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FU-YAU HESSIAN EQUATIONS

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Abstract

We solve the Fu-Yau equation for arbitrary dimension and arbitrary slope α'. Actually we obtain at the same time a solution of the open case $\alpha' > 0$, an improved solution of the known case $\alpha' < 0$, and solutions for a family of Hessian equations which includes the Fu-Yau equation as a special case. The method is based on the introduction of a more stringent ellipticity condition than the usual Γ_k admissible cone condition, and which can be shown to be preserved by precise estimates with scale.

1. Introduction

The main goal of this paper is to solve the following non-linear partial differential equation proposed in 2008 by J.X. Fu and S.T. Yau [**11**],

$$i\partial\bar{\partial}(e^u\hat{\omega} - \alpha' e^{-u}\rho) \wedge \hat{\omega}^{n-2} + \alpha' i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u \wedge \hat{\omega}^{n-2} + \mu\,\hat{\omega}^n = 0. \tag{1.1}$$

Here the unknown is a scalar function u on a compact n-dimensional Kähler manifold $(X, \hat{\omega})$, and the given data is a real $(1,1)$ form ρ, a function μ, and a number $\alpha' \in \mathbf{R}$ called the slope. A key innovation in the solution is the introduction of an ellipticity condition which is more restrictive than the usual cone conditions for fully non-linear second order partial differential equations, but which can be shown to be preserved by the continuity method using some precise estimates with scale. This innovation may be useful for other equations as well, and we shall illustrate this by using it to solve a whole family of Hessian equations in which the equation (1.1) fits as only the simplest example.

The equation (1.1) is a generalization of an equation in complex dimension 2, which was shown in [**11**] to arise from the Hull-Strominger system [**17**, **18**, **27**]. The Hull-Strominger system is an extension of a proposal of Candelas, Horowitz, Strominger, and Witten [**5**] for supersymmetric compactifications of the heterotic string. It poses new geometric difficulties as it involves quadratic expressions in the curvature

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tensor, but it can potentially lead to a new notion of canonical metric in non-Kähler geometry. From our point of view, the equation (1.1) is of particular interest as a model equation for an eventual extension of the classical theory of Monge-Ampère equations of Yau [**34**] and Hessian equations of Caffarelli, Nirenberg, and Spruck [**4**], to more general equations mixing the unknown, its gradient, and several Hessians.

When the dimension of X is $n = 2$, the equation (1.1) was solved by Fu and Yau in two separate papers, [**11**] for the case when $\alpha' > 0$, and [**12**] for the case when $\alpha' < 0$ (when $\alpha' = 0$, the equation poses no difficulty as it reduces essentially to the Laplacian). As we shall discuss below, in the approach of [**11**, **12**], the required estimates in the two cases $\alpha' > 0$ and $\alpha' < 0$ are quite different. In an earlier paper [**22**], we had solved the equation (1.1) for general dimension n when $\alpha' < 0$. However, the case $\alpha' > 0$ for general dimension n remained open, as a key lower bound for the Hessian could not be established [**19**]. In this paper, we shall simultaneously solve the open case $\alpha' > 0$ for general dimension n, improve on the solution found in [**22**] for the case $\alpha' < 0$, and do it actually for more general equations where the factor $(i\partial\bar\partial u)^2$ in (1.1) is replaced by higher powers of $i\partial\bar\partial u$.

More precisely, let $(X, \hat\omega)$, ρ, μ, α' be as above. For each fixed integer k, $1 \leq k \leq n-1$ and each real number $\gamma > 0$, we consider the equation
(1.2)
$$i\partial\bar\partial \left\{ e^{ku}\hat\omega - \alpha' e^{(k-\gamma)u}\rho \right\} \wedge \hat\omega^{n-2} + \alpha'(i\partial\bar\partial u)^{k+1} \wedge \hat\omega^{n-k-1} + \mu\,\hat\omega^n = 0.$$
Clearly, when $k = 1$ and $\gamma = 2$, this equation reduces to the Fu-Yau equation (1.1). We shall refer to (1.2) as Fu-Yau Hessian equations. Our main result is then the following:

Theorem 1.1. *Let $\alpha' \in \mathbf{R}$, $\rho \in \Omega^{1,1}(X, \mathbf{R})$, and $\mu : X \to \mathbf{R}$ be a smooth function such that $\int_X \mu\,\hat\omega^n = 0$. Define the set Υ_k by*

$$\Upsilon_k = \left\{ u \in C^2(X, \mathbf{R}) : e^{-\gamma u} < \delta,\ |\alpha'||e^{-u} i\partial\bar\partial u|^k_{\hat\omega} < \tau \right\}, \tag{1.3}$$

where $0 < \delta, \tau \ll 1$ are explicit fixed constants depending only on $(X, \hat\omega), \alpha', \rho, \mu, n, k, \gamma$, whose expressions are given in (2.5, 2.6) below. Then there exists $M_0 \gg 1$ depending on $(X, \hat\omega)$, α', n, k, γ, μ and ρ, such that for each $M \geq M_0$, there exists a unique smooth function $u \in \Upsilon_k$ with normalization $\int_X e^u\,\hat\omega^n = M$ solving the Fu-Yau Hessian equation (1.2).

We outline now the key differences between the earlier approaches and the approach of the present paper.

The earlier approaches [**11**, **12**, **19**, **22**] were based on rewriting the equation (1.1) as

$$\hat\sigma_2(\omega') = \frac{n(n-1)}{2}(e^{2u} - 4\alpha' e^u |\nabla u|^2) + \nu, \tag{1.4}$$

where ν is a combination of known functions, u and ∇u, ω' is defined by $\omega' = e^u\hat{\omega} + \alpha' e^{-u}\rho + 2n\alpha' i\partial\bar{\partial}u$, and $\hat{\sigma}_k(\omega')$ is the k-th symmetric function of the eigenvalues of ω' with respect to $\hat{\omega}$. We look then for solutions u satisfying the condition $\omega' \in \Gamma_2$, where Γ_2 is defined by the conditions $\hat{\sigma}_1(\omega') > 0$ and $\hat{\sigma}_2(\omega') > 0$. The left hand side is then positive. When $\alpha' > 0$, this implies immediately an upper bound on $|\nabla u|$. However, the difficulty is then to derive a positive lower bound for $\hat{\sigma}_2(\omega')$, and the arguments of [**11**] worked only when $n = 2$. On the other hand, when $\alpha' < 0$, such a lower bound turns out to hold because there is no cancellation in the expression $e^{2u} - 4\alpha' e^u|\nabla u|^2$. The estimate for $|\nabla u|$ and $|\hat{\sigma}_2(\omega')|$ can then be obtained respectively by applying the techniques of Dinew-Kolodziej [**9**], and Chou-Wang [**6**], Hou-Ma-Wu [**16**], Guan [**15**], and the authors [**21**].

The approach in the present paper relies instead on a different strategy.

First, the equation (1.1) corresponds to the case $k = 1$, $\gamma = 2$ of the Fu-Yau Hessian equations. As stated in Theorem 1.1, we look for solutions $u \in \Upsilon_1$, which is a more stringent condition than $\omega' \in \Gamma_2$. The set Υ_1 and its condition $e^{-u}|\alpha' i\partial\bar{\partial}u|_{\hat{\omega}} < \tau$ are inspired by the condition $|\alpha' Rm(\omega)| \ll 1$ in [**20**, **23**] which guarantees the parabolicity of the geometric flows introduced in these papers.[1] In the method of continuity, the given equation (1.1) is realized as the end point of a family of equations for each $t \in [0, 1]$. The condition $u \in \Upsilon_1$ implies that the diffusion operator $F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$ governing the evolution of $|\nabla u|^2$ and $|\alpha' i\partial\bar{\partial}u|^2$ is a controllable perturbation of the Laplacian $\Delta = g^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$. The main problem is then to show that, if $u \in \Upsilon_1$ at time $t = 0$, it will stay in Υ_1 at all times.

This is accomplished by establishing a priori estimates, which we shall refer to as "estimates with scale", which are more precise and delicate than the usual ones. Indeed, a priori estimates for $|u|$, $|\nabla u|$, and $|\alpha' i\partial\bar{\partial}u|$ are usually required only to be independent of $z \in X$ and $t \in [0, 1]$. In the present situation, the normalization as given in Theorem 1.1

$$\int_X e^u \hat{\omega}^n = M \tag{1.5}$$

sets effectively a scale M, and the estimates with scale that we need are estimates for $|u|$, $|\nabla u|$, and $|\alpha' i\partial\bar{\partial}u|$ in terms of some specific powers of M. An example of such an estimate is the C^0 estimate stated in Theorem 3.1 below, $C^{-1}M \le e^u \le C\,M$, which is a version in the present context of similar C^0 estimates established earlier in [**11**, **12**, **19**, **22**]. The hardest part of the paper resides in the proof of similar estimates with scale for $|\nabla u|$ and $|i\partial\bar{\partial}u|$, as stated in Theorems 5.1 and

[1] In these flows, a Hermitian metric ω evolves with time, and $Rm(\omega)$ is the curvature of the Chern unitary connection of ω. The condition $|\alpha' Rm(\omega)| \ll 1$ was subsequently also used in [**10**].

6.1. Neither the set Υ_1 nor the estimates with scale depend on the sign of α', which is why both cases $\alpha' > 0$ and $\alpha' < 0$ can be treated simultaneously. Furthermore, we obtain a solution $u \in \Upsilon_1$, which is better than a solution in Γ_2. A vital clue that a strategy based on Υ_1 and estimates with scale could work was provided by the authors' earlier alternative proof [**23**] by flow methods of the Fu-Yau theorem [**11**, **12**] in dimension $n = 2$.

The power of the new method is even more evident when it comes to the general Fu-Yau Hessian equation (1.2). For $k \geq 2$, it is no longer possible to express the equation (1.2) in terms of a single Hessian $\hat{\sigma}_{k+1}(\omega')$ for some $(1,1)$-form ω' as in (1.4). Rather, the equation leads to a combination of several Hessians, which makes it non-concave, and prevents a derivation of C^2 and $C^{2,\alpha}$ estimates by standard techniques of concave PDE's. On the other hand, the method of an ellipticity condition Υ_k preserved by estimates with scale works seamlessly in all cases of $1 \leq k \leq n-1$. In fact the C^3 estimates that we obtain appear to be the first C^3 estimates established in the literature for any general class of Hessian equations besides the Laplacian and the Monge-Ampère equations.

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2. Proof of Theorem 1: a priori estimates

In our study of (1.2), we will assume that $\mathrm{Vol}(X, \hat{\omega}) = 1$, which can be achieved by scaling $\hat{\omega} \mapsto \lambda\hat{\omega}$, $\alpha' \mapsto \lambda^k \alpha'$, $\rho \mapsto \lambda^{-k+1}\rho$, $\mu \mapsto \lambda^{-1}\mu$. Since the equation (1.2) reduces to the Laplace equation when $\alpha' = 0$, we assume from now on that $\alpha' \neq 0$. We will use the notation $C_n^\ell = \frac{n!}{\ell!(n-\ell)!}$ and $\hat{\sigma}_\ell(i\partial\bar{\partial}u)\,\hat{\omega}^n = C_n^\ell\,(i\partial\bar{\partial}u)^\ell \wedge \hat{\omega}^{n-\ell}$. Given ρ, we define the differential operator L_ρ acting on functions by

$$L_\rho f\, \hat{\omega}^n = n i\partial\bar{\partial}(f\rho) \wedge \hat{\omega}^{n-2}. \tag{2.1}$$

For each fixed $k \in \{1, 2, 3, \ldots, n-1\}$ and a real number $\gamma > 0$, the Fu-Yau Hessian equation (1.2) can be rewritten as

$$\frac{1}{k}\,\Delta_{\hat{g}} e^{ku} + \alpha' \Big\{ L_\rho e^{(k-\gamma)u} + \hat{\sigma}_{k+1}(i\partial\bar{\partial}u) \Big\} = \mu. \tag{2.2}$$

We note that we adjusted our conventions compared to the introduction by redefining μ, ρ, and α' up to a constant. From this point on, we only work with the present conventions (2.2). The standard Fu-Yau equation can be recovered by letting $k = 1$, $\gamma = 2$. We remark that this equation is already of interest in the case when $\rho \equiv 0$, in which case the term $L_\rho e^{(k-\gamma)u}$ vanishes.

We can also write L_ρ as

$$L_\rho = a^{j\bar{k}}\partial_j\partial_{\bar{k}} + b^i\partial_i + b^{\bar{i}}\partial_{\bar{i}} + c, \tag{2.3}$$

where $a^{j\bar{k}}$ is a Hermitian section of $(T^{1,0}X)^*\otimes(T^{0,1}X)^*$, b^i is a section of $(T^{1,0}X)^*$, and c is a real function. All these coefficients are characterized by the following equations

$$ni\partial\bar{\partial}f \wedge \rho \wedge \hat{\omega}^{n-2} = a^{j\bar{k}}\partial_j\partial_{\bar{k}}f\,\hat{\omega}^n, \quad ni\partial f \wedge \bar{\partial}\rho \wedge \hat{\omega}^{n-2} = b^i\partial_i f\,\hat{\omega}^n,$$
$$ni\partial\bar{\partial}\rho \wedge \hat{\omega}^{n-2} = c\hat{\omega}^n,$$

for an arbitrary function f, and can be expressed explicitly in terms of ρ and $\hat{\omega}$ if desired. We will use the constant Λ depending on ρ defined by

$$-\Lambda\hat{g}^{j\bar{k}} \leq a^{j\bar{k}} \leq \Lambda\hat{g}^{j\bar{k}}, \quad \hat{\omega} = \hat{g}_{\bar{k}j}idz^j \wedge d\bar{z}^k, \quad \hat{g}^{j\bar{k}} = (\hat{g}_{\bar{k}j})^{-1}. \tag{2.4}$$

We will look for solutions in the region

$$\Upsilon_k = \left\{u \in C^2(X,\mathbf{R}) : e^{-\gamma u} < \delta,\ |\alpha'||e^{-u}i\partial\bar{\partial}u|^k_{\hat{\omega}} < \tau\right\}, \quad \tau = \frac{2^{-7}}{C^k_{n-1}}, \tag{2.5}$$

where $0 < \delta \ll 1$ is a fixed small constant depending only on $(X,\hat{\omega})$, α', ρ, μ, k, n, γ. More precisely, it suffices for δ to satisfy the inequality

$$\delta \leq \min\left\{1, \frac{2^{-13}}{|\alpha'|(k+\gamma)^3\Lambda}, \left(\frac{\theta}{2C_X\left(\|\mu\|_{L^\infty} + \|\alpha' c\|_{L^\infty}\right)}\right)^{\gamma/\gamma'}\right\}, \tag{2.6}$$

where

$$\theta = \frac{1}{2C_1 - 1}, \quad \gamma' = \min\{k,\gamma\}, \quad C_1 = \{2(C_X+1)(\gamma+k)\}^n\left(\frac{n}{n-1}\right)^{n^2}. \tag{2.7}$$

Here C_X is the maximum of the constants appearing in the Poincaré inequality and Sobolev inequality on $(X,\hat{\omega})$. The proof of Theorem 1.1 is based on the following a priori estimates:

Theorem 2.1. *Let $u \in \Upsilon_k$ be a $C^{5,\beta}(X)$ function with normalization $\int_X e^u\,\hat{\omega}^n = M$ solving the k-th Fu-Yau Hessian equation (2.2). Then*

$$C^{-1}M \leq e^u \leq CM, \quad e^{-u}|\nabla\bar{\nabla}u|_{\hat{\omega}} \leq CM^{-1/2}, \quad e^{-3u}|\nabla\bar{\nabla}\nabla u|^2_{\hat{\omega}} \leq C, \tag{2.8}$$

where $C > 1$ only depends on $(X,\hat{\omega})$, α', k, γ, n, ρ, and μ.

Assuming Theorem 2.1, we can prove Theorem 1.1. Both the existence and uniqueness statements will be proved by the continuity method.

We begin with the existence. Fix $\alpha' \in \mathbf{R}\backslash\{0\}$, $\gamma > 0$, $1 \le k \le (n-1)$, $\rho \in \Omega^{1,1}(X, \mathbf{R})$ and $\mu : X \to \mathbf{R}$ such that $\int_X \mu\,\hat\omega^n = 0$, and define the set Υ_k as above. For a real parameter t, we consider the family of equations

$$\frac{1}{k}\,\Delta_{\hat g} e^{ku_t} + \alpha' \left\{ t L_\rho e^{(k-\gamma)u_t} + \hat\sigma_{k+1}(i\partial\bar\partial u_t) \right\} = t\mu. \tag{2.9}$$

As equations of differential forms, this family can be expressed as

$$i\partial\bar\partial \left\{ \frac{e^{ku}}{k}\hat\omega + \alpha' t e^{(k-\gamma)u}\rho \right\} \wedge \hat\omega^{n-2} + \alpha' \frac{C_{n-1}^k}{k+1}(i\partial\bar\partial u)^{k+1} \wedge \hat\omega^{n-k-1} - t\frac{\mu}{n}\hat\omega^n = 0. \tag{2.10}$$

We introduce the following spaces

$$B_M = \{u \in C^{5,\beta}(X, \mathbf{R}) : \int_X e^u\,\hat\omega^n = M\}, \tag{2.11}$$

$$B_1 = \{(t,u) \in [0,1] \times B_M : u \in \Upsilon_k\}, \tag{2.12}$$

$$B_2 = \{\psi \in C^{3,\beta}(X, \mathbf{R}) : \int_X \psi\,\hat\omega^n = 0\} \tag{2.13}$$

and define the map $\Psi : B_1 \to B_2$ by

$$\Psi(t,u) = \frac{1}{k}\,\Delta_{\hat g} e^{ku} + \alpha' t L_\rho e^{(k-\gamma)u} + \alpha' \hat\sigma_{k+1}(i\partial\bar\partial u) - t\mu. \tag{2.14}$$

We consider

$$I = \{t \in [0,1]: \text{there exists}\, u \in B_M \text{ such that } (t,u) \in B_1 \text{ and } \Psi(t,u) = 0\}. \tag{2.15}$$

First, $0 \in I$: indeed the constant function $u_0 = \log M - \log \int_X \hat\omega^n$ is in Υ_k when $M \gg 1$, and u_0 solves the equation at $t = 0$. In particular I is non-empty.

