

(1) the order

$$\mathcal{O}_x := K \cap g\mathcal{O}_B g^{-1} = \mathcal{O}_F + c(x)\mathcal{O}_K, \tag{3.7}$$

where $c(x)$ is an ideal of $\mathcal{O}_F$ called the conductor of x;

(2) the type (K, S_K), where S_K is the set of the complex embeddings of K so that the action of a $t \in K^\times$ on the tangent space of X at h has eigenvalues given by

$$\sigma\left(\frac{t}{\bar{t}}\right), \quad \sigma \in S_K. \tag{3.8}$$

When $d > 0$, x is defined over an abelian extension of a CM-subfield of $\mathbb{C}$ which is given by

$$\widetilde{K} = \mathbb{Q}\left(\sum_{\sigma \in S_K} \sigma(t), \ \forall t \in K\right). \tag{3.9}$$

More precisely, there is a homomorphism

$$r_x : \mathrm{Gal}\left(\bar{\mathbb{Q}}/\widetilde{K}\right) \longrightarrow T(\mathbb{Q})\backslash T\left(\widehat{\mathbb{Q}}\right)/\widehat{\mathcal{O}}_x \tag{3.10}$$

such that for $\gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\widetilde{K})$, the conjugate γx is a CM-point represented by $(h, r(\gamma)g)$. Let $O_{cm}(x)$ denote the CM-orbit of x consisting of points represented by (h, tg) with $t \in T(\widehat{\mathbb{Q}})$, and let $O_{gl}(x)$ denote the Galois orbit of x under $\mathrm{Gal}(\bar{\mathbb{Q}}/\widetilde{K})$. Then, $O_{gl}(x) \subset O_{cm}(x)$. When $d = 1$, we have $O_{gl}(x) = O_{cm}(x)$. In general, they are different. In fact, the Galois orbit is included in the Hodge orbit $O_{hg}(x)$ defined to be the set of points represented by (h, tg) with $t \in H(\widehat{\mathbb{Q}})$, where $H \subset T$ is the Mumford-Tate subgroup of x.

By a CM-*suborbit* $O(x)$ of a CM-point we mean an orbit $O(x) \subset O_{cm}(x)$ under an open subgroup of $T(\mathbb{Q})\backslash T(\widehat{\mathbb{Q}})$. Its *conductor* $c(O(x))$ is defined to be the largest ideal c so that $(1 + \widehat{c}\mathcal{O}_K)^\times$ stabilizes $O(x)$.

The equidistribution conjecture implies the equidistribution of $O_{cm}(x)$.

Theorem 3.1. Let x_i be a sequence of CM-points on M_U. Then the CM-points $O_{cm}(x_i)$ are equidistributed. □

This is a direct generalization of Duke's result [11]. The proof of this theorem uses some bounds on Hecke eigenvalues by Kim and Shahidi for GL_2-forms and on the central value of the L-series $L(1/2, f \otimes \epsilon_{K/F})$ by Cogdell et al. for holomorphic f, and by Venkatesh [29] for general f, where $\epsilon_{K/F}$ is the quadratic character of $\widehat{F}^\times$ associated to the extension K/F. In the following, we want to extend this result to certain suborbits of $O_{cm}(x)$ under the following assumption.

δ-bound. Let δ be a positive number. There are constants C and A such that for any eigenform $f \in C^\infty(X(\mathbb{C}))$, the following two conditions are verified.

(1) The local parameters α_v of L-series $L(s, f)$ are bounded as follows:

$$\left|\alpha_v\right| \le C\lambda(f)^A q_v^\delta, \tag{3.11}$$

where $\lambda(f)$ is the eigenvalue of f under the Laplacian operator on M_U, and q_v is the cardinality of the residue field of $\mathcal{O}_F$.

(2) For any imaginary quadratic extension K of F with absolute discriminant $\mathrm{disc}(K)$, and any finite character χ of the group $K^\times\backslash\widehat{K}^\times/\widehat{F}^\times$,

$$\left|L\left(\frac{1}{2},\chi,f\right)\right| \le C\lambda(f)^A\,\mathrm{disc}(\chi)^\delta, \tag{3.12}$$

where $L(s,f,\chi)$ is the Rankin-Selberg convolution of $L_F(s,f)$ and $L_K(s,\chi)$, and $\mathrm{disc}(\chi)$ denotes $N_{F/\mathbb{Q}}(c(\chi)^2\,\mathrm{disc}_F(K))$.

Remarks. (1) The δ-bounds assumption always holds for $\delta = 1/2+\epsilon$, which is called a *convexity bound*; any bound with $\delta < 1/2$ is called a *subconvexity bound*.

(2) For the first inequality, the Peterson-Ramanujan conjecture says that the absolute value of α_v is always 1. So δ could be any positive number. The recent work of Kim and Shahidi [19] implies that the inequality holds for $\delta = 1/9+\epsilon$.

(3) By GRH, we should have second inequality for any $\delta > 0$. When $F = \mathbb{Q}$, the subconvexity has been proven by Michel [20] for holomorphic forms f with $\delta = 1/2 - 1/1145$, and by Harcos and Michel [16] for mass forms f with $\delta = 1/2 - 1/2491$. For general F and quadratic χ, the subconvexity has been proven by Cogdell et al. [8] for holomorphic forms with $\delta = 1/2 - 7/130$ and for nonholomorphic forms by Venkatesh [29].

(4) The work of Venkatesh actually holds for any family of L-series of a fixed GL_2-form twisted by *central characters* over any number field. In particular, when f and K are fixed, the subconvexity bound for L-series holds. Indeed, let g be the base change of f over $GL_2(K)$. Then,

$$L_F(s,\chi,f) = L_K(s, g\otimes\chi). \tag{3.13}$$

Theorem 3.2. Let δ be a positive number such that the δ-bounds hold. Let f be a function on $M_U(\mathbb{C})$ which has integral 0 on each connected component and which is constant outside of a compact subset. Then for any $\epsilon > 0$, there is a constant $C(f,\epsilon)$ such that

$$\left|\sum_{y\in O(x)} f(x)\right| \le C(f,\epsilon)\,\mathrm{disc}(x)^{1/4+\delta/2+\epsilon} \tag{3.14}$$

for any CM-suborbits $O(x)$. □

Remarks. (1) The equality is nontrivial only if $O(x)$ has size biger than $\mathrm{disc}(x)^{\delta/2+1/4}$.

(2) For the proof, we only need δ to satisfy the δ-assumption for χ trivial on $O(x)$.

Corollary 3.3. Let δ be a positive number such that the δ-bounds hold. Let $O(x_n)$ be a sequence of CM-suborbits in a connected component M of M_U satisfying the equality

$$\#O(x_n) \geq \text{disc}\,(x_n)^{\delta/2+1/4+\epsilon} \tag{3.15}$$

for some fixed $\epsilon > 0$. Then the $O(x_n)$ are equidistributed on $M(\mathbb{C})$. $\square$

Remarks. (1) Theorem 3.1 follows from Corollary 3.3, since we have the Brauer-Siegel theorem:

$$\lim_{n\to\infty} \frac{\log \#O_{\text{cm}} x_n}{\log \text{disc}\, x_n} = \frac{1}{2}. \tag{3.16}$$

(2) If we assume the Riemann hypothesis, then we take $\delta = 0$ in the second assumption to get the exponent $1/4 + \epsilon$. This is essentially optimal. To give an example, we assume that F is a real quadratic field, $B = M_2(F)$, x_n are in a single modular curve C, and that $O(x_n)$ is the full CM-orbits on C. Since the discriminant of x_n on M is the square of that on C, then the Brauer-Siegel theorem gives

$$\lim_{n\to\infty} \frac{\log \#O_{\text{cm}} x_n}{\log \#O(x_n)} = \frac{1}{4}. \tag{3.17}$$

Equidistribution of Galois orbits. In the following, we want to give some examples of CM-points whose Galois orbits are equidistributed by Corollary 3.3.

Recall that S is the set of all real embeddings of F over which B is split. Let F_0 be the subfield of F on which the restrictions of all embeddings in S give the same embeddings. Let $\widetilde{F}$ be a Galois closure of F over F_0.

Theorem 3.4. Assume that $d = 2$ and that $[\widetilde{F} : F_0]$ is a power of 2. Then the subconvexity bound implies the equidistribution of Galois orbits of CM-points x with the equality $K_0(x) = F_0$. Here, when x has CM-type $(K, \{\sigma_1, \sigma_2\})$, $K_0(x)$ denotes the subfield of K of elements satisfying $\sigma_1(x) = \sigma_2(\bar{x})$. $\square$

Remarks. (1) If we only consider CM-points with fixed CM-field K, then the condition on $[\widetilde{F} : F_0]$ can be dropped.

