

STABILITY OF TORSION-FREE G_2 STRUCTURES ALONG THE LAPLACIAN FLOW

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ABSTRACT. We prove that torsion-free G_2 structures are (weakly) dynamically stable along the Laplacian flow for closed G_2 structures. More precisely, given a torsion-free G_2 structure $\bar{\varphi}$ on a compact 7-manifold M , the Laplacian flow with initial value in $[\bar{\varphi}]$, sufficiently close to $\bar{\varphi}$, will converge to a point in the $\text{Diff}^0(M)$ -orbit of $\bar{\varphi}$. We deduce, from fundamental work of Joyce [18], that the Laplacian flow starting at any closed G_2 structure with sufficiently small torsion will exist for all time and converge to a torsion-free G_2 structure.

1. INTRODUCTION

In this article we focus on the Laplacian flow for closed G_2 structures, whose stationary points are torsion-free G_2 structures on a compact 7-manifold. This geometric flow thus gives a potential means for studying the challenging problem of existence of Ricci-flat metrics with exceptional holonomy G_2 .

A desirable property of a geometric flow is (weak) dynamical stability of its stationary points: if one starts the flow close to a stationary point then it will converge to a (possibly different) stationary point. Such a property gives one hope to obtain further long-time existence and convergence results. We prove that weak dynamical stability holds for torsion-free G_2 structures along the Laplacian flow, building on our work in [23], thus confirming an expectation of Bryant [2]

As a consequence, we deduce from Joyce's fundamental perturbative existence theory for torsion-free G_2 structures [18] that the Laplacian flow starting at a closed G_2 structure with torsion sufficiently close to 0, in a suitable sense, will exist for all time and converge to a torsion-free G_2 structure. This provides key evidence that the Laplacian flow may indeed provide a useful tool for finding holonomy G_2 metrics.

A G_2 structure on a 7-manifold M is defined by a positive 3-form $\varphi \in \Omega_+^3(M)$, where the positivity is a natural nondegeneracy condition. A positive 3-form exists if and only if M is orientable and spinnable. To each $\varphi \in \Omega_+^3(M)$, one can associate a unique Riemannian metric g_φ and an orientation on M . If ∇_φ is the Levi-Civita connection of g_φ , we can interpret $\nabla_\varphi\varphi$ as the torsion of the G_2 structure φ , so if $\nabla_\varphi\varphi = 0$ we say φ is torsion-free and that (M, φ) is a G_2 manifold. Equivalently, if $*_\varphi$ is the Hodge star

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operator determined by g_φ and the orientation, then φ is torsion-free if and only if $d\varphi = d*_\varphi\varphi = 0$.

The key property of torsion-free G_2 structures is that the holonomy group $\text{Hol}(g_\varphi) \subseteq G_2$, and thus G_2 manifolds (M, φ) are Ricci-flat. Moreover, one can characterise the compact G_2 manifolds with $\text{Hol}(g_\varphi) = G_2$ as those with finite fundamental group. Thus understanding torsion-free G_2 structures is crucial for constructing Riemannian manifolds with holonomy G_2 .

Bryant [1] used the theory of exterior differential systems to demonstrate the local existence of many metrics with holonomy G_2 , and then Bryant-Salamon [4] constructed complete non-compact manifolds with holonomy G_2 , which are the spinor bundle of S^3 and the bundles of anti-self-dual 2-forms on S^4 and $\mathbb{C}P^2$. In [18], Joyce constructed the first examples of compact 7-manifolds with holonomy G_2 and many further compact examples have now been constructed [7, 22].

Since Hamilton [14] introduced the Ricci flow in 1982, geometric flows have been an important tool in studying geometric structures on manifolds. In 1992, Bryant (see [2]) proposed the Laplacian flow for closed G_2 structures

$$\begin{cases} \frac{\partial}{\partial t}\varphi &= \Delta_\varphi\varphi, \\ d\varphi &= 0, \\ \varphi(0) &= \varphi_0, \end{cases} \quad (1.1)$$

where $\Delta_\varphi = dd_\varphi^* + d_\varphi^*d$ is the Hodge Laplacian with respect to g_φ and φ_0 is an initial closed G_2 structure. The stationary points of the flow are harmonic φ , which on a compact manifold are precisely the torsion-free G_2 structures, so the Laplacian flow provides a tool for studying the existence of torsion-free G_2 structures on a manifold admitting closed G_2 structures. We remark that there are other proposed flows which also have torsion-free G_2 structures as stationary points (e.g. [13, 21, 27]).

The goal is to understand the long-time behavior of the Laplacian flow on compact manifolds M ; specifically, to understand conditions under which the flow will converge to a torsion-free G_2 structure. A fundamental result of Joyce [18] states that a closed G_2 structure with sufficiently small torsion (in a suitable sense) can be perturbed to a torsion-free G_2 structure in its cohomology class. A reasonable first conjecture is therefore that a dynamic version of Joyce's result holds: namely, that if the initial G_2 structure φ_0 on M is closed and has small torsion, then the Laplacian flow (1.1) exists for all time and converges to a torsion-free G_2 structure. We show that such a conjecture is true in Corollary 1.4 below.

As an essential first step in studying (1.1), a short-time existence result was claimed in [2] and proved in detail in [5].

Theorem 1.1. *For a compact 7-manifold M with closed G_2 structure φ_0 , the initial value problem (1.1) has a unique solution for a short time $t \in [0, \epsilon)$.*

Remark 1.2. The existence time ϵ depends on the initial value φ_0 .

To prove Theorem 1.1, Bryant–Xu [5] modified (1.1) by a term $\mathcal{L}_{V(\varphi)}\varphi$ for some vector field $V(\varphi)$ and considered a “Laplacian–DeTurck flow”:

$$\begin{cases} \frac{\partial}{\partial t}\varphi &= \Delta_\varphi\varphi + \mathcal{L}_{V(\varphi)}\varphi, \\ d\varphi &= 0, \\ \varphi(0) &= \varphi_0, \end{cases} \quad (1.2)$$

where $\mathcal{L}_{V(\varphi)}\varphi$ is the Lie derivative of φ in the direction $V(\varphi)$. Linearizing (1.2), one sees that it is parabolic in the direction of closed forms: this is not a typical type of parabolicity, and so standard parabolic theory does not obviously apply. The conclusion therefore follows by applying DeTurck’s trick [8] and the Nash–Moser inverse function theorem (c.f. [14, 15]). This result is particularly surprising since (1.1) uses the Hodge Laplacian and so appears to have the wrong sign to be parabolic, but it turns out that by gauge-fixing as in (1.2) one does indeed obtain a parabolic flow (at least in the required directions given by closed forms).

Hitchin [17] (see also [5]) showed that the Laplacian flow is related to a volume functional, which we now briefly describe. Suppose that $\bar{\varphi} \in \Omega_+^3(M)$ is a closed G_2 structure on M . Let

$$[\bar{\varphi}]_+ = \{\bar{\varphi} + d\beta \in \Omega_+^3(M) : \beta \in \Omega^2(M)\}$$

be the open subset of the cohomology class $[\bar{\varphi}]$ that consists of positive 3-forms. The volume functional $\mathcal{H} : [\bar{\varphi}]_+ \rightarrow \mathbb{R}^+$ is defined by

$$\mathcal{H}(\varphi) = \frac{1}{7} \int_M \varphi \wedge *_\varphi\varphi = \int_M *_\varphi 1.$$

Then $\varphi \in [\bar{\varphi}]_+$ is a critical point of \mathcal{H} if and only if $d*_\varphi\varphi = 0$, i.e. φ is torsion-free. Moreover, the Laplacian flow can be viewed as the gradient flow for \mathcal{H} , with respect to a non-standard L^2 -type metric on $[\bar{\varphi}]_+$ (see e.g. [5]).

If $\bar{\varphi}$ is torsion-free then by calculating the second variation of the functional \mathcal{H} at $\bar{\varphi}$, one sees that the orbit $\text{Diff}^0(M) \cdot \bar{\varphi}$ of $\bar{\varphi}$ under the diffeomorphisms of M that are isotopic to identity is a local maximum of \mathcal{H} on the moduli space $[\bar{\varphi}]_+ / \text{Diff}^0(M)$ (see [2, 17]). Consequently, Bryant [2] expected that for $\varphi_0 \in [\bar{\varphi}]_+$ sufficiently close to $\bar{\varphi}$, the Laplacian flow (1.1) starting at φ_0 will converge to a point in the orbit $\text{Diff}^0(M) \cdot \bar{\varphi}$. Our main result confirms this expectation.

Theorem 1.3. *Let $\bar{\varphi}$ be a torsion-free G_2 structure on a compact 7-manifold M . There is a neighborhood \mathcal{U} of $\bar{\varphi}$ such that for any $\varphi_0 \in [\bar{\varphi}]_+ \cap \mathcal{U}$, the Laplacian flow (1.1) with initial value φ_0 exists for all $t \in [0, \infty)$ and converges smoothly to $\varphi_\infty \in \text{Diff}^0(M) \cdot \bar{\varphi}$ as $t \rightarrow \infty$.*

In other words, torsion-free G_2 structures are (weakly) dynamically stable along the Laplacian flow for closed G_2 structures.

The proof of Theorem 1.3 is inspired by the proof of an analogous result in Ricci flow: Ricci-flat metrics are dynamically stable along the Ricci flow. The idea is to combine arguments for the Ricci flow case [16, 25, 26] with the particulars of the geometry of closed G_2 structures and new higher order estimates for the Laplacian flow derived by the authors [23]. For a precise definition of the neighbourhood \mathcal{U} , which is defined by norms involving derivatives at least up to order 9, see §5.

If we assume Joyce's existence result for torsion-free G_2 structures [18], then Theorem 1.3 has the following immediate corollary.

Corollary 1.4. *Let φ_0 be a closed G_2 structure on a compact 7-manifold M . There exists a C^9 -open neighbourhood \mathcal{U} of 0 in $\Omega^3(M)$ such that if $d_{\varphi_0}^* \varphi_0 = d_{\varphi_0}^* \gamma$ for some $\gamma \in \mathcal{U}$, then the Laplacian flow (1.1) with initial value φ_0 exists for all time and converges to a torsion-free G_2 structure.*

The theory in [18] states that if we control the C^0 and L^2 -norms of γ and the L^{14} -norm of $d_{\varphi_0}^* \gamma = d_{\varphi_0}^* \varphi_0$, we can deform φ_0 in its cohomology class to a unique C^0 -close torsion-free G_2 structure $\bar{\varphi}$. By choosing a neighbourhood \mathcal{U} appropriately, controlling derivatives up to at least order 9, we can ensure that we can apply both the theory of [18] and Theorem 1.3, and thus deduce the corollary. The neighbourhood \mathcal{U} given by Corollary 1.4 is not optimal, and one would like to be able to prove this result directly using the Laplacian flow with optimal conditions and without recourse to [18], but nevertheless, Corollary 1.4 gives significant evidence that the Laplacian flow will play an important role in understanding the problem of existence of torsion-free G_2 structures on 7-manifolds admitting closed G_2 structures.

Our results also motivate us to study an approach to the following problem¹. The work of Joyce [18] shows that the natural map from the moduli space \mathcal{M} of torsion-free G_2 structures (that is, torsion-free G_2 structures up to the action of $\text{Diff}^0(M)$) to $H^3(M)$ given by $\text{Diff}^0(M) \cdot \bar{\varphi} \mapsto [\bar{\varphi}]$ is locally injective, but the question of whether this map is globally injective, raised by Joyce (c.f. [19]), is still open. Equivalently, it can be phrased as asking whether the Lagrangian immersion of \mathcal{M} into $T^*H^3(M) \cong H^3(M) \times H^4(M)$ given by

$$\text{Diff}^0(M) \cdot \bar{\varphi} \mapsto ([\bar{\varphi}], [*_{\bar{\varphi}} \bar{\varphi}])$$

is an embedding. Suppose we have two torsion-free G_2 structures $\bar{\varphi}_0$ and $\bar{\varphi}_1$ which lie in the same cohomology class, so we can write $\bar{\varphi}_1 = \bar{\varphi}_0 + d\eta$ for some 2-form η . We would like to see whether $\bar{\varphi}_1 \in \text{Diff}^0(M) \cdot \bar{\varphi}_0$. By our main theorem (Theorem 1.3) we know that the Laplacian flow starting at $\varphi_0(s) = \bar{\varphi}_0 + sd\eta$ (which is closed) will exist for all time and converge to $\phi_s^* \bar{\varphi}_0$ for some $\phi_s \in \text{Diff}^0(M)$ when s is sufficiently small. Similarly, the Laplacian flow starting at $\varphi_0(s)$ for s near 1 will also exist for all time and now converge to $\Phi_s^* \bar{\varphi}_1$ for some $\Phi_s \in \text{Diff}^0(M)$. The aim would be to study long-time existence and convergence of the flow starting at any $\varphi_0(s)$ and, if this occurs, use Joyce's local injectivity result to show the G_2 moduli space \mathcal{M} embeds into $T^*H^3(M)$, or otherwise find obstructions.

Generally, one cannot expect the Laplacian flow (1.1) will converge to a torsion-free G_2 structure, even if it has long-time existence. There are compact 7-manifolds with closed G_2 structures that cannot admit holonomy G_2 metrics for topological reasons (c.f. [9, 10]), and Bryant [2] showed that the Laplacian flow starting with a particular one of these examples will exist for all time but it does not converge; for instance, the volume of the associated metrics will increase without bound. However, we note that recently Fernández-Fino-Manero [12] constructed some non-compact solutions for the Laplacian flow which exist for all time and do converge, but to a flat G_2

¹The authors are grateful to Thomas Walpuski for pointing this out.

structure, and Bryant [3] has informed the first author of non-trivial examples of non-compact steady solitons for the Laplacian flow (i.e. solutions which exist for all time and only vary within their diffeomorphism orbit).

