QUERMASSINTEGRAL PRESERVING CURVATURE FLOW IN HYPERBOLIC SPACE

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Abstract. We consider the quermassintegral preserving flow of closed *h-convex* hypersurfaces in hyperbolic space with the speed given by any positive power of a smooth symmetric, strictly increasing, and homogeneous of degree one function f of the principal curvatures which is inverse concave and has dual f_* approaching zero on the boundary of the positive cone. We prove that if the initial hypersurface is *h-convex*, then the solution of the flow becomes strictly *h-convex* for $t > 0$, the flow exists for all time and converges to a geodesic sphere exponentially in the smooth topology.

1. Introduction

Let $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a closed smooth hypersurface in the hyperbolic space $\mathbb{H}^{n+1}$. We consider the smooth family of immersions $X : M^n \times [0, T) \to \mathbb{H}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} X(x,t) = (\phi(t) - \Psi(\mathcal{W}(x,t)))\nu(x,t), \\ X(\cdot, 0) = X_0(\cdot), \end{cases} \tag{1.1}$$

where $\nu(x,t)$ is the unit outward normal of $M_t = X(M,t)$, and $\Psi(\mathcal{W}) = F^\alpha(\mathcal{W})$ where F is a smooth invariant function of the Weingarten matrix $\mathcal{W} = (h_i^j)$ of M_t. The global term $\phi(t)$ is chosen to keep one of the Quermassintegrals of the hypersurface constant (we will explain this below). We assume that $\alpha > 0$ and F satisfies the following conditions:

Assumption 1.1. (i) *$F(\mathcal{W}) = f(\kappa(\mathcal{W}))$, where $\kappa(\mathcal{W})$ gives the eigenvalues of $\mathcal{W}$ and f is a smooth symmetric function on*

$$\Gamma_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \ x_i > 0, \ i = 1, \cdots, n\}.$$

(ii) *f is strictly increasing, i.e., $\dot{f}^i = \partial f / \partial \kappa_i > 0$ on Γ_+, $\forall\ i = 1, \cdots, n$.*
(iii) *f is homogeneous of degree 1, i.e., $f(k\kappa) = kf(\kappa)$ for any $k > 0$.*
(iv) *f is strictly positive on Γ_+ and is normalized such that $f(1, \cdots, 1) = 1$.*
(v) *f is inverse concave, i.e., the function*

$$f_*(x_1, \cdots, x_n) = f(x_1^{-1}, \cdots, x_n^{-1})^{-1} \tag{1.2}$$

is concave.
(vi) *f_* approaches zero on the boundary of Γ_+.*

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To describe the global term $\phi(t)$ in (1.1), we first recall the (normalized) k-th mean curvature E_k of a smooth closed hypersurface M and the quermassintegrals $W_k(\Omega)$ of the bounded domain Ω enclosed by M: If Ω is a (geodesically) convex domain in $\mathbb{H}^{n+1}$, then the quermassintegrals of Ω are defined as follows (see [30, 31, 33]):

$$W_k(\Omega) = \frac{(n+1-k)\omega_{k-1}\cdots\omega_0}{(n+1)\omega_{n-1}\cdots\omega_{n-k}}\int_{\mathcal{L}_k}\chi(L_k\cap\Omega)dL_k, \quad k=1,\cdots,n, \tag{1.3}$$

where $\mathcal{L}_k$ is the space of k-dimensional affine subspaces L_k in $\mathbb{H}^{n+1}$. The function χ is defined to be 1 if $L_k\cap\Omega\neq\emptyset$ and to be 0 otherwise. In particular, we have

$$W_0(\Omega)=|\Omega|, \quad W_{n+1}(\Omega)=\frac{\omega_n}{n+1}, \quad W_1(\Omega)=\frac{1}{n+1}|\partial\Omega|.$$

If the boundary $M=\partial\Omega$ is smooth (at least of class C^2), we can define the principal curvatures $\kappa=(\kappa_1,\cdots,\kappa_n)$ as the eigenvalues of the Weingarten matrix $\mathcal{W}$ of M. For each $k\in\{1,\cdots,n\}$ the k-th mean curvature E_k of M is then defined as the normalized k-th elementary symmetric functions of the principal curvatures of M:

$$E_k=\binom{n}{k}^{-1}\sum_{1\leq i_1<\cdots<i_k\leq n}\kappa_{i_1}\cdots\kappa_{i_k}.$$

These include $E_1=H/n=(\kappa_1+\cdots+\kappa_n)/n$ (the normalized mean curvature) and $E_n=\kappa_1\cdots\kappa_n$ (the Gauss curvature). The *curvature integrals* of Ω are then defined by

$$V_{n-k}(\Omega) \;=\; \int_{\partial\Omega}E_k, \quad k=0,1,\cdots,n. \tag{1.4}$$

The quermassintegrals and curvature integrals of smooth convex domain Ω in $\mathbb{H}^{n+1}$ are related as follows:

$$V_{n-k}(\Omega)=(n+1)\left(W_{k+1}(\Omega)+\frac{k}{n+2-k}W_{k-1}(\Omega)\right), \quad k=1,\cdots,n \tag{1.5}$$

$$V_n(\Omega)=(n+1)W_1(\Omega)=|\partial\Omega|. \tag{1.6}$$

Besides the (geodesic) convexity, there is a stronger notion of convexity for regions in $\mathbb{H}^{n+1}$: *horospherical convexity*. A domain Ω in $\mathbb{H}^{n+1}$ is called *h-convex* (or, *horospherically convex*) if at every boundary point $p\in M=\partial\Omega$ there is a horoball $\mathcal{H}$ of $\mathbb{H}^{n+1}$ which contains Ω and touches at p (i.e. p is a boundary point of $\mathcal{H}$). Recall that a horoball in $\mathbb{H}^{n+1}$ is the union of those geodesics balls which have centre on a given geodesic ray from p and which have p as a boundary point. A smooth domain Ω is *h-convex* in $\mathbb{H}^{n+1}$ if and only if all the principal curvatures of $M=\partial\Omega$ are bounded from below by 1, and Ω is strictly *h-convex* if all the principal curvatures of $M=\partial\Omega$ are strictly bigger than 1. In this paper, when we say a hypersurface M is (strictly) *h-convex*, we mean that the domain Ω enclosed by M is (strictly) *h-convex*.

Fix an integer $k=1,\cdots,n$. If we define the function $\phi(t)$ in (1.1) by

$$\phi(t)=\frac{\int_{M_t}E_k\Psi d\mu_t}{\int_{M_t}E_k d\mu_t}, \tag{1.7}$$

then the quermassintegral $W_k(\Omega_t)$ of Ω_t remains constant along the flow (1.1) (see §2). We call flows of the form (1.1) with $\phi(t)$ given by (1.7) *quermassintegral preserving curvature flows*. In particular, in the case $k=0$, these are *volume preserving curvature flows*. The main result of this paper is the following:

Theorem 1.2. *Let $k \in \{1, \cdots, n\}$ and $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a closed h-convex hypersurface in $\mathbb{H}^{n+1}$. Then for any power $\alpha > 0$, the flow (1.1) with F satisfying Assumption (1.1) and the global term $\phi(t)$ given by (1.7) has a smooth h-convex solution M_t for all time $t \in [0, \infty)$, and M_t converges smoothly and exponentially to a geodesic sphere of radius r_∞ determined by $W_k(B_{r_\infty}) = W_k(\Omega_0)$ as $t \to \infty$.*

Remark 1.3. Some important examples of functions satisfying the Assumption 1.1 include: (i). $f = E_k^{1/k}$ for all $k = 1, \cdots, n$; (ii) the power-means $H_r = (\frac{1}{n}\sum_i x_i^r)^{1/r}$ for all $r > 0$; and (iii) any function of the form $E_1^\sigma G^{1-\sigma}$ where $0 < \sigma < 1$ and G is homogeneous degree one, normalised, increasing in each argument, and inverse-concave. See §2 for further discussion.

Constrained curvature flows have been studied extensively in recent years. In 1987, Huisken [20] studied the volume preserving mean curvature flow in Euclidean space $\mathbb{R}^{n+1}$, and proved that starting from any strictly convex hypersurface a solution exists for all time $t \in [0, \infty)$ and converges smoothly to a round sphere. In 2001, the first named author [3] studied volume preserving anisotropic mean curvature flows in $\mathbb{R}^{n+1}$ and obtained a similar result. Later, McCoy [24, 25, 26, 27] studied some mixed volume preserving curvature flow driven by homogeneous of degree one curvature functions. For higher homogeneity, by imposing a strong pinching assumption on the initial hypersurface, Cabezas-Rivas and Sinestrari [14] proved convergence results for the flow (1.1) in $\mathbb{R}^{n+1}$ with $\Psi = E_k^{\alpha/k}$ where $k = 1, \cdots, n$ and $\alpha > 1$. Using the monotonicity of the isoperimetric ratio, Sinestrari [29] proved a convergence result for the flow (1.1) with $\Psi = H^\alpha$ in $\mathbb{R}^{n+1}$ and for any positive power $\alpha > 0$. This was generalized in [11] for volume (and area) preserving non-homogeneous mean curvature flow in $\mathbb{R}^{n+1}$. Very recently, the authors [8] removed the pinching assumption in [14] and proved that the flow (1.1) in $\mathbb{R}^{n+1}$ with $\Psi = E_k^{\alpha/k}$ for $k = 1, \cdots, n$ and any $\alpha > 0$, will deform any strictly convex hypersurface to a round sphere smoothly.

The volume preserving flow in hyperbolic space $\mathbb{H}^{n+1}$ was firstly studied by Cabezas-Rivas and Miquel [13] in 2007. By imposing *h-convexity* on the initial hypersurface, they proved that the flow (1.1) with $\Psi = H$ exists for all time and converges smoothly to a geodesic sphere. This result was generalized recently by Bertini and Pipoli [12] to volume preserving non-homogeneous mean curvature flow. In particular, their result includes the flow with velocity given by $\Psi = H^\alpha$ with $\alpha > 0$. Some mixed volume preserving flows were considered in [28, 33] with Ψ given by some homogeneous of degree one curvature function F. By assuming *h-convexity* and strong pinching on the initial hypersurface, Guo-Li-Wu [17] proved the convergence of the flow (1.1) with $\Psi = E_k^{\alpha/k}, k = 1, \cdots, n$ and power $\alpha > 1$ by following the same procedure as the Euclidean case in [14].