Next, we show that I is open. Let $(t_0, u_0) \in B_1$, and let $L = (D_u\Psi)_{(t_0,u_0)}$ be the linearized operator at (t_0, u_0),

$$L : \left\{ h \in C^{5,\beta}(X, \mathbf{R}) : \int_X h e^{u_0}\,\hat\omega^n = 0 \right\} \to \left\{ \psi \in C^{3,\beta}(X, \mathbf{R}) : \int_X \psi\,\hat\omega^n = 0 \right\}, \tag{2.16}$$

defined by

$$\begin{aligned} L(h)\hat\omega^n &= i\partial\bar\partial\{e^{ku_0} h\,\hat\omega + \alpha'(k-\gamma)t_0 e^{(k-\gamma)u_0} h\,\rho\} \wedge \hat\omega^{n-2} \\ &\quad + \alpha' C_{n-1}^k i\partial\bar\partial h \wedge (i\partial\bar\partial u_0)^k \wedge \hat\omega^{n-k-1}. \end{aligned} \tag{2.17}$$

The leading order terms are

$$L(h)\hat\omega^n = e^{ku_0}\chi_{(t_0,u_0)} \wedge \hat\omega^{n-k-1} \wedge i\partial\bar\partial h + \cdots, \tag{2.18}$$

where

$$\chi_{(t,u)} = \hat{\omega}^k + \alpha'(k-\gamma)te^{-\gamma u}\,\rho\wedge\hat{\omega}^{k-1} + \alpha' C^k_{n-1}(e^{-u}i\partial\bar{\partial}u)^k. \tag{2.19}$$

Since $u_0 \in \Upsilon_k$, we see from the conditions (2.5) that $\chi_{(t_0,u_0)} > 0$ as a (k,k) form and hence L is elliptic. The L^2 adjoint L^* is readily computed by integrating by parts:

$$\begin{aligned}\int_X \psi L(h)\,\hat{\omega}^n &= \int_X h\,e^{ku_0}\chi_{(t_0,u_0)}\wedge\hat{\omega}^{n-k-1}\wedge i\partial\bar{\partial}\psi\\ &= \int_X hL^*(\psi)\,\hat{\omega}^n.\end{aligned} \tag{2.20}$$

Since L^* is an elliptic operator with no zeroth order terms, by the strong maximum principle the kernel of L^* consists of constant functions. An index theory argument (see e.g. [**21**] or [**11**] for full details) shows that the kernel of L is spanned by a function of constant sign. It follows that L is an isomorphism. By the implicit function theorem, there exists a unique solution (t,u_t) for t sufficiently close to t_0, with $u_t\in\Upsilon_k$ since Υ_k is open. We conclude that I is open.

Finally, we apply Theorem 2.1 to show that I is closed. Consider a sequence $t_i\in I$ such that $t_i\to t_\infty$, and denote $u_{t_i}\in\Upsilon_k\cap B_M$ the associated $C^{5,\beta}$ functions such that $\Psi(t_i,u_{t_i})=0$. By differentiating the equation $e^{-ku_{t_i}}\Psi(t_i,u_{t_i})=0$ with the Chern connection $\hat{\nabla}$ of the Kähler metric $\hat{\omega}$, we obtain

$$\begin{aligned}0 \;=\;& \frac{\chi_{(t_i,u_{t_i})}\wedge\hat{\omega}^{n-k-1}\wedge i\partial\bar{\partial}(\partial_\ell u_{t_i})}{\hat{\omega}^n/n}\\ &+\hat{\nabla}_\ell\{\alpha' t_i e^{-\gamma u_{t_i}}((k-\gamma)^2 a^{p\bar{q}}\partial_p u_{t_i}\,\partial_{\bar{q}}u_{t_i} + (k-\gamma)b^k\partial_k u_{t_i}\\ &+(k-\gamma)b^{\bar{k}}\partial_{\bar{k}}u_{t_i}+c)\} + \hat{\nabla}_\ell(\alpha' t_i e^{-\gamma u_{t_i}}(k-\gamma)a^{p\bar{q}})\partial_p\partial_{\bar{q}}u_{t_i}\\ &+k\partial_\ell|\nabla u_{t_i}|^2_{\hat{g}} - \alpha' k e^{-ku_{t_i}}\hat{\sigma}_{k+1}(i\partial\bar{\partial}u_{t_i})\partial_\ell u_{t_i} - t_i\partial_\ell\{e^{-ku_{t_i}}\mu\}.\end{aligned} \tag{2.21}$$

Since the equations (2.9) are of the form (2.2) with uniformly bounded coefficients ρ and μ, Theorem 2.1 applies to give uniform control of $|u_{t_i}|$ and $|\partial\bar{\partial}\partial u_{t_i}|_{\hat{\omega}}$ along this sequence. Therefore $\hat{\Delta}u_{t_i}$ is uniformly controlled in $C^\beta(X)$ for any $0<\beta<1$. By Schauder estimates, we have $\|u_{t_i}\|_{C^{2,\beta}}\le C$.

Thus the differentiated equation (2.21) is a linear elliptic equation for $\partial_\ell u_{t_i}$ with C^β coefficients. This equation is uniformly elliptic along the sequence, since $\chi_{(t_i,u_{t_i})}\ge\frac{1}{2}\hat{\omega}^k$ by (2.8) when $M\gg 1$. By Schauder estimates, we have uniform control of $\|\nabla u_{t_i}\|_{C^{2,\beta}}$. A bootstrap argument shows that we have uniform control of $\|u_{t_i}\|_{C^{6,\beta}}$, hence we may extract a subsequence converging to $u_\infty\in C^{5,\beta}$. Furthermore, for $M\ge M_0\gg 1$ large enough, we see from (2.8) that

$$e^{-u_\infty} \ll 1, \quad |e^{-u} i\partial\bar\partial u_\infty|_{\hat\omega} \ll 1, \tag{2.22}$$

hence $u_\infty \in \Upsilon_k$. Thus I is closed.

Hence $I = [0,1]$ and consequently there exists a $C^{5,\beta}$ function $u \in \Upsilon_k$ with normalization $\int_X e^u \, \hat\omega^n = M$ solving the Fu-Yau equation (2.2). By applying Schauder estimates and a bootstrap argument to the differentiated equation (2.21), we see that u is smooth.

We complete now the proof of Theorem 1.1 with the proof of uniqueness.

First, we show that the only solutions of the equation

$$\frac{1}{k} i\partial\bar\partial e^{ku} \wedge \hat\omega^{n-1} + \alpha' \frac{C^k_{n-1}}{k+1} (i\partial\bar\partial u)^{k+1} \wedge \hat\omega^{n-k-1} = 0 \tag{2.23}$$

with $|\alpha'| C^k_{n-1} |e^{-u} i\partial\bar\partial u|^k_{\hat\omega} < 2^{-7}$ are constant functions. Multiplying by u and integrating, we see that

$$0 = \int_X i\partial u \wedge \bar\partial u \wedge \left\{ e^{ku}\hat\omega^k + \alpha' \frac{C^k_{n-1}}{k+1} (i\partial\bar\partial u)^k \right\} \wedge \hat\omega^{n-k-1}, \tag{2.24}$$

and hence u must be constant since $e^{ku}\hat\omega^k + \alpha' \frac{C^k_{n-1}}{k+1} (i\partial\bar\partial u)^k > 0$ as a (k,k) form.

Now suppose there are two distinct solutions $u \in \Upsilon_k$ and $v \in \Upsilon_k$ satisfying (2.2) under the normalization $\int_X e^u \, \hat\omega^n = \int_X e^v \, \hat\omega^n = M$ with $M \geq M_0$. For $t \in [0,1]$, define

$$\begin{aligned} \Phi(t,u) &= i\partial\bar\partial \left\{ \frac{e^{ku}}{k}\hat\omega + \alpha'(1-t) e^{(k-\gamma)u} \rho \right\} \wedge \hat\omega^{n-2} \\ &\quad + \alpha' \frac{C^k_{n-1}}{k+1} (i\partial\bar\partial u)^{k+1} \wedge \hat\omega^{n-k-1} - (1-t)\frac{\mu}{n}\hat\omega^n, \end{aligned} \tag{2.25}$$

and consider the path $t \mapsto u_t$ satisfying $\Phi(t,u_t) = 0$, $u_t \in \Upsilon_k$, $\int_X e^{u_t} \hat\omega^n = M$ with initial condition $u_0 = u$.

The same argument which shows that I is open also shows that the path u_t exists for a short-time: there exists $\epsilon > 0$ such that u_t is defined on $[0,\epsilon)$. By our estimates (2.8), we may extend the path to be defined for $t \in [0,1]$. By uniqueness of the equation with $t = 1$, we know that $u_1 = \log M - \log \int_X \hat\omega^n$. The same argument gives a path $t \mapsto v_t$ satisfying $\Phi(t, v_t) = 0$, $v_t \in \Upsilon_k$, $\int_X e^{v_t} \hat\omega^n = M$ with $v_0 = v$ and $v_1 = \log M - \log \int_X \hat\omega^n$. But then at the first time $0 < t_0 \leq 1$ when $u_{t_0} = v_{t_0}$, we contradict the local uniqueness of $\Phi(t,u_t) = 0$ given by the implicit function theorem.

It follows from our discussion that in order to prove Theorem 1.1, it remains to establish the a priori estimates (2.8).

3. The uniform estimate

Theorem 3.1. *Suppose $u \in \Upsilon_k$ solves (2.2) subject to the normalization $\int_X e^u \hat{\omega}^n = M$. Then*

$$C^{-1}M \le e^u \le CM, \tag{3.1}$$

where C only depends on $(X, \hat{\omega})$, k, and γ.

We first note the following general identity which holds for any function u.

$$\begin{aligned} 0 &= \alpha'(p-k)\int_X e^{(p-k)u} i\partial u \wedge \bar{\partial} u \wedge (i\partial\bar{\partial} u)^k \wedge \hat{\omega}^{n-k-1} \\ &\quad + \alpha' \int_X e^{(p-k)u} (i\partial\bar{\partial} u)^{k+1} \wedge \hat{\omega}^{n-k-1}. \end{aligned} \tag{3.2}$$

Substituting the Fu-Yau Hessian equation (2.10) with $t = 1$, we obtain

$$\begin{aligned} 0 &= \alpha' \frac{C_{n-1}^k}{k+1}(p-k)\int_X e^{(p-k)u} i\partial u \wedge \bar{\partial} u \wedge (i\partial\bar{\partial} u)^k \wedge \hat{\omega}^{n-k-1} \\ &\quad + \int_X e^{(p-k)u} \mu \frac{\hat{\omega}^n}{n} - \int_X e^{(p-k)u} i\partial\bar{\partial}\left\{ \frac{e^{ku}}{k}\hat{\omega} + \alpha' e^{(k-\gamma)u}\rho \right\} \wedge \hat{\omega}^{n-2}. \end{aligned} \tag{3.3}$$

We integrate by parts to derive

$$\begin{aligned} 0 &= \alpha' \frac{C_{n-1}^k}{k+1}(p-k)\int_X e^{(p-k)u} i\partial u \wedge \bar{\partial} u \wedge (i\partial\bar{\partial} u)^k \wedge \hat{\omega}^{n-k-1} \\ &\quad + \int_X e^{(p-k)u} \mu \frac{\hat{\omega}^n}{n} + (p-k)\int_X e^{pu}\, i\partial u \wedge i\bar{\partial} u \wedge \hat{\omega}^{n-1} \\ &\quad + (p-k)\alpha' \int_X e^{(p-k)u}\, i\partial u \wedge i\bar{\partial}(e^{(k-\gamma)u}\rho) \wedge \hat{\omega}^{n-2}. \end{aligned} \tag{3.4}$$

Integrating by parts again gives

$$\begin{aligned} &(p-k)\int_X e^{pu} i\partial u \wedge \bar{\partial} u \wedge \hat{\omega}^{n-k-1} \wedge \chi' \\ &= -\int_X e^{(p-k)u} \mu \frac{\hat{\omega}^n}{n} + \frac{p-k}{p-\gamma}\alpha' \int_X e^{(p-\gamma)u} \wedge i\partial\bar{\partial}\rho \wedge \hat{\omega}^{n-2}, \end{aligned} \tag{3.5}$$

where we now assume $p > \gamma$ and we define

$$\chi' = \hat{\omega}^k + \alpha'(k-\gamma)e^{-\gamma u}\rho \wedge \hat{\omega}^{k-1} + \alpha' \frac{C_{n-1}^k}{k+1}(e^{-u} i\partial\bar{\partial} u)^k. \tag{3.6}$$

Next, we estimate

$$
\begin{aligned}
i\partial u \wedge \bar\partial u \wedge \hat\omega^{n-k-1} \wedge \chi' &= \frac{|\nabla u|^2_{\hat\omega}}{n}\,\hat\omega^n + \alpha'(k-\gamma)e^{-\gamma u}\frac{a^{i\bar j}u_i u_{\bar j}}{n}\,\hat\omega^n \\
&\quad + \alpha'\frac{C^k_{n-1}}{k+1} i\partial u \wedge \bar\partial u \wedge (e^{-u} i\partial\bar\partial u)^k \wedge \hat\omega^{n-k-1} \\
&\geq \frac{|\nabla u|^2_{\hat\omega}}{n}\,\hat\omega^n - |\alpha'\Lambda(k-\gamma)|\delta\frac{|\nabla u|^2_{\hat\omega}}{n}\,\hat\omega^n \\
&\quad - |\alpha'|\frac{C^k_{n-1}}{k+1}|e^{-u} i\partial\bar\partial u|^k_{\hat\omega}\frac{|\nabla u|^2_{\hat\omega}}{n}\,\hat\omega^n. \qquad (3.7)
\end{aligned}
$$

Since $u \in \Upsilon_k$, by (2.5) and (2.6) the positive term dominates the expression and we can conclude

$$
i\partial u \wedge \bar\partial u \wedge \hat\omega^{n-k-1} \wedge \chi' \geq \frac{1}{2}\frac{|\nabla u|^2_{\hat\omega}}{n}\,\hat\omega^n. \qquad (3.8)
$$

The proof of Theorem 3.1 will be divided into three propositions. We note that in the following arguments we will omit the background volume form $\hat\omega^n$ when integrating scalar functions.

Proposition 3.2. *Suppose $u \in \Upsilon_k$ solves (2.2) subject to normalization $\int_X e^u = M$. There exists $C_1 > 0$ such that*

$$
e^u \leq C_1 M, \qquad (3.9)
$$

where C_1 only depends on $(X, \hat\omega)$, n, k and γ. In fact, C_1 is given by (2.7).

Proof. Combining (3.5) and (3.8) gives

$$
\begin{aligned}
&\frac{1}{2}(p-k)\int_X e^{pu}|\nabla u|^2_{\hat\omega} \\
&\leq -\int_X e^{(p-k)u}\mu + \frac{p-k}{p-\gamma}n\alpha'\int_X e^{(p-\gamma)u} \wedge i\partial\bar\partial\rho \wedge \hat\omega^{n-2}. \qquad (3.10)
\end{aligned}
$$

We estimate

$$
\int_X |\nabla e^{\frac{p}{2}u}|^2_{\hat\omega} \leq \frac{p^2}{2(p-k)}\left\{\|\mu\|_{L^\infty}\int_X e^{(p-k)u} + \frac{p-k}{p-\gamma}\|\alpha' c\|_{L^\infty}\int_X e^{(p-\gamma)u}\right\}. \qquad (3.11)
$$

For any $p \geq 2\max\{\gamma, k\}$, there holds $\frac{p^2}{2(p-k)} \leq p$ and $\frac{p-k}{p-\gamma} \leq 2$. Using $e^{-\gamma u} \leq \delta \leq 1$ and (2.6), we conclude that

$$
\begin{aligned}
\int_X |\nabla e^{\frac{p}{2}u}|^2_{\hat\omega} &\leq 2(\|\mu\|_{L^\infty} + \|\alpha' c\|_{L^\infty})\delta^{\frac{\min\{k,\gamma\}}{\gamma}} p \int_X e^{pu} \\
&\leq \frac{\theta}{C_X} p \int_X e^{pu} \leq \frac{p}{C_X}\int_X e^{pu}, \qquad (3.12)
\end{aligned}
$$

for any $p \geq 2(\gamma+k)$. Let $\beta = \frac{n}{n-1}$. The Sobolev inequality gives us

$$\left(\int_X e^{\beta p u}\right)^{1/\beta} \leq C_X\left(\int_X |\nabla e^{\frac{p}{2}u}|^2_{\hat\omega} + \int_X e^{pu}\right). \tag{3.13}$$

Therefore for all $p \geq 2(\gamma+k)$,

$$\|e^u\|_{L^{p\beta}} \leq (C_X+1)^{1/p} p^{1/p} \|e^u\|_{L^p}. \tag{3.14}$$

Iterating this inequality gives

$$\|e^u\|_{L^{p\beta^{(k+1)}}} \leq \{(C_X+1)p\}^{\frac{1}{p}\sum_{i=0}^k \frac{1}{\beta^i}} \cdot \beta^{\frac{1}{p}\sum_{i=1}^k \frac{i}{\beta^i}} \|e^u\|_{L^p}. \tag{3.15}$$

Letting $k \to \infty$, we obtain

$$\sup_X e^u \leq C_1' \|e^u\|_{L^{2(\gamma+k)}}, \tag{3.16}$$

where $C_1' = \{2(C_X+1)(\gamma+k)\}^{\frac{1}{2(\gamma+k)}\sum_{i=0}^\infty \frac{1}{\beta^i}} \cdot \beta^{\frac{1}{2(\gamma+k)}\sum_{i=1}^\infty \frac{i}{\beta^i}}$.