We now briefly summarise the contents of this article. In §2 we recall some basic properties of closed G_2 structures and the Laplacian flow, mainly from [2] but also with some new observations from [23]. For the remainder of the article we assume that $(M, \bar{\varphi})$ is a compact G_2 manifold. In §3, we study the Laplacian–DeTurck flow (1.2) for $\varphi_0 \in [\bar{\varphi}]_+$ and show that if φ_0 is sufficiently close to $\bar{\varphi}$ then the flow will exist at least up to time 1 and remain close to $\bar{\varphi}$. In §4 we show that the Laplacian–DeTurck flow exponentially decays in L^2 to $\bar{\varphi}$ and that we can use Sobolev estimates to show that as long as the flow exists, it is uniformly controlled by the initial error $\varphi_0 - \bar{\varphi}$. In §5, we first prove dynamical stability of torsion-free G_2 structures under the Laplacian–DeTurck flow (1.2) for closed G_2 structures, and then use this to prove our main result, Theorem 1.3.

2. PRELIMINARIES

In this section, we collect some facts and results on closed G_2 structures and the Laplacian flow. For more detail, we refer the reader to [2, 5, 19, 23, 24].

Let $\{e_1, e_2, \dots, e_7\}$ denote the standard basis of \mathbb{R}^7 and let $\{e^1, e^2, \dots, e^7\}$ be its dual basis. Write $e^{ijk} = e^i \wedge e^j \wedge e^k$ for simplicity and define a 3-form ϕ by

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

The subgroup of $GL(7, \mathbb{R})$ fixing ϕ is the exceptional Lie group G_2 , which is a compact, connected, simple Lie subgroup of $SO(7)$ of dimension 14. Note that G_2 acts irreducibly on \mathbb{R}^7 and preserves the metric and orientation for which $\{e_1, e_2, \dots, e_7\}$ is an oriented orthonormal basis. If $*_\phi$ denotes the Hodge star on \mathbb{R}^7 , then G_2 also preserves the 4-form

$$*_\phi \phi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

Let M be a 7-manifold. For $x \in M$ we let

$$\Lambda_+^3(M)_x = \{\varphi_x \in \Lambda^3 T_x^* M \mid \exists u \in \text{Hom}(T_x M, \mathbb{R}^7), u^* \phi = \varphi_x\},$$

and thus obtain a bundle $\Lambda_+^3(M) = \sqcup_x \Lambda_+^3(M)_x$, which is an open subbundle of $\Lambda^3 T^* M$, with fibre $GL(7, \mathbb{R})/G_2$. We call a section φ of $\Lambda_+^3(M)$ a positive 3-form on M and we denote the space of sections of $\Lambda_+^3(M)$ by $\Omega_+^3(M)$. There is a 1-1 correspondence between G_2 structures and positive 3-forms, because given $\varphi \in \Omega_+^3(M)$, the subbundle of the frame bundle whose fibre at x consists of $u \in \text{Hom}(T_x M, \mathbb{R}^7)$ such that $u^* \phi = \varphi_x$ defines a principal subbundle with fibre G_2 , i.e. a G_2 structure on M . Thus we usually call $\varphi \in \Omega_+^3(M)$ a G_2 structure on M . Note that the existence of G_2 structures is a purely topological fact, namely that M is oriented and spin.

The G_2 structure induces a splitting of the bundle of k -forms ($2 \leq k \leq 5$) into direct summands, which we denote by $\Lambda_l^k(T^* M, \varphi)$ so that l indicates the rank of the bundle. If we let $\Omega_l^k(M)$ be the space of sections

of $\Lambda_l^k(T^*M, \varphi)$ then we have that

$$\begin{aligned}\Omega^2(M) &= \Omega_7^2(M) \oplus \Omega_{14}^2(M), \\ \Omega^3(M) &= \Omega_1^3(M) \oplus \Omega_7^3(M) \oplus \Omega_{27}^3(M),\end{aligned}$$

where, if we let $\psi = *_\varphi\varphi$,

$$\begin{aligned}\Omega_7^2(M) &= \{\beta \in \Omega^2(M) \mid \beta \wedge \varphi = 2 *_\varphi \beta\} = \{X \lrcorner \varphi \mid X \in \Gamma(TM)\}, \\ \Omega_{14}^2(M) &= \{\beta \in \Omega^2(M) \mid \beta \wedge \varphi = - *_\varphi \beta\} = \{\beta \in \Omega^2(M) \mid \beta \wedge \psi = 0\}\end{aligned}$$

and

$$\begin{aligned}\Omega_1^3(M) &= \{f\varphi \mid f \in C^\infty(M)\}, \\ \Omega_7^3(M) &= \{X \lrcorner \psi \mid X \in \Gamma(TM)\}, \\ \Omega_{27}^3(M) &= \{\gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0 = \gamma \wedge \psi\}.\end{aligned}$$

By Hodge duality, we have a similar decomposition of $\Omega^4(M)$ and $\Omega^5(M)$.

To study the Laplacian flow, it is convenient to write the relevant quantities for study in local coordinates. We write a k -form α locally using summation convention and index notation as follows:

$$\alpha = \frac{1}{k!} \alpha_{i_1 i_2 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where $\{x^1, \dots, x^7\}$ give local coordinates on M .

We define the operator $i_\varphi : S^2 T^*M \rightarrow \Lambda^3 T^*M$ as in [2] (up to a factor of $\frac{1}{2}$) by

$$i_\varphi(h) = \frac{1}{2} h_i^l \varphi_{ljk} dx^i \wedge dx^j \wedge dx^k,$$

for $h = h_{ij} dx^i dx^j$ locally. Then $\Lambda_{27}^3(T^*M, \varphi) = i_\varphi(S_0^2 T^*M)$, where $S_0^2 T^*M$ denotes the trace-free symmetric 2-tensors on M , and $i_\varphi(g) = 3\varphi$. We also have the map j_φ on 3-forms:

$$j_\varphi(\gamma)(u, v) = *_\varphi((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \gamma) \quad (2.1)$$

which is an isomorphism between $\Omega_1^3(M) \oplus \Omega_{27}^3(M)$ and the symmetric 2-tensors on M that provides an inverse of i_φ . Then we have

$$j_\varphi(i_\varphi(h)) = 4h + 2tr_g(h)g.$$

for any $h \in S^2(M)$, and $j_\varphi(\varphi) = 6g$.

Given any G_2 structure $\varphi \in \Omega_+^3(M)$, there exist unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega_{14}^2(M)$ and $\tau_3 \in \Omega_{27}^3(M)$ such that $d\varphi$ and $d\psi$ can be expressed as follows:

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3, \quad (2.2)$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \quad (2.3)$$

We call the four forms $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ the intrinsic torsion forms of a G_2 structure. The full torsion tensor is a 2-tensor T_{ij} satisfying

$$\nabla_i \varphi_{jkl} = T_i^m \psi_{mjkl}, \quad (2.4)$$

$$T_i^j = \frac{1}{24} \nabla_i \varphi_{lmn} \psi^{jlmn}, \quad (2.5)$$

and

$$\nabla_m \psi_{ijkl} = -(T_{mi}\varphi_{jkl} - T_{mj}\varphi_{ikl} - T_{mk}\varphi_{jil} - T_{ml}\varphi_{jki}). \quad (2.6)$$

Then the full torsion tensor T_{ij} is related with the torsion forms by the following (see [20]):

$$T_{ij} = \frac{\tau_0}{4}g_{ij} - (\tau_1^\# \lrcorner \varphi)_{ij} - (\bar{\tau}_3)_{ij} - \frac{1}{2}(\tau_2)_{ij},$$

where $(\tau_1^\# \lrcorner \varphi)_{ij} = (\tau_1)^l \varphi_{lij}$ and $(\bar{\tau}_3)_{ij}$ is the traceless symmetric 2-tensor such that $\tau_3 = i_\varphi(\bar{\tau}_3)$.

If φ is closed, then (2.2) implies that τ_0, τ_1 and τ_3 are all zero, so the only non-zero torsion form is $\tau_2 = \frac{1}{2}(\tau_2)_{ij} dx^i \wedge dx^j$. The full torsion tensor satisfies $T_{ij} = -T_{ji} = -\frac{1}{2}(\tau_2)_{ij}$, which is a skew-symmetric 2-tensor. In the majority of this article, we will only consider closed G_2 structures and will thus write $\tau = \tau_2$ for simplicity.

To study the Laplacian flow we have calculated $\Delta_\varphi \varphi$ for closed φ in [23]:

$$\Delta_\varphi \varphi = d\tau = i_\varphi(h) \quad (2.7)$$

for a symmetric 2-tensor h satisfying

$$h_{ij} = -(\nabla_m T_{ni})\varphi_j^{mn} - \frac{1}{3}|T|^2 g_{ij} - T_i^l T_{lj}. \quad (2.8)$$

Thus the Laplacian flow (1.1) can be rewritten as

$$\frac{\partial}{\partial t} \varphi = i_\varphi(h). \quad (2.9)$$

Under the flow (2.9), the associated metric $g(t)$ of $\varphi(t)$ evolves by (see [20])

$$\frac{\partial}{\partial t} g(t) = 2h(t).$$

Then substituting (2.8), we have

$$\frac{\partial}{\partial t} g_{ij} = -2(\nabla_k T_{li})\varphi_j^{kl} - \frac{2}{3}|T|^2 g_{ij} - 2T_i^k T_{kj}. \quad (2.10)$$

Since $d\varphi = 0$, the Ricci tensor of the associated metric g is equal to (see [20])

$$Ric_{ij} = (\nabla_k T_{li})\varphi_j^{kl} - T_i^k T_{kj}, \quad (2.11)$$

and so we can recover the following formula from [2]:

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij} - \frac{2}{3}|T|^2 g_{ij} - 4T_i^k T_{kj}. \quad (2.12)$$

Thus the leading term of the metric flow corresponds to the Ricci flow. From (2.10), the volume form vol_g evolves by

$$\frac{\partial}{\partial t} vol_g = \frac{1}{2} tr_g \left(\frac{\partial}{\partial t} g(t) \right) vol_g = \frac{2}{3} |T|^2 vol_g. \quad (2.13)$$

Therefore along the Laplacian flow for closed G_2 structures, the volume of M with respect to g will increase (in fact, the volume form increases pointwise).

Notice, as in [2], that taking the trace of (2.11) gives

$$R = -|T|^2, \quad (2.14)$$

where R is the scalar curvature. So, 7-manifolds with closed G_2 structures necessarily have metrics with non-positive scalar curvature, but this is a weak curvature condition amenable to the h-principle.

In [23], we also calculated the evolution of the full torsion tensor T ,

$$\frac{\partial}{\partial t}T = \Delta T + Rm * T + Rm * T * \psi + \nabla T * T * \varphi + T * T * T, \quad (2.15)$$

where we will continually use $*$ to mean some contraction using the metric $g(t)$ associated with $\varphi(t)$, and the evolution equation of the Riemann curvature tensor as

$$\frac{\partial}{\partial t}Rm = \Delta Rm + Rm * Rm + Rm * T * T + \nabla^2 T * T + \nabla T * \nabla T. \quad (2.16)$$

Here Δ denotes the usual (analyst's) Laplacian, which is a negative operator. Notice that Δ_φ on the other hand is the Hodge Laplacian and so is a positive operator.

We end this section with the following useful estimate on the existence time of the Laplacian flow (1.1) from [23].

Proposition 2.1. *If φ_0 is a closed G_2 structure on a compact 7-manifold M and $K \in \mathbb{R}$ so that*

$$\Lambda(x) = (|Rm(x)|^2 + |\nabla T(x)|^2)^{\frac{1}{2}} \leq K, \quad (2.17)$$

for all $x \in M$, then the unique solution $\varphi(t)$ of the Laplacian flow (1.1) will exist at least for time $t \in [0, c/K]$, where $c > 0$ is a uniform constant, independent of φ_0 .

Observe that a bound on Rm gives a bound on T for closed G_2 structures by (2.14).

3. ESTIMATES FOR $t \leq 1$

In this section, we show that if $\bar{\varphi}$ is a torsion-free G_2 structure and we are given a neighbourhood U of $\bar{\varphi}$, then we can find a smaller neighbourhood U_0 of $\bar{\varphi}$ so that if the initial value $\varphi_0 \in [\bar{\varphi}]_+$ lies in U_0 , then the Laplacian–DeTurck flow (1.2) will exist at least for $t \in [0, 1]$ and remain in U .

Lemma 3.1 (Estimate for $t \leq 1$). *Let $(M, \bar{\varphi})$ be a compact G_2 manifold. For any $k \geq 3$, $\epsilon > 0$, there exists a constant $\delta = \delta(M, \bar{\varphi}, k, \epsilon) > 0$ such that: if $\varphi_0 \in [\bar{\varphi}]_+$ and $\|\varphi_0 - \bar{\varphi}\|_{C_{\bar{g}}^{k+4}} < \delta$, then the solution $\tilde{\varphi}(t)$ to the Laplacian–DeTurck flow (1.2) exists on $[0, 1]$ and satisfies*

$$\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_{\bar{g}}^k} < \epsilon \quad \forall t \in [0, 1],$$

where the $C_{\bar{g}}^k$ -norms are defined via the metric \bar{g} determined by $\bar{\varphi}$.

Proof. Let $k \geq 3$ and $\epsilon > 0$. By making $\epsilon > 0$ smaller, we can assume that if φ is a G_2 structure satisfying $\|\varphi - \bar{\varphi}\|_{C_{\bar{g}}^k} \leq \epsilon$ and g is the associated metric of φ , then the C^k -norms via g and \bar{g} differ at most by a factor 2.

