TAUTOLOGICAL SHEAVES ON HILBERT SCHEMES OF POINTS

ZHILAN WANG, JIAN ZHOU

Abstract. We propose some conjectures on the generating series of (equivariant) Euler characteristics of some vector bundles constructed from the tautological bundles on Hilbert schemes of points on affine k-spaces. We establish the surface case of these conjectures and present some verifications of the higher dimensional cases.

1. Introduction

It is an interesting fact that many invariants of the Hilbert schemes of points on a projective surface can be determined explicitly by the corresponding invariants of the surface. These include the Betti numbers [10], Hodge numbers [12], cobordism classes [7], and elliptic genus [4]. In this paper we extend such results to the (equivariant) Euler characteristics of some naturally defined vector bundles related to the tautological vector bundles on the Hilbert schemes $X^{[n]}$ of points in a projective or quasi-projective variety X.

Let X be a smooth projective or quasi-projective k-variety. Let $\mathcal{Z}_n \subset X \times X^{[n]}$ be the universal family of subschemes parameterized by $X^{[n]}$. Denote by $p_1 : \mathcal{Z}_n \to X$ and $\pi : \mathcal{Z}_n \to X^{[n]}$ the projection onto the X and $X^{[n]}$ respectively. For any locally free sheaf F on X let $F^{[n]} = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* F)$. With this notation we write $\xi_n = \xi_n^X = \mathcal{O}_X^{[n]}$. We make the following conjectures:

Conjecture 1. For an arbitrary smooth k-dimensional projective variety X and arbitrary holomorphic line bundle F on X, one has

$$\sum_{n\geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*}) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(X, \Lambda_{-u^n} L \otimes \Lambda_{-v^n} L^*). \tag{1}$$

Conjecture 2. For two birationally equivalent smooth k-dimensional projective varieties X and Y, one has

$$\sum_{n\geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} \xi_n^X \otimes \Lambda_{-v} \xi_n^{X*}) = \sum_{n\geq 0} Q^n \chi(Y^{[n]}, \Lambda_{-u} \xi_n^Y \otimes \Lambda_{-v} \xi_n^{Y*}). \tag{2}$$

Here we have used the following notation: For a rank n holomorphic vector bundle $\pi : E \to M$, denote by $\Lambda^i E$ the i-th exterior power of E, and $\Lambda_t E = \sum_{i=0}^{n} t^i \Lambda^i E$.

We will establish these conjectures for projective surfaces. Our strategy is to reduce to the cases of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, using a result due to Ellingsrud-Göttsche-Lehn [7] on the universal formula for integrals of multiplicative classes on Hilbert schemes. These two surfaces are toric surfaces, so one can use their natural torus actions and the induced actions on their Hilbert schemes to further reduce to the equivariant case of $\mathbb{C}^2$. Then we use a localization calculation to establish a connection with the theory of Macdonald polynomials and complete the proof by the summation formula for Macdonald polynomials.

Let $T^k = (\mathbb{C}^*)^k$ act on $\mathbb{C}^k$ whose actions on the linear coordinates $z_1, \dots, z_k$ are given by

$$(t_1, \dots, t_k) \cdot z_j = t_j z_j.$$

This action induces actions on $(\mathbb{C}^k)^{[n]}$ and ξ_n. Since $(\mathbb{C}^k)^{[n]}$ are quasi-projective, one can consider the equivariant indices of equivariant coherent sheaves on them. For a vector $A = (a_1, \dots, a_k) \in \mathbb{Z}^k$, denote by $\mathcal{O}^A_{\mathbb{C}^k}$ the T^k-equivariant line bundle on $\mathbb{C}^k$ with weight A. Recall the universal family $\mathcal{Z}_n$ lies in $\mathbb{C}^k \times (\mathbb{C}^k)^{[n]}$, denote by $p_1 : \mathcal{Z}_n \to \mathbb{C}^k$ the projection onto the first factor. Let $\xi^A_n = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* \mathcal{O}^A_{\mathbb{C}^k})$.

Conjecture 3. The following identity holds for $k \geq 2$:

$$\begin{aligned} &\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \Lambda_{-u}\xi^A_n \otimes \Lambda_{-v}(\xi^A_n)^*)(t_1, \dots, t_k) \\ &= \exp\Big(\sum_{n=1}^\infty \frac{(1-u^n t^{nA})(1-v^n t^{-nA})Q^n}{n\prod_{i=1}^k (1-t_i^n)}\Big), \end{aligned} \tag{3}$$

where

$$t^{nA} = t_1^{na_1} \cdots t_k^{na_k}.$$

Special cases of this conjecture for $k = 2$ were proved in an earlier version of this paper by the second author based on some new relationship with Macdonald polynomials. We will present the proof of the $k = 2$ case using essentially the same method. Note that various authors have established relationship between equivariant homology/cohomology classes or K-theory classes to the Jack or Macdonald polynomials [14, 27, 22]. In our case we relate the contributions of fixed points by localizations to specializations of Macdonald polynomials, and use summation formula for them to derive our results. We will also present some computations that verify this conjecture in higher dimensions. We have checked by Maple for some small values of k and n (see the end of Section 6).

By standard series manipulations one can rewrite the right-hand side of (3) as an infinite product, which suggests some action of an infinite dimensional Heiseinberg-Clifford algebra. When $k = 2$, $A = (0, \dots, 0)$, by setting $t_1 = t_2^{-1} = q$, $u = v^{-1} = t$, and $Q = 1$, the right-hand side of (3) becomes:

$$\prod_{n=1}^\infty \frac{(1-q^n)^{2n}}{(1-tq^n)^n(1-t^{-1}q^n)^n}.$$

This is a Gopakumar-Vafa type infinite product which might correspond to the topological string partition function of some local Calabi-Yau geometry. A related function

$$\prod_{n=1}^\infty \frac{(1-q^n)^2}{(1-tq^n)(1-t^{-1}q^n)}$$

appeared in helicity supertrace in string theory (see e.g. [17]).

Recall the Hilbert schemes of points on an algebraic surface are the moduli spaces of torsion free sheaves of rank one. It will be interesting to see whether it is possible to make extensions of our results to moduli spaces of stable sheaves of ranks > 1. For a closely related work, see [21].

Acknowledgements. The second author thanks Professors Kefeng Liu, Hiraku Nakajima, Zhenbo Qin and Weiqiang Wang for email correspondences which have improved this paper. Originally the author only considered the $k = 2$ case of (3). The considerations of the $k > 2$ cases were prompted by discussions with Professors Jean-Michel Bismut, Francois Labourie and Xiaonan Ma during a visit to Orsay in 2006, where the computer verification of the $k > 2$ cases mentioned in the end of Section 6 was carried out. This work is partially supported by research grants from NSFC and by NKBRPC (2006CB805905).

2. Motivations and Conjectures

In this section we first recall some results for Hilbert schemes of points in surfaces as motivations, and then present our conjectures. Special cases of the conjectures are also given.

2.1. **Some well-known results for Hilbert schemes of points in surfaces.** As motivations, let us recall some well-known results for the tangent bundles for the Hilbert schemes $S^{[n]}$ of points in an algebraic surface S. By a theorem of Forgarty [8], $S^{[n]}$ are nonsingular projective varieties of dimension $2n$, and each Hilbert-Chow morphism $\pi_n : S^{[n]} \to S^{(n)}$ to the n-symmetric product $S^{(n)}$ is a resolution of singularities. A general phenomenon is that many invariants of $S^{[n]}$ are identical to the corresponding *orbifold* invariants of $S^{(n)}$. This leads to nice expressions for the generating series of these invariants. For example, for the Betti numbers one has Göttsche's formula [10]:

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q^n \sum_{i=0}^{4n} b_i(S^{[n]})(-t)^i \\
&= \prod_{m=0}^{\infty} \frac{(1 - t^{2m+1}Q^{m+1})^{b_1(S)}(1 - t^{2m+3}Q^{m+1})^{b_3(S)}}{(1 - t^{2m}Q^{m+1})^{b_0(S)}(1 - t^{2m+2}Q^{m+1})^{b_2(S)}(1 - t^{2m+4}Q^{m+1})^{b_4(S)}} \\
&= \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1 - t^{2n}Q^n)} \sum_{i=0}^{4} (-t^n)^i b_i(S).
\end{aligned} \tag{4}$$

For the Hodge numbers one has [12]:

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q^n \sum_{i=0}^{2n} h^{i,j}(S^{[n]})(-x)^i(-y)^j \\
&= \prod_{m=0}^{\infty} \prod_{0 \le i,j \le 2} (1 - x^{i+m}y^{j+m}Q^{m+1})^{(-1)^{i+j+1}h^{i,j}(S)} \\
&= \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1 - x^n y^n Q^n)} \sum_{0 \le i,j \le 2} (-x^n)^i(-y^n)^j h^{i,j}(S).
\end{aligned} \tag{5}$$

These formulas can be understood either by the Heisenberg algebra structure introduced by Nakajima [25] and Grojnowski [13], or by the orbifold Betti and Hodge numbers of the symmetric products [11, 31].