The paper is organized as follows: In section 2, we collect some preliminaries on hyperbolic geometry, the evolution equations along the flow (1.1), and examples of the function satisfying the Assumption 1.1. In section 3, we prove that the *h-convexity* is preserved along the flow (1.1) for inverse concave f and for all power $\alpha > 0$. To show this, we will apply the tensor maximum principle proved by the first named author in [4] (which generalized Hamilton's [19] theorem). In section 4, the preservation of $W_k(\Omega_t)$ and the *h-convexity* will be used to estimate the inner radius and outer radius of Ω_t. Then we adapt Tso's [32] technique to prove a uniform upper bound on F. In section 5, by the assumption that f_* approaches zero on the boundary of Γ_+ and the upper bound on F, we derive a uniform upper bound on the principal curvatures. The h-convexity together with the boundedness of principal curvatures makes the evolution equation uniformly parabolic. By projecting the flow solution into the unit ball in $\mathbb{R}^{n+1}$ and using the Gauss map parametrization, we write the flow (1.1) as a scalar parabolic partial differential equation for the support function which is concave with respect to the second spatial derivatives. Then the Hölder estimate of Krylov-Evans [21] and the parabolic Schauder theory [22] can be applied to derive the higher order derivative estimates of the solution M_t. From this we conclude that the solution of (1.1) exists for all time $t \in [0, \infty)$.

In previous work, the convergence of solutions as $t \to \infty$ was deduced using either the monotonicity of curvature pinching ratios [20, 24, 25, 26, 27, 13, 14, 17, 28] or of isoperimetric ratios [3, 29, 11, 12, 8]. In our situation for general F and α, neither of these arguments is available. Instead, we apply the Alexandrov reflection method to prove that the hypersurfaces approach a sphere, and a linearisation argument to prove exponential convergence.

Remark 1.4. In the case where $F = E_k^{1/k}$, we have as in [8] that the quermassintegral $W_k(\Omega_t)$ is monotone decreasing along the volume preserving flow for any $\alpha > 0$. This can also be used to deduce the smooth convergence to the geodesic sphere. We describe this alternative argument in the appendix.

Remark 1.5. As in [11, 12], a result similar to Theorem 1.2 is also true for non-homogeneous constrained flows (1.1) with $\Psi = \psi(f)$, where f is a function satisfying Assumption 1.1, and $\psi : [0, +\infty) \to \mathbb{R}$ is in $C^0([0, \infty)) \cap C^2((0, \infty))$ and satisfies

(i) $\psi(r) > 0$, $\psi'(r) > 0$ for all $r > 0$;
(ii) $\lim_{r\to\infty} \psi(r) = \infty$;
(iii) $\lim_{r\to\infty} \frac{\psi'(r)r^2}{\psi(r)} = \infty$;
(iv) $\psi''(r)r + 2\psi'(r) \geq 0$ for all $r > 0$

In fact, the flow is parabolic due to item (i); item (iv) is used to show that the *h-convexity* is preserved along the flow (see Remark 3.3); items (ii)–(iii) are used to estimate the upper bound of f. The remaining proof is similar.

2. Preliminaries

In this section, we collect some preliminary results concerning hyperbolic geometry, the properties of inverse concave functions, and examples of functions satisfying Assumption 1.1. We also collect the evolution equations for several geometric quantities of the solution M_t of the flow (1.1).

2.1. **Hyperbolic geometry.** Let M be a smooth closed hypersurface in $\mathbb{H}^{n+1}$. We denote by g_{ij}, h_{ij} and ν the induced metric, the second fundamental form and unit outward normal vector of M. The Weingarten map is denoted by $\mathcal{W} = (h_i^j)$, where $h_i^j = h_{ik}g^{kj}$. The principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ of M are defined as the eigenvalues of $\mathcal{W}$. As mentioned before, M is *h-convex* if and only if $\kappa_i \geq 1$ for all $i = 1, \cdots, n$.

A remarkable property of an *h-convex* hypersurface $M = \partial\Omega$ in $\mathbb{H}^{n+1}$ is that its inner radius and outer radius are comparable. Recall that the inner radius ρ_- and outer radius

ρ_+ of a bounded domain Ω are defined by

$$\rho_- = \sup\{\rho : B_\rho(p) \subset \Omega \text{ for some } p \in \mathbb{H}^{n+1}\}$$

and

$$\rho_+ = \inf\{\rho : \Omega \subset B_\rho(p) \text{ for some } p \in \mathbb{H}^{n+1}\},$$

where $B_\rho(p)$ denotes the geodesic ball of radius ρ about p in $\mathbb{H}^{n+1}$. The following results can be found in [9, 10, 13, 28].

Theorem 2.1. *Let Ω be a compact* h-convex *domain in $\mathbb{H}^{n+1}$ and denote the center of an inball by o and its inner radius by ρ_-. Then we have*

(1) *The maximum of the distance $\mathrm{d}_{\mathbb{H}}(o,\cdot)$ between o and the points on $\partial\Omega$ satisfies*

$$\max_{p\in\partial\Omega} \mathrm{d}_{\mathbb{H}}(o,p) \leq \rho_- + \ln \frac{(1+\sqrt{\tanh \rho_-/2})^2}{1+\tanh \rho_-/2} < \rho_- + \ln 2. \tag{2.1}$$

Therefore there exists a constant $c > 0$ such that the outer radius

$$\rho_+ \leq c(\rho_- + \rho_-^{1/2}). \tag{2.2}$$

(2) *For any interior point p of Ω, and any boundary point $q \in \partial\Omega$,*

$$Dr_p(\nu(q)) \geq \tanh(\mathrm{d}_{\mathbb{H}}(p,\partial\Omega)), \tag{2.3}$$

where $r_p(x) = \mathrm{d}_{\mathbb{H}}(p,x)$.

For smooth *h-convex* domains in $\mathbb{H}^{n+1}$, inequalities of Alexandrov-Fenchel type for quermassintegrals were proved by Wang-Xia in [33]. See also [18, 23] for related Alexandrov-Fenchel type inequalities for the curvature integrals (1.4).

Theorem 2.2 ([33]). *For any smooth bounded domain Ω in $\mathbb{H}^{n+1}$ with* h-convex *boundary $\partial\Omega$, and $0 \leq l < k \leq n$, we have*

$$W_k(\Omega) \geq f_k \circ f_l^{-1}(W_l(\Omega)) \tag{2.4}$$

with equality if and only if Ω is a geodesic ball. Here the function $f_k : [0,\infty) \to \mathbb{R}_+$ is increasing and is defined by $f_k(r) = W_k(B_r)$, with B_r a geodesic ball in $\mathbb{H}^{n+1}$. f_l^{-1} is the inverse function of f_l.

2.2. **Inverse concave functions.** For a smooth symmetric function $F(A) = f(\kappa(A))$, where $A = (A_{ij}) \in \mathrm{Sym}(n)$ is a symmetric matrix and $\kappa(A) = (\kappa_1, \cdots, \kappa_n)$ gives the eigenvalues of A, we denote by $\dot{F}^{ij}$ and $\ddot{F}^{ij,kl}$ the first and second derivatives of F with respect to the components of its argument, so that

$$\frac{\partial}{\partial s} F(A+sB)\Big|_{s=0} = \dot{F}^{ij}(A)B_{ij}$$

and

$$\frac{\partial^2}{\partial s^2} F(A+sB)\Big|_{s=0} = \ddot{F}^{ij,kl}(A)B_{ij}B_{kl}$$

for any two symmetric matrices A, B. We also use the notation

$$\dot{f}^i(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa), \quad \ddot{f}^{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\kappa).$$

for the derivatives of f with respect to κ. At any diagonal A with distinct eigenvalues, the second derivative $\ddot{F}$ of F in direction $B \in \mathrm{Sym}(n)$ is given in terms of $\dot{f}$ and $\ddot{f}$ by (see [1, 4]):

$$\ddot{F}^{ij,kl}B_{ij}B_{kl} = \sum_{i,k}\ddot{f}^{ik}B_{ii}B_{kk} + 2\sum_{i>k}\frac{\dot{f}^i - \dot{f}^k}{\kappa_i - \kappa_k}B_{ik}^2. \tag{2.5}$$

This formula makes sense as a limit in the case of any repeated values of κ_i. Since $\Psi(A) = F^\alpha(A)$, we will use the same notations and formulas for the derivatives of Ψ and $\psi = f^\alpha$.

For any positive definite symmetric matrix $A \in \mathrm{Sym}(n)$ with eigenvalues $\kappa(A) \in \Gamma_+$, define $F_*(A) = F(A^{-1})^{-1}$. Then $F_*(A) = f_*(\kappa(A))$, where f_* is defined in (1.2). Since f is defined on the positive definite cone Γ_+, the following lemma characterizes the inverse concavity of f and F.