Recall Λ from (2.17). Define a finite constant $K = K(\epsilon, k) \geq 0$ by:

$$\begin{aligned} K = & \sup\{|\nabla^i Rm(x)|^2 + |\nabla^{i+1} T(x)|^2 : \|\varphi - \bar{\varphi}\|_{C_{\bar{g}}^{k+4}} \leq \epsilon, x \in M, i \leq k+2\} \\ & + \sup\{|\Lambda(x)| : \|\varphi - \bar{\varphi}\|_{C_{\bar{g}}^k} \leq \epsilon, x \in M\}, \end{aligned} \quad (3.1)$$

where ∇ , Rm and T denote the Levi-Civita connection, Riemann curvature tensor and torsion tensor of φ .

Let φ_0 satisfy $\|\varphi_0 - \bar{\varphi}\|_{C_g^{k+4}} \leq \epsilon$ and let $\varphi(t)$ be the solution to the Laplacian flow (1.1) starting at φ_0 . Suppose that $\varphi(t)$ exists on some time interval $[0, \eta]$ (where $\eta > 0$ by Theorem 1.1). From (2.15)-(2.16) and calculations in [23, §4], we can compute the evolution of the pointwise norms $|\nabla^i Rm|$ and $|\nabla^{i+1} T|$ with respect to $g(t)$.

First we have

$$\begin{aligned}
& \frac{\partial}{\partial t} (|\nabla^i Rm|^2 + |\nabla^{i+1} T|^2) \\
& \leq \Delta (|\nabla^i Rm|^2 + |\nabla^{i+1} T|^2) - 2|\nabla^{i+1} Rm|^2 - 2|\nabla^{i+2} T|^2 \\
& + \sum_{j=0}^i \nabla^i Rm * \nabla^{i-j} Rm * \nabla^j (Rm + T * T) \\
& + \sum_{j=0}^{i+1} \nabla^i Rm * \nabla^j T * \nabla^{i+2-j} T + \sum_{j=0}^{i+1} \nabla^{i+1} T * \nabla^{i+1-j} T * \nabla^j (Rm + T * T) \\
& + \sum_{j=0}^{i+1} \nabla^{i+1} T * (\nabla^{i+1-j} (Rm * T) * \nabla^j \psi + \nabla^{i+1-j} (\nabla T * T) * \nabla^j \varphi). \quad (3.2)
\end{aligned}$$

Note that the pointwise norms of $\nabla^j \varphi$ and $\nabla^j \psi$ can be estimated by a constant depending on the upper bounds of $|T|, |\nabla T|, \dots, |\nabla^{j-1} T|$ and that (3.2) for $i = 0$ gives an evolution inequality for Λ^2 .

By definition of K in (3.1) we have that if the Laplacian flow $\varphi(t)$ starting from φ_0 exists on the time interval $[0, \eta]$ and $\|\varphi(t) - \bar{\varphi}\|_{C_g^k} \leq \epsilon$ for $\forall t \in [0, \eta]$, then

$$\begin{cases} \sup_{x \in M} \Lambda(x, t) \leq K, & \forall t \in [0, \eta], \\ |\nabla^i Rm(x, 0)|^2 + |\nabla^{i+1} T(x, 0)|^2 \leq K, & \forall x \in M, i \leq k + 2. \end{cases} \quad (3.3)$$

Applying the maximum principle to (3.2) and using an induction argument, we claim that if $\eta \leq 1$, then there is a constant $\tilde{K} = \tilde{K}(K, k) < \infty$ such that

$$|\nabla^i Rm(x, t)|^2 + |\nabla^{i+1} T(x, t)|^2 \leq \tilde{K}, \forall x \in M, t \in [0, \eta], i \leq k + 2. \quad (3.4)$$

We now prove this claim.

For $i = 1$, using (2.14) we can derive from (3.2) that

$$\begin{aligned}
\frac{\partial}{\partial t} (|\nabla Rm|^2 + |\nabla^2 T|^2) & \leq \Delta (|\nabla Rm|^2 + |\nabla^2 T|^2) - 2|\nabla^2 Rm|^2 - 2|\nabla^3 T|^2 \\
& + C|\nabla^3 T| |Rm|^{\frac{1}{2}} (|\nabla Rm| + |\nabla^2 T|) \\
& + C|\nabla^2 T| |\nabla^2 Rm| |Rm|^{\frac{1}{2}} \\
& + C(|\nabla Rm|^2 + |\nabla^2 T|^2) (|\nabla T| + |Rm|) \\
& + C(|\nabla Rm| + |\nabla^2 T|) |Rm|^{\frac{1}{2}} (|Rm|^2 + |\nabla T|^2), \quad (3.5)
\end{aligned}$$

where C are uniform constants. Young's inequality implies that $2ab \leq \frac{1}{\epsilon}a^2 + \epsilon b^2$ for all $\epsilon > 0$ and $a, b \geq 0$, so for all $\epsilon > 0$ we have

$$\begin{aligned} 2|\nabla^3 T||Rm|^{\frac{1}{2}}(|\nabla Rm| + |\nabla^2 T|) &\leq \frac{2}{\epsilon}|Rm|(|\nabla Rm|^2 + |\nabla^2 T|^2) + \epsilon|\nabla^3 T|^2, \\ 2|\nabla^2 T||\nabla^2 Rm||Rm|^{\frac{1}{2}} &\leq \frac{1}{\epsilon}|Rm||\nabla^2 T|^2 + \epsilon|\nabla^2 Rm|^2. \end{aligned}$$

Substituting these estimates into (3.5) and choosing ϵ sufficiently small then yields the parabolic inequality

$$\begin{aligned} \frac{\partial}{\partial t}(|\nabla Rm|^2 + |\nabla^2 T|^2) &\leq \Delta(|\nabla Rm|^2 + |\nabla^2 T|^2) \\ &\quad + C(|\nabla Rm|^2 + |\nabla^2 T|^2)(|\nabla T| + |Rm|) \\ &\quad + C(|\nabla Rm| + |\nabla^2 T|)|Rm|^{\frac{1}{2}}(|Rm|^2 + |\nabla T|^2). \end{aligned} \quad (3.6)$$

If we denote

$$\Lambda_1(t) = \max_{x \in M}(|\nabla Rm(x, t)|^2 + |\nabla^2 T(x, t)|^2),$$

then applying the maximum principle to (3.6) and using the first line of the condition (3.3), we have

$$\frac{d}{dt}\Lambda_1(t) \leq C_1 K \Lambda_1(t) + C_2 K^2, \quad (3.7)$$

where C_1, C_2 are two uniform constants. Then integrating (3.7) yields that

$$\Lambda_1(t) + \frac{C_2}{C_1} K \leq e^{C_1 K t} (\Lambda_1(0) + \frac{C_2}{C_1} K).$$

Since $t \leq 1$ and $\Lambda_1(0) \leq K$ by the second line of (3.3), the above inequality implies that

$$\Lambda_1(t) \leq e^{C_1 K} K + (e^{C_1 K} - 1) \frac{C_2}{C_1} K := \tilde{K}. \quad (3.8)$$

This proves the $i = 1$ case of (3.4).

For $1 < i \leq k+2$, the conclusion (3.4) follows from an induction argument starting from (3.8). More concretely, suppose that there is some $1 < l \leq k+2$ so that (3.4) holds for all $1 \leq i < l$. Consider (3.2) for $i = l$. By the inductive hypothesis, any term on the right-hand side of (3.2) which can be estimated by $|\nabla^i Rm|$ and $|\nabla^{i+1} T|$ for $i < l$ can be bounded by a constant depending only on \tilde{K} . With this observation, (3.2) for $i = l$ then yields

$$\begin{aligned} \frac{\partial}{\partial t}(|\nabla^l Rm|^2 + |\nabla^{l+1} T|^2) &\leq \Delta(|\nabla^l Rm|^2 + |\nabla^{l+1} T|^2) - 2|\nabla^{l+1} Rm|^2 - 2|\nabla^{l+2} T|^2 \\ &\quad + C|\nabla^{l+2} T|(|\nabla^l Rm| + |\nabla^{l+1} T|) + C|\nabla^{l+1} T||\nabla^{l+1} Rm| \\ &\quad + C(|\nabla^l Rm|^2 + |\nabla^{l+1} T|^2) + C(|\nabla^l Rm| + |\nabla^{l+1} T|), \end{aligned}$$

where the constants C depend on \tilde{K} . Using Young's inequality as before we obtain the parabolic inequality

$$\begin{aligned} \frac{\partial}{\partial t}(|\nabla^l Rm|^2 + |\nabla^{l+1} T|^2) &\leq \Delta(|\nabla^l Rm|^2 + |\nabla^{l+1} T|^2) \\ &\quad + C(|\nabla^l Rm|^2 + |\nabla^{l+1} T|^2) + C(|\nabla^l Rm| + |\nabla^{l+1} T|), \end{aligned}$$

to which we may apply the maximum principle to conclude that (3.4) holds for $i = l$ as required.

Next, using calculations in [23, §4] again, we have

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^i T|^2 &\leq \Delta |\nabla^i T|^2 - 2 |\nabla^{i+1} T|^2 + \sum_{j=0}^i \nabla^i T * \nabla^{i-j} T * \nabla^j (Rm + T * T) \\ &+ \sum_{j=0}^i \nabla^i T * \nabla^{i-j} (Rm * T) * \nabla^j \psi + \sum_{j=0}^i \nabla^i T * \nabla^{i-j} (\nabla T * T) * \nabla^j \varphi. \end{aligned} \quad (3.9)$$

We now choose $\tilde{\epsilon} > 0$, which we will impose smallness conditions on later. Applying the maximum principle to (3.9) and using (3.4), there is a constant $\tilde{\delta} = \tilde{\delta}(K, \tilde{\epsilon}, k) > 0$ such that

$$\begin{aligned} |\nabla^i T(x, 0)| &\leq \tilde{\delta} \quad \forall x \in M, i \leq k+2 \\ &\Rightarrow |\nabla^i T(x, t)| \leq \tilde{\epsilon} \quad \forall x \in M, t \in [0, \eta], i \leq k+2. \end{aligned} \quad (3.10)$$

Note that in (3.10) we have control of one less derivative of T than in (3.4), because on the right-hand side of (3.9) there are terms involving $\nabla^i Rm$ and $\nabla^{i+1} T$ which need to be estimated.

Since $\|\varphi_0 - \tilde{\varphi}\|_{C_g^{k+4}} \leq \epsilon$, by (3.1) we have $\Lambda(x, 0) \leq K < \infty$, and hence we see from Proposition 2.1 that the maximal existence time T_0 of $\varphi(t)$ has a uniform lower bound $T_0 \geq c/K > 0$ depending only on K . We know that the solution $\tilde{\varphi}(t)$ to the Laplacian–DeTurck flow (1.2), with the same initial value φ_0 , is given by

$$\tilde{\varphi}(t) = \phi(t)^* \varphi(t)$$

for a family of diffeomorphisms $\phi(t)$ on M generated by vector fields $V(\varphi(t))$ determined by $\varphi(t)$. Hence $\tilde{\varphi}(t)$ will also exist at least on $[0, T_0)$.

We also know that $V(\varphi(t))$ depends on $\varphi(t)$ and its first derivative, so a bound on $|\nabla^i \varphi(t)|_{g(t)}$ for $i \leq k+2$ will yield a bound on $|\tilde{\nabla}^i \tilde{\varphi}|_{\tilde{g}(t)}$ for $i \leq k$, where $\tilde{g}(t)$ and $\tilde{\nabla}$ are the metric and Levi-Civita connection determined by $\tilde{\varphi}(t)$. Therefore, since $\varphi(t)$ is bounded in C_g^{k+4} we have that $\tilde{\varphi}(t)$ is bounded in $C_{\tilde{g}}^{k+2}$ and thus

$$|\tilde{\nabla}^i \widetilde{Rm}|_{\tilde{g}}^2 + |\tilde{\nabla}^{i+1} \tilde{T}|_{\tilde{g}}^2$$

is finite for $i \leq k$, where \widetilde{Rm} and \tilde{T} are the Riemann curvature and torsion tensors of $\tilde{\varphi}(t)$ respectively. By diffeomorphism invariance, (3.4) and (3.10) show that

$$\begin{aligned} |\tilde{\nabla}^i \widetilde{Rm}|_{\tilde{g}}^2 + |\tilde{\nabla}^{i+1} \tilde{T}|_{\tilde{g}}^2 &= |\nabla^i Rm|_g^2 + |\nabla^{i+1} T|_g^2 \\ &\leq \tilde{K}, \quad \forall x \in M, t \in [0, \eta], i \leq k, \end{aligned} \quad (3.11)$$

$$\begin{aligned} |\tilde{\nabla}^i \tilde{T}|_{\tilde{g}} &= |\nabla^i T|_g \\ &\leq \tilde{\epsilon}, \quad \forall x \in M, t \in [0, \eta], i \leq k+1. \end{aligned} \quad (3.12)$$

Now let $\delta \in (0, \epsilon)$ be sufficiently small so that for all $x \in M$, $i \leq k+2$ we have

$$\|\varphi - \tilde{\varphi}\|_{C_g^{k+4}} \leq \delta \Rightarrow \|\varphi - \tilde{\varphi}\|_{C_g^k} \leq \epsilon/4 \text{ and } |\nabla^i T(x)| \leq \tilde{\delta}.$$

Assume that $\|\varphi_0 - \bar{\varphi}\|_{C_g^{k+4}} \leq \delta < \epsilon$, let $\eta \leq T_0$ be the maximal existence time of the Laplacian–DeTurck flow (1.2) with the initial value φ_0 such that

$$\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_g^k} < \epsilon \quad \forall t \in [0, \eta].$$