By taking $t = 1$ in (4) one gets

$$\sum_{n=0}^{\infty} Q^n \chi(S^{[n]}) = \frac{1}{\prod_{n=1}^{\infty}(1 - Q^n)^{\chi(S)}} = \exp \sum_{n=1}^{\infty} \frac{1}{n} \frac{Q^n}{1 - Q^n} \chi(S). \tag{6}$$

By taking $x=0$, $y=1$ in (5) one gets

$$\sum_{n=0}^{\infty} Q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = \frac{1}{(1-Q)^{\chi(S,\mathcal{O}_S)}} = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(S, \mathcal{O}_S). \tag{7}$$

By taking $y=1$ and change x to y in (5) one gets

$$\begin{aligned} \sum_{n=0}^{\infty} Q^n \chi(S^{[n]}, \Lambda_{-y} T^* S^{[n]}) &= \prod_{m=0}^{\infty} \prod_{i=0}^{2} (1 - y^{i+m} Q^{m+1})^{(-1)^{i+1} \chi(S, \Lambda^i T^* S)} \\ &= \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-y^n Q^n)} \chi(S, \Lambda_{-y^n} T^* S). \end{aligned} \tag{8}$$

2.2. **Equivariant versions.** In the previous subsection we have only recalled results for Hilbert schemes of points on projective surfaces. There are versions for noncompact spaces such as $\mathbb{C}^2$, but one then has to work in the equivariant setting. We use the natural T^2-action on $\mathbb{C}^2$ given by

$$(t_1, t_2) \cdot z_i = t_i z_i$$

on the linear coordinates (z_1, z_2). This action induces natural actions on $(\mathbb{C}^2)^{[n]}$. By results of Thomason [30] one can define equivariant indices for coherent sheaves on $(\mathbb{C}^2)^{[n]}$ and compute them by Lefschetz formula. One has the following analogues of (7) and (8) (cf. [28, 20]):

$$\sum_{n=0}^{\infty} Q^n \chi((\mathbb{C}^2)^{[n]}, \mathcal{O}_{(\mathbb{C}^2)^{[n]}})(t_1, t_2) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2})(t_1^n, t_2^n), \tag{9}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} Q^n \chi((\mathbb{C}^2)^{[n]}, \Lambda_{-y} T^*(\mathbb{C}^2)^{[n]})(t_1, t_2) \\ = &\exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-y^n Q^n)} \chi(\mathbb{C}^2, \Lambda_{-y^n} T^* \mathbb{C}^2)(t_1^n, t_2^n). \end{aligned} \tag{10}$$

Based on these formulas, we make the following:

Conjecture 4. The following identity holds:

$$\begin{aligned} &\sum_{n=0}^{\infty} Q^n \chi((\mathbb{C}^2)^{[n]}, \Lambda_{-y} T(\mathbb{C}^2)^{[n]})(t_1, t_2) \\ = &\exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-y^n Q^n t_1^{-n} t_2^{-n})} \chi(\mathbb{C}^2, \Lambda_{-y^n} T\mathbb{C}^2)(t_1^n, t_2^n). \end{aligned} \tag{11}$$

Conjecture 5. For a smooth projective surface, the following identity holds:

$$\begin{aligned} \sum_{n=0}^{\infty} Q^n \chi(S^{[n]}, \Lambda_{-y} T S^{[n]}) &= \prod_{m=0}^{\infty} \prod_{i=0}^{2} (1 - y^{i+m} Q^{m+1})^{(-1)^{i+1} \chi(S, \Lambda^i T S)} \\ &= \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-y^n Q^n)} \chi(S, \Lambda_{-y^n} T S). \end{aligned} \tag{12}$$

2.3. **The conjectures for tautological bundles.** In (10) the cotangent bundles of Hilbert schemes are involved. We now generalize (10) to some conjectures regarding for tautological bundles presented in the Introduction.

Let X be a smooth projective or projective k-variety. Let $\mathcal{Z}_n \subset X \times X^{[n]}$ be the universal family of subschemes parameterized by $X^{[n]}$. Denote by $p_1 : \mathcal{Z}_n \to S$ and $\pi : \mathcal{Z}_n \to X^{[n]}$ the projection onto the X and $X^{[n]}$ respectively. For any locally free sheaf F on X let $F^{[n]} = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* F)$. In particular, write $\xi_n = \pi_* \mathcal{O}_{\mathcal{Z}_n}$. The cup products with the Chern characters of ξ_n, combined with the Heisenberg algebra structure introduced by Nakajima [25] and Grojnowski [13], have played a very important role in the study of cohomology rings of Hilbert schemes of points in surfaces (cf. [19, 22, 3]).

We conjecture that for an arbitrary smooth k-dimensional projective variety X and arbitrary holomorphic vector bundle F on X, one has

$$\sum_{n\geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} F^{[n]} \otimes \Lambda_{-v} F^{[n]*}) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(X, \Lambda_{-u^n} F \otimes \Lambda_{-v^n} F^*). \tag{13}$$

In particular, when $F = \mathcal{O}_X$,

$$\sum_{n\geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} \xi_n \otimes \Lambda_{-v} \xi_n^*) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(X, \Lambda_{-u^n} \mathcal{O}_X \otimes \Lambda_{-v^n} \mathcal{O}_X^*). \tag{14}$$

Let $T^k = (\mathbb{C}^*)^k$ act on $\mathbb{C}^k$ whose actions on the linear coordinates $z_1, \dots, z_k$ are given by

$$(t_1, \dots, t_k) \cdot z_j = t_j z_j.$$

This action induces actions on $(\mathbb{C}^k)^{[n]}$ and ξ_n. Since $(\mathbb{C}^k)^{[n]}$ are quasi-projective, one can consider the equivariant indices of equivariant coherent sheaves on them. For a vector $A = (a_1, \dots, a_k) \in \mathbb{Z}^k$, denote by $\mathcal{O}^A_{\mathbb{C}^k}$ the T^k-equivariant line bundle on $\mathbb{C}^k$ with weight A. Recall the universal family $\mathcal{Z}_n$ lies in $\mathbb{C}^k \times (\mathbb{C}^k)^{[n]}$, denote by $p_1 : \mathcal{Z}_n \to \mathbb{C}^k$ the projection onto the first factor. Let $\xi_n^A = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* \mathcal{O}^A_{\mathbb{C}^k})$. We conjecture that

$$\begin{aligned} &\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, \dots, t_k) \\ =&\exp(\sum_{n=1}^{\infty} \frac{(1-u^n t^{nA})(1-v^n t^{-nA}) Q^n}{n \prod_{i=1}^k (1-t_i^n)}), \end{aligned} \tag{15}$$

where

$$t^{nA} = t_1^{na_1} \cdots t_k^{na_k}.$$

In particular,

$$\begin{aligned} &\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \Lambda_{-u} \xi_n \otimes \Lambda_{-v} \xi_n^*)(t_1, \dots, t_k) \\ =&\exp(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{(1-u^n)(1-v^n)}{\prod_{i=1}^k (1-t_i^n)}). \end{aligned} \tag{16}$$

2.4. **Special cases.** By taking $u = v = 0$ in (16),

$$\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \mathcal{O}_{(\mathbb{C}^k)^{[n]}})(t_1, \dots, t_k) = \exp(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{1}{\prod_{i=1}^k (1-t_i^n)}). \tag{17}$$

By taking $u = 0$ or $v = 0$ in (16),

$$\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \Lambda_{-u}\xi_n^{\pm})(t_1, \dots, t_k) = \exp(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{(1-u^n)}{\prod_{i=1}^k (1-t_i^n)}). \tag{18}$$

We write $\xi_n^+ = \xi_n$, $\xi_n^- = \xi_n^*$. By taking derivative at $u = 0$ on both sides of (18):

$$\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \xi_n^{\pm})(t_1, \dots, t_k) = \frac{Q}{\prod_{i=1}^k (1-t_i)} \exp(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{1}{\prod_{i=1}^k (1-t_i^n)}). \tag{19}$$

Changing Q to $(-Q/u)$, and taking $u \to \infty$ on both sides of (18):

$$\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \det \xi_n^{\pm})(t_1, \dots, t_k) = \exp(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{(-1)^{n-1}}{\prod_{i=1}^k (1-t_i^n)}). \tag{20}$$

