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L^2-estimates on p-convex Riemannian manifolds ☆

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A B S T R A C T

In this paper, we establish various L^2-estimates for the exterior differential operator on p-convex Riemannian manifolds in the sense of Harvey and Lawson. As applications, we establish a Carleman type estimate which is uniform with respect to both weight functions and domains, and we also obtain topological restrictions for a Riemannian manifold to be p-convex.

Contents

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0. Introduction

The theory of convexity is a cornerstone of geometry, analysis and related areas in mathematics. Recently in a series of articles ([18–20] and references therein), Harvey and Lawson systematically explored the notions of plurisubharmonicity and convexity in the context of differential geometry. It has a long history for the concepts of pseudoconvexity and plurisubharmonicity in several complex analysis and complex geometry, but it has rare attention in more geometric situations until Harvey and Lawson's innovative development in geometric convexity. They also studied potential theory for geometric plurisubharmonic functions and interesting applications to the theory of nonlinear partial differential equations. A number of results in complex analysis and complex geometry turn out to carry over to more general setting. In [20], Harvey and Lawson introduced the notion of p-convexity and p-plurisubharmonicity on Riemannian manifolds. They obtained a deep result which is an analogue of the Levi problem in complex analysis, i.e., local p-convexity implies global p-convexity. This hopefully will enrich the function theory in geometric analysis. For a compact Riemannian manifold with smooth boundary, the concept of p-convexity was first introduced by Sha [29]. In [29], it was proved that any Riemannian manifold with nonnegative sectional curvature and p-convex boundary has the homotopy type of a CW-complex of dimension $< p$. This result was later strengthened by Wu [32]. Note that in [29], the p-convexity of a Riemannian manifold (M, ds^2) with boundary is equivalent to that ∂M is strictly p-convex in the sense of Harvey and Lawson. The notion of p-convexity in the sense of Harvey and Lawson is different from that introduced by Andreotti and Grauert (cf. [1] and [2]) in the context of complex analytic geometry which is defined by certain conditions on the number of negative or positive eigenvalues of the Levi form. The main difference is that the notion of p-convexity in the sense of Andreotti and Grauert only depends on the underlying complex structure (which is used to define the Levi form), while in the Riemannian case the notion of p-convexity of Harvey and Lawson does depend on the given metric, and this feature brings difficulties in introducing complete metric because the p-convex property may not be preserved.

Since the L^2-method has many profound applications in several complex analysis and complex geometry (see [7,13,14,22,23,25,27,30,31] and references therein), we will establish in the present paper various L^2-estimates for the exterior differential operator on p-convex Riemannian manifolds in the sense of Harvey and Lawson. In many situations, the choices of weight functions and estimates for solutions in L^2-method are crucial in applications (see, e.g., [11,13,16,17,25,27]). Hence we will make emphasis on several different types of L^2-estimate. In [21], the author considered the $\bar{\partial}$-problem on (weakly)

q-pseudoconvex domains in $\mathbb{C}^n$, but no effort was made to obtain good estimates for solutions. The method developed here can be used to establish estimates for $\bar{\partial}$-problem on (weakly) q-pseudoconvex Kähler manifolds. To explain the technique clearly, we will first prove L^2-estimates and existence results in Euclidean spaces, and then we will show how the technique still works on Riemannian manifolds. We also discuss geometric applications of the L^2-method on p-convex Riemannian manifolds. We only consider the problems of existence and interior regularity (for the minimal L^2-solutions) in the present paper, we plan to consider the problems of extension of closed forms, boundary regularity of minimal solutions and more geometric applications in subsequent work.

This paper is arranged as follows. In Section 1, we will recall related notions of p-convexity and p-plurisubharmonicity in the sense of Harvey and Lawson and prove some results of exterior algebra which will be used later in our estimate. This section is ended with a lemma concerning the choices of weight functions. Section 2 is devoted to proving a theorem on the existence of certain defining functions which shows that we also have the Diederich–Fornæss type exponent in this case. From this result, we can reproduce a theorem due to Harvey and Lawson [20] which says that boundary p-convexity implies p-convexity. In Section 3, we will establish the basic L^2-estimate and existence theorem for the exterior differential operator on p-convex open sets in $\mathbb{R}^n$. Based on the a priori estimate obtained in Section 3, we prove a Berndtsson type result in Section 4. This kind of estimate involves two p-plurisubharmonic weights with opposite signs in the exponent. Such an estimate for $\bar{\partial}$-problem on pseudoconvex domains was originally obtained by Donnelly and Fefferman (see [16,4,8,10]). In Section 5, we discuss the minimal L^2-solution and estimate for the minimal L^2-solution with respect to a fixed weight function. In Section 6, we establish an estimate by using non-plurisubharmonic weights, the idea of our proof goes back to [5]. In Section 7, these L^2-estimates obtained in Sections 2–6 will be generalized to p-convex Riemannian manifolds in the sense of Harvey and Lawson. As geometric applications, we consider topological restrictions for a Riemannian manifold to be p-convex in the last section. We will prove vanishing and finiteness theorems for the de Rham cohomology groups for p-convex Riemannian manifolds (without additional curvature assumptions). A uniform estimate of Carleman type (Lemma 8.4) plays an important role in establishing these results. Following Hörmander's idea [22] and using a uniform Gårding type estimate, we prove this Carleman type estimate which is uniform with respect to domains and weights. Lemma 8.4 is different from Hörmander's original estimate in the complex analytic case which was proved on a fixed domain. This estimate allows us to prove, without using the approximation theorem for closed forms, a finiteness theorem for non-compact manifolds which are strictly p-convex at infinity (not only for relatively compact domains with strictly p-convex boundary, and the underlying metric is not assumed to be complete). In fact, Lemma 8.4 applied to a fixed weight function and an increasing sequence of domains gives the finiteness theorem (Theorem 8.1), by a similar argument, Lemma 8.4 applied to a fixed domain and an increasing sequence of weight functions also gives the approximation theorem for closed forms (Theorem 8.2).

1. Preliminaries

In this section, we will collect some facts on exterior algebra for later use and recall the notions of p-convexity and p-plurisubharmonicity in the sense of Harvey and Lawson [18–20].

Here and throughout this paper, the convention is adopted for summation over pairs of repeated indices. Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional Euclidean space, then we denote by $\{e_1, \ldots, e_n\}$ an orthonormal basis of $(V, \langle\cdot,\cdot\rangle)$ and by $\{\omega^1, \ldots, \omega^n\}$ its dual basis. For any multi-index $J = (j_1, \ldots, j_p)$, we set $\omega^J = \omega^{j_1} \wedge \cdots \wedge \omega^{j_p}$.

Definition 1.1. A quadratic form $\theta = \theta_{ij}\omega^i \otimes \omega^j \in V^* \otimes V^*$ is called p-positive (semi-)definite if any sum of p eigenvalues of the symmetric matrix (θ_{ij}) is positive (nonnegative) where $1 \leqslant p \leqslant n$.

By using the inner product $\langle\cdot,\cdot\rangle$, we identify the space of symmetric endomorphisms of V with the space of quadratic forms. Then, a self-adjoint endomorphism F is p-positive definite (resp., semi-definite) if and only if for any p-plane $W \subseteq V$, the W-trace $\mathrm{tr}_W F := \mathrm{tr}(F|_W)$ is positive (resp., nonnegative).

Denote by $\bigwedge^p$ the linear space of p-forms on V. For any quadratic form $\theta = \theta_{ij}\omega^i \otimes \omega^j$, we introduce a self-adjoint linear operator on $\bigwedge^p$ by setting

$$F_\theta = \theta_{jk}\omega^k \wedge e_j \lrcorner \tag{1.1}$$

where $\lrcorner$ means the interior product. It follows directly from the definition of F_θ that

$$\begin{aligned} \theta_{jk} g_{jK} g_{kK} &= (\theta_{jk} e_j \lrcorner g)_K \cdot (e_k \lrcorner g)_K \\ &= \langle \theta_{jk} e_j \lrcorner g, e_k \lrcorner g\rangle \\ &= \langle F_\theta g, g\rangle, \end{aligned} \tag{1.2}$$

for any $g = g_J\omega^J \in \bigwedge^p$ where K runs over all strictly increasing multi-indices of length $p-1$.

Now we compute the eigenvalues of F_θ in terms of those of θ. Let us denote the eigenvalues of (θ_{ij}) by

$$\lambda_1 \leqslant \cdots \leqslant \lambda_n,$$

after an orthogonal transformation, we have

$$F_\theta = \sum_{j=1}^n \lambda_j \omega^j \wedge e_j \lrcorner.$$

For any multi-index J with $|J| = p$, set

$$\lambda_J = \sum_{j\in J}\lambda_j, \tag{1.3}$$

then we have

$$\begin{aligned} F_\theta\omega^J &= \sum_{j=1}^n \lambda_j\omega^j \wedge e_j \lrcorner \omega^J \\ &= \sum_{j=1}^n \lambda_j\omega^j \wedge \sum_{a=1}^p (-1)^{a-1}\delta_{jj_a}\omega^{j_1}\wedge\cdots\wedge\widehat{\omega^{j_a}}\wedge\cdots\wedge\omega^{j_p} \\ &= \sum_{a=1}^p \lambda_{j_a}\omega^J = \lambda_J\omega^J \end{aligned}$$

where δ_{jj_1} is the Kronecker delta and the circumflex over a term means that it is to be omitted. Therefore, we have

$$\text{the set of eigenvalues of } F_\theta \text{ are given by } \{\lambda_J \mid |J| = p\}. \tag{1.4}$$

Let $F : \bigwedge^p \to \bigwedge^p$ be a self-adjoint linear map, then we have the following orthogonal decomposition

$$\bigwedge^p = \operatorname{Ker} F \oplus \operatorname{Im} F, \tag{1.5}$$

which implies that F induces an isomorphism $F|_{\operatorname{Im} F} : \operatorname{Im} F \to \operatorname{Im} F$. We can therefore define

$$F^{-1} := (F|_{\operatorname{Im} F})^{-1} : \operatorname{Im} F \to \operatorname{Im} F \tag{1.6}$$

for any self-adjoint linear map F. Notice that F itself is not required to be invertible in the above definition.

The following lemma records the basic estimate concerning the self-adjoint operator F_θ.

Lemma 1.1. *Let $\theta = \theta_{ij}\omega^i \otimes \omega^j$ be a quadratic form. If $\theta - \tau\otimes\tau$ is p-positive semi-definite where $\tau = \tau_i\omega^i$ is a 1-form on V and $1 \leqslant p \leqslant n$, then*

$$\tau\wedge\xi \in \operatorname{Im} F_\theta$$

for any $(p-1)$-form ξ and we have the following estimate

$$\langle F_\theta^{-1} f, \tau\wedge\xi\rangle \leqslant \langle F_\theta^{-1} f, f\rangle^{\frac{1}{2}} |\xi|$$

for any p-form $f \in \operatorname{Im} F_\theta$*, in particular*

$$\left\langle F_\theta^{-1}(\tau \wedge \xi), \tau \wedge \xi \right\rangle \leqslant |\xi|^2.$$

Proof. By definition, we have $F_{\tau\otimes\tau} = \tau \wedge X_\tau \lrcorner$ where $X_\tau := \tau_i e_i$. Let $\eta, \tilde{\eta}$ be arbitrary p-forms, it is clear that

$$\langle F_{\tau\otimes\tau}\eta, \tilde{\eta}\rangle = \langle X_\tau \lrcorner \eta, X_\tau \lrcorner \tilde{\eta}\rangle.$$

Now we assume $F_\theta \eta = 0$, since the quadratic form $\theta - \tau \otimes \tau$ is p-positive semi-definite, we obtain

$$0 = \langle F_\theta \eta, \eta\rangle \geqslant |X_\tau \lrcorner \eta|^2$$

which implies $X_\tau \lrcorner \eta = 0$. Therefore, we get

$$\langle \tau \wedge \xi, \eta\rangle = \langle \xi, X_\tau \lrcorner \eta\rangle = 0.$$

Altogether, we have proved that

$$\tau \wedge \xi \in (\operatorname{Ker} F_\theta)^\perp = \operatorname{Im} F_\theta.$$

According to (1.6), $F_\theta^{-1}(\tau \wedge \xi)$ is well-defined.

Finally, we turn to the desired inequality. The Cauchy–Schwarz inequality gives

$$\begin{aligned}
\left\langle F_\theta^{-1}(\tau \wedge \xi), \tau \wedge \xi \right\rangle &= \left\langle X_\tau \lrcorner F_\theta^{-1}(\tau \wedge \xi), \xi \right\rangle \\
&\leqslant \left\langle X_\tau \lrcorner F_\theta^{-1}(\tau \wedge \xi), X_\tau \lrcorner F_\theta^{-1}(\tau \wedge \xi) \right\rangle^{\frac{1}{2}} |\xi| \\
&= \left\langle F_{\tau\otimes\tau} \circ F_\theta^{-1}(\tau \wedge \xi), F_\theta^{-1}(\tau \wedge \xi) \right\rangle^{\frac{1}{2}} |\xi| \\
&\leqslant \left\langle F_\theta \circ F_\theta^{-1}(\tau \wedge \xi), F_\theta^{-1}(\tau \wedge \xi) \right\rangle^{\frac{1}{2}} |\xi| \\
&= \left\langle \tau \wedge \xi, F_\theta^{-1}(\tau \wedge \xi) \right\rangle^{\frac{1}{2}} |\xi|.
\end{aligned}$$

Dividing both sides by $\langle F_\theta^{-1}(\tau \wedge \xi), \tau \wedge \xi\rangle^{\frac{1}{2}}$ gives

$$\left\langle F_\theta^{-1}(\tau \wedge \xi), \tau \wedge \xi \right\rangle^{\frac{1}{2}} \leqslant |\xi|.$$

This is the second inequality claimed in this lemma. Since $\langle F_\theta^{-1} \cdot, \cdot\rangle$ defines a positive semi-definite bilinear form on $\operatorname{Im} F_\theta \cap \bigwedge^p$, for any p-form $f \in \operatorname{Im} F_\theta$, we have

$$\begin{aligned}
\left\langle F_\theta^{-1} f, \tau \wedge \xi \right\rangle &\leqslant \left\langle F_\theta^{-1} f, f \right\rangle^{\frac{1}{2}} \left\langle F_\theta^{-1}(\tau \wedge \xi), \tau \wedge \xi \right\rangle^{\frac{1}{2}} \\
&\leqslant \left\langle F_\theta^{-1} f, f \right\rangle^{\frac{1}{2}} |\xi|,
\end{aligned}$$

which implies the first inequality, and the proof is complete. □

Now let us recall the notions of p-plurisubharmonicity and p-convexity in the sense of Harvey and Lawson.

Let (M, ds^2) be an n-dimensional oriented Riemannian manifold. Let $\{e_1, \ldots, e_n\}$ be locally orthonormal frame with dual coframe $\{\omega^1, \ldots, \omega^n\}$. With the Levi-Civita connection D, the Hessian of a function φ on M is given by $D^2\varphi(X, Y) = XY\varphi - D_X Y\varphi$.

Definition 1.2. A smooth function φ defined on an open set $\Omega \subset M$ is said to be p-plurisubharmonic if its Hessian $D^2\varphi$ is p-positive semi-definite on Ω and we call φ strictly p-plurisubharmonic if $D^2\varphi$ is p-positive definite on Ω.

It is easy to see that for a Kähler manifold (M, ds^2) the notion of p-plurisubharmonicity is defined by the Levi form of the given function which only depends on the underlying complex structure. In the general case, it depends on the given Riemannian metric.

In [20], it was proved that a smooth function φ is p-plurisubharmonic if and only if the restriction of φ to any p-dimensional minimal submanifold is subharmonic. In what follows, (strict) plurisubharmonicity means (strict) 1-plurisubharmonicity.

Given a smooth function φ, we denote

$$F_\varphi = F_{D^2\varphi} = \varphi_{jk}\omega^k \wedge e_j \lrcorner$$

where $D^2\varphi := \varphi_{ij}\omega^i \otimes \omega^j$ is the Hessian of φ. It is easy to show that the operator F_φ, acting on differential forms, is exactly given by the difference of the Lie derivative and covariant derivative with respect to the gradient of φ (see Lemma 7.1). This observation will be useful to allow us to carry out Morrey's trick handling the boundary term.

Due to (1.4), we have the following criterion for p-plurisubharmonicity of a smooth function: *φ is p-plurisubharmonic* (*resp., strictly p-plurisubharmonic*) *on a domain $\Omega \subseteq M$ if and only if F_φ* (*acting on p-forms*) *is positive semi-definite* (*resp., positive definite*) *at each point of Ω.*

If φ is *strictly p-plurisubharmonic* on Ω, by choosing e_i's to be eigenvectors of $D^2\varphi(x)$ at a given point $x \in \Omega$, it follows from (1.4) that

$$\left(F_\varphi^{-1} g\right)_J = \lambda_J^{-1} g_J \tag{1.7}$$

holds for any $g = g_J\omega^J \in \bigwedge^p$ and any given multi-indices J satisfying $|J| = p$ where λ_J is defined by (1.3) with $\theta = D^2\varphi(x)$. If the function φ is further assumed to be *strictly plurisubharmonic*, we denote by (φ^{jk}) the inverse matrix of the Hessian matrix (φ_{jk}), then we have

$$\begin{aligned}\left\langle F_\varphi^{-1} g, g\right\rangle &= \lambda_J^{-1}|g_J|^2 \\ &= \left(\sum_{j\in J}\lambda_j\right)^{-1}|g_J|^2\end{aligned}$$

$$\begin{aligned} &\leqslant \frac{1}{p^2}\sum_{j\in J}\lambda_j^{-1}|g_J|^2 \\ &= \frac{1}{p^2}\varphi^{jk}g_{jK}g_{kK} \end{aligned} \tag{1.8}$$

for arbitrary $g = g_J\omega^J \in \bigwedge^p$ where we have used (1.2) and (1.3) in the last equality.