If $\eta < 1$, then

$$\begin{aligned} \|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_g^k} &\leq \|\varphi_0 - \bar{\varphi}\|_{C_g^k} + \int_0^t \frac{d}{ds} \|\tilde{\varphi}(s) - \bar{\varphi}\|_{C_g^k} ds \\ &\leq \frac{\epsilon}{4} + \int_0^\eta \|\Delta_{\tilde{\varphi}(s)} \tilde{\varphi}(s) + \mathcal{L}_{V(\tilde{\varphi}(s))} \tilde{\varphi}(s)\|_{C_g^k} ds \\ &\leq \frac{\epsilon}{4} + 2 \int_0^\eta \|\Delta_{\tilde{\varphi}(s)} \tilde{\varphi}(s) + \mathcal{L}_{V(\tilde{\varphi}(s))} \tilde{\varphi}(s)\|_{C_{\tilde{g}(s)}^k} ds \\ &\leq \frac{\epsilon}{4} + C \int_0^\eta (\|\tilde{\nabla} \tilde{T}(x, s)\|_{C_{\tilde{g}(s)}^k} + \|\tilde{T}(x, s)\|_{C_{\tilde{g}(s)}^k}^2) ds \\ &\leq \frac{\epsilon}{4} + 2C\tilde{\epsilon} \end{aligned} \tag{3.13}$$

for all $t \in [0, \eta]$, where in the final step we used $\eta < 1$. Here in the fourth inequality of (3.13) we used the facts $\Delta_{\tilde{\varphi}} \tilde{\varphi} = d\tilde{\tau}$ and the vector field $V(\tilde{\varphi})$ involves the first order covariant derivative of $\tilde{\varphi}$, and in the last inequality we used (3.12), noting that we need a small bound on \tilde{T} in $C_{\tilde{g}}^{k+1}$. Since C is a uniform constant, by choosing $\tilde{\epsilon}$ sufficiently small we can ensure that $2C\tilde{\epsilon} < \frac{\epsilon}{4}$, and hence $\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_g^k} \leq \frac{\epsilon}{2}$, contradicting the maximality of η .

Therefore we have $\eta \geq 1$ as desired. \square

4. ESTIMATES FOR $t \geq 1$

In this section, we now turn to studying the long-time estimates for solutions $\tilde{\varphi}(t)$ to the Laplacian–DeTurck flow (1.2); i.e. for $t \geq 1$. We will show that the flow exponentially decays to the torsion-free G_2 structure in L^2 . We then obtain further integral estimates which allow us to deduce, through Sobolev embedding, that the C_g^k -norm of $\tilde{\varphi}(t) - \bar{\varphi}$ is uniformly controlled by the C_g^{k+4} -norm of $\varphi_0 - \bar{\varphi}$.

We suppose we are in the situation of Lemma 3.1 and use the notation introduced there: in particular, we fix $k \geq 3$ and $\epsilon > 0$ and obtain $\delta > 0$ so that the flow exists at least up to time 1 in $[\bar{\varphi}]_+$ where $\bar{\varphi}$ is torsion-free.

Since $\tilde{\varphi}(t)$ lies in the cohomology class $[\bar{\varphi}]$, we can write $\tilde{\varphi}(t) = \bar{\varphi} + \theta(t)$ with $\theta(t)$ taking values in exact forms. We are assuming that $\theta_0 = \theta(0)$ satisfies $\|\theta_0\|_{C_g^{k+4}} < \delta$. Let η be the maximum time such that the solution $\tilde{\varphi}(t)$ of (1.2) starting at φ_0 exists and $\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_g^k} < \epsilon$. Clearly, from Lemma 3.1 we know that $\eta \geq 1$.

Since we shall focus on the Laplacian–DeTurck flow (1.2) in this section, we now recall the definition of the vector field $V(\varphi)$ which appears in the flow. Let $S = \nabla - \bar{\nabla}$, which is a tensor taking values in $TM \otimes S^2 T^*M$. Let A be the constant in [5, Lemma 1.7]. (Although A is not calculated in [5], it arises purely from G_2 representation theory and so by working on \mathbb{R}^7 one can find that $A = -\frac{1}{8}$ in the conventions of [5].) As in [5], we let $V(\varphi)$ be

$$V(\varphi) = -5V_1(\varphi) - V_2(\varphi), \tag{4.1}$$

with V_1, V_2 given locally by

$$V_1(\varphi) = \frac{1}{7}g^{pq}S_{pq}^i e_i, \quad V_2(\varphi) = 2A(g^{kj}S_{ik}^i e_j + 5V_1(\varphi)).$$

In particular, observe that $V(\bar{\varphi}) = 0$. We let $\mathcal{A}_{\bar{\varphi}}(\varphi) = \Delta_{\varphi}\varphi + \mathcal{L}_{V(\varphi)}\varphi$. It was proved in [5] that the linearization of $\mathcal{A}_{\bar{\varphi}}(\varphi)$ at φ applied to a closed variation $\theta = \partial_s|_{s=0}\varphi_s$ (so $d\theta = 0$) is

$$D_{\varphi}\mathcal{A}_{\bar{\varphi}}(\varphi)\theta = -\Delta_{\varphi}\theta + d\Phi(\theta), \quad (4.2)$$

where $\Phi(\theta)$ is an algebraic linear operator on θ with coefficients depending on the torsion of φ . Notice in (4.2) the surprising fact that the sign of the Laplacian has changed in the linearisation, which indicates the parabolicity of the flow discussed in the introduction. We should also note that (4.2) was derived only assuming $\bar{\varphi}$ was a background G_2 structure and not necessarily a torsion-free one.

We now show that we can approximate the evolution of $\theta(t) = \tilde{\varphi}(t) - \bar{\varphi}$.

Lemma 4.1. *There exists $\bar{\epsilon} = \bar{\epsilon}(\bar{\varphi}) > 0$ such that if $\epsilon \in (0, \bar{\epsilon})$, then for all time t for which the solution $\tilde{\varphi}(t)$ of (1.2) exists and $\|\theta(t)\|_{C_g^k} < \epsilon$, we have*

$$\frac{\partial}{\partial t}\theta(t) = -\Delta_{\bar{\varphi}}\theta + dF(\bar{\varphi}, \tilde{\varphi}(t), \theta(t), \bar{\nabla}\theta(t)), \quad (4.3)$$

where F is a 2-form which is smooth in the first two arguments and linear in the last two arguments.

Proof. Recall that $\tilde{\varphi}(t)$ is a solution to the Laplacian–DeTurck flow (1.2) and $\bar{\varphi}$ is a stationary solution of (1.2), so $\mathcal{A}_{\bar{\varphi}}(\bar{\varphi}) = 0$. For all t for which $\tilde{\varphi}(t)$ exists and $\|\theta(t)\|_{C_g^k} = \|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_g^k} < \epsilon$, we have

$$\begin{aligned} \frac{\partial}{\partial t}\theta(t) &= \frac{\partial}{\partial t}(\tilde{\varphi}(t) - \bar{\varphi}) = \mathcal{A}_{\bar{\varphi}}(\tilde{\varphi}(t)) - \mathcal{A}_{\bar{\varphi}}(\bar{\varphi}) \\ &= D_{\bar{\varphi}}\mathcal{A}_{\bar{\varphi}}(\tilde{\varphi})\theta + \tilde{F} = -\Delta_{\bar{\varphi}}\theta + d\Phi(\theta) + \tilde{F} \end{aligned} \quad (4.4)$$

where \tilde{F} is the remaining higher order term. We choose ϵ small enough such that \tilde{F} is small compared to the linear part. By studying the details of the proof of (4.2) in [5], we can conclude that if we linearize $\mathcal{A}_{\bar{\varphi}}(\tilde{\varphi})$ at the torsion-free G_2 structure $\bar{\varphi}$, then $d\Phi(\theta) = 0$, so

$$D_{\bar{\varphi}}\mathcal{A}_{\bar{\varphi}}(\tilde{\varphi})\theta = -\Delta_{\bar{\varphi}}\theta. \quad (4.5)$$

We next estimate the remaining term \tilde{F} . Since $\theta(t)$ is small in the C_g^k -norm, we can write

$$\theta = 3f^0\bar{\varphi} + *_{\bar{\varphi}}(f^1 \wedge \bar{\varphi}) + f^3 \quad (4.6)$$

for $f^0 \in \Omega^0(M)$, $f^1 \in \Omega^1(M)$ and $f^3 \in \Omega_{27}^3(M)$ (with respect to $\bar{\varphi}$) with small C_g^k -norm. Then by [19] we have

$$*_{\bar{\varphi}}\tilde{\varphi} = *_{\bar{\varphi}}\bar{\varphi} + 4f^0 *_{\bar{\varphi}}\bar{\varphi} + f^1 \wedge \bar{\varphi} - *_{\bar{\varphi}}f^3 + Q(\bar{\varphi}, \tilde{\varphi}, \theta), \quad (4.7)$$

where Q is smooth in its first two arguments, and the principal part of Q is quadratic in the third argument. As $\tilde{\varphi}(t)$ is closed, we have

$$\Delta_{\bar{\varphi}}\tilde{\varphi} = *_{\bar{\varphi}}d *_{\bar{\varphi}}d\tilde{\varphi} - d *_{\bar{\varphi}}d *_{\bar{\varphi}}\tilde{\varphi} = -d *_{\bar{\varphi}}d *_{\bar{\varphi}}\tilde{\varphi}.$$

We linearize $\Delta_{\tilde{\varphi}}\tilde{\varphi}$ at $\tilde{\varphi}$,

$$\begin{aligned} D_{\tilde{\varphi}}\Delta_{\tilde{\varphi}}\tilde{\varphi}(\theta) &= -d(*_{\tilde{\varphi}})d*_{\tilde{\varphi}}\tilde{\varphi} - d*_{\tilde{\varphi}}d(4f^0*_{\tilde{\varphi}}\tilde{\varphi} + f^1\wedge\tilde{\varphi} - *_{\tilde{\varphi}}f^3) \\ &= -d*_{\tilde{\varphi}}d(4f^0*_{\tilde{\varphi}}\tilde{\varphi} + f^1\wedge\tilde{\varphi} - *_{\tilde{\varphi}}f^3), \end{aligned}$$

where we used the fact that $\tilde{\varphi}$ is coclosed (as it is torsion-free) to eliminate the first term involving the linearisation of $*_{\tilde{\varphi}}$. We then have

$$\begin{aligned} \Delta_{\tilde{\varphi}}\tilde{\varphi} - \Delta_{\tilde{\varphi}}\tilde{\varphi} - D_{\tilde{\varphi}}\Delta_{\tilde{\varphi}}\tilde{\varphi}(\theta) &= -d*_{\tilde{\varphi}}d*_{\tilde{\varphi}}\tilde{\varphi} + d*_{\tilde{\varphi}}d(4f^0*_{\tilde{\varphi}}\tilde{\varphi} + f^1\wedge\tilde{\varphi} - *_{\tilde{\varphi}}f^3) \\ &= -d(*_{\tilde{\varphi}} - *_{\tilde{\varphi}})d*_{\tilde{\varphi}}\tilde{\varphi} - d*_{\tilde{\varphi}}dQ(\tilde{\varphi}, \tilde{\varphi}, \theta) \\ &= dF_1(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta), \end{aligned} \tag{4.8}$$

where we denote

$$F_1(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta) = -(*_{\tilde{\varphi}} - *_{\tilde{\varphi}}) d *_{\tilde{\varphi}} \tilde{\varphi} - *_{\tilde{\varphi}} d Q(\tilde{\varphi}, \tilde{\varphi}, \theta).$$

Notice that F_1 is dominated by $\theta\bar{\nabla}\theta$ for ϵ small.

Let $\tilde{g} = \bar{g} + h$ be the metric determined by $\tilde{\varphi}$. To calculate the difference between $\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi}$ and its linearization $D_{\tilde{\varphi}}(\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi})(\theta)$, we first calculate the difference between $V_1(\tilde{\varphi}) = \tilde{g}^{pq}S_{pq}^i e_i$ and its linearization. We can express $V_1(\tilde{\varphi})$ in local coordinates as follows:

$$\begin{aligned} V_1(\tilde{\varphi}) &= (\bar{g} + h)^{pq}(\Gamma_{pq}^i(\bar{g} + h) - \Gamma_{pq}^i(\bar{g}))e_i \\ &= \frac{1}{2}(\bar{g} + h)^{pq} \left((\bar{g} + h)^{ij} (\partial_p(\bar{g} + h)_{jq} + \partial_q(\bar{g} + h)_{jp} - \partial_j(\bar{g} + h)_{pq}) \right. \\ &\quad \left. - \bar{g}^{ij} (\partial_p(\bar{g})_{jq} + \partial_q(\bar{g})_{jp} - \partial_j(\bar{g})_{pq}) \right) e_i. \end{aligned} \tag{4.9}$$

Note that $V_1(\tilde{\varphi})$ is a tensor and its linearization is:

$$V_{1*}(\theta) = \frac{1}{2}\bar{g}^{pq}\bar{g}^{ij} (\bar{\nabla}_p h_{jq} + \bar{\nabla}_q h_{jp} - \bar{\nabla}_j h_{pq}) e_i. \tag{4.10}$$

It is well-known (c.f. [2, 19]) that the metric \tilde{g} of $\tilde{\varphi}$ varies infinitesimally by

$$h = 2f^0\bar{g} + \frac{1}{2}j_{\tilde{\varphi}}(f^3), \tag{4.11}$$

where $j_{\tilde{\varphi}}$ is the linear map defined in (2.1). Equations (4.6) and (4.11) show that there is a linear relation between h and θ . Therefore, using the fact that $V_1(\tilde{\varphi}) = 0$, we may write

$$V_1(\tilde{\varphi}) - V_1(\tilde{\varphi}) - V_{1*}(\theta) = F_2(\bar{g}, \tilde{g}, h, \bar{\nabla}h) = F_2(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta),$$

where F_2 is linear in θ and $\bar{\nabla}\theta$. Moreover, it follows from (4.9) and (4.10), and the relation between h and θ , that F_2 can be expressed in the form $f(\tilde{\varphi}, \tilde{\varphi})\theta\bar{\nabla}\theta$, where f is a tensor depending on $\tilde{\varphi}$ and $\tilde{\varphi}$.