In the same fashion one gets from (16)

$$\begin{aligned} &\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \xi_n^{\pm} \otimes \det \xi_n^{\mp})(t_1, \dots, t_k) \\ =&\frac{Q}{\prod_{i=1}^k (1-t_i)} \exp(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} Q^n}{n \prod_{i=1}^k (1-t_i^n)}). \end{aligned} \tag{21}$$

We also conjecture that

$$\begin{aligned} &\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \xi_n \otimes \det \xi_n)(t_1, \dots, t_k) \\ =&\frac{Q}{\prod_{i=1}^k (1-t_i)} \exp(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} Q^n}{n} \frac{1 - \prod_{i=1}^k (1-t_i^n)}{\prod_{i=1}^k (1-t_i^n)}) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \xi_n^* \otimes \det \xi_n^*)(t_1, \dots, t_k) \\ =&\frac{Q}{\prod_{i=1}^k (1-t_i)} \exp(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} Q^n}{n} \frac{1 - \prod_{i=1}^k (1-t_i^{-n})}{\prod_{i=1}^k (1-t_i^n)}). \end{aligned} \tag{23}$$

2.5. **A formula in dimension one.** In the above we have assumed that $k > 1$. It turns out that some of the above formulas still hold when $k = 1$, for example,

$$\sum_{n\geq 0} t^n \chi(\mathbb{C}^{[n]}, \Lambda_{-u}\xi_n)(q) = \exp(\sum_{n=1}^{\infty} \frac{t^n}{n} \frac{1-u^n}{1-q}). \tag{24}$$

This can be proved by the Lefschetz formula as follows. The linear coordinates on $\mathbb{C}^{[n]}$ are given by $p_i(z_1, \dots, z_n) = z_1^i + \cdots + z_n^i$. The induced torus action on $\mathbb{C}^{[n]}$ is given by

$$q \cdot p_i = q^i p_i, \quad q \in \mathbb{C}^*.$$

This action has only one fixed point at $p_1 = \cdots = p_n = 0$, the cotangent bundle and the tautological bundle has the following weight decompositions at this point:

$$T^*\mathbb{C}^{[n]} = q + \cdots + q^n,$$
$$\xi_n = 1 + q + \cdots + q^{n-1},$$

hence by the Lefschetz formula [30] we have

$$\sum_{n=0}^{\infty} t^n \chi(\mathbb{C}^{[n]}, \Lambda_{-y}\xi_n)(q) = \sum_{n=0}^{\infty} t^n \prod_{i=1}^{n} \frac{1 - yq^{i-1}}{1 - q^i}. \tag{25}$$

The right-hand side of this equality is a specialization of the complete symmetric function h_n. Indeed, let

$$H(t) = \sum_{n=0}^{\infty} h_n t^n = \prod_{i=0}^{\infty} \frac{1 - bq^i t}{1 - aq^i t},$$

then we have ([24, p. 27, Example 5]):

$$h_n = \prod_{i=1}^{n} \frac{a - bq^{i-1}}{1 - q^i}, \qquad e_n = \prod_{i=1}^{n} \frac{aq^{i-1} - b}{1 - q^i}, \qquad p_r = \frac{a^n - b^n}{1 - q^n}.$$

These are related to the Cauchy's q-binomial identity (see e.g. [1, Chapeter II]):

$$\sum_{n=0}^{\infty} t^n \prod_{i=1}^{n} \frac{a - bq^{i-1}}{1 - q^i} = \prod_{i=0}^{\infty} \frac{1 - bq^i t}{1 - aq^i t}. \tag{26}$$

By taking $a = 1$ and $b = y$ one gets from (25) the following identity:

$$\sum_{n=0}^{\infty} t^n \chi(\mathbb{C}^{[n]}, \Lambda_{-y}(\xi_n))(q) = \prod_{i=0}^{\infty} \frac{1 - yq^i t}{1 - q^i t}. \tag{27}$$

By a standard series manipulation one can rewrite (27) in the form of (24). Many other hypergeometric series also appear when one considers suitable bundles on $\mathbb{C}^{[n]}$, including some of those appearing in the Rogers-Ramanujan identities. We will report on this in a separate work.

3. The $k = 2$ Case of Conjecture 3

In this section we will prove (3) for $k = 2$. See [15] and [28] for references on equivariant indices on Hilbert schemes of points on $\mathbb{C}^2$.

3.1. **Localizations on Hilbert schemes of the affine plane.** By a theorem of Fogarty [8] the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ is a nonsingular variety of dimension $2n$. The torus action on $\mathbb{C}^2$ given by

$$(t_1, t_2) \cdot x = t_1 x, \ \ (t_1, t_2) \cdot y = t_2 y$$

on linear coordinates induces an action on $(\mathbb{C}^2)^{[n]}$. The fixed points are isolated and parameterized by partitions $\mu = (\mu_1, \ldots, \mu_l)$ of weight n. They correspond to ideals

$$I_\mu = \langle y^{\mu_1}, xy^{\mu_2}, \ldots, x^{l-1}y^{\mu_l}, x^l \rangle.$$

The weight decomposition of the tangent bundle of $T(\mathbb{C}^2)^{[n]}$ at a fixed point μ is given by [6]:

$$\sum_{(i,j)\in\mu}(t_1^{(\mu_j^t-i)}t_2^{-(\mu_i-j+1)}+t_1^{-(\mu_j^t-i+1)}t_2^{(\mu_i-j)}) \tag{28}$$

$$= \sum_{s\in\mu}(t_1^{l(s)}t_2^{-(a(s)+1)}+t_1^{-(l(s)+1)}t_2^{a(s)}), \tag{29}$$

where $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$. The tautological bundle ξ_n on $(\mathbb{C}^2)^{[n]}$ has its weight decomposition at a fixed point I_μ given by [26]:

$$\xi_n|_{I_\mu} = \sum_{(i,j)\in\mu} t_1^{i-1}t_2^{j-1} = \sum_{s\in\mu} t_1^{l'(s)}t_2^{a'(s)}.$$

Here we have used the following notations: Represent a partition μ by its Young diagram. If a box $s \in \mu$ is located at the i-row and the j-th column, then define

$$a_\mu(s) = \mu_i - j, \qquad a'_\mu(s) = j-1,$$
$$l_\mu(s) = \mu_j^t - i, \qquad l'_\mu(s) = i-1.$$

We sometimes omit the subscript μ from our notations when there is no danger of confusions. Note that

$$a_{\mu^t}(s^t) = l_\mu(s), \qquad a'_{\mu^t}(s^t) = l'_\mu(s),$$

where μ^t is obtained from μ by switching the roles of rows and columns, s^t is the box in μ^t that corresponds to s in μ.

For $A = (a_1, a_2) \in \mathbb{Z}^2$,

$$\xi_n^A|_{I_\mu} = \sum_{(i,j)\in\mu} t_1^{i-1}t_2^{j-1}t_1^{a_1}t_2^{a_2} = \sum_{s\in\mu} t_1^{l'(s)}t_2^{a'(s)}t_1^{a_1}t_2^{a_2}.$$

Hence by the holomorphic Lefschetz formula [30] we have

$$\begin{aligned}
&\sum_{n\geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1,t_2)\\
=&\sum_\mu Q^{|\mu|}\prod_{(i,j)\in\mu}\frac{(1-ut^A t_1^{i-1}t_2^{j-1})\cdot(1-vt^{-A}t_1^{-(i-1)}t_2^{-(j-1)})}{(1-t_1^{-(\mu_j^t-i)}t_2^{\mu_i-j+1})(1-t_1^{\mu_j^t-i+1}t_2^{-(\mu_i-j)})}\\
=&\sum_\mu Q^{|\mu|}\prod_{s\in\mu}\frac{(1-ut^A t_1^{l'(s)}t_2^{a'(s)})\cdot(1-vt^{-A}t_1^{-l'(s)}t_2^{-a'(s)})}{(1-t_1^{-l(s)}t_2^{a(s)+1})(1-t_1^{l(s)+1}t_2^{-a(s)})},
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\sum_{n\geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, (\det\xi_n)^m)(t_1,t_2)\\
=&\sum_\mu Q^{|\mu|}\prod_{(i,j)\in\mu}\frac{(t_1^{i-1}t_2^{j-1})^m}{(1-t_1^{-(\mu_j^t-i)}t_2^{\mu_i-j+1})(1-t_1^{\mu_j^t-i+1}t_2^{-(\mu_i-j)})}\\
=&\sum_\mu Q^{|\mu|}\prod_{s\in\mu}\frac{(t_1^{l'(s)}t_2^{a'(s)})^m}{(1-t_1^{-l(s)}t_2^{a(s)+1})(1-t_1^{l(s)+1}t_2^{-a(s)})},
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\sum_{n\geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \xi_n^{\pm})(t_1,t_2) \\
&= \sum_{\mu} Q^{|\mu|} \frac{\sum_{(i,j)\in\mu} t_1^{\pm(i-1)} t_2^{\pm(j-1)}}{\prod_{(i,j)\in\mu}(1-t_1^{-(\mu_j^t-i)} t_2^{\mu_i-j+1})(1-t_1^{\mu_j^t-i+1} t_2^{-(\mu_i-j)})} \qquad (32)\\
&= \sum_{\mu} Q^{|\mu|} \frac{\sum_{s\in\mu} t_1^{\pm l'(s)} t_2^{\pm a'(s)}}{\prod_{s\in\mu}(1-t_1^{-l(s)} t_2^{a(s)+1})(1-t_1^{l(s)+1} t_2^{-a(s)})}.
\end{aligned}$$

3.2. **Relationship with specializations of Macdonald polynomials.** Denote by $Z^A_{\mathbb{C}^2}(u,v;t_1,t_2)$, $Z^{(m)}_{\mathbb{C}^2}(t_1,t_2)$ and $Z^{\pm}_{\mathbb{C}^2}(t_1,t_2)$ the right-hand sides of (30), (31) and (32) respectively. The main result of this subsection is their relationship with Macdonald polynomials. See [24] for references on the Macdonald polynomials. For the reader's convenience, we have collected some basic definitions, notations and facts about Macdonald polynomials in the Appendix.

