

K-THEORY OF HILBERT SCHEMES AS A FORMAL QUANTUM FIELD THEORY

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ABSTRACT. We define a notion of formal quantum field theory and associate a formal quantum field theory to K-theoretical intersection theories on Hilbert schemes of points on algebraic surfaces. This enables us to find an effective way to compute K-theoretical intersection theories on Hilbert schemes via a connection to Macdonald polynomials and vertex operators.

1. INTRODUCTION

There are some striking similarities between quantum field theory and algebraic topology. In both of these theories one starts with some geometric objects and constructs some algebraic structures associated with them. In a quantum field theory, one constructs a space of observables, defines and computes their correlation functions. In an algebraic topological theory, one constructs a space of cohomology groups or K-groups, etc., and defines and computes their intersection numbers. Both admit some axiomatic characterizations, and both seek for identifications of different constructions. In mathematics, there are various isomorphism theorems that identifies different cohomology theories, e.g. the de Rham theorem. In physics, the identifications of different quantum field theories are often referred to as duality by physicists. This aspect of quantum field theory has produced many striking results relating different branches of mathematics and thus has attracted the attentions of many mathematicians.

Such similarities make some mathematicians to regard quantum field theory as physicists' algebraic topology. It might be fair also to regard algebraic topology as mathematicians' quantum field theory. It has been very fruitful to borrow terminologies and formalisms from quantum field theory. See for example, Atiyah's axioms for topological quantum field theories [7] and Kontsevich-Manin's axioms for Gromov-Witten theory [30].

The construction of a quantum field theory involves integrations over infinite-dimensional spaces that often requires mathematical justifications to become mathematically rigorous. There are some quantum field theories that admit supersymmetries, and physicists apply supersymmetric localizations to reduce to integrals over finite-dimensional spaces. The topological quantum field theories are those supersymmetric quantum field theories which reduce to intersection theories on finite-dimensional moduli spaces in differential geometry and algebraic geometry. For example, Donaldson theory [65], intersection theory on moduli spaces of holomorphic vector bundles on Riemann surfaces [67], and Gromov-Witten theory [66].

If one reverses the direction of the above reduction from topological quantum field theory to intersection theories on moduli spaces, one can start with some intersection theories on some spaces, and seek for suitable topological quantum

field theory that leads to it. Once such a geometric problem has been reformulated as a quantum field theory, one can apply techniques developed in physics literature to its study. In particular, one may find some method to calculate the intersection numbers in a formalism familiar to physicists, but mathematically difficult to find.

Concrete examples of this beautiful idea have been provided by the mathematical work on Donaldson theory, intersection theory on moduli spaces of holomorphic vector bundles on Riemann surfaces, and Gromov-Witten theory. In this paper we will present another example. We will apply this idea to the K-theory of Hilbert schemes of points on algebraic surfaces. We will not actually construct a quantum field theory in its original sense from a Lagrangian formulation as normally done by physicists, but instead we will introduce a notion of *formal quantum field* that preserves only the main features of a quantum field theory such as correlators, partition functions, etc. This formalism avoids the difficult problem of justifications of the mathematical rigor of a physical quantum field theory, but nevertheless it enables us to apply techniques from conformal field theory, such as vertex operators on bosonic Fock space to compute the K-theoretical intersection numbers on Hilbert schemes.

Our starting point is that by a result of Ellingsrud-Göttsche-Lehn [18], one can reduce to the case of Hilbert schemes of points on \mathbb{C}^2 . In this case one can apply localization formula to compute the equivariant K-theoretical intersection numbers by a summation over partitions, hence we are facing a problem in the same spirit of a problem in statistical physics. To reformulate it as a formal quantum field theory, one needs to combine two different connections to Macdonald polynomials discovered by Haiman [28] and Wang-Zhou [62] for different purposes. Furthermore, one needs to reformulate these connections in terms of operators on the bosonic Fock space. For this purpose, we introduce a notion of *vertex realizable operators*, and develop a formalism for calculating their correlation. So this work is a sequel to Wang-Zhou [62] with a broader perspective. In particular, the main result there is used and generalized in this paper.

We arrange the rest of this paper in the following way. In Section 2 we define the notions of formal quantum field theory and operatorial formal quantum field theory. In Section 3 we recall some relevant background results on Hilbert schemes that serve as motivations to our work. We introduce the main object of our study, the equivariant K-theoretical intersection numbers of tautological sheaves on $(\mathbb{C}^2)^{[n]}$, in Section 4, and establish their relationship with Macdonald functions and vertex operators in Section 5. Next, in Section 6, we introduce a notion of vertex realizable operators and develop a formalism for their concrete computations. Applications are made in Section 7 where we associate a formal quantum field theory to K-theory of Hilbert schemes and some correlators are computed. We present our concluding remarks on further questions in Section 8.

2. FORMAL QUANTUM FIELD THEORIES

In this Section we introduce a notion of a formal quantum field theory. Using the bosonic Fock space we present some examples of special formal quantum field theories.

2.1. Formal quantum field theories. By an *observable algebra* we mean a commutative algebra \mathcal{O} with identity 1. We assume \mathcal{O} is generated by $\{\mathcal{O}_i\}_{i \geq 0}$, where

$\mathcal{O}_0 = 1$, and

$$\mathcal{O}_m \mathcal{O}_n = \mathcal{O}_n \mathcal{O}_m,$$

for all $m, n \geq 0$. Then the algebra \mathcal{O} consists of linear combinations of elements of the form $\mathcal{O}_{m_1} \cdots \mathcal{O}_{m_n}$. Elements in \mathcal{O} will be called *observables*. Let R be another commutative ring. By a *formal quantum field theory* with observable algebra \mathcal{O} and with values in R , we mean a sequence of elements in $S^n(\mathcal{O}^*) \otimes R$, one for each $n \geq 1$. It is specified by the *correlators* $\langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle \in R$. The *normalized correlators* $\langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle'$ are defined by

$$(1) \quad \langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle' := \frac{\langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle}{\langle \mathcal{O}_0 \rangle}.$$

A convenient way to encode all the normalized correlators is to introduce formal variables $\{t_0, t_1, \dots, t_n, \dots\}$, and consider the generating series:

$$(2) \quad Z := 1 + \sum_{n \geq 1} \langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle' \frac{t_{m_1} \cdots t_{m_n}}{n!}.$$

It is called the *partition function*. Note:

$$(3) \quad \langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle' = \frac{\partial^n}{\partial t_{m_1} \cdots \partial t_{m_n}} Z|_{t_i=0, i \geq 0}.$$

The *free energy* F is defined by:

$$(4) \quad F := \log Z.$$

The *connected correlators* are defined by:

$$(5) \quad \langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle'_c = \frac{\partial^n}{\partial t_{m_1} \cdots \partial t_{m_n}} F|_{t_i=0, i \geq 0}.$$

By (3) to (5), one can get:

$$\begin{aligned} \langle \mathcal{O}_{m_1} \rangle'_c &= \langle \mathcal{O}_{m_1} \rangle', \\ \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \rangle'_c &= \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \rangle'_c + \langle \mathcal{O}_{m_1} \rangle'_c \langle \mathcal{O}_{m_2} \rangle'_c, \\ \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \mathcal{O}_{m_3} \rangle'_c &= \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \mathcal{O}_{m_3} \rangle'_c + \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \rangle'_c \langle \mathcal{O}_{m_3} \rangle'_c \\ &\quad + \langle \mathcal{O}_{m_1} \mathcal{O}_{m_3} \rangle'_c \langle \mathcal{O}_{m_2} \rangle'_c + \langle \mathcal{O}_{m_2} \mathcal{O}_{m_3} \rangle'_c \langle \mathcal{O}_{m_1} \rangle'_c \\ &\quad + \langle \mathcal{O}_{m_1} \rangle'_c \langle \mathcal{O}_{m_2} \rangle'_c \langle \mathcal{O}_{m_3} \rangle'_c, \end{aligned}$$

and conversely,

$$\begin{aligned} \langle \mathcal{O}_{m_1} \rangle'_c &= \langle \mathcal{O}_{m_1} \rangle', \\ \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \rangle'_c &= \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \rangle' - \langle \mathcal{O}_{m_1} \rangle'_c \langle \mathcal{O}_{m_2} \rangle', \\ \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \mathcal{O}_{m_3} \rangle'_c &= \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \mathcal{O}_{m_3} \rangle' - \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} \rangle'_c \langle \mathcal{O}_{m_3} \rangle' \\ &\quad - \langle \mathcal{O}_{m_1} \mathcal{O}_{m_3} \rangle'_c \langle \mathcal{O}_{m_2} \rangle' - \langle \mathcal{O}_{m_2} \mathcal{O}_{m_3} \rangle'_c \langle \mathcal{O}_{m_1} \rangle' \\ &\quad + 2 \langle \mathcal{O}_{m_1} \rangle'_c \langle \mathcal{O}_{m_2} \rangle'_c \langle \mathcal{O}_{m_3} \rangle', \end{aligned}$$

etc. These notations borrowed from quantum field theory have been used in mathematical literature, see e.g. Okounkov [52]. One can also define the *entropy* by

$$(6) \quad G = \sum_{n \geq 0} t_n \frac{\partial F}{\partial t_n} - F.$$

2.2. Operatorial formal quantum field theory. Let H be a space with a scalar product with values in the ring R , following physicists, a vector in H will be denoted by $|v\rangle$. For a linear operator $A : H \rightarrow H$, the scalar product of $A|v\rangle$ with $|w\rangle$ will be denoted by: $\langle w|A|v\rangle$. Let $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_n, \dots$ be a sequence of operators on H , such that $\mathcal{O}_0 = \text{id}_H$, and

$$\mathcal{O}_m \mathcal{O}_n = \mathcal{O}_n \mathcal{O}_m,$$

for all $m, n \geq 0$. By the *operator algebra* generated by $\{\mathcal{O}_n\}_{n \geq 0}$ we mean the commutative algebra generated by these operators. Fix two nonzero vectors $|v_0\rangle$ and $|w_0\rangle$ in H , we define the correlators $\langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle$ is defined by:

$$(7) \quad \langle \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_n} \rangle := \langle w_0 | \mathcal{O}_{m_1} \cdots \mathcal{O}_{m_n} | v_0 \rangle.$$

This gives rise to a formal quantum field theory in the sense of last Subsection. To summarize, given a space H with an inner product, an operator algebra \mathcal{O} generated by a commuting family of operators $\{\mathcal{O}_n\}_{n \geq 0}$ on H with $\mathcal{O}_0 = \text{id}_H$, and two fixed vector $|v_0\rangle$ and $|w_0\rangle$, one can associate a formal quantum field theory by defining correlators as above. We will refer to such a formal quantum field theory as an *operatorial formal quantum field theory*.

2.3. Special operatorial formal quantum field theories. We now consider a special class of formal quantum field theory. Denote by Λ the space of symmetric functions. Let $\{\alpha_n, \alpha_{-n}\}_{n \geq 1}$ be operators on Λ defined by:

$$(8) \quad \alpha_n := n \frac{\partial}{\partial p_n}, \quad \alpha_{-n} := p_n,$$

then the following commutation relations hold:

$$(9) \quad [\alpha_m, \alpha_n] = m \delta_{m, -n}.$$

Denote by $|0\rangle$ the function 1 in Λ . Then Λ is spanned by $\alpha_{-\lambda}|0\rangle = \alpha_{-\lambda_1} \cdots \alpha_{-\lambda_l}|0\rangle$, where $\lambda = (\lambda_1, \dots, \lambda_l)$ runs through all partitions. A scalar product on Λ can be defined such that

$$(10) \quad \langle 0|0\rangle = 1,$$

$$(11) \quad \alpha_n^* = \alpha_{-n}.$$

The space Λ is called the *bosonic Fock space*. A basic computation tool on Λ is the following Weyl commutation relation:

$$(12) \quad \begin{aligned} & \exp \sum_{m \geq 0} \frac{1}{m} a_m \alpha_m \cdot \exp \sum_{n \geq 0} \frac{1}{n} b_n \alpha_{-n} \\ &= \exp \sum_{m \geq 0} \frac{1}{m} a_m b_m \cdot \exp \sum_{n \geq 0} \frac{1}{n} b_n \alpha_{-n} \cdot \exp \sum_{m \geq 0} \frac{1}{m} a_m \alpha_m. \end{aligned}$$

We will actually work with Λ tensored with some coefficient field F , denoted by Λ_F . Fixing two vectors $|v_0\rangle = A|0\rangle$, $|w_0\rangle = B|0\rangle$ in Λ_F , where A and B are two operators on Λ_F , and a family of commuting family of operators $\{\mathcal{O}_n\}_{n \geq 0}$ on Λ_F , one can define a formal quantum field theory. We will refer to it as a *special formal quantum field theory*. In this case, the partition function can be written defined by vacuum expectation values:

$$(13) \quad Z = \frac{\langle 0|B^* \cdot \exp \sum_{n \geq 0} t_n \mathcal{O}_n \cdot A|0\rangle}{\langle 0|B^* \cdot A|0\rangle}.$$

Introduce an operator L defined by:

$$(14) \quad L := B^* \cdot \exp \sum_{n \geq 0} t_n \mathcal{O}_n \cdot A.$$

Then its evolution with respect to t_n is given by:

$$(15) \quad \frac{\partial}{\partial t_n} L = B^* \cdot \mathcal{O}_n \cdot \exp \sum_{m \geq 0} t_m \mathcal{O}_m \cdot A.$$

It can be regarded as the *Lax operator* in this setting if one can find a sequence of operators P_n such that

$$(16) \quad [P_n, L] = B^* \cdot \mathcal{O}_n \cdot \exp \sum_{m \geq 0} t_m \mathcal{O}_m \cdot A.$$

Later in this paper, we will consider special operatorial formal quantum field theories in which \mathcal{O}_n can all be given using vertex operators. They will be called *vertex operatorial formal quantum field theories*.

3. BACKGROUNDS ON HILBERT SCHEMES AND MOTIVATIONS

In this Section we recall some earlier results on Hilbert schemes that motivate this work.

By a well-known theorem of Fogarty [19] in 1968, when S is a nonsingular projective or quasi-projective algebraic surface, the Hilbert scheme $S^{[n]}$ is a nonsingular variety of dimension $2n$. This result makes it possible to apply various methods to study the topology of the Hilbert schemes. Developments since the 1980s have revealed very rich and deep connections with many other branches of mathematics, e.g., modular forms, representation of infinite-dimensional algebras, vertex operators, Boson-Fermion correspondence, symmetric functions, and integrable hierarchies. For standard reference to these materials, see the book by Nakajima [45] and his more recent lecture notes [47]. Based on these earlier developments, we will present in this paper a method to compute K-theoretical intersection numbers and cohomological intersection numbers on Hilbert schemes based on some connections with the theory of Macdonald polynomials, by applying the vertex operator realizations of Macdonald operators. To better motivate the problem we want to solve, it is instructive to review some of the earlier developments. In 1987, Ellingsrud and Strømme [16] calculated the Betti numbers of the Hilbert schemes of points on the projective plane, the affine plane, and rational ruled surfaces. In 1990, Göttsche [23] computed the generating series of Betti numbers hence also Euler numbers of $S^{[n]}$ by Weil Conjecture. By Göttsche's formula, the generating function of the Poincaré polynomials of $S^{[n]}$ is given by

$$(17) \quad \sum_{n=0}^{\infty} P_t(S^{[n]})q^n = \prod_{m=1}^{\infty} \frac{(1 + t^{2m-1}q^m)^{b_1(S)}(1 + t^{2m+1}q^m)^{b_3(S)}}{(1 - t^{2m-2}q^m)^{b_0(S)}(1 - t^{2m}q^m)^{b_2(S)}(1 - t^{2m+2}q^m)^{b_4(S)}},$$

where $b_i(S)$ is the Betti number of S . By taking $t = -1$, one gets the generating series of Euler numbers:

$$(18) \quad \sum_{n \geq 0} \chi(X^{[n]})q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{\chi(S)}},$$

and a manifest connection with modular forms. In 1993, Göttsche and Soergel [25] computed the generating series of Hodge numbers, and there is a similar formula for the generating series of Hodge polynomials of $S^{[n]}$. In 1994, Vafa and Witten [60] put the above formulas in the context of $N = 4$ topologically twisted supersymmetric Yang-Mills on four-manifolds, and conjectured a relation with 2-dimensional conformal field theory and string theory. In particular, by seeing the orbifold cohomology of symmetric products $\bigoplus_{n \geq 0} H_{orb}^*(X^n/S_n)$ as the bosonic Fock space, an action of infinite-dimensional Heisenberg algebra on $\mathbb{H}(X) := \bigoplus_{n \geq 0} H^*(X^{[n]})$ was conjectured. Furthermore, an action of Virasoro algebra was conjectured.

In 1995, Nakajima [44] and Grojnowski [26] discovered geometric realizations of the Heisenberg algebra action on $\mathbb{H}(S)$. In 1998, Lehn [31] constructed geometric realization of Virasoro action on $\mathbb{H}(S)$ and initiated a program that studies the ring structure on $H^*(S^{[n]})$ via the deep interaction with Heisenberg algebra structure. Both in Lehn's work and an earlier work of Eillingsrud-Strømme [17] in 1993, the Chern classes of the tautological sheaves have played an important role. By a combination and generalization of ideas from [31] and [17], Li, Qin and Wang [34] found a set of ring generators of $H^*(S^{[n]})$ for an arbitrary S in 2000. Later these authors [35] found a different set of ring generators which uses only Nakajima's Heisenberg operators. In subsequent works [36, 38] they also established the universality and stability of Hilbert schemes concerning about the relations among the cohomology rings $H^*(S^{[n]})$ when S or n varies. In [37], a W -algebra was constructed geometrically acting on $\mathbb{H}(S)$ by these authors. Qin and Wang [55] axiomatized the results in [34]-[37] and used the axiomatic approach to establish similar results for the orbifold cohomology of the symmetry products of an even-dimensional closed complex manifold. Through all these works, various algebraic structures in the theory of vertex operator algebras have been geometrically constructed via Hilbert schemes $S^{[n]}$.

Another important aspect of the study of Hilbert schemes is various universal formulas in the study of Hilbert schemes. The first kind of universal formulas give universal expressions for characteristic classes of the tangent bundles or tautological bundles on Hilbert schemes. The first example of this type was due to Lehn [31]:

$$(19) \quad \sum_{n \geq 0} c(L^{[n]})z^n = \exp\left(\sum_{m \geq 1} \frac{(-1)^{m-1}}{m} q_m(c(L))z^m\right),$$

where $L \rightarrow S$ is a line bundle, $L^{[n]} \rightarrow S^{[n]}$ is the tautological bundle associated with L . See Boissière and Nieper-Wißkirchen [9] for more examples. One of their results that concerns us is as follows: There are unique rational constants $\alpha_\lambda, \beta_\lambda, \gamma_\lambda, \delta_\lambda$ such that for each surface S and each vector bundle F on S , the generating series of the Chern characters of the tautological bundles of F is given by:

$$(20) \quad \sum_{n \geq 0} \text{ch}(F^{[n]}) = \sum_{\lambda \in \mathcal{P}} \left(\alpha_\lambda q_\lambda(\text{ch}(F)) + \beta_\lambda q_\lambda(e_S \text{ch}(F)) \right. \\ \left. + \gamma_\lambda q_\lambda(K_S \text{ch}(F)) + \delta_\lambda q_\lambda(K_S^2 \text{ch}(F)) \right) |0\rangle.$$

The coefficients α_λ and β_λ are known from the work of Li-Qin-Wang [38], but unfortunately the closed expressions for γ_λ and δ_λ are not known.

Another kind of universality results are due to Ellingrud-Göttsche-Lehn [18]. The following two results will be useful to us. First, let P be a polynomial in the

Chern classes of the tangent bundle $TS^{[n]}$ and the Chern classes of $F_1^{[n]}, \dots, F_l^{[n]}$. Then there is a universal polynomial \tilde{P} , depending only on P , in the Chern classes of TS , the ranks r_1, \dots, r_l and the Chern classes of F_1, \dots, F_l such that

$$\int_{S^{[n]}} P = \int_S \tilde{P}.$$

Secondly, let $\Psi : K \rightarrow H$ be a multiplicative function, also let $\varphi(x) \in \mathbb{Q}[[x]]$ be a formal power series and, for a complex manifold X of dimension n put $\Phi(X) := \varphi(x_1) \cdots \varphi(x_n)$ with x_1, \dots, x_n the Chern roots of TX . Then for each integer r there are universal power series $A_i \in \mathbb{Q}[[z]]$, $i = 1, \dots, 5$, depending only on Ψ , Φ and r , such that for each $x \in K(S)$ of rank r one has

$$\begin{aligned} (21) \quad & \sum_{n \geq 0} z^n \int_{S^{[n]}} \Psi(x^{[n]}) \Phi(S^{[n]}) \\ & = \exp(c_1^2(x)A_1 + c_2(x)A_2 + c_1(x)c_1(S)A_3 + c_1^2(S)A_4 + c_2(S)A_5). \end{aligned}$$

As pointed out in [18], the series $A_i (i = 1, \dots, 5)$ can be determined by doing the computations for $(\mathbb{P}^2, r \cdot 1)$, $(\mathbb{P}^2, \mathcal{O}(1) + (r-1) \cdot 1)$, $(\mathbb{P}^2, \mathcal{O}(2) + (r-1) \cdot 1)$, $(\mathbb{P}^2, 2\mathcal{O}(1) + (r-2) \cdot 1)$, and $(\mathbb{P}^1 \times \mathbb{P}^1, r \cdot 1)$. So it suffices to do computations for sheaves on \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$, which are equivariant sheaves on toric surfaces. The torus action on a toric surface S induces a torus action on $S^{[n]}$, so after an application of localization formula, one can reduce to compute equivariant intersection numbers on $(\mathbb{C}^2)^{[n]}$. This turns out to be an effective method for computing intersection numbers. This is the approach taken by the author and his collaborators in a series of papers [41, 62, 63], of which this paper is a natural continuation.