It follows that

$$\sup_X e^u \leq C_1' (\sup_X e^u)^{1-(2(\gamma+k))^{-1}} \left(\int_X e^u\right)^{1/2(\gamma+k)}, \tag{3.17}$$

and we conclude that

$$\sup_X e^u \leq C_1 \int_X e^u, \quad C_1 = (C_1')^{2(\gamma+k)}. \tag{3.18}$$

This proves the estimate. As it will be needed in the future, we note that the precise form of C_1 agrees with the definition given in (2.7). q.e.d.

Proposition 3.3. *Suppose $u \in \Upsilon_k$ solves (2.2) subject to normalization $\int_X e^u = M$. There exists a constant C only depending on $(X, \hat\omega)$, n, k and γ such that*

$$\int_X e^{-u} \leq C M^{-1}. \tag{3.19}$$

Proof. Setting $p=-1$ in (3.5) gives

$$\begin{aligned}(k+1)&\int_X e^{-u} i\partial u \wedge \bar\partial u \wedge \hat\omega^{n-k-1} \wedge \chi' \\ =& \int_X e^{-(1+k)u}\mu \frac{\hat\omega^n}{n} - \alpha' \frac{1+k}{1+\gamma}\int_X e^{-(1-\gamma)u} i\partial\bar\partial\rho \wedge \hat\omega^{n-2} \\ \leq& \frac{1}{n}\|\mu\|_{L^\infty}\int_X e^{-(1+k)u} + \frac{1+k}{(1+\gamma)n}\|\alpha' c\|_{L^\infty}\int_X e^{-(1+\gamma)u}.\end{aligned} \tag{3.20}$$

Since $u \in \Upsilon_k$, we may use (3.8) and $e^{-\gamma u} \leq \delta \leq 1$ to obtain

$$\int_X e^{-u}|\nabla u|^2_{\hat\omega} \leq 2\delta^{\frac{\min\{k,\gamma\}}{\gamma}} \left(\|\mu\|_{L^\infty} + \|\alpha' c\|_{L^\infty}\right)\int_X e^{-u}. \tag{3.21}$$

By the Poincaré inequality

$$\int_X e^{-u} - \left(\int_X e^{-u/2}\right)^2 \leq C_X \int_X |\nabla e^{-u/2}|^2_{\hat\omega}. \tag{3.22}$$

After using the definition of δ (2.6), it follows that

$$\int_X e^{-u} \leq \frac{1}{1-\frac{\theta}{4}} \left(\int_X e^{-u/2}\right)^2. \tag{3.23}$$

Let $U = \{x \in X : e^u \geq \frac{M}{2}\}$. From Proposition 3.2, and using $\mathrm{Vol}(X, \hat\omega) = 1$,

$$M = \int_X e^u \leq C_1 M|U| + (1-|U|)\frac{M}{2}. \tag{3.24}$$

Hence $|U| \geq \theta > 0$, where we recall that θ was defined in (2.6). Using $|U| \geq \theta$ and (3.23), it was shown in [**21**] that the estimate

$$\int_X e^{-u} \leq \frac{1}{1-\frac{\theta}{4}} \left(1+\frac{2}{\theta}\right)\left(\frac{2}{\theta^2}\right) M^{-1} \tag{3.25}$$

follows. q.e.d.

Proposition 3.4. *Suppose $u \in \Upsilon_k$ solves (2.2) subject to the normalization $\int_X e^u = M$. There exists C such that*

$$\sup_X e^{-u} \leq C M^{-1}, \tag{3.26}$$

where C only depends on $(X, \hat\omega)$, n, k and γ.

Proof. Exchanging p for $-p$ in (3.5) and using (3.8) gives

$$\begin{aligned}(p+k)\int_X e^{-pu} i\partial u \wedge \bar\partial u \wedge \hat\omega^{n-1} \\ \leq\ 2\int_X e^{-(p+k)u}\mu\frac{\hat\omega^n}{n} - 2\alpha'\frac{p+k}{p+\gamma}\int_X e^{-(p+\gamma)u} i\partial\bar\partial\rho\wedge\hat\omega^{n-2}.\end{aligned} \tag{3.27}$$

By using $e^{\gamma u} \leq \delta \leq 1$, we obtain

$$\int_X |\nabla e^{-\frac{p}{2}u}|^2_{\hat\omega} \leq \frac{p^2}{2(p+k)}\delta^{\frac{\min\{k,\gamma\}}{\gamma}}\left(\|\mu\|_{L^\infty} + \frac{p+k}{p+\gamma}\|\alpha' c\|_{L^\infty}\right)\int_X e^{-pu}. \tag{3.28}$$

We may use (2.6) to obtain a constant C depending on $(X, \hat\omega)$, n, k, and γ such that

$$\int_X |\nabla e^{-\frac{p}{2}u}|^2_{\hat\omega} \leq Cp\int_X e^{-pu}, \tag{3.29}$$

for any $p \geq 1$. Using the Sobolev inequality and iterating in a similar way to Proposition 3.2, we obtain

$$\sup_X e^{-u} \leq C\|e^{-u}\|_{L^1}. \tag{3.30}$$

Applying Proposition 3.3 gives the desired estimate. q.e.d.

4. Setup and notation

4.1. The formalism of evolving metrics. We come now to the key steps of establishing the gradient and the C^2 estimates. It turns out that, for these steps, it is more natural to view the equation (2.2) as an equation for the unknown, non-Kähler, Hermitian form

$$\omega = e^u \hat{\omega} \tag{4.1}$$

and to carry out calculations with respect to the Chern unitary connection ∇ of ω. As usual, we identify the metrics $\hat{g}$ and g via $\hat{\omega} = \hat{g}_{\bar{k}j}\, idz^j \wedge d\bar{z}^k$ and $\omega = g_{\bar{k}j}\, idz^j \wedge d\bar{z}^k$, and denote $\hat{g}^{j\bar{k}}$, $g^{j\bar{k}}$ to be the inverse matrix of $\hat{g}_{\bar{k}j}$, $g_{\bar{k}j}$. Then $g_{\bar{k}j} = e^u \hat{g}_{\bar{k}j}$, $g^{j\bar{k}} = e^{-u}\hat{g}^{j\bar{k}}$. Recall that the Chern unitary connection ∇ is defined by

$$\nabla_{\bar{k}} V^j = \partial_{\bar{k}} V^j, \quad \nabla_k V^j = g^{j\bar{m}} \partial_k (g_{\bar{m}p} V^p) \tag{4.2}$$

and its torsion and curvature by

$$[\nabla_\alpha, \nabla_\beta] V^\gamma = R_{\beta\alpha}{}^\gamma{}_\delta V^\delta + T^\delta{}_{\beta\alpha} \nabla_\delta V^\gamma. \tag{4.3}$$

Explicitly,

$$R_{\bar{k}q}{}^j{}_p = -\partial_{\bar{k}}(g^{j\bar{m}} \partial_q g_{\bar{m}p}), \quad T^j{}_{pq} = g^{j\bar{m}}(\partial_p g_{\bar{m}q} - \partial_q g_{\bar{m}p}). \tag{4.4}$$

The curvatures and torsions of the metrics $g_{\bar{k}j}$ and $\hat{g}_{\bar{k}j}$ are then related by

$$R_{\bar{k}j}{}^p{}_i = \hat{R}_{\bar{k}j}{}^p{}_i - u_{\bar{k}j} \delta^p{}_i, \quad T^\lambda{}_{kj} = u_k \delta^\lambda{}_j - u_j \delta^\lambda{}_k. \tag{4.5}$$

This leads to the commutation relations

$$[\nabla_j, \nabla_{\bar{k}}] V_i = -R_{\bar{k}j}{}^p{}_i V_p = -\hat{R}_{\bar{k}j}{}^p{}_i V_p + u_{\bar{k}j} V_i, \tag{4.6}$$

$$[\nabla_j, \nabla_{\bar{k}}] V_{\bar{i}} = R_{\bar{k}j\bar{i}}{}^{\bar{p}} V_{\bar{p}} = \hat{R}_{\bar{k}j\bar{i}}{}^{\bar{p}} V_{\bar{p}} - u_{\bar{k}j} V_{\bar{i}}, \tag{4.7}$$

$$[\nabla_j, \nabla_k] V_i = T^\lambda{}_{kj} \nabla_\lambda V_i = u_k \nabla_j V_i - u_j \nabla_k V_i. \tag{4.8}$$

The following commutation formulas with either 3 or 4 covariant derivatives will be particularly useful,

$$\nabla_j \nabla_p \nabla_{\bar{q}} u = \nabla_p \nabla_{\bar{q}} \nabla_j u + T^m{}_{pj} \nabla_m \nabla_{\bar{q}} u \tag{4.9}$$

and

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j \nabla_p \nabla_{\bar{q}} u &= \nabla_p \nabla_{\bar{q}} \nabla_j \nabla_{\bar{k}} u - R_{\bar{q}p\bar{k}}{}^{\bar{m}} \nabla_{\bar{m}} \nabla_j u + R_{\bar{k}j}{}^m{}_p \nabla_m \nabla_{\bar{q}} u \\ &\quad + T^{\bar{m}}{}_{\bar{q}\bar{k}} \nabla_p \nabla_{\bar{m}} \nabla_j u + T^m{}_{pj} \nabla_{\bar{k}} \nabla_m \nabla_{\bar{q}} u. \end{aligned} \tag{4.10}$$

They reduce in our case to

$$\nabla_j \nabla_p \nabla_{\bar{q}} u = \nabla_p \nabla_{\bar{q}} \nabla_j u + u_p u_{\bar{q}j} - u_j u_{\bar{q}p}, \tag{4.11}$$

and to

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j \nabla_p \nabla_{\bar{q}} u &= \nabla_p \nabla_{\bar{q}} \nabla_j \nabla_{\bar{k}} u + u_p \nabla_{\bar{k}} \nabla_j \nabla_{\bar{q}} u - u_j \nabla_{\bar{k}} \nabla_p \nabla_{\bar{q}} u \\ &\quad + u_{\bar{q}} \nabla_p \nabla_{\bar{k}} \nabla_j u - u_{\bar{k}} \nabla_p \nabla_{\bar{q}} \nabla_j u \\ &\quad + \hat{R}_{\bar{k}j}{}^\lambda{}_p u_{\bar{q}\lambda} - \hat{R}_{\bar{q}p\bar{k}}{}^{\bar{\lambda}} u_{\bar{\lambda}j}. \end{aligned} \tag{4.12}$$

It will also be convenient to use the symmetric functions of the eigenvalues of $i\partial\bar\partial u$ with respect to ω rather than with respect to $\hat\omega$. Thus we define $\sigma_\ell(i\partial\bar\partial u)$ to be the ℓ-th elementary symmetric polynomial of the eigenvalues of the endomorphism $h^i{}_j = g^{i\bar k}u_{\bar k j}$. Explicitly, if λ_i are the eigenvalues of the endomorphism $h^i{}_j = g^{i\bar k}u_{\bar k j}$, then $\sigma_\ell(i\partial\bar\partial u) = \sum_{i_1<\cdots<i_\ell}\lambda_{i_1}\cdots\lambda_{i_\ell}$. Using this formalism, equation (2.2) becomes

(4.13)
$$\Delta_g u + k|\nabla u|_g^2 + \alpha' e^{-(k+1)u}L_\rho e^{(k-\gamma)u} + \alpha'\sigma_{k+1}(i\partial\bar\partial u) - e^{-(k+1)u}\mu = 0.$$

4.2. Differentiating Hessian operators. We define

$$\sigma_\ell^{p\bar q} = \frac{\partial\sigma_\ell}{\partial h^r{}_p}g^{r\bar q},\quad \sigma_\ell^{p\bar q,r\bar s} = \frac{\partial^2\sigma_\ell}{\partial h^a{}_p\partial h^b{}_r}g^{a\bar q}g^{b\bar s}. \tag{4.14}$$

Then the variational formula $\delta\sigma_\ell = \frac{\partial\sigma_\ell}{\partial h^r{}_p}\delta h^r{}_p$ becomes

$$\nabla_i\sigma_\ell = \sigma_\ell^{p\bar q}\nabla_i u_{\bar q p}. \tag{4.15}$$

Similarly,

$$\nabla_{\bar j}\sigma_\ell^{p\bar q} = \sigma_\ell^{p\bar q,r\bar s}\nabla_{\bar j}u_{\bar s r}. \tag{4.16}$$

We will use a general formula for differentiating a function of eigenvalues of a matrix. Let $F(h) = f(\lambda_1,\cdots,\lambda_n)$ be a symmetric function of the eigenvalues of a Hermitian matrix h. Then at a diagonal matrix h, we have (see [**1**, **13**]),

$$\frac{\partial F}{\partial h^i{}_j} = \delta_{ij}f_i, \tag{4.17}$$

$$\sum\frac{\partial^2 F}{\partial h^i{}_j\partial h^r{}_s}T^i{}_jT^r{}_s = \sum f_{ij}T^i{}_iT^j{}_j + \sum_{p\neq q}\frac{f_p - f_q}{\lambda_p-\lambda_q}|T^p{}_q|^2, \tag{4.18}$$

for any Hermitian matrix T. Since $\sigma_\ell(h) = \sum_{i_1<\cdots<i_\ell}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_\ell}$, this formula implies that at a point $p\in X$ where g is the identity and $u_{\bar q p}$ is diagonal, then

$$\sigma_\ell^{p\bar q} = \delta_{pq}\sigma_{\ell-1}(\lambda|p), \tag{4.19}$$

(4.20)
$$\sigma_\ell^{p\bar q,r\bar s}\nabla_i u_{\bar q p}\nabla_{\bar i}u_{\bar s r} = \sum_{p,q}\sigma_{\ell-2}(\lambda|pq)\nabla_i u_{\bar p p}\nabla_{\bar i}u_{\bar q q} - \sum_{p\neq q}\sigma_{\ell-2}(\lambda|pq)|\nabla_i u_{\bar q p}|^2.$$

We introduced the notation $\sigma_m(\lambda|p)$ and $\sigma_m(\lambda|pq)$ for the m-th elementary symmetric polynomial of

$$(\lambda|i) = (\lambda_1,\cdots,\widehat{\lambda_i},\cdots,\lambda_n)\in\mathbf{R}^{n-1}$$

and

$$(\lambda|ij) = (\lambda_1,\cdots,\widehat{\lambda_i},\cdots,\widehat{\lambda_j},\cdots,\lambda_n)\in\mathbf{R}^{n-2}.$$

Lastly, we introduce the tensor $F^{p\bar{q}}$, which will appear in subsequent sections when we differentiate the Fu-Yau equation.

$$F^{p\bar{q}} = g^{p\bar{q}} + \alpha'(k-\gamma)e^{-(1+\gamma)u}a^{p\bar{q}} + \alpha'\sigma_{k+1}^{p\bar{q}}. \tag{4.21}$$

We will prove that for $u \in \Upsilon_k$, $F^{p\bar{q}}$ is close to the metric $g^{p\bar{q}}$. For this, we first note the following elementary estimate.

Lemma 4.1. *Let m be a positive integer and $\ell \in \{1, \dots, m\}$. For any vector $\lambda \in \mathbf{R}^m$,*

$$|\sigma_\ell(\lambda)| \leq \frac{C_m^\ell}{m^{\ell/2}}|\lambda|^\ell \tag{4.22}$$

with $|\lambda| = \left(\sum_{i=1}^n \lambda_i^2\right)^{1/2}$. Here, $\sigma_\ell(\lambda)$ is the ℓ-th elementary symmetric polynomial of λ and $C_m^\ell = \frac{m!}{\ell!(m-\ell)!}$.

Proof. Using the Newton-Maclaurin inequality,

$$|\sigma_\ell(\lambda)| \leq \sigma_\ell(|\lambda_1|, \dots, |\lambda_m|) \leq C_m^\ell \left(\frac{\sum_i^m |\lambda_i|}{m}\right)^\ell. \tag{4.23}$$

The Cauchy-Schwarz inequality now gives the desired estimate. q.e.d.

We can now prove the following simple but important lemma regarding the ellipticity of $F^{p\bar{q}}$.

Lemma 4.2. *If $u \in \Upsilon_k$, then*

$$(1-2^{-6})g^{p\bar{q}} \leq F^{p\bar{q}} \leq (1+2^{-6})g^{p\bar{q}}. \tag{4.24}$$

Proof. First, at a point z where $g^{p\bar{q}} = \delta_{pq}$ and $u_{\bar{q}p}$ is diagonal, the above lemma implies

$$|\alpha'\sigma_{k+1}^{p\bar{p}}| = |\alpha'\sigma_k(\lambda|p)| \leq |\alpha'|\frac{C_{n-1}^k}{(n-1)^{k/2}}|\nabla\bar{\nabla}u|_g^k. \tag{4.25}$$

The condition $u \in \Upsilon_k$ gives $|\alpha'\sigma_{k+1}^{p\bar{p}}(z)| \leq 2^{-7}$. This argument shows that $\alpha'\sigma_{k+1}^{p\bar{q}}$ is on the order of $2^{-7}g^{p\bar{q}}$ in arbitrary coordinates.

Next, $u \in \Upsilon_k$ also implies that $|\alpha'(k-\gamma)e^{-\gamma u}\Lambda| \leq 2^{-7}$. Since $-\Lambda\hat{g}^{p\bar{q}} \leq a^{p\bar{q}} \leq \Lambda\hat{g}^{p\bar{q}}$, we can put everything together and obtain the estimate (4.24). q.e.d.

5. Gradient estimate

The main goal of this section is to establish Theorem 5.1 below, which gives C^1 estimates with scale. A key tool is the test function in (5.3) below, which was introduced in the paper [**23**] on the Anomaly flow.

Theorem 5.1. *Let $u \in \Upsilon_k$ be a $C^3(X,\mathbf{R})$ function solving the Fu-Yau Hessian equation (2.2). Then*

$$|\nabla u|^2_{\hat{g}} \leq C, \tag{5.1}$$

where C only depends on $(X,\hat{\omega})$, α', k, γ, $\|\rho\|_{C^3(X,\hat{\omega})}$ and $\|\mu\|_{C^1(X)}$.