Theorem 3.1. *The following identity holds:*
(33)
$$Z^A_{\mathbb{C}^2}(u,v;t_1,t_2) = \sum_{\mu} (\frac{-Qvt^{-A}}{t_1t_2})^{|\mu|} \epsilon^x_{ut^A,t_1^{-1}} P_\mu(x;t_2,t_1^{-1}) \cdot \epsilon^y_{v^{-1}t^A,t_2^{-1}} P_{\mu^t}(y;t_1,t_2^{-1}).$$

Proof. One can rewrite (30) as follows:

$$\begin{aligned}
& Z_{\mathbb{C}^2}(u,v;t_1,t_2) \\
=\ & \sum_{\mu} (\frac{-Qvt^{-A}}{t_1t_2})^{|\mu|} \prod_{s\in\mu} \frac{(t_1^{-l'(s)} - t_2^{a'(s)} ut^A)(t_2^{-a'(s)} - t_1^{l'(s)} v^{-1} t^A)}{(1-t_2^{a(s)} t_1^{-(l(s)+1)})(1-t_1^{l(s)} t_2^{-(a(s)+1)})} \\
=\ & \sum_{\mu} (\frac{-Qvt^{-A}}{t_1t_2)})^{|\mu|} \epsilon^x_{ut^A,t_1^{-1}} P_\mu(x;t_2,t_1^{-1}) \cdot \epsilon^y_{v^{-1}t^A,t_2^{-1}} P_{\mu^t}(y;t_1,t_2^{-1}).
\end{aligned}$$

In the second equality we have used (75) and (76). □

In the same fashion we also have

$$\begin{aligned}
Z^{(+1)}_{\mathbb{C}^2}(t_1,t_2) &= \sum_{\mu} Q^{|\mu|} \epsilon^x_{t_2} P_{\mu^t}(x;t_1^{-1},t_2) \cdot \epsilon^y_{t_1} P_\mu(y;t_2^{-1},t_1), \\
Z^{(-1)}_{\mathbb{C}^2}(t_1,t_2) &= \sum_{\mu} (t_1^{-1}t_2^{-1}Q)^{|\mu|} \cdot \epsilon^x_{t_2^{-1}} P_{\mu^t}(x;t_1,t_2^{-1}) \cdot \epsilon^y_{t_1^{-1}} P_\mu(y;t_2,t_1^{-1}), \\
Z^{(0)}_{\mathbb{C}^2}(t_1,t_2) &= \sum_{\mu} Q^{|\mu|} \epsilon^x_{t_2} P_{\mu^t}(x;t_1^{-1},t_2) \cdot (-t_1^{-1})^{|\mu|} \epsilon^y_{t_1^{-1}} P_\mu(y;t_2,t_1^{-1}), \\
&= \sum_{\mu} Q^{|\mu|} (-t_2^{-1})^{|\mu|} \epsilon^x_{t_2^{-1}} P_{\mu^t}(x;t_1,t_2^{-1}) \cdot \epsilon^y_{t_1} P_\mu(y;t_2^{-1},t_1),
\end{aligned}$$

$$\begin{aligned}
Z^{+}_{\mathbb{C}^2}(t_1,t_2) =& \frac{t_1^{-1}}{t_2-1} \sum_{\mu} Q^{|\mu|} \epsilon^x_{t_2} P_{\mu^t}(x;t_1^{-1},t_2) \cdot (-t_1^{-1})^{|\mu|} \epsilon^y_{t_1^{-1}} E^y_{t_2,t_1^{-1}} P_\mu(y;t_2,t_1^{-1}) \\
=& \frac{t_2^{-1}}{t_1-1} \sum_{\mu} Q^{|\mu|} (-t_2^{-1})^{|\mu|} \epsilon^x_{t_2^{-1}} E^x_{t_1,t_2^{-1}} P_{\mu^t}(x;t_1,t_2^{-1}) \cdot \epsilon^y_{t_1} P_\mu(y;t_2^{-1},t_1),
\end{aligned}$$

$$\begin{aligned}Z^-_{\mathbb{C}^2}(t_1,t_2) =&\frac{t_2}{t_1^{-1}-1}\sum_\mu Q^{|\mu|}\epsilon^x_{t_2}E^x_{t_1^{-1},t_2}P_{\mu^t}(x;t_1^{-1},t_2)\cdot(-t_1^{-1})^{|\mu|}\epsilon^y_{t_1^{-1}}P_\mu(y;t_2,t_1^{-1})\\ =&\frac{t_1}{t_2^{-1}-1}\sum_\mu Q^{|\mu|}(-t_2^{-1})^{|\mu|}\epsilon^x_{t_2^{-1}}P_{\mu^t}(x;t_1,t_2^{-1})\cdot\epsilon^y_{t_1}E^y_{t_2^{-1},t_1}P_\mu(y;t_2^{-1},t_1).\end{aligned}$$

Here $\epsilon^x_t := \epsilon^x_{t,0}$. See Appendix A for notations. We will not use these identities below. They are presented here to show that there are various situation when one can relate the equivariant indices on $(\mathbb{C}^2)^{[n]}$ to Macdonald polynomials.

As already mentioned in the Introduction, various authors have established relationship between equivariant homology/cohomology classes or K-theory classes to the Jack or Macdonald polynomials [14, 27, 22]. In our case we relate the contributions of fixed points by localizations to specializations of Macdonald polynomials. In the next subsection we will use summation formula for them to establish (3) in the affine plane case.

3.3. **Proof of (3) when $k = 2$.**

Theorem 3.2. *We have the following identities:*

$$Z^A_{\mathbb{C}^2}(u,v;t_1,t_2) = \exp\Big(\sum_{n=1}^\infty\frac{Q^n}{n}\frac{(1-u^nt^{nA})(1-v^nt^{-nA})}{\prod_{i=1}^2(1-t_i^n)}\Big). \tag{34}$$

Proof. By (33),

$$\begin{aligned}&Z^A_{\mathbb{C}^2}(u,v;t_1,t_2)\\ =\;&\epsilon^x_{ut^A,t_1^{-1}}\epsilon^y_{v^{-1}t^A,t_2^{-1}}\sum_\mu\Big(\frac{-Qvt^{-A}}{t_1t_2}\Big)^{|\mu|}P_\mu(x;t_2,t_1^{-1})\cdot P_{\mu^t}(y;t_1,t_2^{-1})\\ =\;&\epsilon^x_{ut^A,t_1^{-1}}\epsilon^y_{v^{-1}t^A,t_2^{-1}}\exp\Big(\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n}\Big(\frac{-Qvt^{-A}}{t_1t_2}\Big)^np_n(x)p_n(y)\Big)\\ =\;&\exp\Big(\sum_{n=1}^\infty\frac{-1}{n}\Big(\frac{Qvt^{-A}}{t_1t_2}\Big)^n\frac{1-u^nt^{nA}}{1-t_1^{-n}}\cdot\frac{1-v^{-n}t^{nA}}{1-t_2^{-n}}\Big)\\ =\;&\exp\Big(\sum_{n=1}^\infty\frac{Q^n}{n}\frac{(1-u^nt^{nA})(1-v^nt^{nA})}{\prod_{i=1}^2(1-t_i^n)}\Big).\end{aligned}$$

In the second identity we have used (73). □

4. Toric Surface Case of Conjecture 1

In this section we extend the above results to Hilbert schemes on a surface S which admits a torus action with isolated fixed points. Here S can be projective or quasi-projective. Throughout this section, suppose that T^2 acts on S with isolated fixed points $p_1,\dots,p_m$, such that the weights of $T_{p_i}S$ are $u_i^{-1} = t_1^{a_i}t_2^{b_i}$ and $v_i^{-1} = t_1^{c_i}t_2^{d_i}$.

