

ON THE MINKOWSKI-TYPE INEQUALITY FOR OUTWARD MINIMIZING HYPERSURFACES IN SCHWARZSCHILD SPACE

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ABSTRACT. Using the weak solution of Inverse mean curvature flow, we prove the sharp Minkowski-type inequality for outward minimizing hypersurfaces in Schwarzschild space.

1. INTRODUCTION

The Schwarzschild space is an n -dimensional ($n \geq 3$) manifold (M^n, g) with boundary ∂M , which is conformal to $\mathbb{R}^n \setminus D_{r_0}$, with the metric

$$g_{ij}(x) = \left(1 + \frac{m}{2}|x|^{2-n}\right)^{\frac{4}{n-2}} \delta_{ij}, \quad x \in \mathbb{R}^n \setminus D_{r_0}, \quad (1.1)$$

where $m > 0$ is a constant, $r_0 = \left(\frac{m}{2}\right)^{\frac{1}{n-2}}$. The coordinate sphere $S_{r_0} = \partial D_{r_0}$ is the horizon of the Schwarzschild space and is outward minimizing. Equivalently, $M = [s_0, \infty) \times \mathbb{S}^{n-1}$ and

$$g = \frac{1}{1 - 2ms^{2-n}} ds^2 + s^2 g_{\mathbb{S}^{n-1}}, \quad (1.2)$$

where s_0 is the unique positive solution of $1 - 2ms_0^{2-n} = 0$ and $g_{\mathbb{S}^{n-1}}$ is the canonical round metric on the unit sphere \mathbb{S}^{n-1} . In this paper, we will denote

$$f(x) = \sqrt{1 - 2ms^{2-n}}, \quad \text{for any } x = (s, \theta) \in M^n, \quad (1.3)$$

which is called the potential function of (M^n, g) . As is well known, the Schwarzschild space is asymptotically flat, and is static in the sense that the potential function f satisfies

$$\nabla^2 f = f \text{Ric}, \quad \Delta f = 0, \quad (1.4)$$

where Ric is the Ricci tensor of (M^n, g) , ∇, ∇^2 and Δ are gradient, Hessian and Laplacian operator with respect to the metric g on M^n . It can be easily checked that the spacetime metric $\hat{g} = -f^2 dt^2 + g$ on $M^n \times \mathbb{R}$ solves the vacuum Einstein equation. In particular, (1.4) implies that (M^n, g) has constant zero scalar curvature R .

Let Ω be a bounded domain with smooth boundary in (M^n, g) . Then there are two cases:

- (i) Ω has only one boundary component $\Sigma = \partial\Omega$ and we say that Σ is null-homologous;
- (ii) Ω has two boundary components $\partial\Omega = \Sigma \cup \partial M$ and we say that Σ is homologous to the horizon ∂M of the Schwarzschild space.

The boundary hypersurface Σ is said to be outward minimizing if whenever E is a domain containing Ω then $|\partial E| \geq |\partial\Omega|$. From the first variational formula for area functional, an outward minimizing hypersurface must be a mean-convex hypersurface.

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The main result of this paper is the following Minkowski-type inequality for outward minimizing hypersurface in Schwarzschild space.

Theorem 1.1. *Let Ω be a bounded domain with smooth and outward minimizing boundary in the Schwarzschild space (M^n, g) . Assume either*

- (1) $n < 8$, or
- (2) $n \geq 8$ and $\Sigma = \partial\Omega \setminus \partial M$ is homologous to the horizon.

Then

$$\frac{1}{(n-1)\omega_{n-1}} \int_{\Sigma} f H d\mu \geq \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} - 2m, \quad (1.5)$$

where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $|\Sigma|$ is the area of Σ with respect to the induced metric from (M^n, g) . Moreover, the equality holds in (1.5) if and only if Σ is a slice $\{s\} \times \mathbb{S}^{n-1}$.