Lemma 2.3 ([4, 7]). (i) *f is inverse concave if and only if the following matrix*

$$\left(\ddot{f}^{kl} + 2\frac{\dot{f}^k}{\kappa_k}\delta_{kl}\right) \geq 0. \tag{2.6}$$

(ii) *F_* is concave if and only if f_* is concave;*
(ii) *F_* is inverse concave if and only if*

$$\left(\ddot{f}^{kl} + 2\frac{\dot{f}^k}{\kappa_k}\delta_{kl}\right) \geq 0, \quad \text{and} \quad \frac{\dot{f}^k - \dot{f}^l}{\kappa_k - \kappa_l} + \frac{\dot{f}^k}{\kappa_l} + \frac{\dot{f}^l}{\kappa_k} \geq 0, \ k \neq l. \tag{2.7}$$

(i) *If f is inverse concave, then*

$$\sum_{i=1}^n \dot{f}^i\kappa_i^2 \geq f^2. \tag{2.8}$$

Since the function $f = E_k^{1/k}$ is inverse concave for all $k = 1, \cdots, n$ and has dual function

$$f_*(z) = \left(\frac{E_n(z)}{E_{n-k}(z)}\right)^{1/k}, \quad z \in \Gamma_+$$

which vanishes on the boundary of Γ_+, we have that $f = E_k^{1/k}$, $k = 1, \cdots, n$, satisfy the Assumption 1.1. We can also easily see that a convex function $f : \Gamma_+ \to \mathbb{R}$ satisfies the Assumption 1.1. Firstly, the inequality (2.6) is obviously true since f is convex and strictly increasing. Secondly, the convexity of f implies that

$$f(x_1, \cdots, x_n) \geq \frac{1}{n}\sum_{i=1}^n x_i.$$

Then the dual function f_* satisfies

$$f_*(z_1, \cdots, z_n) = f\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}\right)^{-1} \leq n\left(\frac{1}{z_1} + \cdots + \frac{1}{z_n}\right)^{-1} = \frac{E_n(z)}{E_{n-1}(z)}$$

Thus f_* approaches zero on the boundary of Γ_+. Other important examples of functions satisfying Assumption 1.1 are the power means $H_r = (\frac{1}{n}\sum_i x_i^r)^{1/r}$, which are inverse-concave for $r \geq -1$, concave for $r \geq 1$, and have f_* approaching zero on $\partial\Gamma_+$ for $r \geq 0$, and so satisfy our requirements for all $r \geq 0$. More examples can be constructed as follows: If G_1 is homogeneous of degree one, increasing in each argument, and inverse-concave, and

G_2 satisfies Assumption 1.1, then $F = G_1^{\sigma} G_2^{1-\sigma}$ satisfies Assumption 1.1for any $0 < \sigma < 1$ (see [4, 6] for more examples of inverse concave or convex functions).

2.3. **Evolution equations.** Along the flow

$$\frac{\partial}{\partial t}X(x,t) = \ (\phi(t) - \Psi(\mathcal{W}(x,t)))\nu(x,t)$$

in hyperbolic space $\mathbb{H}^{n+1}$, we have the following evolution equations (see [23]) for the induced metric g_{ij}, unit outward normal ν, induced area element $d\mu_t$ and Weingarten matrix $\mathcal{W} = (h_i^j)$ of $M_t = X(M^n, t)$:

$$\frac{\partial}{\partial t}g_{ij} = 2(\phi(t) - \Psi)h_{ij} \tag{2.9}$$

$$\frac{\partial}{\partial t}\nu = \nabla\Psi \tag{2.10}$$

$$\frac{\partial}{\partial t}d\mu_t = nE_1(\phi(t) - \Psi)d\mu_t \tag{2.11}$$

$$\frac{\partial}{\partial t}h_i^j = \nabla^j\nabla_i\Psi + (\Psi - \phi(t))(h_i^k h_k^j - \delta_i^j) \tag{2.12}$$

where ∇ denotes the Levi-Civita connection with respect to the induced metric g_{ij} on M_t. As an immediate consequence of (2.12), we have that the curvature function $\Psi = \Psi(\mathcal{W})$ evolves by

$$\frac{\partial}{\partial t}\Psi = \ \dot{\Psi}^{kl}\nabla^k\nabla_l\Psi + (\Psi - \phi(t))(\dot{\Psi}^{ij}h_i^k h_k^j - \dot{\Psi}^{ij}\delta_i^j), \tag{2.13}$$

where $\dot{\Psi}^{kl}$ denotes the derivatives of Ψ with respect to the components of $\mathcal{W} = (h_i^j)$. Throughout this paper we will always evaluate the derivatives of $\Psi = F^{\alpha}$ at $\mathcal{W} = (h_i^j)$ and the derivatives of $\psi = f^{\alpha}$ at $\kappa(\mathcal{W}) = (\kappa_1, \cdots, \kappa_n)$.

The following lemma gives a parabolic type equation of h_i^j.

Lemma 2.4. *Along the flow (1.1), the Weingarten matrix h_i^j of M_t evolves by*

$$\begin{aligned}\frac{\partial}{\partial t}h_i^j =& \dot{\Psi}^{kl}\nabla_k\nabla_l h_i^j + \ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} + (\dot{\Psi}^{kl}h_k^r h_{rl} + \dot{\Psi}^{kl}g_{kl})h_i^j \\ &- \dot{\Psi}^{kl}h_{kl}(h_i^p h_p^j + \delta_i^j) + (\Psi - \phi(t))(h_i^k h_k^j - \delta_i^j),\end{aligned} \tag{2.14}$$

where $\Psi = F^{\alpha}$ and $\dot{\Psi}^{kl}, \ddot{\Psi}^{kl,pq}$ denote the derivatives of Ψ with respect to the components of $\mathcal{W} = (h_i^j)$.

Proof. Firstly, combining the Gauss and Codazzi equations in hyperbolic space gives the following generalized Simons' identity (see [5]):

$$\nabla_{(i}\nabla_{j)}h_{kl} = \nabla_{(k}\nabla_{l)}h_{ij} + (h_k^p h_{pl} + g_{kl})h_{ij} - h_{kl}(h_i^p h_{pj} + g_{ij}), \tag{2.15}$$

where the brackets denote symmetrisation. Then

$$\begin{aligned}\nabla^j\nabla_i\Psi &= \dot{\Psi}^{kl}\nabla^j\nabla_i h_{kl} + \ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} \\ &= \dot{\Psi}^{kl}\nabla_k\nabla_l h_i^j + \ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} \\ &\quad + \dot{\Psi}^{kl}(h_k^p h_{pl} + g_{kl})h_i^j - \dot{\Psi}^{kl}h_{kl}(h_i^p h_p^j + \delta_i^j).\end{aligned} \tag{2.16}$$

The equation (2.14) follows from (2.12) and (2.16) immediately. □

Using the evolution equations (2.11) and (2.12), we can also derive the evolution equation for the curvature integral defined in (1.4)

$$\begin{aligned}\frac{d}{dt}V_{n-k}(\Omega_t) =& \int_{M_t}\left(\frac{\partial}{\partial t}E_k d\mu_t + E_k\frac{\partial}{\partial t}d\mu_t\right)\\ =& \int_{M_t}\left(\frac{\partial E_k}{\partial h_i^j}\nabla^j\nabla_i F^\alpha + (F^\alpha - \phi(t))(\frac{\partial E_k}{\partial h_i^j}(h_i^p h_p^j - \delta_i^j) + nE_1E_k)\right)d\mu_t\\ =& \int_{M_t}((\phi(t) - F^\alpha)((n-k)E_{k+1} + kE_{k-1}))\, d\mu_t, \end{aligned}\tag{2.17}$$

where we used the facts that $\nabla^j(\partial E_k/\partial h_i^j) = 0$ and

$$\frac{\partial E_k}{\partial h_i^j}h_i^p h_p^j = nE_1E_k - (n-k)E_{k+1}, \qquad \frac{\partial E_k}{\partial h_i^j}\delta_i^j = kE_{k-1}.$$

By applying induction argument to (2.17) and (1.5), we have the following evolution equation for the quermassintegrals of Ω_t along the flow (1.1),

$$\frac{d}{dt}W_k(\Omega_t) = \frac{n+1-k}{n+1}\int_{M_t}E_k\left(\phi(t) - F^\alpha\right)d\mu_t, \quad k = 0, \cdots, n, \tag{2.18}$$

which was also derived in [33]. Thus for the function $\phi(t)$ defined in (1.7), the quermassintegral $W_k(\Omega_t)$ remains constant along the flow (1.1).

Remark 2.5. If $\phi(t)$ is defined as

$$\phi(t) = \frac{1}{|M_t|}\int_{M_t}F^\alpha d\mu_t, \tag{2.19}$$

then the volume $|\Omega_t|$ remains constant. The flow (1.1) with $\phi(t)$ given by (2.19) is called the volume preserving curvature flow.

3. Preserving of h-convexity

In this section, we will use the tensor maximum principle to prove that the *h-convexity* is preserved along the flow (1.1) if f is inverse concave. For the convenience of readers, we include here the statement of the tensor maximum principle, which was first proved by Hamilton [19] and was generalized by Andrews [4].

Theorem 3.1 ([4]). *Let S_{ij} be a smooth time-varying symmetric tensor field on a compact manifold M, satisfying*

$$\frac{\partial}{\partial t}S_{ij} = a^{kl}\nabla_k\nabla_l S_{ij} + u^k\nabla_k S_{ij} + N_{ij}, \tag{3.1}$$

where a^{kl} and u are smooth, ∇ is a (possibly time-dependent) smooth symmetric connection, and a^{kl} is positive definite everywhere. Suppose that

$$N_{ij}v^iv^j + \sup_\Lambda 2a^{kl}\left(2\Lambda_k^p\nabla_l S_{ip}v^i - \Lambda_k^p\Lambda_l^q S_{pq}\right) \geq 0 \tag{3.2}$$

whenever $S_{ij} \geq 0$ and $S_{ij}v^j = 0$. If S_{ij} is positive definite everywhere on M at $t = 0$ and on ∂M for $0 \leq t \leq T$, then it is positive on $M \times [0, T]$.

The main result of this section is the following.

Theorem 3.2. *For any power $\alpha > 0$, if the initial hypersurface M_0 is h-convex and f is inverse concave, then along the flow (1.1) in $\mathbb{H}^{n+1}$ the flow hypersurface M_t is strictly h-convex for $t > 0$.*

Proof. Denote

$$S_{ij} = h_i^j - \delta_i^j.$$

Then the *h-convexity* is equivalent to $S_{ij} \geq 0$. By (2.14), the tensor S_{ij} evolves by

$$\begin{aligned}\frac{\partial}{\partial t} S_{ij} = &\dot{\Psi}^{kl}\nabla_k\nabla_l S_{ij} + \ddot{\Psi}^{kl,pq}\nabla_i h_{kl}\nabla^j h_{pq} + (\dot{\Psi}^{kl}h_k^r h_{rl} + \dot{\Psi}^{kl}g_{kl})S_{ij} \\ &+ (\Psi - \phi(t) - \dot{\Psi}^{kl}h_{kl})(S_{ik}S_{kj} + 2S_{ij}) \\ &+ \dot{\Psi}^{kl}(h_k^r h_{rl} + g_{kl} - 2h_{kl})\delta_i^j \end{aligned} \tag{3.3}$$

To apply the tensor maximum principle in Theorem 3.1, we need to show the inequality (3.2) whenever $S_{ij} \geq 0$ and $S_{ij}v^j = 0$ (so that v is a null vector of S). Let (x_0, t_0) be the point where S_{ij} has a null vector v. By continuity we can assume that h_i^j has all eigenvalues distinct and in increasing order at (x_0, t_0), that is $\kappa_n > \kappa_{n-1} > \cdots > \kappa_1$. The null eigenvector condition $S_{ij}v^j = 0$ implies that $v = e_1$ and $S_{11} = \kappa_1 - 1 = 0$ at (x_0, t_0).