In view of Theorem 3.1, this estimate is equivalent to

$$|\nabla u|^2_{g} \leq CM^{-1}, \tag{5.2}$$

where C only depends on $(X,\hat{\omega})$, α', k, γ, $\|\rho\|_{C^3(X,\hat{\omega})}$ and $\|\mu\|_{C^1(X)}$. We will prove this estimate by applying the maximum principle to the following test function

$$G = \log |\nabla u|^2_g + (1+\sigma)u, \tag{5.3}$$

for a parameter $0 < \sigma < 1$. Though there is a range of values of σ which makes the argument work, to be concrete we will take $\sigma = 2^{-7}$.

5.1. Estimating the leading terms. Suppose G attains a maximum at $p \in X$. Then

$$0 = \frac{\nabla |\nabla u|^2_g}{|\nabla u|^2_g} + (1+\sigma)\nabla u. \tag{5.4}$$

We will compute the operator $F^{p\bar{q}}\nabla_p \nabla_{\bar{q}}$ acting on G at p.

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}} G = {} & \frac{1}{|\nabla u|^2_g} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|^2_g \\ & - \frac{1}{|\nabla u|^4_g} F^{p\bar{q}}\nabla_p|\nabla u|^2_g \nabla_{\bar{q}}|\nabla u|^2_g + (1+\sigma)F^{p\bar{q}}u_{\bar{q}p}. \end{aligned} \tag{5.5}$$

By direct computation

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|^2_g &= F^{p\bar{q}}g^{j\bar{i}}\nabla_p\nabla_{\bar{q}}\nabla_j u\nabla_{\bar{i}}u + F^{p\bar{q}}g^{j\bar{i}}\nabla_j u\nabla_p\nabla_{\bar{q}}\nabla_{\bar{i}}u \\ & \quad + |\nabla\bar{\nabla}u|^2_{Fg} + |\nabla\nabla u|^2_{Fg}, \end{aligned} \tag{5.6}$$

where $|\nabla\nabla u|^2_{Fg} = F^{p\bar{q}}g^{j\bar{i}}\nabla_p\nabla_j u\nabla_{\bar{q}}\nabla_{\bar{i}}u$ and $|\nabla\bar{\nabla}u|^2_{Fg} = F^{p\bar{q}}g^{j\bar{i}}u_{\bar{q}j}u_{\bar{i}p}$. Commuting derivatives according to the relation

$$[\nabla_j, \nabla_{\bar{\ell}}]u_{\bar{i}} = R_{\bar{\ell}j\bar{i}}{}^{\bar{p}}u_{\bar{p}} = \hat{R}_{\bar{\ell}j\bar{i}}{}^{\bar{p}}u_{\bar{p}} - u_{\bar{\ell}j}u_{\bar{i}}, \tag{5.7}$$

we obtain

$$\begin{aligned} F^{p\bar{q}}g^{j\bar{i}}\nabla_j u\nabla_p\nabla_{\bar{q}}\nabla_{\bar{i}}u = {} & \overline{F^{p\bar{q}}g^{j\bar{i}}\nabla_p\nabla_{\bar{q}}\nabla_j u\nabla_{\bar{i}}u} \\ & + F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} - F^{p\bar{q}}g^{j\bar{i}}u_j u_{\bar{q}p}u_{\bar{i}}. \end{aligned} \tag{5.8}$$

Thus

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|^2_g &= 2\,\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}\nabla_p\nabla_{\bar{q}}\nabla_j u\nabla_{\bar{i}}u\} + F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} \\ & \quad - F^{p\bar{q}}g^{j\bar{i}}u_j u_{\bar{q}p}u_{\bar{i}} + |\nabla\bar{\nabla}u|^2_{Fg} + |\nabla\nabla u|^2_{Fg}. \end{aligned} \tag{5.9}$$

Next, we use the equation. Expanding $L_\rho = a^{p\bar{q}}\partial_p\partial_{\bar{q}} + b^i\partial_i + \bar{b}^{\bar{i}}\partial_{\bar{i}} + c$, equation (4.13) becomes

$$\begin{aligned} 0 &= \Delta_g u + \alpha' \Big\{ (k-\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{q}p} + \sigma_{k+1}(i\partial\bar{\partial}u) \Big\} + k|\nabla u|_g^2 \\ & \quad + \alpha'(k-\gamma)^2 e^{-(1+\gamma)u} a^{p\bar{q}} u_p u_{\bar{q}} + 2\alpha'(k-\gamma) e^{-(1+\gamma)u} \,\mathrm{Re}\{b^i u_i\} \\ & \quad + \alpha' e^{-(1+\gamma)u} c - e^{-(k+1)u}\mu. \end{aligned} \tag{5.10}$$

We covariantly differentiate equation (5.10), using (4.15) to differentiate σ_{k+1} and using the notation $F^{p\bar{q}}$ introduced in (4.21). This leads to

$$0 = F^{p\bar{q}} \nabla_j \nabla_p \nabla_{\bar{q}} u + k \nabla_j |\nabla u|_g^2 + \mathcal{E}_j, \tag{5.11}$$

where

$$\begin{aligned} \mathcal{E}_j &= \alpha'(k-\gamma) e^{-(1+\gamma)u} \Big\{ -\gamma a^{p\bar{q}} u_{\bar{q}p} u_j + \hat{\nabla}_j a^{p\bar{q}} u_{\bar{q}p} \Big\} \\ & \quad + \alpha'(k-\gamma)^2 e^{-(1+\gamma)u} \Big\{ -\gamma a^{p\bar{q}} u_p u_{\bar{q}} u_j + \hat{\nabla}_j a^{p\bar{q}} u_p u_{\bar{q}} + a^{p\bar{q}} \nabla_j \nabla_p u u_{\bar{q}} \\ & \quad + a^{p\bar{q}} u_p u_{\bar{q}j} \Big\} + \alpha'(k-\gamma) e^{-(1+\gamma)u} \Big\{ -2(1+\gamma) \,\mathrm{Re}\{b^i u_i\} u_j + \hat{\nabla}_j b^i u_i \\ & \quad + u_j b^i u_i + \partial_j \bar{b}^{\bar{i}} u_{\bar{i}} + b^i \nabla_j \nabla_i u + \bar{b}^{\bar{i}} u_{\bar{i}j} \Big\} \\ & \quad - (1+\gamma) \alpha' e^{-(1+\gamma)u} c u_j + \alpha' e^{-(1+\gamma)u} \partial_j c \\ & \quad + (k+1) e^{-(k+1)u} \mu u_j - e^{-(k+1)u} \partial_j \mu. \end{aligned} \tag{5.12}$$

We used $\nabla_i W^j = \hat{\nabla}_i W^j + u_i W^j$ to replace ∇ by $\hat{\nabla}$ in the above calculation. We will eventually see that the terms $\mathcal{E}_j$ play a minor role when $u \in \Upsilon_k$, and will only perturb the coefficients of the leading terms. Commuting covariant derivatives using (4.11), we obtain

$$F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} \nabla_j u = -F^{p\bar{q}} u_p u_{\bar{q}j} + F^{p\bar{q}} u_j u_{\bar{q}p} - k \nabla_j |\nabla u|_g^2 - \mathcal{E}_j. \tag{5.13}$$

Substituting (5.13) into (5.9), an important partial cancellation occurs, and we obtain

$$\begin{aligned} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla u|_g^2 &= -2 \,\mathrm{Re}\{F^{p\bar{q}} g^{j\bar{i}} u_{\bar{i}} u_p u_{\bar{q}j}\} + |\nabla u|_g^2 F^{p\bar{q}} u_{\bar{q}p} \\ & \quad -2k \,\mathrm{Re}\{g^{j\bar{i}} \nabla_{\bar{i}} u \nabla_j |\nabla u|_g^2\} - 2\,\mathrm{Re}\{g^{j\bar{i}} \mathcal{E}_j u_{\bar{i}}\} \\ & \quad + F^{p\bar{q}} g^{j\bar{i}} u_j \hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}} u_{\bar{\lambda}} + |\nabla \bar{\nabla} u|_{Fg}^2 + |\nabla \nabla u|_{Fg}^2. \end{aligned} \tag{5.14}$$

We note the identity

$$F^{p\bar{q}} u_{\bar{q}p} = \Delta_g u + \alpha'(k-\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{q}p} + (k+1)\alpha' \sigma_{k+1}(i\partial\bar{\partial}u). \tag{5.15}$$

Substituting the equation (5.10) into the identity (5.15), we obtain

$$F^{p\bar{q}} u_{\bar{q}p} = -k|\nabla u|_g^2 + \tilde{\mathcal{E}}, \tag{5.16}$$

where

$$\begin{aligned}\tilde{\mathcal{E}} &= k\alpha'\sigma_{k+1}(i\partial\bar{\partial}u) - \alpha'(k-\gamma)^2 e^{-(1+\gamma)u} a^{p\bar{q}} u_p u_{\bar{q}} \\ &\quad -2\alpha'(k-\gamma)e^{-(1+\gamma)u}\,\mathrm{Re}\{b^i u_i\} - \alpha' e^{-(1+\gamma)u} c + e^{-(k+1)u}\mu, \end{aligned} \tag{5.17}$$

will turn out to be another perturbative term. Substituting (5.14) and (5.16) into (5.5)

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G &= \frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla}u|^2_{Fg} + \frac{1}{|\nabla u|_g^2}|\nabla\nabla u|^2_{Fg} \\ &\quad -\frac{2}{|\nabla u|_g^2}\,\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_{\bar{i}}u_p u_{\bar{q}j}\} \\ &\quad -\frac{1}{|\nabla u|_g^4}F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla u|_g^2 \\ &\quad -2k\frac{1}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar{i}}u_{\bar{i}}\nabla_j|\nabla u|_g^2\} \\ &\quad -(2+\sigma)k|\nabla u|_g^2 + \frac{1}{|\nabla u|_g^2}F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} \\ &\quad -\frac{2}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}\} + (2+\sigma)\tilde{\mathcal{E}}. \end{aligned} \tag{5.18}$$

Using the critical equation (5.4),

$$\begin{aligned} &-\frac{1}{|\nabla u|_g^4}F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla u|_g^2 - 2k\frac{1}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar{i}}u_{\bar{i}}\nabla_j|\nabla u|_g^2\} \\ &= -(1+\sigma)^2|\nabla u|_F^2 + 2(1+\sigma)k|\nabla u|_g^2. \end{aligned} \tag{5.19}$$

Here we introduced the notation $|\nabla f|_F^2 = F^{p\bar{q}}f_p f_{\bar{q}}$ for a real-valued function f. The critical equation (5.4) can also be written as

$$\frac{g^{j\bar{i}}\nabla_p u_j u_{\bar{i}}}{|\nabla u|_g^2} = -\frac{g^{j\bar{i}}u_j u_{\bar{i}p}}{|\nabla u|_g^2} - (1+\sigma)u_p. \tag{5.20}$$

We now combine this identity with the Cauchy-Schwarz inequality, which will lead to a partial cancellation of terms. This idea is also used to derive a C^1 estimate for the complex Monge-Ampère equation, [**2**, **15**, **24**, **25**, **36**].

$$\begin{aligned} \frac{1}{|\nabla u|_g^2}|\nabla\nabla u|^2_{Fg} &\geq \left|\frac{g^{j\bar{i}}\nabla u_j u_{\bar{i}}}{|\nabla u|_g^2}\right|_F^2 \\ &= \frac{1}{|\nabla u|_g^4}|g^{j\bar{i}}u_j\nabla u_{\bar{i}}|_F^2 + (1+\sigma)^2|\nabla u|_F^2 \\ &\quad +\frac{2(1+\sigma)}{|\nabla u|_g^2}\,\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_j u_{\bar{i}p}u_{\bar{q}}\}. \end{aligned} \tag{5.21}$$

Let $\varepsilon > 0$. Combining (5.19) and (5.21) and dropping a nonnegative term,

$$\begin{aligned}(5.22)\qquad & -\frac{1}{|\nabla u|_g^4} F^{p\bar q}\nabla_p|\nabla u|_g^2\nabla_{\bar q}|\nabla u|_g^2 - \frac{2k}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar i}u_{\bar i}\nabla_j|\nabla u|_g^2\} \\ & +(1-\varepsilon)\frac{1}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \\ \geq\ & -(1+\sigma)^2\varepsilon|\nabla u|_F^2 + 2(1+\sigma)k|\nabla u|_g^2 \\ & +\frac{2(1+\sigma)(1-\varepsilon)}{|\nabla u|_g^2}\,\mathrm{Re}\{F^{p\bar q}g^{j\bar i}u_j u_{\bar i p}u_{\bar q}\}.\end{aligned}$$

Substituting this inequality into (5.18), partial cancellation occurs and we are left with

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}G \ \geq\ & \frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_{Fg}^2 + \frac{\varepsilon}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \\ & +\{2\sigma - 2\varepsilon(1+\sigma)\}\frac{1}{|\nabla u|_g^2}\,\mathrm{Re}\{F^{p\bar q}g^{j\bar i}u_{\bar i}u_p u_{\bar q j}\} \\ & +\sigma k|\nabla u|_g^2 - (1+\sigma)^2\varepsilon|\nabla u|_F^2 + \frac{1}{|\nabla u|_g^2}F^{p\bar q}g^{j\bar i}u_j \hat R_{\bar q p\bar i}{}^{\bar\lambda}u_{\bar\lambda} \\ (5.23)\qquad & -\frac{2}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar i}\mathcal{E}_j u_{\bar i}\} + (2+\sigma)\tilde{\mathcal{E}}.\end{aligned}$$

Since $u \in \Upsilon_k$, we now use (4.24) in Lemma 4.2 to pass the norms with respect to $F^{p\bar q}$ to $g^{p\bar q}$ up to an error of order 2^{-6}. We choose

$$\varepsilon = (1+\sigma)^{-2}(1+2^{-6})^{-1}\frac{\sigma}{2}. \tag{5.24}$$

Then

$$(1+\sigma)^2\varepsilon|\nabla u|_F^2 \leq \frac{\sigma}{2}|\nabla u|_g^2, \tag{5.25}$$

and

$$\frac{\varepsilon}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \geq \frac{\sigma}{2(1+\sigma)^2}\frac{1-2^{-6}}{1+2^{-6}}\frac{1}{|\nabla u|_g^2}|\nabla\nabla u|_g^2. \tag{5.26}$$

Since $\sigma = 2^{-7}$, we have the inequality of numbers $\frac{1}{2}\frac{1-2^{-6}}{(1+\sigma)^2(1+2^{-6})} \geq \frac{1}{4}$. Thus

$$\frac{\varepsilon}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \geq \frac{\sigma}{4}\frac{1}{|\nabla u|_g^2}|\nabla\nabla u|_g^2. \tag{5.27}$$

We also note the inequalities

$$\frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_{Fg}^2 \geq (1-2^{-6})\frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_g^2, \tag{5.28}$$

and

$$\begin{aligned}&\{2\sigma - 2\varepsilon(1+\sigma)\}\frac{1}{|\nabla u|_g^2}\,\mathrm{Re}\{F^{p\bar q}g^{j\bar i}u_{\bar i}u_p u_{\bar q j}\}\\ \geq\ & -\{2-(1+\sigma)^{-1}(1+2^{-6})^{-1}\}\sigma(1+2^{-6})|\nabla\bar\nabla u|_g\\ \geq\ & -2\sigma(1+2^{-6})|\nabla\bar\nabla u|_g. \end{aligned}\tag{5.29}$$

The main inequality (5.23) becomes

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}G \ \geq\ & (1-2^{-6})\frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_g^2 + \frac{\sigma}{4}\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2}\\ & -2\sigma(1+2^{-6})|\nabla\bar\nabla u|_g + \frac{\sigma}{2}|\nabla u|_g^2\\ & +\frac{1}{|\nabla u|_g^2}F^{p\bar q}g^{j\bar i}u_j\hat R_{\bar q p\bar i}{}^{\bar\lambda}u_{\bar\lambda}\\ & -\frac{2}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar i}\mathcal{E}_j u_{\bar i}\} + (2+\sigma)\tilde{\mathcal{E}}. \end{aligned}\tag{5.30}$$

5.2. Estimating the perturbative terms.

5.2.1. The $\mathcal{E}_j$ terms. Recall the constant Λ is such that $-\Lambda\hat g^{j\bar i} \leq a^{j\bar i} \leq \Lambda\hat g^{j\bar i}$. We will go through each term in the definition of $\mathcal{E}_j$ (5.12) and estimate the terms appearing in $\frac{2}{|\nabla u|_g^2}\,\mathrm{Re}\{g^{j\bar i}\mathcal{E}_j u_{\bar i}\}$ by groups. In the following, we will use C to denote constants possibly depending on α', k, γ, $a^{p\bar q}$, b^i, c, μ, and their derivatives.