By the same method, we can express the difference between the vector field $\tilde{g}^{kj}S_{ik}^i e_j$ and its linearization as a smooth map which is linear in θ and $\bar{\nabla}\theta$, and thus obtain a similar description for $V_2(\tilde{\varphi}) - V_2(\tilde{\varphi}) - V_{2*}(\theta)$.

Combining these expressions for $V_1(\tilde{\varphi})$ and $V_2(\tilde{\varphi})$ we obtain

$$V(\tilde{\varphi}) - V_*(\theta) = V(\tilde{\varphi}) - V(\tilde{\varphi}) - V_*(\theta) = F_3(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta), \tag{4.12}$$

where $V(\tilde{\varphi}) = 0$, $V_*(\theta)$ is the linearization of the vector field $V(\tilde{\varphi})$ applied to θ and F_3 is linear in θ and $\bar{\nabla}\theta$. Notice that by (4.10), a similar expression

for $V_{2*}(\theta)$ and the linear relation between h and θ , we have that $V_*(\theta)$ is a linear function of $\bar{\nabla}\theta$.

Since $d\tilde{\varphi} = 0$, we see from Cartan's formula that

$$\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi} = d(V(\tilde{\varphi})\lrcorner\tilde{\varphi}).$$

Moreover, as the exterior derivative is a linear map independent of θ , and $V(\tilde{\varphi}) = 0$, we have

$$\begin{aligned} D_{\tilde{\varphi}}(\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi})(\theta) &= D_{\tilde{\varphi}}(d(V(\tilde{\varphi})\lrcorner\tilde{\varphi}))(\theta) \\ &= d(D_{\tilde{\varphi}}V(\tilde{\varphi})(\theta)\lrcorner\tilde{\varphi}) \\ &= d(V_*(\theta)\lrcorner\tilde{\varphi}). \end{aligned}$$

These observations together with (4.12) imply that

$$\begin{aligned} \mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi} - \mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi} - D_{\tilde{\varphi}}(\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi})(\theta) &= d(V(\tilde{\varphi})\lrcorner\tilde{\varphi} - V_*(\theta)\lrcorner\tilde{\varphi}) \\ &= d((V(\tilde{\varphi}) - V_*(\theta))\lrcorner\tilde{\varphi} + V_*(\theta)\lrcorner\theta) \\ &= d(F_3(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta)\lrcorner\tilde{\varphi} + V_*(\theta)\lrcorner\theta). \end{aligned}$$

Since $V_*(\theta)$ depends linearly on $\bar{\nabla}\theta$, we deduce that

$$\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi} - \mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi} - D_{\tilde{\varphi}}(\mathcal{L}_{V(\tilde{\varphi})}\tilde{\varphi})(\theta) = dF_4(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta), \quad (4.13)$$

where F_4 is smooth in its arguments and linear in θ and $\bar{\nabla}\theta$. Combining (4.8) and (4.13) gives:

$$\tilde{F} = d\left(F_1(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta) + F_4(\tilde{\varphi}, \tilde{\varphi}, \theta, \bar{\nabla}\theta)\right) = dF(\tilde{\varphi}, \tilde{\varphi}(t), \theta(t), \bar{\nabla}\theta(t)), \quad (4.14)$$

where F is a 2-form which is smooth in the first two arguments and linear in the last two arguments. Then (4.3) follows from (4.4), (4.5) and (4.14). \square

From Lemma 3.1 we know that $\theta(t)$ exists at least for $t \in [0, \eta]$ with $\eta \geq 1$ and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$. We next improve this estimate in a sequence of results which give estimates on $W^{m,2}$ norms of $\theta(t)$.

In the rest of the paper, we use the same symbol C to denote various positive constants which are uniformly bounded and depend at most on $\tilde{\varphi}$, ϵ , k and Sobolev embedding constants. Unless otherwise stated, all norms, inner products and integrals will be calculated with respect to the fixed metric \bar{g} .

We begin with an L^2 estimate, where we get exponential decay.

Lemma 4.2. *There exists $\bar{\epsilon} = \bar{\epsilon}(\tilde{\varphi}) > 0$ such that if $\epsilon \in (0, \bar{\epsilon})$, then for all time t for which the solution $\tilde{\varphi}(t)$ of (1.2) exists and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$, we have*

$$\int_M |\theta(t)|^2 dv_{\bar{g}} \leq e^{-\lambda_1 t/2} \int_M |\theta(0)|^2 dv_{\bar{g}}, \quad (4.15)$$

where λ_1 is the first eigenvalue of the Hodge-Laplacian $\Delta_{\tilde{\varphi}}$ on exact 3-forms, which is positive due to the Hodge decomposition theorem.

Proof. We take the \bar{g} -inner product of (4.3) with $\theta(t)$ and integrate over M :

$$\begin{aligned}
\frac{d}{dt} \int_M |\theta(t)|^2 dv_{\bar{g}} &= 2 \int_M \theta(t) \cdot \frac{\partial}{\partial t} \theta(t) dv_{\bar{g}} \\
&= 2 \int_M \theta \cdot (-\Delta_{\bar{\varphi}} \theta + dF(\bar{\varphi}, \bar{\varphi}, \theta, \bar{\nabla} \theta)) dv_{\bar{g}} \\
&= -2 \int_M |d^* \theta|^2 dv_{\bar{g}} + 2 \int_M d^* \theta \cdot F dv_{\bar{g}} \\
&\leq -2 \int_M |d^* \theta|^2 dv_{\bar{g}} + 2C\epsilon \int_M |d^* \theta| |\bar{\nabla} \theta| dv_{\bar{g}} \\
&\leq -2 \int_M |d^* \theta|^2 dv_{\bar{g}} + C\epsilon \int_M (|d^* \theta|^2 + |\bar{\nabla} \theta|^2) dv_{\bar{g}}, \quad (4.16)
\end{aligned}$$

where $d^* = d_{\bar{\varphi}}^*$ is defined with respect to $\bar{\varphi}$ and in the first inequality of (4.16) we used the estimate $|F| \leq C|\theta| |\bar{\nabla} \theta| \leq C\epsilon |\bar{\nabla} \theta|$. Recall the Weitzenböck formula:

$$\Delta_{\bar{\varphi}} \theta = -\bar{\Delta} \theta + \mathcal{R}(\theta), \quad (4.17)$$

where \mathcal{R} is some combination of curvature operators with respect to \bar{g} and $\bar{\Delta} = tr \bar{\nabla}^2$ is called the rough (or connection) Laplacian.

Note that θ is closed (as it is exact). Taking the inner product with θ on both sides of (4.17) and integrating over M , we have

$$\int_M |d^* \theta|^2 dv_{\bar{g}} = \int_M (|\bar{\nabla} \theta|^2 + \theta \cdot \mathcal{R}(\theta)) dv_{\bar{g}}, \quad (4.18)$$

which implies

$$\int_M |\bar{\nabla} \theta|^2 dv_{\bar{g}} \leq \int_M (|d^* \theta|^2 + C|\theta|^2) dv_{\bar{g}}, \quad (4.19)$$

where C depends on the bound of the curvature tensor of \bar{g} . Since λ_1 is the first eigenvalue of $\Delta_{\bar{\varphi}}$ on exact 3-forms, we have:

$$\int_M \theta \cdot \Delta_{\bar{\varphi}} \theta dv_{\bar{g}} = \int_M |d^* \theta|^2 dv_{\bar{g}} \geq \lambda_1 \int_M |\theta|^2 dv_{\bar{g}}. \quad (4.20)$$

Substituting (4.19) into (4.16) and using (4.20), we have

$$\begin{aligned}
\frac{d}{dt} \int_M |\theta(t)|^2 dv_{\bar{g}} &\leq -2(1 - C\epsilon) \int_M |d^* \theta|^2 dv_{\bar{g}} + C\epsilon \int_M |\theta|^2 dv_{\bar{g}} \\
&\leq -(\lambda_1 - C\epsilon) \int_M |\theta|^2 dv_{\bar{g}} \\
&\leq -\frac{\lambda_1}{2} \int_M |\theta|^2 dv_{\bar{g}}, \quad (4.21)
\end{aligned}$$

where we have chosen ϵ small such that $C\epsilon < \min\{\frac{1}{2}, \frac{\lambda_1}{2}\}$. Then (4.15) follows from integrating (4.21) in t . \square

We now proceed to look at higher order estimates, which will follow from the next three lemmas.

Lemma 4.3. *There exists $\bar{\epsilon} = \bar{\epsilon}(\bar{\varphi}) > 0$ such that if $\epsilon \in (0, \bar{\epsilon})$, then for all time t for which the solution $\bar{\varphi}(t)$ of (1.2) exists and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$, we have*

$$\int_0^t \int_M |\bar{\nabla} \theta(s)|^2 dv_{\bar{g}} ds \leq \left(1 + \frac{C}{\lambda_1}\right) \int_M |\theta(0)|^2 dv_{\bar{g}}, \quad (4.22)$$

where λ_1 is the first eigenvalue of $\Delta_{\bar{\varphi}}$ on exact 3-forms.

Proof. Substituting the Weitzenböck formula (4.17) into the evolution equation (4.3), we have

$$\frac{\partial}{\partial t}\theta(t) = \bar{\Delta}\theta(t) - \mathcal{R}(\theta(t)) + dF. \quad (4.23)$$

Taking the \bar{g} -inner product of (4.23) with $\theta(t)$ and integrating over M , we can use the estimates for \mathcal{R} and F to compute:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_M |\theta(t)|^2 dv_{\bar{g}} &\leq - \int_M |\bar{\nabla}\theta|^2 dv_{\bar{g}} + C \int_M |\theta|^2 dv_{\bar{g}} + \int_M d^*\theta \cdot F dv_{\bar{g}} \\ &\leq - \int_M |\bar{\nabla}\theta|^2 dv_{\bar{g}} + C \int_M |\theta|^2 dv_{\bar{g}} \\ &\quad + \frac{1}{2} C \epsilon \int_M (|d^*\theta|^2 + |\bar{\nabla}\theta|^2) dv_{\bar{g}} \\ &\leq - (1 - C\epsilon) \int_M |\bar{\nabla}\theta|^2 dv_{\bar{g}} + C \int_M |\theta|^2 dv_{\bar{g}} \\ &\leq - \frac{1}{2} \int_M |\bar{\nabla}\theta|^2 dv_{\bar{g}} + C \int_M |\theta|^2 dv_{\bar{g}}, \end{aligned} \quad (4.24)$$

where in the third inequality we used

$$\int_M |d^*\theta|^2 dv_{\bar{g}} \leq \int_M (|\bar{\nabla}\theta|^2 + C|\theta|^2) dv_{\bar{g}},$$

which follows from (4.18). Integrating the above inequality in t gives:

$$\begin{aligned} \frac{1}{2} \int_0^t \int_M |\bar{\nabla}\theta(s)|^2 dv_{\bar{g}} ds + \frac{1}{2} \int_M |\theta(t)|^2 dv_{\bar{g}} \\ \leq \frac{1}{2} \int_M |\theta(0)|^2 dv_{\bar{g}} + C \int_0^t \int_M |\theta(s)|^2 dv_{\bar{g}} ds \\ \leq \left(\frac{1}{2} + \frac{2C}{\lambda_1} (1 - e^{-\frac{\lambda_1}{2}t}) \right) \int_M |\theta(0)|^2 dv_{\bar{g}}, \end{aligned}$$

where in the last inequality we used the estimate (4.15). The result follows. \square

Lemma 4.4. *There exists $\bar{\epsilon} = \bar{\epsilon}(\bar{\varphi}) > 0$ such that if $\epsilon \in (0, \bar{\epsilon})$, then for all time t for which the solution $\tilde{\varphi}(t)$ of (1.2) exists and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$, we have*

$$\begin{aligned} \int_0^t \int_M \left| \frac{\partial}{\partial s} \theta(s) \right|^2 dv_{\bar{g}} ds + \int_M |\bar{\nabla}\theta(t)|^2 dv_{\bar{g}} + \frac{1}{2} \int_0^t \int_M |\bar{\nabla}^2 \theta(s)|^2 dv_{\bar{g}} ds \\ \leq \int_M |\bar{\nabla}\theta(0)|^2 dv_{\bar{g}} + C \int_M |\theta(0)|^2 dv_{\bar{g}}. \end{aligned} \quad (4.25)$$

Proof. We rewrite the equation (4.23) as

$$\frac{\partial}{\partial t}\theta(t) - \bar{\Delta}\theta(t) = -\mathcal{R}(\theta(t)) + dF.$$

Squaring both sides with \bar{g} and integrating over M , we can use the estimates of \mathcal{R} and F to see that

$$\int_M \left(\left| \frac{\partial}{\partial t} \theta \right|^2 - 2 \frac{\partial}{\partial t} \theta \cdot \bar{\Delta} \theta + |\bar{\Delta} \theta|^2 \right) dv_{\bar{g}} \leq \int_M 2(C|\theta|^2 + |dF|^2) dv_{\bar{g}}. \quad (4.26)$$