4.1. **Localization on $S^{[n]}$.**

The T^2-action on S induces a natural T^2-action on $S^{[n]}$. It is easy to see that the fixed points on $S^{[n]}$ are parameterized by m-tuples of partitions $(\mu^1,\dots,\mu^m)$, where μ^i can be an empty partition, such that

$$|\mu^1|+\cdots+|\mu^m| = n.$$

Furthermore, the weight decomposition of the cotangent space at the fixed point is given by:

$$\sum_{i=1}^{m}\sum_{s^i\in\mu^i}(u_i^{-l(s^i)}v_i^{a(s^i)+1}+u_i^{l(s^i)+1}v_i^{-a(s^i)}),$$

Similarly, the tautological bundle ξ_n on $S^{[n]}$ has the following weight decomposition:

$$\sum_{i=1}^{m}\sum_{s^i\in\mu^i}u_i^{l'(s^i)}v_i^{a'(s^i)}.$$

Suppose that L is an equivariant line bundle on S such that

$$L|_{p_i}=t^{A^i}=t_1^{a_1^i}t_2^{a_2^i}. \tag{35}$$

The weights of $L^{[n]}$ are those of ξ_n twisted by the corresponding t^{A^i}, i.e.

$$\sum_{i=1}^{m}t^{A^i}\sum_{s^i\in\mu^i}u_i^{l'(s^i)}v_i^{a'(s^i)}.$$

Theorem 4.1. *With the above notations, we have*

$$\begin{aligned}&\sum_{n\geq 0}Q^n\chi(S^{[n]},\Lambda_{-u}L^{[n]}\otimes\Lambda_{-v}L^{[n]*})(t_1,t_2)\\ =&\exp(\sum_{n=1}^{\infty}\frac{Q^n}{n}\chi(S,\Lambda_{-u^n}L\otimes\Lambda_{-v^n}L^*)(t_1^n,t_2^n)).\end{aligned} \tag{36}$$

Proof. By the holomorphic Lefschetz formula we have

$$\begin{aligned}&\sum_{n\geq 0}Q^n\chi(S^{[n]},\Lambda_{-u}L^{[n]}\otimes\Lambda_{-v}L^{[n]*})(t_1,t_2)\\ =&\prod_{i=1}^{m}\sum_{\mu^i}Q^{|\mu^i|}\prod_{s^i\in\mu^i}\frac{(1-ut^AA^iu_i^{l'(s^i)}v_i^{a'(s^i)})(1-vt^{-A^i}u_i^{-l'(s^i)}v_i^{-a'(s^i)})}{(1-u_i^{-l(s^i)}v_i^{a(s^i)+1})(1-u_i^{l(s^i)+1}v_i^{-a(s^i)})}\\ =&\prod_{i=1}^{m}Z_{\mathbb{C}^2}^{A^i}(u,v;u_i,v_i)\\ =&\exp(\sum_{n=1}^{\infty}\frac{Q^n}{n}\sum_{i=1}^{m}\frac{(1-u^nt^{nA^i})(1-v^nt^{-nA^i})}{(1-u_i^n)(1-v_i^n)})\\ =&\exp(\sum_{n=1}^{\infty}\frac{Q^n}{n}\chi(S,\Lambda_{-u^n}L\otimes\Lambda_{-v^n}L^*)(t_1^n,t_2^n)).\end{aligned}$$

□

By taking nonequivariant limit one then gets:

Corollary 4.2. *Let L be an equivariant holomorphic line bundle on a projective surface S that admits a torus action with isolated fixed points. Then the following formula holds:*

$$\sum_{n\geq 0}Q^n\chi(S^{[n]},\Lambda_{-u}L^{[n]}\otimes\Lambda_{-v}L^{[n]*})=\exp(\sum_{n=1}^{\infty}\frac{Q^n}{n}\chi(S,\Lambda_{-u^n}L\otimes\Lambda_{-v^n}L^*)). \tag{37}$$

Example 4.1. Let $\mathbb{Z}_r$ be the cyclic group of order r, regarded as the following diagonal subgroup of $SL_2(\mathbb{C})$:

$$\left\{\begin{pmatrix}\omega_r^i & 0\\ 0 & \omega_r^{-i}\end{pmatrix}, 0 \leq i < r\right\},$$

where $\omega_r = e^{2\pi\sqrt{-1}/r}$. Let $\pi : X_r \to \mathbb{C}^2/\mathbb{Z}_r$ be the minimal desingularization of $\mathbb{C}^2/\mathbb{Z}_r$. Following [16] and [29], we identify X_r with the subvariety of $(\mathbb{C}^2)^{[r]}$ which consists of the points corresponding to the following $\mathbb{Z}_r$-invariant ideals in $\mathbb{C}[z_1, z_2]$:

$$I(O) \quad := \quad = (z_1^r - a_1^r, z_1z_2 - a_1a_2, z_2^r - a_2^r), \tag{38}$$

$$I_i(p_i : q_i) \quad := \quad (p_iz_1^i - q_iz_2^{r-i}, z_1z_2, z_1^{i+1}.z_2^{r+1-i}) \tag{39}$$

Here O stands for a $\mathbb{Z}_r$-orbit in $\mathbb{C}^2$ disjoint from the origin, and $I(O)$ corresponds to a point in X_r outside the exception divisor. For $1 \leq i \leq r-1$, all the ideals $I_i(p_i : q_i)$ ($[p_i : q_i] \in \mathbb{P}^1$) correspond to the points on a component of the exceptional divisor.

For $0 \leq i < r$, let $(a_{i,1}, a_{i,2})$ stand for points in $X_i \cong \mathbb{C}^2$. Consider the following two-dimensional flat family of $\mathbb{Z}_r$-invariant ideals:

$$\mathfrak{I}_i(a_{i,1}, a_{i,2}) := (z_1^{i+1} - a_{i,1}z_2^{r-i-1}, z_1z_2 - a_{i,1}a_{i,2}, z_2^{r-i} - a_{i,2}z_1^i) \subset \mathbb{C}[z_1, z_2] \tag{40}$$

By the universal property of the Hilbert scheme $(\mathbb{C}^2)^{[r]}$ one gets a map $X_i \hookrightarrow X_r$. This gives an affine open cover

$$X_r = \cup_{0 \leq i \leq r-1} X_i.$$

Note that when $a_{i,1}a_{i,2} \neq 0$, we have $\mathfrak{I}_i(a_{i,1}, a_{i,2}) = I(O)$ where O is the $\mathbb{Z}_r$-orbit of the point $(a_1, a_2) \in \mathbb{C}^2$ with

$$a_1 = a_{i,1}^{(r-i)/r} a_{i,2}^{(r-i-1)/r}, \qquad a_2 = a_{i,1}a_{i,2}/a_1 = a_{i,1}^{i/r} a_{i,2}^{(i+1)/r}$$

i.e.,

$$a_{i,1} = a_1^{i+1}/a_2^{r-i-1}, \qquad a_{i,2} = a_2^{r-i}/a_1^i. \tag{41}$$

Similarly, we obtain

$$\mathfrak{I}_i(a_{i,1}, 0) = I_{i+1}(1 : a_{i,1}), \qquad 0 \leq i \leq r-2 \tag{42}$$

$$\mathfrak{I}_{r-1}(a_{r-1,1}, 0) = (z_1^r - a_{r-1,1}, z_2), \tag{43}$$

$$\mathfrak{I}_i(0, a_{i,2}) = I_i(a_{i,2} : 1), \qquad 1 \leq i \leq r-1 \tag{44}$$

$$\mathfrak{I}_0(0, a_{0,2}) = (z_1, z_2^r - a_{0,2}). \tag{45}$$

Let $\xi_0, \dots, \xi_{r-1} \in X$ be the points corresponding respectively to the ideals:

$$I_1(1:0),\ I_1(0:1) = I_2(1:0), \dots,\ I_{r-2}(0:1) = I_{r-1}(1:0),\ I_{r-1}(0:1). \tag{46}$$

Then we see from (42)–(45) that ξ_i is the origin of the open affine chart X_i.