(2) For Hilbert modular surfaces, the subconvexity bound implies the equidistribution of Galois orbits of CM-points which are not included in any Shimura curves.

The idea of the proof of this proposition is to show that for a CM-point x with CM-field K, the reciprocity map

$$r_x : \text{Gal}\,(\bar{\mathbb{Q}}/\widetilde{K}) \longrightarrow T(\mathbb{Q})\backslash T(\bar{\mathbb{Q}})/\widehat{\mathcal{O}}_x^\times = \frac{\text{Pic}\,(\mathcal{O}_x)}{\text{Pic}\,(\mathcal{O}_F)} \tag{3.18}$$

has cokernel annihilated by some positive integer n depending on the Mumford-Tate group. We may drop the assumption that $[\widetilde{F}:F_0]$ is a power of 2 under the following assumption.

Conjecture 3.5 (ϵ-conjecture)**.** Fix a totally real number field F, a positive integer n, and a positive number ϵ. Then, for any quadratic CM-extension K, and any order $\mathcal{O}$ of K containing $\mathcal{O}_F$, the n-torsion of the class group of $\mathcal{O}_c$ has the following bound:

$$\#\,\mathrm{Pic}\left(\mathcal{O}_c\right)[n] \leq C(\epsilon)\,\mathrm{disc}\left(\mathcal{O}_c\right)^{\epsilon}, \tag{3.19}$$

where $C(\epsilon)$ is a positive constant depending only on ϵ. □

Remarks. (1) We will reduce to the case where $\mathcal{O}_c = \mathcal{O}_K$ is maximal and n is odd (Corollary 6.4).

(2) By the Brauer-Siegel theorem, $1/2+\epsilon$ will be the trivial bound.

(3) When $n = 2$, $K = \mathbb{Q}\sqrt{D}$, with $D \in \mathbb{Z}$ a fundamental discriminant, and the conjecture is true by Gauss′ genus theory. Actually, the 2-torsion of a class group equals 2^{δ}, where δ is the number of prime factors of D. We will prove the conjecture for 2-torsion for an arbitrary CM-extension (Corollary 6.4).

(4) When $n = 3$, Helfgott and Venkatesh obtain the bound $\epsilon = 0.44187$.

(5) By Brumer and Silverman [3], the following stronger bound has been formulated:

$$\log \#\,\mathrm{Cl}(L)[n] \leq \frac{C \log \mathrm{disc}(L)}{\log\log \mathrm{disc}(L)}. \tag{3.20}$$

Theorem 3.6. Assume the ϵ-conjecture (for a positive integer n as specified in Corollary 6.2). Then the following estimate holds for the size of Galois orbits for a CM-point x on M_U with maximal Mumford-Tate group $H = T$:

$$\#O_{\mathrm{gl}}(x) \gg \mathrm{disc}(x)^{1/2-\epsilon}. \tag{3.21}$$

□

Applying Corollary 3.3, we obtain the following corollary.

Corollary 3.7. The ϵ-conjecture and subconvexity bound imply the equidistribution of CM-points with maximal Mumford-Tate group $H = T$. □

Corollary 3.8. The equidistribution holds for Galois CM-points with a fixed maximal Mumford-Tate group $H = T$. In particular, any infinite set of such CM-points are Zariski dense. □

Remarks. (1) In our current setting, the Zariski density in Corollary 3.8 strengthens a theorem of Edixhoven and Yafaev [15] about the finiteness of CM-points with fixed Hodge $\mathbb{Q}$-structure on a non-Hodge curve. Our finiteness holds for any proper subvariety. Of course, their theorem applies to general Shimura varieties.

(2) Theorem 3.4 follows from this corollary and Proposition 7.3, which says that H is the kernel of the norm

$$N_{K/K_0} : K^\times/F^\times \longrightarrow K_0^\times/F_0^\times. \tag{3.22}$$

(3) When $d > 2$, we do not have a general description of CM-points having maximal Mumford-Tate orbits, except for some partial results given in Section 7. In particular, we can show that $H = T$ for all CM-points if d is odd and if F/F_0 is abelian with Galois group verifying that one of the following two conditions is verified:

(i) $[F : F_0]$ is a power of 2 (Corollary 7.6), or
(ii) $\mathrm{Gal}(F/F_0)$ has cyclic 2-Sylow subgroup and $d < p$ for any odd prime factor of $[F : F_0]$ (Corollary 7.8).

Thus, we have equidistribution of Galois orbits of *all CM-points* on these Shimura varieties.

4 Hecke orbits

In this section and the next, we want to prove Theorem 3.2. More precisely, we want to estimate the sum

$$\ell\big(f; O(x)\big) := \sum_{y \in O(x)} f(x) \tag{4.1}$$

for a CM-suborbit $O(x)$ and for a function f on $M_U(\mathbb{C})$ which has integral 0 on each connected component and is constant outside of a compact set. In this section, we want to reduce the computation of this integral to the case where x and $O(x)$ have the same conductor (see Proposition 4.4).

Let Γ be the stabilizer of $O(x)$ in $T(\mathbb{Q})\backslash T(\widehat{\mathbb{Q}})$ with index $i(\Gamma)$. Then, we have

$$\ell\big(f, O(x)\big) = i(\Gamma)^{-1} \sum_{\chi} \ell_\chi(f; x), \tag{4.2}$$

where χ runs through characters of $T(\mathbb{Q})\backslash T(\widehat{\mathbb{Q}})/\Gamma$, and

$$\ell_\chi(f; x) := \sum_{t \in T(\mathbb{Q})\backslash T(\widehat{\mathbb{Q}})/\widehat{\mathcal{O}}_x^\times} \chi^{-1}(t) f(tx). \tag{4.3}$$

Let K be the CM-field defining x. Then the set of CM-points with field K is given by

$$T(\mathbb{Q})\backslash G(\widehat{\mathbb{Q}})/U = K^\times\backslash \widehat{B}^\times/\widehat{F}^\times \widehat{\mathcal{O}}_B^\times. \tag{4.4}$$

For each ideal c, the CM-points of conductor c are represented by $g \in \widehat{B}^\times$ such that

$$g\mathcal{O}_B g^{-1} \cap K = \mathcal{O}_c. \tag{4.5}$$

The set of CM-points with conductor c is a single orbit under left multiplication by $\widehat{K}^\times$. Thus, the value $\ell_\chi(f, x)$ depends only on the conductor of x up to multiple by a root of unity.

Let us define a distinguished CM-point x_c which is represented by g_c with components $g_v \in B_v^\times$ given as follows. If v does not divide c, we take $g_v = 1$. For v dividing c, we have an isomorphism $B_v \simeq M_2(F_v)$ so that $\mathcal{O}_{K,v}$ is embedded into $M_2(\mathcal{O}_v)$. The action of K_v on F_v^2 identifies F_v^2 with K_v as K_v-modules. The map $\alpha \to \alpha(\mathcal{O}_{K_v})$ defines a bijection between the set of $B_v^\times/\mathcal{O}_{B,v}^\times$ and the set of $\mathcal{O}_v$-lattices of K_v. The conductor of α is exactly the conductor of the $\mathcal{O}_v$-endomorphism algebra of the lattices. Thus, we may take g_v such that $g_v(\mathcal{O}_{K,v}) = \mathcal{O}_{c,v}$.

Now fix an anticyclotomic character χ of conductor $c = c(\chi)$. For each ideal n, we define a function γ_n on CM-points with field K supported on set of CM-points of conductor nc and such that

$$\gamma_n(tx_{nc}) = \chi(t), \quad \forall t \in T(\widehat{\mathbb{Q}}). \tag{4.6}$$

Let $r_\chi(m)$ be a function on nonzero ideals of $\mathcal{O}_F$ defined by the formula

$$r_\chi(m) = \begin{cases} \sum\limits_{N(n)=m} \chi(n) & \text{if } (m, c) = 1, \\ 0 & \text{if } (m, c) \neq 1. \end{cases} \tag{4.7}$$

Proposition 4.1. For m an ideal prime to N,

$$T_m\gamma_1 = \sum_n r_\chi\left(\frac{m}{n}\right)\gamma_n. \tag{4.8}$$

□

Proof. It is clear that $T_m\gamma_1$ still has character χ under left multiplication by $T(\widehat{\mathbb{Q}})$. Thus, we have a decomposition

$$T_m\gamma_1 = \sum a_{m,n}\gamma_n. \tag{4.9}$$

The number $a_{m,n}$ can be expressed as follows:

$$a_{m,n} = T_m\gamma_1\big(\mathcal{O}_{nc}\big) = \sum_{\Lambda} \gamma_1(\Lambda). \tag{4.10}$$