Working in the equivariant setting and applying localization formula on Hilbert schemes dates back to the 1987 paper of Ellingsrud-Strømme [16]. In the 1990s, Haiman and his collaborators (see e.g. [28]) related equivariant K-theory of Hilbert schemes $(\mathbb{C}^2)^{[n]}$ to Macdonald polynomials and found deep applications in combinatorics. Nakajima [46] related equivariant cohomology of Hilbert schemes of a line bundle over \mathbb{P}^1 to Jack polynomials. Recall that Jack polynomials depend on some parameter α . In this case, α is the negative of the degree of the line bundle. Motivated by this result, Vasserot [61] identified the ring $H_T^{2n}((\mathbb{C}^2)^{[n]})$ with the class algebra of the symmetric group S_n in 2002. Here the $T = \mathbb{C}^*$ -action on $S = \mathbb{C}^2$ is defined by $t \cdot (x, y) = (tx, t^{-1}y)$. In this setting, the fixed points of the circle action on $(\mathbb{C}^2)^{[n]}$ are parameterized by partitions $\{\lambda \mid |\lambda| = n\}$ of n , and their classes correspond to Schur functions $\{s_\lambda \mid |\lambda| = n\}$. Once a connection with the theory of symmetric functions is set up, one can apply its connection to integrable hierarchies as developed by the Kyoto school [43] and its connection to Gromov-Witten invariants of algebraic curves developed by Okounkov-Pandharipande [53]. Furthermore, one can apply a connection to Bloch-Okounkov formula [10] via boson-fermion correspondence. This was done by Li-Qin-Wang [39] in 2003. In 2004, these authors [40] established another link between equivariant cohomology of Hilbert schemes and Jack polynomials. They considered the following one-torus action on \mathbb{C}^2 : $t \cdot (x, y) = (t^\alpha x, t^{-\beta} y)$. In this case the parameter for the Jack polynomials is β/α . See Nakajima's lecture notes [47] for a review of some results on equivariant cohomology of Hilbert schemes $(\mathbb{C}^2)^{[n]}$.

The above results that we have reviewed so far only concern the conjecture of Vafa-Witten [60] on the relation with 2-dimensional conformal field theories. In

their discussion of $N = 4$ supersymmetric gauge theory in the end of [60], Vafa and Witten have focused on moduli spaces on K3 surfaces and T^4 . They remarked that: “So it is conceivable that K3 could be replaced here by another, perhaps noncompact, space.” As it turns out, the conjectured relation with string theory has gradually been developed into what is called “geometric engineering” of $N = 2$ gauge theories from noncompact toric Calabi-Yau 3-folds first proposed in 1996 by Katz, Klemm and Vafa [29]. Furthermore, while the partition function of $N = 4$ gauge theory is the generating series of the Euler characteristics of the moduli spaces, for $N = 2$ gauge theory on \mathbb{R}^4 one considers the Nekrasov partition function [51] which can be defined by localization on the moduli spaces of framed torsion sheaves on \mathbb{P}^2 with respect to a torus action (see Nakajima-Yoshioka [50] for the mathematical backgrounds). More precisely, let $M(r, k)$ denote the framed moduli space of torsion free sheaves on \mathbb{P}^2 with rank r and $c_2 = k$. By framing we mean a trivialization of the sheaf restricted to the line at infinity. In particular for $r = 1$, $M(1, k) = (\mathbb{C}^2)^{[k]}$. On the gauge theory side we are interested in the generating series of equivariant Riemann-Roch numbers

$$(22) \quad \sum_{k \geq 0} Q^k \chi(M(r, k), K_{r,k}^{1/2} \otimes (\det \mathbb{V})^m)(e_1, \dots, e_r, t_1, t_2)$$

with respect to some $T^r \times T^2$ -action, where $K_{r,k}$ is the canonical line bundle of $M(r, k)$, and \mathbb{V} is some tautological bundle on $M(r, k)$. The idea of geometric engineering is to identify this partition function with generating series of local Gromov-Witten invariants of some toric Calabi-Yau 3-fold which is a fibration over \mathbb{P}^1 with ALE spaces of type A_{r-1} as fibers. This was done by Li-Liu-Zhou [32] in 2004 using the theory of the topological vertex [1, 33], together with some combinatorial results in [68]. When the rank is one, the relevant moduli spaces are the Hilbert schemes $(\mathbb{C}^2)^{[n]}$, the tautological bundle is the tautological bundle ξ_n induced from the trivial bundle $\mathcal{O}_{\mathbb{C}^2}$. The following result was proved in Yang-Zhou [64] in 2011:

$$(23) \quad Z_{\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k-2)}(\lambda, t) = \sum_{n \geq 0} ((-1)^k t)^n \chi((\mathbb{C}^2)^{[n]}, (\det \xi_n)^{k+1})(q^{-1}, q),$$

where $Z_{\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k-2)}(\lambda, t)$ is the generating series of local Gromov-Witten invariants of $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k-2)$. Hence geometric engineering directly relates curve counting to K -theoretical intersection numbers on Hilbert schemes. (For other applications of Hilbert schemes to curve counting problem, see [27] and the references therein.) The motivation behind [32] and [64] is the computation of Gopakumar-Vafa invariants of the corresponding toric Calabi-Yau 3-folds. The idea is to find analogues of Göttsche’s formula for K -theoretical intersection numbers on Hilbert schemes. Motivated by this, in 2006 the author made the following conjecture: The following identity holds for $k \geq 2$:

$$(24) \quad \begin{aligned} & \sum_{n \geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, \dots, t_k) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{(1 - u^n t^{nA})(1 - v^n t^{-nA}) Q^n}{n \prod_{i=1}^k (1 - t_i^n)}\right), \end{aligned}$$

where

$$t^{nA} = t_1^{na_1} \dots t_k^{na_k}.$$

Some special cases were proved then by relating the local contributions from the fixed points to specializations of Macdonald polynomials. In 2011, Zhilan Wang realized that a slight modification can be used to prove the general case, hence another connection to Macdonald polynomials was established in [62] in the setting of equivariant K -theory of $(\mathbb{C}^2)^{[n]}$.

Nekrasov [51] made a conjecture relating his partition functions to Seiberg-Witten prepotential. Two different proofs appeared in 2003. The proof by Nakajima and Yoshioka [48] derived an equation for the partition function called the blowup equation by considering the instanton moduli space on \mathbb{C}^2 blown up at the origin. They also proved the K -theoretic version using the same idea [49]. See also [50]. In the paper by Nekrasov and Okounkov where they gave a different proof, the following result result was proved in the the rank $r = 1$ case:

$$(25) \quad \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{\square \in \lambda} \frac{h(\square)^2 - m^2}{h(\square)^2} = \text{tr } \mathcal{V}_m(1)|_{\mathcal{F}_0} = \prod_{n>0} (1 - q^n)^{m^2 - 1},$$

where \mathcal{F}_0 denotes the charge zero part of the fermionic Fock space, and $\mathcal{V}_m(z)$ is the vertex operator:

$$(26) \quad \mathcal{V}_m(z) := \exp\left(-m \sum_{n>0} \alpha_{-n} \frac{z^n}{n}\right) \exp\left(m \sum_{n>0} \alpha_n \frac{z^{-n}}{n}\right).$$

The geometric realization of this vertex operator on Hilbert schemes has been discovered by Carlsson and Okounkov [11, 12, 13]. With the help of this vertex operator, it is possible to evaluate intersection numbers of the form:

$$(27) \quad F(k_1, \dots, k_N) := \sum_{n \geq 0} q^n \int_{S^{[n]}} \text{ch}_{k_1}(L_1^{[n]}) \cdots \text{ch}_{k_N}(L_N^{[n]}) e(T_m S^{[n]})$$

where $e(T_m S^{[n]}) = \sum_j m^j c_{2n-j}(T S^{[n]})$ can be understood as the Chern polynomial of $T S^{[n]}$. In [11, 12, 13], many examples of type $F(k_1, \dots, k_N)$ have been shown to lead to quasimodular forms. More generally, Okounkov [52] made a conjecture connecting cohomological intersection numbers on Hilbert schemes to q -zeta values. This conjecture has been verified up to lower weight terms by Qin and Yu [57] based on the vertex operator of Carlsson and Okounkov [11, 13] and the Chern character operators [37]. See also the more recent work by Qin and Shen [56].

Inspired by the above work on the Okounkov conjecture, the author [69] related the computations of intersection numbers of the form

$$(28) \quad \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_{S^1}^{[n]}} \prod_{j=1}^N \sum_{k_j \geq 0} z_j^{k_j} \text{ch}_{k_j}(\xi_n^{A_j})_{S^1} \cdot e_{S^1}(T\mathbb{C}^{[n]})$$

to Bloch-Okounkov formula [10] where the circle acts on \mathbb{C}^2 by $t \cdot (x, y) = (tx, t^{-1}y)$, related the two-torus-equivariant version of it, i.e.,

$$\sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{[n]}} \prod_{j=1}^N \sum_{k_j \geq 0} z_j^{k_j} \text{ch}_{k_j}(\xi_n^{A_j})_T \cdot e_T(T\mathbb{C}^{[n]}),$$

to the generalized Bloch-Okounkov correlation functions introduced by Cheng and Wang [15].

Now we can state the problem we want to solve in this paper: We want to find a general algorithm to compute equivariant cohomological intersection numbers of the form:

$$\sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{[n]}} \prod_{j=1}^N \text{ch}_{k_j}(\xi_n^{A_j})_T,$$

or their generating series:

$$\sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)^{[n]}} \prod_{j=1}^N \sum_{k_j \geq 0} z_j^{k_j} \text{ch}_{k_j}(\xi_n^{A_j})_T.$$

We will use a strategy that has been widely used: If the problem does not seem to have a good answer, try a more complicated related problem. For us we first change the problem to a problem in equivariant K -theory by taking each z_i to be a positive integer m_i , then

$$(29) \quad \sum_{k_i \geq 0} m_i^{k_i} \text{ch}_{k_i}(\xi_n^{A_i}) = \text{ch}(\psi_{m_i}(\xi_n^{A_i})).$$

where ψ_m are the Adams operations. So we are interested in the problem of computing equivariant Riemann-Roch numbers:

$$(30) \quad \chi((\mathbb{C}^2)^{[n]}, \psi_{m_1}(\xi_n^{A_1}) \otimes \cdots \otimes \psi_{m_N}(\xi_n^{A_N}))(t_1, t_2),$$

As it turns out, we need to consider an even more complicated problem of computing the equivariant Riemann-Roch numbers of the form:

$$(31) \quad \chi((\mathbb{C}^2)^{[n]}, \psi_{m_1}(\xi_n^{A_1}) \otimes \cdots \otimes \psi_{m_N}(\xi_n^{A_N}) \otimes \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, t_2),$$

and after obtaining results for computing them, take $u = v = 0$. The inclusion of $\Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*$ is clearly inspired by Conjecture 1 and the results in [62]. This is the crucial step that enables us to combine the two connections to Macdonald functions discovered by Haiman [28] and Wang-Zhou [62] in the same framework, and apply some vertex operator realizations of the Macdonald operators.

Our goal is to combine and generalize all the ideas in the works mentioned in this Section together. In the rest of this paper we will report on some progresses in this direction.

4. EQUIVARIANT INTERSECTION NUMBERS ON $(\mathbb{C}^2)^{[n]}$ BY LOCALIZATION FORMULAS

In this Section we review how to compute some equivariant intersection numbers in equivariant K -theory and equivariant cohomology theory of $(\mathbb{C}^2)^{[n]}$ by localizations.

4.1. Preliminaries for localizations on Hilbert schemes of the affine plane. See [45] for references on equivariant indices on Hilbert schemes of points on \mathbb{C}^2 . The torus action on \mathbb{C}^2 given by

$$(t_1, t_2) \cdot (x, y) = (t_1^{-1}x, t_2^{-1}y)$$

on linear coordinates induces an action on $(\mathbb{C}^2)^{[n]}$. The fixed points are isolated and parameterized by partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ of weight n . The weight decomposition

of the tangent bundle of $T(\mathbb{C}^2)^{[n]}$ at a fixed point I_λ indexed by a partition λ is given by [45]:

$$(32) \quad \sum_{s \in \lambda} (t_1^{l(s)} t_2^{-(a(s)+1)} + t_1^{-(l(s)+1)} t_2^{a(s)}).$$

It follows that the equivariant Euler class is given at I_λ by:

$$(33) \quad e_T(T(\mathbb{C}^2)^{[n]})|_{I_\lambda} = \prod_{s \in \lambda} (l(s)w_1 - (a(s) + 1)w_2)(-l(s) + 1)w_1 + a(s)w_2).$$

For a vector $A = (a, b) \in \mathbb{Z}^2$, denote by $\mathcal{O}_{\mathbb{C}^2}^A$ the T^2 -equivariant line bundle on \mathbb{C}^2 with weight A , and let ξ_n^A be its associated tautological bundles. Then one has the following weight decomposition at the fixed points:

$$\xi_n^A|_{I_\lambda} = \sum_{s \in \lambda} t_1^{l'(s)} t_2^{a'(s)} t_1^{a,b}.$$

So their equivariant Chern characters at the fixed points are given by the following formula:

$$(34) \quad \text{ch}(\xi_n^A)_T|_{I_\lambda} = \sum_{s \in \lambda} e^{(l'(s)+a)w_1 + (a'(s)+b)w_2}.$$

In particular,

$$(35) \quad \text{ch}_k(\xi_n^A)_T|_{I_\lambda} = \frac{1}{k!} \sum_{s \in \lambda} ((l'(s) + a)w_1 + (a'(s) + b)w_2)^k.$$

In this Subsection, the following notations have been used:

$$\begin{aligned} a_\mu(s) &= \mu_i - j, & a'_\mu(s) &= j - 1, \\ l_\mu(s) &= \mu_j^t - i, & l'_\mu(s) &= i - 1. \end{aligned}$$

4.2. Some equivariant K-theoretical intersection numbers. For the equivariant K-theoretical case, we are interested in the equivariant Riemann-Roch numbers:

$$(36) \quad \chi((\mathbb{C}^2)^{[n]}, \xi_n^{A_1} \otimes \cdots \otimes \xi_n^{A_N})(t_1, t_2),$$

where $A_j = (a_j, b_j) \in \mathbb{Z}^2$. By holomorphic Lefschetz formula [8],

$$(37) \quad \begin{aligned} & \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \xi_n^{A_j})(t_1, t_2) \\ &= \sum_{\mu} Q^{|\mu|} \prod_{j=1}^N \sum_{s \in \mu} t_1^{l'(s)+a_j} t_2^{a'(s)+b_j} \\ & \quad \cdot \prod_{s \in \mu} \frac{1}{(1 - t_1^{-l(s)} t_2^{a(s)+1})(1 - t_1^{l(s)+1} t_2^{-a(s)})}. \end{aligned}$$

Inspired by the proof of Theorem 1 in [62], we will instead first compute the more complicated equivariant Riemann-Roch numbers:

$$(38) \quad \chi((\mathbb{C}^2)^{[n]}, \xi_n^{A_1} \otimes \cdots \otimes \xi_n^{A_N} \otimes \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, t_2),$$

where $A = (a, b)$, and then take $u = v = 0$. By holomorphic Lefschetz formula,

$$\begin{aligned}
(39) \quad & \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \xi_n^{A_j} \otimes \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, t_2) \\
&= \sum_{\mu} Q^{|\mu|} \prod_{j=1}^N \sum_{s \in \mu} t_1^{l'(s)+a_j} t_2^{a'(s)+b_j} \\
&\quad \cdot \prod_{s \in \mu} \frac{(1 - ut^A t_1^{l'(s)} t_2^{a'(s)}) \cdot (1 - vt^{-A} t_1^{-l'(s)} t_2^{-a'(s)})}{(1 - t_1^{-l(s)} t_2^{a(s)+1}) (1 - t_1^{l(s)+1} t_2^{-a(s)})}.
\end{aligned}$$

This will enable us to exploit the connection to Macdonald polynomials as in [62].

We make some digressions on symmetric functions before we discuss more K-theoretical intersection numbers.

4.3. Basis of the space Λ of symmetric functions. As a vector space, Λ has several natural additive basis. For our purpose, we will use the following three of them: $\{e_\lambda\}_{\lambda \in \mathcal{P}}$, $\{h_\lambda\}_{\lambda \in \mathcal{P}}$, $\{p_\lambda\}_{\lambda \in \mathcal{P}}$. Here for a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}$, $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$, where e_n , h_n and p_n are the *elementary*, *complete*, and *Newton* symmetric functions, respectively. Their generating series are defined by:

$$(40) \quad E(t) = \sum_{n \geq 0} e_n t^n,$$

$$(41) \quad H(t) = \sum_{n \geq 0} h_n t^n,$$

$$(42) \quad P(t) = \sum_{n \geq 1} \frac{p_n}{n} t^n.$$

They are related to each other as follows:

$$(43) \quad H(t) = \frac{1}{E(-t)}, \quad P(t) = -\log E(-t) = \log H(t).$$

4.4. Action of the space Λ on K -theory. There is an action of Λ on $K(X)$ using the power operations on $K(X)$ (see e.g. Atiyah [6]). First, define a map $\Lambda \rightarrow \text{End}(K(X))$ by $p_\mu \mapsto \prod_{i=1}^l \psi^{\mu_i} : K(X) \rightarrow K(X)$ where ψ^n are the Adams operations. Under this map $e_\mu \mapsto \lambda^\mu = \lambda^{\mu_1} \cdots \lambda^{\mu_l}$, where $\lambda^i : K(X) \rightarrow K(X)$ is defined as follows:

$$(44) \quad \sum_{i \geq 0} t^i \lambda^i(E - F) = \sum_{i \geq 0} t^i \Lambda^i(E) \sum_{j \geq 0} (-t)^j S^j(F).$$

Similarly, $h_\mu \mapsto \sigma^\mu = \sigma^{\mu_1} \cdots \sigma^{\mu_l}$, where $\sigma^i K(X) \rightarrow K(X)$ is defined by

$$(45) \quad \sum_{i \geq 0} t^i \sigma^i(E - F) = \sum_{i \geq 0} t^i S^i(E) \sum_{j \geq 0} (-t)^j \Lambda^j(F).$$

Indeed, (191) now becomes

$$(46) \quad \sum_{i \geq 0} t^i \sigma^i = \frac{1}{\sum_{i \geq 0} (-t)^i \lambda^i}, \quad \sum_{i \geq 0} \frac{t^i}{i} \psi^i = -\log \sum_{i \geq 0} (-t)^i \lambda^i = \log \sum_{i \geq 0} t^i \sigma^i.$$

These operations can also be defined in equivariant K -theory.

4.5. **More K-theoretical intersection numbers.** Note one has

$$(47) \quad \text{ch}(\psi^m(\xi_n^A))_T|_{I_\lambda} = \sum_{s \in \lambda} e^{m[(l'(s)+a)w_1+(a'(s)+b)w_2]}.$$

In particular,

$$(48) \quad \begin{aligned} \text{ch}_k(\psi^m(\xi_n^A))_T|_{I_\lambda} &= \frac{m^k}{k!} \sum_{s \in \lambda} ((l'(s) + a)w_1 + (a'(s) + b)w_2)^k \\ &= m^k \cdot \text{ch}(\xi_n^A)_T|_{I_\lambda}. \end{aligned}$$

We are interested in the equivariant Riemann-Roch numbers:

$$(49) \quad \chi((\mathbb{C}^2)^{[n]}, \psi^{m_1}(\xi_n^{A_1}) \otimes \cdots \otimes \psi^{m_N}(\xi_n^{A_N}))(t_1, t_2),$$

where $A_j = (a_j, b_j) \in \mathbb{Z}^2$. Again we will instead first compute the equivariant Riemann-Roch numbers:

$$(50) \quad \chi((\mathbb{C}^2)^{[n]}, \psi^{m_1}(\xi_n^{A_1}) \otimes \cdots \otimes \psi^{m_N}(\xi_n^{A_N}) \otimes \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2).$$

By holomorphic Lefschetz formula,

$$(51) \quad \begin{aligned} &\sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \psi^{m_j}(\xi_n^{A_j}) \otimes \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2) \\ &= \sum_{\mu} Q^{|\mu|} \prod_{j=1}^N \sum_{s \in \mu} e^{m_j[(l'(s)+a_j)w_1+(a'(s)+b_j)w_2]} \\ &\quad \cdot \prod_{s \in \mu} \frac{(1 - ut^A t_1^{l'(s)} t_2^{a'(s)}) \cdot (1 - vt^{-A} t_1^{-l'(s)} t_2^{-a'(s)})}{(1 - t_1^{-l(s)} t_2^{a(s)+1})(1 - t_1^{l(s)+1} t_2^{-a(s)})}. \end{aligned}$$

We also consider other intersection numbers by changing ψ^{m_j} to λ^{m_j} or σ^{m_j} . We will also establish a connection of all these to Macdonald polynomials as in [62].

One can also consider the generating series of these series:

$$\begin{aligned} &\sum_{m_1, \dots, m_N \geq 0} \frac{\prod_{j=1}^N a_{m_j}^{A_j}}{N!} \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \psi^{m_j}(\xi_n^{A_j}) \otimes \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2), \\ &\sum_{m_1, \dots, m_N \geq 0} \frac{\prod_{j=1}^N b_{m_j}^{A_j}}{N!} \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \lambda^{m_j}(\xi_n^{A_j}) \otimes \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2), \\ &\sum_{m_1, \dots, m_N \geq 0} \frac{\prod_{j=1}^N c_{m_j}^{A_j}}{N!} \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \sigma^{m_j}(\xi_n^{A_j}) \otimes \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2), \end{aligned}$$

where $\{a_m^A, b_m^A, c_m^A\}_{m \geq 0}$ are some formal variables, indexed by $A \in \mathbb{Z}^2$. These generating series are related to each other if we require

$$(52) \quad \sum_{\mu \in \mathcal{P}} a_\mu^A \psi^\mu = \sum_{\mu \in \mathcal{P}} b_\mu^A \lambda^\mu = \sum_{\mu \in \mathcal{P}} c_\mu^A \sigma^\mu$$

for each $A \in \mathbb{Z}^2$.