Definition 1.3. A Riemannian manifold (M, ds^2) is called (strictly) p-convex if it admits a smooth (strictly) p-plurisubharmonic proper exhaustion function. It is called strictly p-convex at infinity if it admits a proper exhaustion function which is strictly p-plurisubharmonic outside a compact subset of M.

Let $\Omega \subset M$ be a compact domain with smooth boundary $\partial\Omega$. Let $II_{\partial\Omega}(X,Y) = \langle D_X Y, \nu\rangle$ be the second fundamental form of the boundary with respect to the inward pointing unit normal vector ν.

Definition 1.4. The boundary $\partial\Omega$ is said to be p-convex if $\mathrm{tr}_W\, II_x \geqslant 0$ for any tangential p-plane $W \subseteq T_x(\partial\Omega)$ and any $x \in \partial\Omega$. If the above inequality is strict for any tangential p-plane W, $\partial\Omega$ will said to be strict p-convex.

The notion of boundary convexity can be described in terms of a defining function as follows. Let ρ be a defining function for Ω, by (1.2) and (1.4), we know that $\partial\Omega$ is p-convex if and only if

$$\rho_{ij}g_{iK}g_{jK} \geqslant 0$$

holds on $\partial\Omega$ for every p-form $g = g_J\omega^J$ which satisfies

$$\sum_{i=1}^n \rho_i g_{iK} = 0$$

for all multi-indices K with $|K| = p-1$. In [20], it was proved that *if the boundary $\partial\Omega$ is p-convex, then the domain Ω is p-convex* (this also follows from our Theorem 2.1 below).

The following lemma is useful for choosing weight functions in various applications of L^2-estimates.

Lemma 1.2. *Let (M, ds^2) be an n-dimensional Riemannian manifold and ω be a continuous function on M. We have the following conclusions*:

(i) *If (M, ds^2) is strictly p-convex, then there is a strictly p-plurisubharmonic proper exhaustion function $\varphi \in C^\infty(M)$ such that $F_\varphi + \omega\,\mathrm{Id}$ is p-positive definite on M.*
(ii) *If (M, ds^2) is strictly p-convex at infinity, then there is a p-plurisubharmonic proper exhaustion function $\varphi \in C^\infty(M)$ such that $F_\varphi + \omega\,\mathrm{Id}$ is p-positive definite outside some compact subset of M. In particular, (M, ds^2) is p-convex.*

(iii) *Let $\varphi \in C^\infty(M)$ be a p-plurisubharmonic proper exhaustion function. For any constant $c \in \mathbb{R}$ and $\eta \in L^2(M, \mathrm{Loc})$, there is a function $\psi \in C^\infty(M)$ such that $0 \leqslant \psi - \varphi$ is p-plurisubharmonic, $\varphi \equiv \psi$ when $\varphi < c$ and $\int_M |\eta|^2 e^{-\psi} < +\infty$.*

Proof. (i) Let us begin with any strictly p-plurisubharmonic exhaustion function $\phi \in C^\infty(M)$. Set

$$\Lambda_\phi := \lambda_1 + \cdots + \lambda_p$$

where $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ are the eigenvalue functions of the Hessian $D^2\phi$ with respect to the underlying metric ds^2, then we know by definition that $\Lambda_\phi > 0$. Assume, without loss of generality, $\inf_M \phi = 0$ and denote

$$\Omega_\nu := \{x \in M \mid \phi(x) < \nu\} \quad \text{for } \nu = 1, 2, \ldots.$$

Since the functions $\Lambda_\phi > 0$ and ω are both continuous on M, one can always find for each $\nu = 1, 2, \ldots$ a positive constant σ_ν such that

$$\sigma_\nu \Lambda_\phi + p\omega > 0 \quad \text{holds on } \Omega_{\nu+1} \setminus \Omega_\nu. \tag{1.9}$$

Now we choose a function $\kappa \in C^\infty[0, +\infty)$ such that

$$\kappa'(t) > 0, \qquad \kappa''(t) > 0 \quad \text{for } t \geqslant 0, \qquad \kappa'(\nu) > \sigma_\nu \quad \text{for } \nu = 1, 2, \ldots, \tag{1.10}$$

and

$$\kappa'(0) \inf_{\Omega_1} \Lambda_\phi + p \sup_{\Omega_1} \omega > 0. \tag{1.11}$$

Set $\varphi = \kappa \circ \phi$, then $D^2\varphi = \kappa' \circ \phi \cdot D^2\phi + \kappa'' \circ \phi \cdot d\phi \otimes d\phi$ and consequently we have

$$\Lambda_\varphi \geqslant \kappa' \circ \phi \cdot \Lambda_\phi.$$

The construction of κ implies that $F_\varphi + \omega \,\mathrm{Id}$ is p-positive definite on M.

(ii) In this case, the proof is a slight modification of the proof given above and we will keep the notations the same as above. By definition, we have a proper exhaustion function ϕ and a compact subset $S \subseteq M$ such that ϕ is strictly p-convex in $M \setminus S$. Without loss of generality, we assume $S \subseteq \Omega_{\frac{1}{2}}$. Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi'(t) > 0, \qquad \chi''(t) > 0 \quad \text{for } t > \frac{1}{2} \quad \text{and} \quad \chi(t) = 0 \quad \text{for } t \leqslant \frac{1}{2}.$$

It is easy to see that $\chi \circ \phi$ is a p-plurisubharmonic proper exhaustion function and strictly p-plurisubharmonic outside $\overline{\Omega_{\frac{1}{2}}}$, in particular, we have proved that (M, ds^2) is p-convex.

Let $\kappa \in C^\infty[0,\infty)$ be a function which satisfies (1.9) and (1.10) with ϕ being replaced by $\chi \circ \phi$, then it is easy to check that $\varphi := \kappa \circ \chi \circ \phi$ is a p-plurisubharmonic proper exhaustion function and that $F_\varphi + \omega \,\mathrm{Id}$ is p-positive definite outside Ω_1 (note that in this case, we can not have (1.11) because $\inf_{\Omega_1} \Lambda_\phi$ is not necessarily positive).

(iii) Choose a smooth function γ defined on $\mathbb{R}$ such that

$$\gamma'(t) \geqslant 0, \qquad \gamma''(t) \geqslant 0 \quad \text{for } t \in \mathbb{R}, \qquad \gamma(t) \equiv 0 \quad \text{for } t < c$$

and

$$\gamma(c+\nu) > \nu + \log \int\limits_{\Omega_{c+\nu+1}} |\eta|^2 \quad \text{for } \nu = 1, 2, \ldots$$

where $\Omega_{c+\nu+1}$'s are sub-level sets of φ. Set $\phi = \gamma \circ \varphi$, then we know by definition that $0 \leqslant \phi$ is p-plurisubharmonic and

$$\begin{aligned}
\int\limits_M |\eta|^2 e^{-\phi} &= \Bigg(\int\limits_{\Omega_{c+1}} + \sum_{\nu \geqslant 1} \int\limits_{\Omega_{c+\nu+1} \setminus \Omega_{c+\nu}} \Bigg) |\eta|^2 e^{-\phi} \\
&\leqslant \int\limits_{\Omega_{c+1}} |\eta|^2 e^{-\phi} + \sum_{\nu \geqslant 1} e^{-\gamma(c+\nu)} \int\limits_{\Omega_{c+\nu+1} \setminus \Omega_{c+\nu}} |\eta|^2 \\
&\leqslant \int\limits_{\Omega_{c+1}} |\eta|^2 e^{-\phi} + \sum_{\nu \geqslant 1} e^{-\nu} < +\infty.
\end{aligned}$$

It is easy to see that $\psi := \varphi + \phi$ is a desired function. The proof is complete. □

2. The Diederich–Fornæss type exponent

In this section, we prove a Diederich–Fornæss type result on the defining function for p-convex open set with smooth boundary.

Theorem 2.1. *Let $\Omega \Subset \mathbb{R}^n$ be a p-convex open set with smooth boundary and let $r \in C^\infty(\overline{\Omega})$ be a defining function for Ω. Then for any strictly p-plurisubharmonic function $\varphi \in C^\infty(\overline{\Omega})$, there exist constants $K > 0, \eta_0 \in (0,1)$ such that for any $\eta \in (0,\eta_0)$ the function $\rho := -(-re^{-K\varphi})^\eta$ is strictly p-plurisubharmonic on Ω.*

Proof. It suffices to show that $\langle F_\rho g, g\rangle > 0$ for any $0 \neq g \in \bigwedge^p$.

By direct computation, we obtain

$$\begin{aligned}
\langle F_\rho g, g\rangle = \eta(-r)^{\eta-2} e^{-K\eta\varphi} \big[& Kr^2 \big(\langle F_\varphi g, g\rangle - K\eta \langle \nabla\varphi \lrcorner g, \nabla\varphi \lrcorner g\rangle \big) \\
& + (-r)\langle F_r g, g\rangle + (1-\eta)\langle \nabla r \lrcorner g, \nabla r \lrcorner g\rangle + 2K\eta r \langle \nabla r \lrcorner g, \nabla\varphi \lrcorner g\rangle \big]. \quad (2.1)
\end{aligned}$$

Throughout the proof, we denote by $A_1, A_2, \dots$ various constants which are independent of η, K.

Since the boundary of Ω is assumed to be smooth, for any sufficiently small $\varepsilon > 0$ there is a smooth map $\pi : N_\varepsilon \to \partial\Omega$ such that

$$\operatorname{dist}(x, \partial\Omega) = \bigl|x - \pi(x)\bigr|, \quad \forall x \in N_\varepsilon, \tag{2.2}$$

where $N_\varepsilon := \{x \in \Omega \mid r(x) > -\varepsilon\}$. As the function $r \in C^\infty(\overline{\Omega})$ is a defining function for Ω, there exists a constant $A_1 > 0$ which only depends on ε such that

$$\operatorname{dist}(x, \partial\Omega) \leqslant -A_1 r(x), \qquad A_1 \leqslant \bigl|\nabla r(x)\bigr|, \quad \forall x \in N_\varepsilon. \tag{2.3}$$

For any $g \in \bigwedge^p$, $x \in N_\varepsilon$, we define p-forms $g_1(x), g_2(x)$ as follows:

$$g_1(x) = \frac{1}{|\nabla r(x)|^2} \nabla r(x) \lrcorner dr(x) \wedge g$$

and

$$g_2(x) = \frac{1}{|\nabla r(x)|^2} dr(x) \wedge \nabla r(x) \lrcorner g.$$

It is easy to see that

$$g = g_1(x) + g_2(x), \qquad |g|^2 = \bigl|g_1(x)\bigr|^2 + \bigl|g_2(x)\bigr|^2$$

and

$$\nabla r(x) \lrcorner g_1(x) = 0, \qquad \bigl|g_2(x)\bigr| \leqslant \frac{1}{|\nabla r(x)|} \bigl|\nabla r(x) \lrcorner g\bigr| \tag{2.4}$$

for every $x \in N_\varepsilon$. From (2.2) and the first inequality in (2.3), there is a constant $A_2 > 0$ such that

$$\begin{aligned} \bigl|\langle F_r g_1, g_1\rangle(x) - \langle F_r g_1, g_1\rangle\bigl(\pi(x)\bigr)\bigr| &= \left| \int_0^1 \frac{d}{dt} \langle F_r g_1, g_1\rangle\bigl(tx + (1-t)\pi(x)\bigr)\, dt \right| \\ &\leqslant -A_2 r(x)|g|^2 \end{aligned} \tag{2.5}$$

holds for any $x \in N_\varepsilon$. By using the identity in (2.4) and the definition of p-convexity, we get

$$\langle F_r g_1, g_1\rangle\bigl(\pi(x)\bigr) \geqslant 0, \quad \forall x \in N_\varepsilon.$$

Therefore, for any $x \in N_\varepsilon$, the following estimate follows from (2.5)

$$\langle F_r g_1, g_1\rangle(x) \geqslant A_2 r(x)|g|^2.$$

Taking into account the inequality in (2.4) and $|g_1(x)| \leqslant |g|$, the above estimate implies that

$$\langle F_r g, g\rangle(x) \geqslant A_2 r(x)|g|^2 - \frac{A_3}{|\nabla r(x)|}\left|\nabla r(x) \lrcorner g\right| \cdot |g| \tag{2.6}$$

holds for any $x \in N_\varepsilon$ where $A_3 > 0$ is another constant.

Since φ is strictly p-plurisubharmonic on $\overline{\Omega}$, there is a constant $\sigma > 0$ such that

$$\langle F_\varphi g, g\rangle(x) - \eta K\left|\nabla\varphi(x)\lrcorner g\right|^2 \geqslant (\sigma - A_4\eta K)|g|^2 \tag{2.7}$$

holds for any $x \in \Omega$ where $A_4 := \sup_\Omega |\nabla\varphi|^2$. From (2.1) and (2.7), there exists a constant $A_5 > 0$ such that

$$\langle F_\rho g, g\rangle(x) \geqslant \eta(-r)^{\eta-2} e^{-\eta K\varphi}\left[K r^2(x)\left(\sigma - \frac{\eta}{1-\eta}A_4 K\right) - A_5\right]|g|^2 \tag{2.8}$$

holds for any $x \in \Omega$.

When $K > \frac{4A_5}{\sigma\varepsilon^2}$ and $\eta \in (0, \frac{\sigma}{2A_4K+\sigma})$, (2.8) implies that

$$\langle F_\rho g, g\rangle \geqslant \frac{1}{4}\eta(-r)^{\eta-2}e^{-\eta K\varphi}K\varepsilon^2\sigma|g|^2 \tag{2.9}$$

holds on $\Omega \setminus N_\varepsilon$.

Similarly, for any constants $\eta \in (0, \frac{\sigma}{2A_4K})$ and $K > \frac{4}{\sigma}(A_2 + \frac{\sigma^2}{4A_4} + 2A_6^2 + \sigma^2)$, $A_6 := \frac{A_3}{2A_1}$, from (2.1), (2.6) and (2.7) it follows that the following inequality holds on N_ε

$$\begin{aligned}
\langle F_\rho g, g\rangle &\geqslant \eta(-r)^{\eta-2}e^{-\eta K\varphi}\Big\{\big[K(\sigma - A_4\eta K) - A_2\big]r^2|g|^2 \\
&\qquad + 2(A_6 + A_4\eta K)|\nabla r\lrcorner g||r||g| + (1-\eta)|\nabla r\lrcorner g|^2\Big\} \\
&\geqslant \eta(-r)^\eta e^{-\eta K\varphi}\left[K(\sigma - A_4\eta K) - A_2 - \frac{2A_6^2 + 2A_4^2\eta^2K^2}{1-\eta}\right]|g|^2 \\
&\geqslant \eta(-r)^\eta e^{-\eta K\varphi}\left(\frac{K\sigma}{2} - A_2 - 4A_6^2 - \sigma^2\right)|g|^2 \\
&\geqslant \frac{1}{4}\eta(-r)^\eta e^{-\eta K\varphi}K\sigma|g|^2 \\
&= \frac{K\eta\sigma}{4}(-\rho)|g|^2.
\end{aligned} \tag{2.10}$$

By combining (2.9) and (2.10), we know Theorem 2.1 is true for any constant $K > \frac{4}{\sigma}(A_2 + \frac{\sigma^2}{4A_4} + \frac{A_5}{\varepsilon^2} + 2A_6^2 + \sigma^2)$ and $\eta_0 := \frac{\sigma}{2A_4K+\sigma}$. □

Remark 2.1. (i) The constant η is an analogue of the Diederich–Fornæss exponent in several complex variables (see [15]).

(ii) By Theorem 3.1, we know that $\psi := -\log(-\rho)$ is a strictly p-plurisubharmonic proper exhaustion function on Ω, and this implies Theorem 3.10 in [20].

3. The L^2-existence theorem

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset, $\varphi \in C^1(\Omega)$. Following [22], the weighted L^2-Hermitian inner product of p-forms will be denoted by $(\cdot,\cdot)_\varphi$ and the corresponding Hilbert space will be denoted by $L^2_p(\Omega,\varphi)$. We will still denote by d the maximal (weak) differential operator (from $L^2_{p-1}(\Omega,\varphi)$ to $L^2_p(\Omega,\varphi)$) of the exterior differential. It is easy to see that the formal adjoint of d with respect to the weight φ is given by $\delta_\varphi := e^\varphi \circ \delta \circ e^{-\varphi}$ where δ is the codifferential operator on $\mathbb{R}^n$. If $\Omega \Subset \mathbb{R}^n$ has smooth boundary and $\varphi \in C^1(\overline{\Omega})$, integration by parts shows that $C^\infty_p(\overline{\Omega}) \cap \operatorname{Dom}(d^*_\varphi) = \{g \in C^\infty_p(\overline{\Omega}) \mid \nabla\rho \lrcorner g = 0 \text{ on } \partial\Omega\}$ where d^*_φ is the Hilbert space adjoint of d with respect to the weight φ and ρ is a defining function of Ω.