The lower order terms in (3.3) involving S_{ij} and $S_{ik}S_{kj}$ satisfy the null vector condition and can be ignored: In particular for the last term in (3.3), we have

$$\begin{aligned}\dot{\Psi}^{kl}(h_k^r h_{rl} + g_{kl} - 2h_{kl}) &= \sum_k \dot{\psi}^k \left(\kappa_k^2 + 1 - 2\kappa_k\right) \\ &= \sum_k \dot{\psi}^k (\kappa_k - 1)^2 \ \geq \ 0. \end{aligned} \tag{3.4}$$

Thus it remains to show that

$$Q_1 := \ddot{\Psi}^{kl,pq}\nabla_1 h_{kl}\nabla_1 h_{pq} + 2\sup_{\Lambda} \dot{\Psi}^{kl}\left(2\Lambda_k^p \nabla_l S_{1p} - \Lambda_k^p\Lambda_l^q S_{pq}\right) \ \geq \ 0. \tag{3.5}$$

Note that $S_{11} = 0$ and $\nabla_k S_{11} = 0$ at (x_0, t_0), the supremum over Λ can be computed exactly as follows:

$$\begin{aligned}&2\dot{\Psi}^{kl}\left(2\Lambda_k^p\nabla_l S_{1p} - \Lambda_k^p\Lambda_l^q S_{pq}\right) \\ &\quad = 2\sum_{k=1}^n\sum_{p=2}^n \dot{\psi}^k\left(2\Lambda_k^p\nabla_k S_{1p} - (\Lambda_k^p)^2 S_{pp}\right) \\ &\quad = 2\sum_{k=1}^n\sum_{p=2}^n \dot{\psi}^k\left(\frac{(\nabla_k S_{1p})^2}{S_{pp}} - \left(\Lambda_k^p - \frac{\nabla_k S_{1p}}{S_{pp}}\right)^2 S_{pp}\right).\end{aligned}$$

It follows that the supremum is obtained by choosing $\Lambda_k^p = \frac{\nabla_k S_{1p}}{S_{pp}}$. The required inequality for Q_1 becomes:

$$Q_1 = \ddot{\Psi}^{kl,pq}\nabla_1 h_{kl}\nabla_1 h_{pq} + 2\sum_{k=1}^n\sum_{p=2}^n \dot{\psi}^k \frac{(\nabla_k S_{1p})^2}{S_{pp}} \ \geq \ 0. \tag{3.6}$$

Using (2.5) to express the second derivatives of Ψ and noting that $\psi = f^\alpha$, $\nabla_1 S_{1p} = \nabla_1 h_{1p} = \nabla_p h_{11} = 0$ at (x_0, t_0), we have

$$Q_1 = \alpha f^{\alpha-1}\ddot{f}^{kl}\nabla_1 h_{kk}\nabla_1 h_{ll} + \alpha(\alpha-1)f^{\alpha-2}(\nabla_1 F)^2 + 2\alpha f^{\alpha-1}\sum_{k>l}\frac{\dot{f}^k-\dot{f}^l}{\kappa_k-\kappa_l}(\nabla_1 h_{kl})^2 + 2\alpha f^{\alpha-1}\sum_{k>1,l>1}\frac{\dot{f}^k}{\kappa_l-1}(\nabla_1 h_{kl})^2. \tag{3.7}$$

To make use the inverse concavity of f, let $\tau_i = 1/\kappa_i$ and $f_*(\tau) = f(\kappa)^{-1}$. We compute that

$$\begin{aligned}\dot{f}^k =& f_*^{-2}\frac{\partial f_*}{\partial \tau_k}\frac{1}{\kappa_k^2}\\ \ddot{f}^{kl} =& -f_*^{-2}\frac{\partial^2 f_*}{\partial\tau_k\partial\tau_l}\frac{1}{\kappa_k^2\kappa_l^2} + 2f_*^{-3}\frac{\partial f_*}{\partial\tau_k}\frac{1}{\kappa_k^2}\frac{\partial f_*}{\partial\tau_l}\frac{1}{\kappa_l^2} - 2f_*^{-2}\frac{\partial f_*}{\partial\tau_k}\frac{1}{\kappa_k^3}\delta_{kl}\\ =& -f_*^{-2}\frac{\partial^2 f_*}{\partial\tau_k\partial\tau_l}\frac{1}{\kappa_k^2\kappa_l^2} + 2f^{-1}\dot{f}^k\dot{f}^l - 2\frac{\dot{f}^k}{\kappa_k}\delta_{kl}.\end{aligned}$$

By the concavity of f_*, the first term of (3.7) can be estimated as

$$\alpha f^{\alpha-1}\ddot{f}^{kl}\nabla_1 h_{kk}\nabla_1 h_{ll} \geq 2\alpha f^{\alpha-1}\left(f^{-1}(\nabla_1 F)^2 - \sum_k\frac{\dot{f}^k}{\kappa_k}(\nabla_1 h_{kk})^2\right)$$

Then

$$\begin{aligned}\frac{Q_1}{\alpha f^{\alpha-1}} \geq& (\alpha+1)f^{-1}(\nabla_1 F)^2 - 2\sum_k\frac{\dot{f}^k}{\kappa_k}(\nabla_1 h_{kk})^2\\ &+ 2\sum_{k>l}\frac{\dot{f}^k-\dot{f}^l}{\kappa_k-\kappa_l}(\nabla_1 h_{kl})^2 + 2\sum_{k>1,l>1}\frac{\dot{f}^k}{\kappa_l-1}(\nabla_1 h_{kl})^2\\ \geq& (\alpha+1)f^{-1}(\nabla_1 F)^2 - 2\sum_{k>1}\frac{\dot{f}^k}{\kappa_k}(\nabla_1 h_{kk})^2\\ &- 2\sum_{k\neq l>1}\frac{\dot{f}^k}{\kappa_l}(\nabla_1 h_{kl})^2 + 2\sum_{k>1,l>1}\frac{\dot{f}^k}{\kappa_l-1}(\nabla_1 h_{kl})^2\\ =& (\alpha+1)f^{-1}(\nabla_1 F)^2 + 2\sum_{k>1,l>1}\left(\frac{\dot{f}^k}{\kappa_l-1} - \frac{\dot{f}^k}{\kappa_l}\right)(\nabla_1 h_{kl})^2\\ \geq& 0\end{aligned}$$

for all $\alpha > 0$, where we used the second inequality of (2.7) and the fact $\nabla_k h_{11} = 0$. The tensor maximum principle implies that the *h-convexity* is preserved along the flow (1.1).

Finally we show that M_t is strictly *h-convex* for $t > 0$. If this is not true, there exists some interior point (x_0, t_0) such that the smallest principal curvature is 1. By the strong maximum principle there exists a parallel vector field v such that $S_{ij}v^iv^j = 0$ on M_{t_0}. Then the smallest principal curvature is 1 on M_{t_0} everywhere. On the other hand, a standard argument shows that on any closed hypersurface in $\mathbb{H}^{n+1}$, there exists at least one point where all the principal curvatures are strictly bigger than one. This contradiction completes the proof of Theorem 3.2. □

Remark 3.3. The above argument implies that *h-convexity* is also preserved along the flow

$$\frac{\partial}{\partial t}X(x,t) = \ (\phi(t) - \Psi(\mathcal{W}(x,t)))\nu(x,t)$$

in hyperbolic space $\mathbb{H}^{n+1}$ with $\Psi = \psi(f)$, where f is an inverse concave function and $\psi : [0, +\infty) \to \mathbb{R}_+$ satisfies $\psi'(r) > 0$ and $\psi''(r)r + 2\psi'(r) \geq 0$ for all $r > 0$.

4. Upper bound of F

In this section, we will prove that F is uniformly bounded from above along the flow (1.1). Firstly, the preservation of $W_k(\Omega_t)$ and the *h-convexity* of $M_t = \partial\Omega_t$ imply uniform two-sided bounds on inner radius and outer radius of Ω_t.

Lemma 4.1. *Denote by $\rho_-(t)$ and $\rho_+(t)$ the inner and outer radii of the domain Ω_t enclosed by M_t. Then there exist positive constants c_1, c_2 depending only on n, k, M_0 such that*

$$0 < c_1 \leq \rho_-(t) \leq \rho_+(t) \leq c_2 \tag{4.1}$$

for all time $t \in [0, T)$.

Proof. On the one hand, since $W_k(\Omega_t) = W_k(\Omega_0)$, we have

$$W_k(B_{\rho_+(t)}) \geq \ W_k(\Omega_t) = W_k(\Omega_0),$$

where $B_{\rho_+(t)}$ is the geodesic ball of radius $\rho_+(t)$ that encloses Ω_t. Thus

$$\rho_+(t) \geq f_k^{-1}(W_k(\Omega_0)) > 0,$$

where f_k^{-1} is the inverse function of $f_k(r) = W_k(B_r)$. Similarly, $\rho_-(t) \leq f_k^{-1}(W_k(\Omega_0))$. Since each M_t is *h-convex*, the estimate (4.1) follows by the inequality (2.2). ☐

By (4.1), the inner radius of Ω_t is bounded below by a positive constant c_1. This implies that there exists a geodesic ball of radius c_1 contained in Ω_t for each $t \in [0, T)$. The following lemma shows the existence of a geodesic ball with fixed center enclosed by the flow hypersurfaces on a suitable time interval.