Let $T = (\mathbb{C}^*)^2$ act on $\mathbb{C}^2$ by

$$(t_1, t_2)(z_1, z_2) = (t_1z_1, t_2z_2), \quad (t_1, t_2) \in T. \tag{47}$$

This action induces a T-action on $(\mathbb{C}^2)^{[r]}$ which preserves the minimal resolution X_r as a subvariety of $(\mathbb{C}^2)^{[r]}$. By examining (38) and (39), one sees that the set X^T of T-fixed points is

$$X^T = \{\xi_0, \dots, \xi_{r-1}\}.$$

Since the family of ideals in (40) is T-invariant, X_i is T-invariant. Furthermore, by (41), T acts on points $(a_{i,1}, a_{i,2}) \in X_i$ by

$$(t_1, t_2)(a_{i,1}, a_{i,2}) = (t_1^{-i-1} t_2^{r-i-1} a_{i,1}, t_1^i t_2^{i-r} a_{i,2}). \tag{48}$$

In other words, we have in this case:

$$u_i = t_1^{i+1} t_2^{-r+i+1}, \qquad v_i = t_1^{-i} t_2^{r-i}.$$

It follows that we have

$$\chi(X_r, \Lambda_{-u}\mathcal{O}_{X_r} \otimes \Lambda_{-v}\mathcal{O}_{X_r})(t_1, t_2) = \sum_{i=0}^{r-1} \frac{(1-u)(1-v)}{(1 - t_1^{i+1} t_2^{-r+i+1})(1 - t_1^{-i} t_2^{r-i})}. \tag{49}$$

An elementary calculation shows that

$$\chi(X_r, \Lambda_{-u}\mathcal{O}_{X_r} \otimes \Lambda_{-v}\mathcal{O}_{X_r})(t_1, t_2) \quad = \quad \frac{(1-u)(1-v)(1-(t_1t_2)^r)}{(1-t_1^r)(1-t_2^r)(1-t_1t_2)}.$$

Indeed, let $x = t_1 t_2$, $y = t_2^r$, then the right-hand side of (49) becomes $(1-u)(1-v)$ times

$$\begin{aligned}
&\sum_{i=1}^{r-1} \frac{1}{(1-x^{i+1}y^{-1})(1-x^{-i}y)} = \sum_{i=1}^{r-1} \frac{-x^i y}{(y-x^{i+1})(y-x^i)} \\
= \quad & \frac{1}{x-1} \sum_{i=0}^{r-1} \left(\frac{x^i}{y-x^i} - \frac{x^{i+1}}{y-x^{i+1}}\right) = \frac{1}{x-1}\left(\frac{1}{y-1} - \frac{x^r}{y-x^r}\right) \\
= \quad & \frac{1}{x-1} \frac{-(x^r-1)u}{(y-x^r)(y-1)} = \frac{x^r-1}{(y^{-1}x^r-1)(y-1)(x-1)} \\
= \quad & \frac{(t_1t_2)^r - 1}{(t_1^r-1)(t_2^r-1)(t_1t_2-1)}.
\end{aligned}$$

Therefore, by (36)

$$\begin{aligned}
&\sum_{n \geq 0} Q^n \chi(X_r^{[n]}, \Lambda_{-u}\xi_n \otimes \Lambda_{-v}\xi_n^*)(t_1, t_2) \\
&= \exp\Big(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{(1-u^n)(1-v^n)(1-(t_1t_2)^{nr})}{(1-t_1^{nr})(1-t_2^{nr})(1-t_1^n t_2^n)}\Big).
\end{aligned} \tag{50}$$

Example 4.2. Consider the blow up of $\mathbb{C}^2$ at the origin, $\hat{\mathbb{C}}^2$, which corresponds to $\mathcal{O}_{\mathbb{P}^1}(-1)$. In this case we have two fixed points p_1 and p_2, with

$$u_1 = t_1, \qquad v_1 = t_1^{-1} t_2,$$
$$u_2 = t_2, \qquad v_2 = t_1 t_2^{-1}.$$

Hence in this case we have

$$\begin{aligned}
&\chi(\hat{\mathbb{C}}^2, \Lambda_{-u}\mathcal{O}_{\hat{\mathbb{C}}^2} \otimes \Lambda_{-v}\mathcal{O}_{\hat{\mathbb{C}}^2})(t_1, t_2) = \frac{(1-u)(1-v)}{(1-t_1)(1-t_1^{-1}t_2)} + \frac{(1-u)(1-v)}{(1-t_2)(1-t_1t_2^{-1})} \\
= \quad & \frac{(1-u)(1-v)}{(1-t_1)(1-t_2)} = \chi(\mathbb{C}^2, \Lambda_{-u}\mathcal{O}_{\mathbb{C}^2} \otimes \Lambda_{-v}\mathcal{O}_{\mathbb{C}^2})(t_1, t_2),
\end{aligned}$$

it follows from (36) that

$$\begin{aligned}&\sum_{n\geq 0} Q^n\chi((\hat{\mathbb{C}}^2)^{[n]}, \Lambda_{-u}\xi_n\otimes\Lambda_{-v}\xi_n^*)(t_1,t_2)\\ =&\sum_{n\geq 0} Q^n\chi((\mathbb{C}^2)^{[n]}, \Lambda_{-u}\xi_n\otimes\Lambda_{-v}\xi_n^*)(t_1,t_2).\end{aligned} \tag{51}$$

One can then show that

$$\begin{aligned}&\sum_{n\geq 0} Q^n\chi((\hat{S})^{[n]}, \Lambda_{-u}\xi_n\otimes\Lambda_{-v}\xi_n^*)(t_1,t_2)\\ =&\sum_{n\geq 0} Q^n\chi((S)^{[n]}, \Lambda_{-u}\xi_n\otimes\Lambda_{-v}\xi_n^*)(t_1,t_2)\\ =&\exp\sum_{n=1}^{\infty}\frac{Q^n}{n}\chi(S, \Lambda_{-u^n}\mathcal{O}_S\otimes\Lambda_{-v^n}\mathcal{O}_S)(t_1,t_2),\end{aligned} \tag{52}$$

where S is a projective or quasi-projective surface which admits a torus action with isolated fixed points, and $\hat{S}$ is the blow up of S at one of these fixed points. When S is projective, one takes the nonequivariant limit to get:

$$\begin{aligned}&\sum_{n\geq 0} Q^n\chi((\hat{S})^{[n]}, \Lambda_{-u}\xi_n\otimes\Lambda_{-v}\xi_n^*)\\ =&\sum_{n\geq 0} Q^n\chi((S)^{[n]}, \Lambda_{-u}\xi_n\otimes\Lambda_{-v}\xi_n^*)\\ =&\exp\sum_{n=1}^{\infty}\frac{Q^n}{n}\chi(S, \Lambda_{-u^n}\mathcal{O}_S\otimes\Lambda_{-v^n}\mathcal{O}_S).\end{aligned} \tag{53}$$

This establishes Conjecture 2 for toric projective surface.

5. The Surface Case of Conjecture 1 and Conjecture 2

5.1. **Universal formula on Hilbert schemes.** In this subsection we recall a result due to Ellingsrud-Göttsche-Lehn [7]. For a complex n-manifold X, let $\Psi: K(X)\to H^{\times}[u,v]$ be a group homomorphism from the additive group $K(X)$ to the multiplicative group $H^{\times}$ of units of $H(X;\mathbb{Q})$. We require Ψ is functorial with respect to pull-backs and is polynomial in Chern classes of its argument. Also let $\phi(x)\in\mathbb{Q}[[x]]$ be a formal power series and put $\Phi(X) := \phi(x_1)\cdots\phi(x_n)\in H^*(X;\mathbb{Q})$ with $x_1,\cdots,x_n$ the Chern roots of T_X. For $x\in K(X)$, define a power series in $\mathbb{Q}[[z,u,v]]$ as follows:

$$H_{\Psi,\Phi}(S,x) := \sum_{n=0}^{\infty}\int_{S^{[n]}}\Psi(x^{[n]})\Phi(S^{[n]})z^n.$$

Theorem 5.1. [7] *For each integer r there are universal power series $A_i\in\mathbb{Q}[[z,u,v]]$, $i=1,\cdots,5$, depending only on Ψ, Φ and r, such that for each $x\in K(S)$ of rank r we have*

$$H_{\Psi,\Phi}(S,x) = \exp\Big(\int_S(c_1^2(x)A_1 + c_2(x)A_2 + c_1(x)c_1(S)A_3 + c_1^2(S)A_4 + c_2(S)A_5)\Big).$$

5.2. **Conjecture 1 for line bundles over surfaces.**

Theorem 5.2. *For a holomorphic line bundle L on a smooth projective surface S, we have*

$$\sum_{n\geq 0} Q^n \chi(S^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*}) = \exp(\sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(S, \Lambda_{-u^n} L \otimes \Lambda_{-v^n} L^*)). \tag{54}$$