Here, the sum is over sublattices of $\widetilde{\mathcal{O}}_{nc} := \prod_{v|mnc} \mathcal{O}_{nc,v}$ of index m. Notice that $\gamma_1(\Lambda) \neq 0$ if and only if Λ has the form $t\widetilde{\mathcal{O}}_c$ for some $t \in \widetilde{K}$. In this case $\gamma_1(\Lambda) = \chi(t)$. The condition that $\Lambda \subset \widetilde{\mathcal{O}}_{nc}$ with index m is equivalent to $m\widetilde{\mathcal{O}}_{nc} \subset \Lambda = t\widetilde{\mathcal{O}}_c$ with index m, which is equivalent to $mt^{-1}\widetilde{\mathcal{O}}_{nc} \subset \widetilde{\mathcal{O}}_c$ with index m. This last condition is equivalent to

$$mt^{-1} \in \widetilde{\mathcal{O}}_c, \quad N_F\big(mt^{-1}\big)n = m. \tag{4.11}$$

The second condition is $N_F(t) = mn$. Thus, $t^{-1} = \bar{t}(mn)^{-1}$ and the first equation becomes $t \in n\widetilde{\mathcal{O}}_c$. Write $t = ns$, then $N(s) = m/n$, and $\chi(t) = \chi(s)$. Thus, we obtain

$$a_{m,n} = \sum_{s} \chi(s), \tag{4.12}$$

where s runs through elements in $\widetilde{\mathcal{O}}_c/\widetilde{\mathcal{O}}_c^\times$ with norm m/n.

If m/n is not prime to c, there is a prime π dividing both c and s. For each s in the sum above, we may write $s = \pi t u$, where t runs through representatives of $\widetilde{\mathcal{O}}_{c/\pi}/\widetilde{\mathcal{O}}_{c/\pi}^\times$ and u runs through $\widetilde{\mathcal{O}}_{c/\pi}^\times/\widetilde{\mathcal{O}}_c^\times$. Thus, we have

$$a_{m,n} = \chi(\pi)\sum \chi(t)\sum \chi(u). \tag{4.13}$$

As χ has conductor c, the sum of $\chi(u)$ is certainly 0. ■

Write

$$L(s,\chi) = \sum_{n} \frac{\chi(n)}{N(n)^s}. \tag{4.14}$$

By the proposition, we have formally

$$\sum_{m} \frac{T_m\gamma_1}{Nm^s} = L(s,\chi)\cdot \sum \frac{\gamma_n}{Nn^s}. \tag{4.15}$$

It follows that

$$\sum \frac{\gamma_n}{Nn^s} = L(s,\chi)^{-1} \sum_{m} \frac{T_m\gamma_1}{Nm^s}. \tag{4.16}$$

In other words, we have the following corollary.

Corollary 4.2.

$$\gamma_n = \sum_{m|n} s_\chi\left(\frac{n}{m}\right) T_m \gamma_1, \tag{4.17}$$

where $s_\chi(n)$ are coefficients of $L_K(s,\chi)^{-1}$:

$$s_\chi(n) = \sum_{N_{K/F}(a)=n} \chi(a)\cdot\mu(a), \tag{4.18}$$

where $\mu(a)$ is the Möbius function on the ideals of $\mathcal{O}_K$. □

We may express $\ell_\chi(f;x_{nc})$ as an inner product of two functions f and γ_n on CM-points:

$$\ell_\chi(f;x_{nc}) = \#O_{cm}(x_{nc})^{-1}\langle f,\gamma_n\rangle. \tag{4.19}$$

The Hecke operator is certainly selfadjoint for this inner product. Thus, we have the following corollary.

Corollary 4.3. Let f be an eigenform with eigenvalue λ_m under the action by T_m. Then for n prime to c,

$$\ell_\chi(f;x_{nc}) = \left(\sum_{m|n} s_\chi\left(\frac{n}{m}\right)\lambda_m\right)\cdot \ell_\chi(f;x_c). \tag{4.20}$$

□

Proposition 4.4. Assume the δ-bound in Section 3. For any $\epsilon > 0$, there is an $C(\epsilon) > 0$ depending only on $[F:\mathbb{Q}]$ such that

$$|\ell_\chi(f;x_{nc})| \le C(\epsilon) Nn^{1/2+\delta+\epsilon}|\ell_\chi(f;x_c)|. \tag{4.21}$$

□

Proof. The period sum is given by

$$\ell_\chi(f;x_{nc}) = \prod_{\wp^e\|n} \kappa(\wp^e)\cdot \ell_\chi(f;x_c), \tag{4.22}$$

where

$$\kappa(\wp^n) = \sum_{i=0}^{e} s_\chi(\pi^{e-i})\lambda_{\pi^i}. \tag{4.23}$$

Let us compute this number in separate cases.

First, assume that $\wp$ is inert in K. Then,

$$s_\chi\left(\pi^{e-i}\right) = \begin{cases} 1 & \text{if } e = i, \\ -\chi(\pi) & \text{if } e = i+2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.24}$$

Here, it is understood that $\chi(\pi) = 1$ if χ is unramified, and that $\chi(\pi) = 0$ if χ is ramified. It follows that

$$\kappa\left(\pi^e\right) = \lambda_{\pi^e} - \chi(\pi)\lambda_{\pi^{e-2}}. \tag{4.25}$$

Here, it is understood that $\lambda_{\pi^n} = 0$ if $n < 0$.

Now let us treat the case where $\wp$ is ramified in K. Then,

$$s_\chi\left(\pi^{e-i}\right) = \begin{cases} 1 & \text{if } e = i, \\ -\chi\left(\pi_K\right) & \text{if } e = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.26}$$

It follows that

$$\kappa\left(\pi^e\right) = \lambda_{\pi^e} - \chi\left(\pi_K\right)\lambda_{\pi^{e-1}}. \tag{4.27}$$

Finally let us treat the case where $\wp$ is split in $K_\wp$. Then

$$s_\chi\left(\pi^{e-i}\right) = \begin{cases} 1 & \text{if } e = i, \\ -\left(\chi_1(\pi) + \chi_1(\pi)^{-1}\right) & \text{if } e = i+1, \\ 1 & \text{if } e = i+2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

It follows that

$$\kappa\left(\pi^e\right) = \lambda_{\pi^e} - 2\,\mathrm{Re}\left(\chi_1(\pi)\right)\lambda_{\pi^{e-1}} + \lambda_{\pi^{e-2}}. \tag{4.29}$$

Assume that the L-function of ϕ has parameter $\alpha_\wp$. Then, $q^{-n/2}\lambda_{\pi^n}$ is the coefficient of the L-series at q^{-ns}:

$$\left(1 - \alpha_\wp q^{-s}\right)^{-1}\left(1 - \alpha_\wp^{-1} q^{-s}\right)^{-1}. \tag{4.30}$$

It follows that

$$\lambda_{\pi^n} = q^{n/2} \frac{\alpha^{n+1} - \alpha^{-1-n}}{\alpha - \alpha^{-1}}. \tag{4.31}$$

From the δ-bound, $|\alpha^{\pm 1}| \leq q^{\delta}$. Thus we have that

$$\left|\lambda_{\pi^n}\right| \leq q^{(1/2+\delta+\epsilon)n}. \tag{4.32}$$

It follows that in all cases, $\kappa(\pi^e)$ has bound

$$\left|\kappa\left(\pi^e\right)\right| \leq q^{e(\delta+1/2+\epsilon)}. \tag{4.33}$$

In summary, we have

$$\left|\ell_\chi\left(f; x_{nc}\right)\right| \leq C(\epsilon) N n^{1/2+\delta+\epsilon} \left|\ell_\chi\left(f; x_c\right)\right|. \tag{4.34}$$

■

5 Period sums

In this section, we want to finish the proof of Theorem 3.2. By the equality

$$\ell\left(f, O(x)\right) = i(\Gamma)^{-1} \sum_{\chi} \ell_\chi(f; x), \tag{5.1}$$

the question is reduced to estimating the sum $\ell_\chi(f; x)$.