Lemma 4.2. *Let M_t be a smooth h-convex solution of (1.1) on $[0, T)$ with global term $\phi(t)$ given by (1.7). For any $t_0 \in [0, T)$, let $B(p_0, \rho_0)$ be the inball of Ω_{t_0}, where $\rho_0 = \rho_-(t_0)$. Then*

$$B(p_0, \rho_0/2) \subset \Omega_t, \quad t \in [t_0, \min\{T, t_0 + \tau\}) \tag{4.2}$$

for some τ depending only on n, α, k, Ω_0.

Proof. Given p_0, we denote by r_{p_0} the distance function to p_0 in $\mathbb{H}^{n+1}$ and by $\partial_r = \partial_{r_{p_0}}$ the gradient vector of r_{p_0}. For any $x \in M_t$,

$$\begin{aligned}\frac{\partial}{\partial t}\sinh^2 r_{p_0}(x) =& 2\langle \sinh r_{p_0}(x)\partial r, \frac{\partial}{\partial t}(\sinh r_{p_0}(x)\partial_r)\rangle \\ =& 2\sinh r_{p_0}(x)\cosh r_{p_0}(x)(\phi(t) - F^\alpha(x,t))\langle \partial_r, \nu\rangle,\end{aligned} \tag{4.3}$$

where we used the conformal property of the vector field $\sinh r\partial r$, i.e.,

$$\langle \bar{\nabla}_X(\sinh r\partial_r), Y\rangle = \cosh r\langle X, Y\rangle \tag{4.4}$$

for any tangential vector fields X, Y in $\mathbb{H}^{n+1}$ (see, e.g.,[16]). It follows from (4.3) that

$$\frac{\partial}{\partial t} r_{p_0}(x) =(\phi(t) - F^\alpha(x,t))\langle \partial r, \nu\rangle \geq \ -F^\alpha(x,t)\langle \partial_r, \nu\rangle,$$

since $\phi(t) > 0$ and $\langle \partial_r, \nu\rangle > 0$ on M_t. Denote $r(t) = \min_{M_t} r_{p_0}(x)$. At the minimum point, we have $\langle \partial_r, \nu\rangle = 1$ and $\kappa_i \leq \coth r(t)$. Then $F \leq \coth r(t)$ at the minimum point and

$$\frac{d}{dt} r(t) \geq -\coth\left(r(t)\right)^\alpha. \tag{4.5}$$

Note that $0 < c_1 \leq r(t) \leq 2\rho_+(t) \leq 2c_2$, where c_1, c_2 are the constants in (4.1). Then

$$\coth^{\alpha-1} r(t) \ \leq \ \max\{\coth^{\alpha-1}(2c_2), \coth^{\alpha-1}(c_1)\} \ =: \ c_3. \tag{4.6}$$

We deduce from (4.5) and (4.6) that

$$\frac{d}{dt} r(t) \geq -c_3 \coth r(t),$$

from which we solve that

$$\cosh r(t) \ \geq \ \cosh r(0) \exp\{-c_3 t\}.$$

In particular,

$$r(t) \geq \ \frac{r(0)}{2} = \ \frac{\rho_0}{2}$$

provided that

$$t - t_0 \leq \ \frac{1}{c_3} \ln \frac{\cosh r(0)}{\cosh \frac{r(0)}{2}} \ =: \ \tau$$

which depends only on n, α, k, Ω_0. Then $B(p_0, \rho_0/2) \subset \Omega_t$ for $t \in [t_0, \min\{T, t_0 + \tau\})$. □

Consider the support function $u(x,t) = \sinh r_{p_0}(x)\langle \partial_{r_{p_0}}, \nu\rangle$ of M_t with respect to the point p_0. Then by (2.3) and (4.2),

$$u(x,t) \ \geq \ \sinh(\frac{\rho_0}{2}) \tanh(\frac{\rho_0}{2}) \ =: \ 2c \tag{4.7}$$

on M_t for any $t \in [t_0, \min\{T, t_0 + \tau\})$. On the other hand, the estimate (4.1) implies that $u(x,t) \leq \sinh(2c_2)$ on M_t for all $t \in [0, T)$.

Lemma 4.3. *The support function $u(x,t)$ evolves by*

$$\frac{\partial}{\partial t} u =\dot{\Psi}^{kl} \nabla_k \nabla_l u + \cosh r_{p_0}(x) \left(\phi(t) - \Psi - \dot{\Psi}^{kl} h_{kl}\right) + \dot{\Psi}^{ij} h_i^k h_{kj} u. \tag{4.8}$$

Proof. Firstly, by (4.4) and (2.10),

$$\frac{\partial}{\partial t} u = \cosh r_{p_0}(x)\left(\phi(t) - \Psi\right) + \sinh r_{p_0}(x)\langle \partial r, \nabla \Psi\rangle. \tag{4.9}$$

Secondly, the spatial derivatives of u can also be computed using (4.4):

$$\nabla_j u = \ \sinh r_{p_0}(x)\langle \partial r, h_i^k \partial_k\rangle \tag{4.10}$$

$$\nabla_i \nabla_j u = \ \cosh r_{p_0}(x) h_{ij} + \sinh r_{p_0}(x)\langle \partial_r, \nabla^k h_{ij}\partial_k - h_i^k h_{kj}\nu\rangle. \tag{4.11}$$

Then the evolution equation (4.8) follows by a direct computation using (4.9) –(4.11). □

Now we can use the technique that was first introduced by Tso [32] to prove the upper bound of F along the flow (1.1).

Theorem 4.4. *Let M_t be a smooth h-convex solution of (1.1) on $[0,T)$ with global term $\phi(t)$ given by (1.7). Then we have $\max_{M_t} F \leq C$ for any $t \in [0,T)$, where C depends on n, k, α, M_0 but not on T.*

Proof. For any given $t_0 \in [0,T)$, define the auxiliary function

$$W(x,t) = \frac{\Psi(x,t)}{u(x,t)-c}$$

which is well-defined for all $t \in [t_0, \min\{T, t_0+\tau\})$ by (4.7). Combining (2.13) and (4.8), we can compute the evolution equation of the function W

$$\begin{aligned}\frac{\partial}{\partial t}W =& \dot{\Psi}^{ij}\left(\nabla_j\nabla_i W + \frac{2}{u-c}\nabla_i u \nabla_j W\right)\\ &- \frac{\phi(t)}{u-c}\left(\dot{\Psi}^{ij}(h_i^k h_k^j - \delta_i^j) + W\cosh r_{p_0}(x)\right)\\ &+ \frac{\Psi}{(u-c)^2}(\Psi + \dot{\Psi}^{kl}h_{kl})\cosh r_{p_0}(x) - \frac{c\Psi}{(u-c)^2}\dot{\Psi}^{ij}h_i^k h_k^j - W\dot{\Psi}^{ij}\delta_i^j. \end{aligned} \tag{4.12}$$

The *h-convexity* of M_t implies that $\dot{\Psi}^{ij}(h_i^k h_k^j - \delta_i^j) \geq 0$. So the terms involving $\phi(t)$ in (4.12) are non-positive. Since $\Psi = F^\alpha$ where F is inverse concave, we have $\Psi + \dot{\Psi}^{kl}h_{kl} = (1+\alpha)\Psi$ and $\dot{\Psi}^{ij}h_i^k h_k^j \geq \alpha F^{\alpha+1}$. The last term of (4.12) is obviously non-positive. Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}W \leq& \dot{\Psi}^{ij}\left(\nabla_j\nabla_i W + \frac{2}{u-c}\nabla_i u \nabla_j W\right)\\ &+ (\alpha+1)W^2\cosh r_{p_0}(x) - \alpha c W^2 F. \end{aligned} \tag{4.13}$$

Using (4.7) and the upper bound $r_{p_0}(x) \leq 2c_2$, we obtain the following estimate

$$\begin{aligned}\frac{\partial}{\partial t}W \leq& \dot{\Psi}^{ij}\left(\nabla_j\nabla_i W + \frac{2}{u-c}\nabla_i u \nabla_j W\right)\\ &+ W^2\left((\alpha+1)\cosh(2c_2) - \alpha c^{1+\frac{1}{\alpha}}W^{1/\alpha}\right) \end{aligned} \tag{4.14}$$

holds on $[t_0, \min\{T, t_0+\tau\})$. Let $\tilde{W}(t) = \sup_{M_t} W(\cdot, t)$. Then (4.14) implies that

$$\frac{d}{dt}\tilde{W}(t) \leq \tilde{W}^2\left((\alpha+1)\cosh(2c_2) - \alpha c^{1+\frac{1}{\alpha}}\tilde{W}^{1/\alpha}\right)$$

from which it follows using the maximum principle that

$$\tilde{W}(t) \leq \max\left\{\left(\frac{2(1+\alpha)\cosh(2c_2)}{\alpha}\right)^{\alpha} c^{-(\alpha+1)}, \left(\frac{2}{1+\alpha}\right)^{\frac{\alpha}{1+\alpha}} c^{-1}(t-t_0)^{-\frac{\alpha}{1+\alpha}}\right\}. \tag{4.15}$$

Then the upper bound on F follows from (4.15) and the facts that

$$c = \frac{1}{2}\sinh(\frac{\rho_0}{2})\tanh(\frac{\rho_0}{2}) \geq \frac{1}{2}\sinh(\frac{c_1}{2})\tanh(\frac{c_1}{2})$$

and $u - c \leq 2c_2$, where c_1, c_2 are constants in (4.1) depending only on n, k, M_0. □

As an corollary of the upper bound of F and the *h-convexity* of M_t, there exist constants c_4, c_5 depend only on n, k, α, M_0 such that

$$c_4 \leq \phi(t) \leq c_5 \tag{4.16}$$

on $[0,T)$.

5. Long-time existence

Until now, we only used the fact that f is inverse concave. The upper bound on F proved in §4 implies that the dual function f_* of f is bounded from below by a positive constant. Applying condition (vi) in Assumption 1.1 that f_* approaches zero on the boundary of Γ_+, there exists a positive constant C such that $1/\kappa_i \geq C$ for all $i = 1, \cdots, n$, which implies a uniform upper bound on the Weingarten matrix $\mathcal{W} = (h_i^j)$ along the flow (1.1) for all $t \in [0, T)$.