4.6. **Some equivariant cohomological intersection numbers.** Similarly, in the equivariant cohomological case, we are interested in the equivariant intersection numbers:

$$(53) \quad \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_T^{[n]}} \text{ch}_{k_1}(\xi_n^{A_1})_T \cdots \text{ch}_{k_N}(\xi_n^{A_N})_T.$$

In [69] we have considered a different type of equivariant intersection numbers:

$$(54) \quad \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_T^{[n]}} \text{ch}_{k_1}(\xi_n^{A_1})_T \cdots \text{ch}_{k_N}(\xi_n^{A_N})_T \cdot e_T(T\mathbb{C}^{[n]}).$$

The motivation there was Okounkov's Conjecture. By localization formula we have:

$$(55) \quad \begin{aligned} & \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_T^{[n]}} \text{ch}_{k_1}(\xi_n^{A_1})_T \cdots \text{ch}_{k_N}(\xi_n^{A_N})_T \\ &= \sum_{\lambda} q^{|\lambda|} \prod_{j=1}^N \frac{1}{k_j!} \sum_{s \in \lambda} ((l'(s) + a_j)w_1 + (a'(s) + b_j)w_2)^{k_j} \\ & \quad \cdot \prod_{s \in \lambda} \frac{1}{(l(s)w_1 - (a(s) + 1)w_2)(-(l(s) + 1)w_1 + a(s)w_2)} \end{aligned}$$

and for the intersection numbers with the equivariant Euler class:

$$(56) \quad \begin{aligned} & \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_T^{[n]}} \text{ch}_{k_1}(\xi_n^{A_1})_T \cdots \text{ch}_{k_N}(\xi_n^{A_N})_T \cdot e_T(T\mathbb{C}^{[n]}) \\ &= \sum_{\lambda} q^{|\lambda|} \prod_{j=1}^N \frac{1}{k_j!} \sum_{s \in \lambda} ((l'(s) + a_j)w_1 + (a'(s) + b_j)w_2)^{k_j}. \end{aligned}$$

It was explained in [69] that the latter can be computed by first considering the generating series:

$$\begin{aligned} & \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_{S^1}^{[n]}} \prod_{j=1}^N \sum_{k_j \geq 0} z_j^{k_j} \text{ch}_{k_j}(\xi_n^{A_j})_T \cdot e_T(T\mathbb{C}^{[n]}) \\ &= \sum_{\lambda} q^{|\lambda|} \prod_{k=1}^N \sum_{(i,j) \in \lambda} e^{z_k[(i-1+a_k)t_1 + (j-1+b_k)t_2]} \\ &= e^{\sum_{j=1}^N (a_j t_1 + b_j t_2) z_j} \cdot \sum_{\lambda} q^{|\lambda|} \prod_{k=1}^N \sum_{s \in \lambda} e^{z_k[l'_\lambda(s)t_1 + a'_\lambda(s)t_2]}, \end{aligned}$$

then reducing to the deformed n -point function

$$(57) \quad \langle \mathfrak{B}_\lambda(e^{z_1 t_2}, e^{z_1 t_1}) \cdots \mathfrak{B}_\lambda(e^{z_N t_2}, e^{z_N t_1}) \rangle_q$$

introduced by Cheng and Wang [15]. In order to compute (55) using the same ideas, we first compute

$$\begin{aligned}
 & \sum_{n \geq 0} q^n \int_{(\mathbb{C}^2)_T^{[n]}} \text{ch}_{k_1}(\xi_n^{A_1})_T \cdots \text{ch}_{k_N}(\xi_n^{A_N})_T \cdot c_x(\xi_n^A)_T \cdot c_y((\xi_n^A)^*)_T \\
 (58) \quad &= \sum_{\lambda} q^{|\lambda|} \prod_{j=1}^N \frac{1}{k_j!} \sum_{s \in \lambda} ((l'(s) + a_j)w_1 + (a'(s) + b_j)w_2)^{k_j} \\
 & \cdot \prod_{s \in \lambda} \frac{(x + (l'(s) + a)w_1 + (a'(s) + b)w_2) \cdot (y - (l'(s) + a)w_1 - (a'(s) + b)w_2)}{(l(s)w_1 - (a(s) + 1)w_2)(-l(s) + 1)w_1 + a(s)w_2)},
 \end{aligned}$$

then take the coefficients of $q^n x^n y^n$. Here $c_x(E)$ denotes the Chern polynomial of E : $c_x(E) = \sum_{j=0}^r x^{r-j} c_j(E)$. One can understand (59) as a quantization of (58).

5. LOCALIZATION ON HILBERT SCHEMES AND MACDONALD FUNCTIONS

We rewrite (39) as follows:

$$\begin{aligned}
 & \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \xi_n^{A_j} \otimes \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, t_2) \\
 (59) \quad &= \prod_{j=1}^N t_1^{a_j} t_2^{b_j} \cdot \sum_{\mu} (-ut^A Q)^{|\mu|} \prod_{j=1}^N \sum_{s \in \mu} t_1^{l'(s)} t_2^{a'(s)} \\
 & \cdot \prod_{s \in \mu} \frac{(t_2^{a'(s)} - vt^{-A} t_1^{-l'(s)})}{(1 - t_1^{-l(s)} t_2^{a(s)+1})} \frac{(t_1^{l'(s)} - (ut^A)^{-1} t_2^{-a'(s)})}{(1 - t_2^{-a(s)} t_1^{l(s)+1})}.
 \end{aligned}$$

One can recognize the following three expressions on the right-hand side. First the expression

$$(60) \quad \sum_{s \in \mu} t_2^{a'(s)} \cdot t_1^{l'(s)}$$

is related to the eigenvalues of Macdonald operator E , and this fact has been used by Haiman [28] to relate Macdonald polynomials to Hilbert schemes. Secondly, the expressions

$$(61) \quad \prod_{s \in \mu} \frac{(t_2^{a'(s)} - vt^{-A} t_1^{-l'(s)})}{(1 - t_1^{-l(s)} t_2^{a(s)+1})}, \quad \prod_{s \in \mu} \frac{(t_1^{l'(s)} - (ut^A)^{-1} t_2^{-a'(s)})}{(1 - t_2^{-a(s)} t_1^{l(s)+1})}$$

have been recognized by Wang and the author [62] as the specializations of the Macdonald polynomials $P_{\mu}(x; q, t)$ and $Q_{\mu}^t(x; t, q)$ respectively, and hence once again a connection between Macdonald polynomials and Hilbert schemes was observed.

In this Section we will first recall these connections, then reformulate them in term of operators on the bosonic Fock space. This provides a method to compute the right-hand side of (59) and its generalizations obtained by power operations in K-theory.

5.1. Operator formulation of orthogonality of Macdonald polynomials.

Denote by Λ^x the space of symmetric functions in $x = (x_1, \dots, x_n, \dots)$, and let $\Lambda_{q,t}^x = \Lambda^x \otimes \mathbb{C}(q, t)$ the algebra of symmetric functions with coefficients in rational functions in q and t . The Macdonald polynomials $\{P_{\mu} = P_{\mu}(x; q, t)\}$ are suitable normalized symmetric functions in $\Lambda_{q,t}^x$ such that [42, §VI.4]:

- (i) $P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu$, where m_μ are the monomial symmetric functions, and the coefficients $u_{\lambda\mu}$ are rational functions of q and t ;
- (ii) With respect to the scalar product $(\cdot, \cdot)_{q,t}$ on $\Lambda_{q,t}^x$ defined by

$$(62) \quad (p_\lambda, p_\mu)_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}},$$

$\{P_\lambda(x; q, t)\}$ is an orthogonal basis:

$$(63) \quad (P_\lambda, P_\mu)_{q,t} = \delta_{\lambda,\mu} b_\lambda^{-1},$$

$$(64) \quad b_\lambda = \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}.$$

Define another basis $\{Q_\lambda(x; q, t)\}$ by

$$(65) \quad Q_\lambda(x; q, t) = b_\lambda P_\lambda(x; q, t),$$

then $\{Q_\lambda(x; q, t)\}$ is the orthogonal dual basis of $\{P_\lambda(x; q, t)\}$ with respect to the scalar product $(\cdot, \cdot)_{q,t}$.

One can also consider the orthogonal dual basis of $\{P_\lambda(x; q, t)\}$ with respect to the scalar product (\cdot, \cdot) defined by

$$(66) \quad (p_\lambda, p_\mu) = \delta_{\lambda\mu} z_\lambda.$$

By [42, §VI.5],

$$(67) \quad (\omega P_{\lambda^t}(x; q, t), P_\mu(x; t, q)) = \delta_{\lambda\mu},$$

where $\omega : \Lambda \rightarrow \Lambda$ is the involution on Λ defined by:

$$(68) \quad \omega(p_\lambda) = (-1)^{|\lambda| - l(\lambda)} p_\lambda.$$

Equivalently,

$$(69) \quad \sum_{\lambda} P_\lambda(x; q, t) P_{\lambda^t}(y; t, q) = \prod_{i,j} (1 + x_i y_j) = \exp \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y).$$

Now we reinterpret this identity by operators on $\Lambda_{q,t}$. For $n > 0$, let $\alpha_{-n} : \Lambda_{q,t}^x \rightarrow \Lambda_{q,t}^x$ be the operator defined by multiplication by p_n , and let $\alpha_n : \Lambda_{q,t}^x \rightarrow \Lambda_{q,t}^x$ be the operator $n \frac{\partial}{\partial p_n}$. For a partition μ , denote by $|\mu\rangle := s_\mu(x) \in \Lambda_{q,t}^x$, and by $|\mu; q, t\rangle := P_\mu(x; q, t) \in \Lambda_{q,t}^x$. Now (69) can be rewritten as

$$(70) \quad \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(y) \alpha_{-n} \right) |0\rangle = \sum_{\lambda} P_{\lambda^t}(y; t, q) \cdot |\lambda; q, t\rangle.$$

By interchanging x and y we also have:

$$(71) \quad \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(y) \alpha_{-n} \right) |0\rangle = \sum_{\lambda} P_\lambda(y; q, t) \cdot |\lambda^t; t, q\rangle.$$

5.2. Specialization of Macdonald polynomials. Denote by $\epsilon_{u,t}^y : \Lambda_{q,t}^y \rightarrow \mathbb{Q}(q, t)$ the specialization homomorphism defined by

$$\epsilon_{u,t}^y p_n(y) = \frac{1 - u^n}{1 - t^n}$$

for each integer $n \geq 1$. Then we have [42, (6.16), (6.17)]:

$$(72) \quad \epsilon_{u,t}^y P_\lambda(y; q, t) = \prod_{s \in \lambda} \frac{t^{l'(s)} - q^{a'(s)} u}{1 - q^{a(s)} t^{l(s)+1}}.$$

Changing λ to λ^t and u to v , we also have

$$(73) \quad \epsilon_{v,t}^y P_{\lambda^t}(y; q, t) = \prod_{s \in \lambda} \frac{t^{a'(s)} - q^{l'(s)} v}{1 - q^{l(s)} t^{a(s)+1}}.$$

We further interchange q and t to get:

$$(74) \quad \epsilon_{v,q}^y P_{\lambda^t}(y; t, q) = \prod_{s \in \lambda} \frac{q^{a'(s)} - t^{l'(s)} v}{1 - t^{l(s)} q^{a(s)+1}}.$$

Now we apply $\epsilon_{u,t}^y$ on both sides of (71) to get:

$$(75) \quad \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - u^n}{1 - t^n} \alpha_{-n}\right) |0\rangle = \sum_{\lambda} \prod_{s \in \lambda} \frac{t^{l'(s)} - q^{a'(s)} u}{1 - q^{a(s)} t^{l(s)+1}} \cdot |\lambda^t; t, q\rangle.$$

We now change q to q^{-1} and t to t^{-1} to get:

$$(76) \quad \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - u^n}{1 - t^{-n}} \alpha_{-n}\right) |0\rangle = \sum_{\lambda} \prod_{s \in \lambda} \frac{t^{-l'(s)} - q^{-a'(s)} u}{1 - q^{-a(s)} t^{-l(s)+1}} \cdot |\lambda^t; t, q\rangle.$$

In the above we have used the following property of P_λ (c.f. [42, p. 324]):

$$(77) \quad P_\lambda(x; q^{-1}, t^{-1}) = P_\lambda(x; q, t).$$

Similarly, we apply $\epsilon_{v,q}^y$ on both sides of (70) to get:

$$(78) \quad \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - v^n}{1 - q^n} \alpha_{-n}\right) |0\rangle = \sum_{\lambda} \prod_{s \in \lambda} \frac{q^{a'(s)} - t^{l'(s)} v}{1 - t^{l(s)} q^{a(s)+1}} \cdot |\lambda; q, t\rangle.$$

Now we take $q = t_2$, $t = t_1^{-1}$, and change u to $(ut^A)^{-1}$, v to vt^{-A} :

$$(79) \quad \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - v^n t^{-nA}}{1 - t_2^n} \alpha_{-n}\right) |0\rangle = \sum_{\lambda} \prod_{s \in \lambda} \frac{t_2^{a'(s)} - vt^{-A} t_1^{-l'(s)}}{1 - t_1^{-l(s)} t_2^{a(s)+1}} \cdot |\lambda; t_2, t_1^{-1}\rangle.$$

$$(80) \quad \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - (ut^A)^{-n}}{1 - t_1^n} \alpha_{-n}\right) |0\rangle = \sum_{\lambda} \prod_{s \in \lambda} \frac{t_1^{l'(s)} - t_2^{-a'(s)} (ut^A)^{-1}}{1 - t_2^{-a(s)} t_1^{l(s)+1}} |\lambda^t; t_1^{-1}, t_2\rangle.$$

Let K be the operator on Λ such that $K|\mu\rangle := (-ut^A Q)^{|\mu|} |\mu\rangle$. Extend it naturally to $\Lambda_{q,t}$, and it is easy to see that one has $K|\mu; q, t\rangle = (-ut^A)^{|\mu|} |\mu; q, t\rangle$. Apply

operator $K\omega$ on both sides of (79) and take scalar product with both sides of (80):

$$\begin{aligned}
& \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*)(t_1, t_2) \\
&= \sum_{\mu} (-ut^A Q)^{|\mu|} \cdot \prod_{s \in \mu} \frac{(t_2^{a'(s)} - vt^{-A} t_1^{-l'(s)})}{(1 - t_1^{-l(s)} t_2^{a(s)+1})} \frac{(t_1^{l'(s)} - (ut^A)^{-1} t_2^{-a'(s)})}{(1 - t_2^{-a(s)} t_1^{l(s)+1})} \\
&= \langle 0 | \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - (ut^A)^{-n}}{1 - t_1^n} \alpha_n\right) K\omega \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - v^n t^{-nA}}{1 - t_2^n} \alpha_{-n}\right) | 0 \rangle \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-ut^A Q)^n}{n} \frac{1 - (ut^A)^{-n}}{1 - t_1^n} \cdot \frac{1 - v^n t^{-nA}}{1 - t_2^n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{Q^n}{n} \frac{1 - (ut^A)^n}{1 - t_1^n} \cdot \frac{1 - (vt^{-A})^n}{1 - t_2^n}\right).
\end{aligned}$$

This concludes our operator reformulation of the observation in Wang-Zhou [62].

5.3. The Macdonald operators D_n^r . We recall some facts about Macdonald's operators [42, Chapter VI] as summarized in [62, Appendix]. Denote by Λ_n the space of symmetric polynomials in n variables x_1, \dots, x_n . For every function f in x_1, \dots, x_n , $q \in \mathbb{C}$, $1 \leq i \leq n$, define the shift operator T_{q, x_i} by

$$(T_{u, x_i} f)(x_1, \dots, x_n) = f(x_1, \dots, ux_i, \dots, x_n).$$

Define the *Macdonald operators* D_n^r on Λ_n by [42, p.316]:

$$D_n^r = \sum_{I=\{1 \leq i_1 < \dots < i_r \leq n\}} A_I(x_1, \dots, x_n; t) \prod_{i \in I} T_{q, x_i},$$

where

$$A_I(x_1, \dots, x_n; t) = t^{r(r-1)/2} \prod_{i \in I, j \in I^c} \frac{tx_i - x_j}{x_i - x_j}.$$

Here $I^c = \{1, \dots, n\} - I$. The Macdonald polynomials $\{P_{\mu}(x_1, \dots, x_n; q, t)\}$ are suitable normalized symmetric functions such that (cf. [42, Chapter VI, Section 4]):

$$(81) \quad D_n^r P_{\mu}(x_1, \dots, x_n; q, t) = t^{nr} e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) \cdot P_{\mu}(x_1, \dots, x_n; q, t).$$

It turns out that

$$(82) \quad P_{\lambda}(x_1, \dots, x_n; q, t) = m_{\lambda}(x_1, \dots, x_n) + \sum_{\mu < \lambda} u_{\lambda\mu} m_{\mu}(x_1, \dots, x_n),$$

where the coefficients $u_{\lambda\mu}$ are independent of n , and since

$$(83) \quad m_{\mu}(x_1, \dots, x_{n-1}, 0) = m_{\mu}(x_1, \dots, x_{n-1})$$

for $n > l(\mu)$, hence one has

$$(84) \quad P_{\lambda}(x_1, \dots, x_{n-1}, 0; q, t) = P_{\lambda}(x_1, \dots, x_{n-1}; q, t)$$

for $n > l(\lambda)$. Hence one can define $P_{\lambda}(x_1, \dots, x_n, \dots; q, t)$ as a symmetric function in infinitely many variables x_1, \dots, x_n, \dots .

5.4. **The modified Macdonald operator E .** The operators D_n^r do not have stability property, instead,

$$(85) \quad D_n^r|_{x_n=0} = t^r D_{n-1}^r + t^{r-1} D_{n-1}^{r-1},$$

where $D_n^0 = 1$, so one needs to modify them to take $n \rightarrow \infty$. For $r = 1$ this was done by Macdonald [42, Section VI.4]. Define an operator E on Λ whose restriction to Λ_n is given by:

$$E_n = t^{-n} D_n^1(x) - \sum_{i=1}^n t^{-i}.$$

It is easy to see that [15, Lemma 1]:

$$(86) \quad \sum_{i=1}^n q^{\mu_i} t^{-i} - \sum_{i=1}^n t^{-i} = \frac{q-1}{t} \sum_{(i,j) \in \mu} t^{-(i-1)} q^{j-1} = \frac{q-1}{t} \sum_{s \in \mu} t^{-l'(s)} q^{a'(s)}.$$

Hence we have

$$(87) \quad EP_\mu(x; q, t) = \frac{q-1}{t} \sum_{s \in \mu} t^{-l'(s)} q^{a'(s)} \cdot P_\mu(x; q, t).$$

It follows that

$$(88) \quad EP_\mu(x; t_2, t_1^{-1}) = t_1(t_2 - 1) \sum_{s \in \mu} t_1^{l'(s)} t_2^{a'(s)} \cdot P_\mu(x; t_2, t_1^{-1}).$$

5.5. **Vertex operator representation of the modified Macdonald operator E .** The action of the operator D_n^1 on Newton polynomials is given by [20, (63)]:

$$(89) \quad \begin{aligned} D_n^1 p_\mu(x_1, \dots, x_n) &= \frac{1}{1-t} p_\mu(x_1, \dots, x_n) \\ &+ \frac{t^n}{t-1} \left[\prod_{i=1}^n \frac{1-zx_i}{1-zx_i t} \prod_{i=1}^{l(\mu)} \left(p_{\mu_i}(x_1, \dots, x_n) + \frac{q^{\mu_i} - 1}{(tz)^{\mu_i}} \right) \right]_0, \end{aligned}$$

where for a formal power series $f(z) = \sum_{n \in \mathbb{Z}} b_n z^n$, $f(z)_0 = b_0$. This can be seen as follows. Since

$$(90) \quad D_n^1 = \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q, x_i},$$

we have

$$D_n^1 p_\mu(x_1, \dots, x_n) = \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \prod_{k=1}^{l(\mu)} (x_1^{\mu_k} + \dots + x_n^{\mu_k} + (q^{\mu_k} - 1)x_i^{\mu_k}).$$

The idea is to express each term on the right-hand side as the residue of a function and apply Cauchy's residue formula. We take

$$f(z) = \frac{1}{z} \prod_{j=1}^n \frac{1-x_j z}{1-tx_j z} \prod_{k=1}^{l(\mu)} \left(x_1^{\mu_k} + \dots + x_n^{\mu_k} + \frac{q^{\mu_k} - 1}{(tz)^{\mu_k}} \right).$$

This function has poles at $z = 0$, $z = \frac{1}{tx_i}$, $i = 1, \dots, n$. For $R > 0$ large enough, one has

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|z|=R} f(z) dz \\ &= -\frac{1}{2\pi i} \int_{|w|=\frac{1}{R}} w \cdot \prod_{j=1}^n \frac{w-x_j}{w-tx_j} \prod_{k=1}^{l(\mu)} \left(x_1^{\mu_k} + \dots + x_n^{\mu_k} + (q^{\mu_k} - 1)t^{-\mu_k} w^{\mu_k} \right) \frac{-dw}{w^2} \\ &= t^{-n} \prod_{k=1}^{l(\mu)} \left(x_1^{\mu_k} + \dots + x_n^{\mu_k} \right). \end{aligned}$$

On the other hand, by Cauchy Residue formula,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|z|=R} f(z) dz \\ &= \left[\prod_{j=1}^n \frac{1-x_j z}{1-tx_j z} \prod_{k=1}^{l(\mu)} \left(x_1^{\mu_k} + \dots + x_n^{\mu_k} + \frac{q^{\mu_k} - 1}{(tz)^{\mu_k}} \right) \right]_{z^0} \\ &+ \sum_{i=1}^n \frac{t-1}{t^n} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{k=1}^{l(\mu)} \left(x_1^{\mu_k} + \dots + x_n^{\mu_k} + (q^{\mu_k} - 1)x_i^{\mu_k} \right). \end{aligned}$$

This completes the proof of (89). It follows that the action of E on p_μ is given by:

$$\begin{aligned} & Ep_\mu(x_1, \dots, x_n, \dots) \\ (91) \quad &= \frac{1}{t-1} \left[\prod_{i=1}^{\infty} \frac{1-zx_i}{1-zx_i t} \cdot \prod_{i=1}^{l(\mu)} \left(p_{\mu_i}(x_1, \dots, x_n, \dots) + \frac{q^{\mu_i} - 1}{(tz)^{\mu_i}} \right) \right]_{z^0} \\ &- \frac{1}{t-1} p_\mu(x_1, \dots, x_n, \dots). \end{aligned}$$

Since we are taking the coefficients of z^0 , the answer will not be changed if we change z to z/t . Now we are led to consider the operator on $\Lambda_{q,t}$:

$$(92) \quad p_\mu \mapsto \prod_{i=1}^{\infty} \frac{1-t^{-1}x_i z}{1-x_i z} \cdot \prod_{i=1}^{l(\mu)} \left(p_{\mu_i} + \frac{q^{\mu_i} - 1}{z^{\mu_i}} \right).$$

The right-hand side can be written in the following way

$$(93) \quad \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} p_n z^n\right) \cdot \exp\left(-\sum_{n=1}^{\infty} \frac{1-q^n}{n} \cdot n \frac{\partial}{\partial p_n} z^{-n}\right) p_\mu,$$

so Macdonald's operator E can be written in term of vertex operator as follows:

$$(94) \quad E = \frac{1}{t-1} \left[\exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} p_n z^n\right) \cdot \exp\left(-\sum_{n=1}^{\infty} \frac{1-q^n}{n} \cdot n \frac{\partial}{\partial p_n} z^{-n}\right) - 1 \right] \Big|_{z^0}.$$

This vertex operator appears in [4, (32)]. It also appears in a different form in the study of Macdonald polynomials by Garsia and Haiman [20, (73)], as remarked by Cheng and Wang [15, Remark 6] who rephrased it in terms of vertex operator.