Proof. By Hirzebruch-Rimann-Roch formula,

$$\chi(S^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n*}) = \int_{S^{[n]}} \mathrm{ch}(\Lambda_{-u}\xi_n \otimes \Lambda_{-v}\xi_n^*) \,\mathrm{td}(T_n),$$

hence one can apply Theorem 5.1 to see that there are universal power series $A_i \in \mathbb{Q}[[z, u, v]]$, $i = 1, 3, 4, 5$, such that for each holomorphic line bundle L on S,

$$\begin{aligned} & \chi(S^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*}) \\ = \quad & \exp(\int_S (c_1^2(L)A_1 + c_1(L)c_1(S)A_3 + c_1^2(S)A_4 + c_2(S)A_5)). \end{aligned}$$

Now one can apply Corollary 4.2 to choose suitable line bundles on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ to complete the proof. □

5.3. **Conjecture 2 for surfaces.** Because birationally equivalent surfaces are related by blowup and blowdown, it suffices to consider the case of blowup $\hat{S}$ of a smooth projective surface S at a point. Because one has $h^{0,q}(\hat{S}, \mathcal{O}_{\hat{S}}) = h^{0,q}(S, \mathcal{O}_S)$, one has

$$\chi(\hat{S}, \Lambda_{-u}\mathcal{O}_{\hat{S}} \otimes \Lambda_{-v}\mathcal{O}_{\hat{S}}) = \chi(S, \Lambda_{-u}\mathcal{O}_S \otimes \Lambda_{-v}\mathcal{O}_S). \tag{55}$$

Therefore, by Theorem 5.2, we have

Theorem 5.3. *Let $\hat{S}$ be the blowup at a point of a smooth projective surface S. Then one has:*

$$\chi(\hat{S}^{[n]}, \xi_n^{\hat{S}}) = \chi(S^{[n]}, \xi_n^S). \tag{56}$$

6. Some Calculations in Dimensions $k > 2$

Unfortunately when $k > 2$, $(\mathbb{C}^k)^{[n]}$ is in general singular. We expect that there is a suitable equivariant theory in which one can still use the localization techniques, but this is not available to us at present. Nevertheless, when $n < 4$, $(\mathbb{C}^k)^{[n]}$ is always nonsingular, so we will only deal with these cases.

Consider the natural action of the k-torus T^k on $\mathbb{C}^k$ whose action on the linear coordinates is given by

$$(t_1, \ldots, t_k) \cdot z_j = t_j z_j.$$

The induced action on $(\mathbb{C}^k)^{[n]}$ has monomial ideals in $\mathbb{C}[z_1, \ldots, z_k]$ as fixed points hence as in the $k = 2$ case, they correspond to k-dimensional partitions. The $n = 1$ case is trivial, so we will consider the cases $n = 2$ and $n = 3$.

6.1. **The case of $k > 2$, $n = 2$.** When $n = 2$, the fixed points correspond to the monomial ideals I_j^2 generated by $z_1, \dots, z_{j-1}, z_j^2, z_{j+1}, \dots, z_n$. The Zariski tangent space of $(\mathbb{C}^k)^{[2]}$ at I_j^2 is given by

$$\mathrm{Hom}_{\mathbb{C}[z_1,\dots,z_k]}(I_j^2, \mathbb{C}[z_1, \dots, z_k]/I_j^2).$$

Now

$$\mathbb{C}[z_1, \dots, z_k]/I_j^2 \cong \mathbb{C}1 \oplus \mathbb{C}z_j, \tag{57}$$

so a vector in the tangent space is specified by

$$\begin{aligned} f(z_i) &= a_i + b_i z_j, \quad i \neq j, \\ f(z_j^2) &= a_j + b_j z_j. \end{aligned}$$

It follows that the tangent space at the fixed point has the following weight decomposition:

$$\sum_{i \neq j} t_i^{-1}(1 + t_j) + t_j^{-2}(1 + t_j). \tag{58}$$

Note that by (57) the tautological sheaf ξ_2 has the following weight decomposition

$$\xi_2|_{I_j^2} = 1 + t_j. \tag{59}$$

6.2. **The case of $k > 2$, $n = 3$.** When $n = 3$, there are two kinds of fixed points. The fixed points of first kind correspond to monomial ideals I_j^3 generated by z_1, $\dots, z_{j-1}, z_j^3, z_{j+1}, \dots, z_n$. Now

$$\mathbb{C}[z_1, \dots, z_k]/I_j^3 \cong \mathbb{C}1 \oplus \mathbb{C}z_j \oplus \mathbb{C}z_j^2, \tag{60}$$

so a vector in the tangent space is specified by

$$\begin{aligned} f(z_i) &= a_i + b_i z_j + c_i z_j^2, \quad i \neq j, \\ f(z_j^2) &= a_j + b_j z_j + c_j z_j^2. \end{aligned}$$

It follows that the tangent space at the fixed point I_j^3 has the following weight decomposition:

$$\sum_{i \neq j} \left(t_i^{-1}(1 + t_j + t_j^2) + t_j^{-3}(1 + t_j + t_j^2) \right). \tag{61}$$

Note that by (60) the tautological sheaf ξ_3 has the following weight decomposition:

$$\xi_3|_{I_j^3} = 1 + t_j + t_j^2. \tag{62}$$

The fixed points of second kind correspond to monomial ideals $I_{j_1 j_2}^3$ $(j_1 < j_2)$, for example I_{12}^3 is generated by $z_1^2, z_1 z_2, z_2^2, z_3, \dots, z_k$. Now

$$\mathbb{C}[z_1, \dots, z_k]/I_{12}^3 \cong \mathbb{C}1 \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2, \tag{63}$$

so an element in the tangent space is determined by

$$\begin{aligned} f(z_i) &= a_i + b_i z_1 + c_i z_2, \quad i \neq 1, 2, \\ f(z_1^2) &= a_{11} + b_{11} z_1 + c_{11} z_2, \\ f(z_1 z_2) &= a_{12} + b_{12} z_1 + c_{12} z_2, \\ f(z_2^2) &= a_{22} + b_{22} z_1 + c_{22} z_2. \end{aligned}$$

From

$$f(z_1^2 z_2) = z_1 f(z_1 z_2) = z_2 f(z_1^2), \qquad f(z_1 z_2^2) = z_2 f(z_1 z_2) = z_1 f(z_2^2),$$

one sees that

$$a_{11} = a_{12} = a_{22} = 0.$$

Therefore, the tangent space at I_{12}^3 has the following weight decomposition

$$\sum_{i=3}^{k} \big(t_i^{-1}(1 + t_1 + t_2) + (t_1^{-2} + t_1^{-1}t_2^{-1} + t_2^{-2})(t_1 + t_2)\big). \tag{64}$$

Also by (63) we have

$$\xi_3|_{I_{12}^3} = 1 + t_1 + t_2. \tag{65}$$

With this information on the weight decompositions we have checked by Maple the conjectured identity (3) for $n = 2$ and $k \leq 5$, and $n = 3$ and $k \leq 4$.

As in the surface case, assuming (3) one can establish (1) for toric varieties and (2) for toric varieties blown up at a fixed point. The latter involves the following identity:

$$\chi(\hat{\mathbb{C}}^k, \mathcal{O}_{\hat{\mathbb{C}}^k})(t_1, \dots, t_k) = \chi(\mathbb{C}^k, \mathcal{O}_{\mathbb{C}^k})(t_1, \dots, t_k). \tag{66}$$

By holomorphic Lefschetz formula, this is equivalent to

$$\sum_{i=1}^{k} \frac{1}{(1 - t_i)\prod_{j \neq i, 1 \leq j \leq k}(1 - t_j/t_i)} = \frac{1}{(1 - t_1)\cdots(1 - t_k)}. \tag{67}$$

This can be easily proved by using Cauchy's residue theorem to the function $f(z) = \frac{z^{k-1}}{(1-z)\prod_{i=1}^k (z - t_i)}$.

In general, if X and X' are two smooth projective varieties birationally equivalent to each other, then by Hartogs Theorem,

$$H^0(X, \Omega_X^i) \cong H^0(X, \Omega_{X'}^i)$$

one has

$$\chi(X, \mathcal{O}_X) = \chi(X', \mathcal{O}_{X'}) \tag{68}$$

and so

$$\chi(\hat{X}, \Lambda_{-u}\mathcal{O}_X \otimes \Lambda_{-v}\mathcal{O}_X) = \chi(X', \Lambda_{-u}\mathcal{O}_{X'} \otimes \Lambda_{-v}\mathcal{O}_{X'}). \tag{69}$$

Appendix A. Some Summation Formulas for Macdonald Polynomials

In this appendix we recall some definitions and facts about Macdonald polynomials. Our reference is the classical book [24].