The following Kohn–Morrey–Hörmander type identity is crucial in establishing basic a priori estimate.

Proposition 3.1. *Let $\Omega \Subset \mathbb{R}^n$ be a domain with smooth boundary. Assume that the defining function satisfies $|\nabla\rho| = 1$ when restricted to $\partial\Omega$. Then we have the following identity:*

$$\|dg\|^2_\varphi + \|\delta_\varphi g\|^2_\varphi = \int_\Omega |\partial_j g_I|^2 e^{-\varphi} + \int_\Omega \langle F_\varphi g, g\rangle e^{-\varphi} + \int_{\partial\Omega} \langle F_\rho g, g\rangle e^{-\varphi} \tag{3.1}$$

for $g \in C^\infty_p(\overline{\Omega}) \cap \operatorname{Dom}(d^*_\varphi)$ $(1 \leqslant p \leqslant n)$.

Proof. Let $\delta_k = e^\varphi \partial_k(e^{-\varphi}\cdot)$, then it is easy to see that

$$[\delta_k, \partial_j] = \partial_j\partial_k\varphi$$

holds on functions. By definition, we have the following equalities

$$|dg|^2 = |\partial_j g_I|^2 - \partial_j g_{kK}\partial_k g_{jK},$$
$$|\delta_\varphi g|^2 = \delta_j g_{jK}\delta_k g_{kK}.$$

Repeated use of the formula

$$\int_\Omega \partial_j v w e^{-\varphi} = \int_\Omega -v\delta_j w e^{-\varphi} + \int_{\partial\Omega} \partial_j\rho v w e^{-\varphi}$$

gives that

$$\begin{aligned}\int\limits_{\Omega} \partial_j g_{kK} \partial_k g_{jK} e^{-\varphi} &= -\int\limits_{\Omega} g_{kK} \delta_j \partial_k g_{jK} e^{-\varphi} + \int\limits_{\partial\Omega} g_{kK} \partial_k g_{jK} \partial_j \rho e^{-\varphi} \\ &= -\int\limits_{\Omega} g_{kK} \big(\partial_k \delta_j g_{jK} + [\delta_j, \partial_k] g_{jK}\big) e^{-\varphi} + \int\limits_{\partial\Omega} g_{kK} \partial_k g_{jK} \partial_j \rho e^{-\varphi} \\ &= \int\limits_{\Omega} \delta_k g_{kK} \delta_j g_{jK} e^{-\varphi} - \int\limits_{\partial\Omega} g_{kK} \delta_j g_{jK} \partial_k \rho e^{-\varphi} \\ &\quad - \int\limits_{\Omega} g_{kK} \partial_j \partial_k \varphi g_{jK} e^{-\varphi} + \int\limits_{\partial\Omega} g_{kK} \partial_k g_{jK} \partial_j \rho e^{-\varphi}.\end{aligned}$$

From the boundary condition

$$\partial_k \rho g_{kK} = 0 \quad \text{on } \partial\Omega,$$

we know that, for any fixed K with $|K| = p - 1$,

$$g_{jK} \frac{\partial}{\partial x^j} \text{ defines a tangent vector field of } \partial\Omega.$$

Consequently, we obtain

$$0 = \sum_{k=1}^{n} g_{kK} \partial_k \left(\sum_{j=1}^{n} g_{jK} \partial_j \rho \right) = \sum_{j,k=1}^{n} (g_{kK} \partial_k g_{jK} \partial_j \rho + \partial_j \partial_k \rho g_{kK} g_{jK}) \quad \text{on } \partial\Omega.$$

Therefore, we get

$$\begin{aligned}\|dg\|_{\varphi}^2 + \|\delta_{\varphi} g\|_{\varphi}^2 &= \int\limits_{\Omega} |\partial_j g_I|^2 e^{-\varphi} + \int\limits_{\Omega} \partial_j \partial_k \varphi g_{jK} g_{kK} e^{-\varphi} \\ &\quad + \int\limits_{\partial\Omega} \partial_k \rho g_{kK} \delta_j g_{jK} e^{-\varphi} - \int\limits_{\partial\Omega} g_{kK} \partial_k g_{jK} \partial_j \rho e^{-\varphi} \\ &= \int\limits_{\Omega} |\partial_j g_I|^2 e^{-\varphi} + \int\limits_{\Omega} \partial_j \partial_k \varphi g_{jK} g_{kK} e^{-\varphi} + \int\limits_{\partial\Omega} \partial_j \partial_k \rho g_{jK} g_{kK} e^{-\varphi} \\ &= \int\limits_{\Omega} |\partial_j g_I|^2 e^{-\varphi} + \int\limits_{\Omega} \langle F_{\varphi} g, g\rangle e^{-\varphi} + \int\limits_{\partial\Omega} \langle F_{\rho} g, g\rangle e^{-\varphi}\end{aligned}$$

which gives the desired identity (3.1). □

To establish L^2-existence theorem, we also need the following basic lemma from functional analysis due to Hörmander:

Lemma 3.1. *(See Theorem 1.1.4 in [22].) Let $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$ be a complex of closed and densely defined operators between Hilbert spaces and let $L \subseteq H_2$ be a closed subspace which contains $\mathrm{Im}(T)$. For any $f \in L \cap \mathrm{Ker}(S)$ and any constant $C > 0$, the following conditions are equivalent*

1. *there exists some $u \in H_1$ such that $Tu = f$ and $\|u\|_{H_1} \leqslant C$.*
2. *$|(f,g)_{H_2}|^2 \leqslant C^2(\|T^*g\|^2_{H_1} + \|Sg\|^2_{H_3})$ holds for any $g \in L \cap \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S)$.*

Now we can prove an L^2-existence result for the exterior differential operator.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a p-convex domain and $\varphi \in C^2(\Omega)$ be a p-plurisubharmonic function over Ω. Then for any closed p-form $f \in L^2_p(\Omega, \mathrm{Loc})$ satisfying*

$$\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi} < \infty$$

there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi)$ such that

$$du = f, \qquad \|u\|^2_\varphi \leqslant \int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi}$$

where F_φ^{-1} is defined by (1.6) and it is assumed implicitly that $F_\varphi^{-1} f$ is defined almost everywhere in Ω.

Proof. First we suppose $\Omega \Subset \mathbb{R}^n$ has smooth p-convex boundary. Then we have, in formula (3.1), $\langle F_\rho g, g\rangle \geqslant 0$ on $\partial\Omega$, which implies that

$$\|dg\|^2_\varphi + \|\delta_\varphi g\|^2_\varphi \geqslant \int_\Omega \langle F_\varphi g, g\rangle e^{-\varphi} \tag{3.2}$$

holds for any $g \in C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}(d^*_\varphi)$. By Hörmander's density lemma [22,23], the above estimate (3.2) holds for any $g \in \mathrm{Dom}(d^*_\varphi) \cap \mathrm{Dom}(d)$.

We will apply Lemma 3.1 to

$$H_1 = L^2_{p-1}(\Omega, \varphi), \qquad H_2 = L^2_p(\Omega, \varphi), \qquad H_3 = L^2_{p+1}(\Omega, \varphi)$$

and S, T both given by the maximal differential operators of exterior differentials. Since the $\langle F_\varphi \cdot, \cdot\rangle$ is positive semi-definite, it follows from Schwarz inequality that

$$\left|\int_\Omega \langle f, g\rangle e^{-\varphi}\right|^2 = \left|\int_\Omega \langle F_\varphi F_\varphi^{-1} f, g\rangle e^{-\varphi}\right|^2$$
$$\leqslant \left(F_\varphi F_\varphi^{-1} f, F_\varphi^{-1} f f\right)_\varphi (F_\varphi g, g)_\varphi$$

$$= \left(F_\varphi^{-1} f, f\right)_\varphi (F_\varphi g, g)_\varphi$$
$$\leqslant \left(F_\varphi^{-1} f, f\right)_\varphi \left(\left\|T^* g\right\|_{H_1}^2 + \|Sg\|_{H_2}^2\right).$$

Now from Lemma 3.1, it follows that there is a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi)$ such that

$$du = f, \qquad \|u\|_\varphi^2 \leqslant \int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi}.$$

For the general case, by Theorem 3.4 in [20], there exists a sequence of domains Ω_ν $(\nu = 1, 2, \ldots)$ with smooth p-convex boundary such that $\Omega = \bigcup_{\nu \geqslant 1} \Omega_\nu$. For each $\nu \geqslant 1$, we have a solution $u_\nu \in L^2_{p-1}(\Omega_\nu, \varphi)$ of $du_\nu = f$ such that

$$\int_{\Omega_\nu} |u_\nu|^2 e^{-\varphi} \leqslant \int_{\Omega_\nu} \langle F_\varphi^{-1} f, f\rangle e^{-\varphi}.$$

By the estimate on u_ν we obtain the desired solution by taking weak limit. The proof is complete. □

Starting from any strictly p-plurisubharmonic proper exhaustion function φ and then using Lemma 1.2(iii), we have the following corollary of Theorem 3.1.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a p-convex domain. For any closed p-form $f \in L^2_p(\Omega, \mathrm{Loc})$, there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \mathrm{Loc})$ such that $du = f$.*

4. A Berndtsson type estimate

In this section, we will establish a Berndtsson type result which involves two p-plurisubharmonic weights with opposite signs in the exponent. This kind of estimates for $\bar\partial$-problem was first obtained by Berndtsson (see [4,6,8–10] and references therein) and had its root in Donnelly–Fefferman estimate [16]. The key for our proof is to establish the following a priori estimate.

$$\|\delta_{\varphi+\sigma\psi} g\|_{\varphi+\psi}^2 + \|dg\|_{\varphi+\psi}^2 \geqslant \sigma^2 \int_\Omega \langle F_\psi g, g\rangle e^{-\varphi-\psi}, \tag{$*$}$$

for any $g \in \mathrm{Dom}(d^*) \cap C_p^\infty(\overline{\Omega})$ where $\varphi \in C^\infty(\overline{\Omega})$ is a p-plurisubharmonic function, $\psi \in C^\infty(\overline{\Omega})$ with $-e^{-\psi}$ being p-plurisubharmonic and $\sigma \in (0, \frac{1}{2}]$ is a constant. The following proof involves a useful procedure to introduce a twist factor into a known a priori estimate (see also [24] and references therein).

Theorem 4.1. *Let Ω be a p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let φ be a p-plurisubharmonic function on Ω, $\psi \in C^2(\Omega)$ be a function such that $-e^{-\psi}$ is* *p-plurisubharmonic. For any constant $\alpha \in [0,1)$ and any closed p-form $f \in L^2_p(\Omega, \mathrm{Loc})$, if*

$$\int_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi+\alpha\psi} < \infty$$

then there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi - \alpha\psi)$ such that

$$du = f, \qquad \|u\|^2_{\varphi-\alpha\psi} \leqslant \frac{4}{(1-\alpha)^2} \int_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi+\alpha\psi},$$

where F_ψ^{-1} is defined by (1.6) and it is required implicitly that $F_\psi^{-1} f$ is defined almost everywhere in Ω.

Proof. We consider first the case where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $\varphi, \psi \in C^\infty(\overline{\Omega})$.

We will apply Lemma 3.1 to the following weighted L^2-spaces of differential forms

$$H_1 = L^2_{p-1}\left(\Omega, \varphi + \frac{1-\alpha}{2}\psi\right), \qquad H_2 = L^2_p\left(\Omega, \varphi + \frac{1-\alpha}{2}\psi\right),$$
$$H_3 = L^2_{p+1}\left(\Omega, \varphi + \frac{1-\alpha}{2}\psi\right)$$

and

$$T = d \circ e^{-\frac{1+\alpha}{4}\psi}, \qquad S = e^{-\frac{1+\alpha}{4}\psi} \circ d.$$

In order to use Lemma 3.1, we need to show that the following estimate

$$\begin{aligned} \left|(f,g)_{\varphi+\frac{1-\alpha}{2}\psi}\right|^2 \leqslant \frac{4(F_\psi^{-1} f, f)_{\varphi-\alpha\psi}}{(1-\alpha)^2} \Big(&\left\|e^{-\frac{1+\alpha}{4}\psi} \delta_{\varphi+\frac{1-\alpha}{2}\psi} g\right\|^2_{\varphi+\frac{1-\alpha}{2}\psi} \\ &+ \left\|e^{-\frac{1+\alpha}{4}\psi} dg\right\|^2_{\varphi+\frac{1-\alpha}{2}\psi}\Big) \end{aligned} \tag{4.1}$$

holds for arbitrary $g \in \mathrm{Dom}(d^*) \cap C^\infty_p(\overline{\Omega})$.

Let $g \in \mathrm{Dom}(d^*) \cap C^\infty_p(\overline{\Omega})$, then the basic estimate with $\phi = \varphi + \psi$ gives

$$\|dg\|^2_{\varphi+\psi} + \|\delta_{\varphi+\psi} g\|^2_{\varphi+\psi} \geqslant \int_\Omega \langle F_{\varphi+\psi} g, g\rangle e^{-\varphi-\psi}. \tag{4.2}$$

Since

$$\delta_{\varphi+\psi} g = \delta_{\varphi+\frac{1-\alpha}{2}\psi} g + \frac{1+\alpha}{2} \nabla\psi \lrcorner g,$$

it follows that

$$\|\delta_{\varphi+\psi}g\|^2_{\varphi+\psi} \leqslant \frac{1+\epsilon}{\epsilon}\|\delta_{\varphi+\frac{1-\alpha}{2}\psi}g\|^2_{\varphi+\psi} + \frac{(1+\epsilon)(1+\alpha)^2}{4}\|\nabla\psi\lrcorner g\|^2_{\varphi+\psi}$$

for any positive constant ϵ.

By choosing

$$\epsilon = \frac{1-\alpha}{1+\alpha},$$

the above inequality becomes

$$\|\delta_{\varphi+\psi}g\|^2_{\varphi+\psi} \leqslant \frac{2}{1-\alpha}\|\delta_{\varphi+\frac{1-\alpha}{2}\psi}g\|^2_{\varphi+\psi} + \frac{1+\alpha}{2}\|\nabla\psi\lrcorner g\|^2_{\varphi+\psi}. \tag{4.3}$$

Since $-e^{-\psi}$ is p-plurisubharmonic and

$$\begin{aligned} F_{-e^{-\psi}} &= e^{-\psi}(\psi_{jk} - \psi_j\psi_k)dx^k \wedge \frac{\partial}{\partial x^j}\lrcorner \\ &= e^{-\psi}(F_\psi - d\psi \wedge \nabla\psi\lrcorner), \end{aligned}$$

we know that $F_\psi - d\psi \wedge \nabla\psi\lrcorner$ defines a positive semi-definite operator on the space of p-forms. This implies

$$\int_\Omega \langle F_\psi g, g\rangle e^{-\varphi-\psi} \geqslant \|\nabla\psi\lrcorner g\|^2_{\varphi+\psi}. \tag{4.4}$$

Substituting (4.3), (4.4) into (4.2), the p-plurisubharmonicity of φ gives

$$\frac{2}{1-\alpha}\|\delta_{\varphi+\frac{1-\alpha}{2}\psi}g\|^2_{\varphi+\psi} + \|dg\|^2_{\varphi+\psi} \geqslant \frac{1-\alpha}{2}\int_\Omega \langle F_\psi g, g\rangle e^{-\varphi-\psi}$$

which further implies the desired Donnelly–Fefferman type estimate $(*)$ with the constant $\sigma = \frac{1-\alpha}{2}$ as follows

$$\begin{aligned} \left\|e^{-\frac{1+\alpha}{4}\psi}\delta_{\varphi+\frac{1-\alpha}{2}\psi}g\right\|^2_{\varphi+\frac{1-\alpha}{2}\psi} + \left\|e^{-\frac{1+\alpha}{4}\psi}dg\right\|^2_{\varphi+\frac{1-\alpha}{2}\psi} &= \|\delta_{\varphi+\frac{1-\alpha}{2}\psi}g\|^2_{\varphi+\psi} + \|dg\|^2_{\varphi+\psi} \\ &\geqslant \|\delta_{\varphi+\frac{1-\alpha}{2}\psi}g\|^2_{\varphi+\psi} + \frac{1-\alpha}{2}\|dg\|^2_{\varphi+\psi} \\ &\geqslant \frac{(1-\alpha)^2}{4}\int_\Omega \langle F_\psi g, g\rangle e^{-\varphi-\psi}. \end{aligned}$$