Consider the spectral decomposition

$$f = \sum c_n f_n + \int_{\Omega} c_\mu E_\mu \, d\mu, \tag{5.2}$$

where the f_n are discrete (cuspidal or residual) eigenforms under Hecke operators with norm 1, and the E_μ are Eisenstein series indexed by characters μ of $F^\times \backslash \mathbb{A}_F^\times$ modulo equivalence $\mu \sim \mu^{-1}$, which exist only when $B = M_2(F)$. The measure $d\mu$ is induced from a Haar measure on the topological group of idele class characters of F. In this case, for discrete spectrum, we may take f_n to be

$$f_n = \left\|f_n^{\text{new}}\right\|^{-1} f_n^{\text{new}} \tag{5.3}$$

with f_n^{new} a newform. For the continuous spectrum, we may take E_μ to be

$$E_\mu = \left|L\left(1, \mu^2\right)\right|^{-1} E_\mu^{\text{new}}, \tag{5.4}$$

where E_μ^{new} is the newform in $\pi(\mu,\mu^{-1})$. Here a newform φ^{new} means a Hecke eigenform with minimal level and normalized so that

$$L(s,\Pi)=\mathrm{disc}(F)^{1/2-s}\int_{F^\times\backslash\mathbb{A}_F^\times}\left(\varphi^{\mathrm{new}}-C_{\varphi^{\mathrm{new}}}\right)\begin{pmatrix}a&0\\0&1\end{pmatrix}|a|^{1-1/2}d^\times a, \tag{5.5}$$

where Π is the automorphic representation of $GL_2(\mathbb{A}_F)$ generated by φ^{new}, and $C_{\varphi^{\mathrm{new}}}$ is the constant part in the Fourier expansion with respect characters of the unipotent group of matrixes $\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$.

Since f is compactly support, we have

$$\|f\|^2=\sum|c_n|^2+\int_\Omega|c_\mu|^2d\mu<\infty. \tag{5.6}$$

Moreover, for Δ the Laplacien operator on $M_U(\mathbb{C})$, $\Delta^m f$ is still compactly supported for any positive integer m,

$$\left\|\Delta^m f\right\|^2=\sum\left|c_n\lambda_n^m\right|^2+\int_\Omega\left|c_\mu\lambda_\mu^m\right|^2d\mu<\infty, \tag{5.7}$$

where λ_i (resp., λ_μ) are eigenvalues of f_i (resp., E_μ) under D. Thus c_n (resp., c_μ) decays faster than any negative power of λ_n (resp., λ_μ).

It can be shown that $\|\phi_n\|_{\sup}$ is bounded by a polynomial function of λ_n. Thus, the sum of the right-hand side of (5.2) is absolutely convergent pointwisely. Similarly, for a fixed compact domain E of $M(\mathbb{C})$, it can be shown that $\sup_{x\in E}|E_\mu(x)|$ is bounded by a polynomial function of λ_μ. Thus, the integral of the right-hand side of (5.2) is absolutely and uniformly convergent on E. See Clozel and Ullmo [6, Lemmas 7.2–7.4] for a complete proof. It follows that

$$\ell(f;x)=\sum c_n\ell\left(f_n;x\right)+\int_\Omega c_\mu\ell\left(E_\mu;x\right)d\mu. \tag{5.8}$$

Thus, for the proof of Theorem 3.2, it suffices to show the following proposition.

Proposition 5.1. For any $\epsilon>0$, there are positive numbers C, A such that for any Hecke eigen form f of norm 1 (which is either cuspidal or Eisenstein),

$$\left|\ell_X(f;x)\right|\le C\cdot\lambda(f)^A\cdot\mathrm{disc}(x)^{1/4+\delta/2+\epsilon}, \tag{5.9}$$

where $\|f\|=|L(1,\mu^2)|$ if $f\in\pi(\mu,\mu^{-1})$. □

Let c be the conductor of χ and let $\mathfrak{n}c$ be the conductor of x. Then by Proposition 4.4,

$$\left|\ell_\chi(f;x)\right| \ll N(\mathfrak{n})^{\delta+1/2+\epsilon}\left|\ell_\chi(f;x_0)\right|, \tag{5.10}$$

where x_0 is a CM-point of conductor c with the same CM-group. Now the question is reduced to estimating $\ell_\chi(f;x_0)$. In the following, we will show that this special case follows from the central value formula proved in [30, 35] and the subconvexity bound.

By Jacquet-Langlands theory, there is a unique newform φ on $GL_2(\mathbb{A}_F)$ of weight 0 and level N which has the same Hecke eigenvalues as f. Notice that when $N = \mathcal{O}_F$, $B = M_2(F)$ and φ is a multiple of f.

Theorem 5.2 (see [30, 35]). Let χ be a character of $T(\mathbb{Q})\backslash T(\widehat{\mathbb{Q}})$ with the same conductor as x,

$$L\left(\frac{1}{2},\Pi\otimes\chi\right) = \frac{2^{[F:\mathbb{Q}]+d}}{\sqrt{\operatorname{disc}(x_0)}}\cdot\|\varphi\|^2\cdot\left|\ell_\chi(f;x_0)\right|^2. \tag{5.11}$$

Here, $L(s,\Pi\otimes\chi)$ is the Rankin-Selberg convolution of $L(s,\Pi)$ and $L(s,\chi)$. □

Proof. When the conductor c of χ is prime to the relative discriminant d of K, this is proved in [35]. For Eisenstein series Π without coprime condition $(c,D) = 1$, this can be proved easily by the same method in [35]. For cusp form Π without coprime condition $(c,d) = 1$, the formula can be deduced from Waldspurger's formula in [30, Proposition 7, page 222]. ■

Now Proposition 5.1 for $x = x_0$ follows from Theorem 5.2 and following well-known estimate

$$\lambda(f)^{\varepsilon} \gg \|\varphi\| \gg \lambda(f)^{-\varepsilon} \tag{5.12}$$

for λ big and any $\varepsilon > 0$.

6 Galois orbits

In this section, we are going to prove Theorem 3.6 about an estimate of the sizes of Galois orbits CM-points with maximal Mumford-Tate group.

Let x be a CM-point with CM-group $T = K^\times/F^\times$ and Mumford-Tate group $H \subset T$. By the Shimura theory, the CM-orbit $O_{cm}(x)$ as a reduced subscheme is defined over the reflex field $\widetilde{K} \subset \mathbb{C}$ generated over $\mathbb{Q}$ by $\sum_{\sigma \in S_K} \sigma(x)$ for all $x \in F$. Moreover, the Galois action is given by a homomorphism

$$r_x : \mathrm{Gal}\left(\bar{\mathbb{Q}}/\widetilde{K}\right) \longrightarrow K^\times \backslash \widehat{K}^\times / \widehat{F}^\times \widehat{\mathcal{O}}_x^\times = \frac{\mathrm{Pic}\left(\mathcal{O}_x\right)}{\mathrm{Pic}\left(\mathcal{O}_F\right)}, \tag{6.1}$$

where $\mathcal{O}_x$ is the order of x defined in Section 3.

The Galois action factors through the maximal abelian quotient, thus it is determined by the homomorphism

$$\widetilde{K}^\times \backslash \widehat{\widetilde{K}}^\times \longrightarrow K^\times \backslash \widehat{K}^\times / \widehat{F}^\times \widehat{\mathcal{O}}_x^\times. \tag{6.2}$$

By Shimura's theory, this is induced by a homomorphism of algebraic groups

$$N : \widetilde{K}^\times \longrightarrow K^\times, \qquad \tau N(x) = \prod_{\tau \in \sigma \circ S_K} \sigma(x), \tag{6.3}$$

where τ is a any fixed embedding $K \to \mathbb{C}$, and σ runs through the coset $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})/\mathrm{Gal}(\mathbb{C}/\widetilde{K})$. Notice that the definition does not depend on the choice of τ. The restriction of N to the totally real subgroup $\widetilde{F}^\times$ takes values in $F^\times$. Thus, N induces a homomorphism on the quotient which is also denoted by N:

$$N : \widetilde{T} := \widetilde{K}^\times / \widetilde{F}^\times \longrightarrow T = K^\times / F^\times. \tag{6.4}$$

Let F_0 be the subfield of F over which all embeddings of S have the same restriction. Then $K^\times/F^\times$ and $\widetilde{K}^\times/\widetilde{F}^\times$ can be viewed as algebraic groups T_0 and $\widetilde{T}_0$ over F_0, and the morphism N is induced by a morphism over F_0:

$$N_0 : \widetilde{T}_0 \longrightarrow T_0. \tag{6.5}$$

The assumption that $H = T$ means that N_0 is surjective. Taking L to be a Galois closure of K over F_0, these groups are split over L. First, we want to show that N_0 has a section up to isogeny.

Lemma 6.1. Let $\alpha : T_1 \to T_2$ be a surjective homomorphism of tori over F_0 which are split over L. Let n be a positive integer which is a product of $[L : F_0]$ and an integer m annihilating the components group of $\ker \alpha$. Then there is a homomorphism $\beta : T_2 \to T_1$ such that $\alpha \circ \beta = n$ on T_2. □

Proof. Let $\alpha^* : X(T_2) \to X(T_1)$ be the corresponding injection of $\mathrm{Gal}\,(\bar{\mathbb{Q}}/F_0)$-modules of characters. It suffices to show that there is a homomorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}/F_0)$-modules $\phi : X(T_1) \to X(T_2)$ such that $\alpha^* \circ \phi = n$. Since both T_i are split over L, the $\mathrm{Gal}(\bar{\mathbb{Q}}/F_0)$-module structures on $X(T_i)$ descend to Δ-module structures, where $\Delta = \mathrm{Gal}(L/F_0)$.