For mean convex and star-shaped hypersurface in Schwarzschild space, the inequality (1.5) was obtained by Brendle-Hung-Wang [3] as the limit case of their inequality in Anti-de Sitter-Schwarzschild space. Note that a star-shaped hypersurface must be homologous to the horizon of the Schwarzschild space. Our result does not require the hypersurface to be star-shaped. The inequality (1.5) is a natural generalization of the classical Minkowski inequality for convex hypersurface Σ in \mathbb{R}^n , which states that

$$\int_{\Sigma} H d\mu \geq (n-1)\omega_{n-1}^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}}. \quad (1.6)$$

The inequality (1.6) was originally proved for convex hypersurfaces using the theory of convex geometry and was proved recently by Guan-Li [7] for mean convex and star-shaped hypersurfaces using the smooth solution of inverse mean curvature flow (IMCF). Huisken recently applied the weak solution of IMCF in [10] to show that the inequality (1.6) also holds for outward minimizing hypersurfaces in \mathbb{R}^n (see [9]). The proof of this result was also given by Freire-Schwartz [6]. By letting $m \rightarrow 0$, the Schwarzschild metric reduces to the Euclidean metric $g = ds^2 + s^2 g_{\mathbb{S}^{n-1}}$ and the potential function f approaches to 1. Thus Theorem 1.1 generalizes the result of Huisken and Freire-Schwartz to that for outward-minimizing hypersurfaces in Schwarzschild space.

To prove Theorem 1.1, we use the standard procedure in proving geometric inequalities using the hypersurface curvature flows (see e.g., [3, 6, 7, 10]). We will employ the weak solution of IMCF, which was developed by Huisken-Ilmanen in [10] and was applied to prove the Riemannian Penrose inequality for asymptotically flat 3-manifold with nonnegative scalar curvature. The weak solution of IMCF has also been applied in many other problems, see for example [1, 2, 6, 12]. In our case, if Σ is homologous to the horizon, then starting from Σ there exists the weak solution of IMCF which is given by the level sets $\Sigma_t = \partial\Omega_t = \partial\{u < t\}$ of a proper locally Lipschitz function $u : \Omega^c \rightarrow \mathbb{R}^+$, where Ω^c denotes the compliments of Ω in M . Each Σ_t is $C^{1,\alpha}$ away from a closed singular set Z of Hausdorff dimension at most $n-8$ and Σ_t will become $C^{1,\alpha}$ close to a large coordinate sphere as $t \rightarrow \infty$. On each Σ_t we define the following quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right), \quad (1.7)$$

where $|\Sigma_t|$ is the area of Σ_t . $Q(t)$ is well-defined because each Σ_t is $C^{1,\alpha}$ with small singular set, the weak mean curvature of Σ_t can be defined as a locally L^1 function using the first variation formula for area. We will prove that $Q(t)$ is monotone non-increasing along the weak solution of IMCF. If $\Sigma = \partial\Omega$ is null-homologous, we first fill-in the region W bounded by the horizon ∂M

to obtain a new manifold \tilde{M} and then run the weak IMCF in \tilde{M} with initial condition Σ . When the flow $\Sigma_t = \partial\Omega_t$ nearly touches the horizon ∂M , we jump to the strictly minimizing hull F of the union $\Omega_t \cup W$. Assume that $n < 8$, we show that

$$|\partial F| \geq |\Sigma_t|, \quad \int_{\Sigma_t} fHd\mu_t \geq \int_{\partial F} fHd\mu,$$

which implies that $Q(t)$ does not increase during the jump. Then we restart the flow from ∂F . The restriction $n < 8$ on the dimension in this case is due to that we need to jump to the strictly minimizing hull F before we restart the flow, and ∂F is only known to be smooth, more precisely $C^{1,1}$ for $n < 8$. In summary, under the assumption of Theorem 1.1 we can prove that the quantity $Q(t)$ is monotone decreasing in time along the weak solution of IMCF.

Once we have the monotonicity of $Q(t)$, the next step is to estimate the limit when $t \rightarrow \infty$. For this, we will use the property that the weak solution becomes $C^{1,\alpha}$ close to a large coordinate sphere as $t \rightarrow \infty$ as shown in [10, §7]. The estimate that we will prove is the following:

$$\lim_{t \rightarrow \infty} Q(t) = (n-1)\omega_{n-1}^{\frac{1}{n-1}}. \quad (1.8)$$

Then the main inequality (1.5) follows immediately from the monotonicity of $Q(t)$ and the estimate (1.8) on its limit. To complete the proof of Theorem 1.1, we need to show the rigidity of the inequality. If the equality holds in (1.5), from the proof of the monotonicity in §4 we know that Σ must be homologous to the horizon and Σ_t is umbilic a.e. for almost all time. This can be used to show that Σ is an Euclidean sphere if it is considered as a hypersurface in $\mathbb{R}^n \setminus D_{r_0}$ with respect to the Euclidean metric. The last step is to show that Σ is a sphere centered at the origin. We will use the property that a hypersurface to be umbilic is invariant under the conformal change of the ambient metric and the totally umbilic of Σ_t for almost all time.