Since ψ is p-plurisubharmonic, the Cauchy–Schwarz inequality applied to the positive semi-definite Hermitian form $(F_\psi\cdot,\cdot)_{\varphi+\psi}$ gives

$$
\begin{aligned}
\left|(f,g)_{\varphi+\frac{1-\alpha}{2}\psi}\right|^2 &= \left|\left(F_\psi \circ F_\psi^{-1} e^{\frac{1+\alpha}{2}\psi} f, g\right)_{\varphi+\psi}\right|^2 \\
&\leqslant \left(e^{\frac{1+\alpha}{2}} f, e^{\frac{1+\alpha}{2}} F_\psi^{-1} f\right)_{\varphi+\psi} (F_\psi g, g)_{\varphi+\psi} \\
&\leqslant \frac{4(F_\psi^{-1} f, f)_{\varphi-\alpha\psi}}{(1-\alpha)^2} \left(\left\|e^{-\frac{1+\alpha}{4}\psi} \delta_{\varphi+\frac{1-\alpha}{2}\psi} g\right\|^2_{\varphi+\frac{1-\alpha}{2}\psi} + \left\|e^{-\frac{1+\alpha}{4}\psi} dg\right\|^2_{\varphi+\frac{1-\alpha}{2}\psi}\right)
\end{aligned}
$$

where F_ψ^{-1} is defined by (1.6). Thus the estimate (4.2) has been proved for $g \in \mathrm{Dom}(d^*) \cap C_p^\infty(\overline{\Omega})$. By using the density lemma (Proposition 1.2.4 in [22]), we know that (4.2) holds for any $g \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S)$. Consequently, by Lemma 3.1, there exists some $v \in L^2_{p-1}(\Omega, \varphi + \frac{1-\alpha}{2}\psi)$ such that

$$
Tv = f, \qquad \|v\|^2_{\varphi+\frac{1-\alpha}{2}\psi} \leqslant \frac{4}{(1-\alpha)^2} \left(F_\psi^{-1} f, f\right)_{\varphi-\alpha\psi}.
$$

Set $u = e^{-\frac{1+\alpha}{4}\psi} v$, then we get $u \in L^2_{p-1}(\Omega, \varphi - \alpha\psi)$ and

$$
du = f, \qquad \|u\|^2_{\varphi-\alpha\psi} = \|v\|^2_{\varphi+\frac{1-\alpha}{2}\psi} \leqslant \frac{4}{(1-\alpha)^2} \left(F_\psi^{-1} f, f\right)_{\varphi-\alpha\psi}. \tag{4.5}
$$

Theorem 4.1 now follows, in its full generality, from (4.5) and the standard argument of smooth approximation followed by taking weak limit as we did in the proof of Theorem 3.1. □

Corollary 4.1. *Let Ω be a p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let φ be a p-plurisubharmonic function on Ω, $\psi \in C^2(\Omega)$ be a strictly plurisubharmonic function such that $-e^{-\psi}$ is p-plurisubharmonic. For any constant $\alpha \in [0,1)$ and closed p-form $f \in L^2_p(\Omega, \mathrm{Loc})$, if*

$$
\int_\Omega \psi^{jk} f_{jK} f_{kK} e^{-\varphi+\alpha\psi} < \infty
$$

then there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi - \alpha\psi)$ such that

$$
du = f, \qquad \|u\|^2_{\varphi-\alpha\psi} \leqslant \frac{4}{p^2(1-\alpha)^2} \int_\Omega \psi^{jk} f_{jK} f_{kK} e^{-\varphi+\alpha\psi}
$$

where $(\psi^{jk}) := (\psi_{jk})^{-1}$.

Proof. Corollary 4.1 follows directly from Theorem 4.1 and the pointwise inequality (1.8). □

As a consequence of Theorem 4.1, we have the following analogue of the Donnelly–Fefferman estimate [16].

Theorem 4.2. *Let Ω be a p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let φ be a p-plurisubharmonic function on Ω, $\psi \in C^2(\Omega)$ be a strictly p-plurisubharmonic function such that $-e^{-\psi}$ is p-plurisubharmonic. For any closed p-form $f \in L^2_p(\Omega, \mathrm{Loc})$, if*

$$\int_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi} < \infty$$

then there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi)$ such that

$$du = f, \qquad \|u\|^2_\varphi \leqslant 4 \int_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi}.$$

Proof. Theorem 4.2 follows directly from Theorem 4.1 by choosing the constant α to be 0. □

Corollary 4.2. *Let Ω be a bounded p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let φ be a p-plurisubharmonic function on Ω. For any closed p-form $f \in L^2_p(\Omega, \varphi)$, there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi)$ such that*

$$du = f, \qquad \|u\|_\varphi \leqslant \frac{2D}{p}\|f\|_\varphi,$$

where D is the diameter of Ω.

Proof. Without loss of generality, we assume that Ω contains the origin of $\mathbb{R}^n$. Let

$$\psi = \frac{p|x|^2}{2D^2},$$

then (1.7) implies that

$$F_\psi^{-1} = \frac{D^2}{p^2}\,\mathrm{Id} \quad \text{holds on } p\text{-forms.}$$

Since the Hessian of $-e^{-\psi}$ is given by

$$\frac{p}{D^2} e^{-\psi}\left(dx^i \otimes dx^i - \frac{p}{D^2} x^i\, dx^i \otimes x^j\, dx^j\right),$$

we know that any sum of p eigenvalues of the Hessian of $-e^{-\psi}$ is no less than

$$\frac{p}{D^2} e^{-\psi}\left[\left(1 - \frac{p}{D^2}|x|^2\right) + p - 1\right] = \frac{p^2}{D^2} e^{-\psi}\left(1 - \frac{|x|^2}{D^2}\right) \geqslant 0.$$

So $-e^{-\psi}$ is, by definition, a p-plurisubharmonic function on Ω (but not plurisubharmonic). Applying Theorem 4.2 with the weight function $\psi = \frac{p|x|^2}{2D^2}$, we obtain the following estimate for the solution u

$$\|u\|_\varphi^2 \leqslant \frac{4D^2}{p^2}\|f\|_\varphi^2.$$

This completes the proof of Corollary 4.2. □

5. Minimal L^2-solutions

Let Ω be an open subset of $\mathbb{R}^n$ and $\varphi \in L^\infty(\Omega, \mathrm{Loc})$, then the de Rham complex induces the following complex of closed and densely defined operators

$$\cdots \to L^2_{p-2}(\Omega,\varphi) \xrightarrow{d_{p-2}} L^2_{p-1}(\Omega,\varphi) \xrightarrow{d_{p-1}} L^2_p(\Omega,\varphi) \to \cdots,$$

where d_ℓ's denote the maximal (weak) differential operators defined by the exterior derivatives. Then we have

$$\operatorname{Ker} d^*_{p-2,\varphi} \supseteq \operatorname{Ker} d^\perp_{p-1} \tag{5.1}$$

and since $\operatorname{Ker} d_{p-1}$ is a closed subspace of $L^2_{p-1}(\Omega,\varphi)$ we also have the following orthogonal decomposition

$$L^2_{p-1}(\Omega,\varphi) = \operatorname{Ker} d^\perp_{p-1} \oplus \operatorname{Ker} d_{p-1}. \tag{5.2}$$

Given a d-closed form $f \in L^2_p(\Omega, \mathrm{Loc})$, if there is a p-form $u \in L^2_p(\Omega,\varphi)$ such that $du = f$, we can decompose u according to (5.2)

$$u = u_0 + u_1 \in \big(\mathrm{Dom}(d_{p-1}) \cap \operatorname{Ker} d^\perp_{p-1}\big) \oplus \operatorname{Ker} d_{p-1} \tag{5.3}$$

which, together with (5.1) above, implies that

$$d_{p-1}u_0 = f, \qquad d^*_{p-1,\varphi}u_0 = 0. \tag{5.4}$$

We will call the solution u_0 constructed in (5.3) the **minimal solution** of $du = f$ in $L^2_{p-1}(\Omega,\varphi)$.

Remark 5.1. (i) For any p-convex open subset $\Omega \subseteq \mathbb{R}^n$ and any closed p-form $f \in L^2_p(\Omega, \mathrm{Loc})$, by Corollary 3.1, we can find some $u \in L^2_{p-1}(\Omega, \mathrm{Loc})$ such that $du = f$. Let $\varphi \in L^\infty(\Omega, \mathrm{Loc})$ and $\Omega' \Subset \Omega$, previous decomposition (5.3) applied to $L^2_{p-1}(\Omega',\varphi)$ gives the minimal solution of $du = f$ in $L^2_{p-1}(\Omega',\varphi)$.

(ii) It is easy to see the uniqueness of minimal solution, to be more precise, by using (5.3) we have that $\|u_0\|_\varphi \leqslant \|u\|_\varphi$ holds for any $u \in L^2_p(\Omega, \varphi)$ satisfying $du = f$, and the equality holds if and only if $u = u_0$.

(iii) As an easy consequence of (ii), we have the following monotonicity of L^2-solutions. Let $\Omega_1 \subseteq \Omega_2$ be open subsets of $\mathbb{R}^n$ and $\varphi \in L^\infty(\Omega_2, \mathrm{Loc})$, for the minimal solution u_i of $du = f$ in $L^2_{p-1}(\Omega_i)$ $(i = 1, 2)$, we have

$$\int_{\Omega_1} |u_1|^2 e^{-\varphi} \leqslant \int_{\Omega_2} |u_2|^2 e^{-\varphi}.$$

Similarly, for any open set $\Omega \subset \mathbb{R}^n, \varphi_i \in L^\infty(\Omega, \mathrm{Loc})$ and the minimal solution u_i of $du = f$ in $L^2_{p-1}(\Omega, \varphi_i)$ $(i = 1, 2)$, if $\varphi_1 \leqslant \varphi_2$ holds on Ω then we have

$$\int_{\Omega} |u_1|^2 e^{-\varphi_1} \geqslant \int_{\Omega} |u_2|^2 e^{-\varphi_2}.$$

The minimal L^2-solution enjoys the following interior regularity property.

Proposition 5.1. *Under the conditions of Theorem 3.1, for any $q \geqslant p$ and any closed q-form $f \in L^2_q(\Omega, \varphi)$ with $\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi} < \infty$, $du = f$ has a unique minimal solution u_0 in $L^2_{q-1}(\Omega, \varphi)$, moreover if f and the weight φ are both smooth then $u_0 \in C^\infty_{q-1}(\Omega)$. The same conclusion holds for Theorem 4.1.*

Proof. The existence and uniqueness of minimal L^2-solution follows from the decomposition (5.3) and Theorem 3.1. By (5.4), we obtain

$$du_0 = f, \qquad \delta_\varphi u_0 = 0$$

in the sense of distribution. This can be rewritten as $(d\delta_\varphi + \delta_\varphi d)u_0 = \delta_\varphi f \in C^\infty_{q-1}(\Omega)$. Now the smoothness of the minimal solution u_0 follows from the interior elliptic regularity of the Hodge Laplace operator $d\delta_\varphi + \delta_\varphi d$. □

If Ω is a strictly p-convex open set with smooth boundary, it was proved (for compact Riemannian manifolds with smooth p-convex boundary) in [29] and [32] that $\overline{\Omega}$ has the homotopy type of CW complex of dimension $< p$. As an application of L^2-method we obtain the following vanishing result of de Rham cohomology groups. Note that this result was also obtained in [3]. We will generalize this result in Section 8 (see Proposition 8.1 and Remark 8.1(ii)).

Corollary 5.1. *For any p-convex open subset $\Omega \subseteq \mathbb{R}^n$ $(1 \leqslant p \leqslant n)$, the de Rham cohomology groups $H^q(\Omega, \mathbb{R}) = 0$, $p \leqslant q \leqslant n$.*

Proof. Let $f \in C_q^\infty(\Omega)$ $(p \leqslant q \leqslant n)$ be a closed form. Since p-convexity implies q-convexity, there exists a q-plurisubharmonic proper exhaustion function φ such that $\int_\Omega |f|e^{-\varphi} < \infty$. One can therefore find, by Theorem 3.1 (with the weight $\varphi(x) + |x|^2$) and Proposition 5.1, a $(q-1)$-form $u \in C_{q-1}^\infty(\Omega)$ which solves the equation $du = f$ and this completes the proof of $H^q(\Omega, \mathbb{R}) = 0$. □

By the argument in [5] (or [7]), Prekopa's minimal principle follows from the estimate in Theorem 3.1 (with $n = p = 1$) applied to the L^2-minimal solution given by Proposition 5.1.

Corollary 5.2. *Let $\varphi(x, y)$ be a convex function in $\mathbb{R}_x^n \times \mathbb{R}_y^m$. Define $\tilde{\varphi}$ by*

$$\tilde{\varphi}(x) = -\log \int\limits_{\mathbb{R}^m} e^{-\varphi(x,y)}\, dy.$$

Then $\tilde{\varphi}$ is a convex function on $\mathbb{R}^n$.

We end this section by proving an estimate for L^2-minimal solutions. The difference between this estimate and Theorem 4.1 is that the minimal solution here only depends on one of the weights. The idea of the following proof goes back to [6] and [8] (see also [11,12] and references therein).

Theorem 5.1. *Let Ω be a p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let φ be a p-plurisubharmonic function on Ω, $\psi \in C^2(\Omega)$. If we assume, in addition, that there are a function $0 \leqslant \omega < 1$ and a constant $\alpha \in [0, 1)$ such that the quadratic form $\omega^2 D^2\psi - d\psi \otimes d\psi$ is p-positive semi-definite on Ω and that $\omega \leqslant \alpha$ holds on $\operatorname{supp} f$, where $f \in L_p^2(\Omega, \mathrm{Loc})$ is a closed p-form, then the minimal solution, denoted by u_φ, of $du = f$ in $L_{p-1}^2(\Omega, \varphi)$ satisfies*

$$\int\limits_\Omega (1 - \omega^2)|u_\varphi|^2 e^{-\varphi+\psi} \leqslant \frac{1+\alpha}{1-\alpha} \int\limits_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi+\psi}$$

where $D^2\psi := \psi_{ij}\, dx^i \otimes dx^j$ is the Hessian of ψ.

Proof. By the monotonicity discussed in Remark 5.1(iii) and the standard argument of approximation followed by taking weak limit, we can assume in addition that Ω is a bounded open set with smooth boundary and that φ, ψ are both smooth up to the boundary of Ω. Set

$$u = e^\psi u_\varphi,$$

by (5.3), u is the minimal solution of $du = e^\psi(d\psi \wedge u_\varphi + f) := e^\psi g$ in $L_{p-1}^2(\Omega, \varphi + \psi)$. Since the quadratic form $\omega^2 D^2\psi - d\psi \otimes d\psi$ is p-positive semi-definite and $\omega \leqslant \alpha$ on $\operatorname{supp} f$, by using Lemma 1.1 to

$$\theta = \omega^2 D^2\psi \quad \text{and} \quad \tau = d\psi,$$

it follows that $F_\psi^{-1}(d\psi \wedge u_\varphi)$ is well-defined and

$$\begin{aligned}\langle F_{\varphi+\psi}^{-1} g, g\rangle &\leqslant \langle F_\psi^{-1}\, d\psi \wedge u_\varphi, d\psi \wedge u_\varphi\rangle + 2\langle F_\psi^{-1} f, d\psi \wedge u_\varphi\rangle + \langle F_\psi^{-1} f, f\rangle \\ &\leqslant \omega^2 |u_\varphi|^2 + 2\langle f, F_\psi^{-1} f\rangle^{\frac{1}{2}} \cdot \alpha |u_\varphi| + \langle F_\psi^{-1} f, f\rangle \\ &\leqslant \frac{\alpha+\omega^2}{1+\alpha}|u_\varphi|^2 + \frac{1}{1-\alpha}\langle F_\psi^{-1} f, f\rangle. \end{aligned} \tag{5.5}$$

Since $\varphi + \psi$ is p-plurisubharmonic and $du = e^\psi g$, we can apply Theorem 3.1 to get

$$\begin{aligned}\int_\Omega |u_\varphi|^2 e^{-\varphi+\psi} &= \|u\|^2_{\varphi+\psi} \\ &\leqslant \int_\Omega \langle F_{\varphi+\psi}^{-1}(e^\psi g), e^\psi g\rangle e^{-\varphi-\psi} \\ &= \int_\Omega \langle F_{\varphi+\psi}^{-1} g, g\rangle e^{-\varphi+\psi} \\ &\leqslant \int_\Omega \frac{\alpha+\omega^2}{1+\alpha}|u_\varphi|^2 e^{-\varphi+\psi} + \frac{1}{1-\alpha}\int_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi+\psi} \end{aligned} \tag{5.6}$$

where we have used the inequality (5.5). Now the desired L^2-estimate follows directly from (5.6). □

Remark 5.2. Theorem 5.1 could be used to deduce a weaker version of Theorem 4.1. Let φ be a p-plurisubharmonic function on Ω, $\psi \in C^2(\Omega)$ be a function such that $-e^{-\psi}$ is p-plurisubharmonic. Then for any constant $\alpha \in [0, 1)$, $\alpha\psi$ satisfies the conditions assumed in Theorem 5.1 with ω given by the constant $\sqrt{\alpha}$, and consequently we obtain

$$\|u\|^2_{\varphi-\alpha\psi} \leqslant \frac{1}{\alpha(1-\sqrt{\alpha})^2}\int_\Omega \langle F_\psi^{-1} f, f\rangle e^{-\varphi+\alpha\psi}.$$

6. Non-plurisubharmonic weights

Next we prove a theorem which has the feature of allowing non-plurisubharmonic weights. This kind of result will provide more flexibility in choosing weights for L^2-estimates. Such an estimate for $\bar\partial$-problem was proved by Błocki [11,12].