By integration by parts and the Ricci identity, we compute

$$- \int_M 2 \frac{\partial}{\partial t} \theta \cdot \bar{\Delta} \theta dv_{\bar{g}} = \frac{d}{dt} \int_M |\bar{\nabla} \theta|^2 dv_{\bar{g}}, \quad (4.27)$$

$$\int_M |\bar{\Delta} \theta|^2 dv_{\bar{g}} = \int_M |\bar{\nabla}^2 \theta|^2 dv_{\bar{g}} + \int_M Rm(\bar{g}) * \bar{\nabla} \theta * \bar{\nabla} \theta dv_{\bar{g}}, \quad (4.28)$$

where $*$ means a contraction of tensors using \bar{g} . Recall that, by the proof of Lemma 4.1, we can write $F = f(\bar{\varphi}, \tilde{\varphi})\theta\bar{\nabla}\theta$ for some tensor f depending on $\bar{\varphi}$ and $\tilde{\varphi}$. Moreover, $\tilde{\varphi} = \bar{\varphi} + \theta$ and $\bar{\nabla}\bar{\varphi} = 0$ so $\bar{\nabla}\tilde{\varphi} = \bar{\nabla}\theta$. Hence, differentiating the expression for F yields

$$|dF| \leq |\bar{\nabla} F| \leq C (|\theta|\bar{\nabla}\theta|^2 + |\bar{\nabla}\theta|^2 + |\theta|\bar{\nabla}^2\theta|),$$

where the first term on the right-hand side arises from differentiating f . Using the facts that $|\theta| < \epsilon$ and $|\bar{\nabla}\theta| < \epsilon$, we obtain the estimate

$$|dF| \leq C\epsilon(|\bar{\nabla}\theta| + |\bar{\nabla}^2\theta|). \quad (4.29)$$

Combining (4.26)-(4.29) gives that

$$\begin{aligned} \int_M \left| \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla} \theta|^2 dv_{\bar{g}} + \int_M |\bar{\nabla}^2 \theta|^2 dv_{\bar{g}} \\ \leq \int_M C (|\theta|^2 + |\bar{\nabla} \theta|^2 + \epsilon^2 |\bar{\nabla}^2 \theta|^2) dv_{\bar{g}}. \end{aligned}$$

Choosing ϵ small such that $C\epsilon^2 < 1/2$, we have

$$\begin{aligned} \int_M \left| \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla} \theta|^2 dv_{\bar{g}} + \frac{1}{2} \int_M |\bar{\nabla}^2 \theta|^2 dv_{\bar{g}} \\ \leq \int_M C (|\theta|^2 + |\bar{\nabla} \theta|^2) dv_{\bar{g}}. \end{aligned} \quad (4.30)$$

Integrating (4.30) and using the estimates in Lemmas 4.2-4.3 gives the result. \square

Taking covariant derivative with respect to \bar{g} in (4.23), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \bar{\nabla} \theta &= \bar{\nabla} \bar{\Delta} \theta - \mathcal{R}(\bar{\nabla} \theta) - (\bar{\nabla} \mathcal{R})\theta - \bar{\nabla} dF \\ &= \bar{\Delta} \bar{\nabla} \theta - \mathcal{R}_1(\bar{\nabla} \theta) - (\bar{\nabla} \mathcal{R}_1)\theta - \bar{\nabla} dF, \end{aligned} \quad (4.31)$$

where in the second equality we used the Ricci identity

$$\bar{\nabla} \bar{\Delta} \theta - \bar{\Delta} \bar{\nabla} \theta = \bar{\nabla} Rm(\bar{g}) * \theta + Rm(\bar{g}) * \bar{\nabla} \theta,$$

and $\mathcal{R}, \mathcal{R}_1$ are some combinations of curvature operators with respect to the metric \bar{g} . Recall that we can write $F = f(\bar{\varphi}, \tilde{\varphi})\theta\bar{\nabla}\theta$ and that $\bar{\nabla}\bar{\varphi} = 0$ and $\bar{\nabla}\tilde{\varphi} = \bar{\nabla}\theta$. By taking a second derivative of this expression for F we deduce an estimate on $\bar{\nabla}dF$:

$$|\bar{\nabla}dF| \leq |\bar{\nabla}^2 F| \leq C(|\theta|\bar{\nabla}\theta|^3 + |\bar{\nabla}\theta|^3 + |\bar{\nabla}^2\theta|\bar{\nabla}\theta| + |\theta|\bar{\nabla}^2\theta|\bar{\nabla}\theta| + |\theta|\bar{\nabla}^3\theta|).$$

Then, using $|\theta| < \epsilon$ and $|\bar{\nabla}\theta| < \epsilon$, we obtain

$$|\bar{\nabla}dF| \leq C\epsilon(|\bar{\nabla}\theta| + |\bar{\nabla}^2\theta| + |\bar{\nabla}^3\theta|). \quad (4.32)$$

Lemma 4.5. *There exists $\bar{\epsilon} = \bar{\epsilon}(\bar{\varphi}) > 0$ such that if $\epsilon \in (0, \bar{\epsilon})$, then for all time t for which the solution $\tilde{\varphi}(t)$ of (1.2) exists and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$, we have*

$$\begin{aligned} & \int_0^t \int_M \left| \bar{\nabla} \frac{\partial}{\partial s} \theta \right|^2 dv_{\bar{g}} ds + \int_M |\bar{\nabla}^2 \theta(t)|^2 dv_{\bar{g}} + \frac{1}{2} \int_0^t \int_M |\bar{\nabla}^3 \theta(s)|^2 dv_{\bar{g}} ds \\ & \leq \int_M |\bar{\nabla}^2 \theta(0)|^2 dv_{\bar{g}} + C \int_M |\bar{\nabla} \theta(0)|^2 dv_{\bar{g}} + C \int_M |\theta(0)|^2 dv_{\bar{g}}. \end{aligned}$$

Proof. We rewrite (4.31) as

$$\frac{\partial}{\partial t} \bar{\nabla} \theta - \bar{\Delta} \bar{\nabla} \theta = -\mathcal{R}_1(\bar{\nabla} \theta) - (\bar{\nabla} \mathcal{R}_1) \theta - \bar{\nabla} dF. \quad (4.33)$$

In the usual way, squaring both sides of (4.33) and integrating over M gives

$$\begin{aligned} & \int_M \left| \bar{\nabla} \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla}^2 \theta|^2 dv_{\bar{g}} + \int_M |\bar{\Delta} \bar{\nabla} \theta|^2 dv_{\bar{g}} \\ & \leq C \int_M |\bar{\nabla} \theta|^2 dv_{\bar{g}} + C \int_M |\theta|^2 dv_{\bar{g}} + 3 \int_M |\bar{\nabla} dF|^2 dv_{\bar{g}}. \end{aligned} \quad (4.34)$$

Note that curvature identities give that

$$\int_M |\bar{\Delta} \bar{\nabla} \theta|^2 dv_{\bar{g}} = \int_M |\bar{\nabla}^3 \theta|^2 dv_{\bar{g}} + \int_M Rm(\bar{g}) * \bar{\nabla}^2 \theta * \bar{\nabla}^2 \theta dv_{\bar{g}}. \quad (4.35)$$

Substituting (4.32) and (4.35) into (4.34) gives

$$\begin{aligned} & \int_M \left| \bar{\nabla} \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla}^2 \theta|^2 dv_{\bar{g}} + \int_M |\bar{\nabla}^3 \theta|^2 dv_{\bar{g}} \\ & \leq C \int_M (|\bar{\nabla}^2 \theta|^2 + |\bar{\nabla} \theta|^2 + |\theta|^2) dv_{\bar{g}} + 3 \int_M |\bar{\nabla} dF|^2 dv_{\bar{g}} \\ & \leq C \int_M (|\bar{\nabla}^2 \theta|^2 + |\bar{\nabla} \theta|^2 + |\theta|^2) dv_{\bar{g}} + C\epsilon^2 \int_M |\bar{\nabla}^3 \theta|^2 dv_{\bar{g}}. \end{aligned} \quad (4.36)$$

Choose ϵ small such that $C\epsilon^2 < 1/2$. Integrating (4.36) in t , we get

$$\begin{aligned} & \int_0^t \int_M \left| \bar{\nabla} \frac{\partial}{\partial s} \theta \right|^2 dv_{\bar{g}} ds + \int_M |\bar{\nabla}^2 \theta(t)|^2 dv_{\bar{g}} + \frac{1}{2} \int_0^t \int_M |\bar{\nabla}^3 \theta(s)|^2 dv_{\bar{g}} ds \\ & \leq \int_M |\bar{\nabla}^2 \theta(0)|^2 dv_{\bar{g}} + C \int_0^t \int_M (|\bar{\nabla}^2 \theta(s)|^2 + |\bar{\nabla} \theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}} ds \\ & \leq \int_M |\bar{\nabla}^2 \theta(0)|^2 dv_{\bar{g}} + C \int_M |\bar{\nabla} \theta(0)|^2 dv_{\bar{g}} + C \int_M |\theta(0)|^2 dv_{\bar{g}} \end{aligned}$$

as desired. Note that we used Lemmas 4.3 and 4.4 in the derivation. \square

Combining Lemmas 4.2, 4.4 and 4.5, we have our Sobolev estimates:

$$\int_M |\bar{\nabla}^m \theta(t)|^2 dv_{\bar{g}} \leq C \|\theta(0)\|_{W_{\bar{g}}^{m,2}}, \quad m = 0, 1, 2. \quad (4.37)$$

It is evident that we can continue repeating a similar procedure as follows. Taking $(m-1)$ covariant derivatives of (4.23) with respect to \bar{g} , we have

$$\begin{aligned}\bar{\nabla}^{m-1} \frac{\partial}{\partial t} \theta &= \bar{\nabla}^{m-1} \bar{\Delta} \theta - \bar{\nabla}^{m-1} (\mathcal{R}(\theta)) + \bar{\nabla}^{m-1} dF \\ &= \bar{\Delta} \bar{\nabla}^{m-1} \theta + \sum_{j=0}^{m-1} \bar{\nabla}^j \mathcal{R} * \bar{\nabla}^{m-1-j} \theta + \bar{\nabla}^{m-1} dF.\end{aligned}$$

We rewrite the above equation as

$$\bar{\nabla}^{m-1} \frac{\partial}{\partial t} \theta - \bar{\Delta} \bar{\nabla}^{m-1} \theta = \sum_{j=0}^{m-1} \bar{\nabla}^j \mathcal{R} * \bar{\nabla}^{m-1-j} \theta + \bar{\nabla}^{m-1} dF.$$

Squaring both sides and integrating over M , we obtain

$$\begin{aligned}\int_M \left| \bar{\nabla}^{m-1} \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla}^m \theta|^2 dv_{\bar{g}} + \int_M |\bar{\Delta} \bar{\nabla}^{m-1} \theta|^2 dv_{\bar{g}} \\ \leq C \sum_{j=0}^{m-1} \int_M |\bar{\nabla}^j \theta|^2 dv_{\bar{g}} + (m+1) \int_M |\bar{\nabla}^{m-1} dF|^2 dv_{\bar{g}},\end{aligned}\quad (4.38)$$

where C only depends on $\bar{\varphi}$ and m . Once again using the expression $F = f(\bar{\varphi}, \tilde{\varphi}) \theta \bar{\nabla} \theta$, together with $\bar{\nabla} \bar{\varphi} = 0$ and $\bar{\nabla} \tilde{\varphi} = \bar{\nabla} \theta$, we estimate $|\bar{\nabla}^{m-1} dF| \leq |\bar{\nabla}^m F|$ as follows:

$$\begin{aligned}|\bar{\nabla}^m F| &\leq C \sum_{j=0}^m |\bar{\nabla}^j \theta| |\bar{\nabla}^{m+1-j} \theta| + C \sum_{l=1}^m |\bar{\nabla}^l \theta| \sum_{j=0}^{m-l} |\bar{\nabla}^j \theta| |\bar{\nabla}^{m-l+1-j} \theta| \\ &\leq C \epsilon \sum_{j=\lfloor \frac{m}{2} \rfloor}^{m+1} |\bar{\nabla}^j \theta|,\end{aligned}\quad (4.39)$$

where we need $|\bar{\nabla}^j \theta| < \epsilon$ for $0 \leq j \leq \lfloor \frac{m}{2} \rfloor + 1$, which will hold if $k \geq \lfloor \frac{m}{2} \rfloor + 1$. Using integration by parts and the formula for commuting covariant derivatives, we also have

$$\int_M |\bar{\Delta} \bar{\nabla}^{m-1} \theta|^2 dv_{\bar{g}} = \int_M |\nabla^{m+1} \theta|^2 dv_{\bar{g}} + \int_M Rm(\bar{g}) * \bar{\nabla}^m \theta * \bar{\nabla}^m \theta dv_{\bar{g}}.\quad (4.40)$$

Substituting (4.39) and (4.40) into (4.38), we obtain

$$\begin{aligned}\int_M \left| \bar{\nabla}^{m-1} \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla}^m \theta|^2 dv_{\bar{g}} + \int_M |\bar{\nabla}^{m+1} \theta|^2 dv_{\bar{g}} \\ \leq C \sum_{j=1}^m \int_M |\bar{\nabla}^j \theta|^2 dv_{\bar{g}} + C \epsilon^2 \int_M |\bar{\nabla}^{m+1} \theta|^2 dv_{\bar{g}},\end{aligned}\quad (4.41)$$

where $C = C(\bar{\varphi}, m)$. Choosing ϵ such that $C \epsilon^2 < 1/2$, we have

$$\begin{aligned}\int_M \left| \bar{\nabla}^{m-1} \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla}^m \theta|^2 dv_{\bar{g}} + \frac{1}{2} \int_M |\bar{\nabla}^{m+1} \theta|^2 dv_{\bar{g}} \\ \leq C \sum_{j=1}^m \int_M |\bar{\nabla}^j \theta|^2 dv_{\bar{g}}.\end{aligned}\quad (4.42)$$

Starting from (4.37), an induction argument applied to (4.42) gives that

$$\begin{aligned} \int_0^t \int_M \left| \bar{\nabla}^{m-1} \frac{\partial}{\partial s} \theta \right|^2 dv_{\bar{g}} ds + \int_M |\bar{\nabla}^m \theta(t)|^2 dv_{\bar{g}} + \frac{1}{2} \int_0^t \int_M |\bar{\nabla}^{m+1} \theta(s)|^2 dv_{\bar{g}} ds \\ \leq \int_M |\bar{\nabla}^m \theta(0)|^2 dv_{\bar{g}} + C \sum_{j=1}^{m-1} \int_M |\bar{\nabla}^j \theta(0)|^2 dv_{\bar{g}}. \end{aligned} \quad (4.43)$$

So we have

$$\int_M |\bar{\nabla}^m \theta(t)|^2 dv_{\bar{g}} \leq C \|\theta(0)\|_{W_{\bar{g}}^{m,2}} \quad (4.44)$$

whenever $[\frac{m}{2}] \leq k - 1$.