Let $X(T_1) = Y_1 + Y_2$ be a direct sum decomposition of $\mathbb{Z}$-modules such that Y_2 is the subgroup of elements x such that some positive multiple $mx \in X(T_2)$. Then, Y_2 is a Δ-submodule and $Y_2/X(T_2)$ is annihilated by n. Let $\pi' \in \mathrm{End}(X(T_1))$ denote the projection of $X(T_1)$ onto Y_1 with respect to this decomposition and let

$$\pi := m \cdot \sum_{\delta\in\Delta} \delta^{-1} \circ \pi' \circ \delta. \tag{6.6}$$

Then π is Δ-homomorphism with values in $X(T_2)$, and for $x \in X(T_2)$, $\pi(x) = nx$. ■

Corollary 6.2. Let n be the product of $[L : F_0]$ and the smallest positive integer annihilating the components group of $\ker N_0$. The cokernel of r_x is annihilated by n. □

Proof. By Lemma 6.1, N_0 will have a section up to multiplication by n. Thus for any F_0-algebra A, the morphism on any A-points of T_i will have cokernel annihilated by n. ■

Proof of Theorem 3.6. The corollary implies that the image of the homomorphism r_x in (6.1) has order bounded below by

$$\frac{\#\left(\dfrac{\mathrm{Pic}\,\mathcal{O}_x}{\mathrm{Pic}\,\mathcal{O}_F}\right)}{\#\left(\dfrac{\mathrm{Pic}\,\mathcal{O}_x}{\mathrm{Pic}\,\mathcal{O}_F}\right)[n]}. \tag{6.7}$$

Now Theorem 3.6 follows from the Brauer-Siegel estimate

$$\#\,\mathrm{Pic}\left(\mathcal{O}_x\right) \gg \mathrm{disc}(x)^{1/2-\epsilon}. \tag{6.8}$$

In the rest of section, we want to estimate ℓ-torsion in the anticyclotomic extension in two special cases. For an abelian group M and prime ℓ, let rank_ℓ denote the ℓ-rank of M:

$$\mathrm{rank}_\ell\, M := \mathrm{rank}_{\mathbb{Z}/\ell\mathbb{Z}}\, M \otimes \mathbb{Z}/\ell\mathbb{Z} = \mathrm{rank}_{\mathbb{Z}/\ell\mathbb{Z}}\, M[\ell]. \tag{6.9}$$

It is easy to see that $\mathrm{rank}_\ell\, M$ is the minimal number of generators for the ℓ-Sylow subgroup of M.

Proposition 6.3. Let $\mathcal{O}_c = \mathcal{O}_F + c\mathcal{O}_K$ be an order of a CM-field K, where F is its totally real subfield.

(1) Let μ be the number of prime factors of c. Consider the map

$$\alpha : \mathrm{Pic}\left(\mathcal{O}_c\right) \longrightarrow \mathrm{Pic}\left(\mathcal{O}_K\right). \tag{6.10}$$

Then

$$\mathrm{rank}_\ell \,\mathrm{Ker}(\alpha) \leq g\mu. \tag{6.11}$$

(2) Let δ be the number of primes of $\mathcal{O}_K$ ramified over F, then

$$\mathrm{rank}_2\left(\mathrm{Pic}\left(\mathcal{O}_K\right)\right) \leq 2\,\mathrm{rank}_2\left(\mathrm{Pic}\,\mathcal{O}_F\right) + g + \delta. \tag{6.12}$$

□

Proof. It is easy to see that $\ker\alpha$ has an expression

$$\ker\alpha = \widehat{\mathcal{O}}_K^\times/\mathcal{O}_K^\times \cdot \widehat{\mathcal{O}}_K^\times = \left(\mathcal{O}_k/c\right)^\times/\mathcal{O}_K^\times\left(\mathcal{O}_F/c\right)^\times. \tag{6.13}$$

Thus, $\mathrm{rank}_\ell \ker\alpha$ is additive over prime decomposition of c. So we need only to estimate the $\mathrm{rank}_\ell \ker\alpha$ for $c = \pi^n$ a positive power of prime π of $\mathcal{O}_F$. Consider the exact sequence

$$1 \longrightarrow \frac{1+\pi\mathcal{O}_K}{1+\pi\mathcal{O}_F+\pi^n\mathcal{O}_K} \longrightarrow \frac{\left(\mathcal{O}_K/\pi^n\right)^\times}{\left(\mathcal{O}_F/\pi^n\right)^\times} \longrightarrow \frac{\left(\mathcal{O}_K/\pi\right)^\times}{\left(\mathcal{O}_F/\pi\right)^\times} \longrightarrow 1. \tag{6.14}$$

This induces an exact sequence

$$1 \longrightarrow \frac{1+\pi\mathcal{O}_K}{1+\pi\mathcal{O}_F+\pi^n\mathcal{O}_K}[\ell] \longrightarrow \frac{\left(\mathcal{O}_K/\pi^n\right)^\times}{\left(\mathcal{O}_F/\pi^n\right)^\times}[\ell] \longrightarrow \frac{\left(\mathcal{O}_K/\pi\right)^\times}{\left(\mathcal{O}_F/\pi\right)^\times}[\ell]. \tag{6.15}$$

If the characteristic of $\mathcal{O}_F/\pi$ is not ℓ (resp., ℓ), the ℓ-rank of the first group is 0 (resp., bounded by g) while the last group is bounded by 1 (resp., 0). Thus

$$\mathrm{rank}_\ell\left(\left(\mathcal{O}_K/\pi^n\right)^\times/\left(\mathcal{O}_F/\pi^n\right)^\times\right) \leq g. \tag{6.16}$$

This proves the first part.

Now we assume that $\mathcal{O}_c = \mathcal{O}_K$ and let $\xi \in \mathrm{Pic}(\mathcal{O}_K)[2]$ be a class. Then, we have the homomorphism

$$\beta : \mathrm{Pic}\left(\mathcal{O}_K\right) \longrightarrow \mathrm{Pic}\left(\mathcal{O}_F\right), \qquad \beta(\xi) = \xi \cdot \bar{\xi}. \tag{6.17}$$

Thus

$$\mathrm{rank}_2\left(\mathrm{Pic}\,\mathcal{O}_K\right) \le \mathrm{rank}_2\left(\mathrm{Pic}\,\mathcal{O}_F\right) + \mathrm{rank}_2 \ker\beta. \tag{6.18}$$

Now assume that $\xi \in \ker\beta$. Then both ξ^2, $\xi\bar{\xi}$ are trivial, and so is $\xi/\bar{\xi}$. Let ξ be represented by an $x \in \widehat{K}^\times$:

$$\mathrm{Pic}\left(\mathcal{O}_K\right) = \widehat{K}^\times/K^\times\widehat{\mathcal{O}}_K^\times. \tag{6.19}$$

Then, we have an expression

$$x/\bar{x} = tu, \quad t \in K^\times,\ u \in \widehat{\mathcal{O}}_K^\times. \tag{6.20}$$

Taking norm $N_{K/F}$ on both sides, we find that $N(t) \in \mathcal{O}_F^\times$. As t is uniquely determined modulo $\mathcal{O}_K^\times$, the norm $N_{K/F}(t) \in \mathcal{O}_F^\times$ is uniquely determined modulo $N(\mathcal{O}_K^\times)$. Thus, we have homomorphism

$$\gamma : \ker\beta \longrightarrow \mathcal{O}_F^\times/N_{K/F}\left(\mathcal{O}_K^\times\right). \tag{6.21}$$

As $(\mathcal{O}_F^\times)^2 \subset N(\mathcal{O}_K)$, the second group has 2-rank bounded by g. Thus, we have

$$\mathrm{rank}_2 \ker\beta \le g + \mathrm{rank}_2 \ker\gamma. \tag{6.22}$$

Now we assume that $\xi \in \ker\gamma$. Then we may take $t \in K^\times$ so that $N_{K/F}(t) = 1$. By Hilbert 90, there is an $s \in K^\times$ such that $t = \bar{s}/s$. Now replacing x by sx which does not change the class of ξ, we may assume that $t = 1$ in the expression in (6.20). Let I denote the elements in $\widehat{K}^\times$ which are invariant under conjugation modulo $\widehat{\mathcal{O}}_K^\times$. Then, we have

$$\ker\gamma = I/\left(I \cap K^\times\right)\widehat{\mathcal{O}}_K^\times[2]. \tag{6.23}$$

Now we consider the homomorphism

$$\theta : \ker\gamma \longrightarrow I/\widehat{F}^\times(I \cap K)^\times\widehat{\mathcal{O}}_K^\times. \tag{6.24}$$

As the second group is a quotient of the genus group of K and generated by ramified primes of $\mathcal{O}_K$, we have

$$\mathrm{rank}_2 \ker \gamma = \mathrm{rank}_2 \ker \theta + \delta, \tag{6.25}$$

where δ is number of ramified primes of $\mathcal{O}_K$ over $\mathcal{O}_F$.