Theorem 6.1. *Let Ω be a p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let $\varphi \in C^2(\Omega)$ be a p-plurisubharmonic function on Ω and $\psi \in C^1(\Omega)$. There are a function $0 \leqslant \omega < 2$*

and a constant $\alpha \in [0,2)$ such that the quadratic form $\omega^2 D^2\varphi - d\psi \otimes d\psi$ is p-positive semi-definite on Ω and that $\omega \leqslant \alpha$ on $\operatorname{supp} f$ where $f \in L^2_p(\Omega, \mathrm{Loc})$ is a closed p-form. If

$$\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi} < \infty,$$

then there exists a $(p-1)$-form $u \in L^2_{p-1}(\Omega, \varphi - \psi)$ such that

$$du = f, \qquad \int_\Omega \left(1 - \frac{\omega^2}{4}\right)|u|^2 e^{-\varphi+\psi} \leqslant \frac{2+\alpha}{2-\alpha}\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi},$$

where F_φ^{-1} is defined by (1.6) and it is required implicitly that $F_\varphi^{-1} f$ is defined almost everywhere in Ω, $D^2\varphi := \varphi_{ij}\, dx^i \otimes dx^j$ is the Hessian of φ.

Proof. By the standard argument used in the proof of Theorem 3.1, we may assume, without loss of generality, that Ω is a bounded open set with smooth p-convex boundary and that φ, ψ are both smooth up to the boundary. In this case, there exists a unique minimal solution, denoted by u_0, of $du = f$ in $L^2_{p-1}(\Omega, \varphi - \frac{1}{2}\psi)$. For u_0, we have

$$\int_\Omega \langle u_0, v\rangle e^{-\varphi+\frac{1}{2}\psi} = 0 \tag{6.1}$$

for any closed $(p-1)$-form $v \in L^2_{p-1}(\Omega, \varphi - \frac{1}{2}\psi)$. Set

$$u = e^{\frac{1}{2}\psi} u_0$$

then (6.1) implies that u is the minimal solution of $du = g$ in $L^2_{p-1}(\Omega, \varphi)$ where g is the closed p-form given by

$$g = e^{\frac{1}{2}\psi}\left(\frac{1}{2}\, d\psi \wedge u_0 + f\right).$$

By Lemma 1.1, $F_\varphi g$ is well-defined and we have the following pointwise inequality

$$\begin{aligned}\langle F_\varphi^{-1} g, g\rangle &= \left(\frac{1}{4}\langle F_\varphi^{-1}\, d\psi \wedge u_0, d\psi \wedge u_0\rangle + \langle F_\varphi^{-1} f, d\psi \wedge u_0\rangle + \langle F_\varphi^{-1} f, f\rangle\right)e^{\psi} \\ &\leqslant \left(\frac{\omega^2}{4}|u_0|^2 + \langle f, F_\varphi^{-1} f\rangle^{\frac{1}{2}} \cdot \alpha|u_0| + \langle F_\varphi^{-1} f, f\rangle\right)e^{\psi} \end{aligned} \tag{6.2}$$

where we have used the assumptions that $\omega^2 D^2\varphi - d\psi \otimes d\psi$ is p-positive semi-definite and that $\omega \leqslant \alpha$ holds on $\operatorname{supp} f$.

Since φ is by assumption a p-plurisubharmonic function, from Theorem 3.1 it follows that

$$\int_\Omega |u_0|^2 e^{-\varphi+\psi} = \|u\|_\varphi^2 \leqslant \int_\Omega \langle F_\varphi^{-1} g, g\rangle e^{-\varphi},$$

which, together with (6.2), implies

$$\begin{aligned}
&\int_\Omega \left(1-\frac{\omega^2}{4}\right)|u_0|^2 e^{-\varphi+\psi} \\
&\quad\leqslant \int_\Omega \left(\langle f, F_\varphi^{-1} f\rangle^{\frac{1}{2}} \cdot \alpha|u_0| + \langle F_\varphi^{-1} f, f\rangle\right) e^{-\varphi+\psi} \\
&\quad\leqslant \epsilon \int_\Omega \left(1-\frac{\omega^2}{4}\right)|u_0|^2 e^{-\varphi+\psi} + \int_\Omega \left[1+\frac{\alpha^2}{(4-\omega^2)\epsilon}\right]\langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi} \\
&\quad\leqslant \epsilon \int_\Omega \left(1-\frac{\omega^2}{4}\right)|u_0|^2 e^{-\varphi+\psi} + \int_\Omega \left[1+\frac{\alpha^2}{(4-\alpha^2)\epsilon}\right]\langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi}
\end{aligned}$$

where $0 < \epsilon < 1$ is any constant. Set

$$\epsilon = \frac{\alpha}{2+\alpha},$$

the above inequality gives

$$\int_\Omega \left(1-\frac{\omega^2}{4}\right)|u_0|^2 e^{-\varphi+\psi} \leqslant \frac{2+\alpha}{2-\alpha}\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi},$$

hence u_0 is the desired solution. □

As an immediate consequence of the above theorem, if the function ω is constant we have the following corollary.

Corollary 6.1. *Let Ω be a p-convex domain in $\mathbb{R}^n$ $(1 \leqslant p \leqslant n)$ and let $\varphi \in C^2(\Omega)$ be a p-plurisubharmonic function on Ω and $\psi \in C^1(\Omega)$. There is a constant $\alpha \in [0, 2)$ such that the symmetric bilinear form $\alpha^2 D^2\varphi - d\psi \otimes d\psi$ is p-positive semi-definite on Ω. For any closed p-form $f \in L_p^2(\Omega, \mathrm{Loc})$, if*

$$\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi} < \infty,$$

then there exists a $(p-1)$-form $u \in L_{p-1}^2(\Omega, \varphi-\psi)$ such that

$$du = f, \qquad \|u\|_{\varphi-\psi}^2 \leqslant \frac{4}{(2-\alpha)^2}\int_\Omega \langle F_\varphi^{-1} f, f\rangle e^{-\varphi+\psi}.$$

Remark 6.1. (i) If we choose the constant $\alpha = 0$ and the weight function $\psi = 0$, then Corollary 6.1 recovers Theorem 3.1.

(ii) We can give an alternative proof of Theorem 4.1 by using Corollary 6.1 in the following way. Let $\varphi_1 = \varphi + \psi$ and $\psi_1 = (1+\alpha)\psi$, then φ_1 is p-plurisubharmonic. Since

$$(1+\alpha)^2 D^2\varphi_1 - d\psi_1 \otimes d\psi_1 = (1+\alpha)^2\big[D^2\varphi + e^{\psi} D^2\big(-e^{-\psi}\big)\big],$$

the assumption that φ and $-e^{-\psi}$ are both p-plurisubharmonic functions implies that $(1+\alpha)^2 D^2\varphi_1 - d\psi_1 \otimes d\psi_1$ is p-positive semi-definite. Applying Corollary 6.1 to the weights φ_1 and ψ_1, we obtain Theorem 4.1.

(iii) The proof of Theorem 4.1 given in (ii) does not indicate the estimate $(*)$ in Section 4. Actually, Corollary 6.1 also follows from the following estimate whose proof is an imitation of that of $(*)$. Let $\varphi \in C^\infty(\overline{\Omega})$ be a p-plurisubharmonic function and let $\psi \in C^\infty(\overline{\Omega})$ be a function such that the symmetric form $\alpha D^2\varphi - d\psi \otimes d\psi$ is p-positive semi-definite for some constant $\alpha \in [0, 2)$, we have the following a priori estimate

$$\|\delta_{\varphi - \frac{1}{2}\psi} g\|_\varphi^2 + \|dg\|_\varphi^2 \geqslant \frac{(2-\alpha)^2}{4} \int_\Omega \langle F_\varphi g, g\rangle e^{-\varphi}, \tag{$**$}$$

for any p-form $g \in \mathrm{Dom}(d^*) \cap C_p^\infty(\overline{\Omega})$ on p-convex domains with smooth boundary.

7. L^2-estimates on p-convex Riemannian manifolds

We will generalize the results established in Sections 2–6 to Riemannian manifolds. To this end, we only need to take care of the curvature term which enters the a priori estimate and we will focus on such modifications.

Let (M, ds^2) be an oriented Riemannian manifold of dimension n. We denote by $R_{XY} = D_X D_Y - D_Y D_X - D_{[X,Y]}$ the curvature of the Levi-Civita connection D. Let $\{e_1, \ldots, e_n\}$ be locally defined orthonormal frame field of the tangent bundle and $\{\omega^1, \ldots, \omega^n\}$ be its dual coframe field. Since D is torsion free, the exterior differential operator d and its formal adjoint δ satisfy

$$d = \omega^i \wedge D_{e_i}, \qquad \delta = -e_i \lrcorner D_{e_i}. \tag{7.1}$$

For any $\varphi \in C^\infty(M)$, we denote as before

$$F_\varphi = \varphi_{ij}\omega^j \wedge e_i \lrcorner$$

where φ_{ij}'s are given by the Hessian $D^2\varphi := \varphi_{ij}\omega^i \otimes \omega^j$ of φ.

For our later use, we collect here some easy geometric computations.

Lemma 7.1. *For any $\varphi \in C^\infty(M)$ and any p-form $g \in C_p^\infty(M)$, we have the following identity*

$$L_{\nabla\varphi} g = D_{\nabla\varphi} g + F_\varphi g \tag{7.2}$$

where $L_{\nabla\varphi} = d\nabla\varphi\lrcorner + \nabla\varphi\lrcorner d$ *is the Lie derivative and* $\nabla\varphi$ *is the gradient of* φ*.*

Proof. By repeated use of the first formula in (7.1), we have

$$\begin{aligned} \nabla\varphi\lrcorner dg &= \nabla\varphi\lrcorner\omega^i \wedge D_{e_i} g \\ &= \langle\nabla\varphi, e_i\rangle D_{e_i} g - \omega^i \wedge \nabla\varphi\lrcorner D_{e_i} g \\ &= D_{\nabla\varphi} g - \omega^i \wedge \left[D_{e_i}(\nabla\varphi\lrcorner g) - (D_{e_i}\nabla\varphi)\lrcorner g\right] \\ &= D_{\nabla\varphi} g - \omega^i \wedge D_{e_i}(\nabla\varphi\lrcorner g) + \varphi_{ij}\omega^i \wedge e_j\lrcorner g \\ &= D_{\nabla\varphi} g + F_\varphi g - d\nabla\varphi\lrcorner g. \end{aligned}$$

The proof is complete. □

Lemma 7.2. *Let* $\Omega \Subset M$ *be an open subset with smooth boundary. For any differential forms* $f \in C^\infty_{p+1}(\overline{\Omega})$, $g \in C^\infty_p(\overline{\Omega})$, *we have the following identities*

$$\int_\Omega \langle f, dg\rangle = \int_\Omega \langle\delta f, g\rangle + \int_{\partial\Omega} \langle\nabla\rho\lrcorner f, g\rangle \frac{1}{|\nabla\rho|}, \tag{7.3}$$

$$\int_\Omega \langle\triangle g, g\rangle = -\int_\Omega |Dg|^2 + \int_{\partial\Omega} \langle D_{\nabla\rho} g, g\rangle \frac{1}{|\nabla\rho|} \tag{7.4}$$

where $\triangle := \operatorname{tr} D^2$ *is the Laplacian,* ρ *is a defining function for* Ω*, i.e.,* $\rho \in C^\infty(\overline{\Omega})$ *satisfying* $\rho < 0$ *in* Ω, $\rho = 0$ *and* $\nabla\rho \neq 0$ *on* $\partial\Omega$*.*

Proof. Set

$$X = \langle g, e_i\lrcorner f\rangle e_i, \qquad Y = \langle g, D_{e_i} g\rangle e_i,$$

it is obvious that X, Y are both well-defined smooth vector fields on $\overline{\Omega}$. By using (7.1), we see that

$$\operatorname{div} X = \langle dg, f\rangle - \langle g, \delta f\rangle \tag{7.5}$$

and that

$$\operatorname{div} Y = \langle\triangle g, g\rangle + |Dg|^2. \tag{7.6}$$

Now the divergence theorem gives the required identities (7.3), (7.4) by integrating (7.5) and (7.6) respectively. □

We use the same notation d to denote the maximal (weak) differential operator $d : L^2_{p-1}(\Omega) \to L^2_p(\Omega)$ where $\Omega \Subset M$ is a smooth open subset. We also denote the adjoint of the closed and densely defined operator by d^*. From (7.3), it is easy to see that

$$C^\infty_p(\overline{\Omega}) \cap \operatorname{Dom}\bigl(d^*\bigr) = \bigl\{g \in C^\infty_p(\overline{\Omega}) \bigm| \nabla\rho \lrcorner g = 0 \text{ on } \partial\Omega\bigr\} \tag{7.7}$$

where $1 \leqslant p \leqslant n$.

To establish the basic estimate in Section 3 on Riemannian manifolds, we first compute the integral $\int_\Omega |dg|^2 + |\delta g|^2$ for any $g \in C^\infty_p(\overline{\Omega}) \cap \operatorname{Dom}(d^*)$. From (7.7) and Lemma 7.2, it follows that

$$\begin{aligned} \int_\Omega |dg|^2 + |\delta g|^2 &= \int_\Omega \bigl\langle (d\delta + \delta d)g, g\bigr\rangle + \int_{\partial\Omega} \bigl(\langle \nabla\rho \lrcorner dg, g\rangle - \langle \nabla\rho \lrcorner g, \delta g\rangle\bigr)\frac{1}{|\nabla\rho|} \\ &= \int_\Omega \bigl\langle (d\delta + \delta d)g, g\bigr\rangle + \int_{\partial\Omega} \langle \nabla\rho \lrcorner dg, g\rangle \frac{1}{|\nabla\rho|}. \end{aligned} \tag{7.8}$$

Let us choose the orthonormal frame field $\{e_1, \ldots, e_n\}$ to be adapted to $\partial\Omega$ with

$$e_n = \frac{\nabla\rho}{|\nabla\rho|},$$

then we know by (7.7) that

$$\langle d\nabla\rho \lrcorner g, g\rangle = \sum_{\nu=1}^{n-1} \bigl\langle \omega^\nu \wedge D_{e_\nu}(\nabla\rho \lrcorner g), g\bigr\rangle + \bigl\langle D_{e_n}(\nabla\rho \lrcorner g), e_n \lrcorner g\bigr\rangle = 0 \tag{7.9}$$

holds on the boundary $\partial\Omega$. Combining (7.2), (7.8) and (7.9) gives the next identity

$$\begin{aligned} \int_\Omega |dg|^2 + |\delta g|^2 &= \int_\Omega \bigl\langle (d\delta + \delta d)g, g\bigr\rangle + \int_{\partial\Omega} \langle L_{\nabla\rho} g, g\rangle \frac{1}{|\nabla\rho|} \\ &= \int_\Omega \bigl\langle (d\delta + \delta d)g, g\bigr\rangle + \int_{\partial\Omega} \langle D_{\nabla\rho} g + F_\rho g, g\rangle \frac{1}{|\nabla\rho|}. \end{aligned} \tag{7.10}$$

To handle the first term on the right hand side of (7.10), we use the Bochner–Weitzenböck formula

$$(d\delta + \delta d)g = -\triangle g + \omega^j \wedge e_i \lrcorner R_{e_i e_j} g. \tag{7.11}$$

Recall that the curvature operator $\mathfrak{R} : \bigwedge^2 T^*M \to \bigwedge^2 T^*M$ is defined as a self-adjoint linear map by

$$\mathfrak{R}\bigl(\omega^i \wedge \omega^j\bigr) := R_{ij\ell k}\omega^k \wedge \omega^\ell$$

where $R_{ijk\ell} := \langle R_{e_i e_j} e_k, e_\ell \rangle$. It is known that (cf. [33])

$$\left\langle \omega^j \wedge e_i \lrcorner R_{e_i e_j} g, g \right\rangle = \sum_{i_1 < \cdots < i_p} \left\langle \mathfrak{R} \xi^g_{i_1 \cdots i_p}, \xi^g_{i_1 \cdots i_p} \right\rangle \tag{7.12}$$

where $\xi^g_{i_1 \cdots i_p}$ is the 2-form given by

$$\xi^g_{i_1 \cdots i_p} = \sum_{a=1}^{p} \sum_{i=1}^{n} g_{i_1 \cdots (i)_a \cdots i_p} \omega^i \wedge \omega^{i_a}. \tag{7.13}$$

By (7.4), (7.11) and (7.12), it is easy to see the following equality

$$\int_\Omega \left\langle (d\delta + \delta d) g, g \right\rangle = \int_\Omega |Dg|^2 + \sum_{i_1 < \cdots < i_p} \left\langle \mathfrak{R} \xi^g_{i_1 \cdots i_p}, \xi^g_{i_1 \cdots i_p} \right\rangle - \int_{\partial\Omega} \langle D_{\nabla\rho} g, g \rangle \frac{1}{|\nabla\rho|}. \tag{7.14}$$

Substituting (7.14) into (7.10) implies that

$$\int_\Omega |dg|^2 + |\delta g|^2 = \int_\Omega |Dg|^2 + \sum_{i_1 < \cdots < i_p} \left\langle \mathfrak{R} \xi^g_{i_1 \cdots i_p}, \xi^g_{i_1 \cdots i_p} \right\rangle + \int_{\partial\Omega} \langle F_\rho g, g \rangle \frac{1}{|\nabla\rho|} \tag{7.15}$$

holds for any $g \in C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}(d^*)$.

Now we commence introducing a weight function into the identity (7.15).