First, using $2ab \leq a^2+b^2$ and $e^{-\gamma u} \leq \delta$, we estimate the terms involving $\nabla\bar\nabla u$

$$\begin{aligned} & \frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar i}u_{\bar i}(-\gamma a^{p\bar q}u_{\bar q p}u_j + \hat\nabla_j a^{p\bar q}u_{\bar q p}\\ & +(k-\gamma)a^{p\bar q}u_p u_{\bar q j} + \bar b^q u_{\bar q j})|\\ \leq\ & 2|\alpha'\Lambda(k-\gamma)(k+2\gamma)|e^{-\gamma u}|\nabla\bar\nabla u|_g + Ce^{-\gamma u}e^{-u/2}\frac{|\nabla\bar\nabla u|_g}{|\nabla u|_g}\\ \leq\ & 2\Big\{|\alpha'\Lambda|^{1/2}(k-\gamma)|\delta^{1/2}|\nabla u|_g\Big\}\Big\{\delta^{1/2}(k+2\gamma)|\Lambda\alpha'|^{1/2}\frac{|\nabla\bar\nabla u|_g}{|\nabla u|_g}\Big\}\\ & +Ce^{-u/2}\frac{|\nabla\bar\nabla u|_g}{|\nabla u|_g}\\ \leq\ & |\alpha'|\Lambda(k-\gamma)^2\delta|\nabla u|_g^2 + 4|\Lambda\alpha'|(k+\gamma)^2\delta\frac{|\nabla\bar\nabla u|_g^2}{|\nabla u|_g^2} + \sigma\frac{|\nabla\bar\nabla u|_g^2}{|\nabla u|_g^2}\\ & +C(\sigma)e^{-u}. \end{aligned}$$

Second, we estimate the terms involving $\nabla\nabla u$

$$\begin{aligned}
&\frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar{i}}u_{\bar{i}}\{(k-\gamma)a^{p\bar{q}}\nabla_j\nabla_p u\, u_{\bar{q}}+b^p\nabla_j\nabla_p u\}| \\
\leq\ & 2|\alpha'|(k-\gamma)^2\Lambda e^{-\gamma u}|\nabla\nabla u|_g \\
&+2\left\{\frac{C}{|\alpha'\Lambda|^{1/2}}e^{-(1+\gamma)u/2}\right\}\left\{|\alpha'\Lambda|^{1/2}|k-\gamma|e^{-\gamma u/2}\frac{|\nabla\nabla u|_g}{|\nabla u|_g}\right\} \\
\leq\ & |\alpha'|(k-\gamma)^2\Lambda\delta\left\{\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2}+|\nabla u|_g^2\right\}+|\alpha'\Lambda|(k-\gamma)^2e^{-\gamma u}\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} \\
&+\frac{C^2}{|\alpha'\Lambda|}e^{-(1+\gamma)u} \\
\leq\ & 2|\alpha'|\Lambda(k-\gamma)^2\delta\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2}+\delta|\alpha'|(k-\gamma)^2\Lambda|\nabla u|_g^2+Ce^{-u}.
\end{aligned}$$

Third, we estimate the terms involving ∇u quadratically

$$\begin{aligned}
&\frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar{i}}u_{\bar{i}}\{(k-\gamma)\hat{\nabla}_j a^{p\bar{q}}u_p u_{\bar{q}} \\
&-2(1+\gamma)\mathrm{Re}\{b^p u_p\}u_j+u_j b^p u_p\}| \\
\leq\ & Ce^{-\gamma u}e^{-u/2}|\nabla u|_g\leq\frac{\sigma}{16}|\nabla u|_g^2+C(\sigma)e^{-(1+2\gamma)u} \\
\leq\ & \frac{\sigma}{16}|\nabla u|_g^2+Ce^{-u}.
\end{aligned}$$

Finally, for all the other terms in $\mathcal{E}_j$, we can estimate

$$\begin{aligned}
&\frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar{i}}u_{\bar{i}}\{-\gamma(k-\gamma)a^{p\bar{q}}u_p u_{\bar{q}}u_j+\hat{\nabla}_j b^p u_p+\partial_j\bar{b}^q u_{\bar{q}}\}| \\
&+\frac{2}{|\nabla u|_g^2}|g^{j\bar{i}}u_{\bar{i}}\{-(1+\gamma)\alpha' c e^{-(1+\gamma)u}u_j+\alpha' e^{-(1+\gamma)u}\partial_j c \\
&+(k+1)e^{-(k+1)u}\mu u_j-e^{-(k+1)u}\partial_j\mu\}| \\
&\leq 2|\alpha'|\Lambda(k-\gamma)^2\gamma e^{-\gamma u}|\nabla u|_g^2+Ce^{-(1+\gamma)u}+Ce^{-(1+\gamma)u}\frac{e^{-u/2}}{|\nabla u|_g} \\
&+Ce^{-(k+1)u}+Ce^{-(k+1)u}\frac{e^{-u/2}}{|\nabla u|_g} \\
&\leq 2|\alpha'|\Lambda(k-\gamma)^2\gamma\delta|\nabla u|_g^2+Ce^{-u}+Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}.
\end{aligned}$$

Putting everything together, we obtain the following estimate for the terms coming from $\mathcal{E}_j$.

$$\begin{aligned}\frac{2}{|\nabla u|_g^2}|g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}| \;\leq\;& \left\{2|\alpha'|\Lambda(k-\gamma)^2(1+\gamma)\delta+\frac{\sigma}{16}\right\}|\nabla u|_g^2+Ce^{-u}\\ &+Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}+\{4|\alpha'|\Lambda(k+\gamma)^2\delta+\sigma\}\frac{|\nabla\bar{\nabla}u|_g^2}{|\nabla u|_g^2}\\ &+2|\alpha'|\Lambda(k-\gamma)^2\delta\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2}.\end{aligned}\tag{5.31}$$

5.2.2. The $\tilde{\mathcal{E}}$ terms. Next, estimating $\tilde{\mathcal{E}}$ defined in (5.17) gives

$$\begin{aligned}(2+\sigma)|\tilde{\mathcal{E}}| \leq\;& k(2+\sigma)|\alpha'||\sigma_{k+1}(i\partial\bar{\partial}u)|+(2+\sigma)|\alpha'\Lambda|(k-\gamma)^2e^{-\gamma u}|\nabla u|_g^2\\ &+2\|\alpha'(k-\gamma)b^i\|_{L^\infty}e^{-\gamma u}e^{-u/2}|\nabla u|_g\\ &+Ce^{-(1+\gamma)u}+Ce^{-(k+1)u}.\end{aligned}$$

Using $e^{-\gamma u}\leq\delta\leq 1$ and

$$2\|\alpha'(k-\gamma)b\|_{L^\infty}e^{-\gamma u}e^{-u/2}|\nabla u|_g\leq\frac{\sigma}{16}|\nabla u|_g^2+C(\sigma)e^{-u}e^{-2\gamma u},\tag{5.32}$$

we obtain

$$\begin{aligned}(2+\sigma)|\tilde{\mathcal{E}}| \;\leq\;& k(2+\sigma)|\alpha'||\sigma_{k+1}(i\partial\bar{\partial}u)|\\ &+(2+\sigma)|\alpha'\Lambda|(k-\gamma)^2\delta|\nabla u|_g^2+\frac{\sigma}{16}|\nabla u|_g^2+Ce^{-u}.\end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned}k|\alpha'||\sigma_{k+1}(i\partial\bar{\partial}u)| \;\leq\;& k|\alpha'|\frac{C_n^{k+1}}{n^{1/2}n^{k/2}}|\nabla\bar{\nabla}u|_g^k|\nabla\bar{\nabla}u|_g\\ \leq\;& \{|\alpha'|C_{n-1}^k|\nabla\bar{\nabla}u|_g^k\}|\nabla\bar{\nabla}u|_g.\end{aligned}$$

Since $u\in\Upsilon_k$, we have $|\alpha'|C_{n-1}^k|\nabla\bar{\nabla}u|_g^k\leq 2^{-7}$. Thus

$$\begin{aligned}(2+\sigma)|\tilde{\mathcal{E}}| \;\leq\;& \left\{(2+\sigma)|\alpha'\Lambda|(k-\gamma)^2\delta+\frac{\sigma}{16}\right\}|\nabla u|_g^2\\ &+2^{-7}(2+\sigma)|\nabla\bar{\nabla}u|_g+Ce^{-u}.\end{aligned}\tag{5.33}$$

5.3. Completing the estimate. Combining (5.31) and (5.33),

$$\begin{aligned}&\frac{2}{|\nabla u|_g^2}|g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}|+(2+\sigma)|\tilde{\mathcal{E}}|\\ \leq\;& \left\{5|\alpha'|\Lambda(k-\gamma)^2(1+\gamma)\delta+\frac{\sigma}{8}\right\}|\nabla u|_g^2\\ &+2|\alpha'\Lambda|(k-\gamma)^2\delta\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2}+\{4|\alpha'|\Lambda(k+\gamma)^2\delta+\sigma\}\frac{|\nabla\bar{\nabla}u|_g^2}{|\nabla u|_g^2}\\ &+2^{-7}(2+\sigma)|\nabla\bar{\nabla}u|_g+Ce^{-u}+Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}.\end{aligned}\tag{5.34}$$

Since $\sigma = 2^{-7}$ and $(k-\gamma)^2(1+\gamma) \leq (k+\gamma)^3$, the definition (2.6) of δ implies

$$5|\alpha'|\Lambda(k-\gamma)^2(1+\gamma)\delta \leq \frac{\sigma}{8}; \qquad 4|\alpha'\Lambda|(k+\gamma)^2\delta \leq 2^{-7}.$$

Then, we have

$$\begin{aligned} &\frac{2}{|\nabla u|_g^2}|g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}| + (2+\sigma)|\tilde{\mathcal{E}}| \\ \leq\ & \frac{\sigma}{4}|\nabla u|_g^2 + \frac{\sigma}{4}\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} + 2^{-6}\frac{|\nabla\bar{\nabla} u|_g^2}{|\nabla u|_g^2} + 2^{-7}(2+\sigma)|\nabla\bar{\nabla} u|_g \\ & + Ce^{-u} + Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}. \end{aligned} \tag{5.35}$$

Using (5.35), the main inequality (5.30) becomes

$$\begin{aligned} & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \\ \geq\ & (1-2^{-5})\frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla} u|_g^2 - \left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}|\nabla\bar{\nabla} u|_g \\ & + \frac{\sigma}{4}|\nabla u|_g^2 + \frac{1}{|\nabla u|_g^2}F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} - Ce^{-u} - Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}. \end{aligned} \tag{5.36}$$

By our choice $\sigma = 2^{-7}$, we have the inequality of numbers

$$\left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}^2 \frac{1}{1-2^{-5}} \leq \frac{\sigma}{2}. \tag{5.37}$$

Thus

$$\begin{aligned} & \left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}|\nabla\bar{\nabla} u|_g \\ \leq\ & (1-2^{-5})\frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla} u|_g^2 \\ & + \frac{1}{4}\left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}^2 \frac{1}{1-2^{-5}}|\nabla u|_g^2 \\ \leq\ & (1-2^{-5})\frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla} u|_g^2 + \frac{\sigma}{8}|\nabla u|_g^2. \end{aligned}$$

We may also estimate

$$\frac{1}{|\nabla u|_g^2}F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} \geq -Ce^{-u}. \tag{5.38}$$

Putting everything together, at p there holds

$$0 \geq F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \geq \frac{\sigma}{8}|\nabla u|_g^2 - \frac{Ce^{-u}e^{-u/2}}{|\nabla u|_g} - Ce^{-u}. \tag{5.39}$$

From this inequality, we can conclude

$$|\nabla u|_g^2(p) \leq Ce^{-u(p)}. \tag{5.40}$$

By definition $G(x) \leq G(p)$, and we have

$$|\nabla u|_g^2 \leq Ce^{-u(p)} e^{(1+\sigma)(u(p)-u)} \leq CM^{-1}, \tag{5.41}$$

since $e^{u(p)}e^{-u} \leq C$ and $e^{-u} \leq CM^{-1}$ by Theorem 3.1. This completes the proof of Theorem 5.1.

6. Second order estimate

The main goal of this section is to establish Theorem 6.1 below, which gives C^2 estimates with scale. A key tool is the test function in (6.19) below, which was indeed introduced in the paper [**23**] on the Anomaly flow.

Theorem 6.1. *Let $u \in \Upsilon_k$ be a $C^4(X)$ function with normalization $\int_X e^u \, \hat{\omega}^n = M$ solving the Fu-Yau equation (2.2). Then*

$$|\nabla\bar{\nabla} u|_g^2 \leq CM^{-1}, \tag{6.1}$$

where C only depends on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^4(X,\hat{\omega})}$ and $\|\mu\|_{C^2(X)}$.

We begin by noting the following elementary estimate.

Lemma 6.2. *Let $\ell \in \{2, 3, \ldots, n\}$. The following estimate holds:*

$$|g^{j\bar{i}}\sigma_\ell^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_{\bar{i}} u_{\bar{s}r}| \leq C_{n-2}^{\ell-2}|\nabla\bar{\nabla} u|_g^{\ell-2}|\nabla\bar{\nabla}\nabla u|_g^2. \tag{6.2}$$

Proof. Since the inequality is invariant, we may work at a point $p \in X$ where g is the identity and $u_{\bar{q}p}$ is diagonal. At p, we can use (4.20) and conclude

$$|g^{j\bar{i}}\sigma_\ell^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_{\bar{i}} u_{\bar{s}r}| \leq \sum_i \sum_{p,q} |\sigma_{\ell-2}(\lambda|pq)||\nabla_i u_{\bar{q}p}|^2. \tag{6.3}$$

By Lemma 4.1,

$$|\sigma_{\ell-2}(\lambda|pq)| \leq \frac{C_{n-2}^{\ell-2}}{(n-2)^{(\ell-2)/2}}|\nabla\bar{\nabla} u|_g^{\ell-2}. \tag{6.4}$$

This inequality proves the Lemma. q.e.d.

6.1. Differentiating the norm of second derivatives.

Lemma 6.3. *Let $u \in \Upsilon_k$ be a $C^4(X)$ function solving (2.2) with normalization $\int_X e^u = M$. There exists a constant $C > 0$ depending only on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^4(X,\hat{\omega})}$ and $\|\mu\|_{C^2(X)}$ such that*

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\bar{\nabla} u|_g^2 \geq\ & 2(1-2^{-5})|\nabla\bar{\nabla}\nabla u|_g^2 - (1+2k)|\alpha'|^{-1/k}\tau^{1/k}|\nabla\nabla u|_g^2 \\ & - (1+2k)|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla} u|_g^2 \\ & - CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g - CM^{-1}|\nabla\nabla u|_g - CM^{-1}. \end{aligned} \tag{6.5}$$

Proof. We start by differentiating $F^{p\bar{q}}$ (4.21) by using (4.16).

$$\nabla_{\bar{i}} F^{p\bar{q}} = -\alpha'(k-\gamma)(1+\gamma)e^{-(1+\gamma)u} u_{\bar{i}} a^{p\bar{q}} + \alpha'(k-\gamma)e^{-(1+\gamma)u}\nabla_{\bar{i}} a^{p\bar{q}} + \alpha' \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_{\bar{i}} u_{\bar{s}r}. \tag{6.6}$$

Differentiating the Fu-Yau Hessian equation twice corresponds to differentiating (5.11), which gives

$$\begin{aligned} 0 &= \alpha' \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_{\bar{i}} u_{\bar{s}r} \nabla_j u_{\bar{q}p} + F^{p\bar{q}} \nabla_{\bar{i}} \nabla_j \nabla_p \nabla_{\bar{q}} u \\ &\quad + k \nabla_{\bar{i}} \nabla_j |\nabla u|_g^2 - \alpha'(k-\gamma)(1+\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{i}} \nabla_j \nabla_p \nabla_{\bar{q}} u \\ &\quad + \alpha'(k-\gamma) e^{-(1+\gamma)u} \nabla_{\bar{i}} a^{p\bar{q}} \nabla_j \nabla_p \nabla_{\bar{q}} u + \nabla_{\bar{i}} \mathcal{E}_j. \end{aligned} \tag{6.7}$$

Next, we use (4.12) to commute covariant derivatives and conclude

$$\begin{aligned} & F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} u_{\bar{i}j} \\ = & -\alpha' \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_j u_{\bar{q}p} \nabla_{\bar{i}} u_{\bar{s}r} \\ & - F^{p\bar{q}} [u_p \nabla_{\bar{i}} \nabla_j \nabla_{\bar{q}} u - u_j \nabla_{\bar{i}} \nabla_p \nabla_{\bar{q}} u + u_{\bar{q}} \nabla_p \nabla_{\bar{i}} \nabla_j u - u_{\bar{i}} \nabla_p \nabla_{\bar{q}} \nabla_j u] \\ & - F^{p\bar{q}} \hat{R}_{\bar{i}j}{}^{\lambda}{}_{p} u_{\bar{q}\lambda} + F^{p\bar{q}} \hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}} u_{\bar{\lambda}j} - k \nabla_{\bar{i}} \nabla_j |\nabla u|_g^2 \\ & + \alpha'(k-\gamma)(1+\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{i}} \nabla_j \nabla_p \nabla_{\bar{q}} u \\ & - \alpha'(k-\gamma) e^{-(1+\gamma)u} \nabla_{\bar{i}} a^{p\bar{q}} \nabla_j \nabla_p \nabla_{\bar{q}} u - \nabla_{\bar{i}} \mathcal{E}_j. \end{aligned} \tag{6.8}$$

Direct computation gives

$$F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla \bar{\nabla} u|_g^2 = 2 g^{s\bar{i}} g^{j\bar{r}} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} u_{\bar{i}j} u_{\bar{r}s} + 2 |\nabla \bar{\nabla} \nabla u|_{Fgg}^2. \tag{6.9}$$

Recall (4.24) that we can pass from $F^{p\bar{q}}$ to the metric $g^{p\bar{q}}$ up to an error of order 2^{-6}. Substituting (6.8) into (6.9) and estimating terms gives

$$\begin{aligned} & F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla \bar{\nabla} u|_g^2 \\ \geq & \; 2 \Big\{ (1-2^{-6}) |\nabla \bar{\nabla} \nabla u|_g^2 - |\alpha' g^{m\bar{i}} g^{j\bar{n}} \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_j u_{\bar{q}p} \nabla_{\bar{i}} u_{\bar{s}r} u_{\bar{n}m}| \Big\} \\ & - C |\nabla \bar{\nabla} u|_g |\nabla \bar{\nabla} \nabla u|_g \Big\{ |\nabla u|_g + e^{-\gamma u} |\nabla u|_g + e^{-\gamma u} e^{-\frac{1}{2} u} \Big\} \\ & - C |\nabla \bar{\nabla} u|_g \Big\{ e^{-u} |\nabla \bar{\nabla} u|_g \Big\} \\ & - 2k \Big| g^{s\bar{i}} g^{j\bar{r}} \nabla_{\bar{i}} \nabla_j |\nabla u|_g^2 u_{\bar{r}s} \Big| - 2 \Big| g^{s\bar{i}} g^{j\bar{r}} \nabla_{\bar{i}} \mathcal{E}_j u_{\bar{r}s} \Big|. \end{aligned} \tag{6.10}$$

The condition $u \in \Upsilon_k$ (2.5) together with $k \leq (n-1)$ gives

$$C_{n-2}^{k-1} |\alpha'| |\nabla \bar{\nabla} u|_g^k \leq |\alpha'| C_{n-1}^k |\nabla \bar{\nabla} u|_g^k \leq 2^{-7}. \tag{6.11}$$