The rest of this paper is organized as follows. In §2, we review some properties of the weak solution of IMCF. For more detail, we refer the readers to Huisken-Ilmanen's original paper [10]. In §3, we show how to derive the monotonicity of $Q(t)$ in the case that the flow is smooth. In §4, we use the approximation argument to show the monotonicity of $Q(t)$ under the weak IMCF. In the last section, we estimate the limit of $Q(t)$ as $t \rightarrow \infty$ and complete the proof of Theorem 1.1.

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2. WEAK SOLUTION OF IMCF

Let (M^n, g) be the Schwarzschild space. The classical solution of IMCF is a smooth family $x : \Sigma \times [0, T) \rightarrow M$ of hypersurfaces $\Sigma_t = x(\Sigma, t)$ satisfying

$$\frac{\partial x}{\partial t} = \frac{1}{H}\nu, \quad x \in \Sigma_t, \quad (2.1)$$

where H, ν are the mean curvature and outward unit normal of Σ_t , respectively. If the initial hypersurface is star-shaped and strictly mean convex, the smooth solution of (2.1) exists for all time $t \in [0, \infty)$, and the flow hypersurfaces Σ_t converge to large coordinate sphere in exponentially fast, see [13, 17]. In general, without some special assumption on the initial hypersurface, the smoothness may not be preserved, the mean curvature may tend to zero at some points and

the singularities develop. See for example the thin torus in Euclidean space ([10, §1]), i.e. the boundary of an ϵ -neighborhood of a large round circle. The mean curvature is positive on this thin torus, so the smooth solution of (2.1) exists for at least a short time. By deriving the upper bound of the mean curvature along the flow, we can see that the torus will steadily fatten up and the mean curvature will become negative in the donut hole in finite time.

In [10], Huisken-Ilmanen used the level-set approach and developed the weak solution of IMCF to overcome this problem. The evolving hypersurfaces are given by the level-sets of a scalar function $u : M^n \rightarrow \mathbb{R}$ via

$$\Sigma_t = \partial\{x \in M : u(x) < t\}.$$

Whenever u is smooth with non vanishing gradient $\nabla u \neq 0$, the flow (2.1) is equivalent to the following degenerate elliptic equation

$$\operatorname{div}_M \left(\frac{\nabla u}{|\nabla u|} \right) = |\nabla u|. \quad (2.2)$$

Using the minimization principle and elliptic regularization, Huisken-Ilmanen proved the existence, uniqueness, compactness and regularity properties of the weak solution of (2.2). The existence result only require mild growth assumption on the underlying manifold, and applies in particular to the Schwarzschild space here. We summaries their results in the following.

Theorem 2.1 ([10]). *Let Ω be a bounded domain with smooth boundary in the Schwarzschild space (M^n, g) with $n < 8$ and $\Sigma = \partial\Omega \setminus \partial M$. In case that Σ is null-homologous, we fill-in the region W bounded by the horizon. Then there exists a proper, locally Lipschitz function $u \geq 0$ on $\Omega^c = M \setminus \Omega$, called the weak solution of IMCF with initial condition Σ , satisfying*

- (a) $u|_{\Sigma} = 0$, $\lim_{x \rightarrow \infty} u = \infty$. For $t > 0$, $\Sigma_t = \partial\{u < t\}$ and $\Sigma'_t = \partial\{u > t\}$ define increasing families of $C^{1,\alpha}$ hypersurfaces.
- (b) The hypersurfaces Σ_t (resp. Σ'_t) minimize (resp. strictly minimize) area among hypersurfaces homologous to Σ_t in the region $\{u \geq t\}$. The hypersurface $\Sigma' = \partial\{u > 0\}$ strictly minimizes area among hypersurfaces homologous to Σ in Ω^c .
- (c) For $t > 0$, we have

$$\Sigma_s \rightarrow \Sigma_t \quad \text{as } s \nearrow t, \quad \Sigma_s \rightarrow \Sigma'_t \quad \text{as } s \searrow t \quad (2.3)$$

locally in $C^{1,\beta}$ in Ω^c , $\beta < \alpha$. The second convergence also holds as $s \searrow 0$.

- (d) For almost all $t > 0$, the weak mean curvature of Σ_t is defined and equals to $|\nabla u|$, which is positive and bounded for almost all $x \in \Sigma_t$.
- (e) For each $t > 0$, $|\Sigma_t| = e^t |\Sigma'|$, and $|\Sigma_t| = e^t |\Sigma|$ if Σ is outward minimizing.