Lemma 7.3. *Let $\Omega \Subset M$ be an open subset with a defining function $\rho \in C^\infty(\overline{\Omega})$, $\varphi \in C^\infty(\overline{\Omega})$. Then for any differential form $g \in C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}(d^*)$, we have the following identity*

$$\begin{aligned} \int_\Omega \left(|dg|^2 + |\delta_\varphi g|^2\right) e^{-\varphi} = {} & \int_\Omega \left(|Dg|^2 + \sum_{i_1 < \cdots < i_p} \left\langle \mathfrak{R} \xi^g_{i_1 \cdots i_p}, \xi^g_{i_1 \cdots i_p} \right\rangle + \langle F_\varphi g, g \rangle \right) e^{-\varphi} \\ & + \int_{\partial\Omega} \langle F_\rho g, g \rangle \frac{e^{-\varphi}}{|\nabla\rho|} \end{aligned} \tag{7.16}$$

where $\delta_\varphi := e^\varphi \circ \delta \circ e^{-\varphi}$ is the formal adjoint of d with respect to the weight φ and $\xi^g_{i_1 \cdots i_p}$ is defined by (7.13).

Proof. For any $g \in C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}(d^*)$, set

$$h = e^{-\frac{\varphi}{2}} g$$

then we know by (7.7) $h \in C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}(d^*)$. The equality (7.15) applied to h gives

$$\begin{aligned}\int\limits_{\Omega}\left(|dg|^2+|\delta_\varphi g|^2\right)e^{-\varphi} &= \int\limits_{\Omega}\left|dh+\frac{1}{2}\,d\varphi\wedge h\right|^2+\left|\delta h+\frac{1}{2}\nabla\varphi\lrcorner h\right|^2\\ &= \int\limits_{\Omega}|dh|^2+|\delta h|^2+\langle L_{\nabla\varphi}h,h\rangle+\frac{1}{4}\left(|d\varphi\wedge h|^2+|\nabla\varphi\lrcorner h|^2\right)\\ &= \int\limits_{\Omega}|Dh|^2+\sum_{i_1<\cdots<i_p}\left\langle\Re\xi^h_{i_1\cdots i_p},\xi^h_{i_1\cdots i_p}\right\rangle+\langle F_\varphi h,h\rangle\\ &\quad+\int\limits_{\Omega}\langle D_{\nabla\varphi}h,h\rangle+\frac{1}{4}|d\varphi|^2|h|^2+\int\limits_{\partial\Omega}\langle F_\rho h,h\rangle\frac{1}{|\nabla\rho|}\end{aligned} \tag{7.17}$$

where we have also used (7.3) to get the second equality, (7.2) and the Lagrange identity to get the last equality. By substituting $h=e^{\frac{-\varphi}{2}}g$ into (7.17), we obtain the desired identity as follows

$$\begin{aligned}\int\limits_{\Omega}\left(|dg|^2+|\delta_\varphi g|^2\right)e^{-\varphi} &= \int\limits_{\Omega}\left(\left|Dg-\frac{1}{2}\,d\varphi\otimes g\right|^2+\sum_{i_1<\cdots<i_p}\left\langle\Re\xi^g_{i_1\cdots i_p},\xi^g_{i_1\cdots i_p}\right\rangle\right)e^{-\varphi}\\ &\quad+\int\limits_{\Omega}\left(\langle F_\varphi g,g\rangle+\left\langle D_{\nabla\varphi}g-\frac{1}{2}|\nabla\varphi|^2g,g\right\rangle\right)e^{-\varphi}\\ &\quad+\frac{1}{4}\int\limits_{\Omega}|d\varphi|^2|g|^2e^{-\varphi}+\int\limits_{\partial\Omega}\langle F_\rho g,g\rangle\frac{e^{-\varphi}}{|\nabla\rho|}\\ &= \int\limits_{\Omega}\left(|Dg|^2+\sum_{i_1<\cdots<i_p}\left\langle\Re\xi^g_{i_1\cdots i_p},\xi^g_{i_1\cdots i_p}\right\rangle+\langle F_\varphi g,g\rangle\right)e^{-\varphi}\\ &\quad+\int\limits_{\Omega}\left(\left\langle D_{\nabla\varphi}g-\frac{1}{2}|\nabla\varphi|^2g,g\right\rangle-\langle Dg,d\varphi\otimes g\rangle\right)e^{-\varphi}\\ &\quad+\frac{1}{2}\int\limits_{\Omega}|d\varphi|^2|g|^2e^{-\varphi}+\int\limits_{\partial\Omega}\langle F_\rho g,g\rangle\frac{e^{-\varphi}}{|\nabla\rho|}\\ &= \int\limits_{\Omega}\left(|Dg|^2+\sum_{i_1<\cdots<i_p}\left\langle\Re\xi^g_{i_1\cdots i_p},\xi^g_{i_1\cdots i_p}\right\rangle+\langle F_\varphi g,g\rangle\right)e^{-\varphi}\\ &\quad+\int\limits_{\partial\Omega}\langle F_\rho g,g\rangle\frac{e^{-\varphi}}{|\nabla\rho|}.\end{aligned}$$

The proof is complete. □

Before we prove the L^2-existence theorem on (M,ds^2), we need to bound the curvature term in (7.16). Set

$$\begin{aligned}
&\lambda_{\mathfrak{R}}(x) := \text{the smallest eigenvalue of } \mathfrak{R}(x), \\
&\Lambda_{\mathfrak{R}}(x) := \text{the largest eigenvalue of } \mathfrak{R}(x)
\end{aligned} \tag{7.18}$$

for any $x \in M$. Then we have, for any p-form g, the following pointwise inequalities for the curvature term $\sum_{i_1<\cdots<i_p} \langle \mathfrak{R}\xi^g_{i_1\cdots i_p}, \xi^g_{i_1\cdots i_p} \rangle$ in (7.16).

Lemma 7.4.

$$p(n-p)\lambda_{\mathfrak{R}}|g|^2 \leqslant \sum_{i_1<\cdots<i_p} \langle \mathfrak{R}\xi^g_{i_1\cdots i_p}, \xi^g_{i_1\cdots i_p} \rangle \leqslant p(n-p)\Lambda_{\mathfrak{R}}|g|^2 \tag{7.19}$$

where the $\xi^g_{i_1\cdots i_p}$'s are defined by (7.13).

Proof. By the definition of $\xi^g_{i_1\cdots i_p}$, we get

$$\begin{aligned}
&\sum_{i_1<\cdots<i_p} \langle \mathfrak{R}\xi^g_{i_1\cdots i_p}, \xi^g_{i_1\cdots i_p} \rangle \\
&\geqslant \lambda_{\mathfrak{R}} \sum_{i_1<\cdots<i_p} \left[\sum_{a=1}^{p}\sum_{i=1}^{n} g^2_{i_1\cdots(i)_a\cdots i_p} - \sum_{a,b=1}^{p} g_{i_1\cdots(i_b)_a\cdots i_p} g_{i_1\cdots(i_a)_b\cdots i_p} \right] \\
&= \lambda_{\mathfrak{R}} \sum_{i_1<\cdots<i_p} \left[\sum_{a=1}^{p}\sum_{i=1}^{n} g^2_{i_1\cdots(i)_a\cdots i_p} - p g^2_{i_1\cdots i_p} \right] \\
&= \lambda_{\mathfrak{R}} \sum_{i_1<\cdots<i_p} \sum_{a=1}^{p} \sum_{i\neq i_1,\ldots,i_p} g^2_{i_1\cdots(i)_a\cdots i_p} \\
&= \lambda_{\mathfrak{R}} \sum_{j_1<\cdots<j_p} \sum_{i_1<\cdots<i_p} \sum_{a=1}^{p} \sum_{i\neq i_1,\ldots,i_p} \operatorname{sgn} \begin{pmatrix} i_1\cdots(i)_a\cdots i_p \\ j_1\cdots\cdots\cdots\cdots j_p \end{pmatrix}^2 g^2_{j_1\cdots j_p},
\end{aligned}$$

where sgn denotes the signature of permutation and we have used the identity

$$g_{i_1\cdots(i)_a\cdots i_p} = \sum_{j_1<\cdots<j_p} \operatorname{sgn} \begin{pmatrix} i_1\cdots(i)_a\cdots i_p \\ j_1\cdots\cdots\cdots\cdots j_p \end{pmatrix} g_{j_1\cdots j_p}.$$

For fixed $j_1 < \cdots < j_p$ we have

$$\sum_{i_1<\cdots<i_p} \sum_{a=1}^{p} \sum_{i\neq i_1,\ldots,i_p} \operatorname{sgn} \begin{pmatrix} i_1\cdots(i)_a\cdots i_p \\ j_1\cdots\cdots\cdots\cdots j_p \end{pmatrix}^2 = p(n-p),$$

because only terms given by $\{i_1,\ldots,i_p\} = \{j_1,\ldots,\widehat{j_a},\ldots,j_p,k\}$ ($k \neq j_1,\ldots,j_p$ and $1 \leqslant a \leqslant p$) contribute to the sum. We can therefore rewrite the above inequality as

$$\sum_{i_1<\cdots<i_p} \langle \mathfrak{R}\xi^g_{i_1\cdots i_p}, \xi^g_{i_1\cdots i_p} \rangle \geqslant p(n-p)\lambda_{\mathfrak{R}} \sum_{j_1<\cdots<j_p} g^2_{j_1\cdots j_p}.$$

The second inequality can be proved in the same manner, and the proof is complete. □

In order to establish L^2-existence theorem, we will use $d : L^2_{p-1}(\Omega,\varphi) \to L^2_p(\Omega,\varphi)$, the maximal (weak) differential operator between the weighted L^2-spaces. Let $d^*_\varphi : L^2_p(\Omega,\varphi) \to L^2_{p-1}(\Omega,\varphi)$ be the adjoint operator. As mentioned before, we know by (7.3) that the formal adjoint of d with respect to the weight is given by δ_φ, and consequently we have

$$\begin{aligned} C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}\big(d^*_\varphi\big) &= C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}\big(d^*\big) \\ &= \big\{g \in C^\infty_p(\overline{\Omega}) \bigm| \nabla\rho \lrcorner g = 0 \text{ on } \partial\Omega\big\}. \end{aligned} \tag{7.20}$$

Now we are in the position to prove the main result of this section.

Theorem 7.1. *Let (M, ds^2) be an n-dimensional oriented p-convex Riemannian manifold. Let $\varphi \in C^2(M)$ be a p-plurisubharmonic function on M. If $F_\varphi + p(n-p)\lambda_{\mathfrak{R}}\,\mathrm{Id}$ is p-positive semi-definite on M, then for any closed p-form $f \in L^2_p(M, \mathrm{Loc})$ with*

$$\int_M \langle [F_\varphi + p(n-p)\lambda_{\mathfrak{R}}\,\mathrm{Id}]^{-1} f, f\rangle e^{-\varphi} < \infty,$$

there exists some $(p-1)$-form $u \in L^2_{p-1}(M,\varphi)$ such that

$$du = f \quad \text{and} \quad \int_M |u|^2 e^{-\varphi} \leqslant \int_M \langle [F_\varphi + p(n-p)\lambda_{\mathfrak{R}}\,\mathrm{Id}]^{-1} f, f\rangle e^{-\varphi}$$

where $1 \leqslant p \leqslant n$, $[F_\varphi + p(n-p)\lambda_{\mathfrak{R}}\,\mathrm{Id}]^{-1}$ is defined by (1.6) and $\lambda_{\mathfrak{R}}$ is given by (7.18). Moreover, if f and φ are both assumed additionally to be smooth then we can choose u to be a smooth form.

Proof. It has been proved in [20] that M admits a smooth p-plurisubharmonic proper exhaustion function, so M itself can be exhausted by compact open subsets with smooth p-convex boundary. Since the resulting L^2-estimate enables us to apply the standard argument of approximation to take weak limit, we only need to work on a smooth domain $\Omega \Subset M$ which has p-convex boundary. From (7.16), (7.19) and (7.20), it follows that

$$\int_\Omega \big(|dg|^2 + |\delta_\varphi g|^2\big) e^{-\varphi} \geqslant \int_\Omega \langle [F_\varphi + p(n-p)\lambda_{\mathfrak{R}}] g, g\rangle e^{-\varphi} \tag{7.21}$$

where $g \in C^\infty_p(\overline{\Omega}) \cap \mathrm{Dom}(d^*_\varphi)$. By Hörmander's density lemma (see [22] or [23]), we know that (7.20) holds for any $g \in \mathrm{Dom}(d^*_\varphi)$. Now the desired result follows from the estimate (7.21) and Lemma 3.1. For the regularity, we can apply the procedure in Section 5 to get the minimal solution in $L^2_{p-1}(M,\varphi)$ and the interior regularity then follows from the ellipticity of $d\delta_\varphi + \delta_\varphi d$. $\square$

Remark 7.1. By results in [26] and [28], we know that the curvature term

$$\sum_{i_1<\cdots<i_p} \langle \mathfrak{R}\xi^g_{i_1\cdots i_p}, \xi^g_{i_1\cdots i_p}\rangle \geqslant 0$$

for $p = 2$ when (M, ds^2) is assumed to have nonnegative complex sectional curvature (isotropic sectional curvature when n is even). In this case, we have instead of (7.21) the following a priori estimate

$$\int_\Omega \big(|dg|^2 + |\delta_\varphi g|^2\big)e^{-\varphi} \geqslant \int_\Omega \langle F_\varphi g, g\rangle e^{-\varphi} \tag{7.22}$$

for any $g \in C^\infty_2(\overline{\Omega}) \cap \mathrm{Dom}(d^*_\varphi)$. The same argument for Theorem 7.1 also implies the following result:

Let (M, ds^2) be an n-dimensional oriented 2-convex Riemannian manifold. Let $\varphi \in C^2(M)$ be a 2-plurisubharmonic function on M. If (M, ds^2) has nonnegative complex sectional curvature (isotropic sectional curvature when n is even), then for any closed 2-form $f \in L^2_2(M, \mathrm{Loc})$ with

$$\int_M \langle F^{-1}_\varphi f, f\rangle e^{-\varphi} < \infty,$$

there exists some 1-form $u \in L^2_1(M,\varphi)$ such that

$$du = f \quad \text{and} \quad \int_M |u|^2 e^{-\varphi} \leqslant \int_M \langle F^{-1}_\varphi f, f\rangle e^{-\varphi}.$$

Moreover, if f and φ are both assumed additionally to be smooth then we can choose u to be a smooth form.

As an easy corollary, we have the following result which is a generalization of Theorem 3.1 to Riemannian manifolds with nonnegative curvature operator.

Corollary 7.1. *Assume that (M, ds^2) is p-convex and has nonnegative curvature operator. Let $\varphi \in C^2(M)$ be a p-plurisubharmonic function on M. Then for any closed p-form $f \in L^2_p(M, \mathrm{Loc})$ with*

$$\int_M \langle F^{-1}_\varphi f, f\rangle e^{-\varphi} < \infty,$$

there exists some $(p-1)$*-form* $u \in L^2_{p-1}(M,\varphi)$ *such that*

$$du = f \quad and \quad \int_M |u|^2 e^{-\varphi} \leqslant \int_M \langle F_\varphi^{-1} f, f\rangle e^{-\varphi}.$$

Moreover, if f *and* φ *are both assumed to be smooth then we can choose* u *to be a smooth form. When* $p = 2$*, it is enough to assume* (M, ds^2) *has nonnegative complex sectional curvature* (*isotropic sectional curvature when* n *is even*)*.*

Remark 7.2. All the results in Sections 2–6 can be established on Riemannian manifolds without any additional difficulty. For Theorem 2.1, the minor difference is that the Levi-Civita connection D enters the derivatives and the gradient is taken with respect to the underlying metric. To prove, on Riemannian manifolds, these L^2-estimates obtained in Sections 3–6, the only modification is to use the estimate (7.21) to replace (3.2) (or use Theorem 7.1 to replace Theorem 3.1).

8. Geometric applications

In this section, we will prove vanishing and finiteness theorems for de Rham cohomology groups. The key is to control the curvature term (in the basic estimate (7.21)) by choosing appropriate weight functions. The main tool is a Carleman type estimate (Lemma 8.4) which is uniform with respect to both of weights and domains. To establish such an estimate, we will first prove a Gårding type estimate (Lemma 8.1) which is also uniform w.r.t. domains and weights. Since the notion of p-convexity depends on the underlying metric, we do not have the flexibility in the way of modifying the metric as the complex analytic case (cf. [2] and [14]).

Solving $du = f$ in appropriate weighted L^2-space, we have the following immediate corollary of Theorem 7.1.