Lemma 5.1. *There exists a constant $C > 0$ depending only on $n, k, , \alpha, M_0$ such that along the flow (1.1), the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_i)$ of the solution M_t satisfy*

$$1 \leq \kappa_i \leq C, \quad i = 1, \cdots, n$$

for all $t \in [0, T)$.

In the following, we will derive higher order estimates on the solution M_t of the flow (1.1) and prove that the solution M_t exists for all time $t \in [0, \infty)$.

Let us denote by $\mathbb{R}^{1,n+1}$ the Minkowski spacetime, that is the vector space $\mathbb{R}^{n+2}$ endowed with the Minkowski spacetime metric $\langle \cdot, \cdot \rangle$ given by

$$\langle X, X \rangle \; = \; -X_0^2 + \sum_{i=1}^{n} X_i^2$$

for any vector $X = (X_0, X_1, \cdots, X_n) \in \mathbb{R}^{n+2}$. The hyperbolic space $\mathbb{H}^{n+1}$ is then

$$\mathbb{H}^{n+1} = \; \{X \in \mathbb{R}^{1,n+1}, \; \langle X, X \rangle = -1, \; X_0 > 0\}$$

An embedding $X : M^n \to \mathbb{H}^{n+1}$ induces an embedding $Y : M^n \to B_1(0) \subset \mathbb{R}^{n+1}$ by

$$X \; = \; \frac{(1, Y)}{\sqrt{1 - |Y|^2}}, \tag{5.1}$$

where $Y \in B_1(0) \subset \mathbb{R}^{n+1}$. Let $\{x_i\}, i = 1, \cdots, n$ be a local coordinate system on M and $\{\partial_i\}$ be the corresponding coordinate vectors. Then

$$\partial_i X = \; X_*(\partial_i) = \; \frac{(0, \partial_i Y)}{\sqrt{1 - |Y|^2}} + \frac{X}{1 - |Y|^2} \langle Y, \partial_i Y \rangle \tag{5.2}$$

and

$$\begin{aligned} \partial_i \partial_j X = & \frac{(0, \partial_i \partial_j Y)}{\sqrt{1 - |Y|^2}} + \frac{(0, \partial_j Y)}{(1 - |Y|^2)^{3/2}} \langle Y, \partial_i Y \rangle + \partial_i X \frac{\langle Y, \partial_i Y \rangle}{1 - |Y|^2} \\ & + X \partial_i \left(\frac{\langle Y, \partial_i Y \rangle}{1 - |Y|^2} \right). \end{aligned}$$

Let $\nu \in T\mathbb{H}^{n+1}, h_{ij}^X$ and $N \in \mathbb{R}^{n+1}, h_{ij}^Y$ be the unit normal vectors and the second fundamental forms of $X(M^n) \subset \mathbb{H}^{n+1}$ and $Y(M^n) \subset \mathbb{R}^{n+1}$ respectively. Note that $\langle \nu, X \rangle = \langle \nu, \partial_i X \rangle = 0$. Taking the inner product of (5.2) with ν, we also have

$$\langle \nu, (0, \partial_i Y) \rangle = 0 \tag{5.3}$$

Therefore,

$$
\begin{aligned}
h_{ij}^X =& - \langle \partial_i \partial_j X, \nu\rangle = -\frac{1}{\sqrt{1-|Y|^2}}\langle (0, \partial_i\partial_j Y), \nu\rangle \\
=& \ - \frac{\langle \partial_i\partial_j Y, N\rangle}{\sqrt{1-|Y|^2}}\langle (0,N),\nu\rangle = \frac{h_{ij}^Y}{\sqrt{1-|Y|^2}}\langle (0,N),\nu\rangle
\end{aligned} \tag{5.4}
$$

By (5.3), we can write $\nu = a_1(0,N) + a_2(1,0)$ where a_1, a_2 can be computed as follows:

$$
\begin{aligned}
1 =& \langle \nu,\nu\rangle = \ a_1^2 - a_2^2 \\
0 =& \langle \nu, X\rangle = \ \frac{1}{\sqrt{1-|Y|^2}}(a_1\langle N, Y\rangle - a_2)
\end{aligned}
$$

from which we deduce that

$$
\langle (0,N),\nu\rangle = \ a_1 = \frac{1}{\sqrt{1-\langle N, Y\rangle^2}}.
$$

Substituting this into (5.4) yields

$$
h_{ij}^X = \frac{h_{ij}^Y}{\sqrt{(1-|Y|^2)(1-\langle N,Y\rangle^2)}}. \tag{5.5}
$$

From (5.2), we also have that the induced metrics g_{ij}^X and g_{ij}^Y of $X(M^n)\subset \mathbb{H}^{n+1}$ and $Y(M^n)\subset\mathbb{R}^{n+1}$ are related by

$$
\begin{aligned}
g_{ij}^X =& \langle \partial_i X, \partial_j X\rangle = \frac{\langle \partial_i Y, \partial_j Y\rangle}{1-|Y|^2} + \frac{\langle Y, \partial_i Y\rangle\langle Y, \partial_j Y\rangle}{(1-|Y|^2)^2} \\
=& \frac{1}{1-|Y|^2}\left(g_{ij}^Y + \frac{\langle Y, \partial_i Y\rangle\langle Y, \partial_j Y\rangle}{(1-|Y|^2)}\right)
\end{aligned}
$$

Suppose $X : M^n\times[0,T)\to\mathbb{H}^{n+1}$ is a solution to the flow (1.1). We next derive the evolution equation of the corresponding $Y : M^n\times[0,T)\to\mathbb{R}^{n+1}$ related by (5.1). Denote by $\mathcal{W}^X = (h_{ik}^X g_X^{kj})$ the Weingarten matrix of $X(M^n,t)\subset\mathbb{H}^{n+1}$, where (g_X^{kj}) denotes the inverse matrix of the induce metric g_{kj}^X. Then

$$
\begin{aligned}
\phi(t) - F^\alpha(\mathcal{W}^X) =& \ \langle \partial_t X, \nu\rangle \\
=& \ \frac{\langle (0,\partial_t Y),\nu\rangle}{\sqrt{1-|Y|^2}} + \langle X,\nu\rangle\frac{\langle Y, \partial_t Y\rangle}{1-|Y|^2} \\
=& \ \langle \partial_t Y, N\rangle\frac{\langle (0,N),\nu\rangle}{\sqrt{1-|Y|^2}} \\
=& \ \langle \partial_t Y, N\rangle\frac{1}{\sqrt{(1-|Y|^2)(1-\langle N,Y\rangle^2)}}
\end{aligned}
$$

where we used the facts $\langle X,\nu\rangle = 0$ and $\langle (0,\partial_i Y),\nu\rangle = 0$ in the third equality. Thus up to a tangential diffeomorphism, $Y : M^n\times[0,T)\to\mathbb{R}^{n+1}$ satisfies the following evolution equation:

$$
\partial_t Y = \ \sqrt{(1-|Y|^2)(1-\langle N,Y\rangle^2)}\left(\phi(t) - F^\alpha(\mathcal{W}^X)\right)N. \tag{5.6}
$$

Here $\mathcal{W}^X$ is the Weingarten matrix of $X(M^n,t)\subset\mathbb{H}^{n+1}$, which we next relate to the geometry of Y: In local coordinates, the inverse matrix $\mathcal{W}_X^{-1}$ of $\mathcal{W}^X$ satisfies

$$\begin{aligned}(\mathcal{W}_X^{-1})_{ij} =&(h_X^{-1})^{jk}g_{ki}^X\\ =&(h_Y^{-1})^{kj}\left(g_{ki}^Y+\frac{\langle Y,\partial_iY\rangle\langle Y,\partial_kY\rangle}{(1-|Y|^2)}\right)\sqrt{\frac{1-\langle N,Y\rangle^2}{1-|Y|^2}}\end{aligned}\tag{5.7}$$

By the estimate (4.1), X stays in a bounded subset of $\mathbb{H}^{n+1}$. Then there exists a positive constant $c>0$ depending only on n,k,M_0 such that

$$0<c\leq 1-|Y|^2\leq 1,\quad 0<c\leq 1-\langle N,Y\rangle^2\leq 1.$$

Since each M_t is *h-convex* in $\mathbb{H}^{n+1}$, the equation (5.5) implies that each $Y_t=Y(M^n,t)$ is strictly convex in $\mathbb{R}^{n+1}$. We can parametrise Y_t using the Gauss map and the support function $s(z):=\langle Y(N^{-1}(z)),z\rangle$, where $N^{-1}:\mathbb{S}^n\to M^n$ is the inverse map of the Gauss map which exists due to the strict convexity of Y_t. Then Y is given by the embedding $Y:\mathbb{S}^n\to\mathbb{R}^{n+1}$ with (cf. [2, 7])

$$Y(z)=\ s(z)z+\bar\nabla s$$

where $\bar\nabla$ is the gradient with respect to the standard round metric $\bar g$ on $\mathbb{S}^n$. The derivative of this map is given by

$$\partial_iY=\ \tau_{ik}\bar g^{kl}\partial_lz$$

in local coordinates, where τ_{ij} is given as follows

$$\tau_{ij}=\ \bar\nabla_i\bar\nabla_js+s\bar g_{ij}.$$

The eigenvalues τ_i of the matrix τ_{ij} with respect to the metric $\bar g$ are the inverse of the principal curvatures κ_i, i.e., $\tau_i=1/\kappa_i$, and are called the principal radii of curvature. Then

$$g_{ki}^Y=\ \langle\partial_kY,\partial_iY\rangle=\tau_{kp}\bar g^{pq}\tau_{qi}$$

and

$$\frac{\langle Y,\partial_iY\rangle\langle Y,\partial_kY\rangle}{(1-|Y|^2)}=\ \tau_{kp}\frac{\langle\bar g^{pa}\bar\nabla_as,\bar g^{qb}\bar\nabla_bs\rangle}{1-s^2-|\bar\nabla s|^2}\tau_{qi}$$