5.6. Higher modified Macdonald operators. From (81) it is natural to consider $t^{-rn}D_n^r$, then (85) becomes

$$(95) \quad t^{-rn}D_n^r|_{x_n=0} = t^{-r(n-1)}D_{n-1}^r + t^{-n} \cdot t^{-(r-1)(n-1)}D_{n-1}^{r-1},$$

So one is led to consider a combination of the form

$$(96) \quad E_n^r := \sum_{j=0}^r c_{j,n}(t)t^{-(r-j)n}D_n^{r-j},$$

where $c_{j,n}(t)$ is a function in t , parameterized by j, n , and $c_{0,n}(t) = 1$. If we require that

$$(97) \quad E_n^r|_{x_n=0} = E_{n-1}^r,$$

then from

$$\begin{aligned} E_n^r|_{x_n=0} &= (t^{-rn}D_n^r + \sum_{j=1}^r c_{j,n}(t)t^{-(r-j)n}D_n^{r-j})|_{x_n=0} \\ &= \sum_{j=0}^{r-1} c_{j,n}(t)(t^{-(r-j)(n-1)}D_{n-1}^{r-j} + t^{-n} \cdot t^{-(r-j-1)(n-1)}D_{n-1}^{r-j-1}) + c_{r,n}(t) \\ &= t^{-r(n-1)}D_{n-1}^r + \sum_{j=1}^r (c_{j,n}(t) + t^{-n}c_{j-1,n}(t))t^{-(r-j)(n-1)}D_{n-1}^{r-j} \\ &= D_{n-1}^r + \sum_{j=0}^{r-1} c_{j,n-1}(t) \cdot t^{-jr}D_{n-1}^j. \end{aligned}$$

One has the following recursion relation:

$$(98) \quad c_{j,n}(t) = c_{j,n-1}(t) - t^{-n}c_{j-1,n}(t),$$

for $j = 0, \dots, r-1$. This can be solved with the initial values $c_{r,n}(t) = 1$ and $c_{j,0}(t) = 0$. The following are some examples:

$$\begin{aligned} c_{1,n}(t) &= -(t^{-1} + \dots + t^{-n}) = -t^{-1}e_1(1, t^{-1}, \dots, t^{-(n-1)}), \\ c_{2,n}(t) &= t^{-1} \cdot t^{-1} + t^{-2} \cdot (t^{-1} + t^{-2}) + \dots + t^{-n}(t^{-1} + \dots + t^{-n}) \\ &= t^{-1}e_2(1, t^{-1}, \dots, t^{-n}), \\ c_{3,n}(t) &= -t^{-1} \cdot t^{-1}e_2(1, t^{-1}) - t^{-2} \cdot t^{-1}e_2(1, t^{-1}, t^{-2}) \\ &\quad - \dots - t^{-n} \cdot t^{-1}e_2(1, t^{-1}, \dots, t^{-n}) \\ &= -e_3(1, t^{-1}, \dots, t^{-(n+1)}), \\ c_{4,n}(t) &= t^{-1} \cdot e_3(1, t^{-1}, t^{-2}) + t^{-2} \cdot e_3(1, t^{-1}, \dots, t^{-3}) + \dots \\ &\quad + t^{-n} \cdot e_3(1, t^{-1}, \dots, t^{-(n+1)}) \\ &= t^2 \cdot e_4(1, t^{-1}, \dots, t^{-(n+2)}), \\ c_{5,n}(t) &= -t^{-1} \cdot t^2e_4(1, t^{-1}, t^{-3}) - t^{-2} \cdot t^2e_4(1, t^{-1}, \dots, t^{-4}) + \dots \\ &\quad - t^{-n} \cdot t^2e_4(1, t^{-1}, \dots, t^{-(n+2)}) \\ &= -t^5 \cdot e_5(1, t^{-1}, \dots, t^{-(n+3)}). \end{aligned}$$

In general

$$(99) \quad c_{j,n}(t) = (-1)^j t^{(j^2-3j)/2} e_j(1, t^{-1}, \dots, t^{-(n+j-2)}).$$

So we have proved the following:

Proposition 5.1. *Let the operator E_n^r be defined by:*

$$(100) \quad E_n^r := \sum_{j=0}^r (-1)^j t^{(j^2-3j)/2} e_j(1, t^{-1}, \dots, t^{-(n+j-2)}) \cdot t^{-(r-j)n} D_n^{r-j}.$$

Then for n large enough one has

$$(101) \quad E_n^r|_{x_n=0} = E_{n-1}^r.$$

The Macdonald polynomials $P_\mu(x_1, \dots, x_n; q, t)$ are eigenvectors of E_n^r with eigenvalues:

$$(102) \quad e_n^r(\mu) := \sum_{j=0}^r (-1)^j t^{\frac{j^2-3j}{2}} e_j(1, t^{-1}, \dots, t^{-(n+j-2)}) \cdot e_{r-j}(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}),$$

and the Macdonald functions $P_\mu(x_1, \dots, x_n, \dots; q, t)$ are eigenvectors of E^r with eigenvalues:

$$(103) \quad e^r(\mu) := \sum_{j=0}^r (-1)^j t^{\frac{j^2-3j}{2}} e_j(1, t^{-1}, \dots, t^{-n}, \dots) \cdot e_{r-j}(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}, \dots).$$

We now understand the eigenvalues of E_n^r and E_n by some basic results in basic hypergeometric series. Recall Gauss q -binomial formula:

$$(104) \quad \prod_{j=0}^{n-1} (1 + q^j z) = \sum_{j=0}^n q^{j(j-1)/2} \binom{n}{j}_q z^j,$$

$$(105) \quad \prod_{j=0}^{n-1} \frac{1}{1 - q^j z} = \sum_{j=0}^{\infty} \binom{n+j-1}{j}_q z^j,$$

where the coefficients on the right-hand side are defined by:

$$(106) \quad \binom{n}{j}_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-j+1})}{(1 - q) \cdots (1 - q^j)},$$

one gets:

$$(107) \quad e_j(1, q, \dots, q^{n-1}) = q^{j(j-1)/2} \cdot \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-j+1})}{(1 - q) \cdots (1 - q^j)}.$$

It follows that

$$\begin{aligned} & (-1)^j t^{(j^2-3j)/2} e_j(1, t^{-1}, \dots, t^{-(n+j-2)}) \\ &= (-1)^j t^{-j} \cdot \frac{(1 - t^{-(n+j-2)}) \cdots (1 - t^{-(n-1)})}{(1 - t^{-1}) \cdots (1 - t^{-j})} \\ &= (-1)^j t^{-j} \binom{n+j-2}{j}_{t^{-1}}. \end{aligned}$$

It follows that

$$\begin{aligned}
 \sum_{r=0}^{\infty} e_n^r(\mu) z^r &= \sum_{r=0}^{\infty} z^r \sum_{j=0}^r (-1)^j t^{-j} \binom{n+j-2}{j}_{t^{-1}} \cdot e_{r-j}(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) \\
 &= \sum_{j=0}^{\infty} (-1)^j t^{-j} \binom{n+j-2}{j}_{t^{-1}} z^j \cdot \sum_{k=0}^n e_k(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) z^k \\
 &= \frac{\prod_{j=1}^n (1 + q^{\mu_j} t^{-j} z)}{\prod_{j=1}^{n-1} (1 + t^{-j} z)}.
 \end{aligned}$$

Taking $n \rightarrow \infty$:

$$(108) \quad \sum_{r=0}^{\infty} e^r(\mu) z^r = \prod_{j=1}^{\infty} \frac{1 + q^{\mu_j} t^{-j} z}{1 + t^{-j} z}.$$

One can also require that the operators E_n^r has the Macdonald polynomials $P_{\mu}(x; q, t)$ as eigenvectors, with eigenvalues $e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}, \dots)$. I.e.,

$$\sum_{j=0}^r c_{j,n}(t) \cdot e_{r-j}(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) = e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}, \dots).$$

This is very easy to achieve, it suffices to take $c_{j,n}(t) = e_j(t^{-n-1}, t^{-n-2}, \dots)$. In fact,

$$\begin{aligned}
 &\sum_{r=0}^{\infty} z^r \sum_{j=0}^r e_j(t^{-n-1}, t^{-n-2}, \dots) \cdot e_{r-j}(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) \\
 &= \sum_{j=0}^{\infty} z^j e_j(t^{-n-1}, t^{-n-2}, \dots) \cdot \sum_{k=0}^{\infty} e_k(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}) \\
 &= \prod_{j=n+1}^{\infty} (1 + z t^{-j}) \cdot \prod_{k=1}^n (1 + z q^{\mu_k} t^{-k}) \\
 &= \prod_{j=1}^{\infty} (1 + z q^{\mu_j} t^{-j}) \\
 &= \sum_{r=0}^{\infty} z^r e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}, \dots).
 \end{aligned}$$

So we have recovered [2, (C.5),(C.7)]:

Proposition 5.2. *Let the operator \tilde{E}_n^r be defined by:*

$$(109) \quad \tilde{E}_n^r := \sum_{j=0}^r e_j(t^{-n-1}, t^{-n-2}, \dots) \cdot t^{-(r-j)n} D_n^{r-j}.$$

Then for n large enough, the Macdonald polynomials $P_{\mu}(x; q, t)$ are eigenvectors of these operators, with eigenvalues $e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}, \dots)$.

5.7. Vertex operator realization of \tilde{E}_n^r . In this subsection we will recall the vertex operator realization of \tilde{E}_n^r first derived by Shiraishi [58] and reformulated by Awata and Kanno [2]. We will make a new derivation by combining some of their ideas. Our derivation is completely elementary.

The generating series of the eigenvalues of \tilde{E}_n^r is

$$(110) \quad \sum_{r=0}^{\infty} z^r e_r(q^{\mu_1} t^{-1}, \dots, q^{\mu_n} t^{-n}, \dots) = \prod_{j=1}^{\infty} (1 + q^{\mu_j} t^{-j} z),$$

hence by comparing with (108), we have

$$(111) \quad \sum_{r=0}^{\infty} z^r \tilde{E}^r = \prod_{j=1}^{\infty} (1 + t^{-j} z^j) \cdot \sum_{s=0}^{\infty} z^s E^s,$$

and so

$$(112) \quad \tilde{E}^r = \sum_{j=0}^r e_j(t^{-1}, t^{-2}, \dots) \cdot E^{r-j}.$$

By Euler's formula (see e.g. [5]):

$$(113) \quad \prod_{n=1}^{\infty} (1 + q^n z) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{\prod_{j=1}^n (1 - q^j)} z^n,$$

one gets

$$(114) \quad e_n(q, q^2, \dots) = \frac{q^{n(n+1)/2}}{\prod_{j=1}^n (1 - q^j)},$$

so one has

$$(115) \quad e_j(t^{-1}, t^{-2}, \dots) = \frac{t^{-j(j+1)/2}}{\prod_{j=1}^n (1 - t^{-j})} = \prod_{j=1}^n \frac{1}{t^j - 1},$$

and so

$$(116) \quad \tilde{E}^r = \sum_{j=0}^r \prod_{j=1}^n \frac{1}{t^j - 1} \cdot E^{r-j}.$$

One also has

$$\begin{aligned} e_j(t^{-n-1}, t^{-n-2}, \dots) &= t^{-jn} e_j(t^{-1}, t^{-2}, \dots) = t^{-jn} \cdot \frac{t^{-j(j+1)/2}}{\prod_{a=1}^j (1 - t^{-a})} \\ &= \frac{t^{-jn}}{\prod_{a=1}^j (t^a - 1)}. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{E}_n^r &= \frac{t^{-rn}}{\prod_{a=1}^r (t^a - 1)} \cdot \sum_{k=0}^r \prod_{i=0}^{k-1} (t^{r-i} - 1) \cdot D_n^k \\ &= t^{-rn} e_r(t^{-1}, t^{-2}, \dots) \cdot \sum_{k=0}^r \prod_{i=0}^{k-1} (t^{r-i} - 1) \cdot D_n^k. \end{aligned}$$

(See [58, (78)] and [2, (D.9)].) From this one can now derive the vertex operator realization of \tilde{E}^r in [58, 2]. The idea is to consider the action of $t^{-rn} \cdot \sum_{k=0}^r \prod_{i=0}^{k-1} (t^{r-i} -$

1) $\cdot D_n^k$ on p_μ and rewrite it as the constant terms of some Laurent series. From the definition of D_n^r , we have

$$\begin{aligned}
 & t^{-rn} \sum_{k=0}^r \prod_{i=0}^{k-1} (t^{r-i} - 1) \cdot D_n^k p_\mu(x_1, \dots, x_n) \\
 (117) \quad & = t^{-rn} \sum_{k=0}^r \prod_{i=0}^{k-1} (t^{r-i} - 1) \cdot \sum_{I=\{1 \leq i_1 < \dots < i_k \leq n\}} t^{k(k-1)/2} \prod_{i \in I, j \in I^c} \frac{tx_i - x_j}{x_i - x_j} \\
 & \cdot \prod_{a=1}^{l(\mu)} (x_1^{\mu_a} + \dots + x_n^{\mu_a} + (q^{\mu_a} - 1) \sum_{j=1}^k x_{i_j}^{\mu_a}).
 \end{aligned}$$

Consider the constant term of

$$\begin{aligned}
 f_r(z_1, \dots, z_r) & = \prod_{\alpha=1}^r \prod_{j=1}^n \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^r z_\alpha^{-\mu_a} \right).
 \end{aligned}$$

where $\prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)}$ is understood as the series

$$(118) \quad \prod_{1 \leq \alpha < \beta \leq r} \left(1 + \sum_{n=1}^{\infty} t^{-n} (1-t) (z_\alpha / z_\beta)^n \right).$$

The constant term can be computed recursively as follows. First,

$$\begin{aligned}
 & f_r(z_1, \dots, z_r) \Big|_{z_r^0} \\
 & = \operatorname{Res}_{z_r=\infty} \left(f_r(z_1, \dots, z_r) \frac{dz_r}{z_r} \right) - \sum_{i=1}^n \operatorname{Res}_{z_r=1/x_i} \left(f_r(z_1, \dots, z_r) \frac{dz_r}{z_r} \right).
 \end{aligned}$$

Since one has

$$\begin{aligned}
 f_r(z_1, \dots, z_r) & = \frac{1}{z_r} \prod_{j=1}^n \frac{x_j - t/z_r}{x_j - 1/z_r} \cdot \prod_{1 \leq \alpha < r} \frac{1 - z_\alpha / z_r}{1 - z_\alpha / (z_r t)} \\
 & \cdot \prod_{\alpha=1}^{r-1} \left(\frac{1}{z_\alpha} \prod_{j=1}^n \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \right) \cdot \prod_{\alpha < \beta < r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) z_r^{-\mu_a} \right),
 \end{aligned}$$

and so

$$\begin{aligned}
 \operatorname{Res}_{z_r=\infty} f(z_1, \dots, z_r) \frac{dz_r}{z_r} & = \prod_{\alpha=1}^{r-1} \left(\frac{1}{z_\alpha} \prod_{j=1}^n \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \right) \cdot \prod_{\alpha < \beta < r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} \right) \\
 & = f_{r-1}(z_1, \dots, z_{r-1}).
 \end{aligned}$$

We also have

$$\begin{aligned}
& \sum_{i=1}^n \operatorname{Res}_{z_r=1/x_i} f(z_1, \dots, z_r) \frac{dz_r}{z_r} \\
&= (1-t) \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-1} \frac{1 - x_i z_\alpha}{1 - x_i z_\alpha / t} \cdot \prod_{\alpha=1}^{r-1} \prod_{j=1}^n \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{\alpha < \beta < r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
&\quad \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) x_i^{\mu_a} \right) \\
&= t^{r-1} (1-t) \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-1} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{\alpha < \beta < r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
&\quad \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) x_i^{\mu_a} \right).
\end{aligned}$$

Therefore, one has the following recursion relation:

$$\begin{aligned}
& f_r(z_1, \dots, z_r) \Big|_{z_r^0} = f_{r-1}(z_1, \dots, z_{r-1}) \\
&+ t^{r-1} (t-1) \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-1} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-1} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
&\quad \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) x_i^{\mu_a} \right).
\end{aligned}$$

Repeating the above computations for z_{r-1}, \dots, z_1 , one gets:

$$\begin{aligned}
& f_r(z_1, \dots, z_r) \Big|_{z_r^0 | z_{r-1}^0} = f_{r-2}(z_1, \dots, z_{r-2}) \\
&+ t^{r-2} (t-1) \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-2} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
&\quad \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) x_i^{\mu_a} \right) \\
&+ t^{r-1} (t-1) \cdot \left[\sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-2} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \right. \\
&\quad \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) x_i^{\mu_a} \right) \\
&+ \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \sum_{k \neq i} (t-1) \prod_{j \neq i, k} \frac{t - x_j / x_k}{1 - x_j / x_k} \prod_{\alpha=1}^{r-2} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \\
&\quad \cdot \prod_{1 \leq \alpha < \beta < r-2} \frac{1 - x_k z_\alpha}{1 - x_k z_\alpha / t} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\
&\quad \left. \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1) (x_k^{\mu_a} + x_i^{\mu_a}) \right) \right].
\end{aligned}$$

After some simplifications one gets:

$$\begin{aligned}
 & f_r(z_1, \dots, z_r)|_{z_r^0|z_{r-1}^0} = f_{r-2}(z_1, \dots, z_{r-2}) \\
 + & (t^{r-1} + t^{r-2})(t-1) \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-2} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1)x_i^{\mu_a} \right) \\
 + & t^{r-1} t^{r-2} (t-1)^2 \cdot \sum_{1 \leq i \neq k \leq n} \frac{tx_i - x_k}{x_i - x_k} \cdot \prod_{j \neq i, k} \frac{tx_i - x_j}{x_i - x_j} \frac{tx_k - x_j}{x_k - x_j} \\
 & \cdot \prod_{\alpha=1}^{r-2} \prod_{j \neq i, k} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1)(x_k^{\mu_a} + x_i^{\mu_a}) \right).
 \end{aligned}$$

For the summation $\sum_{1 \leq i \neq k \leq n}$, one considers two cases: When $i < k$, set $i_1 = i$ and $i_2 = k$; when $i > k$, set $i_1 = k$ and $i_2 = i$. Note

$$(119) \quad \frac{tx_{i_1} - x_{i_2}}{x_{i_1} - x_{i_2}} + \frac{tx_{i_2} - x_{i_1}}{x_{i_2} - x_{i_1}} = t + 1.$$

So we get:

$$\begin{aligned}
 & f_r(z_1, \dots, z_r)|_{z_r^0|z_{r-1}^0} = f_{r-2}(z_1, \dots, z_{r-2}) \\
 + & (t^{r-1} + t^{r-2})(t-1) \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-2} \prod_{j \neq i} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1)x_i^{\mu_a} \right) \\
 + & t^{r-1} t^{r-2} (t-1)^2 (t+1) \cdot \sum_{1 \leq i_1 < i_2 \leq n} \prod_{j \neq i_1, i_2} \frac{tx_{i_1} - x_j}{x_{i_1} - x_j} \frac{tx_{i_2} - x_j}{x_{i_2} - x_j} \\
 & \cdot \prod_{\alpha=1}^{r-2} \prod_{j \neq i_1, i_2} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r-2} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \\
 & \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-2} z_\alpha^{-\mu_a} + (q^{\mu_a} - 1)(x_{i_1}^{\mu_a} + x_{i_2}^{\mu_a}) \right).
 \end{aligned}$$

One can continue this procedure by induction. For this purpose, let us introduce some notations. For a subset $I \subset \{1, 2, \dots, n\}$, denote by $|I|$ the number of elements

in I and by $I^c = \{1, \dots, n\} - I$. Define:

$$\begin{aligned} f_{r,I}(z_1, \dots, z_r) : &= \prod_{j \in I^c} \left(\prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^r \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \right) \cdot \prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\ &\cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^r z_\alpha^{-\mu_a} + \sum_{i \in I} (q^{\mu_a} - 1) x_i^{\mu_a} \right). \end{aligned}$$

When I is the empty set \emptyset , write $f_{r,\emptyset}(z_1, \dots, z_r) = f_r(z_1, \dots, z_r)$. We also have

$$\begin{aligned} &\text{Res}_{z_r=\infty} f_{r,I}(z_1, \dots, z_r) \frac{dz_r}{z_r} \\ &= \prod_{j \in I^c} \left(\prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-1} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \right) \cdot \prod_{1 \leq \alpha < \beta \leq r-1} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\ &\cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} + \sum_{i \in I} (q^{\mu_a} - 1) x_i^{\mu_a} \right), \\ &= f_{r-1,I}(z_1, \dots, z_{r-1}). \end{aligned}$$

For $k \in I^c$,

$$\begin{aligned} &\text{Res}_{z_r=1/x_k} f_{r,I}(z_1, \dots, z_r) \frac{dz_r}{z_r} \\ &= -(t-1) \prod_{j \in (I \cup \{k\})^c} \frac{t - x_j / x_k}{1 - x_j / x_k} \cdot \prod_{\alpha=1}^{r-1} \frac{1 - x_k z_\alpha}{1 - x_k z_\alpha / t} \\ &\cdot \prod_{j \in I^c} \left(\prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{\alpha=1}^{r-1} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \right) \cdot \prod_{1 \leq \alpha < \beta \leq r-1} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \\ &\cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-\mu_a} + \sum_{i \in I \cup \{k\}} (q^{\mu_a} - 1) x_i^{\mu_a} \right) \\ &= -t^{r-1}(t-1) \prod_{i \in I} \frac{tx_i - x_k}{x_i - x_k} \cdot f_{r-1, I \cup \{k\}}(z_1, \dots, z_{r-1}). \end{aligned}$$

The constant term of $f_{r,I}(z_1, \dots, z_r)$ in z_r :

$$\begin{aligned} &f_{r,I}(z_1, \dots, z_r)|_{z_r^0} \\ &= \text{Res}_{z_r=\infty} (f_{r,I}(z_1, \dots, z_r) \frac{dz_r}{z_r}) - \sum_{k \in I^c} \text{Res}_{z_r=1/x_k} (f_{r,I}(z_1, \dots, z_r) \frac{dz_r}{z_r}) \\ &= f_{r-1,I}(z_1, \dots, z_{r-1}) \\ &+ t^{r-1}(t-1) \sum_{k \in I^c} \prod_{i \in I} \frac{tx_i - x_k}{tx_i - x_k} \cdot f_{r-1, I \cup \{k\}}(z_1, \dots, z_{r-1}). \end{aligned}$$