Proposition 8.1. *Let* (M, ds^2) *be a strictly* p*-convex* n*-dimensional Riemannian manifold,* $1 \leqslant p \leqslant n$*. Then for any closed* $f \in L^2_q(M, \mathrm{Loc})$ $(p \leqslant q \leqslant n)$ *there exists some* $(q-1)$*-form* $u \in L^2_{q-1}(M, \mathrm{Loc})$ *such that* $du = f$*. In particular, the de Rham cohomology group* $H^q(M, \mathbb{R}) = 0$ *for every* $p \leqslant q \leqslant n$*.*

Proof. Since strict p-convexity implies strict $(p+1)$-convexity, it suffices to consider the case $q = p$. By using Lemma 1.2(i) with $\omega = p(n-p)\lambda_{\mathfrak{R}}$, one can find a p-plurisubharmonic proper exhaustion function $\varphi \in C^\infty(M)$ such that $F_\varphi + p(n-p)\lambda_{\mathfrak{R}} \operatorname{Id}$ is p-positive definite on M. Then $\langle [F_\varphi + p(n-p)\lambda_{\mathfrak{R}} \operatorname{Id}]^{-1} f, f\rangle$ is a continuous function on M. By Lemma 1.2(iii), one can find a function $\psi \in C^\infty(M)$ such that $\psi - \varphi$ is p-plurisubharmonic and $\int_M \langle [F_\varphi + p(n-p)\lambda_{\mathfrak{R}} \operatorname{Id}]^{-1} f, f\rangle e^{-\psi} < \infty$. Consequently, we have

$$\int_M \big\langle \big[F_\psi + p(n-p)\lambda_{\mathfrak{R}} \operatorname{Id}\big]^{-1} f, f\big\rangle e^{-\psi} \leqslant \int_M \big\langle \big[F_\varphi + p(n-p)\lambda_{\mathfrak{R}} \operatorname{Id}\big]^{-1} f, f\big\rangle e^{-\psi} < \infty.$$

It follows from Theorem 7.1 that there exists some $(p-1)$-form $u \in L^2_{p-1}(M,\psi)$ such that $du = f$. To see the vanishing of $H^p(M,\mathbb{R})$, it is enough to consider the minimal solution of $du = f$ in $L^2_{p-1}(M,\psi)$ which is smooth provided $f \in C^\infty_p(M)$. The proof is complete. □

Remark 8.1. (i) As observed by Harvey and Lawson, Proposition 8.1 also follows from Morse theory (see Theorem 4.16 in [19]).

(ii) By making an additional assumption on the sectional curvature, we can prove the following vanishing result for Riemannian manifolds which are strictly p-convex at infinity. Since (M, ds^2) is strictly p-convex at infinity, M can be exhausted by open subsets with strictly p-convex boundary $\Omega_1 \Subset \Omega_2 \Subset \cdots$. When (M, ds^2) is assumed to have nonnegative sectional curvature, by the main theorem in [29], we obtain

$$H^q(\Omega_\nu,\mathbb{R}) = 0 \quad \text{for each } \nu \geqslant 1 \text{ and } p \leqslant q \leqslant n.$$

Taking the inverse limit implies that

$$H^q(M,\mathbb{R}) \cong \varprojlim H^q(\Omega_\nu,\mathbb{R}) = 0 \quad \text{for } p \leqslant q \leqslant n.$$

This is a generalization of Corollary 5.1.

Combining the inequalities (8.2) and (8.3) below, we will get a Gårding type estimate which is **uniform** with respect to both p-convex domains $\Omega \Subset M$ and p-plurisubharmonic weight functions $\varphi \in C^2(M)$ satisfying the condition (8.1) below. The existence of such a weight is given by Lemma 1.2(ii). In the sequel, we will denote by $(d|_\Omega)^*_\varphi$ the adjoint of the maximal differential operator $d|_\Omega : L^2_q(\Omega,\varphi) \to L^2_{q+1}(\Omega,\varphi)$.

Lemma 8.1. *Let M be an oriented n-dimensional manifold. Let φ be a C^2 function which is p-plurisubharmonic on M and satisfies*

$$F_\varphi + \Big[\max_{p\leqslant \ell\leqslant n} \ell(n-\ell)\lambda_{\mathfrak{R}} - 1\Big]\,\mathrm{Id} \text{ is } p\text{-positive outside a compact subset } S \subseteq M. \tag{8.1}$$

For any bounded open set Ω with p-convex boundary and any open neighborhoods $U \Subset U_1 \subseteq \Omega$ of S in Ω, there is a constant $A = A(S,U,U_1) > 0$ such that

$$\int_\Omega \big(|dg|^2 + \big|(d|_\Omega)^*_\varphi g\big|^2 + |g|^2\big)e^{-\varphi} \geqslant A\int_U |Dg|^2 e^{-\varphi} \tag{8.2}$$

and

$$\int_\Omega \big(|dg|^2 + \big|(d|_\Omega)^*_\varphi g\big|^2\big)e^{-\varphi} + \int_U |g|^2 e^{-\varphi} \geqslant A\int_\Omega |g|^2 e^{-\varphi} \tag{8.3}$$

hold for every $g \in \mathrm{Dom}(d|_\Omega)^*_\varphi \cap \mathrm{Dom}(d|_\Omega) \subseteq L^2_q(\Omega,\varphi)$, $p \leqslant q \leqslant n$.

Proof. Let $g \in C_q^\infty(\overline{\Omega})$, $p \leqslant q \leqslant n$. Choose a cut-off function $\chi_1 \in C^\infty(\Omega)$ such that

$$\chi_1|_U \equiv 1 \quad \text{and} \quad \operatorname{Supp}\chi_1 \subseteq U_1.$$

It follows from (7.16) and (7.19) that

$$\begin{aligned}\int_\Omega \left(\left|d(\chi_1 g)\right|^2 + \left|\delta_\varphi(\chi_1 g)\right|^2\right)e^{-\varphi} &\geqslant \int_\Omega \left(\left|D(\chi_1 g)\right|^2 - q(n-q)A_1\left|(\chi_1 g)\right|^2\right)e^{-\varphi} \\ &\geqslant \int_U |Dg|^2 e^{-\varphi} - q(n-q)A_1 \int_\Omega |\chi_1 g|^2 e^{-\varphi}\end{aligned}$$

where $A_1 > 0$ is a constant such that $\lambda_{\mathfrak{R}} \geqslant -A_1$ on U_1. Therefore, we obtain

$$\int_\Omega \left\{|dg|^2 + |\delta_\varphi g|^2 + A_2^2\left[q(n-q)A_1 + 2\right]|g|^2\right\}e^{-\varphi} \geqslant \frac{1}{2}\int_U |Dg|^2 e^{-\varphi} \tag{8.4}$$

where $A_2 := \sup_\Omega\left(\frac{|\chi_1|}{\sqrt{2}} + |\nabla\chi_1|\right)$ and $g \in C_q^\infty(\overline{\Omega})$.

Let χ_2 be a smooth function on Ω satisfying

$$\chi_2|_S \equiv 0 \quad \text{and} \quad \chi_2|_{\Omega\setminus U} \equiv 1.$$

Set $A_3 := \sup_\Omega |\nabla\chi_2|$. For any $g \in C_q^\infty(\overline{\Omega}) \cap \operatorname{Dom}((d|_\Omega)_\varphi^*)$, by using (7.16) and (7.19) again, we have

$$\begin{aligned}\int_\Omega \left(\left|d(\chi_2 g)\right|^2 + \left|\delta_\varphi(\chi_2 g)\right|^2\right)e^{-\varphi} &\geqslant \int_\Omega \left\langle\left(F_\varphi + q(n-q)\lambda_{\mathfrak{R}}\operatorname{Id}\right)\chi_2 g, \chi_2 g\right\rangle e^{-\varphi} \\ &\geqslant \int_\Omega |\chi_2 g|^2 e^{-\varphi} \\ &\geqslant \int_{\Omega\setminus U} |g|^2 e^{-\varphi}\end{aligned}$$

which implies that

$$\int_\Omega \left(|dg|^2 + |\delta_\varphi g|^2\right)e^{-\varphi} + 2A_3^2\int_U |g|^2 e^{-\varphi} \geqslant \frac{1}{2}\int_{\Omega\setminus U} |g|^2 e^{-\varphi}$$

and consequently,

$$\int_\Omega \left(|dg|^2 + |\delta_\varphi g|^2\right)e^{-\varphi} + \left(2A_3^2 + \frac{1}{2}\right)\int_U |g|^2 e^{-\varphi} \geqslant \frac{1}{2}\int_\Omega |g|^2 e^{-\varphi}. \tag{8.5}$$

By Hörmander's density lemma (cf. [22] or [23]), the estimates (8.4) and (8.5) are both valid for $g \in \mathrm{Dom}(d|_\Omega)^*_\varphi \cap \mathrm{Dom}(d|_\Omega) \subseteq L^2_q(\Omega, \varphi), p \leqslant q \leqslant n$. □

By a compactness argument, the next result follows from Lemma 8.1.

Lemma 8.2. *Let $\varphi \in C^2(M)$ be a p-plurisubharmonic function satisfying (8.1). For any bounded open set Ω with p-convex boundary which contains the subset S in (8.1), $\mathrm{Ker}(d|_\Omega)^*_\varphi \cap \mathrm{Ker}(d|_\Omega) \subseteq L^2_q(\Omega, \varphi)$ is finite dimensional and we have the orthogonal decomposition*

$$\mathrm{Ker}(d|_\Omega) = \big(\mathrm{Ker}(d|_\Omega)^*_\varphi \cap \mathrm{Ker}(d|_\Omega)\big) \oplus \mathrm{Im}(d|_\Omega) \subseteq L^2_q(\Omega, \varphi), \quad p \leqslant q \leqslant n. \tag{8.6}$$

Proof. Fix open neighborhoods $U \Subset U_1 \subseteq \Omega$ of S in Ω such that U has smooth boundary. Let $\{g_\nu\} \subseteq \mathrm{Ker}(d|_\Omega)^*_\varphi \cap \mathrm{Ker}(d|_\Omega)$ be a sequence of q-forms with $\|g_\nu\|_\varphi$ bounded and $\|dg_\nu\|_\varphi \to 0, \|(d|_\Omega)^*_\varphi g_\nu\|_\varphi \to 0$. In view of (8.2) and the Rellich–Kondrachov theorem, we can pass to a subsequence and thereby assume that $\{g_\nu|_{\Omega_1}\}$ converges in $L^2_q(U, \varphi)$. On the other hand, (8.3) implies that $\{g_\nu\}$ is a Cauchy sequence in $L^2_q(\Omega, \varphi)$. Therefore, there exists a $g \in L^2_q(\Omega, \varphi)$ such that $g_\nu \to g$ in $L^2_q(\Omega, \varphi)$. By applying Lemma 8.3 below to the weighted L^2-de Rham complex

$$\cdots \to L^2_{q-1}(\Omega, \varphi) \xrightarrow{T=d|_\Omega} L^2_q(\Omega, \varphi) \xrightarrow{S=d|_\Omega} L^2_q(\Omega, \varphi) \to \cdots$$

we get the desired results. □

In the proof of Lemma 8.2, we have used the following result.

Lemma 8.3. *(See Theorems 1.12 and 1.13 in [22].) Let $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$ be a complex of closed and densely defined operators between Hilbert spaces. Assume that from every sequence $g_\nu \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S)$ with $\|g_\nu\|_{H_2}$ bounded and $T^* g_\nu \to 0$ in H_1, $Sg_\nu \to 0$ in H_3, one can select a strongly convergent subsequence. Then there exists a constant $C > 0$ such that*

$$\|g\|^2_{H_2} \leqslant C^2 \big(\big\|T^* g\big\|^2_{H_1} + \|Sg\|^2_{H_3}\big) \tag{8.7}$$

holds for any $g \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S) \cap (\mathrm{Ker}\, T^* \cap \mathrm{Ker}\, S)^\perp$ and $\mathrm{Ker}\, T^* \cap \mathrm{Ker}\, S$ is finite dimensional. Moreover, when the above estimate (8.7) holds, we also have the following orthogonal decomposition

$$\mathrm{Ker}\, S = \big(\mathrm{Ker}\, T^* \cap \mathrm{Ker}\, S\big) \oplus \mathrm{Im}\, T. \tag{8.8}$$

Remark 8.2. Since $L^2_*(\Omega, \varphi) = L^2_*(\Omega)$, in the orthogonal decomposition (8.6), the left hand side and the second summand on the right hand side are independent of the choice of φ. Different choices of φ result in different complementary subspaces of $\mathrm{Im}(d|_\Omega)$ in $\mathrm{Ker}(d|_\Omega)$.

We can deduce from Lemma 8.1 a Carleman type inequality which is uniform with respect to an increasing sequence of open subsets and weight functions. To formulate such estimates, we introduce an increasing sequence of convex increasing functions $\chi_\nu \in C^2(\mathbb{R})$, $\nu = 1, 2, \ldots$, such that

$$\chi_\nu(t) \equiv 0 \quad \text{for } t \leqslant 0 \text{ and } \nu = 1, 2, \ldots, \qquad \lim_{\nu \to +\infty} \chi_\nu(t) = +\infty \quad \text{for } t > 0. \tag{8.9}$$

Lemma 8.4. *Let $\varphi \in C^2(M)$ be a p-plurisubharmonic function satisfying (8.1). Assume that the subset S in (8.1) is contained in $U := \{x \in M \mid \varphi(x) < 0\}$ and that U has smooth boundary. Then for any sequence $\Omega_1 \Subset \Omega_2 \Subset \cdots \subseteq M$ of smooth open subsets with p-convex boundary such that*

$$U \Subset \bigcup_{\nu \geqslant 1} \Omega_\nu, \tag{8.10}$$

there exist constants $C > 0$ and $\nu_0 > 0$ such that

$$\int_{\Omega_\mu} |f|^2 e^{-\varphi - \chi_\nu \circ \varphi} \leqslant C^2 \int_{\Omega_\mu} \big(\big| (d|_{\Omega_\mu})^*_{\varphi + \chi_\nu \circ \varphi} f \big|^2 + |df|^2 \big) e^{-\varphi - \chi_\nu \circ \varphi} \tag{8.11}$$

for every $\mu, \nu \geqslant \nu_0$ and every $f \in \mathrm{Dom}((d|_{\Omega_\mu})^*_{\varphi + \chi_\nu \circ \varphi}) \cap \mathrm{Dom}(d|_{\Omega_\mu}) \subseteq L^2_q(\Omega_\mu, \varphi + \chi_\nu \circ \varphi)$ which satisfies

$$\int_U \langle f, g \rangle e^{-\varphi} = 0, \quad \forall g \in \mathrm{Ker}\big((d|_U)^*_\varphi\big) \cap \mathrm{Ker}(d|_U) \subseteq L^2_q(U, \varphi), \tag{8.12}$$

where $p \leqslant q \leqslant n$ and $\{\chi_\nu\}$ is any increasing sequence which consists of convex increasing functions satisfying (8.9).

Proof. We proceed by contradiction. Since $U \Subset \bigcup_{\nu \geqslant 1} \Omega_\nu$, we can assume, without loss of generality, that $U \Subset \Omega_1$.

It is easy to see that each $\varphi + \chi_\nu \circ \varphi$ ($\nu \geqslant 1$) satisfies the condition (8.1) with the same subset S. By Lemma 8.1, we know that (8.2) and (8.3) hold for all subsets Ω_μ and weight functions $\varphi + \chi_\nu \circ \varphi$ ($\mu, \nu \geqslant 1$). It is easy to see, by fixing an open set U_1 such that $U \Subset U_1 \subseteq \Omega_1$, that the constant A in Lemma 8.1 is independent of $\mu, \nu \geqslant 1$.