We can express (5.7) using $s,\bar\nabla s$ and the matrix τ_{ij} as follows:

$$\begin{aligned}(\mathcal{W}_X^{-1})_{ij} =&(h_Y^{-1})^{kj}\left(g_{ki}^Y+\frac{\langle Y,\partial_iY\rangle\langle Y,\partial_kY\rangle}{(1-|Y|^2)}\right)\sqrt{\frac{1-\langle N,Y\rangle^2}{1-|Y|^2}}\\ =&\tau_{lj}(\tau^{-1})^{ls}\bar g_{st}(\tau^{-1})^{tk}\tau_{kp}\left(\bar g^{pq}+\frac{\langle\bar g^{pa}\bar\nabla_as,\bar g^{qb}\bar\nabla_bs\rangle}{1-s^2-|\bar\nabla s|^2}\right)\tau_{qi}\sqrt{\frac{1-s^2}{1-s^2-|\bar\nabla s|^2}}\\ =&\left(\bar g^{jq}+\frac{\langle\bar g^{ja}\bar\nabla_as,\bar g^{qb}\bar\nabla_bs\rangle}{1-s^2-|\bar\nabla s|^2}\right)\tau_{qi}\sqrt{\frac{1-s^2}{1-s^2-|\bar\nabla s|^2}}\end{aligned}\tag{5.8}$$

where τ^{-1} denotes the inverse matrix of τ_{ij}. The solution of the evolution equation (5.6) is then given up to a tangential diffeomorphism by solving the following scalar parabolic equation

$$\partial_ts=\ \sqrt{(1-s^2-|\bar\nabla s|^2)(1-s^2)}\left(\phi(t)-F_*^{-\alpha}((\mathcal{W}_X^{-1})_{ij})\right)\tag{5.9}$$

for the support function $s(z,t)$, where $(\mathcal{W}_X^{-1})_{ij}$ is the matrix given in (5.8) in terms of $s,\bar\nabla s$ and the matrix $\tau_{ij}=\bar\nabla_i\bar\nabla_js+s\bar g_{ij}$.

By (4.1) and Lemma 5.1, we already have uniform C^2 estimates on the support function $s(z,t)$. Denote the right hand side of (5.9) by $G(\bar{\nabla}^2 s, \bar{\nabla} s, s, z, t)$. Then

$$\begin{aligned}\dot{G}^{ij} = \frac{\partial G}{\partial(\bar{\nabla}^2_{ij} s)} =& \ \alpha F_*^{-\alpha-1}\sqrt{(1-s^2-|\bar{\nabla} s|^2)(1-s^2)}\dot{F}_*^{pq}\frac{\partial(\mathcal{W}_X^{-1})_{pq}}{\partial\bar{\nabla}^2_{ij}s}\\ =& \ \alpha F_*^{-\alpha-1}(1-s^2)\dot{F}_*^{pq}\left(\bar{g}^{pi}+\frac{\langle \bar{g}^{pa}\bar{\nabla}_a s, \bar{g}^{ib}\bar{\nabla}_b s\rangle}{1-s^2-|\bar{\nabla} s|^2}\right)\bar{g}_{qj},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt{(1-s^2-|\bar{\nabla} s|^2)(1-s^2)}}\dot{G}^{ij,kl} =& \ \alpha F_*^{-\alpha-1}\dot{F}_*^{pq,rt}\frac{\partial(\mathcal{W}_X^{-1})_{pq}}{\partial\bar{\nabla}^2_{ij}s}\frac{\partial(\mathcal{W}_X^{-1})_{rt}}{\partial\bar{\nabla}^2_{kl}s}\\ &-\alpha(\alpha+1)F_*^{-\alpha-2}\dot{F}_*^{pq}\frac{\partial(\mathcal{W}_X^{-1})_{pq}}{\partial\bar{\nabla}^2_{ij}s}\dot{F}_*^{rt}\frac{\partial(\mathcal{W}_X^{-1})_{rt}}{\partial\bar{\nabla}^2_{kl}s} \quad (5.10)\end{aligned}$$

The uniform bound on F and Lemma 5.1 imply that there exists a constant $C>0$ such that $0<C^{-1}I \le \left(\dot{F}_*^{ij}\right) \le CI$. The estimates on $s, \bar{\nabla} s$ and F then imply that

$$\lambda I \le (\dot{G}^{ij}) \le \Lambda I$$

for some constants $\lambda, \Lambda > 0$. By the concavity of F_* and $\alpha>0$, from (5.10) the operator G is concave with respect to $\bar{\nabla}^2 s$. Since we have uniform C^2 estimates on s in space-time, we can apply the Hölder estimate of [21] (as in [8, 14]) to obtain the $C^{2,\alpha}$ estimate on s and C^α estimate on $\partial_t s$ for some $\alpha\in(0,1)$ in space-time. By the parabolic Schauder theory [22], we can derive estimates on all higher order derivatives of s. A standard continuation argument yields that the flow (1.1) exists for all time $[0,\infty)$.

Proposition 5.2. *Let M_t be a smooth h-convex solution to the flow (1.1) with $\alpha>0$ and $\phi(t)$ given by (1.7). If f is inverse concave and f_* approaches zero on the boundary of Γ_+, then the solution M_t exists for all time $t\in[0,\infty)$.*

6. Smooth convergence

In this section, we will use the Alexandrov reflection method to show that the solution of (1.1) converges smoothly to a geodesic sphere as $t\to\infty$.

Let γ be a geodesic line in $\mathbb{H}^{n+1}$, and let $H_{\gamma(s)}$ be the totally geodesic hyperbolic n-plane in $\mathbb{H}^{n+1}$ which is perpendicular to γ at $\gamma(s), s\in\mathbb{R}$. We use the notation H_s^+ and H_s^- for the half-spaces in $\mathbb{H}^{n+1}$ determined by $H_{\gamma(s)}$:

$$H_s^+ := \bigcup_{s'\ge s} H_{\gamma(s')}, \qquad H_s^- := \bigcup_{s'\le s} H_{\gamma(s')}$$

For a bounded domain Ω in $\mathbb{H}^{n+1}$, denote

$$\Omega^+(s) = \Omega\cap H_s^+, \qquad \Omega^-(s) = \Omega\cap H_s^-.$$

The reflection map across $H_{\gamma(s)}$ is denoted by $R_{\gamma,s}$. We define

$$S_\gamma^+(\Omega) := \inf\{s\in\mathbb{R} \mid R_{\gamma,s}(\Omega^+(s))\subset\Omega^-(s)\}. \quad (6.1)$$

Lemma 6.1. *For any geodesic line γ in $\mathbb{H}^{n+1}$, $S_\gamma^+(\Omega_t)$ is strictly decreasing along the flow (1.1) unless $R_{\gamma,\bar{s}}(\Omega_t)=\Omega_t$ for some $\bar{s}\in\mathbb{R}$ (in which case $S_\gamma^+(\Omega_t)=\bar{s}$ for all t).*

Proof. Fix $t_0 \in [0,\infty)$, and set $\bar{s} = S^+_\gamma(\Omega_{t_0})$. By definition we have $R_{\gamma,\bar{s}}(\Omega^+_{t_0}(\bar{s})) \subset \Omega^-_{t_0}(\bar{s})$, and since s cannot be decreased below $\bar{s}$ we must have one of two possibilities: Either (i) there is a point $\bar{x} \in R_{\gamma,\bar{s}}(\partial\Omega^+_{t_0}(\bar{s})) \cap \partial\Omega^-_{t_0}(\bar{s})$ which is not in $H_{\gamma(\bar{s})}$, or (ii) there is a point $\bar{x}$ in $\partial\Omega_{t_0} \cap H_{\gamma(\bar{s})}$ such that $T_{\bar{x}}\partial\Omega_{t_0}$ is mapped to itself by $R_{\gamma,\bar{s}}$.

For case (i), since both $M^-_t(s)$ and $R_{\gamma(s)}(M^+_t(s))$ are strictly *h-convex*, we locally express them as graphs of functions $r^-(\theta,t)$ and $r^+(\theta,t)$ over a domain U of a geodesic sphere for t sufficiently close to t_0. Define $\omega(\theta,t) := r^-(\theta,t) - r^+(\theta,t)$. Then $\omega(\theta,t_0)$ is nonnegative for $\theta \in U$ and there exists a point $\theta_0 \in U$ where the minimum $\omega(\theta_0,t_0) = 0$ is achieved. We will argue below using the strong maximum principle that ω vanishes identically, and it follows from this that $M^-_{t_0}(s)$ coincides with $R_{\gamma(s)}(M^+_{t_0}(s))$ and hence M_{t_0} is reflection symmetric across the totally geodesic plane $H_{\gamma(s)}$ at $s = S^+_\gamma(\Omega_{t_0})$.

In order to apply the strong maximum principle we first recall the graphical representation of hypersurfaces in $\mathbb{H}^{n+1}$. Let $M \subset \mathbb{H}^{n+1}$ be a hypersurface which can be written as a radial graph over a sphere S^n, i.e., $M = \{(\theta, r(\theta)) :\ \theta \in S^n\}$ for a smooth function r on S^n. Let $\{\theta^i\}, i = 1,\cdots,n$ be a local coordinate system on S^n. The induced metric on M from $\mathbb{H}^{n+1}$ takes the form

$$g_{ij} \ = \ D_i r D_j r + \sinh^2 r \sigma_{ij},$$

where σ_{ij} denotes the standard metric on S^n. Denote

$$v = \sqrt{1 + |Dr|^2/\sinh^2 r}, \quad \text{and} \quad \tilde{\sigma}^{jk} \ = \ \sigma^{jk} - \frac{D^j r D^k r}{v^2 \sinh^2 r} \tag{6.2}$$

where σ^{jk} is the inverse matrix of σ_{jk} and $D^j r = \sigma^{jk} D_k r$. Then the Weingarten matrix (h_i^j) can be expressed as (cf. [15])

$$h_i^j = \ \frac{\coth r}{v}\delta_i^j + \frac{\coth r}{v^3 \sinh^2 r} D^j r D_i r - \frac{\tilde{\sigma}^{jk}}{v \sinh^2 r} D_k D_i r. \tag{6.3}$$

Up to a tangential diffeomorphism, the flow equation (1.1) is equivalent to the following scalar parabolic PDE

$$\frac{\partial r}{\partial t} = \ (\phi(t) - F^\alpha(h_i^j))\sqrt{1 + |Dr|^2/\sinh^2 r}. \tag{6.4}$$

Denote the right hand side of (6.4) by $\Phi(D^2 r, Dr, r, \theta, t)$.