In these notations, we have proved above the following identities:

$$\begin{aligned}
 f_r(z_1, \dots, z_r)|_{z_r^0} &= f_{r-1}(z_1, \dots, z_{r-1}) + t^{r-1}(t-1) \sum_{k=1}^n f_{r-1, \{i\}}(z_1, \dots, z_{r-1}), \\
 f_r(z_1, \dots, z_r)|_{z_r^0|z_{r-1}^0} &= f_{r-2}(z_1, \dots, z_{r-2}) \\
 + (t^{r-1} + t^{r-2})(t-1) &\sum_{|I|=1} f_{r-2, I}(z_1, \dots, z_{r-2}) \\
 + t^{r-1}t^{r-2}(t-1)^2(t+1) &\cdot \sum_{|I|=2} f_{r-2, I}(z_1, \dots, z_{r-2}).
 \end{aligned}$$

Inductively, one can show that

$$f_r(z_1, \dots, z_r)|_{z_r^0|z_{r-1}^0 \cdots |z_{r-s}^0} = \sum_{l=0}^s \alpha_j^{(s)} \sum_{|I|=l} f_{r-s, I}(z_1, \dots, z_{r-s}).$$

Taking the constant terms z_{r-s-1}^0 on both sides:

$$\begin{aligned}
 &\sum_{l=0}^{s+1} \alpha_l^{(s+1)} \sum_{|I|=l} f_{r-s-1, I}(z_1, \dots, z_{r-s-1}) \\
 &= f_r(z_1, \dots, z_r)|_{z_r^0|z_{r-1}^0 \cdots |z_{r-s}^0|z_{r-s-1}^0} \\
 &= \sum_{l=0}^s \alpha_l^{(s)} \sum_{|I|=l} f_{r-s, I}(z_1, \dots, z_{r-s})|_{z_{r-s-1}^0} \\
 &= \sum_{l=0}^s \alpha_l^{(s)} \sum_{|I|=l} \left(f_{r-s-1, I}(z_1, \dots, z_{r-s-1}) \right. \\
 &+ \left. t^{r-s-1}(t-1) \sum_{k \in I^c} \prod_{i \in I} \frac{tx_i - x_k}{x_i - x_k} \cdot f_{r-s-1, I \cup \{k\}}(z_1, \dots, z_{r-s-1}) \right) \\
 &= \sum_{l=0}^s \alpha_l^{(s)} \sum_{|I|=l} f_{r-s-1, I}(z_1, \dots, z_{r-s-1}) \\
 &+ t^{r-s-1}(t-1) \sum_{l=0}^s \alpha_l^{(s)} \sum_{|I|=l} \sum_{k \in I^c} \prod_{i \in I} \frac{tx_i - x_k}{x_i - x_k} \cdot f_{r-s-1, I \cup \{k\}}(z_1, \dots, z_{r-s-1}).
 \end{aligned}$$

Now note:

$$\begin{aligned}
 &\sum_{|I|=l} \sum_{k \in I^c} \prod_{i \in I} \frac{tx_i - x_k}{x_i - x_k} \cdot f_{r-s-1, I \cup \{k\}}(z_1, \dots, z_{r-s-1}) \\
 &= \sum_{|J|=l+1} \left(\sum_{k \in J} \prod_{i \in J - \{k\}} \frac{tx_i - x_k}{x_i - x_k} \right) \cdot f_{r-s-1, J}(z_1, \dots, z_{r-s-1}) \\
 &= \frac{1 - t^{l+1}}{1 - t} \cdot \sum_{|J|=l+1} f_{r-s-1, J}(z_1, \dots, z_{r-s-1}),
 \end{aligned}$$

where in the last equality we have used an identity:

$$(120) \quad \sum_{i=1}^n \prod_{1 \leq j \leq n, j \neq i} \frac{tx_j - x_i}{x_j - x_i} = \frac{1 - t^n}{1 - t}.$$

So we have

$$\begin{aligned}
& \sum_{l=0}^{s+1} \alpha_l^{(s+1)} \sum_{|I|=l} f_{r-s-1,I}(z_1, \dots, z_{r-s-1}) \\
&= \sum_{l=0}^s \alpha_l^{(s)} \sum_{|I|=l} f_{r-s-1,I}(z_1, \dots, z_{r-s-1}) \\
&+ t^{r-s-1} \sum_{l=0}^s \alpha_l^{(s)} \sum_{|J|=l+1} (t^{l+1} - 1) \cdot f_{r-s-1,J}(z_1, \dots, z_{r-s-1}).
\end{aligned}$$

So we get a recursion relation:

$$\begin{aligned}
\alpha_0^{(s+1)} &= \alpha_0^{(s)}, \\
\alpha_l^{(s+1)} &= \alpha_l^{(s)} + t^{r-s-1} (t^l - 1) \alpha_{l-1}^{(s)}, \quad l = 1, \dots, s+1,
\end{aligned}$$

with the initial value $\alpha_0^{(0)} = 1$. An easy solution is given by the elementary symmetric functions:

$$(121) \quad \alpha_l^{(s)} = \prod_{j=1}^l (t^j - 1) \cdot e_l(t^{r-1}, t^{r-2}, \dots, t^{r-s}).$$

Hence we get

$$(122) \quad \alpha_l^{(r)} = \prod_{j=1}^l (t^j - 1) \cdot e_r(t^{r-1}, t^{r-2}, \dots, t, 1) = t^{l(l-1)/2} \cdot \prod_{j=1}^l (t^{r-l+1} - 1),$$

where in the second equality we have used (104). To summarize, we have shown that

$$\begin{aligned}
& \tilde{E}_n^r p_\mu(x_1, \dots, x_n) \\
&= e_r(t^{-1}, t^{-2}, \dots) \cdot t^{-rn} \sum_{k=0}^r \prod_{i=0}^{k-1} (t^{r-i} - 1) \cdot D_n^k p_\mu(x_1, \dots, x_n) \\
(123) \quad &= e_r(t^{-1}, t^{-2}, \dots) \cdot \left(\prod_{\alpha=1}^r \prod_{j=1}^n \frac{1 - x_j z_\alpha / t}{1 - x_j z_\alpha} \cdot \prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \right. \\
& \quad \left. \cdot \prod_{a=1}^{l(\mu)} \left(\sum_{j=1}^n x_j^{\mu_a} + (q^{\mu_a} - 1) \sum_{\alpha=1}^r z_\alpha^{-\mu_a} \right) \right) \Big|_{z_r^0 z_{r-1}^0 \dots z_1^0}.
\end{aligned}$$

It can be reformulated in terms of vertex operators:

$$\begin{aligned}
(124) \quad \tilde{E}^r &= e_r(t^{-1}, t^{-2}, \dots) \cdot \left[\prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha / z_\beta}{1 - z_\alpha / (z_\beta t)} \cdot \exp \left(\sum_{n=1}^{\infty} \frac{1 - t^{-n}}{n} p_n \sum_{\alpha=1}^r z_\alpha^n \right) \right. \\
& \quad \left. \cdot \exp \left(- \sum_{n=1}^{\infty} \frac{1 - q^n}{n} \sum_{\alpha=1}^r z_\alpha^{-n} \cdot n \frac{\partial}{\partial p_n} \right) \right] \Big|_{z_r^0 z_{r-1}^0 \dots z_1^0}.
\end{aligned}$$

This was obtained by Awata-Kanno [2]. Here we use the notation $|_{z_r^0 z_{r-1}^0 \dots z_1^0}$ to indicate the order of taking the constant term: One first take the constant term in z_r , then in z_{r-1} , and so on. A symmetrized version was given earlier by Shiraishi [58]. Our derivation follow their ideas with some differences in presentations.

6. VERTEX REALIZABLE OPERATORS AND THEIR CORRELATIONS FUNCTIONS

In this Section we introduce a notion of vertex realizable operators on $\Lambda_{q,t}$ and define their correlation functions by introducing (u, v) -bracket. We also develop a method to compute the correlators of the modified Macdonald operators \tilde{E}^r .

6.1. **Vertex realizable operators.** Now we introduce the following

Definition 6.1. An operator $\mathcal{A} : \Lambda_{q,t} \rightarrow \Lambda_{q,t}$ with $\{P_\mu(x; q, t)\}_{\mu \in \mathcal{P}}$ as eigenvectors with eigenvalues $a_\mu(q, t)$ is said to be *vertex realizable* of weight $r \geq 1$ if and a Laurent series $K_{\mathcal{A}}(z_r, \dots, z_1)$ with coefficients in $\mathbb{C}(q, t)$ such that

$$(125) \quad \mathcal{A} = \left[K_{\mathcal{A}}(z_r, \dots, z_1) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} p_n \sum_{\alpha=1}^r z_\alpha^n\right) \cdot \exp\left(-\sum_{n=1}^{\infty} \frac{1-q^n}{n} \sum_{\alpha=1}^r z_\alpha^{-n} \cdot n \frac{\partial}{\partial p_n}\right) \right] \Big|_{z_r^0 z_{r-1}^0 \dots z_1^0}.$$

We will refer to $K_{\mathcal{A}}(z_r, \dots, z_1)$ is the *kernel* of \mathcal{A} . A linear combination of vertex realizable operators of different weights will also be called vertex realizable.

Proposition 6.1. Suppose that \mathcal{A} and \mathcal{B} are two vertex realizable operators of weights r and s , with kernels $K_{\mathcal{A}}(z_r, \dots, z_1)$ and $K_{\mathcal{B}}(z_s, \dots, z_1)$, respectively. Then their composition $\mathcal{A}\mathcal{B}$ is a vertex realizable operator of weight $r+s$ with kernel

$$(126) \quad \exp\left(\sum_{n=1}^{\infty} \frac{(1-t^{-n})(1-q^n)}{n} \sum_{\alpha=s+1}^{r+s} z_\alpha^{-n} \cdot \sum_{\beta=1}^s z_\beta^n\right) \cdot K_{\mathcal{A}}(z_{r+s}, \dots, z_{s+1}) K_{\mathcal{B}}(z_s, \dots, z_1).$$

Proof. This is a straightforward consequence of the identity:

$$(127) \quad \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} \cdot n \frac{\partial}{\partial p_n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n} p_n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{a_n b_n}{n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} \cdot n \frac{\partial}{\partial p_n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n} p_n\right).$$

□

6.2. **The (u, v) -bracket and correlation functions.** For a vertex realizable operator \mathcal{A} as in the above Definition, we define the (u, v) -bracket of \mathcal{A} by:

$$(128) \quad \langle \mathcal{A} \rangle_{u,v} := \sum_{\mu} (-uQ)^{|\mu|} \cdot \prod_{s \in \mu} \frac{(q^{a'(s)} - vt^{l'(s)})}{(1 - t^{l(s)} q^{a(s)+1})} \cdot a_\mu(q, t) \cdot \prod_{s \in \mu} \frac{(t^{l'(s)} - u^{-1} q^{-a'(s)})}{(1 - q^{-a(s)} t^{-(l(s)+1)})}.$$

This is clearly motivated by §5.2: If one changes u to $ut^{-a}q^b$ and v to $vt^a q^{-b}$, then one gets the expressions as in §5.2. We also define the normalized (u, v) -bracket of \mathcal{A} by:

$$(129) \quad \langle \mathcal{A} \rangle'_{u,v} := \frac{\langle \mathcal{A} \rangle_{u,v}}{\langle 1 \rangle_{u,v}}.$$

For vertex realizable operators $\mathcal{A}_1, \dots, \mathcal{A}_n$, define their (u, v) -correlation function by:

$$(130) \quad \langle \mathcal{A}_1 \cdots \mathcal{A}_n \rangle_{u,v}.$$

We also define

$$(131) \quad \langle \mathcal{A}_1 \cdots \mathcal{A}_n \rangle'_{u,v} := \frac{\langle \mathcal{A}_1 \cdots \mathcal{A}_n \rangle_{u,v}}{\langle 1 \rangle_{u,v}},$$

and the connected correlations functions:

$$\begin{aligned} \langle \mathcal{A}_1 \mathcal{A}_2 \rangle_{u,v}^\circ &:= \langle \mathcal{A}_1 \mathcal{A}_2 \rangle'_{u,v} - \langle \mathcal{A}_1 \rangle'_{u,v} \cdot \langle \mathcal{A}_2 \rangle'_{u,v}, \\ \langle \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \rangle_{u,v}^\circ &:= \langle \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \rangle'_{u,v} - \langle \mathcal{A}_1 \mathcal{A}_2 \rangle'_{u,v} \cdot \langle \mathcal{A}_3 \rangle'_{u,v} \\ &\quad - \langle \mathcal{A}_1 \mathcal{A}_3 \rangle'_{u,v} \cdot \langle \mathcal{A}_2 \rangle'_{u,v} - \langle \mathcal{A}_2 \mathcal{A}_3 \rangle'_{u,v} \cdot \langle \mathcal{A}_1 \rangle'_{u,v} \\ &\quad + 2 \langle \mathcal{A}_1 \rangle'_{u,v} \cdot \langle \mathcal{A}_2 \rangle'_{u,v} \cdot \langle \mathcal{A}_3 \rangle'_{u,v}, \end{aligned}$$

etc. These notations borrowed from quantum field theory have been used in mathematical literature, see e.g. Okounkov [52].

Proposition 6.2. *Let \mathcal{A} be a vertex realizable operator as in the above definition, then we have:*

$$(132) \quad \begin{aligned} \langle \mathcal{A} \rangle'_{u,v} &= \left[K_{\mathcal{A}}(z_r, \dots, z_1) \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (u^n - 1) \sum_{a=1}^r z_a^n Q^n \right) \right. \\ &\quad \left. \cdot \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1 - v^n) \sum_{a=1}^r z_a^{-n} \right) \right] \Big|_{z_r^0 z_{r-1}^0 \cdots z_1^0}. \end{aligned}$$

Proof. By slightly generalizing the computations in §5.2, we have:

$$\begin{aligned} \langle \mathcal{A} \rangle_{u,vz} &= \langle 0 | \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - u^{-n}}{1 - t^{-n}} \alpha_n \right) \\ &\quad K \omega \mathcal{A} \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - v^n}{1 - q^n} \alpha_{-n} \right) | 0 \rangle \\ &= \langle 0 | \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - u^{-n}}{1 - t^{-n}} (-uQ)^n \alpha_n \right) \\ &\quad \cdot \left[K_{\mathcal{A}}(z_r, \dots, z_1) \cdot \exp \left(\sum_{n=1}^{\infty} \frac{1 - t^{-n}}{n} \sum_{a=1}^r z_a^n \alpha_{-n} \right) \right. \\ &\quad \left. \cdot \exp \left(- \sum_{n=1}^{\infty} \frac{1 - q^n}{n} \sum_{a=1}^r z_a^{-n} \cdot \alpha_n \right) \right] \Big|_{z_r^0 z_{r-1}^0 \cdots z_1^0} \\ &\quad \cdot \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1 - v^n}{1 - q^n} \alpha_{-n} \right) | 0 \rangle. \end{aligned}$$

Then (132) can be obtained by applying (127). \square

6.3. Explicit computations of correlation functions of \tilde{E}^r . Using Proposition 6.1 and Proposition 6.2, it is possible to compute $\langle \tilde{E}^{r_1} \cdots \tilde{E}^{r_n} \rangle'_{u,v}$. For example, by (124) and (132) we get:

$$\begin{aligned} \langle \tilde{E}^r \rangle'_{u,v} &= e_r(t^{-1}, t^{-2}, \dots) \cdot \left[\prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \right. \\ &\quad \cdot \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (u^n - 1) \sum_{a=1}^r z_a^n Q^n \right) \\ &\quad \left. \cdot \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1 - v^n) \sum_{a=1}^r z_a^{-n} \right) \right] \Big|_{z_r^0 z_{r-1}^0 \cdots z_1^0}. \end{aligned}$$

Note we have

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (u^n - 1) z^n Q^n \right) &= \frac{1 + zQ}{1 + uzQ} = 1 + \sum_{n=1}^{\infty} (-1)^n u^{n-1} (u-1) Q^n z^n, \\ \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1 - v^n) z^{-n} \right) &= \frac{1 + vz^{-1}}{1 + z^{-1}} = 1 + \sum_{n=1}^{\infty} (-1)^n (1-v) z^{-n}. \end{aligned}$$

For the purpose of simplifying the notations in the computations of $\langle \tilde{E}^r \rangle$, we introduce the following notation: Let f be formal Laurent series in z_1, \dots, z_r , define:

$$(133) \quad \{f\}_r := \left[f \cdot \prod_{1 \leq \alpha < \beta \leq r} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \cdot \prod_{a=1}^r \frac{1 + z_a Q}{1 + uz_a Q} \cdot \prod_{a=1}^r \frac{1 + vz_a^{-1}}{1 + z_a^{-1}} \right] \Big|_{z_r^0 \cdots z_1^0}.$$

Expressions on the right-hand side of the above definition are not understood as rational function, but as series as follows:

$$\begin{aligned} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} &= 1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_\alpha^k z_\beta^{-k}, \\ \frac{1 + z_a Q}{1 + uz_a Q} &= 1 + \sum_{n=1}^{\infty} (-1)^{n-1} u^{n-1} (1-u) Q^n z_a^n, \\ \frac{1 + vz_a^{-1}}{1 + z_a^{-1}} &= 1 + \sum_{n=1}^{\infty} (-1)^n (1-v) z_a^{-n}. \end{aligned}$$

It is also useful to define:

$$(134) \quad [f]_r := \left[f \cdot \frac{1 + z_r Q}{1 + uz_r Q} \cdot \frac{1 + vz_r^{-1}}{1 + z_r^{-1}} \right] \Big|_{z_r^0}.$$

Note for a formal Laurent series f in z_1, \dots, z_r ,

$$(135) \quad \{f\}_r = \left[\cdots \left[\left[f \cdot \prod_{1 \leq \alpha < r} \frac{1 - z_\alpha/z_r}{1 - z_\alpha/(z_r t)} \right]_r \cdot \prod_{1 \leq \alpha < r-1} \frac{1 - z_\alpha/z_{r-1}}{1 - z_\alpha/(z_{r-1} t)} \right]_{r-1} \cdots \right]_1.$$

In particular, for a formal Laurent series f in z_1 ,

$$(136) \quad \{f\}_1 = [f]_1,$$

and so we have

$$(137) \quad [z_r^k]_r = [z_1^k]_1 = \{z_1^k\}_1.$$

So the computations of $\{f\}_r$ can be reduced to the computations of $[z_1^k]_1 = \{z_1^k\}_1$. This is our method for the computations of correlators of \tilde{E}^r .

6.3.1. Computations of $[z_1^k]_1$.

Proposition 6.3. *The following identities hold:*

$$(138) \quad [1]_1 = 1 - Q \frac{(1-u)(1-v)}{1-uQ},$$

$$(139) \quad [z_1^k]_1 = (-1)^k \cdot \frac{(1-v)(1-Q)}{1-uQ},$$

$$(140) \quad [z_1^{-k}]_1 = (-uQ)^{k-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ},$$

where $k > 0$.

Proof. These can also be obtained by direct computations.

$$\begin{aligned} [1]_1 &= \left[\frac{1+z_1Q}{1+uz_1Q} \cdot \frac{1+vz_1^{-1}}{1+z_1^{-1}} \right] \Big|_{z_1^0} \\ &= \left[\left(1 + \sum_{n=1}^{\infty} (-1)^{n-1} u^{n-1} (1-u) Q^n z_a^n \right) \cdot \left(1 + \sum_{n=1}^{\infty} (-1)^n (1-v) z_a^{-n} \right) \right] \Big|_{z_2^0} \\ &= 1 - \sum_{n=1}^{\infty} (1-u)(1-v) u^{n-1} Q^n = 1 - Q \frac{(1-u)(1-v)}{1-uQ}. \\ [z_1^k]_1 &= \left[z_2^k \cdot \left(1 + \sum_{n=1}^{\infty} (-1)^{n-1} u^{n-1} (1-u) Q^n z_a^n \right) \cdot \left(1 + \sum_{n=1}^{\infty} (-1)^n (1-v) z_a^{-n} \right) \right] \Big|_{z_2^0} \\ &= (-1)^k (1-v) + \sum_{n=1}^{\infty} (-1)^{k-1} u^{n-1} (1-u)(1-v) Q^n \\ &= (-1)^k (1-v) \cdot \left(1 - \frac{1-u}{1-uQ} Q \right) = (-1)^k \cdot \frac{(1-v)(1-Q)}{1-uQ}. \\ [z_2^{-k}]_2 &= \left[z_2^{-k} \cdot \left(1 + \sum_{n=1}^{\infty} (-1)^{n-1} u^{n-1} (1-u) Q^n z_a^n \right) \cdot \left(1 + \sum_{n=1}^{\infty} (-1)^n (1-v) z_a^{-n} \right) \right] \Big|_{z_2^0} \\ &= (-1)^{k-1} u^{k-1} (1-u) Q^k + \sum_{n=1}^{\infty} (-1)^{k-1} u^{k+n-1} (1-u)(1-v) Q^{k+n} \\ &= (-1)^{k-1} u^{k-1} (1-u) Q^k \left(1 + (1-v) \sum_{n=1}^{\infty} u^n Q^n \right) \\ &= (-1)^{k-1} u^{k-1} (1-u) Q^k \left(1 + (1-v) \cdot \frac{uQ}{1-uQ} \right) \\ &= Q (-uQ)^{k-1} \cdot \frac{(1-u)(1-uvQ)}{1-uQ}. \end{aligned}$$

□

As a corollary,

$$(141) \quad \langle \tilde{E}^1 \rangle_{u,v} = \frac{t^{-1}}{1-t^{-1}} \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right).$$

6.3.2. *One-point function of \tilde{E}^2 .* Our result for $\langle \tilde{E}^2 \rangle'_{u,v}$ is

$$(142) \quad \langle \tilde{E}^2 \rangle'_{u,v} = \frac{t^{-3}}{(1-t^{-1})(1-t^{-2})} \left[\left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 + (1-t^{-1})Q \frac{(1-Q)(1-u)(1-v)(1-uvQ)}{(1-t^{-1}uQ)(1-uQ)^2} \right].$$

First we have:

$$\langle \tilde{E}^2 \rangle'_{u,v} = e_2(t^{-1}, t^{-2}, \dots) \cdot \left[\frac{1-z_1/z_2}{1-z_1/(z_2t)} \cdot \prod_{a=1}^2 \frac{1+z_aQ}{1+uz_aQ} \cdot \prod_{a=1}^2 \frac{1+vz_a^{-1}}{1+z_a^{-1}} \right] \Big|_{z_2^0 z_1^0},$$

and so

$$\langle \tilde{E}^2 \rangle'_{u,v} = \frac{t^{-3}}{(1-t^{-1})(1-t^{-2})} \cdot \{1\}_2.$$

We use the expansions

$$\frac{1-z_1/z_2}{1-z_1/(z_2t)} = 1 + (t^{-1}-1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k z_2^{-k},$$

to get:

$$\begin{aligned} \{1\}_2 &= \left[\left[\frac{1-z_1/z_2}{1-z_1/(z_2t)} \right]_2 \right]_1 = \left[[1]_2 + (t^{-1}-1) \left[\sum_{k=1}^{\infty} t^{-(k-1)} z_1^k z_2^{-k} \right]_2 \right]_1 \\ &= \{1\}_1^2 + (t^{-1}-1) \cdot \left\{ \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k \cdot (-uQ)^{k-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \right\}_1 \\ &= \{1\}_1^2 + (t^{-1}-1)Q \frac{(1-u)(1-uvQ)}{1-uQ} \cdot \left\{ \sum_{k=1}^{\infty} (-t^{-1}uQ)^{k-1} z_1^k \right\}_1. \end{aligned}$$

We rewrite it in the following form:

$$(143) \quad \{1\}_2 = \{1\}_1^2 + (t^{-1}-1)Q \frac{(1-u)(1-uvQ)}{1-uQ} \cdot \left\{ \frac{z_1}{1+t^{-1}uQz_1} \right\}_1.$$

Now note

$$(144) \quad \left\{ \frac{z_1}{1+t^{-1}uQz_1} \right\}_1 = -\frac{1}{1-t^{-1}uQ} \cdot \frac{(1-v)(1-Q)}{1-uQ}.$$

Indeed,

$$\begin{aligned} \left\{ \frac{z_1}{1+t^{-1}uQz_1} \right\}_1 &= \sum_{k=1}^{\infty} (-t^{-1}u)^{k-1} Q^{k-1} \cdot [z_1^k]_1 \\ &= -\sum_{k=1}^{\infty} t^{-(k-1)} u^{k-1} Q^{k-1} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\ &= -\frac{1}{1-t^{-1}uQ} \cdot \frac{(1-v)(1-Q)}{1-uQ}. \end{aligned}$$

This finishes the computation for $\langle \tilde{E}^2 \rangle'_{u,v}$.