If the conclusion were false, by passing to subsequences of $\{\Omega_\mu\}_{\mu \geqslant 1}$ and $\{\chi_\nu\}_{\nu \geqslant 1}$ (as the conditions (8.9) and (8.10) are both fulfilled for any subsequence), we would assume that there exists a sequence of $f_\nu \in \mathrm{Dom}((d|_{\Omega_\nu})^*_{\varphi + \chi_\nu \circ \varphi}) \cap \mathrm{Dom}(d|_{\Omega_\nu}) \subseteq L^2_q(\Omega_\nu, \varphi + \chi_\nu \circ \varphi)(\nu \geqslant 1)$ with the following properties

$$\int_{\Omega_\nu} |f_\nu|^2 e^{-\varphi - \chi_\nu \circ \varphi} = 1, \tag{8.13}$$

$$\int_{\Omega_\nu} \left(\left|(d|_{\Omega_\nu})^*_{\varphi+\chi_\nu\circ\varphi} f_\nu\right|^2 + |df_\nu|^2\right) e^{-\varphi-\chi_\nu\circ\varphi} \leqslant \nu^{-1}, \tag{8.14}$$

$$\int_U \langle f_\nu, g\rangle e^{-\varphi} = 0, \quad \forall g \in \operatorname{Ker}\left((d|_U)^*_\varphi\right) \cap \operatorname{Ker}(d|_U) \subseteq L^2_q(U,\varphi). \tag{8.15}$$

By (8.2), (8.13) and (8.14), we get

$$\begin{aligned}
\int_U |Df_\nu|^2 e^{-\varphi} &= \int_U |Df_\nu|^2 e^{-\varphi-\chi_\nu\circ\varphi} \\
&\leqslant A^{-1} \int_{\Omega_\nu} \left(|f_\nu|^2 + \left|(d|_{\Omega_\nu})^*_{\varphi+\chi_\nu\circ\varphi} f_\nu\right|^2 + |df_\nu|^2\right) e^{-\varphi-\chi_\nu\circ\varphi} \\
&\leqslant A^{-1}\left(1+\nu^{-1}\right)
\end{aligned}$$

and

$$\int_U |f_\nu|^2 e^{-\varphi} = \int_U |f_\nu|^2 e^{-\varphi-\chi_\nu\circ\varphi} \leqslant \int_{\Omega_\nu} |f_\nu|^2 e^{-\varphi-\chi_\nu\circ\varphi} = 1.$$

The Rellich–Kondrachov theorem implies that we may assume, by passing to a subsequence, that

$$\lim_{\nu\to+\infty} f_\nu = f \quad \text{in } L^2_q(U,\varphi). \tag{8.16}$$

Taking into account (8.14), we also have

$$\lim_{\nu\to+\infty} df_\nu = 0 \quad \text{in } L^2_q(U,\varphi)$$

which implies that

$$f \in \operatorname{Ker}(d|_U) \subseteq L^2_q(U,\varphi). \tag{8.17}$$

Taking limit in (8.15), we obtain

$$\int_U \langle f, g\rangle e^{-\varphi} = 0, \quad \forall g \in \operatorname{Ker}\left((d|_U)^*_\varphi\right) \cap \operatorname{Ker}(d|_U) \subseteq L^2_q(U,\varphi). \tag{8.18}$$

Now set

$$g_\nu = e^{-\chi_\nu\circ\varphi} f_\nu \quad (\nu \geqslant 1).$$

By using (8.13) and (8.14) respectively, we have

$$\int_{\Omega_1} |g_\nu|^2 e^{-\varphi+\chi_\mu\circ\varphi} \leqslant \int_{\Omega_\nu} |g_\nu|^2 e^{-\varphi+\chi_\nu\circ\varphi} = \int_{\Omega_\nu} |f_\nu|^2 e^{-\varphi-\chi_\nu\circ\varphi} = 1, \tag{8.19}$$

and

$$\int_{\Omega_1} \left|(d|_{\Omega_\nu})^*_\varphi g_\nu\right|^2 e^{-\varphi} \leqslant \int_{\Omega_\nu} \left|(d|_{\Omega_\nu})^*_\varphi g_\nu\right|^2 e^{-\varphi+\chi_\nu\circ\varphi}$$
$$= \int_{\Omega_\nu} \left|(d|_{\Omega_\nu})^*_{\varphi+\chi_\nu\circ\varphi} f_\nu\right|^2 e^{-\varphi-\chi_\nu\circ\varphi} \leqslant \nu^{-1} \tag{8.20}$$

for any $\nu \geqslant \mu \geqslant 1$. By (8.19), we may assume

$$g_\nu \xrightarrow{\mathrm{w}} g \quad \text{in } L^2_q(\Omega_1, \varphi - \chi_\mu \circ \varphi) \text{ as } \nu \to +\infty \tag{8.21}$$

for any $\mu \geqslant 1$. Combining (8.19) and (8.21) gives

$$\int_{\Omega_1} |g|^2 e^{-\varphi+\chi_\mu\circ\varphi} \leqslant 1$$

for any $\mu \geqslant 1$, which implies that

$$\operatorname{Supp} g \subseteq U. \tag{8.22}$$

By (8.21), we know that

$$g_\nu|_{\Omega_1} \to g \quad \text{in the sense of distribution} \tag{8.23}$$

as $\nu \to +\infty$. Consequently,

$$\delta_\varphi g_\nu|_{\Omega_1} \to \delta_\varphi g \tag{8.24}$$

in the sense of distribution, as $\nu \to +\infty$. Meanwhile, we know by (8.20) that

$$\delta_\varphi g_\nu|_{\Omega_1} \to 0 \quad \text{in } L^2_q(\Omega_1, \varphi) \text{ as } \nu \to +\infty. \tag{8.25}$$

Combining (8.24) and (8.25), we get

$$\delta_\varphi g = 0 \quad \text{on } \Omega_1 \tag{8.26}$$

in the sense of distribution. From (8.22) and (8.26), it follows that

$$g|_U \in \operatorname{Ker}\big((d|_U)^*_\varphi\big). \tag{8.27}$$

By the definition of g_ν, we know

$$g_\nu = f_\nu \quad \text{on } U$$

which, together with (8.16), (8.17) and (8.23), implies that

$$g|_U = f \in \operatorname{Ker}(d|_U) \subseteq L^2_q(U, \varphi). \tag{8.28}$$

From (8.18), (8.27) and (8.28), it follows that

$$\lim_{\nu \to +\infty} f_\nu|_U = f = 0 \quad \text{in } L^2_q(U, \varphi). \tag{8.29}$$

On the other hand, by (8.3), (8.13) and (8.14), we have

$$\begin{aligned} \nu^{-1} + \int\limits_U |f_\nu|^2 e^{-\varphi} &\geqslant \int\limits_{\Omega_\nu} \left(\left| (d|_{\Omega_\nu})^*_{\varphi+\chi_\nu\circ\varphi} f_\nu \right|^2 + |df_\nu|^2 \right) e^{-\varphi-\chi_\nu\circ\varphi} + \int\limits_U |f_\nu|^2 e^{-\varphi} \\ &\geqslant A \int\limits_{\Omega_\nu} |f_\nu|^2 e^{-\varphi-\chi_\nu\circ\varphi} = A. \end{aligned}$$

Letting $\nu \to +\infty$ and using (8.29), we get the contradiction $0 \geqslant A$ which completes the proof. □

Theorem 8.1. *Let (M, ds^2) be an n-dimensional Riemannian manifold which is strictly p-convex at infinity $(1 \leqslant p \leqslant n)$. Then the de Rham cohomology group $H^q(M, \mathbb{R})$ is finite dimensional for every $p \leqslant q \leqslant n$.*

Proof. Let $\pi : \widetilde{M} \to M$ be the orientation covering of M, then we know by definition that $\widetilde{M}$, endowed with the pulled back metric $\pi^* ds^2$, is again strictly p-convex at infinity. Since $\pi : \widetilde{M} \to M$ is a double covering, the induced homomorphism $\pi^* : H^q(M, \mathbb{R}) \to H^q(\widetilde{M}, \mathbb{R})$ is injective for every q. By passing to $\widetilde{M}$, we may assume without loss of generality that M is oriented. We will deduce Theorem 8.1 as a consequence of Lemmas 8.2 and 8.4 in the following three steps.

Step 1. By Lemma 1.2(ii), there is a proper exhaustion function $\varphi \in C^\infty(M)$ satisfying the hypotheses of Lemma 8.4 where $\Omega_\nu := \{x \in M \mid \varphi(x) < \nu\}$, $\nu = 1, 2, \ldots$. From Lemma 8.2 (choose Ω to be the subset U in Lemma 8.4), it is sufficient to prove that the natural homomorphism from the de Rham cohomology $H^q(M, \mathbb{R})$ to the L^2-cohomology $L^2H^q(U) := \frac{\{f \in L^2_q(U) | df = 0\}}{\{du \in L^2_q(U) | u \in L^2_{q-1}(U)\}}$, given by the pullback of the inclusion map, is injective for any $p \leqslant q \leqslant n$.

Step 2. By Corollary 3.1, we have the following fine resolution of the constant sheaf $\mathbb{R}$

$$0 \to \mathbb{R} \to \mathscr{A}_0 \xrightarrow{d} \mathscr{A}_1 \xrightarrow{d} \mathscr{A}_2 \xrightarrow{d} \cdots$$

where $\mathscr{A}_q(V) := \{f \in L_q^2(V, \mathrm{Loc}) \mid df \in L_{q+1}^2(V, \mathrm{Loc})\}$ for any open subset $V \subseteq M$ and $0 \leqslant q \leqslant n$. Hence it suffices to show that for any closed q-form $h \in L_q^2(M, \mathrm{Loc})$ if

$$h|_U = du \quad \text{where } u \in L_{q-1}^2(U) \tag{8.30}$$

then there exists a $(q-1)$-form $\widetilde{u} \in L_{q-1}^2(M, \mathrm{Loc})$ such that $d\widetilde{u} = h$ holds on M in the sense of distribution.

Step 3. By Lemma 1.2(iii), one can find some function $\psi \in C^\infty(M)$ such that $\varphi \equiv \psi$ when $\varphi < 1$, $\psi - \varphi$ is p-plurisubharmonic and that

$$\int_M |h|^2 e^{-\psi} < +\infty. \tag{8.31}$$

It is easy to see that ψ still satisfies the hypotheses of Lemma 8.4 with the same S and U. By Lemma 8.4, there are constants $C > 0$ and $\nu_0 > 0$ such that

$$\int_{\Omega_\nu} |f|^2 e^{-\psi - \chi_{\nu_0} \circ \psi} \leqslant C^2 \int_{\Omega_\nu} \left(\left|(d|_{\Omega_\nu})^*_{\psi + \chi_{\nu_0} \circ \psi} f\right|^2 + |df|^2\right) e^{-\psi - \chi_{\nu_0} \circ \psi}$$

holds for every $f \in \mathrm{Dom}((d|_{\Omega_\nu})^*_{\psi + \chi_{\nu_0} \circ \psi}) \cap \mathrm{Dom}(d|_{\Omega_\nu}) \subseteq L_q^2(\Omega_\nu, \psi + \chi_{\nu_0} \circ \psi)$ satisfying (8.12) where $\nu = \nu_0, \nu_0 + 1, \nu_0 + 2, \ldots$.

From the above estimate and Lemma 3.1, we know that for any closed q-form $h_\nu \in L_q^2(\Omega_\nu, \psi + \chi_{\nu_0} \circ \psi)$ satisfying (8.12), there exists, for each $\nu \geqslant \nu_0$, some $(q-1)$-form $u_\nu \in L_{q-1}^2(\Omega_\nu, \psi + \chi_{\nu_0} \circ \psi)$ such that

$$du_\nu = h_\nu, \qquad \int_{\Omega_\nu} |u_\nu|^2 e^{-\psi - \chi_{\nu_0} \circ \psi} \leqslant C^2 \int_{\Omega_\nu} |h_\nu|^2 e^{-\psi - \chi_{\nu_0} \circ \psi} \leqslant C^2 \int_{\Omega_\nu} |h_\nu|^2 e^{-\psi}. \tag{8.32}$$

By (8.30), (8.31) and (8.32), we get some $u_\nu \in L_{q-1}^2(\Omega_\nu, \psi + \chi_{\nu_0} \circ \psi)$ such that

$$du_\nu = h|_{\Omega_\nu}, \qquad \int_{\Omega_\nu} |u_\nu|^2 e^{-\psi - \chi_{\nu_0} \circ \psi} \leqslant C^2 \int_{\Omega_\nu} |h|^2 e^{-\psi} < +\infty \tag{8.33}$$

for each $\nu \geqslant \nu_0$. Now the desired solution $\widetilde{u} \in L_{q-1}^2(M, \psi + \chi_{\nu_0} \circ \psi) \subseteq L_{q-1}^2(M, \mathrm{Loc})$ follows from using (8.33) to take weak limits. □

As an intermediate consequence, we have

Corollary 8.1. *Let (M, ds^2) be an oriented n-dimensional Riemannian manifold, and $\Omega \Subset M$ be an open subset with strictly p-convex boundary, then the de Rham cohomology group $H^q(\Omega, \mathbb{R})$ is finite dimensional for every $p \leqslant q \leqslant n$.*

Proof. By Lemma 3.17 in [20], we know that Ω is strictly p-convex at infinity w.r.t. the induced metric from M. Thus the finiteness result follows from Theorem 8.1. □

In the above proof of Theorem 8.1, Lemma 8.4 is applied to a fixed weight function and a sequence of domains. If we apply Lemma 8.4 to a fixed domain and a sequence of weight functions, then we achieve the following approximation result.

Theorem 8.2. *Let $\varphi \in C^2(M)$ be a p-plurisubharmonic function satisfying (8.1). Assume that the subset S in (8.1) is contained in $U := \{x \in M \mid \varphi(x) < 0\}$ and that U has smooth boundary. Let $\Omega \Subset M$ be an open subset with p-convex boundary such that $U \Subset \Omega$. Then for any closed $(q-1)$-form $u \in L^2_{q-1}(U)$ there exists a sequence of closed $(q-1)$-forms $u_\nu \in L^2_{q-1}(\Omega)$ such that $u_\nu|_U \to u$ in $L^2_{q-1}(U)$ where $p \leqslant q \leqslant n$.*

Proof. It is easy to see that

$$\operatorname{Ker}(d|_\Omega) \subseteq \operatorname{Ker}(d|_U) \subseteq L^2_{q-1}(U).$$

The desired conclusion is $\overline{\operatorname{Ker}(d|_\Omega)} \supseteq \operatorname{Ker}(d|_U)$ where $\bar{\cdot}$ means taking the closure in $L^2_{q-1}(U)$. Since $\operatorname{Ker}(d|_U) \subseteq L^2_{q-1}(U)$ is closed, it suffices to show

$$\operatorname{Ker}(d|_\Omega)^\perp \subseteq \operatorname{Ker}(d|_U)^\perp$$

where $\cdot^\perp$ means taking the orthogonal complement in the Hilbert space $L^2_{q-1}(U)$.

Let $u \in \operatorname{Ker}(d|_\Omega)^\perp \subseteq L^2_{q-1}(U)$, then we extend u to be an element $\widetilde{u} \in L^2_{q-1}(\Omega)$ by setting $\widetilde{u} = 0$ outside U. The condition $u \in \operatorname{Ker}(d|_\Omega)^\perp \subseteq L^2_{q-1}(U)$ implies that $\widetilde{u}$ lies in the orthogonal complement of $\operatorname{Ker}(d|_\Omega)$ in $L^2_{q-1}(\Omega)$. Let $\{\chi_\nu\}$ be an increasing sequence which consists of convex increasing functions satisfying (8.9), then we know that

$$\widetilde{u}e^{\varphi+\chi_\nu\circ\varphi} \in \operatorname{Ker}(d|_\Omega)^\perp \subseteq L^2_{q-1}(\Omega, \varphi + \chi_\nu \circ \varphi),$$

as before, $\cdot^\perp$ means taking the orthogonal complement in the Hilbert space $L^2_{q-1}(\Omega, \varphi + \chi_\nu \circ \varphi)$. By Lemma 8.2, it follows that

$$\operatorname{Ker}(d|_\Omega)^\perp = \operatorname{Im}\bigl((d|_\Omega)^*_{\varphi+\chi_\nu\circ\varphi}\bigr) \subseteq L^2_{q-1}(\Omega, \varphi + \chi_\nu \circ \varphi).$$

Hence we can find a unique $f_\nu \in \operatorname{Dom}((d|_\Omega)^*_{\varphi+\chi_\nu\circ\varphi}) \cap \operatorname{Ker}((d|_\Omega)^*_{\varphi+\chi_\nu\circ\varphi})^\perp \subseteq L^2_q(\Omega, \varphi + \chi_\nu \circ \varphi)$ such that

$$(d|_\Omega)^*_{\varphi+\chi_\nu\circ\varphi} f_\nu = e^{\varphi+\chi_\nu\circ\varphi}\widetilde{u} \quad \text{on } \Omega \tag{8.34}$$

for each $\nu \geqslant 1$.

Since $\operatorname{Ker}((d|_\Omega)^*_{\varphi+\chi_\nu\circ\varphi})^\perp$ (in $L^2_{q-1}(\Omega, \varphi + \chi_\nu \circ \varphi)$) $\subseteq \operatorname{Ker}((d|_U)^*_\varphi)^\perp$ (in $L^2_{q-1}(U, \varphi)$), by Lemma 8.4 and (8.34), there are constants $C > 0$ and $\nu_0 > 0$ such that

$$\int_{\Omega} |f_\nu|^2 e^{-\varphi-\chi_\nu\circ\varphi} \leqslant C^2 \int_{\Omega} \left|e^{\varphi+\chi_\nu\circ\varphi}\widetilde{u}\right|^2 e^{-\varphi-\chi_\nu\circ\varphi}$$
$$= C^2 \int_U |u|^2 e^{\varphi} \tag{8.35}$$

for any $\nu \geqslant \nu_0$. Set $g_\nu = e^{-\varphi-\chi_\nu\circ\varphi} f_\nu$ $(\nu \geqslant 1)$, then by (8.34) and (8.35) we get

$$(d|_\Omega)^* g_\nu = \widetilde{u} \tag{8.36}$$

and

$$\int_{\Omega} |g_\nu|^2 e^{\varphi+\chi_\nu\circ\varphi} \leqslant C^2 \int_U |u|^2 e^{\varphi} < +\infty \tag{8.37}$$

for any $\nu \geqslant \nu_0$. Estimate (8.37) implies that

$$\int_{\Omega} |g_\nu|^2 e^{\varphi+\chi_\mu\circ\varphi} \leqslant C^2 \int_U |u|^2 e^{\varphi} < +\infty \tag{8.38}$$

for any $\nu \geqslant \mu \geqslant \nu_0$. Therefore there is a weak limit, denoted by g, of g_ν in $L^2_{q-1}(\Omega, \varphi + \chi_\mu \circ \varphi)$ for any $\mu \geqslant \nu_0$ (note that g is independent of μ). It follows from (8.36) that

$$\delta g = \widetilde{u} \quad \text{on } \Omega \tag{8.39}$$

in the sense of distribution. Letting $\nu \to +\infty$, (8.38) gives

$$\int_{\Omega} |g|^2 e^{\varphi+\chi_\mu\circ\varphi} \leqslant C^2 \int_U |u|^2 e^{\varphi} < +\infty$$

for any $\mu \geqslant \nu_0$. Taking limit as $\mu \to +\infty$ yields

$$\operatorname{Supp} g \subseteq U. \tag{8.40}$$

Combining (8.39) and (8.40) shows that $g \in \operatorname{Dom}((d|_U)^*)$ and consequently $u \in \operatorname{Im}((d|_U)^*) = \operatorname{Ker}(d|_U)^\perp \subseteq L^2_{q-1}(U)$. The proof is complete. □

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