It remains to estimate $\ker \theta$, which is certainly a quotient of $\mathrm{Pic}(\mathcal{O}_F)$. It follows that

$$\mathrm{rank}_2 \ker \theta \leq \mathrm{rank}_2 \left(\mathrm{Pic}\left(\mathcal{O}_F\right)\right). \tag{6.26}$$

■

Corollary 6.4. Let n be a fixed positive integer with decomposition $n = 2^t m$ with m odd. Then for any $\epsilon > 0$,

$$\#\,\mathrm{Pic}\,\mathcal{O}_c[n] \ll \mathrm{disc}\left(\mathcal{O}_c\right)^{\epsilon} \cdot \#\,\mathrm{Pic}\left(\mathcal{O}_K\right)[m]. \tag{6.27}$$

□

Proof. Consider the morphism $\alpha : \mathrm{Pic}(\mathcal{O}_c) \to \mathrm{Pic}(\mathcal{O}_K)$. Then, we have

$$\begin{aligned} \#\,\mathrm{Pic}\,\mathcal{O}_c[n] &\leq \#\,\mathrm{Pic}\left(\mathcal{O}_K\right)[n] \cdot \#\,\mathrm{Ker}\,\alpha[n] \\ &= \#\,\mathrm{Pic}\left(\mathcal{O}_K\right)[m] \cdot \#\,\mathrm{Pic}\left(\mathcal{O}_K\right)\left[2^t\right] \cdot \#\,\mathrm{Ker}\,\alpha[n]. \end{aligned} \tag{6.28}$$

It remains to estimate the last two terms. Write $n = \prod p_i^{n_i}$,

$$\begin{aligned} &\#\left(\mathrm{Pic}\,\mathcal{O}_K\right)\left[2^t\right] \leq 2^{t\,\mathrm{rank}_2\,\mathrm{Pic}\,\mathcal{O}_K} \leq 2^{t(2\,\mathrm{rank}_2\,\mathcal{O}_F + g + \delta)} \ll \mathrm{disc}\left(\mathcal{O}_K\right)^{\epsilon}, \\ &\#\,\mathrm{Ker}\,\alpha[n] \leq \prod_i p_i^{n_i \cdot \mathrm{rank}_{p_i} \cdot \mathrm{Ker}\,\alpha} \leq n^{g\mu} \ll N\left(c^2\right)^{\epsilon}. \end{aligned} \tag{6.29}$$

■

7 Mumford-Tate groups

In this section, we will compute the Mumford-Tate group for CM-points on M_U. When $d = 1$ or 2, our result is complete. When $d > 2$, we will give some examples where every CM-point has maximal Mumford-Tate group.

Let us fix a CM-type (K, S_K). Let Σ_K denote the set of all complex σ_0-embeddings of K which admit an action by $\mathrm{Gal}(\bar{\mathbb{Q}}/F_0)$ by composition. The character group of the algebraic torus $K^\times$ over F_0 is the group $\mathbb{Z}[\Sigma_K]$ of divisors on Σ_K with left action. We may also view $\mathbb{Z}[\Sigma_K]$ as the space of functions ϕ on Σ_K under the correspondence

$$\phi \longrightarrow \sum_{\sigma \in \Sigma_K} \phi(\sigma)[\sigma]. \tag{7.1}$$

With this convention, the group of characters of the CM-group $T_0 = K^\times/F^\times$ is the Galois submodule $\mathbb{Z}[\Sigma]^-$ of functions annihilated by $1+[c]$, where c is complex conjugation acting on Σ_K. Let $\Sigma_{\widetilde{K}}$ denote the set of all complex σ_0-embeddings of $\widetilde{K}$ equipped with action by $\mathrm{Gal}(\bar{\mathbb{Q}}/F_0)$. Then $\Sigma_{\widetilde{K}}$ can be identified with the set

$$\left\{gS_K,\ g \in \mathrm{Gal}\left(\bar{\mathbb{Q}}/F_0\right)\right\} \tag{7.2}$$

of subsets of Σ_K. Again, the groups of characters of the torus $\widetilde{T} := \widetilde{K}^\times/\widetilde{F}^\times$ can be identified with $\mathbb{Z}[\Sigma_{\widetilde{K}}]^-$. Recall that we have a norm morphism

$$N_0 : \widetilde{T}_0 \longrightarrow T_0, \qquad \tau N_0(x) = \prod_{\tau\in\sigma\circ S_K} \sigma(x) \tag{7.3}$$

for any $\tau \in \Sigma_K$. The Mumford-Tate group H is the restriction of scalars of the image H_0 of N_0. Let N_0^* denote the induced homomorphism of Galois modules of characters:

$$N_0^* : \mathbb{Z}\left[\Sigma_K\right]^- \longrightarrow \mathbb{Z}\left[\Sigma_{\widetilde{K}}\right]^-. \tag{7.4}$$

Proposition 7.1. With notation as above, the following assertions hold.

(1) For any $\phi \in \mathbb{Z}[\Sigma_K]^-$,

$$N_0^*(\phi)(gS) = \sum_{\sigma\in S_K} \phi(gs). \tag{7.5}$$

(2) Let $\Phi = \ker N_0^*$. The group of characters of the Mumford-Tate group H_0 is

$$X\left(H_0\right) = \mathbb{Z}\left[\Sigma_K\right]^-/\Phi. \tag{7.6}$$

(3) The order of the component group of $\ker N_0$ is bounded by a constant independent of K and S_K.

(4) Let p be a prime. Then p does not divide the component group of $\ker N_0$ if and only if

$$\ker\left(N_0^* \otimes \mathbb{F}_p\right) = \left(\ker N_0^*\right) \otimes \mathbb{F}_p. \tag{7.7}$$

□

Proof. The first assertion follows from a direct computation:

$$N_0^*\phi = \phi \circ N_0 = \sum_{g\in\Sigma_{\widetilde{K}}} \phi(\sigma) \sum_{\sigma\in g\circ S_K} \left[gS_K\right] = \sum_{g\in\Sigma_{\widetilde{K}}} \left(\sum_{s\in S_K} \phi(gs)\right)\left[gS_K\right]. \tag{7.8}$$

The second assertion follows from the decomposition of N_0:

$$\widetilde{T}_0 \twoheadrightarrow H \rightarrowtail T_0, \tag{7.9}$$

which induces a decomposition of the character groups

$$X(T_0) \twoheadrightarrow X(H_0) \rightarrowtail X(T_0). \tag{7.10}$$

The third assertion follows from the fact that the group of components of $\ker N_0$ is dual to the maximal torsion subgroup of $\mathrm{coker} N_0^*$ and the fact that there are only finitely many isomorphic classes of homomorphism N_0^* of $\mathbb{Z}$-modules.

For the last assertion, we notice that p does not divide the order of the component group if and only if the induced homomorphism

$$X(H_0) \otimes \mathbb{F}_p \longrightarrow X(\widetilde{T}_0) \otimes \mathbb{F}_p \tag{7.11}$$

remains injective. This is equivalent to the following identity:

$$X(H_0) \otimes \mathbb{F}_p = \mathbb{F}_p[\Sigma_K]^- / \operatorname{Ker}(N_0^* \otimes \mathbb{F}_p), \tag{7.12}$$

which is equivalent to

$$(\operatorname{Ker} N_0^*) \otimes \mathbb{F}_p = \operatorname{Ker}(N_0^* \otimes \mathbb{F}_p). \tag{7.13}$$

■

In the following, we want to compute H_0 in some special cases. The case where M_U has dimension 1 is easy.

Proposition 7.2. If S_K consists of a single element, then $\widetilde{K} = K$, $H = T$ and the reciprocity map $N_0^* : \widetilde{T}_0 \to T_0$ is the identity map. □

Now we consider the case where S_K has two elements.