6.3.3. *One-point function of \tilde{E}^3 .* Now we come to compute $\langle \tilde{E}^3 \rangle'_{u,v}$ in the same fashion. Recall

$$\begin{aligned} \langle \tilde{E}^3 \rangle'_{u,v} &= e_3(t^{-1}, t^{-2}, \dots) \\ &\cdot \left[\prod_{1 \leq \alpha < \beta \leq 3} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/(z_\beta t)} \cdot \prod_{a=1}^3 \frac{1 + z_a Q}{1 + u z_a Q} \cdot \prod_{a=1}^3 \frac{1 + v z_a^{-1}}{1 + z_a^{-1}} \right] \Big|_{z_3^0 z_2^0 z_1^0}. \end{aligned}$$

We first get:

$$\begin{aligned} &\left[\frac{1 - z_1/z_3}{1 - z_1/(z_3 t)} \cdot \frac{1 - z_2/z_3}{1 - z_2/(z_3 t)} \right]_3 \\ &= \left[(1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k z_3^{-k}) \cdot (1 + (t^{-1} - 1) \sum_{l=1}^{\infty} t^{-(l-1)} z_2^l z_3^{-l}) \right]_3 \\ &= [1]_3 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k \left[z_3^{-k} \right]_3 + (t^{-1} - 1) \sum_{l=1}^{\infty} t^{-(l-1)} z_2^l \left[z_3^{-l} \right]_3 \\ &+ (t^{-1} - 1)^2 \sum_{k,l=1}^{\infty} t^{-(k-1)} z_1^k \cdot t^{-(l-1)} z_2^l \left[z_3^{-k-l} \right]_3 \\ &= \{1\}_1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k (-uQ)^{k-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \\ &+ (t^{-1} - 1) \sum_{l=1}^{\infty} t^{-(l-1)} z_2^l (-uQ)^{l-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \\ &+ (t^{-1} - 1)^2 \sum_{k,l=1}^{\infty} t^{-(k-1)} z_1^k \cdot t^{-(l-1)} z_2^l (-uQ)^{k+l-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ}. \end{aligned}$$

We rewrite it as follows:

$$\begin{aligned} \{1\}_3 &= \{1\}_1 \cdot \{1\}_2 \\ &+ Q(t^{-1} - 1) \frac{(1-u)(1-uvQ)}{1-uQ} \cdot \left\{ \frac{z_1}{1 + t^{-1}uQz_1} \right\}_2 \\ &+ Q(t^{-1} - 1) \frac{(1-u)(1-uvQ)}{1-uQ} \cdot \left\{ \frac{z_2}{1 + t^{-1}uQz_2} \right\}_2 \\ &- uQ^2(t^{-1} - 1)^2 \frac{(1-u)(1-uvQ)}{1-uQ} \cdot \left\{ \frac{z_1}{1 + t^{-1}uQz_1} \cdot \frac{z_2}{1 + t^{-1}uQz_2} \right\}_2. \end{aligned}$$

Computations for $\left\{ \frac{z_1}{1+t^{-1}uQz_1} \right\}_2$. From

$$\begin{aligned}
 & [z_1^l(1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k z_2^{-k})]_2 \\
 &= z_1^l([1]_2 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k [z_2^{-k}]_2) \\
 &= z_1^l(\{1\}_1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k \{z_1^{-k}\}_1) \\
 &= z_1^l \left(\{1\}_1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k \cdot (-uQ)^{k-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \right) \\
 &= \{1\}_1 z_1^l + (t^{-1} - 1) Q \frac{(1-u)(1-uvQ)}{1-uQ} \cdot z_1^l \cdot \frac{z_1}{1+ut^{-1}Qz_1},
 \end{aligned}$$

we get:

$$\begin{aligned}
 \left\{ \frac{z_1}{1+t^{-1}uQz_1} \right\}_2 &= \{1\}_1 \cdot \left\{ \frac{z_1}{1+t^{-1}uQz_1} \right\}_1 \\
 &+ (t^{-1} - 1) \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \cdot \left\{ \frac{z_1}{1+t^{-1}uQz_1} \cdot \frac{z_1}{1+t^{-1}uQz_1} \right\}_1.
 \end{aligned}$$

We have

$$\begin{aligned}
 & \left\{ \frac{z_1}{1+t^{-1}uQz_1} \cdot \frac{z_1}{1+t^{-1}uQz_1} \right\}_1 \\
 &= \left\{ \sum_{l \geq 1} (-t^{-1}uQ)^{l-1} z_1^l \cdot \sum_{k=1}^{\infty} (-t^{-1}uQ)^{k-1} z_1^k \right\}_1 \\
 &= \sum_{k, l \geq 1} (-t^{-1}uQ)^{l-1} \cdot (-t^{-1}uQ)^{k-1} \cdot \{z_1^{k+l}\}_1 \\
 &= \sum_{k, l \geq 1} (-t^{-1}uQ)^{l-1} \cdot (-t^{-1}uQ)^{k-1} \cdot (-1)^{k+l} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\
 &= \frac{1}{(1-ut^{-1}Q)^2} \cdot \frac{(1-v)(1-Q)}{1-uQ}.
 \end{aligned}$$

In the same fashion one can prove the following identity:

$$(145) \quad \left\{ \left(\frac{z_1}{1+t^{-1}uQz_1} \right)^m \right\}_1 = \frac{(-1)^m}{(1-t^{-1}uQ)^m} \cdot \frac{(1-v)(1-Q)}{(1-uQ)}.$$

Computations for $\left\{ \frac{z_2}{1+t^{-1}uQz_2} \right\}_2$. We need to first compute

$$\begin{aligned}
 & \sum_{l \geq 1} (-t^{-1}uQ)^{l-1} \left[z_2^l (1 + (t^{-1} - 1) \sum_{k=1}^{\infty} t^{-(k-1)} z_1^k z_2^{-k}) \right]_2 \\
 &= \sum_{l \geq 1} (-t^{-1}uQ)^{l-1} [z_2^l]_2 + (t^{-1} - 1) \sum_{k, l \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k [z_2^{l-k}]_2.
 \end{aligned}$$

We have two parts to consider. Part one:

$$\begin{aligned} & \sum_{l \geq 1} (-t^{-1}uQ)^{l-1} [z_2^l]_2 = \sum_{l \geq 1} (-t^{-1}uQ)^{l-1} \cdot \{z_1^l\}_1 \\ &= \left\{ \frac{z_1}{1 + ut^{-1}Qz_1} \right\}_1 = -\frac{1}{1 - t^{-1}uQ} \cdot \frac{(1-v)(1-Q)}{(1-uQ)}. \end{aligned}$$

Part two without a factor of $(t^{-1} - 1)$:

$$\begin{aligned} & \sum_{k, l \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k [z_2^{l-k}]_2 = \sum_{k, l \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k \{z_1^{l-k}\}_1 \\ &= \sum_{k \geq 1} (-t^{-1}uQ)^{k-1} t^{-(k-1)} z_1^k \{1\}_1 + \sum_{l > k \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k \{z_1^{l-k}\}_1 \\ &+ \sum_{k > l \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k \{z_1^{l-k}\}_1 \\ &= \{1\}_1 \cdot \frac{z_1}{1 + ut^{-1}Qz_1} + \sum_{l > k \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k \cdot (-1)^{l-k} \frac{(1-v)(1-Q)}{(1-uQ)} \\ &+ \sum_{k > l \geq 1} (-t^{-1}uQ)^{l-1} t^{-(k-1)} z_1^k \cdot (-uQ)^{k-l-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ}. \end{aligned}$$

After simplification it becomes:

$$\begin{aligned} & \{1\}_1 \cdot \frac{z_1}{1 + ut^{-1}Qz_1} + \sum_{l > k \geq 1} (-1)^{k-1} t^{-k-l+2} u^{l-1} Q^{l-1} z_1^k \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\ &+ \sum_{k > l \geq 1} (-1)^k t^{-k-l+2} u^{k-2} Q^{k-2} z_1^k \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ}. \end{aligned}$$

Take $[\cdot]_1$:

$$\begin{aligned} & \{1\}_1 \cdot \frac{1}{1 - t^{-1}uQ} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\ &+ \sum_{l > k \geq 1} (-1)^{k-1} t^{-k-l+2} u^{l-1} Q^{l-1} \cdot (-1)^k \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\ &+ \sum_{k > l \geq 1} (-1)^k t^{-k-l+2} u^{k-2} Q^{k-2} \cdot (-1)^k \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \\ &= \{1\}_1 \cdot \frac{1}{1 - t^{-1}uQ} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\ &- \frac{1}{1 - ut^{-2}Q} \cdot \frac{ut^{-1}Q}{1 - ut^{-1}Q} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\ &+ \frac{t^{-1}}{1 - ut^{-2}Q} \cdot \frac{1}{1 - ut^{-1}Q} \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ}. \end{aligned}$$

In the above we have used the following summations:

$$\begin{aligned}
 \sum_{l>k\geq 1} t^{-k-l+2} u^{l-1} Q^{l-1} &= \sum_{j,k\geq 1} t^{-j-2k+2} u^{j+k-1} Q^{j+k-1} \\
 &= \frac{1}{1-ut^{-2}Q} \cdot \frac{ut^{-1}Q}{1-ut^{-1}Q}, \\
 \sum_{k>l\geq 1} t^{-k-l+2} u^{k-2} Q^{k-2} &= \sum_{m,l\geq 1} t^{-m-2l+2} u^{l+m-2} Q^{l+m-2} \\
 &= \frac{t^{-1}}{1-ut^{-2}Q} \cdot \frac{1}{1-ut^{-1}Q}.
 \end{aligned}$$

The computation of $\left\{ \frac{z_1}{1+t^{-1}uQz_1} \cdot \frac{z_2}{1+t^{-1}uQz_2} \right\}_2$ is similar and will be omitted. From this example it is clear that the complexity the computation increase very rapidly.

6.3.4. *Computations of $\langle \tilde{E}^1 \tilde{E}^1 \rangle$.* Now we compute the two-point function of \tilde{E}^1 :

$$\begin{aligned}
 \langle \tilde{E}^1 \tilde{E}^1 \rangle_{u,v} &= \left(\frac{t^{-1}}{1-t^{-1}} \right)^2 \langle 0 | \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1-u^{-n}}{1-t^{-n}} \alpha_n \right) K\omega \\
 &\quad \cdot \left[\exp \left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} \alpha_{-n} z_1^n \right) \cdot \exp \left(- \sum_{n=1}^{\infty} \frac{1-q^n}{n} z_1^{-n} \cdot \alpha_n \right) \right] \Big|_{z_1^0} \\
 &\quad \cdot \left[\exp \left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} \alpha_{-n} z_2^n \right) \cdot \exp \left(- \sum_{n=1}^{\infty} \frac{1-q^n}{n} z_2^{-n} \cdot \alpha_n \right) \right] \Big|_{z_2^0} \\
 &\quad \cdot \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1-v^n}{1-q^n} \alpha_{-n} \right) |0\rangle \\
 &= \left(\frac{t^{-1}}{1-t^{-1}} \right)^2 \langle 0 | \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1-u^{-n}}{1-t^{-n}} (-uQ)^n \alpha_n \right) \\
 &\quad \cdot \exp \left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} \alpha_{-n} z_1^n \right) \cdot \exp \left(- \sum_{n=1}^{\infty} \frac{1-q^n}{n} z_1^{-n} \cdot \alpha_n \right) \\
 &\quad \cdot \exp \left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} \alpha_{-n} z_2^n \right) \cdot \exp \left(- \sum_{n=1}^{\infty} \frac{1-q^n}{n} z_2^{-n} \cdot \alpha_n \right) \\
 &\quad \cdot \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1-v^n}{1-q^n} \alpha_{-n} \right) |0\rangle \Big|_{z_2^0 z_1^0}.
 \end{aligned}$$

It follows that

$$\begin{aligned} \langle \tilde{E}^1 \tilde{E}^1 \rangle_{u,v} &= \left(\frac{t^{-1}}{1-t^{-1}} \right)^2 \\ &\cdot \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (1-u^{-n})(-uQ)^n z_2^n \right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (1-u^{-n})(-uQ)^n z_1^n \right) \\ &\cdot \exp\left(- \sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{n} z_1^{-n} z_2^n \right) \\ &\cdot \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1-v^n) z_1^{-n} \right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1-v^n) z_2^{-n} \right) \Big|_{z_2^0 z_1^0}. \end{aligned}$$

We rewrite as

$$\begin{aligned} \langle \tilde{E}^1 \tilde{E}^1 \rangle_{u,v} &= \left(\frac{t^{-1}}{1-t^{-1}} \right)^2 \\ &\cdot \prod_{a=1}^2 \frac{1+z_a Q}{1+u z_a Q} \cdot \prod_{a=1}^2 \frac{1+v z_a^{-1}}{1+z_a^{-1}} \cdot \frac{(1-z_2 z_1^{-1})(1-qt^{-1}z_2 z_1^{-1})}{(1-qz_2 z_1^{-1})(1-t^{-1}z_2 z_1^{-1})} \Big|_{z_2^0 z_1^0}. \end{aligned}$$

We use the following expansions:

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (u^n - 1) z_a^n Q^n \right) &= \frac{1+z_a Q}{1+u z_a Q} = 1 - \sum_{n=1}^{\infty} (-1)^n u^{n-1} (1-u) Q^n z_a^n, \\ \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1-v^n) z_a^{-n} \right) &= \frac{1+v z_a^{-1}}{1+z_a^{-1}} = 1 + \sum_{n=1}^{\infty} (-1)^n (1-v) z_a^{-n}, \\ \exp\left(- \sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{n} z_2^n z_1^{-n} \right) &= \frac{(1-z_2 z_1^{-1})(1-qt^{-1}z_2 z_1^{-1})}{(1-qz_2 z_1^{-1})(1-t^{-1}z_2 z_1^{-1})} \\ &= 1 - \sum_{n=1}^{\infty} (1-q)(1-t^{-1}) \frac{q^n - t^{-n}}{q-t^{-1}} z_2^n z_1^{-n}. \end{aligned}$$

The last equality is an application of the following identity:

$$(146) \quad \frac{(1-x)(1-t_1 t_2 x)}{(1-t_1 x)(1-t_2 x)} = 1 - \sum_{n=1}^{\infty} (1-t_1)(1-t_2) \frac{t_1^n - t_2^n}{t_1 - t_2} x^n.$$

We get:

$$\begin{aligned} &\left[\frac{(1-z_2 z_1^{-1})(1-qt^{-1}z_2 z_1^{-1})}{(1-qz_2 z_1^{-1})(1-t^{-1}z_2 z_1^{-1})} \right]_2 \\ &= \left[1 - \sum_{n=1}^{\infty} (1-q)(1-t^{-1}) \frac{q^n - t^{-n}}{q-t^{-1}} z_2^n z_1^{-n} \right]_2 \\ &= [1]_2 - \sum_{n=1}^{\infty} (1-q)(1-t^{-1}) \frac{q^n - t^{-n}}{q-t^{-1}} z_1^{-n} [z_2^n]_2 \\ &= \{1\}_1 - \sum_{n=1}^{\infty} (1-q)(1-t^{-1}) \frac{q^n - t^{-n}}{q-t^{-1}} z_1^{-n} (-1)^n \cdot \frac{(1-v)(1-Q)}{(1-uQ)}. \end{aligned}$$

Take $\{\cdot\}_1$:

$$\begin{aligned}
 & \{1\}_1 \cdot [1]_1 - \sum_{n=1}^{\infty} (1-q)(1-t^{-1}) \frac{q^n - t^{-n}}{q - t^{-1}} [z_1^{-n}]_1 \cdot (-1)^n \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\
 = & \{1\}_1^2 - \sum_{n=1}^{\infty} (1-q)(1-t^{-1}) \frac{q^n - t^{-n}}{q - t^{-1}} \cdot (-uQ)^{n-1} \cdot Q \frac{(1-u)(1-uvQ)}{1-uQ} \\
 & \cdot (-1)^n \cdot \frac{(1-v)(1-Q)}{(1-uQ)} \\
 = & \left(1 - Q \frac{(1-u)(1-v)}{1-uQ}\right)^2 \\
 + & Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \cdot (1-u)(1-v) \cdot \frac{(1-q)(1-t^{-1})}{(1-uqQ)(1-ut^{-1}Q)}.
 \end{aligned}$$

So we get

$$\begin{aligned}
 (147) \quad \langle \tilde{E}^1 \tilde{E}^1 \rangle_{u,v} = & \left(\frac{t^{-1}}{1-t^{-1}}\right)^2 \cdot \left[\left(1 - Q \frac{(1-u)(1-v)}{1-uQ}\right)^2 \right. \\
 & \left. + Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \cdot (1-u)(1-v) \cdot \frac{(1-q)(1-t^{-1})}{(1-uqQ)(1-ut^{-1}Q)} \right].
 \end{aligned}$$

6.4. Generalizations of (86). In order to get more vertex realizable operators we will first generalize (86). By changing q to q^m and t to t^m in (86), we have

$$\begin{aligned}
 (148) \quad \sum_{i=1}^{\infty} q^{m\mu_i} t^{-mi} - \sum_{i=1}^{\infty} t^{-mi} &= \frac{q^m - 1}{t^m} \sum_{(i,j) \in \mu} t^{-m(i-1)} q^{m(j-1)} \\
 &= \frac{q^m - 1}{t^m} \sum_{s \in \mu} t^{-ml'(s)} q^{ma'(s)}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 (149) \quad \sum_{s \in \mu} t^{-ml'(s)} q^{ma'(s)} &= \frac{t^m}{q^m - 1} \left(\sum_{i=1}^{\infty} q^{m\mu_i} t^{-mi} - \sum_{i=1}^{\infty} t^{-mi} \right) \\
 &= \frac{t^m}{q^m - 1} (p_m(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) - p_m(t^{-1}, t^{-2}, \dots)).
 \end{aligned}$$

Recall the following relationship between Newton symmetric functions and elementary symmetric functions:

$$(150) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n z^n = \log \sum_{n=0}^{\infty} e_n z^n.$$

Write the right-hand side as

$$(151) \quad \log \sum_{n=0}^{\infty} e_n z^n = \sum_{\lambda \in \mathcal{P}} \alpha_{\lambda} \cdot e_{\lambda} z^{|\lambda|}$$

for some coefficients α_{λ} . The first few terms are:

$$(152) \quad e_1 z + (e_2 - \frac{1}{2} e_1^2) z^2 + (e_3 - e_1 e_2 + \frac{1}{3} e_1^3) z^3 + (e_4 - e_1 e_3 - \frac{1}{2} e_2^2 + e_2 e_1^2 - \frac{1}{4} e_1^4) z^4 + \dots$$

In other words,

$$(153) \quad \alpha_{(1)} = 1, \quad \alpha_{(2)} = 1, \quad \alpha_{(1,1)} = -\frac{1}{2},$$

$$(154) \quad \alpha_{(3)} = 1, \quad \alpha_{(2,1)} = -1, \quad \alpha_{(1,1,1)} = \frac{1}{3}.$$

One can also rewrite (150) as

$$(155) \quad \sum_{n=0}^{\infty} e_n z^n = \exp \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n z^n.$$