Therefore by (6.2)

$$|\alpha' g^{m\bar{i}} g^{j\bar{n}} \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_j u_{\bar{q}p} \nabla_{\bar{i}} u_{\bar{s}r} u_{\bar{n}m}| \leq 2^{-7} |\nabla \bar{\nabla} \nabla u|_g^2. \tag{6.12}$$

In the coming estimates, we will often use the C^0 and C^1 estimates, and the condition $u \in \Upsilon_k$ (2.5), which we record here for future reference.

$$e^{-u} \leq CM^{-1}, \quad |\nabla u|_g^2 \leq CM^{-1}, \quad |\nabla\bar\nabla u|_g \leq |\alpha'|^{-1/k}\tau^{1/k}, \tag{6.13}$$

where $\tau = (C_{n-1}^k)^{-1}2^{-7}$. Since $u \in \Upsilon_k$, we have $M = \int_X e^u \hat\omega^n \geq 1$, and so we will often only keep the leading power of M since $M \geq 1$. Applying all this to (6.10), we have

$$\begin{aligned} & F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla u|_g^2 \\ \geq\ & 2(1-2^{-5})|\nabla\bar\nabla\nabla u|_g^2 - CM^{-1/2}|\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g \\ & -CM^{-1}|\nabla\bar\nabla u|_g|\nabla\bar\nabla u|_g \\ & -2k\left|g^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s}\right| - 2\left|g^{s\bar i}g^{j\bar r}\nabla_{\bar i}\mathcal{E}_j u_{\bar r s}\right|. \end{aligned} \tag{6.14}$$

We will now estimate the two last terms. We compute the first of these directly, using (4.5) to commute derivatives.

$$\begin{aligned} 2kg^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s} = \ & 2kg^{s\bar i}g^{j\bar r}\Big\{ g^{p\bar q}u_{\bar q}\nabla_j\nabla_{\bar i}\nabla_p u + g^{p\bar q}u_p\nabla_{\bar i}\nabla_j\nabla_{\bar q}u \\ & +g^{p\bar q}\nabla_j\nabla_p u\nabla_{\bar i}\nabla_{\bar q}u + g^{p\bar q}u_{\bar i p}u_{\bar q j} \\ & +g^{p\bar q}u_{\bar q}\hat R_{\bar i j}{}^{\ell}{}_p u_\ell - g^{p\bar q}u_{\bar q}u_{\bar i j}u_p\Big\}u_{\bar r s}. \end{aligned} \tag{6.15}$$

We estimate

$$\begin{aligned} & \left|2kg^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s}\right| \\ & \quad \leq k\Big\{4|\nabla\bar\nabla\nabla u|_g|\nabla u|_g + 2|\nabla\bar\nabla u|_g^2 + 2|\nabla\nabla u|_g^2 \\ & \qquad + Ce^{-u}|\nabla u|_g^2 + 2|\nabla u|_g^2|\nabla\bar\nabla u|_g\Big\}|\nabla\bar\nabla u|_g. \end{aligned} \tag{6.16}$$

We will use (6.13). Then

$$\begin{aligned} & \left|2kg^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s}\right| \\ & \quad \leq 2k|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla u|_g^2 + 2k|\alpha'|^{-1/k}\tau^{1/k}|\nabla\nabla u|_g^2 \\ & \qquad + CM^{-1/2}|\nabla\bar\nabla\nabla u|_g + CM^{-2} + CM^{-1}. \end{aligned} \tag{6.17}$$

Next, using the definition (5.12) of $\mathcal{E}_j$, we keep track of the order of each term and obtain the estimate

$$\begin{aligned}
& |g^{s\bar{i}} g^{j\bar{r}} \nabla_{\bar{i}} \mathcal{E}_j u_{\bar{r}s}| \\
\leq\ & C(a,b,c,\alpha')|\nabla\bar{\nabla}u|_g |\nabla\bar{\nabla}\nabla u|_g \Big\{ e^{-\gamma u} e^{-u/2} + e^{-\gamma u}|\nabla u|_g \Big\} \\
& + C(a,b,c)|\nabla\bar{\nabla}u|_g^2 \Big\{ e^{-\gamma u}|\nabla u|_g^2 + e^{-\gamma u}e^{-u/2}|\nabla u|_g + e^{-(1+\gamma)u} \Big\} \\
& + C(a,b,c,\alpha')|\nabla\bar{\nabla}u|_g |\nabla\nabla u|_g \\
& \times \Big\{ e^{-\gamma u}|\nabla u|_g^2 + e^{-\gamma u}e^{-u/2}|\nabla u|_g + e^{-(1+\gamma)u} \Big\} \\
& + C(a,b,c,\alpha')|\nabla\bar{\nabla}u|_g \Big\{ e^{-(2+\gamma)u} + e^{-(1+\gamma)u}e^{-u/2}|\nabla u|_g \\
& + e^{-(1+\gamma)u}|\nabla u|_g^2 + e^{-\gamma u}e^{-u/2}|\nabla u|_g^3 + e^{-\gamma u}|\nabla u|_g^4 \Big\} \\
& + C(\mu)|\nabla\bar{\nabla}u|_g \Big\{ e^{-(k+1)u}|\nabla\bar{\nabla}u|_g + e^{-(k+1)u}|\nabla u|_g^2 \\
& + e^{-(k+1)u}e^{-u/2}|\nabla u|_g + e^{-(k+2)u} \Big\} \\
& + (k-\gamma)^2 g^{s\bar{k}} g^{j\bar{r}} |(\alpha' e^{-(1+\gamma)u} a^{p\bar{q}} \nabla_{\bar{k}} \nabla_j \nabla_p u u_{\bar{q}}) u_{\bar{r}s}| \\
& + |k-\gamma| g^{s\bar{k}} g^{j\bar{r}} |(\alpha' e^{-(1+\gamma)u} b^i \nabla_{\bar{k}} \nabla_j \nabla_i u) u_{\bar{r}s}| \\
& + |k-\gamma| g^{s\bar{k}} g^{j\bar{r}} |(\alpha' e^{-(1+\gamma)u} \gamma a^{p\bar{q}} u_{\bar{q}p}) u_{\bar{k}j} u_{\bar{r}s}| \\
& + (k-\gamma)^2 g^{s\bar{k}} g^{j\bar{r}} |(\alpha' e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{k}p} u_{\bar{q}j}) u_{\bar{r}s}| \\
& + (k-\gamma)^2 g^{s\bar{k}} g^{j\bar{r}} |(\alpha' e^{-(1+\gamma)u} a^{p\bar{q}} \nabla_j \nabla_p u \nabla_{\bar{k}} \nabla_{\bar{q}} u) u_{\bar{r}s}|.
\end{aligned}$$

We will use our estimates (6.13). We also recall the notation $-\Lambda \hat{g}^{p\bar{q}} \leq a^{p\bar{q}} \leq \Lambda \hat{g}^{p\bar{q}}$. We use these estimates and commute covariant derivatives to obtain

$$\begin{aligned}
& |g^{s\bar{k}} g^{j\bar{r}} \nabla_{\bar{k}} \mathcal{E}_j u_{\bar{r}s}| \\
\leq\ & CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g + CM^{-1}|\nabla\nabla u|_g + CM^{-1} + CM^{-2} \\
& + CM^{-(k+1)} + CM^{-(k+2)} \\
& + (k-\gamma)^2 e^{-(1+\gamma)u} g^{s\bar{k}} g^{j\bar{r}} |(\alpha' a^{p\bar{q}} \nabla_j \nabla_{\bar{k}} \nabla_p u u_{\bar{q}} + \alpha' a^{p\bar{q}} R_{\bar{k}j}{}^{\lambda}{}_{p} u_\lambda u_{\bar{q}}) u_{\bar{r}s}| \\
& + |k-\gamma| e^{-(1+\gamma)u} g^{s\bar{k}} g^{j\bar{r}} |(\alpha' b^i \nabla_j \nabla_{\bar{k}} \nabla_i u + \alpha' b^i R_{\bar{k}j}{}^{\lambda}{}_{i} u_\lambda) u_{\bar{r}s}| \\
& + 2e^{-\gamma u}|\alpha'|\Lambda(k+\gamma)^2 |\nabla\bar{\nabla}u|_g |\nabla\bar{\nabla}u|_g^2 \\
& + e^{-\gamma u}|\alpha'|\Lambda(k+\gamma)^2 |\nabla\bar{\nabla}u|_g |\nabla\nabla u|_g^2.
\end{aligned}$$

Since $u \in \Upsilon_k$, we have $2|\alpha'|\Lambda(k+\gamma)^2 e^{-\gamma u} \leq 1$. It follows that

$$\begin{aligned} (6.18)\quad & |g^{s\bar{k}} g^{j\bar{r}} \nabla_{\bar{k}} \mathcal{E}_j u_{\bar{r}s}| \\ \leq\ & |\alpha'|^{-1/k}\tau^{1/k}|\nabla\nabla u|_g^2 + |\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla} u|_g^2 + CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g \\ & +CM^{-1}|\nabla\nabla u|_g + CM^{-1}. \end{aligned}$$

Substituting (6.17) and (6.18) into (6.14) and keeping the leading orders of M, we arrive at (6.5). q.e.d.

6.2. Using a test function. Let

$$G = |\nabla\bar{\nabla} u|_g^2 + \Theta|\nabla u|_g^2, \tag{6.19}$$

where $\Theta \gg 1$ is a large constant depending on n, k, α'. To be precise, we let

$$\Theta = (1-2^{-6})^{-1}\{(1+2k)|\alpha'|^{-1/k}\tau^{1/k} + 1\}. \tag{6.20}$$

By (5.9),

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 \ \geq\ & |\nabla\bar{\nabla} u|_{Fg}^2 + |\nabla\nabla u|_{Fg}^2 - 2|\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g \\ (6.21)\quad & -|\nabla u|_g^2|\nabla\bar{\nabla} u|_g - Ce^{-u}|\nabla u|_g^2. \end{aligned}$$

Applying (6.13) and converting $F^{p\bar{q}}$ to $g^{p\bar{q}}$ yields

$$\begin{aligned} (6.22)\quad & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 \\ \geq\ & (1-2^{-6})|\nabla\bar{\nabla} u|_g^2 + (1-2^{-6})|\nabla\nabla u|_g^2 - CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g \\ & -CM^{-1}|\nabla\bar{\nabla} u|_g - CM^{-2}. \end{aligned}$$

Combining (6.5) and (6.22), we have

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}} G \ \geq\ & 2(1-2^{-5})|\nabla\bar{\nabla}\nabla u|_g^2 + |\nabla\bar{\nabla} u|_g^2 + |\nabla\nabla u|_g^2 \\ (6.23)\quad & -CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g - CM^{-1}|\nabla\nabla u|_g - CM^{-1}. \end{aligned}$$

We will split the linear terms into quadratic terms by applying

$$CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g \leq \frac{1}{2}|\nabla\bar{\nabla}\nabla u|_g^2 + \frac{C^2}{2}M^{-1}, \tag{6.24}$$

$$CM^{-1}|\nabla\nabla u|_g \leq \frac{C^2}{4}M^{-2} + |\nabla\nabla u|_g^2. \tag{6.25}$$

Applying these estimates, we may discard the remaining quadratic positive terms and (6.23) becomes

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}} G \geq \frac{1}{2}|\nabla\bar{\nabla} u|_g^2 - CM^{-1}. \tag{6.26}$$

Let $p \in X$ be a point where G attains its maximum. From the maximum principle, $|\nabla\bar{\nabla} u|_g^2(p) \leq CM^{-1}$. We conclude from $G \leq G(p)$ that

$$|\nabla\bar{\nabla} u|_g^2 \leq CM^{-1}, \tag{6.27}$$

establishing Theorem 6.1.

We note that many equations involving the derivative of the unknown and/or several Hessians have been studied recently in the literature (see e.g. [**3**, **4**, **7**, **8**, **14**, **26**, **28**, **30**, **31**, **35**, **36**] and references therein). It would be very interesting to determine when estimates with scale hold.

7. Third order estimate

The goal of this section is to establish C^3 estimates for general Fu-Yau Hessian equations. A key tool is the test function (7.2) below. Note that it is different from the test function used for C^3 estimates for Monge-Ampère equations. Rather, it is inspired by the test function used by Fu and Yau [**11**, **12**], although we apply it here to Hessian equations rather than to Monge-Ampère equations.

Theorem 7.1. *Let $u \in \Upsilon_k$ be a $C^5(X)$ function solving equation (2.2). Then*

$$|\nabla\bar{\nabla}\nabla u|_g^2 \leq C, \tag{7.1}$$

where C only depends on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^5(X,\hat{\omega})}$ and $\|\mu\|_{C^3(X)}$.

To prove this estimate, we will apply the maximum principle to the test function

$$G = (|\nabla\bar{\nabla}u|_g^2 + \eta)|\nabla\bar{\nabla}\nabla u|_g^2 + B(|\nabla u|_g^2 + A)|\nabla\nabla u|_g^2, \tag{7.2}$$

where $A, B \gg 1$ are large constants to be specified later and $\eta = m\tau^{2/k}|\alpha'|^{-2/k}$. We will specify $m \gg 1$ later and $\tau = (C_{n-1}^k)^{-1}2^{-7}$. The condition (2.5) $u \in \Upsilon_k$ implies

$$|\alpha'|^{1/k}|\nabla\bar{\nabla}u|_g \leq \tau^{1/k}. \tag{7.3}$$

Our choice of constants ensures that η and $|\nabla\bar{\nabla}u|_g^2$ are of the same α' scale.

As noted earlier, if $u \in \Upsilon_k$ then M must be greater than 1. By our work thus far, as long as $M \geq 1$ we may estimate by C any term involving e^{-u}, $|\nabla u|_g$, $|\nabla\bar{\nabla}u|_g$, $|Rm|_g$ or $|T|_g$, where $|Rm|_g$ and $|T|_g$ are the norms of the curvature and torsion of $g = e^u\hat{g}$. Also, since

$$\nabla_\ell u_{\bar{i}j} = \partial_\ell u_{\bar{i}j} - \hat{\Gamma}^\lambda{}_{\ell j}u_{\bar{i}\lambda} - u_\ell u_{\bar{i}j}, \quad \hat{\Gamma}^\lambda{}_{\ell j} = \hat{g}^{\lambda\bar{p}}\partial_\ell\hat{g}_{\bar{p}j}, \tag{7.4}$$

we note that Theorem 7.1 proves the third order estimate (2.8) in Theorem 2.1.

7.1. Quadratic second order term.

Lemma 7.2. *Let $u \in \Upsilon_k$ be a $C^4(X)$ function solving equation (2.2). Then for all $A \gg 1$ larger than a fixed constant only depending on $|\nabla u|_g$*

and for all $B > 0$,

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla u|_g^2+A)|\nabla\nabla u|_g^2\right\} \geq \frac{A}{2}|\nabla\nabla\nabla u|_g^2+(1-2^{-5})|\nabla\nabla u|_g^4 - \frac{1}{2^5B}|\nabla\bar{\nabla}\nabla u|_g^4 - C(A,B), \tag{7.5}$$

where $C(A,B)$ only depends on A, B, $(X,\hat{\omega})$, α', k, γ, $\|\rho\|_{C^4(X,\hat{\omega})}$ and $\|\mu\|_{C^2(X)}$.

Proof. Differentiating (5.11) gives

$$\begin{aligned} F^{p\bar{q}}\nabla_\ell\nabla_j\nabla_p\nabla_{\bar{q}}u &= -\alpha'(k-\gamma)\nabla_\ell(e^{-(1+\gamma)u}a^{p\bar{q}})\nabla_j u_{\bar{q}p} \\ &\quad -\alpha'(\nabla_\ell\sigma_{k+1}^{p\bar{q}})\nabla_j u_{\bar{q}p} - k\nabla_\ell\nabla_j|\nabla u|_g^2 - \nabla_\ell\mathcal{E}_j. \end{aligned} \tag{7.6}$$

Commuting derivatives

$$\begin{aligned} & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\nabla_\ell\nabla_j u \\ = \;& F^{p\bar{q}}\nabla_\ell\nabla_j\nabla_p\nabla_{\bar{q}}u + F^{p\bar{q}}\nabla_p(\hat{R}_{\bar{q}\ell}{}^{\lambda}{}_j\nabla_\lambda u - u_{\bar{q}\ell}u_j) \\ & -F^{p\bar{q}}T^{\lambda}{}_{p\ell}\nabla_\lambda\nabla_j\nabla_{\bar{q}}u - F^{p\bar{q}}\nabla_\ell(u_p\nabla_j\nabla_{\bar{q}}u - u_j\nabla_p\nabla_{\bar{q}}u). \end{aligned} \tag{7.7}$$

We compute directly and commute derivatives to derive

$$\begin{aligned} & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\nabla u|_g^2 \\ = \;& 2\,\mathrm{Re}\{g^{\ell\bar{b}}g^{j\bar{d}}F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\nabla_\ell\nabla_j u\nabla_{\bar{b}}\nabla_{\bar{d}}u\} \\ & +g^{\ell\bar{b}}g^{j\bar{d}}\nabla_\ell\nabla_j uF^{p\bar{q}}R_{\bar{q}p\bar{b}}{}^{\bar{\lambda}}\nabla_{\bar{\lambda}}\nabla_{\bar{d}}u + g^{\ell\bar{b}}g^{j\bar{d}}\nabla_\ell\nabla_j uF^{p\bar{q}}R_{\bar{q}p\bar{d}}{}^{\bar{\lambda}}\nabla_{\bar{b}}\nabla_{\bar{\lambda}}u \\ & +F^{p\bar{q}}g^{\ell\bar{b}}g^{j\bar{d}}\nabla_p\nabla_\ell\nabla_j u\nabla_{\bar{q}}\nabla_{\bar{b}}\nabla_{\bar{d}}u + F^{p\bar{q}}g^{\ell\bar{b}}g^{j\bar{d}}\nabla_{\bar{q}}\nabla_\ell\nabla_j u\nabla_p\nabla_{\bar{b}}\nabla_{\bar{d}}u. \end{aligned} \tag{7.8}$$