A.1. **The Macdonald operators D_n^k.** Denote by Λ_n the space of symmetric polynomials in n variables $x_1, \dots, x_n$. For every function f in $x_1, \dots, x_n$, $q \in \mathbb{C}$, $1 \le i \le n$, define the shift operator T_{q,x_i} by

$$(T_{u,x_i} f)(x_1, \dots, x_n) = f(x_1, \dots, ux_i, x_n).$$

Define the *Macdonald operators* D_n^r on Λ_n by [24, p.316]:

$$D_n^r = \sum_{I=\{1\le i_1<\cdots<i_r\le n\}} A_I(x_1, \dots, x_n; t) \prod_{i\in I} T_{q,x_i},$$

where

$$A_I(x_1, \dots x_n; t) = t^{r(r-1)/2} \prod_{i\in I, j\in I^c} \frac{tx_i - x_j}{x_i - x_j}.$$

Here $I^c = \{1, \dots, n\} - I$. The operator D_n^1 on Newton polynomials is given by [9]:

$$\begin{aligned} D_n^1 p_\mu(x_1, \dots, x_n) &= \frac{1}{1-t} p_\mu(x_1, \dots, x_n) \\ &+ \frac{t^n}{t-1} \left[\prod_{i=1}^n \left((p_{\mu_i}(x) + (\frac{q-1}{tz})^{\mu_i}) \frac{1 - zx_i}{1 - zx_i t} \right) \right]_0, \end{aligned} \tag{70}$$

where for a formal power series $f(z) = \sum_{n\in\mathbb{Z}} b_n z^n$, $f(z)_0 = b_0$.

A.2. **The Macdonald polynomials.** Denote by Λ the space of symmetric functions in $x = (x_1, \dots, x_n, \dots)$. Let $F = \mathbb{C}(q, t)$ be the algebra of rational functions in q and t, and let $\Lambda_F = \Lambda \otimes F$ be the F-algebras of symmetric functions with coefficients in F. The Macdonald polynomials $\{P_\mu = P_\mu(x; q, t)\}$ are suitable normalized symmetric functions such that (cf. [24, Chapter VI, Section 4]):

$$D_n^r P_\mu(x_1, \dots, x_n) = t^{nr} e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) \cdot P_\mu(x_1, \dots, x_n). \tag{71}$$

The Macdonald polynomials have the following properties [24, Chapter VI]:

$$P_\mu(x; q^{-1}, t^{-1}) = P_\mu(x; q, t), \tag{72}$$

$$\sum_\mu v^{|\mu|} P_\mu(x; q, t) P_{\mu^t}(y; t, q) = \prod_{j,k} (1 + vx_j y_k). \tag{73}$$

A.3. **The operator E.** Define an operator E on Λ whose restriction to Λ_n is given by:

$$E_n = t^{-n} D_n^1(x) - \sum_{i=1}^n t^{-i}.$$

It is easy to see that [5, Lemma 1]:

$$\sum_{i=1}^n q^{\mu_i} t^{-i} - \sum_{i=1}^n t^{-i} = \frac{q-1}{t} \sum_{(i,j)\in\mu} t^{-(i-1)} q^{j-1} = \frac{q-1}{t} \sum_{s\in\mu} t^{-l'(s)} q^{a'(s)}.$$

Hence we have

$$EP_\mu(x; q, t) = \frac{q-1}{t} \sum_{s\in\mu} t^{-l'(s)} q^{a'(s)} \cdot P_\mu(x; q, t). \tag{74}$$

A.4. **Specializations of Macdonald polynomials.** Denote by $\epsilon^x_{u,t} : \Lambda_F \to F$ the specialization homomorphism defined by

$$\epsilon^x_{u,t} p_n(x) = \frac{1-u^n}{1-t^n}$$

for each integer $n \geq 1$. Then we have [24, (6.16), (6.17)]:

$$\epsilon^x_{u,t} P_\mu(x;q,t) = \prod_{s\in\mu} \frac{t^{l'(s)} - q^{a'(s)}u}{1-q^{a(s)}t^{l(s)+1}}. \tag{75}$$

Changing μ to μ^t, we also have

$$\epsilon^x_{u,t} P_{\mu^t}(x;q,t) = \prod_{s\in\mu} \frac{t^{a'(s)} - q^{l'(s)}u}{1-q^{l(s)}t^{a(s)+1}}. \tag{76}$$

References

[1] G. Andrews, The theory of partitions, Addison-Wiley, Reading, Massachusetts, 1976.

[2] M. Atiyah, I. Singer, *The index of elliptic operators. III.*, Ann. of Math. (2) **87** (1968), 546–604.

[3] S. Boissiere, M.A. Nieper-Wisskirchen, *Universal formulas for characteristic classes on the Hilbert schemes of points on surfaces*, J. Algebra 315 (2007), no. 2, 924C953, math.AG/0507470.

[4] L. Borisov, A. Libgober, *McKay correspondence for elliptic genera*, Ann. Math, Second Series, Vol. 161 (2005), no. 3, 1521-1569.

[5] S.-J. Cheng, W. Wang, *The correlation funcitons of vertex operators and Macdonald polynomials*, math.CO/0512064.

[6] G. Ellisngsrud, S.A. Strømme, *On the homology of the Hilbert schemes of points in the plane*, Invent. Math. **87** (1987), 343-352.

[7] G. Ellisngsrud, L. Göttsche, M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, ArXiv:math/9904095.

[8] Fogarty, *Algebraic families on an algebraic surface*. Amer. J. Math **90** (1968), 511–521.

[9] A.M. Garsia, M. Haiman, M. *A remarkable q, t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combin. **5** (1996), no. 3, 191–244.

[10] L. Göttsche, *The Betti numbers of the Hilbert schemes of points of a smooth projective surface*, Math. Ann. **286** (1990), 193-207.

[11] L. Göttsche, Orbifold-Hodge numbers of Hilbert schemes, in: Parameter spaces (Warsaw, 1994), Polish Acad. Sci., Warsaw, 1996, pp. 83-87.

[12] L. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. **296** (1993), 235-245.

[13] I. Grojnowski, *Instantons and affine algebras. I. The Hilbert scheme and vertex operators.* Math. Res. Lett. **3** (1996), no. 2, 275–291.

[14] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture.* J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006.

[15] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), 371–407.

[16] Y. Ito, I. Nakamura, *McKay corespondence and Hilbert schemes*, Proc. Japan Acad. Ser. **A** **72** (1996), 275-291.

[17] E. Kiritsis, *Introduction to non-perturbative string theory*, Trends in theoretical physics (La Plata, 1997), 265C308, AIP Conf. Proc., 419, Amer. Inst. Phys., Woodbury, NY, 1998, hep-th/9708130.

[18] A. Lascoux, J.-Y. Thibon, *Vertex operators and the class algebras of symmetric groups.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 283 (2001), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6, 156–177, 261; translation in J. Math. Sci. (N. Y.) 121 (2004), no. 3, 2380–2392. math.CO/0102041.

[19] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces.* Invent. Math. **136** (1999), no. 1, 157–207.

[20] J. Li, K. Liu, J. Zhou, *Topological string partition functions as equivariant indices*, Asian J. Math. 10 (2006), no. 1, 81C114, math.AG/math.AG/0412089.
[21] W.P. Li, Z. Qin, *Donaldson-Thomas invariants of certain calabi-Yau 3-folds*, arXiv:1002.4080.
[22] W.P. Li, Z. Qin, W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, Math. Ann. **324** (2002), no. 1, 105–133.
[23] W.P. Li, Z. Qin, W. Wang, *The cohomology rings of Hilbert schemes via Jack polynomials*, in Algebraic structures and moduli spaces, 249–258, CRM Proc. Lecture Notes, **38**, Amer. Math. Soc., Providence, RI, 2004.
[24] I.G. Macdonald, Symmetric functions and Hall polynomials. Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[25] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces.* Ann. of Math. (2) **145** (1997), no. 2, 379–388.
[26] H. Nakajima, Lectures on Hilbert schemes of points on surfaces. University Lecture Series, **18**. American Mathematical Society, Providence, RI, 1999.
[27] H. Nakajima, *Jack polynomials and Hilbert schemes of points on surfaces*, alg-geom/9610021.
[28] H. Nakajima, K. Yoshioka, *Instanton counting on blowup. I.*, Invent. Math. 162 (2005), no. 2, 313C355. math.AG/0306198.
[29] Z. Qin, W. Wang, *Hilbert schemes of points on the minimal resolution and soliton equations*, Lie algebras, vertex operator algebras and their applications, 435C462, Contemp. Math., 442, Amer. Math. Soc., Providence, RI, 2007, math.QA/0404540.
[30] R. Thomason, *Une formule de Lefschetz en K-théorie équivariante algebébrique*, Duke math. J. **68** (1992), 447-462.
[31] J. Zhou, *Delocalized equivariant cohomology of symmetric products*, math.DG/9910028.

Department of Mathematical Sciences, Tsinghua University, Beijing, China
E-mail address: `jzhou@math.tsinghua.edu.cn`