This expresses the elementary symmetric functions in terms of Newton functions. From this one sees that that $e_k(\{t^{-l'(s)}q^{a'(s)}\}_{s \in \mu})$ and $p_m(\{t^{-l'(s)}q^{a'(s)}\}_{s \in \mu})$ can be expressed as a polynomial in $e_r(q^{\mu_1}t^{-1}, q^{\mu_2}t^{-2}, \dots)$:

$$\begin{aligned} & \sum_{k \geq 0} e_k(\{t^{-l'(s)}q^{a'(s)}\}_{s \in \mu}) z^k = \exp \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} p_m(\{t^{-l'(s)}q^{a'(s)}\}_{s \in \mu}) z^m \\ &= \exp \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{t^m}{q^m - 1} (p_m(q^{\mu_1}t^{-1}, q^{\mu_2}t^{-2}, \dots) - p_m(t^{-1}, t^{-2}, \dots)) \right) z^m \\ &= \exp \sum_{n=0}^{\infty} \sum_{m \geq 1} \frac{(-1)^m}{m} p_m(q^{\mu_1}t^{-1}, q^{\mu_2}t^{-2}, \dots) (tq^n z)^m \\ & \quad \cdot \exp \sum_{n=0}^{\infty} \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p_m(t^{-1}, t^{-2}, \dots) (tq^n z)^m \\ &= \prod_{n \geq 0} \left(\sum_{r \geq 0} e_r(q^{\mu_1}t^{-1}, q^{\mu_2}t^{-2}, \dots) (tq^n z)^r \right)^{-1} \cdot \prod_{n \geq 0} \sum_{r \geq 0} e_r(t^{-1}, t^{-2}, \dots) (tq^n z)^r \\ &= \prod_{n \geq 0} \left(\sum_{r \geq 0} e_r(q^{\mu_1}t^{-1}, q^{\mu_2}t^{-2}, \dots) (tq^n z)^r \right)^{-1} \cdot \prod_{n \geq 0} \prod_{j=1}^{\infty} (1 + t^{1-j} q^n z). \end{aligned}$$

To compute the series expansions, we need to generalize Euler's formulas (113) and (156) below:

$$(156) \quad \prod_{n=0}^{\infty} \frac{1}{1 - q^n z} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(1 - q) \cdots (1 - q^n)}.$$

Proposition 6.4. *Write*

$$(157) \quad \prod_{n \geq 0} \sum_{r \geq 0} a_r q^{nr} z^r = \sum_{m \geq 0} b_m z^m,$$

where $a_0 = 1$, $\{a_r\}_{r \geq 1}$ are formal variables, then each b_m is a weighted homogeneous polynomial of $\{a_r\}_{r \geq 1}$ if we assign $\deg a_r = r$. Furthermore, the sequence $\{b_m\}_{m \geq 0}$ satisfies the following recursion relations:

$$(158) \quad b_0 = 1,$$

$$(159) \quad b_m = \frac{1}{1 - q^m} \sum_{r=1}^m q^{m-r} a_r b_{m-r}.$$

Proof. To get the first statement, change z to λz on both sides of (157) to get:

$$\prod_{n \geq 0} \sum_{r \geq 0} (\lambda^r a_r) q^{nr} z^r = \sum_{m \geq 0} (\lambda^m b_m) z^m.$$

This means when $\{a_r\}_{r \geq 1}$ is changed to $\{\lambda^r a_r\}_{r \geq 1}$, $\{b_m\}_{m \geq 1}$ is changed to $\{\lambda^m b_m\}_{m \geq 1}$. To get the second statement, note

$$\begin{aligned} \sum_{m \geq 0} b_m z^m &= \prod_{n \geq 0} \sum_{r \geq 0} a_r q^{nr} z^r = \sum_{r \geq 0} a_r q^{nr} z^r \cdot \prod_{n \geq 0} \sum_{r \geq 0} a_r q^{nr} (qz)^r \\ &= \sum_{r \geq 0} a_r z^r \cdot \sum_{m \geq 0} q^m b_m z^m, \end{aligned}$$

and so

$$(160) \quad b_m = \sum_{r=0}^m a_r q^{m-r} b_{m-r}.$$

□

The following are the first few terms of b_m :

$$\begin{aligned} b_1 &= \frac{1}{1-q} a_1, \\ b_2 &= \frac{1}{1-q^2} a_2 + \frac{q}{(1-q)(1-q^2)} a_1^2, \\ b_3 &= \frac{1}{1-q^3} a_3 + \frac{q+2q^2}{(1-q^2)(1-q^3)} a_2 a_1 + \frac{q^3}{(1-q)(1-q^2)(1-q^3)} a_1^3, \\ b_4 &= \frac{1}{1-q^4} a_4 + \frac{q+q^2+2q^3}{(1-q^3)(1-q^4)} a_3 a_1 + \frac{q^2}{(1-q^2)(1-q^4)} a_2^2 \\ &\quad + \frac{q^3+2q^4+3q^5}{(1-q^2)(1-q^3)(1-q^4)} a_2 a_1^2 + \frac{q^6}{(1-q)(1-q^2)(1-q^3)} a_1^4. \end{aligned}$$

Proposition 6.5. Write

$$(161) \quad \prod_{n \geq 0} \left(\sum_{r \geq 0} a_r q^{nr} z^r \right)^{-1} = \sum_{m \geq 0} c_m z^m,$$

where $a_0 = 1$, $\{a_r\}_{r \geq 1}$ are formal variables, then each c_m is a weighted homogeneous polynomial of $\{a_r\}_{r \geq 1}$ if we assign $\deg a_r = r$. Furthermore, the sequence $\{c_m\}_{m \geq 0}$ satisfies the following recursion relations:

$$(162) \quad c_0 = 1,$$

$$(163) \quad c_m = -\frac{1}{1-q^m} \sum_{r=1}^m a_r c_{m-r}.$$

Proof. The first statement can be proved by the same argument as above. To get the second statement, note

$$\begin{aligned} \sum_{m \geq 0} c_m z^m &= \prod_{n \geq 0} \left(\sum_{r \geq 0} a_r q^{nr} z^r \right)^{-1} = \left(\sum_{r \geq 0} a_r q^{nr} z^r \right)^{-1} \cdot \prod_{n \geq 0} \left(\sum_{r \geq 0} a_r q^{nr} (qz)^r \right)^{-1} \\ &= \left(\sum_{r \geq 0} a_r z^r \right)^{-1} \cdot \sum_{m \geq 0} q^m c_m z^m, \end{aligned}$$

and so

$$(164) \quad \sum_{m \geq 0} q^m c_m z^m = \sum_{r \geq 0} a_r z^r \cdot \sum_{m \geq 0} c_m z^m,$$

therefore,

$$(165) \quad q^m c_m = \sum_{r=0}^m a_r c_{m-r}.$$

□

The following are the first few terms of c_m :

$$\begin{aligned} c_1 &= -\frac{1}{1-q} a_1, \\ c_2 &= -\frac{1}{1-q^2} a_2 - \frac{1}{(1-q)(1-q^2)} a_1^2, \\ c_3 &= -\frac{1}{1-q^3} a_3 - \frac{q+2}{(1-q^2)(1-q^3)} a_2 a_1 - \frac{1}{(1-q)(1-q^2)(1-q^3)} a_1^3, \\ c_4 &= -\frac{1}{1-q^4} a_4 - \frac{q^2+q+2}{(1-q^3)(1-q^4)} a_3 a_1 - \frac{1}{(1-q^2)(1-q^4)} a_2^2 \\ &\quad - \frac{q^2+2q}{(1-q)(1-q^3)(1-q^4)} a_2 a_1^2 - \frac{q^3}{(1-q)(1-q^2)(1-q^3)} a_1^4. \end{aligned}$$

We will write

$$(166) \quad b_n = \sum_{|\mu|=n} \beta_\mu a_\mu, \quad c_n = \sum_{|\mu|=n} \gamma_\mu a_\mu,$$

where for a partition $\mu = (\mu_1, \dots, \mu_l)$, $a_\mu := a_{\mu_1} \cdots a_{\mu_l}$, β_μ and γ_μ are rational functions in q , with poles only at roots of unity.

With the above preparations, we now know how to compute the series expansion of the right-hand side of the second of the following equations:

$$\begin{aligned} \sum_{k \geq 0} e_k(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) z^k &= \exp \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} p_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) z^m \\ &= \prod_{n \geq 0} \left(\sum_{r \geq 0} e_r(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) (tq^n z)^r \right)^{-1} \cdot \prod_{n \geq 0} \sum_{r \geq 0} e_r(t^{-1}, t^{-2}, \dots) (tq^n z)^r. \end{aligned}$$

We formulate the result in the following:

Proposition 6.6. *For a partition λ , the following identity holds:*

$$(167) \quad e_k(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \lambda}) = \sum_{\substack{\mu, \nu \in \mathcal{P} \\ |\mu| + |\nu| = k}} \gamma_\mu \cdot e_\mu(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) \cdot \beta_\nu \cdot e_\nu(\{t^{-i+1}\}_{i \geq 1}).$$

Proof. This is an easy consequence of the following expansions:

$$\begin{aligned} \prod_{n \geq 0} \left(\sum_{r \geq 0} e_r(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) q^{rn} z^r \right)^{-1} &= \sum_{\mu \in \mathcal{P}} \gamma_\mu \cdot e_\mu(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) \cdot z^{|\mu|}, \\ \prod_{n \geq 0} \sum_{r \geq 0} e_r(\{t^{-i+1}\}_{i \geq 1}) q^{rn} z^r &= \sum_{\nu \in \mathcal{P}} \beta_\nu \cdot e_\nu(\{t^{-i+1}\}_{i \geq 1}) \cdot z^{|\nu|}. \end{aligned}$$

□

For example,

$$\begin{aligned}
 e_1(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) q^{rn} &= \frac{1}{1-q} e_1(\{t^{-i+1}\}_{i \geq 1}) - \frac{1}{1-q} e_1(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}), \\
 e_2(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) q^{rn} &= \frac{1}{1-q^2} e_2(\{t^{-i+1}\}_{i \geq 1}) + \frac{q}{(1-q)(1-q^2)} e_1^2(\{t^{-i+1}\}_{i \geq 1}) \\
 &\quad - \frac{1}{(1-q)^2} e_1(\{t^{-i+1}\}_{i \geq 1}) e_1(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) \\
 &\quad - \frac{1}{1-q^2} e_2(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) - \frac{q}{(1-q)(1-q^2)} e_1^2(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}).
 \end{aligned}$$

One can also get:

$$\begin{aligned}
 &\sum_{m=1}^{\infty} \frac{(-1)^m}{m} p_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) z^m \\
 &= \sum_{n \geq 0} \log \sum_{r \geq 0} e_r(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) (tq^n z)^r - \sum_{n \geq 0} \log \sum_{r \geq 0} e_r(t^{-1}, t^{-2}, \dots) (tq^n z)^r \\
 &= \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}} \alpha_\lambda \left(e_\lambda(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) - e_\lambda(t^{-1}, t^{-2}, \dots) \right) \cdot (tq^n z)^{|\lambda|} \\
 &= \sum_{\lambda \in \mathcal{P}} \alpha_\lambda \left(e_\lambda(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) - e_\lambda(t^{-1}, t^{-2}, \dots) \right) \cdot \frac{(tz)^{|\lambda|}}{1 - q^{|\lambda|}}.
 \end{aligned}$$

Note we have

$$\begin{aligned}
 &\sum_{m \geq 0} \sum_{|\lambda|=m} \alpha_\lambda \cdot e_\lambda(t^{-1}, t^{-2}, \dots) z^m = \log \sum_{m=0}^{\infty} e_m(t^{-1}, t^{-2}, \dots) z^m \\
 &= \log \prod_{n \geq 1} (1 + t^{-n} z) = \sum_{n \geq 1} \log(1 + t^{-n} z) \\
 &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{n=1}^{\infty} t^{-mn} z^m = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \frac{t^{-m}}{1 - t^{-m}} z^m.
 \end{aligned}$$

Therefore, we get the following generalization of (86):

Proposition 6.7. *For a partition μ and $m \geq 1$, the following identity holds:*

$$\begin{aligned}
 &p_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) \\
 (168) \quad &= \frac{(-1)^m m t^m}{1 - q^m} \cdot \sum_{|\lambda|=m} \alpha_\lambda \cdot e_\lambda(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) + \frac{1}{(1 - q^m)(1 - t^{-m})},
 \end{aligned}$$

where α_λ is defined in (151).

For example,

$$\begin{aligned}
 &p_2(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) \\
 (169) \quad &= \frac{2t^2}{1 - q^2} \cdot \left(e_2(\{q^{\mu_i} t^{-i}\}_{i \geq 1}) - \frac{1}{2} e_1^2(\{q^{\mu_i} t^{-i}\}_{i \geq 1}) \right) + \frac{1}{(1 - q^2)(1 - t^{-2})}.
 \end{aligned}$$

In the above we have related $e_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu})$ and $p_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu})$ to the eigenvalues of the operators \tilde{E}^r . We can also relate them to the eigenvalues

of the operators E^r . Recall the generating series of the eigenvalues $e^r(\mu)$ of the operator E^r is given by (108):

$$(170) \quad \sum_{r=0}^{\infty} e^r(\mu) z^r = \prod_{j=1}^{\infty} \frac{1 + q^{\mu_j} t^{-j} z}{1 + t^{-j} z}.$$

From the above calculation we have:

$$\begin{aligned} \sum_{k \geq 0} e_k(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) z^k &= \exp \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} p_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) z^m \\ &= \prod_{n \geq 0} \left(\sum_{r \geq 0} e_r(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) (tq^n z)^r \right)^{-1} \cdot \prod_{n \geq 0} \prod_{j=1}^{\infty} (1 + t^{1-j} q^n z) \\ &= \prod_{n \geq 0} \left[\prod_{j \geq 1} [(1 + q^{\mu_j} t^{-j} \cdot (tq^n z))^{-1} (1 + t^{-j} \cdot (tq^n z))] \right] \\ &= \prod_{n \geq 0} \left(\sum_{r \geq 0} e^r(\mu) (tq^n z)^r \right)^{-1} \cdot \prod_{n \geq 0} \prod_{j=1}^{\infty} (1 + t^{1-j} q^n z). \end{aligned}$$

It is then natural to consider:

$$(171) \quad \begin{aligned} \left(\sum_{k \geq 0} e_k(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) z^k \right)^{-1} &= \prod_{n \geq 0} \prod_{j \geq 1} \frac{1 + q^{\mu_j} t^{-j} \cdot (tq^n z)}{1 + t^{-j} \cdot (tq^n z)} \\ &= \prod_{n \geq 0} \sum_{r \geq 0} e_r(q^{\mu_1} t^{-1}, q^{\mu_2} t^{-2}, \dots) (tq^n z)^r \cdot \prod_{m, n \geq 0} \frac{1}{1 + t^{-m} q^n z}. \end{aligned}$$

Recall the following relationship between elementary symmetric functions $\{e_k\}$ and complete symmetric functions $\{h_k\}$:

$$(172) \quad \left(\sum_{k \geq 0} e_k z^k \right)^{-1} = \sum_{k \geq 0} (-1)^k h_k z^k.$$

Proposition 6.8. *For a partition λ , the following identity holds:*

$$(173) \quad h_k(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \lambda}) = (-1)^k \sum_{\substack{\mu, \nu \in \mathcal{P} \\ |\mu| + |\nu| = k}} \beta_{\mu} e_{\mu}(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) \gamma_{\nu} e_{\nu}(\{t^{-i+1}\}_{i \geq 1}).$$

Proof. This is an easy consequence of the following expansions:

$$\begin{aligned} \prod_{n \geq 0} \sum_{r \geq 0} e_r(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) q^{rn} z^r &= \sum_{\mu \in \mathcal{P}} \beta_{\mu} \cdot e_{\mu}(\{q^{\lambda_i} t^{-i+1}\}_{i \geq 1}) \cdot z^{|\mu|}, \\ \prod_{n \geq 0} \left(\sum_{r \geq 0} e_r(\{t^{-i+1}\}_{i \geq 1}) q^{rn} z^r \right)^{-1} &= \sum_{\nu \in \mathcal{P}} \gamma_{\nu} \cdot e_{\nu}(\{t^{-i+1}\}_{i \geq 1}) \cdot z^{|\nu|}. \end{aligned}$$

□

There is another way to compute $\prod_{n \geq 0} \prod_{j=1}^{\infty} (1 + t^{1-j} q^n z)$. Consider the expansion:

$$\prod_{m, n \geq 0} (1 + q_1^m q_2^n z) = \sum_{n \geq 0} c_n z^n.$$

$$\begin{aligned}
 \sum_{n \geq 0} c_n z^n &= \prod_{m, n \geq 0} (1 + q_1^m q_2^n z) \\
 &= \frac{1}{1+z} \prod_{m \geq 0} (1 + q_1^m z) \cdot \prod_{n \geq 0} (1 + q_2^n z) \cdot \prod_{m, n \geq 0} (1 + q_1^m q_2^n (q_1 q_2 z)) \\
 &= \sum_{l \geq 0} e_l(\{q_1^{n+1}, q_2^n\}_{n \geq 0}) z^l \cdot \sum_{m \geq 0} c_m \cdot (q_1 q_2)^m z^m \\
 &= \sum_{l, m \geq 0} e_l(\{q_1^{n+1}, q_2^n\}_{n \geq 0}) c_m \cdot (q_1 q_2)^m z^{l+m}.
 \end{aligned}$$

It follows that

$$(174) \quad c_n = \sum_{l+m=n} e_l(\{q_1^{n+1}, q_2^n\}_{n \geq 0}) (q_1 q_2)^m c_m$$

and therefore,

$$(175) \quad c_n = \frac{1}{1 - (q_1 q_2)^n} \sum_{l=0}^{n-1} (q_1 q_2)^{m-l} e_l(\{q_1^n, q_2^n\}).$$

6.5. More examples of vertex realizable operators. Define three operator $\Lambda^m(q, t), \Sigma^m(q, t), \Psi^m(q, t) : \Lambda_{q,t} \rightarrow \Lambda_{q,t}$ such that

$$(176) \quad \Lambda^m(q, t) P_\mu(x; q, t) = e_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) P_\mu(x; q, t),$$

$$(177) \quad \Sigma^m(q, t) P_\mu(x; q, t) = h_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) P_\mu(x; q, t),$$

$$(178) \quad \Psi^m(q, t) P_\mu(x; q, t) = p_m(\{t^{-l'(s)} q^{a'(s)}\}_{s \in \mu}) P_\mu(x; q, t).$$

By the discussion in last subsection, one easily get the following:

Proposition 6.9. *The operators Λ^m, Σ^m and Ψ^m can all be expressed as weighted homogeneous polynomials in operators \tilde{E}^r which are of weight r , hence they are all vertex realizable.*

Therefore, one can use the results for correlation functions for \tilde{E}^r to get correlations functions of these operators. For example, by (168) we have

$$(179) \quad \Psi^m = \frac{(-1)^m m t^m}{1 - q^m} \cdot \sum_{|\lambda|=m} \alpha_\lambda \cdot \tilde{E}^\lambda + \frac{1}{(1 - q^m)(1 - t^{-m})},$$

where for $\lambda = (\lambda_1, \dots, \lambda_l)$, $\tilde{E}^\lambda = \tilde{E}^{\lambda_1} \dots \tilde{E}^{\lambda_l}$. In particular,

$$(180) \quad \Psi^1 = \frac{-t}{1 - q} \tilde{E}^1 + \frac{1}{(1 - q)(1 - t^{-1})},$$

$$(181) \quad \Psi^2 = \frac{2t^2}{1 - q^2} \left(\tilde{E}^2 - \frac{1}{2} (\tilde{E}^1)^2 \right) + \frac{1}{(1 - q^2)(1 - t^{-2})},$$

$$(182) \quad \Psi^3 = \frac{-3t^3}{1 - q^3} \left(\tilde{E}^3 - \tilde{E}^2 \tilde{E}^1 + \frac{1}{3} (\tilde{E}^1)^3 \right) + \frac{1}{(1 - q^3)(1 - t^{-3})}.$$

It follows that we have:

$$\begin{aligned}
\langle \Psi^1 \rangle'_{u,v} &= \frac{-t}{1-q} \langle \tilde{E}^1 \rangle'_{u,v} + \frac{1}{(1-q)(1-t^{-1})} \\
&= \frac{-t}{1-q} \cdot \frac{t^{-1}}{1-t^{-1}} \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right) + \frac{1}{(1-q)(1-t^{-1})} \\
&= Q \frac{(1-u)(1-v)}{(1-q)(1-t^{-1})(1-uQ)}.
\end{aligned}$$

$$\begin{aligned}
\langle \Psi^2 \rangle'_{u,v} &= -\frac{2t^2}{1-q^2} \left(\frac{1}{2} \langle (\tilde{E}^1)^2 \rangle'_{u,v} - \langle \tilde{E}^2 \rangle'_{u,v} \right) + \frac{1}{(1-q^2)(1-t^{-2})} \\
&= -\frac{t^2}{1-q^2} \left(\left(\frac{t^{-1}}{1-t^{-1}} \right)^2 \cdot \left[\left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \right. \right. \\
&\quad \left. \left. + Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \cdot (1-u)(1-v) \cdot \frac{(1-q)(1-t^{-1})}{(1-uqQ)(1-ut^{-1}Q)} \right] \right. \\
&\quad \left. - 2 \frac{t^{-3}}{(1-t^{-1})(1-t^{-2})} \left[\left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \right. \right. \\
&\quad \left. \left. + (1-t^{-1})Q \frac{(1-Q)(1-u)(1-v)(1-uvQ)}{(1-t^{-1}uQ)(1-uQ)^2} \right] \right) + \frac{1}{(1-q^2)(1-t^{-2})} \\
&= \frac{1}{(1-q^2)(1-t^{-2})} - \frac{1}{(1-q^2)(1-t^{-2})} \cdot \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \\
&\quad + \frac{-1+q+t^{-1}+qt^{-1}-2uqt^{-1}Q}{(1-q^2)(1-t^{-2})} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-uqQ)(1-ut^{-1}Q)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\langle (\Psi^1)^2 \rangle'_{u,v} &= \frac{t^2}{(1-q)^2} \langle (\tilde{E}^1)^2 \rangle'_{u,v} - \frac{2t}{(1-q)^2(1-t^{-1})} \langle \tilde{E}^1 \rangle'_{u,v} + \frac{1}{(1-q)^2(1-t^{-1})^2} \\
&= \frac{t^2}{(1-q)^2} \cdot \left(\frac{t^{-1}}{1-t^{-1}} \right)^2 \cdot \left[\left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \right. \\
&\quad \left. + Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \cdot (1-u)(1-v) \cdot \frac{(1-q)(1-t^{-1})}{(1-uqQ)(1-ut^{-1}Q)} \right] \\
&\quad - \frac{2t}{(1-q)^2(1-t^{-1})} \cdot \frac{t^{-1}}{1-t^{-1}} \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right) + \frac{1}{(1-q)^2(1-t^{-1})^2} \\
&= \left(Q \frac{(1-u)(1-v)}{(1-q)(1-t^{-1})(1-uQ)} \right)^2 \\
&\quad + \frac{(1-u)(1-v)}{(1-q)(1-t^{-1})} \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2(1-uqQ)(1-ut^{-1}Q)}.
\end{aligned}$$

Since we have

$$(183) \quad \Lambda^2 = \frac{1}{2} ((\Psi^1)^2 - \Psi^2),$$

we get the following formula:

$$\begin{aligned}
 2\langle \Lambda^2 \rangle'_{u,v} &= \left(Q \frac{(1-u)(1-v)}{(1-q)(1-t^{-1})(1-uQ)} \right)^2 \\
 &+ \frac{(1-u)(1-v)}{(1-q)(1-t^{-1})} \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2(1-ugQ)(1-ut^{-1}Q)} \\
 &- \frac{1}{(1-q^2)(1-t^{-2})} + \frac{1}{(1-q^2)(1-t^{-2})} \cdot \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \\
 &- \frac{-1+q+t^{-1}+qt^{-1}-2ugt^{-1}Q}{(1-q^2)(1-t^{-2})} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-ugQ)(1-ut^{-1}Q)} \\
 &= \left(Q \frac{(1-u)(1-v)}{(1-q)(1-t^{-1})(1-uQ)} \right)^2 \\
 &- \frac{1}{(1-q^2)(1-t^{-2})} + \frac{1}{(1-q^2)(1-t^{-2})} \cdot \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \\
 &+ \frac{2+2ugt^{-1}Q}{(1-q^2)(1-t^{-2})} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-ugQ)(1-ut^{-1}Q)}.
 \end{aligned}$$

7. APPLICATIONS TO K-THEORETICAL INTERSECTION NUMBERS ON HILBERT SCHEMES

In this Section we return to the discussions in Section 4 and the beginning of Section 5.