Proposition 7.3. Assume that the set S_K consists of two elements σ_1, σ_2. Let K_0 (resp., F_0) be the subfield of K (resp., F) consisting of elements x such that $\sigma_1(x) = \bar{\sigma}_2(x)$. Then the torus H_0 is isomorphic to the kernel of the norm map

$$N_{K/K_0} : K^\times / F^\times \longrightarrow K_0^\times / F_0^\times. \tag{7.14}$$

Moreover, the kernel of the morphism

$$N_0 : \widetilde{T}_0 \longrightarrow T_0 \tag{7.15}$$

is connected. □

Proof. Let us fix an embedding of K into $\mathbb{C}$ by the element σ_1 in S_K and let L be a Galois closure of K over F_0 in $\mathbb{C}$. Write $\Delta = \mathrm{Gal}(L/F_0)$ and $\Delta_M = \mathrm{Gal}(L/M)$ for an extension M of F_0 in L. Then, we have inclusions

$$\Delta \supset \Delta_F = (\Delta_K, c) \supset \Delta_K, \tag{7.16}$$

where $c \in \Delta$ is complex conjugation. The set Σ_K is naturally identified with the cosets Δ/Δ_K. We lift $\sigma_1, \sigma_2 \in S_K$ to $e, s \in \Delta$, respectively. Then, Δ is generated by Δ_F and the set s.

With above notations, $\mathbb{Z}[\Sigma_K]^-$ can be identified with the set of functions on Δ right invariant by Γ_K and with eigenvalue -1 under c. The space Φ in Proposition 7.1 becomes the space of functions with the above property and such that

$$\phi(g) + \phi(gs) = 0, \quad \forall g \in \Delta, \tag{7.17}$$

or equivalently

$$\phi(g) = \phi(gcs). \tag{7.18}$$

This is equivalent to saying that ϕ is invariant under the subgroup $(\Delta_K, sc) = \Delta_{K_0}$. Then Φ is the subspace of functions on Δ invariant by Δ_{K_0}, and having eigenvalue -1 under c. Thus Φ is the character group of $K_0^\times/F_0^\times$.

On the other hand the exact sequence

$$1 \longrightarrow \ker N_{K/K_0} \longrightarrow K^\times/F^\times \longrightarrow K_0^\times/F_0^\times \longrightarrow 1 \tag{7.19}$$

induces a morphism of groups of characters

$$1 \longrightarrow X(K_0^\times/F_0^\times) \longrightarrow X(K^\times/F^\times) \longrightarrow X(\ker N_{K/K_0}) \longrightarrow 1. \tag{7.20}$$

If we identified the first two groups of characters as functions on Δ invariant under right translation by Δ_{K_0} and Δ_K, respectively, then the map is the natural inclusion. Thus, we have shown that $X(H) = X(\ker N_{K/K_0})$. It follows that $H = \ker N_{K/K_0}$.

To show that N_0 has connected kernel, we want to verify part 3 of Proposition 7.1. Notice that $\ker(N_0^* \otimes \mathbb{F}_p)$ is the set of $\mathbb{F}_p$-valued functions ψ satisfying

$$\psi(g) + \psi(gs) = 0. \tag{7.21}$$

The same proof as above shows that this is equivalent to that ψ is invariant under Δ_{K_0} and has eigenvalue -1 under c. Thus ψ is a reduction of a $\mathbb{Z}$-valued function invariant under Δ_{K_0}. Thus the equality of part 4 of Proposition 7.1 holds. ∎

Remarks. (1) If $K_0 = F_0$, then $H = T$.

(2) If $K_0 \neq F_0$, then $K = K_0 \cdot F$.

(3) Conversely, let K_0 be an imaginary quadratic extension of F_0 and take $K = K_0 \cdot F$. There always exists a lifting of S to S_K such that K_0 is not fixed by the elements σ_1, σ_2 in S_K. Then K_0 must be fixed by $\sigma_1, c\sigma_2$. By **Proposition 7.3**, H is the kernel of $K^\times/F^\times \to K_0^\times/F_0^\times$.

It seems hard to write a general description of the Mumford-Tate groups for any quaternion Shimura variety of dimension 3 or higher. In the following, we give a statement for cases where

(i) every CM-point has maximal Mumford-Tate group;

(ii) the subconvexity bound and ϵ conjecture imply the equidistribution of Galois orbits of CM-points.

Proposition 7.4. Let Σ be the set of σ_0-embeddings of F equipped with a natural action by $\mathrm{Gal}(\bar{\mathbb{Q}}/F_0)$. Assume that there is no nonzero function

$$\psi : \Sigma \longrightarrow \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z} \tag{7.22}$$

such that

$$\sum_{s \in S} \psi(gs) = 0, \quad \forall g \in \mathrm{Gal}\left(\bar{\mathbb{Q}}/F_0\right). \tag{7.23}$$

Then for any CM-point on M_U, $H = T$. □

Proof. The reduction $(\mathrm{mod}\, 2)$ of any $\phi \in \Phi$ will be invariant under complex conjugation, and thus define an $\mathbb{F}_2$-valued function on Σ. The assumption then implies that $\phi \equiv 0 \bmod 2$. Thus $\Phi/2\Phi = 0$ and then $\Phi = 0$. ■

We may apply the proposition when F/F_0 is abelian.

Proposition 7.5. Assume that F/F_0 is abelian with Galois group Γ satisfying that there is no character $\chi : \Gamma \to \bar{\mathbb{F}}_2^\times$ such that

$$\sum_{s \in S} \chi(s) = 0. \tag{7.24}$$

Then for any CM-point on M_U, $H = T$. □

Proof. We want to show that the condition of **Proposition 7.4** is satisfied. The proof is divided into two steps. In the first step, we reduce the proof to the case where $\#\Gamma$ is odd. Then we prove the lemma when $\#\Gamma$ is odd.

Without loss of generality, we assume that $e \in S$. Then the equation in the proposition gives

$$\psi(g) = \sum_{s \neq e} \psi(gs), \quad \forall g \in \Gamma. \tag{7.25}$$

Taking this equation with g replaced by gs, we then obtain

$$\psi(g) = \sum_{s \neq 1, t \neq 1} \psi(gst) = \sum_{s \neq 1} \psi\left(gs^2\right). \tag{7.26}$$

We may repeat this step to obtain

$$\psi(g) = \sum_{s \neq 1} \psi\left(gs^{2^n}\right) \tag{7.27}$$

for any $n \in \mathbb{N}$. Let $\Gamma = \mathbb{Z}/2^m \times \Gamma'$ be the decomposition of Γ with Γ' a commutative group of odd order. Take an n so that $g \to g^{2^n}$ is the projection $\Gamma \to \Gamma'$. Let S' be the set of elements in Γ' whose preimage has odd cardinality in the projection $S \to \Gamma'$. For any $h \in \Gamma$, the function $\psi_h(g') := \psi(gh)$ $(g' \in \Gamma)$ will satisfy the equation

$$\sum_{s \in S'} \psi_h(g's') = 0, \quad \forall g' \in \Gamma'. \tag{7.28}$$

Thus, we are reduced to proving that $\psi_h = 0$ on Γ' for all h. Since S' also has odd cardinality, we are in the case where Γ is odd.

Assume that Γ is odd. Let Ψ be the space of functions ψ on Γ satisfying the equation in the lemma. Then $\Psi \otimes \bar{\mathbb{F}}_2$ will be a direct sum of characters. Thus we need to show that there is no character $\chi : \Gamma \to \bar{\mathbb{F}}_2^\times$ such that

$$\sum_{s \in S} \chi(s) = 0. \tag{7.29}$$

■

Corollary 7.6. Assume that $\#S$ is odd, and assume that F is abelian over F_0 such that the order of $\mathrm{Gal}(F/F_0)$ is a power of 2. Then, $H = T$ for every CM-point on M_U. □

We have some further statements for abelian case.

Proposition 7.7. Assume that F is abelian over F_0 with Galois group Γ such that the following conditions are verified:

(1) Γ has cyclic 2-primary part $\Gamma[2^\infty]$;

(2) for all characters $\alpha : \Gamma \to \mu_{p^n}$ of order a positive power of an odd prime p, $\alpha(S)$ does not contain any coset of μ_p.

Then $H = T$ for any CM-point on M_U. □

Corollary 7.8. Assume the following conditions are verified:

(1) $\#S$ is odd and smaller than p for all odd prime factor p of $[F : F_0]$;

(2) the Galois group $\mathrm{Gal}(F/F_0)$ is commutative with cyclic 2-primary 2.