Combining (7.6), (7.7), (7.8) and converting $F^{p\bar{q}}$ to $g^{p\bar{q}}$ using Lemma 4.2, we estimate

$$\begin{aligned} & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\nabla u|_g^2 \\ \geq \;& (1-2^{-6})|\nabla\nabla\nabla u|_g^2 + (1-2^{-6})|\bar{\nabla}\nabla\nabla u|_g^2 \\ & -2\alpha'\,\mathrm{Re}\{g^{\ell\bar{b}}g^{j\bar{d}}\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_\ell u_{\bar{s}r}\nabla_j u_{\bar{q}p}\nabla_{\bar{b}}\nabla_{\bar{d}}u\} \\ & -2\,\mathrm{Re}\{g^{\ell\bar{b}}g^{j\bar{d}}\nabla_\ell\mathcal{E}_j\nabla_{\bar{b}}\nabla_{\bar{d}}u\} \\ & -C|\nabla\nabla u|_g(|\nabla\nabla\nabla u|_g + |\nabla\bar{\nabla}\nabla u|_g + |\nabla\nabla u|_g + 1). \end{aligned}$$

Next, using (6.2) we estimate

$$\begin{aligned} & -2\,\mathrm{Re}\{\alpha' g^{\ell\bar{b}}g^{j\bar{d}}\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_\ell u_{\bar{s}r}\nabla_j u_{\bar{q}p}\nabla_{\bar{b}}\nabla_{\bar{d}}u\} \\ \geq \;& -2C_{n-2}^{k-1}|\alpha'||\nabla\bar{\nabla}u|_g^{k-1}|\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g^2 \\ \geq \;& -2C_{n-2}^{k-1}\tau^{1-(1/k)}|\alpha'|^{1/k}|\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g^2 \end{aligned}$$

and using (5.12) we estimate

$$|g^{\ell\bar{b}}g^{j\bar{d}}\nabla_\ell\mathcal{E}_j\nabla_{\bar{b}}\nabla_{\bar{d}}u| \leq C|\nabla\nabla u|_g\{1+|\nabla\nabla u|_g + |\nabla\bar{\nabla}\nabla u|_g + |\nabla\nabla\nabla u|_g\}.$$

Thus

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\nabla u|_g^2 \geq (1-2^{-6})|\nabla\nabla\nabla u|_g^2+(1-2^{-6})|\bar{\nabla}\nabla\nabla u|_g^2 -C|\nabla\nabla u|_g\{|\nabla\bar{\nabla}\nabla u|_g^2+|\nabla\nabla\nabla u|_g+|\nabla\bar{\nabla}\nabla u|_g+|\nabla\nabla u|_g+1\}. \tag{7.9}$$

By (5.14),

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 \geq (1-2^{-6})|\nabla\bar{\nabla}u|_g^2+(1-2^{-6})|\nabla\nabla u|_g^2-C|\nabla\nabla u|_g-C. \tag{7.10}$$

Direct computation gives

$$\begin{aligned} &F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla u|_g^2+A)|\nabla\nabla u|_g^2\right\} \\ =\;& (|\nabla u|_g^2+A)F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\nabla u|_g^2+|\nabla\nabla u|_g^2F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 \\ &+2\operatorname{Re}\{F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla\nabla u|_g^2\}. \end{aligned} \tag{7.11}$$

We estimate

$$\begin{aligned} &2\left|F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla\nabla u|_g^2\right| \\ \leq\;& 2(1+2^{-6})|\nabla\nabla u|_g^2|\nabla u|_g|\bar{\nabla}\nabla\nabla u|_g+2(1+2^{-6})|\nabla\nabla u|_g^2|\nabla u|_g|\nabla\nabla\nabla u|_g \\ &+C|\nabla\nabla u|_g\{|\nabla\bar{\nabla}\nabla u|_g+|\nabla\nabla\nabla u|_g+1\}. \end{aligned} \tag{7.12}$$

Substituting (7.9), (7.10), (7.12) into (7.11),

$$\begin{aligned} &F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\{(|\nabla u|_g^2+A)|\nabla\nabla u|_g^2\} \\ \geq\;& A(1-2^{-6})\left\{|\nabla\nabla\nabla u|_g^2+|\bar{\nabla}\nabla\nabla u|_g^2\right\}+(1-2^{-6})|\nabla\nabla u|_g^4 \\ &-3|\nabla\nabla u|_g^2|\nabla u|_g\left\{|\bar{\nabla}\nabla\nabla u|_g+|\nabla\nabla\nabla u|_g\right\} \\ &-C(A)|\nabla\nabla u|_g\Big\{|\nabla\bar{\nabla}\nabla u|_g^2+|\nabla\nabla\nabla u|_g+|\nabla\bar{\nabla}\nabla u|_g \\ &+|\nabla\nabla u|_g^2+|\nabla\nabla u|_g+1\Big\}. \end{aligned} \tag{7.13}$$

Using $2ab\leq a^2+b^2$,

$$3|\nabla\nabla u|_g^2|\nabla u|_g|\bar{\nabla}\nabla\nabla u|\leq 2^{-7}|\nabla\nabla u|_g^4+2^5 3^2|\nabla u|_g^2|\bar{\nabla}\nabla\nabla u|_g^2, \tag{7.14}$$

$$3|\nabla\nabla u|_g^2|\nabla u|_g|\nabla\nabla\nabla u|\leq 2^{-7}|\nabla\nabla u|_g^4+2^5 3^2|\nabla u|_g^2|\nabla\nabla\nabla u|_g^2, \tag{7.15}$$

$$C(A)|\nabla\nabla\nabla u|_g|\nabla\nabla u|_g\leq|\nabla\nabla\nabla u|_g^2+\frac{C(A)^2}{4}|\nabla\nabla u|_g^2, \tag{7.16}$$

$$C(A)|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g\leq\frac{1}{2^5B}|\nabla\bar{\nabla}\nabla u|_g^4+2^3C(A)^2B|\nabla\nabla u|_g^2 \tag{7.17}$$

for a constant $B \gg 1$ to be determined later. Then

$$
\begin{aligned}
(7.18)\qquad & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla u|_g^2 + A)|\nabla\nabla u|_g^2\right\} \\
\geq\ & \left\{A(1-2^{-6}) - 2^6 3^2|\nabla u|_g^2 - 1\right\}|\nabla\nabla\nabla u|_g^2 \\
& + \left\{A(1-2^{-6}) - 2^6 3^2|\nabla u|_g^2 - 1\right\}|\bar{\nabla}\nabla\nabla u|_g^2 \\
& + (1-2^{-5})|\nabla\nabla u|_g^4 - \frac{1}{2^5 B}|\nabla\bar{\nabla}\nabla u|_g^4 \\
& - C(A,B)\bigg\{|\nabla\nabla u|_g + |\nabla\nabla u|_g^2 + |\nabla\nabla u|_g^3\bigg\}.
\end{aligned}
$$

The terms $|\nabla\nabla u|_g + |\nabla\nabla u|_g^2 + |\nabla\nabla u|_g^3$ can be absorbed into $|\nabla\nabla u|_g^4$ by Young's inequality. For $A \gg 1$, obtain (7.5). q.e.d.

7.2. Third order term.

Lemma 7.3. *Let $u \in \Upsilon_k$ be a $C^5(X)$ function solving equation (2.2). Then*

$$
\begin{aligned}
(7.19)\qquad & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla\bar{\nabla}u|_g^2 + \eta)|\nabla\bar{\nabla}\nabla u|_g^2\right\} \\
\geq\ & \frac{1}{16}|\nabla\bar{\nabla}\nabla u|_g^4 \\
& - C|\nabla\nabla\nabla u|_g\bigg\{|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g + |\nabla\bar{\nabla}\nabla u|_g + |\nabla\nabla u|_g\bigg\} \\
& - C\bigg\{|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2 + |\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g \\
& + |\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g^2 + |\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g + 1\bigg\},
\end{aligned}
$$

where C only depends on $(X,\hat{\omega})$, α', k, γ, $\|\rho\|_{C^5(X,\hat{\omega})}$ and $\|\mu\|_{C^3(X)}$.

Proof. To start this computation, we differentiate (6.8).

$$
\begin{aligned}
(7.20)\ & F^{p\bar{q}}\nabla_i\nabla_p\nabla_{\bar{q}}u_{\bar{\ell}j} \\
=\ & -\alpha'\nabla_i(\sigma_{k+1}^{p\bar{q},r\bar{s}})\nabla_j u_{\bar{q}p}\nabla_{\bar{\ell}}u_{\bar{s}r} - \alpha'\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_i\nabla_j u_{\bar{q}p}\nabla_{\bar{\ell}}u_{\bar{s}r} \\
& -\alpha'\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_i\nabla_{\bar{\ell}}u_{\bar{s}r} + \nabla_i\left[-F^{p\bar{q}}u_p\nabla_{\bar{\ell}}u_{\bar{q}j} + F^{p\bar{q}}u_j\nabla_{\bar{\ell}}u_{\bar{q}p}\right] \\
& +\nabla_i\left[-F^{p\bar{q}}u_{\bar{q}}\nabla_p u_{\bar{\ell}j} + F^{p\bar{q}}u_{\bar{\ell}}\nabla_p u_{\bar{q}j}\right] \\
& +\nabla_i[F^{p\bar{q}}\hat{R}_{\bar{q}p\bar{\ell}}{}^{\bar{\lambda}}u_{\bar{\lambda}j} - F^{p\bar{q}}\hat{R}_{\bar{\ell}j}{}^{\lambda}{}_p u_{\bar{q}\lambda}] \\
& -k\nabla_i\bigg[g^{p\bar{q}}u_{\bar{q}}\nabla_j u_{\bar{\ell}p} + g^{p\bar{q}}u_p\nabla_{\bar{\ell}}u_{\bar{q}j} + g^{p\bar{q}}\nabla_j\nabla_p u\nabla_{\bar{\ell}}\nabla_{\bar{q}}u + g^{p\bar{q}}u_{\bar{\ell}p}u_{\bar{q}j} \\
& +g^{p\bar{q}}u_{\bar{q}}\hat{R}_{\bar{\ell}j}{}^{\lambda}{}_p u_\lambda - g^{p\bar{q}}u_{\bar{q}}u_{\bar{\ell}j}u_p\bigg] \\
& +\nabla_i[\alpha'(k-\gamma)(1+\gamma)e^{-(1+\gamma)u}a^{p\bar{q}}u_{\bar{\ell}}\nabla_j u_{\bar{q}p}] \\
& -\nabla_i[\alpha'(k-\gamma)e^{-(1+\gamma)u}\nabla_{\bar{\ell}}a^{p\bar{q}}\nabla_j u_{\bar{q}p}] - \nabla_i\nabla_{\bar{\ell}}\mathcal{E}_j.
\end{aligned}
$$

Our conventions (4.3) imply the following commutator identities for any tensor $W_{\bar{k}j}$.

$$\nabla_p \nabla_{\bar{q}} W_{\bar{k}j} = \nabla_{\bar{q}} \nabla_p W_{\bar{k}j} + R_{\bar{q}p\bar{k}}{}^{\bar{\lambda}} W_{\bar{\lambda}j} - R_{\bar{q}p}{}^{\lambda}{}_j W_{\bar{k}\lambda}, \tag{7.21}$$

$$\nabla_p \nabla_{\bar{q}} \nabla_i W_{\bar{k}j} = \nabla_i \nabla_p \nabla_{\bar{q}} W_{\bar{k}j} + T^{\lambda}{}_{ip} \nabla_\lambda W_{\bar{k}j} - \nabla_p [R_{\bar{q}i\bar{k}}{}^{\bar{\lambda}} W_{\bar{\lambda}j} - R_{\bar{q}i}{}^{\lambda}{}_j W_{\bar{k}\lambda}]. \tag{7.22}$$

Thus commuting derivatives gives

$$\begin{aligned} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} \nabla_i u_{\bar{k}j} &= F^{p\bar{q}} \nabla_i \nabla_p \nabla_{\bar{q}} u_{\bar{k}j} + F^{p\bar{q}} u_i \nabla_p \nabla_{\bar{q}} u_{\bar{k}j} - F^{p\bar{q}} u_p \nabla_i \nabla_{\bar{q}} u_{\bar{k}j} \\ &\quad + F^{p\bar{q}} \nabla_p [R_{\bar{q}i}{}^{\lambda}{}_j u_{\bar{k}\lambda} - R_{\bar{q}i\bar{k}}{}^{\bar{\lambda}} u_{\bar{\lambda}j}]. \end{aligned} \tag{7.23}$$

We compute the expression for $F^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$ acting on $|\nabla \bar{\nabla} \nabla u|_g^2$, and exchange covariant derivatives to obtain

$$\begin{aligned} & F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla \bar{\nabla} \nabla u|_g^2 \\ &= 2 \,\mathrm{Re}\{ g^{i\bar{d}} g^{a\bar{k}} g^{j\bar{b}} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} \nabla_i u_{\bar{k}j} \nabla_{\bar{d}} u_{\bar{b}a} \} \\ &\quad + F^{p\bar{q}} g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_p \nabla_a u_{\bar{b}c} \nabla_{\bar{q}} \nabla_{\bar{d}} u_{\bar{f}e} + F^{p\bar{q}} g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_a \nabla_{\bar{q}} u_{\bar{b}c} \nabla_{\bar{d}} \nabla_p u_{\bar{f}e} \\ &\quad + F^{p\bar{q}} g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_a \nabla_{\bar{q}} u_{\bar{b}c} R_{\bar{d}p\bar{f}}{}^{\bar{\lambda}} u_{\bar{\lambda}e} - F^{p\bar{q}} g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_a \nabla_{\bar{q}} u_{\bar{b}c} R_{\bar{d}p}{}^{\lambda}{}_e u_{\bar{f}\lambda} \\ &\quad - F^{p\bar{q}} g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} R_{\bar{q}a\bar{b}}{}^{\bar{\lambda}} u_{\bar{\lambda}c} \nabla_p \nabla_{\bar{d}} u_{\bar{f}e} + F^{p\bar{q}} g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} R_{\bar{q}a}{}^{\lambda}{}_c u_{\bar{b}\lambda} \nabla_p \nabla_{\bar{d}} u_{\bar{f}e} \\ &\quad + g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_a u_{\bar{b}c} F^{p\bar{q}} R_{\bar{q}p\bar{d}}{}^{\bar{\lambda}} \nabla_{\bar{\lambda}} u_{\bar{f}e} + g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_a u_{\bar{b}c} F^{p\bar{q}} R_{\bar{q}p\bar{f}}{}^{\bar{\lambda}} \nabla_{\bar{d}} u_{\bar{\lambda}e} \\ &\quad - g^{a\bar{d}} g^{e\bar{b}} g^{c\bar{f}} \nabla_a u_{\bar{b}c} F^{p\bar{q}} R_{\bar{q}p}{}^{\lambda}{}_e \nabla_{\bar{d}} u_{\bar{f}\lambda}. \end{aligned} \tag{7.24}$$

Substituting (7.20) and (7.23) into (7.24), and using Lemma 4.2 to convert $F^{p\bar{q}}$ into $g^{p\bar{q}}$, we have

$$\begin{aligned} & F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla \bar{\nabla} \nabla u|_g^2 \\ \geq\ & (1 - 2^{-6}) |\nabla \nabla \bar{\nabla} \nabla u|_g^2 + (1 - 2^{-6}) |\nabla \bar{\nabla} \nabla \bar{\nabla} u|_g^2 \\ & -2\alpha' \,\mathrm{Re}\{ g^{i\bar{d}} g^{a\bar{k}} g^{j\bar{b}} \nabla_i (\sigma_{k+1}^{p\bar{q},r\bar{s}}) \nabla_j u_{\bar{q}p} \nabla_{\bar{k}} u_{\bar{s}r} \nabla_{\bar{d}} u_{\bar{b}a} \} \\ & -2\alpha' \,\mathrm{Re}\{ g^{i\bar{d}} g^{a\bar{k}} g^{j\bar{b}} \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_i \nabla_j u_{\bar{q}p} \nabla_{\bar{k}} u_{\bar{s}r} \nabla_{\bar{d}} u_{\bar{b}a} \} \\ & -2\alpha' \,\mathrm{Re}\{ g^{i\bar{d}} g^{a\bar{k}} g^{j\bar{b}} \sigma_{k+1}^{p\bar{q},r\bar{s}} \nabla_j u_{\bar{q}p} \nabla_i \nabla_{\bar{k}} u_{\bar{s}r} \nabla_{\bar{d}} u_{\bar{b}a} \} \\ & -C \Big\{ (|\nabla \bar{\nabla} \nabla \bar{\nabla} u|_g + |\nabla \nabla \bar{\nabla} \nabla u|_g) |\nabla \bar{\nabla} \nabla u|_g + |\nabla \bar{\nabla} \nabla \bar{\nabla} u|_g \\ & + (|\nabla \nabla \nabla u|_g + |\bar{\nabla} \nabla \nabla u|_g + 1) |\nabla \nabla u|_g |\nabla \bar{\nabla} \nabla u|_g \\ & + |\nabla \bar{\nabla} \nabla u|_g^3 + |\nabla \bar{\nabla} \nabla u|_g^2 + |\nabla \bar{\nabla} \nabla u|_g \Big\} \\ & -2 \,\mathrm{Re}\{ g^{i\bar{d}} g^{a\bar{k}} g^{j\bar{b}} \nabla_i \nabla_{\bar{k}} \mathcal{E}_j \nabla_{\bar{d}} u_{\bar{b}a} \}. \end{aligned} \tag{7.25}$$