7.1. Formal quantum field theory associated to K -theory on Hilbert schemes.

Let X be a regular algebraic surface. For $F, F_1, \dots, F_N \in K(X)$, define the following normalized correlator by

$$\begin{aligned}
 (184) \quad &\langle \Psi^{m_1}(F_1) \cdots \Psi^{m_N}(F_N) \rangle'_{u,v,F} \\
 &:= \frac{\sum_{n \geq 0} Q^n \chi(X, \bigotimes_{j=1}^N \psi^{m_j}(F_j^{[n]}) \otimes \lambda_{-u} F^{[n]} \otimes \lambda_{-v}(F^{[n]})^*)}{\sum_{n \geq 0} Q^n \chi(X, \lambda_{-u} F^{[n]} \otimes \lambda_{-v}(F^{[n]})^*)}.
 \end{aligned}$$

We can also define other correlators: When Ψ^m is replaced by Σ^m or Λ^m , ψ^m is replaced by σ^m or λ^m . The connected normalized correlators will be denoted by $\langle \cdot \rangle'_{u,v,F,c}$.

When X is toric with a T -action and we work with equivariant K -theory, the normalized correlators will be denoted by $\langle \cdot \rangle'_{u,v,F;T}$. The connected normalized correlators will be denoted by $\langle \cdot \rangle'_{u,v,F,c;T}$.

7.2. Equivariant K -theory of Hilbert schemes of \mathbb{C}^2 as a vertex operatorial formal quantum field theory.

We rewrite (51) as follows:

$$\begin{aligned}
 (185) \quad &\sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \psi^{m_j}(\xi_n^{A_j}) \otimes \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2) \\
 &= \prod_{j=1}^N e^{m_j(a_j w_1 + b_j w_2)} \cdot \sum_{\mu} (-ut^A) Q^{|\mu|} \prod_{j=1}^N \sum_{s \in \mu} e^{m_j[l'(s)w_1 + a'(s)w_2]} \\
 &\cdot \prod_{s \in \mu} \frac{(t_1^{l'(s)} - (ut^A)^{-1} t_2^{-a'(s)}) \cdot (t_2^{a'(s)} - (vt^{-A}) t_1^{-l'(s)})}{(1 - t_1^{-l(s)} t_2^{a(s)+1})(1 - t_1^{l(s)+1} t_2^{-a(s)})}.
 \end{aligned}$$

After taking

$$(186) \quad t_2 = q, \quad t_1 = t^{-1},$$

we find

$$(187) \quad \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \bigotimes_{j=1}^N \psi^{m_j}(\xi_n^{A_j}) \otimes \Lambda_{-u} \xi_n^A \otimes \Lambda_{-v} (\xi_n^A)^*) (t_1, t_2) \\ = \prod_{j=1}^N e^{m_j(a_j w_1 + b_j w_2)} \cdot \langle \Psi^{m_1} \dots \Psi^{m_N} \rangle_{ut^A, vt^{-A}}.$$

So we have reached our main result:

Theorem 7.1. *The equivariant K-theory on Hilbert schemes on \mathbb{C}^2 can be reformulated as a vertex operatorial formal quantum field theory and can be computed using the vertex operator realizations of the Macdonald operators.*

7.3. Generalization to toric surfaces. In this Subsection we extend the above result to Hilbert schemes on a projective or quasi-projective surface X which admits a torus action with isolated fixed points. Suppose that T^2 acts on X with isolated fixed points p_1, \dots, p_m , such that the weights of $T_{p_i} S = t_{1,i}^{-1} + t_{2,i}^{-1}$. The T^2 -action on S induces a natural T^2 -action on $S^{[n]}$. The fixed points on $X^{[n]}$ are parameterized by m -tuples of partitions (μ^1, \dots, μ^m) , where μ^i can be an empty partition, such that

$$|\mu^1| + \dots + |\mu^m| = n.$$

Furthermore, the weight decomposition of the cotangent space at the fixed point is given by:

$$\sum_{i=1}^m \sum_{s^i \in \mu^i} (t_{1,i}^{l(s^i)} t_{2,i}^{-(a(s^i)+1)} + t_{1,i}^{-(l(s^i)+1)} t_{2,i}^{a(s^i)}),$$

Suppose that L is an equivariant holomorphic line bundle on S such that

$$(188) \quad L|_{p_i} = t^{A^i} = t_1^{a_1^i} t_2^{a_2^i}.$$

The weights of $L^{[n]}$ are:

$$\sum_{i=1}^m t^{A^i} \sum_{s^i \in \mu^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}.$$

By holomorphic Lefschetz formula one then gets:

Theorem 7.2. *Let X be a surface as above and let L, L_1, \dots, L_N be holomorphic line bundles such that*

$$(189) \quad L|_{p_i} = t^{A^i} = t_1^{a_1^i} t_2^{a_2^i}, \quad L_j|_{p_i} = t^{A_j^i} = t_1^{a_{1,j}^i} t_2^{a_{j,2}^i}.$$

Then we have

$$\begin{aligned}
 & \sum_{n \geq 0} Q^n \chi(X^{[n]}, \bigotimes_{j=1}^N \Lambda_{x_j} L_j^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2) \\
 (190) \quad &= \prod_{i=1}^m \sum_{\mu^i} Q^{|\mu^i|} \prod_{s^i \in \mu^i} \prod_{j=1}^N (1 + x_j t^{A_j^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}) \\
 & \cdot \prod_{s^i \in \mu^i} \frac{(1 - ut^{A^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}) (1 - vt^{-A^i} t_{1,i}^{-l'(s^i)} t_{2,i}^{-a'(s^i)})}{(1 - t_{1,i}^{-l(s^i)} t_{2,i}^{a(s^i)+1}) (1 - t_{1,i}^{l(s^i)+1} t_{2,i}^{-a(s^i)})},
 \end{aligned}$$

and similarly for symmetric powers,

$$\begin{aligned}
 & \sum_{n \geq 0} Q^n \chi(X^{[n]}, \bigotimes_{j=1}^N S_{x_j} L_j^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2) \\
 (191) \quad &= \prod_{i=1}^m \sum_{\mu^i} Q^{|\mu^i|} \prod_{s^i \in \mu^i} \prod_{j=1}^N \frac{1}{1 - x_j t^{A_j^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}} \\
 & \cdot \prod_{s^i \in \mu^i} \frac{(1 - ut^{A^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}) (1 - vt^{-A^i} t_{1,i}^{-l'(s^i)} t_{2,i}^{-a'(s^i)})}{(1 - t_{1,i}^{-l(s^i)} t_{2,i}^{a(s^i)+1}) (1 - t_{1,i}^{l(s^i)+1} t_{2,i}^{-a(s^i)})}.
 \end{aligned}$$

7.4. Reduction of K -theory on Hilbert schemes to vertex operatorial formal quantum field theory. By this Theorem, we can reduce the K -theoretical intersection numbers on Hilbert schemes to the special formal quantum field theory calculations we develop in last Section. The strategy, as in Wang-Zhou [62], is to apply the results of Ellingsrud-Göttsche-Lehn [18] to reduce first to toric surfaces then to \mathbb{C}^2 . We illustrate the idea by some examples in this Subsection.

If we take the coefficient of x_1 on both sides of (190), we get the following equations for toric surface X and equivariant line bundle L, L_1 :

$$\begin{aligned}
 & \frac{\sum_{n \geq 0} Q^n \chi(X^{[n]}, L_1^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)}{\sum_{n \geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)} \\
 &= \sum_{i=1}^m \frac{\sum_{\mu^i} Q^{|\mu^i|} \sum_{s^i \in \mu^i} t^{A_1^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)} \cdot \prod_{s^i \in \mu^i} \frac{(1 - ut^{A^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}) (1 - vt^{-A^i} t_{1,i}^{-l'(s^i)} t_{2,i}^{-a'(s^i)})}{(1 - t_{1,i}^{-l(s^i)} t_{2,i}^{a(s^i)+1}) (1 - t_{1,i}^{l(s^i)+1} t_{2,i}^{-a(s^i)})}}{\sum_{\mu^i} Q^{|\mu^i|} \prod_{s^i \in \mu^i} \frac{(1 - ut^{A^i} t_{1,i}^{l'(s^i)} t_{2,i}^{a'(s^i)}) (1 - vt^{-A^i} t_{1,i}^{-l'(s^i)} t_{2,i}^{-a'(s^i)})}{(1 - t_{1,i}^{-l(s^i)} t_{2,i}^{a(s^i)+1}) (1 - t_{1,i}^{l(s^i)+1} t_{2,i}^{-a(s^i)})}} \\
 &= \sum_{i=1}^m t^{A_1^i} \langle \Lambda^1 \rangle'_{ut^{A^i}, vt^{-A^i}; t_{1,i}, t_{2,i}} \\
 &= Q \sum_{i=1}^m t^{A_1^i} \frac{(1-u)(1-v)}{(1-uQ)(1-t_{1,i})(1-t_{2,i})} \\
 &= \frac{(1-u)(1-v)Q}{1-uQ} \cdot \chi(S, L_1)(t_1, t_2).
 \end{aligned}$$

Here $\langle \cdot \rangle'_{u,v;t_{1,i},t_{2,i}}$ means the following specialization of the parameters q and t :

$$(192) \quad q = t_{2,i}, \quad t = t_{1,i}^{-1},$$

After taking the nonequivariant limit,

$$\frac{\sum_{n \geq 0} Q^n \chi(X^{[n]}, L_1^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})}{\sum_{n \geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})} = \frac{(1-u)(1-v)Q}{1-uQ} \cdot \chi(X, L_1).$$

Combined with the following formula proved in Wang-Zhou [62]:

$$(193) \quad \sum_{n \geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*}) = \exp\left(\sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(X, \Lambda_{-u^n} L \otimes \Lambda_{-v^n} L^*)\right),$$

we then get:

$$(194) \quad \sum_{n \geq 0} Q^n \chi(X^{[n]}, L_1^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*}) \\ = \frac{(1-u)(1-v)Q}{1-uQ} \cdot \chi(X, L_1) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(X, \Lambda_{-u^n} L \otimes \Lambda_{-v^n} L^*)\right).$$

By the results of Ellingsrud-Göttsche-Lehn [18], this formula holds for all projective surfaces. In the notation of §7.1,

$$(195) \quad \langle \lambda^1(L_1) \rangle'_{u,v,L} = \frac{(1-u)(1-v)Q}{1-uQ} \cdot \chi(X, L_1).$$

If we take the coefficient of $x_1 x_2$ on both sides of (190), we get the following equations for toric surface X and equivariant line bundle L, L_1, L_2 :

$$\begin{aligned} & \frac{\sum_{n \geq 0} Q^n \chi(X^{[n]}, L_1^{[n]} \otimes L_2^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)}{\sum_{n \geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)} \\ &= \sum_{i=1}^m t^{A_1^i} t^{A_2^i} \langle (\Lambda^1)^2 \rangle'_{ut^{A^i}, vt^{-A^i}; t_{1,i}, t_{2,i}} + \sum_{1 \leq i \neq j \leq m} t^{A_1^i} \langle \Lambda^1 \rangle'_{ut^{A^i}, vt^{-A^i}} \cdot t^{A_2^j} \langle \Lambda^1 \rangle'_{ut^{A^j}, vt^{-A^j}; t_{1,i}, t_{2,i}} \\ &= \sum_{i=1}^n t^{A_1^i} t^{A_2^i} \left[\left(Q \frac{(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})(1-uQ)} \right)^2 \right. \\ & \quad \left. + \frac{(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})} \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2(1-ut_{1,i}Q)(1-ut_{2,i}Q)} \right] \\ & \quad + \sum_{1 \leq i \neq j \leq m} t^{A_1^i} t^{A_2^j} Q \frac{(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})(1-uQ)} \cdot Q \frac{(1-u)(1-v)}{(1-t_{1,j})(1-t_{2,j})(1-uQ)} \\ &= \sum_{i=1}^m t^{A_1^i} Q \frac{(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})(1-uQ)} \cdot \sum_{j=1}^m t^{A_2^j} Q \frac{(1-u)(1-v)}{(1-t_{1,j})(1-t_{2,j})(1-uQ)} \\ & \quad + \sum_{i=1}^m \frac{(1-u)(1-v)}{(1-t_{1,j})(1-t_{2,j})} \cdot Q \frac{(1-Q)(1-uvQ)t^{A_1^i} t^{A_2^i}}{(1-uQ)^2(1-ut_{1,j}Q)(1-ut_{2,j}Q)} \\ &= \prod_{j=1}^2 \left(\frac{Q(1-u)(1-v)}{1-uQ} \chi(X, L_j)(t_1, t_2) \right) \\ & \quad + (1-u)(1-v) \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \chi(X, L_1 \otimes L_2 \otimes S_{uQ} T^* X)(t_1, t_2). \end{aligned}$$

After taking the nonequivariant limit:

$$\begin{aligned}
 & \frac{\sum_{n \geq 0} Q^n \chi(X^{[n]}, L_1^{[n]} \otimes L_2^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})}{\sum_{n \geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})} \\
 &= \prod_{j=1}^2 \left(\frac{Q(1-u)(1-v)}{1-uQ} \chi(X, L_j) \right) \\
 &+ (1-u)(1-v) \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \chi(X, L_1 \otimes L_2 \otimes S_{uQ} T^* X).
 \end{aligned}$$

In the notation of §7.1,

$$\begin{aligned}
 (196) \quad & \langle \lambda^1(L_1) \lambda^1(L_2) \rangle'_{u,v,L} = \prod_{j=1}^2 \left(\frac{Q(1-u)(1-v)}{1-uQ} \chi(X, L_j) \right) \\
 &+ (1-u)(1-v) \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \chi(X, L_1 \otimes L_2 \otimes S_{uQ} T^* X) \\
 &= \langle \lambda^1(L_1) \rangle'_{u,v,L} \cdot \langle \lambda^1(L_2) \rangle'_{u,v,L} \\
 &+ (1-u)(1-v) \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \chi(X, L_1 \otimes L_2 \otimes S_{uQ} T^* X).
 \end{aligned}$$

Therefore, the following formula for connected corrlator holds:

$$\begin{aligned}
 (197) \quad & \langle \lambda^1(L_1) \lambda^1(L_2) \rangle'_{u,v,L,c} \\
 &= (1-u)(1-v) \cdot Q \frac{(1-Q)(1-uvQ)}{(1-uQ)^2} \cdot \chi(X, L_1 \otimes L_2 \otimes S_{uQ} T^* X).
 \end{aligned}$$

If we take the coefficient of x_1^2 on both sides of the above formulas, we get:

$$\begin{aligned}
& \frac{\sum_{n \geq 0} Q^n \chi(S^{[n]}, \Lambda^2 L_1^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)}{\sum_{n \geq 0} Q^n \chi(S^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)} \\
&= \sum_{i=1}^n (t^{A_i})^2 \langle \Lambda^2 \rangle'_{ut^{A_i}, vt^{-A_i}} + \sum_{1 \leq i < j \leq n} t^{A_i} \langle \Lambda^1 \rangle'_{ut^{A_i}, vt^{-A_i}} \cdot t^{A_j} \langle \Lambda^1 \rangle'_{ut^{A_j}, vt^{-A_j}} \\
&= \frac{1}{2} \sum_{i=1}^n (t^{A_i})^2 \left[\left(Q \frac{(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})(1-uQ)} \right)^2 \right. \\
&\quad - \frac{1}{(1-t_{1,i}^2)(1-t_{2,i}^2)} + \frac{1}{(1-t_{1,i}^2)(1-t_{2,i}^2)} \cdot \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \\
&\quad + \frac{2+2uqt^{-1}Q}{(1-t_{1,i}^2)(1-t_{2,i}^2)} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-ut_{1,i}Q)(1-ut_{2,i}Q)} \\
&\quad \left. + \frac{1}{(1-t_{1,i})(1-t_{2,i})} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-ut_{1,i}Q)(1-ut_{2,i}Q)} \right] \\
&\quad + \sum_{1 \leq i < j \leq n} t^{A_i} t^{A_j} \frac{Q(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})(1-uQ)} \cdot \frac{Q(1-u)(1-v)}{(1-t_{1,j})(1-t_{2,j})(1-uQ)} \\
&= \frac{1}{2} \left(\sum_{i=1}^n Q t^{A_i} \frac{(1-u)(1-v)}{(1-t_{1,i})(1-t_{2,i})(1-uQ)} \right)^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^n (t^{A_i})^2 \left[\left(-\frac{1}{(1-t_{1,i}^2)(1-t_{2,i}^2)} + \frac{1}{(1-t_{1,i}^2)(1-t_{2,i}^2)} \cdot \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \right. \right. \\
&\quad \left. + \frac{2+2ut_{i,1}t_{i,2}Q}{(1-t_{1,i}^2)(1-t_{2,i}^2)} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-ut_{1,i}Q)(1-ut_{2,i}Q)} \right. \\
&\quad \left. + \frac{1}{(1-t_{1,i})(1-t_{2,i})} \cdot \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2(1-ut_{1,i}Q)(1-ut_{2,i}Q)} \right].
\end{aligned}$$

We rewrite it in the following form:

$$\begin{aligned}
& \frac{\sum_{n \geq 0} Q^n \chi(S^{[n]}, \Lambda^2 L_1^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)}{\sum_{n \geq 0} Q^n \chi(S^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})(t_1, t_2)} \\
&= \frac{1}{2} \left(Q \frac{(1-u)(1-v)}{(1-uQ)} \cdot \chi(X, L_1)(t_1, t_2) \right)^2 \\
&\quad - \frac{1}{2} \chi(X, L_1)(t_1^2, t_2^2) + \frac{1}{2} \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \chi(X, L_1)(t_1^2, t_2^2) \\
&\quad + \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2} \\
&\quad \cdot \chi(X, L_1^2 \otimes (\mathcal{O}_X + uQK_X) \otimes S_\epsilon(T^*X) \otimes S_{uQ}(T^*X))(t_1, t_2)|_{\epsilon=-1} \\
&\quad + \frac{1}{2} \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2} \chi(X, L_1^2 \otimes S_{uQ}(T^*X))(t_1, t_2).
\end{aligned}$$

By taking nonequivariant limit one then gets:

$$\begin{aligned}
 & \frac{\sum_{n \geq 0} Q^n \chi(S^{[n]}, \Lambda^2 L_1^{[n]} \otimes \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})}{\sum_{n \geq 0} Q^n \chi(S^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*})} \\
 &= \frac{1}{2} \left(Q \frac{(1-u)(1-v)}{(1-uQ)} \cdot \chi(X, L_1) \right)^2 \\
 &- \frac{1}{2} \chi(X, L_1) + \frac{1}{2} \left(1 - Q \frac{(1-u)(1-v)}{1-uQ} \right)^2 \chi(X, L_1) \\
 &+ \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2} \\
 &\cdot \chi(X, L_1^2 \otimes (\mathcal{O}_X + uQK_X) \otimes S_\epsilon(T^*X) \otimes S_{uQ}(T^*X))|_{\epsilon=-1} \\
 &+ \frac{1}{2} \frac{Q(1-Q)(1-u)(1-v)(1-uvQ)}{(1-uQ)^2} \chi(X, L_1^2 \otimes S_{uQ}(T^*X)).
 \end{aligned}$$

This gives a formula to compute $\langle \lambda^2(L_1) \rangle'_{u,v,L}$.

8. CONCLUDING REMARKS

In this paper we have made a first step towards reformulating the K-theoretical intersection theory on Hilbert schemes of points as a quantum field theory. We have not attempted to construct a quantum field theory as physicists do, but instead reformulate it as a formal quantum field theory. This not only facilitates the computations of these intersection number, but also it suffices as a portal to further investigations as a physicist normally will do to an authentic quantum field theory. One can at least consider the following four types of problems.

Problem 1. Computations and properties of the correlators. There are two directions to pursue for this question. Symmetries intrinsic to a quantum field theory lead to symmetries of the correlators, in the form of what physicists called Ward identities. Such symmetries can also manifest themselves in the form of an integrable hierarchy of which the partition function is a tau-function. Instead of compute the correlators for the formal quantum field theory associated with K-theory directly as in this paper, one can search for the constraints or the integrable hierarchies satisfied by them.

Problem 2. Finding other field theories dual to our theory in this paper. At least in the case of Hilbert schemes of \mathbb{C}^2 , our theory is expected to be dual to a refined topological string theory on the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It is interesting to extend to the theories on toric surfaces to see whether they are related to some refined topological string theory on some toric Calabi-Yau 3-folds.

Problem 3. Study of relationship between Problem 1 and Problem 2. In Gromov-Witten theory toric Calabi-Yau 3-folds are conjecturally mirror to some plane curves on which after quantization one can define constrains and integrable hierarchies [3]. We expect this can be generalized to our theory defined using K-theory on Hilbert schemes.

Problem 4. Study the relationship to Cherednik's double affine Hecke algebras. It will be interesting to relate tautological sheaves to the work of Gordon and Stafford [21, 22]. We expect a quantized version of the work by Costello-Gronjowski [14] describes the equivariant K-theory of the Hilbert schemes of points in \mathbb{C}^2 . It should

be related to but different from the quantum differential operators discovered by Okounkov and Pandhriapnde [54].

We expect that works in the field of noncommutative algebraic geometry will be very helpful in solving these problems.

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