Then $H = T$ for every CM-point on M_U. □

Proof of Proposition 7.7. Let us fix an embedding of K into $\mathbb{C}$ by an element in S_K, and let L be a Galois closure of K over F_0 in $\mathbb{C}$. Write $\Delta = \mathrm{Gal}(L/F_0)$ and $\Lambda = \mathrm{Gal}(L/F)$, $\Lambda_K = \mathrm{Gal}(L/K)$, then we have inclusions

$$\Delta \supset \Lambda = \left(\Lambda_K, c\right) \supset \Lambda_K, \tag{7.30}$$

where $c \in \Delta$ is complex conjugation. The following lemma gives an almost-commutative-lifting of Γ.

Lemma 7.9. Consider an exact sequence of finite groups:

$$1 \longrightarrow \Lambda \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow 1. \tag{7.31}$$

Assume the following properties are satisfied.

(1) Γ is commutative and fits in an exact sequence

$$1 \longrightarrow \Gamma_2 \longrightarrow \Gamma \longrightarrow \Gamma_1 \longrightarrow 1 \tag{7.32}$$

so that one of Γ_i is odd and one is 2-primary and cyclic.

(2) Λ is commutative and has order a power of 2.

Then Δ contains a commutative subgroup $\widetilde{\Gamma}$ mapping surjectively onto Γ. □

Proof. Indeed, the extension

$$0 \longrightarrow \Lambda \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow 0 \tag{7.33}$$

is given by an element $\xi \in H^2(\Gamma, \Lambda)$ which can be computed using the spectral sequence $H^i(\Gamma_1, H^j(\Gamma_2, \Lambda))$. As Λ has order a power of 2, the cohomology of odd groups vanishes. Thus, we have

$$H^2(\Gamma, \Lambda) = \begin{cases} H^2\left(\Gamma_1, H^0\left(\Gamma_2, \Lambda\right)\right) & \text{if } \Gamma_2 \text{ is odd,} \\ H^0\left(\Gamma_1, H^2\left(\Gamma_1, \Lambda\right)\right) & \text{if } \Gamma_1 \text{ is odd.} \end{cases} \tag{7.34}$$

Notice that the H^2 cohomology of a cyclic group is equal to Tate's cohomology group $\widehat{H}^0$. Thus, we have

$$\begin{aligned} H^2(\Gamma, \Lambda) &= \begin{cases} \widehat{H}^0\left(\Gamma_1, H^0\left(\Gamma_2, \Lambda\right)\right) & \text{if } \Gamma_2 \text{ is odd,} \\ H^0\left(\Gamma_1, \widehat{H}^0\left(\Gamma_2, \Lambda\right)\right) & \text{if } \Gamma_1 \text{ is odd,} \end{cases} \\ &= \begin{cases} H^0(\Gamma, \Lambda)/N_{\Gamma_1} H^0\left(\Gamma_2, \Lambda\right) & \text{if } \Gamma_2 \text{ is odd,} \\ H^0(\Gamma, \Lambda)/N_{\Gamma_2} H^0\left(\Gamma_1, \Lambda\right) & \text{if } \Gamma_1 \text{ is odd,} \end{cases} \end{aligned} \tag{7.35}$$

where N_{Γ_i} is the norm defined by Γ_i. Let x be the element in $H^0(\Gamma, \Lambda)$ representing ξ. Then there is commutative subgroup $\widetilde{\Gamma}$ of Δ which maps surjectively onto Γ with kernel X generated by x. ■

The set Σ_K is naturally identified with the cosets Δ/Λ_K. We pick up liftings of $S_K \subset \Sigma_K$ to $S_L \subset \widetilde{\Gamma}$ containing the unit element

$$e \in S_L \simeq S_K \simeq S. \tag{7.36}$$

Then Δ is generated by Λ and the set S_L.

With above notations, $\mathbb{Z}[\Sigma_K]^-$ can be identified with the set of functions on Δ invariant under the right by Λ_K and with eigenvalue -1 under c. The space Φ in Proposition 7.1 becomes the space of functions with the above property and such that

$$\sum_{\sigma \in S_L} \phi(g\sigma) = 0, \quad \forall g \in \Delta. \tag{7.37}$$

Since Δ is generated by S_L and Λ and Λ is normal in Δ, we have an expression

$$\Delta = \widetilde{\Gamma} \cdot \Lambda_K. \tag{7.38}$$

Thus the restriction on $\widetilde{\Gamma}$ defines injections,

$$\mathbb{Z}\left[\Delta/\Lambda_K\right]^- \longrightarrow \mathbb{Z}[\widetilde{\Gamma}], \qquad \Phi \longrightarrow \widetilde{\Phi}, \tag{7.39}$$

where $\widetilde{\Phi}$ is the set of elements in $\mathbb{Z}[\widetilde{\Gamma}]$ satisfying the same (7.37). Since $\widetilde{\Gamma}$ is abelian, $\widetilde{\Phi} \otimes \mathbb{C}$ is generated by characters. Now we apply the following lemma to complete the proof of Proposition 7.7.

Lemma 7.10. Let Γ be a finite commutative group and let W be a subset of Γ of odd order. Assume that for each nontrivial character $\alpha : \Gamma \to \mu_{p^n}$ whose order is a power of an odd prime, the image of W does not contain any coset of μ_p. Then there is no character χ of Γ satisfying the equation

$$\sum_{w \in \Gamma} \chi(w) = 0. \tag{7.40}$$

□

Proof. First we want to reduce the proof to the case where Γ has no prime bigger than $\#W$.

Let p be an odd prime dividing $\#\Gamma$. Let χ be a character on Γ which has decomposition $\chi = \chi_1 \cdot \chi_2$ into characters with orders prime to p and a power of p, respectively. Let $\Gamma_i = \ker \chi_i$, then $\Gamma / \ker \chi = \Gamma_1 \cdot \Gamma_2$. The equation in the proposition becomes

$$0 = \sum_{w \in W} \chi(w) = \sum_{\zeta \in \chi_2(W)} \phi(\zeta, \chi_1)\zeta, \tag{7.41}$$

where

$$\phi(\zeta, \chi_1) = \sum_{w \in W_\zeta} n_\zeta(w)\chi_1(w), \tag{7.42}$$

and where W_ζ is the projection of $\chi_2^{-1}(\zeta) \cap W$ onto Γ_1, and $n_\zeta(w) \in \mathbb{N}$ such that

$$\sum_{\zeta} \sum_{w \in W_\zeta} n_\zeta(w) = \#W. \tag{7.43}$$

Let K be the cyclotomic field generated by values of $\phi(\delta, \chi_1)$. Then K is disjoint with the field generated by χ_2. If χ_2 is nontrivial, then the above equation implies that

$$\phi(\zeta, \chi_1) = \phi(\zeta\eta, \chi_1), \quad \forall \eta \in \mu_p. \tag{7.44}$$

If one of the $\phi(\zeta, \chi_1) \neq 0$, then $\chi_2(W)$ contains a coset of μ_p which contradicts the assumption of the lemma.

By induction on the number of odd prime factors of $\#\Gamma$, we reduce to the case where $\#\Gamma = 2^n$ is a power of 2, and the equation is

$$\sum_{w \in W} n(w)\chi(w) = 0, \tag{7.45}$$

where $n(w) \in \mathbb{N}$ such that $\sum n(w)$ is odd. This is impossible as the only nontrivial relation among 2^nth root (like $\chi(w)$) of unity is $\zeta + (-\zeta) = 0$. ■

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DISCRETE OSCILLATION THEORY

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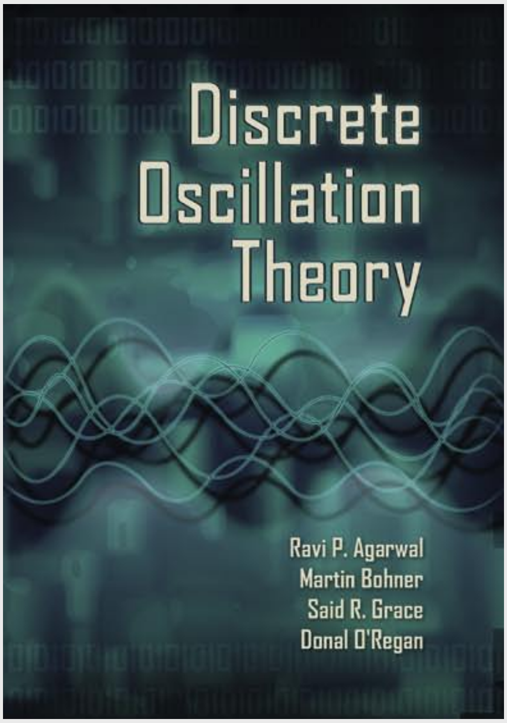

This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

The book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists. Besides serving as a reference tool for researchers in difference equations, this book can also be easily used as a textbook for undergraduate or graduate classes. It is written at a level easy to understand for college students who have had courses in calculus.