We used (6.6) to expand and estimate terms involving $\nabla_i F^{p\bar{q}}$. For the following steps, we will use that $|\alpha'|^{1/k}|\nabla\bar{\nabla}u|_g \leq \tau^{1/k}$ for any $u \in \Upsilon_k$, where $\tau = (C^k_{n-1})^{-1}2^{-7}$. We also recall that we use the notation $C^\ell_m = \frac{m!}{\ell!(m-\ell)!}$. If $k > 1$, we can estimate

$$\begin{aligned} (7.26)\qquad & 2|\alpha' g^{i\bar{d}} g^{a\bar{\ell}} g^{j\bar{b}} \nabla_i(\sigma^{p\bar{q},r\bar{s}}_{k+1}) \nabla_j u_{\bar{q}p} \nabla_{\bar{\ell}} u_{\bar{s}r} \nabla_{\bar{d}} u_{\bar{b}a}| \\ \leq\ & 2|\alpha'| C^{k-2}_{n-3} |\nabla\bar{\nabla}u|^{k-2} |\nabla\bar{\nabla}\nabla u|^4_g \\ \leq\ & (2C^k_{n-1}\tau)|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar{\nabla}\nabla u|^4_g \\ =\ & 2^{-6}|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar{\nabla}\nabla u|^4_g. \end{aligned}$$

We used $C^{k-2}_{n-3} \leq C^k_{n-1}$. If $k = 1$, the term on the left-hand side vanishes and the inequality still holds. Using the same ideas, we can also estimate

$$\begin{aligned} & -2\alpha' \operatorname{Re} g^{i\bar{d}} g^{a\bar{\ell}} g^{j\bar{b}} \sigma^{p\bar{q},r\bar{s}}_{k+1} \{\nabla_i \nabla_j u_{\bar{q}p} \nabla_{\bar{\ell}} u_{\bar{s}r} \nabla_{\bar{d}} u_{\bar{b}a} + \nabla_j u_{\bar{q}p} \nabla_i \nabla_{\bar{\ell}} u_{\bar{s}r} \nabla_{\bar{d}} u_{\bar{b}a}\} \\ \geq & -2|\alpha'| C^{k-1}_{n-2} |\nabla\bar{\nabla}u|^{k-1}_g |\nabla\bar{\nabla}\nabla u|^2_g \Big\{ |\nabla\nabla\bar{\nabla}\nabla u|_g + |\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g \Big\} \\ \geq & -(2C^k_{n-1}\tau)|\alpha'|^{1/k}\tau^{-1/k}|\nabla\bar{\nabla}\nabla u|^2_g \Big\{ |\nabla\nabla\bar{\nabla}\nabla u|_g + |\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g \Big\} \\ = & -2^{-6}|\alpha'|^{1/k}\tau^{-1/k}|\nabla\bar{\nabla}\nabla u|^2_g \Big\{ |\nabla\nabla\bar{\nabla}\nabla u|_g + |\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g \Big\}. \end{aligned}$$

The perturbative terms can be estimated roughly by using the definition (5.12) of $\mathcal{E}_j$ and keeping track of the orders of terms that we do not yet control.

$$\begin{aligned} (7.27)\qquad & -2\operatorname{Re}\{g^{i\bar{d}} g^{a\bar{k}} g^{j\bar{b}} \nabla_i \nabla_{\bar{k}} \mathcal{E}_j \nabla_{\bar{d}} u_{\bar{b}a}\} \\ \geq\ & -C|\nabla\bar{\nabla}\nabla u|_g \Big\{ |\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g + |\nabla\bar{\nabla}\nabla\nabla u|_g \\ & +(|\nabla\bar{\nabla}\nabla u|_g + |\nabla\nabla\nabla u|_g)|\nabla\nabla u|_g + |\nabla\bar{\nabla}\nabla u|_g + |\nabla\nabla\nabla u|_g \\ & +|\nabla\nabla u|^2_g + |\nabla\nabla u|_g + 1 \Big\}. \end{aligned}$$

Applying these estimates leads to

$$\begin{aligned} (7.28)\ & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\bar{\nabla}\nabla u|^2_g \\ \geq\ & (1-2^{-6})\left[|\nabla\nabla\bar{\nabla}\nabla u|^2_g + |\nabla\bar{\nabla}\nabla\bar{\nabla}u|^2_g\right] - 2^{-6}|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar{\nabla}\nabla u|^4_g \\ & -2^{-6}|\alpha'|^{1/k}\tau^{-1/k}|\nabla\bar{\nabla}\nabla u|^2_g\left[|\nabla\nabla\bar{\nabla}\nabla u|_g + |\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g\right] \\ & -C\mathcal{P}, \end{aligned}$$

where

$$
\begin{aligned}
\mathcal{P} \;=\;& |\nabla\bar\nabla\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g+|\nabla\nabla\bar\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g+|\nabla\bar\nabla\nabla\bar\nabla u|_g \\
&+|\nabla\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g+|\nabla\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g \\
&+|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g+|\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g^2+|\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g \\
&+|\nabla\nabla\nabla u|_g|\nabla\nabla u|_g+|\nabla\bar\nabla\nabla u|_g^3+|\nabla\bar\nabla\nabla u|_g^2+|\nabla\bar\nabla\nabla u|_g.
\end{aligned}
\tag{7.29}
$$

We used the fact that the difference between $|\nabla\bar\nabla\nabla\nabla u|_g$ and $|\nabla\nabla\bar\nabla\nabla u|_g$ is a lower order term according to the commutation formula (7.21).

Next, we apply (6.5) to obtain

$$
\begin{aligned}
& F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla u|_g^2 \\
\geq\;& |\nabla\bar\nabla\nabla u|_g^2-C|\nabla\bar\nabla\nabla u|_g-C|\nabla\nabla u|_g^2-C|\nabla\nabla u|_g-C.
\end{aligned}
\tag{7.30}
$$

We directly compute

$$
\begin{aligned}
& F^{p\bar q}\nabla_p\nabla_{\bar q}\left\{(|\nabla\bar\nabla u|_g^2+\eta)|\nabla\bar\nabla\nabla u|_g^2\right\} \\
=\;& |\nabla\bar\nabla\nabla u|_g^2F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla u|_g^2+(|\nabla\bar\nabla u|_g^2+\eta)F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla\nabla u|_g^2 \\
&+2\,\mathrm{Re}\{F^{p\bar q}\nabla_p|\nabla\bar\nabla u|_g^2\nabla_{\bar q}|\nabla\bar\nabla\nabla u|_g^2\}.
\end{aligned}
\tag{7.31}
$$

We can estimate

$$
\begin{aligned}
& 2\,\mathrm{Re}\{F^{p\bar q}\nabla_p|\nabla\bar\nabla u|_g^2\nabla_{\bar q}|\nabla\bar\nabla\nabla u|_g^2\} \\
\geq\;& -4(1+2^{-6})|\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g^2|\nabla\bar\nabla\nabla\bar\nabla u|_g \\
&-4(1+2^{-6})|\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla\bar\nabla\nabla u|_g \\
\geq\;& -4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla\nabla u|_g^2|\nabla\bar\nabla\nabla\bar\nabla u|_g \\
&-4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla\bar\nabla\nabla u|_g.
\end{aligned}
\tag{7.32}
$$

Combining (7.28), (7.30), (7.32) with (7.31), setting $\eta=m|\alpha'|^{-2/k}\tau^{2/k}$ and using $|\nabla\bar\nabla u|_g^2\leq|\alpha'|^{-2/k}\tau^{2/k}$ leads to

$$
\begin{aligned}
& F^{p\bar q}\nabla_p\nabla_{\bar q}\left\{(|\nabla\bar\nabla u|_g^2+\eta)|\nabla\bar\nabla\nabla u|_g^2\right\} \\
\geq\;& m(1-2^{-6})|\alpha'|^{-2/k}\tau^{2/k}\left\{|\nabla\nabla\bar\nabla\nabla u|_g^2+|\nabla\bar\nabla\nabla\bar\nabla u|_g^2\right\} \\
&-4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla\nabla u|_g^2\left\{|\nabla\nabla\bar\nabla\nabla u|_g+|\nabla\bar\nabla\nabla\bar\nabla u|_g\right\} \\
&-2^{-6}(m+1)|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla\nabla u|_g^2\left\{|\nabla\nabla\bar\nabla\nabla u|_g+|\nabla\bar\nabla\nabla\bar\nabla u|_g\right\} \\
&+\left\{1-2^{-6}(m+1)\right\}|\nabla\bar\nabla\nabla u|_g^4-C|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g^2-C\mathcal{P}.
\end{aligned}
\tag{7.33}
$$

Using $2ab \le a^2 + b^2$, we estimate

$$4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla\nabla u|_g^2\{|\nabla\bar\nabla\nabla\bar\nabla u|_g + |\nabla\nabla\bar\nabla\nabla u|_g\}$$
$$\le 16(1+2^{-6})^2|\alpha'|^{-2/k}\tau^{2/k}\{|\nabla\bar\nabla\nabla\bar\nabla u|_g^2 + |\nabla\nabla\bar\nabla\nabla u|_g^2\} + \frac{1}{2}|\nabla\bar\nabla\nabla u|_g^4,$$

and

$$2^{-6}(m+1)|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar\nabla\nabla u|_g^2\{|\nabla\nabla\bar\nabla\nabla u|_g + |\nabla\bar\nabla\nabla\bar\nabla u|_g\}$$
$$\le \frac{1}{2}|\alpha'|^{-2/k}\tau^{2/k}\{|\nabla\nabla\bar\nabla\nabla u|_g^2 + |\nabla\bar\nabla\nabla\bar\nabla u|_g^2\} + 2^{-12}(m+1)^2|\nabla\bar\nabla\nabla u|_g^4.$$

The main inequality becomes

$$\begin{aligned} (7.34)\qquad & F^{p\bar q}\nabla_p\nabla_{\bar q}\left\{(|\nabla\bar\nabla u|_g^2+\eta)|\nabla\bar\nabla\nabla u|_g^2\right\} \\ \ge\ & \{m(1-2^{-6}) - 16(1+2^{-6})^2 - \frac{1}{2}\}|\alpha'|^{-2/k}\tau^{2/k}\Big\{|\nabla\nabla\bar\nabla\nabla u|_g^2 \\ & +|\nabla\bar\nabla\nabla\bar\nabla u|_g^2\Big\} \\ & +\left\{\frac{1}{2} - 2^{-6}(m+1) - 2^{-12}(m+1)^2\right\}|\nabla\bar\nabla\nabla u|_g^4 \\ & -C|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g^2 - C\mathcal{P}. \end{aligned}$$

Next, we estimate terms on the first line in the definition (7.29) of $\mathcal{P}$

$$\begin{aligned} & C\{|\nabla\bar\nabla\nabla\bar\nabla u|_g + |\nabla\nabla\bar\nabla\nabla u|_g\}|\nabla\bar\nabla\nabla u|_g \\ \le\ & \frac{1}{16}|\alpha'|^{-2/k}\tau^{2/k}\{|\nabla\bar\nabla\nabla\bar\nabla u|_g^2 + |\nabla\nabla\bar\nabla\nabla u|_g^2\} \\ & +8C^2|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar\nabla\nabla u|_g^2 \end{aligned}$$

and

$$(7.35)\qquad C|\nabla\bar\nabla\nabla\bar\nabla u|_g \le \frac{1}{16}|\alpha'|^{-2/k}\tau^{2/k}|\nabla\bar\nabla\nabla\bar\nabla u|_g^2 + 4C^2|\alpha'|^{2/k}\tau^{-2/k}$$

and absorb $|\nabla\bar\nabla\nabla u|_g^3 + |\nabla\bar\nabla\nabla u|_g^2 + |\nabla\bar\nabla\nabla u|_g$ into $2^{-12}|\nabla\bar\nabla\nabla u|_g^4$ plus a large constant. We can now let $m = 18$ and drop the positive fourth order terms. We are left with

$$\begin{aligned} (7.36)\quad & F^{p\bar q}\nabla_p\nabla_{\bar q}\left\{(|\nabla\bar\nabla u|_g^2+\eta)|\nabla\bar\nabla\nabla u|_g^2\right\} \\ \ge\ & \left\{\frac{1}{2} - 2^{-6}(m+1) - 2^{-12}(m+1)^2 - 2^{-12}\right\}|\nabla\bar\nabla\nabla u|_g^4 \\ & -C|\nabla\nabla\nabla u|_g\Big\{|\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g + |\nabla\nabla u|_g\Big\} \\ & -C\Big\{|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g^2 + |\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g^2 \\ & +|\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g + 1\Big\}. \end{aligned}$$

Since $m = 18$,

$$\frac{1}{2} - 2^{-6}(m+1) - 2^{-12}(m+1)^2 - 2^{-12} \geq 2^{-4}, \tag{7.37}$$

and we obtain (7.19). q.e.d.

7.3. Using the test function. We have computed $F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$ acting on the two terms of the test function G defined in (7.2). Combining (7.5) and (7.19)

$$\begin{aligned}
(7.38)\qquad & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \\
\geq\ & \frac{1}{32}|\nabla\bar{\nabla}\nabla u|_g^4 + \frac{AB}{2}|\nabla\nabla\nabla u|_g^2 + (1-2^{-5})B|\nabla\nabla u|_g^4 \\
& -C\Big\{|\nabla\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g + |\nabla\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g \\
& +|\nabla\nabla\nabla u|_g|\nabla\nabla u|_g + |\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2 + |\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g \\
& +|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g^2 + |\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g\Big\} - C(A,B).
\end{aligned}$$

The negative terms are readily split and absorbed into the positive terms on the first line. For example,

(7.39)

$$C|\nabla\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g \leq |\nabla\nabla\nabla u|_g^2 + \frac{C^2}{4}|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2,$$

$$C|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2 \leq 2^{-7}|\nabla\bar{\nabla}\nabla u|_g^4 + 2^5C^2|\nabla\nabla u|_g^4, \tag{7.40}$$

$$C|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g \leq 2^{-7}|\nabla\bar{\nabla}\nabla u|_g^4 + 2^5C^2|\nabla\nabla u|_g^2. \tag{7.41}$$

This leads to

$$\begin{aligned}
(7.42)\qquad & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \\
\geq\ & 2^{-7}|\nabla\bar{\nabla}\nabla u|_g^4 + \{\frac{AB}{2} - 1\}|\nabla\nabla\nabla u|_g^2 + \{\frac{B}{2} - C\}|\nabla\nabla u|_g^4 \\
& -C(A,B).
\end{aligned}$$

By choosing $A, B \gg 1$ to be large, we conclude by the maximum principle that at a point p where G attains a maximum, we have

$$|\nabla\bar{\nabla}\nabla u|_g^4(p) \leq C, \quad |\nabla\nabla u|_g^4(p) \leq C. \tag{7.43}$$

Therefore $|\nabla\bar{\nabla}\nabla u|_g$ and $|\nabla\nabla u|_g$ are both uniformly bounded.

7.4. Remark on the case $k = 1$. In the case of the standard Fu-Yau equation ($k = 1$), to prove Theorem 1.1 we can instead appeal to a general theorem of concave elliptic PDE and obtain Hölder estimates for the second order derivatives of the solution. To exploit the concave structure, we must rewrite the Fu-Yau equation into the standard form of complex Hessian equation.

Recall that $\hat\sigma_1(\chi)\,\hat\omega^n = n\chi\wedge\hat\omega^{n-1}$, $\hat\sigma_2(\chi)\,\hat\omega^n = \frac{n(n-1)}{2}\chi^2\wedge\hat\omega^{n-2}$. A direct computation with equation (1.1) gives

$$\begin{aligned}(7.44)\qquad & \hat\sigma_2(e^u\hat\omega+\alpha' e^{-u}\rho+2\alpha' i\partial\bar\partial u)\\ = {} & \frac{n(n-1)}{2}e^{2u}-2(n-1)\alpha' e^u|\nabla u|^2_{\hat\omega}-2(n-1)\alpha'\mu\\ & +2(n-1)(\alpha')^2e^{-u}(a^{j\bar k}u_ju_{\bar k}-b^iu_i-b^{\bar i}u_{\bar i})\\ & +2(n-1)(\alpha')^2e^{-u}c+(n-1)e^{-u}\hat\sigma_1(\alpha'\rho)+e^{-2u}\hat\sigma_2(\alpha'\rho).\end{aligned}$$

We note that the right hand side of the equation involves the given data α', ρ, μ, u and ∇u. Since $u\in\Upsilon_1$, the $(1,1)$-form $\omega'=e^u\hat\omega+\alpha' e^{-u}\rho+2\alpha' i\partial\bar\partial u$ is positive definite, and thus both sides of the above equation have a positive lower bound. Moreover, our previous estimates imply that we have uniform a priori estimates on $\|u\|_{C^{1,\beta}(X)}$ for any $0<\beta<1$. The right hand side is therefore bounded in $C^\beta(X)$. Since $\hat\sigma_2^{1/2}(\chi)$ is a concave uniformly elliptic operator on the space of admissible solutions, we may apply a Evans-Krylov type result of Tosatti-Weinkove-Wang-Yang [**32**] (see also [**33**]) to conclude $\|u\|_{C^{2,\beta}}\le C$.

However, for general $k\ge 2$ it is impossible to re-write equation (2.2) into a standard complex Hessian equation and thus there is no obvious concavity that we can use.

Note: Just as we were about to post this paper, a preprint, *The Fu-Yau equation in higher dimensions* by J. Chu, L. Huang, and X.H. Zhu appeared in the net, arXiv:1801.09351, in which is stated the existence of a solution of the Fu-Yau Hessian equation for $k=1$, $\gamma=2$. In fact they established the same key gradient estimate for the case $k=1$, $\gamma=2$ of Theorem 5.1, using the same test function (5.3). As we had noted in sections §5 and when introducing it, this test functions is the same as the one introduced earlier in our paper [**23**] on the Anomaly flow. The major differences between the special case $k=1$, $\gamma=2$ treated in the Chu-Huang-Zhu paper and the general case $1\le k\le n-1$, $\gamma>0$ treated in Theorem 1.1 of the present paper are, on one hand the considerable technical complications in establishing Theorems 5.1 and 6.1 in this generality, and on the other hand, the C^3 estimates in §7. As we had explained in §7, $C^{2,\alpha}$ estimates can be obtained in the case $k=1$ by an Evans-Krylov type result [**32**] without any additional work. But for $k\ge 2$, Evans-Krylov type results are not available because there is no concavity, and C^3 estimates have to be established separately (Theorem 7.1). In fact, such C^3 estimates for general Hessian equations (different from Monge-Ampère equations) don't seem to have been treated before in the literature, and Theorem 7.1 may be of independent interest.

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