We now come back to the $M^-_t(s)$ and $R_{\gamma(s)}(M^+_t(s))$ and $\omega(\theta,t) = r^-(\theta,t) - r^+(\theta,t)$. Since the flow (1.1) is invariant under reflection, by (6.4) the function $\omega(\theta,t)$ satisfies the following equation

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \Phi(D^2 r^-, Dr^-, r^-, \theta, t) - \Phi(D^2 r^+, Dr^+, r^+, \theta, t)\\ &= \frac{\partial \Phi}{\partial D^2_{ij} r}(\chi,\xi,\eta) D^2_{ij}\omega + \frac{\partial \Phi}{\partial D_i r}(\chi,\xi,\eta) D_i \omega + \frac{\partial \Phi}{\partial r}(\chi,\xi,\eta)\omega,\end{aligned} \tag{6.5}$$

where $(\chi,\xi,\eta) = (\rho D^2 r^- + (1-\rho) D^2 r^+, \rho Dr^- + (1-\rho) Dr^+, \rho r^- + (1-\rho) r^+)$ for some $\rho \in [0,1]$. The uniform estimates in §5 implies that the equation (6.5) is uniformly parabolic, i.e., there exist constants $\lambda, \Lambda > 0$ such that

$$\lambda I \le (\frac{\partial \Phi}{\partial D^2_{ij} r}) \le \Lambda I.$$

The coefficients $\frac{\partial\Phi}{\partial D_i r}$ and $\frac{\partial\Phi}{\partial r}$ are uniformly bounded and smooth. Since $\omega(\theta,t_0)$ is non-negative and is positive somewhere in U, the strong maximum principle applied to the equation (6.5) yields that $\omega>0$ everywhere in U for $t>t_0$, unless it is identically zero. This in turn implies that $S_\gamma^+(\Omega_t)<S_\gamma^+(\Omega_{t_0})$ for $t>t_0$.

The discussion for case (ii) is similar. We again write $M_t^-(s)$ and $R_{\gamma(s)}(M_t^+(s))$ locally as graphs of functions $r^-(\theta,t)$ and $r^+(\theta,t)$ over a domain U of a geodesic sphere for t sufficiently close to t_0, and then apply the boundary strong maximum principle (the Hopf Lemma). □

Now we prove that the flow (1.1) converges smoothly to a geodesic sphere. For any fixed τ, we define the flow $\Omega_\tau(t)$ by $\Omega_\tau(t):=\Omega_{t+\tau}$. The uniform estimates in §5 imply that there exists a sequence of $\tau_k\to\infty$ such that the families $\Omega_{\tau_k}(t)$ converge smoothly to a limiting flow $\Omega_\infty(t),t\in[0,\infty)$ which is again a solution of (1.1). By the monotonicity of $S_\gamma^+(\Omega_t)$ proved in Lemma 6.1, we have

$$S_\gamma^+(\Omega_\infty(t))=\lim_{t'\to\infty}S_\gamma^+(\Omega_{t'}) \tag{6.6}$$

which exists by the monotonicity. The right hand side of (6.6) is independent of t and is finite. We conclude that the limiting flow $\Omega_\infty(t)$ is symmetric under reflection across a totally geodesic hyperplane H_γ which is perpendicular to the geodesic line γ. Since γ is arbitrary, we conclude that $\Omega_\infty(t)$ is a geodesic sphere. This implies the subsequential smooth convergence of Ω_t to a geodesic sphere of radius $r_\infty=f_k^{-1}(W_k(\Omega_0))$.

The linearisation of the flow (1.1) about the geodesic sphere of radius r_∞ can be used to deduce the stronger convergence results. The hypersurface near the geodesic sphere can be written as a graph of a smooth function r over the geodesic sphere. Setting $r=r_\infty(1+\epsilon\eta)$. The linearised equation of the flow (6.4) about the geodesic sphere of radius r_∞ is given by

$$\frac{\partial\eta}{\partial t}=\frac{\alpha\coth^{\alpha-1}r_\infty}{n\sinh^2 r_\infty}\left(\Delta\eta+n\eta-\frac{n}{|S^n|}\int_{S^n}\eta d\sigma\right) \tag{6.7}$$

This equation is the same as that for mixed volume preserving mean curvature flow in [25, Eqn.(21)] (see also [3, §12]). Thus the same argument as in [3] gives that the flow (1.1) converges exponentially in smooth topology to the geodesic sphere with radius r_∞. This completes the proof of Theorem 1.2.

Appendix A. Smooth convergence: $f=E_k^{1/k}$

In this appendix, we provide an alternative approach to the proof of smooth convergence to a geodesic sphere for the volume preserving flow (1.1) with $f=E_k^{1/k}$ and $\alpha>0$. The key ingredient is the monotonicity of the quermassintegral $W_k(\Omega_t)$.

Lemma A.1. *For any integer $k\in\{1,\cdots,n\}$, let M_t be a smooth convex solution of the volume preserving flow (1.1) with $f=E_k^{1/k}$ and $\alpha>0$. Denote Ω_t the domain enclosed by M_t. Then $W_k(\Omega_t)$ is monotone decreasing in time t.*

Proof. The function $\phi(t)$ in (1.1) for the volume preserving flow is defined as in (2.19). From the evolution equation (2.18) for the quermassintegral of Ω_t:

$$\frac{d}{dt}W_k(\Omega_t)=\frac{n+1-k}{n+1}\int_{M_t}E_k\left(\phi(t)-F^\alpha\right)d\mu_t \tag{A.1}$$

and $F = E_k^{1/k}$, we have

$$\frac{d}{dt}W_k(\Omega_t) = \frac{n+1-k}{n+1}\left(\frac{1}{|M_t|}\int_{M_t} E_k \int_{M_t} E_k^{\alpha/k} - \int_{M_t} E_k^{1+\alpha/k} d\mu_t\right).$$

Then the monotonicity of $W_k(\Omega_t)$ follows immediately from the Jensen's inequality

$$\int_{M_t} E_k^{(\alpha+k)/k} d\mu_t \ \geq \ \frac{1}{|M_t|}\int_{M_t} E_k d\mu_t \int_{M_t} E_k^{\alpha/k} d\mu_t. \tag{A.2}$$

□

If the initial hypersurface M_0 is *h-convex*, then the long time existence of the flow (1.1) has been proved in §5. To show the smooth convergence to a geodesic sphere, we need the following lemma. Denote

$$\bar{E}_k = \frac{1}{|M_t|}\int_{M_t} E_k d\mu_t.$$

Lemma A.2. *For any integer $k \in \{1, \cdots, n\}$, let M_t be a smooth h-convex solution of the volume preserving flow (1.1) with $f = E_k^{1/k}$ and $\alpha > 0$. Then there exists a sequence of times $t_i \to \infty$ such that*

$$\int_{M_{t_i}} \left(E_k - \bar{E}_k\right)^2 d\mu_{t_i} \ \to 0, \quad \text{as } i \to \infty \tag{A.3}$$

Proof. Since M_t is *h-convex*, the Alexandrov-Fenchel inequality (2.4) implies that ,

$$W_k(\Omega_t) \ \geq \ f_k \circ f_0^{-1}(W_0(\Omega_t)) = \ f_k \circ f_0^{-1}(W_0(\Omega_0)) \ > \ 0,$$

where we used the condition $|\Omega_t| = |\Omega_0|$. By Lemma A.1, $W_k(\Omega_t)$ is monotone decreasing. Then we can find a sequence of times $t_i \to \infty$ such that

$$\left.\frac{d}{dt}\right|_{t_i} W_k(\Omega_t) \ \to \ 0, \quad \text{as } i \to \infty,$$

Then from the proof of Lemma A.1, equality is attained in Jensen's inequality (A.2) as $t_i \to \infty$, or equivalently

$$\begin{aligned} 0 \leq & \int_{M_{t_i}} E_k^{\alpha/k}\left(E_k - \bar{E}_k\right) d\mu_{t_i} \\ = & \int_{M_{t_i}} \left(E_k^{\alpha/k} - \bar{E}_k^{\alpha/k}\right)\left(E_k - \bar{E}_k\right) d\mu_{t_i} \ \to \ 0, \quad \text{as } i \to \infty. \end{aligned} \tag{A.4}$$

Since

$$\left(E_k^{\alpha/k} - \bar{E}_k^{\alpha/k}\right)\left(E_k - \bar{E}_k\right) \ \geq \ C\left(E_k - \bar{E}_k\right)^2, \tag{A.5}$$

as in the proof of Lemma 6.1 of [8], then (A.3) follows from (A.4) and (A.5) immediately.

□

The uniform estimates on all the higher derivatives of the Weingarten map (h_i^j) implies that for any sequence of times $t_i \to \infty$, there exists a subsequence (still denoted by t_i) such that M_{t_i} converges to a limit hypersurface M_∞ (up to an ambient isometry). Thus by Lemma A.2, we can find a sequence of times $t_i \to \infty$ such that M_{t_i} converges smoothly to a geodesic sphere up to an isometry. For any other sequence of times $t_j \to \infty$, we can also find a subsequence of time (labeled by the same t_j) such that M_{t_j} converges to a limit $\hat{M}_\infty = \partial\hat{\Omega}_\infty$. The monotonicity of $W_k(\Omega_t)$ then yields $W_k(\hat{\Omega}_\infty) = W_k(B^{n+1})$, which implies that $\hat{\Omega}_\infty$ is a geodesic ball by the equality case of Theorem 2.2. Since the above argument works for any sequence, we conclude that the whole family of M_t converges to a geodesic sphere as $t \to \infty$ up to an isometry. The exponential convergence is the same as in §6.

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