Recall that as $\dim M = 7$ if $m - k > \frac{7}{2}$, then the Sobolev embedding theorem says that $W_{\bar{g}}^{m,2}$ embeds continuously in $C_{\bar{g}}^k$ by inclusion. Since $m = k + 4$ satisfies $k + \frac{7}{2} < m$ trivially and $[\frac{m}{2}] \leq k - 1$ for all $k \geq 5$ (since $k \geq 5$ implies $[\frac{k}{2}] \leq k - 3$), we can use (4.44) to obtain the main estimate of this section.

Lemma 4.6. *For any $k \geq 5$ there exists $\bar{\epsilon} > 0$ such that if $\epsilon \in (0, \bar{\epsilon})$, then for all time t for which the solution $\tilde{\varphi}(t)$ of (1.2) exists and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$, we have*

$$\|\theta(t)\|_{C_{\bar{g}}^k} \leq C_1 \|\theta(0)\|_{W_{\bar{g}}^{k+4,2}} \leq C_2 \|\theta(0)\|_{C_{\bar{g}}^{k+4}}, \quad (4.45)$$

where C_1, C_2 are constants depending only on $\bar{\varphi}, \epsilon, k$ and Sobolev embedding constants.

5. PROOF OF THEOREM 1.3

In this section, we prove the main theorem of this paper, Theorem 1.3. First, we show the dynamical stability of torsion-free G_2 structures under the Laplacian–DeTurck flow (1.2) for closed G_2 structures.

Theorem 5.1. *Let $(M, \bar{\varphi})$ be a compact G_2 manifold, let $k \geq 5$ and let $\bar{\epsilon} = \bar{\epsilon}(\bar{\varphi}) > 0$ be such that Lemmas 4.1–4.6 hold. Let $\epsilon \in (0, \bar{\epsilon})$ and let C_1 be the constant given in (4.45).*

There exists $\delta = \delta(M, \bar{\varphi}, k, \epsilon) > 0$ such that if $\varphi_0 \in [\bar{\varphi}]_+$ satisfies

$$\|\varphi_0 - \bar{\varphi}\|_{C_{\bar{g}}^{k+4}} < \delta \quad \text{and} \quad \|\varphi_0 - \bar{\varphi}\|_{W_{\bar{g}}^{k+4,2}} \leq \epsilon/2C_1,$$

then the solution $\tilde{\varphi}(t)$ of the Laplacian–DeTurck flow (1.2) starting at φ_0 exists for all time $t \in [0, \infty)$, satisfies $\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_{\bar{g}}^k} < \epsilon$ for all $t \in [0, \infty)$ and $\tilde{\varphi}(t) \rightarrow \bar{\varphi}$ exponentially in $C_{\bar{g}}^k$ as $t \rightarrow \infty$.

Proof. By Lemma 3.1 there exists $\delta = \delta(M, \bar{\varphi}, k, \epsilon) > 0$ such if the initial value $\varphi_0 \in [\bar{\varphi}]_+$ satisfies $\|\varphi_0 - \bar{\varphi}\|_{C_{\bar{g}}^{k+4}} < \delta$, then the solution $\tilde{\varphi}(t)$ of the flow (1.2) exists at least for time $t \in [0, 1]$. Let $\theta(t) = \tilde{\varphi}(t) - \bar{\varphi}$ as in the previous section.

Let η be the maximum time such that $\tilde{\varphi}(t)$ exists and $\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon$ for all $t \in [0, \eta)$. By Lemma 3.1, we know that $\eta \geq 1$. Suppose towards a contradiction that $\eta < \infty$. From the estimates (4.45), we have that

$$\|\theta(t)\|_{C_{\bar{g}}^k} \leq C_1 \|\theta(0)\|_{W_{\bar{g}}^{k+4,2}} \leq C_1 \frac{\epsilon}{2C_1} = \epsilon/2, \quad (5.1)$$

for all $t \in [0, \eta)$. This contradicts the maximality in the definition of η . Thus $\eta = \infty$ and

$$\|\theta(t)\|_{C_{\bar{g}}^k} < \epsilon \quad \forall t \in [0, \infty). \quad (5.2)$$

To complete the proof, we need to demonstrate the exponential convergence of $\tilde{\varphi}(t) \rightarrow \bar{\varphi}$ as $t \rightarrow \infty$ in $C_{\bar{g}}^k$. First, by Lemma 4.2, $\varphi(t) \rightarrow \bar{\varphi}$ exponentially in $L_{\bar{g}}^2$, so $\theta(t) \rightarrow 0$ in $L_{\bar{g}}^2$. Now, from (4.24) in the proof of Lemma 4.3,

$$\frac{1}{2} \frac{d}{dt} \int_M |\theta(t)|^2 dv_{\bar{g}} \leq -\frac{1}{2} \int_M |\bar{\nabla}\theta|^2 dv_{\bar{g}} + C \int_M |\theta|^2 dv_{\bar{g}}.$$

Integrating this equation in time t , using estimate (4.15) and the fact that $\theta(t) \rightarrow 0$ in L^2 we get

$$\begin{aligned} \frac{1}{2} \int_t^\infty \int_M |\bar{\nabla}\theta(s)|^2 dv_{\bar{g}} ds &\leq \frac{1}{2} \int_M |\theta(t)|^2 dv_{\bar{g}} + C \int_t^\infty \int_M |\theta(s)|^2 dv_{\bar{g}} ds \\ &\leq \left(\frac{1}{2} + \frac{2C}{\lambda_1}\right) e^{-\frac{\lambda_1 t}{2}} \int_M |\theta(0)|^2 dv_{\bar{g}}. \end{aligned} \quad (5.3)$$

Recall (4.30) in the proof of Lemma 4.4:

$$\begin{aligned} \int_M \left| \frac{\partial}{\partial t} \theta \right|^2 dv_{\bar{g}} + \frac{d}{dt} \int_M |\bar{\nabla}\theta|^2 dv_{\bar{g}} + \frac{1}{2} \int_M |\bar{\nabla}^2 \theta|^2 dv_{\bar{g}} \\ \leq \int_M C (|\theta|^2 + |\bar{\nabla}\theta|^2) dv_{\bar{g}}. \end{aligned} \quad (5.4)$$

Let $\xi(s)$ be a cut-off function with $0 \leq \xi(s) \leq 1$ satisfying $\xi(s) = 0$ for $s \leq t - \frac{1}{2}$, $|\xi'(s)| \leq C$ for $s \in [t - \frac{1}{2}, t]$ and $\xi(s) = 1$ for $s \geq t$. Then from (5.4), we have

$$\begin{aligned} \frac{d}{ds} \left(\xi(s) \int_M |\bar{\nabla}\theta(s)|^2 dv_{\bar{g}} \right) \\ = \xi'(s) \int_M |\bar{\nabla}\theta(s)|^2 dv_{\bar{g}} + \xi(s) \frac{d}{ds} \int_M |\bar{\nabla}\theta(s)|^2 dv_{\bar{g}} \\ \leq (\xi'(s) + C\xi(s)) \int_M |\bar{\nabla}\theta(s)|^2 dv_{\bar{g}} + C\xi(s) \int_M |\theta(s)|^2 dv_{\bar{g}} \\ \leq C \int_M (|\bar{\nabla}\theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}}. \end{aligned}$$

Integrating the above inequality from $t - \frac{1}{2}$ to t and using (4.15) and (5.3) we get

$$\begin{aligned} \int_M |\bar{\nabla}\theta(t)|^2 dv_{\bar{g}} &\leq C \int_{t-\frac{1}{2}}^t \int_M (|\bar{\nabla}\theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}} \\ &\leq C \int_{t-\frac{1}{2}}^\infty \int_M (|\bar{\nabla}\theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}} \\ &\leq C e^{-\frac{\lambda_1 t}{2}} \int_M |\theta(0)|^2 dv_{\bar{g}}, \end{aligned} \quad (5.5)$$

where C is a uniform constant. Hence $\bar{\nabla}\theta \rightarrow 0$ in $L^2_{\bar{g}}$ as $t \rightarrow \infty$. Using this fact, Lemma 4.2 and (5.4) imply that

$$\begin{aligned} \int_t^\infty \int_M |\bar{\nabla}^2 \theta(s)|^2 dv_{\bar{g}} ds & \\ & \leq 2 \int_M |\bar{\nabla} \theta(t)|^2 dv_{\bar{g}} + 2C \int_t^\infty \int_M (|\bar{\nabla} \theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}} ds \\ & \leq Ce^{-\frac{\lambda_1 t}{2}} \int_M |\theta(0)|^2 dv_{\bar{g}}. \end{aligned}$$

Using (4.36) and a cut-off argument in a similar way as above, we have

$$\begin{aligned} \int_M |\bar{\nabla}^2 \theta(t)|^2 dv_{\bar{g}} & \leq C \int_{t-\frac{1}{2}}^t \int_M (|\bar{\nabla}^2 \theta(s)|^2 + |\bar{\nabla} \theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}} \\ & \leq C \int_{t-\frac{1}{2}}^\infty \int_M (|\bar{\nabla}^2 \theta(s)|^2 + |\bar{\nabla} \theta(s)|^2 + |\theta(s)|^2) dv_{\bar{g}} \\ & \leq Ce^{-\frac{\lambda_1 t}{2}} \int_M |\theta(0)|^2 dv_{\bar{g}}, \end{aligned}$$

By repeating this procedure and using the induction part in §3, we can obtain

$$\int_M |\bar{\nabla}^m \theta(t)|^2 dv_{\bar{g}} \leq Ce^{-\frac{\lambda_1 t}{2}} \int_M |\theta(0)|^2 dv_{\bar{g}} \quad (5.6)$$

for all m so that $[\frac{m}{2}] \leq k-1$, in particular for $m \leq k+4$ since $k \geq 5$. Then the Sobolev embedding theorem gives that

$$\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C^k_{\bar{g}}} = \|\theta(t)\|_{C^k_{\bar{g}}} \leq Ce^{-\frac{\lambda_1 t}{2}} \quad (5.7)$$

as required. \square

Theorem 5.1 shows that the solution $\tilde{\varphi}(t)$ of the Laplacian–DeTurck flow (1.2) converges to the torsion free G_2 structure $\bar{\varphi}$ exponentially in the $C^k_{\bar{g}}$ -norm, if the initial value φ_0 is sufficiently close to $\bar{\varphi}$. Let $\varphi(t)$ be the solution to the Laplacian flow (1.1) with the same initial value φ_0 . Then $\varphi(t) = \phi_t^* \tilde{\varphi}(t)$ and the associated metric $g(t) = \phi_t^* \tilde{g}(t)$, where ϕ_t is a family of diffeomorphisms satisfying

$$\begin{cases} \frac{\partial}{\partial t} \phi_t(x) & = -V(\tilde{\varphi}(t))|_{\phi_t(x)} \\ \phi_0 & = id \end{cases} \quad (5.8)$$

for $x \in M$, where the vector field $V(\tilde{\varphi}(t))$ is defined in (4.1). Note that $V(\tilde{\varphi}(t))$ involves the first order covariant derivative of $\tilde{\varphi}(t)$. Since $\tilde{\varphi}(t)$ exists for all time $[0, \infty)$ and converges to $\bar{\varphi}$ exponentially in $C^k_{\bar{g}}$ -norm where $k \geq 3$, the vector field $V(\tilde{\varphi}(t))$ also exists for all time $[0, \infty)$ and converges to $V(\bar{\varphi}) = 0$ in the $C^{k-1}_{\bar{g}}$ -norm exponentially. Then the solution ϕ_t of (5.8) exists for all $t \in [0, \infty)$ (cf. [6, Lemma 3.15]). Moreover, ϕ_t converges to a limit map ϕ_∞ in $C^{k-1}_{\bar{g}}$ -norm as $t \rightarrow \infty$.

Recalling that $\tilde{\varphi}(t) - \bar{\varphi} = \theta(t)$ and $\bar{\varphi}$ is torsion-free, we have by (4.8) that

$$\Delta_{\tilde{\varphi}(t)} \tilde{\varphi}(t) = \Delta_{\tilde{\varphi}(t)} \tilde{\varphi}(t) - \Delta_{\bar{\varphi}} \bar{\varphi} = L(\theta(t)) + dF(\theta(t), \bar{\nabla} \theta(t)),$$

where L is a linear second order differential operator and F is dominated by $\theta \bar{\nabla} \theta$ when $|\theta|$ is sufficiently small. Therefore, since the exterior derivative is a linear first order differential operator, we have

$$\|L(\theta(t))\|_{C_{\bar{g}}^{k-2}} \leq C \|\theta(t)\|_{C_{\bar{g}}^k},$$

$$\|dF(\theta(t), \bar{\nabla} \theta(t))\|_{C_{\bar{g}}^{k-2}} \leq C \|F(\theta(t), \bar{\nabla} \theta(t))\|_{C_{\bar{g}}^{k-1}} \leq C \|\theta(t)\|_{C_{\bar{g}}^k}^2 \leq C \|\theta(t)\|_{C_{\bar{g}}^k},$$

when $\|\theta(t)\|_{C_{\bar{g}}^k}$ is sufficiently small. Combining these estimates with (5.7) we obtain

$$\|\Delta_{\bar{\varphi}(t)} \tilde{\varphi}(t)\|_{C_{\bar{g}}^{k-2}} \leq C \|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_{\bar{g}}^k} \leq C e^{-\frac{\lambda_1 t}{2}}.$$

Since $\|\tilde{\varphi}(t) - \bar{\varphi}\|_{C_{\bar{g}}^k} < \epsilon$ for all time $t \in [0, \infty)$, by making ϵ smaller if necessary we can assume that the C^k -norms of \bar{g} and $\tilde{g}(t)$ differ at most by a factor 2 for all $t \in [0, \infty)$. Then we have

$$\|\Delta_{\tilde{\varphi}(t)} \tilde{\varphi}(t)\|_{C_{\tilde{g}(t)}^{k-2}} \leq 2 \|\Delta_{\bar{\varphi}(t)} \tilde{\varphi}(t)\|_{C_{\bar{g}}^{k-2}} \leq 2C e^{-\frac{\lambda_1 t}{2}} \quad (5.9)$$

By diffeomorphism invariance, (5.9) implies that

$$\|\Delta_{\varphi(t)} \varphi(t)\|_{C_{g(t)}^{k-2}} \leq 2C e^{-\frac{\lambda_1 t}{2}}. \quad (5.10)$$

We know that for closed G_2 structures, $\Delta_{\varphi} \varphi = i_{\varphi}(h)$, where h is a symmetric 2-tensor satisfying (2.8). By [20, Proposition 2.9],

$$|\Delta_{\varphi} \varphi|_g^2 = |i_{\varphi}(h)|_g^2 = (\text{tr}_g(h))^2 + 2h_i^k h_k^i. \quad (5.11)$$

From (5.10)-(5.11), we have

$$\|h(t)\|_{C_{g(t)}^{k-2}} \leq \frac{C}{2} e^{-\frac{\lambda_1 t}{2}}. \quad (5.12)$$

Recall that under the Laplacian flow the associated metric $g(t)$ of $\varphi(t)$ evolves by

$$\frac{\partial}{\partial t} g(t) = 2h(t).$$

From (5.12),

$$\int_0^{\infty} \|h(t)\|_{C_{g(t)}^{k-2}} dt \leq \frac{C}{2} \int_0^{\infty} e^{-\frac{\lambda_1 t}{2}} dt = \frac{C}{\lambda_1}. \quad (5.13)$$

We now observe that, for any tangent vector $X \neq 0$ on M , we have

$$\left| \frac{\partial}{\partial t} g_t(X, X) \right| = 2|h_t(X, X)| \leq 2\|h_t\|_{C_{g(t)}^0} g_t(X, X),$$

which implies that

$$\left| \frac{\partial}{\partial t} \ln g_t(X, X) \right| \leq 2\|h_t\|_{C_{g(t)}^0}.$$

Then, by (5.13), we have

$$\left| \ln \frac{g_t(X, X)}{g_0(X, X)} \right| \leq \int_0^t 2\|h_t\|_{C_{g(t)}^0} dt \leq \int_0^{\infty} 2\|h_t\|_{C_{g(t)}^0} dt \leq \frac{2C}{\lambda_1}.$$

Hence, for any $t \in [0, \infty)$,

$$e^{-\frac{2C}{\lambda_1}} g(0) \leq g(t) \leq e^{\frac{2C}{\lambda_1}} g(0). \quad (5.14)$$

Note that $g(0)$ is the associated metric of φ_0 (which is very close to $\bar{\varphi}$), so $g(0)$ is uniformly equivalent to \bar{g} , the associated metric of $\bar{\varphi}$. Therefore, from (5.14) we have $g(t)$ is uniformly equivalent to \bar{g} for all $t \in [0, \infty)$.

Then we deduce from (5.10) that

$$\|\Delta_{\varphi(t)}\varphi(t)\|_{C_{\bar{g}}^0} \leq C\|\Delta_{\varphi(t)}\varphi(t)\|_{C_{g(t)}^0} \leq Ce^{-\frac{\lambda_1 t}{2}}, \quad (5.15)$$

and therefore $\|\frac{d}{dt}\varphi(t)\|_{C_{\bar{g}}^0} \leq Ce^{-\frac{\lambda_1 t}{2}}$. This tells us there exists a limit G_2 structure φ_∞ such that

$$\begin{aligned} \|\varphi(t) - \varphi_\infty\|_{C_{\bar{g}}^0} &\leq \int_t^\infty \left\| \frac{d}{ds}\varphi(s) \right\|_{C_{\bar{g}}^0} ds \\ &= \int_t^\infty \|\Delta_{\varphi(s)}\varphi(s)\|_{C_{\bar{g}}^0} ds \\ &\leq \frac{2C}{\lambda_1} e^{-\frac{\lambda_1 t}{2}}. \end{aligned} \quad (5.16)$$

Note that φ_∞ is a positive 3-form because we can see as in [23] that φ_∞ defines a metric g_∞ which is the limit of the metric $g(t)$. The limit φ_∞ is torsion-free as M is compact and we have $\Delta_{\varphi_\infty}\varphi_\infty = 0$ from (5.15).

We claim that the limit $\varphi_\infty = \phi_\infty^* \bar{\varphi}$. In fact, for any time-independent vector field v on M , we see from (5.8) that the differential ϕ_{t*} of ϕ_t satisfies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\phi_{t*}(v)|_{\bar{g}}^2 &= \bar{g}(\phi_{t*}(v), \frac{d}{dt}\phi_{t*}(v)) \\ &= \bar{g}(\phi_{t*}(v), \nabla_{\phi_{t*}(\partial_t)}\phi_{t*}(v)) \\ &= \bar{g}(\phi_{t*}(v), \nabla_{\phi_{t*}(v)}\phi_{t*}(\partial_t)) \\ &= \bar{g}(\phi_{t*}(v), -\nabla_{\phi_{t*}(v)}V(\tilde{\varphi}(t))). \end{aligned}$$

We deduce that

$$-\|\nabla V(\tilde{\varphi}(t))\|_{C_{\bar{g}}^0} \leq \frac{d}{dt} \ln |\phi_{t*}(v)|_{\bar{g}} \leq \|\nabla V(\tilde{\varphi}(t))\|_{C_{\bar{g}}^0} \quad (5.17)$$

Since $V(\tilde{\varphi}(t))$ converges to zero exponentially in C^{k-1} -norm, then

$$\int_0^t \|\nabla V(\tilde{\varphi}(s))\|_{C_{\bar{g}}^0} ds \leq \int_0^t Ce^{-\frac{\lambda_1 s}{2}} ds = \frac{2C}{\lambda_1} (1 - e^{-\frac{\lambda_1 t}{2}})$$

Note that ϕ_0 is the identity map, which means that $|\phi_{0*}(v)|_{\bar{g}} = |v|_{\bar{g}}$. Integrating the inequalities in (5.17), we have

$$|\phi_{t*}(v)|_{\bar{g}} \leq |\phi_{0*}(v)|_{\bar{g}} e^{\int_0^t \|\nabla V(\tilde{\varphi}(s))\|_{C_{\bar{g}}^0} ds} \leq |v|_{\bar{g}} e^{\frac{2C}{\lambda_1}}$$

and

$$|\phi_{t*}(v)|_{\bar{g}} \geq |\phi_{0*}(v)|_{\bar{g}} e^{-\int_0^t \|\nabla V(\tilde{\varphi}(s))\|_{C_{\bar{g}}^0} ds} \geq |v|_{\bar{g}} e^{-\frac{2C}{\lambda_1}}$$

So the differential ϕ_{t*} of the map ϕ_t is non-degenerate everywhere on M and the norm of ϕ_{t*} is uniformly bounded independently of t . This estimate thus also descends to the limit map ϕ_∞ . The inverse function theorem then implies that ϕ_∞ is a local diffeomorphism. Since $\phi_0 = id$ the identity map, and each ϕ_t is a diffeomorphism which is isotopic to the identity map, we deduce that ϕ_∞ is a surjective local diffeomorphism homotopic to the identity. As M is compact, ϕ_∞ is a covering map. Moreover, ϕ_∞ is homotopic

to the identity so it must have degree one and thus is injective. We deduce that the limit map ϕ_∞ is also a diffeomorphism. Then

$$\begin{aligned} |\varphi_\infty - \phi_\infty^* \bar{\varphi}| &\leq \lim_{t \rightarrow \infty} (|\varphi_\infty - \phi_t^* \tilde{\varphi}(t)| + |\phi_t^*(\tilde{\varphi}(t) - \bar{\varphi})| + |(\phi_t^* - \phi_\infty^*)\bar{\varphi}|) \\ &\leq \lim_{t \rightarrow \infty} (|\varphi_\infty - \varphi(t)| + C|\tilde{\varphi}(t) - \bar{\varphi}| + C|\phi_t^* - \phi_\infty^*|) = 0. \end{aligned}$$

We conclude that the Laplacian flow $\varphi(t)$ converges to $\varphi_\infty = \phi_\infty^* \bar{\varphi}$, which lies in the diffeomorphism orbit of $\bar{\varphi}$. Moreover, the convergence is exponentially in the C^0 -norm.

We next show that $\varphi(t)$ converges to $\varphi_\infty = \phi_\infty^* \bar{\varphi}$ in any C^k -norm. Since $\tilde{\varphi}(t)$ converges to $\bar{\varphi}$ exponentially in C^k -norm ($k \geq 5$), we have

$$|Rm(\tilde{g}(t)) - Rm(\bar{g})|_{\bar{g}} \leq Ce^{-\frac{\lambda_1 t}{2}}, \quad |\tilde{\nabla} \tilde{T}(x, t)|_{\bar{g}} \leq Ce^{-\frac{\lambda_1 t}{2}}.$$

We also note that for t sufficiently large, $\tilde{g}(t)$ is uniformly equivalent to \bar{g} , therefore we have

$$|Rm(\tilde{g}(t))|_{\tilde{g}(t)} + |\tilde{\nabla} \tilde{T}(x, t)|_{\tilde{g}(t)} \leq C,$$

for large t , i.e. for $t \geq t_0$, which gives us

$$|Rm(g(t))|_{g(t)} + |\nabla T(x, t)|_{g(t)} \leq C, \quad (5.18)$$

for $t \geq t_0$ by diffeomorphism invariance. By Shi-type derivative estimates in [23], we get uniform bounds on the covariant derivatives of Rm and ∇T ,

$$|\nabla^k Rm(g(t))|_{g(t)} + |\nabla^{k+1} T(x, t)|_{g(t)} \leq C(k), \quad (5.19)$$

for $t \geq t_0 + 1$ and all $k \geq 1$. The injectivity radius of $g(t)$ also satisfies a uniform lower bound $inj(g(t)) \geq \delta > 0$ since $g(t)$ is uniformly equivalent to \bar{g} (see the proof of [23, Theorem 8.1]). Then from the compactness theorem for G_2 structures in [23], there is a sequence of times $t_i \rightarrow \infty$ such that $(M, \varphi(t_i))$ converges smoothly to $(M_\infty, \hat{\varphi}_\infty)$. Note that, since the metrics $g(t)$ are uniformly equivalent to \bar{g} , the diameters of the manifolds $(M, g(t))$ are uniformly bounded, and thus the compactness theory implies that the limit M_∞ is diffeomorphic to M . We can then pull back $\hat{\varphi}_\infty$ to M and, by abuse of notation, view it as being defined on M . Therefore the closed G_2 structures $\varphi(t_i)$ converge smoothly to a closed G_2 structure $\hat{\varphi}_\infty$ on M as $t_i \rightarrow \infty$. Since we already know that $\varphi(t)$ converges to φ_∞ continuously (i.e. in the C_g^0 -norm) as $t \rightarrow \infty$, the uniqueness of limits implies that the limit G_2 structure $\hat{\varphi}_\infty = \varphi_\infty$.

Moreover, we have this smooth convergence without passing to a subsequence, in the sense that $\varphi(t) \rightarrow \varphi_\infty$ smoothly as $t \rightarrow \infty$. For a contradiction we suppose not, so that there exists an integer k , $\epsilon > 0$ and a sequence of times $t_l \rightarrow \infty$ such that, for all l ,

$$\|\varphi(t_l) - \varphi_\infty\|_{C_g^k} \geq \epsilon.$$

Since $\varphi(t_l)$ satisfies (5.19) and the injectivity radius estimate, we can apply the compactness theorem for G_2 structures in [23] again to obtain a subsequence $t_{l(j)}$ which converges in C_g^k to a limit φ'_∞ on M with

$$\|\varphi'_\infty - \varphi_\infty\|_{C_g^k} \geq \epsilon.$$

Thus $\varphi'_\infty \neq \varphi_\infty$, which contradicts the fact that $\varphi(t_{l(j)})$ converges to φ_∞ continuously.

We conclude that $\varphi(t)$ converges to $\varphi_\infty = \phi_\infty^* \bar{\varphi}$ smoothly as $t \rightarrow \infty$. Since ϕ_0 is the identity map, we have $\varphi_\infty \in \text{Diff}^0(M) \cdot \bar{\varphi}$. This completes the proof of Theorem 1.3.

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