For $n \geq 8$, the regularity and convergence are also true away from a closed singular set Z of dimension at most $n - 8$ and disjoint from $\bar{\Omega}$.

Note that in [8], Heidusch proved the optimal $C^{1,1}$ regularity for the level sets Σ_t and Σ'_t away from the singular set Z . The property (b) says that $\Omega_t = \{u < t\}$ and $\Omega'_t = \operatorname{int}\{u \leq t\}$ are minimizing hull and strictly minimizing hull in $\{u > t\}$. Here we call a set E a minimizing hull in G if E minimizes area on the outside in G , that is, if

$$|\partial^* E \cap K| \leq |\partial^* F \cap K|$$

for any F of locally finite perimeter containing E such that $F \setminus E \subset\subset G$, and any compact set K containing $F \setminus E$. Here $\partial^* F$ denotes the reduced boundary of a set F of locally finite perimeter. E is called a strictly minimizing hull if equality implies that $F \cap G = E \cap G$. Define E' to be the intersection of all strictly minimizing hulls in G that contain E . Up to a set of measure zero, E'

may be realised by a countable intersection, so E' itself is a strictly minimizing hull and open. We call E' the strictly minimizing hull of E in G .

The existence result of weak IMCF in Theorem 2.1 was proved using a minimization principle (see [10, §1]), together with the elliptic regularization. Consider the following perturbed equation

$$\operatorname{div}_M \left(\frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} \right) = \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} \quad (2.4)$$

on a large domain $\Omega_L = \{v < L\}$ defined using a subsolution v of (2.2), with Dirichlet boundary condition $u^\epsilon = 0$ on Σ and $u^\epsilon = L - 2$ on the boundary $\partial\Omega_L \setminus \Sigma$. This equation (2.4) has the geometric interpretation that the downward translating graph

$$\hat{\Sigma}_t^\epsilon := \operatorname{graph} \left(\frac{u^\epsilon(x)}{\epsilon} - \frac{t}{\epsilon} \right)$$

solves the smooth IMCF (2.1) in the manifold $M \times \mathbb{R}$ of one dimension higher. Using the compactness theorem to pass the solutions of (2.4) to limits as $\epsilon_i \rightarrow 0$, we obtain a family of cylinders in $M \times \mathbb{R}$, which sliced by $M \times \{0\}$ gives a family of hypersurfaces weakly solving (2.2). Similar techniques to show existence of weak solutions of geometric flows have been used by various authors, cf. [5, 11, 14, 15, 16].

From the argument in [10, §3], we find that there exists a sequence of smooth function $u_i = u^{\epsilon_i}$ such that $u_i \rightarrow u$ locally uniformly in Ω^c to a function $u \in C^{0,1}(\Omega^c)$. u_i and u are uniformly bounded in $C^{0,1}(\Omega^c)$. For a.e. $t \geq 0$, the hypersurfaces $\hat{\Sigma}_t^i := \hat{\Sigma}_t^{\epsilon_i}$ converges to the cylinder $\hat{\Sigma}_t := \Sigma_t \times \mathbb{R}$ locally in $C^{1,\alpha}$ away from the singular set $Z \times \mathbb{R}$. Moreover, as in [10, §5], the mean curvature $H_{\hat{\Sigma}_t^i}$ of $\hat{\Sigma}_t^i$ converges to the weak mean curvature $H_{\hat{\Sigma}_t}$ of the cylinder $\hat{\Sigma}_t$ locally in L^2 sense for a.e. $t \geq 0$. Precisely,

$$\int_{\hat{\Sigma}_t^i} \phi H_{\hat{\Sigma}_t^i}^2 \rightarrow \int_{\hat{\Sigma}_t} \phi H_{\hat{\Sigma}_t}^2, \quad \text{a.e.}, t \geq 0 \quad (2.5)$$

for any cut-off function $\phi \in C_c^0(\Omega^c \times \mathbb{R})$. The weak second fundamental form $A_{\hat{\Sigma}_t}$ exists on $\hat{\Sigma}_t$ in L^2 and the lower semicontinuity implies

$$\int_{\hat{\Sigma}_t} |A_{\hat{\Sigma}_t}|^2 \leq \liminf_{i \rightarrow \infty} \int_{\hat{\Sigma}_t^i} |A_{\hat{\Sigma}_t^i}|^2 < \infty \quad (2.6)$$

for a.e. $t \geq 0$. Slicing this families $\hat{\Sigma}_t^i$ by $M \times \{0\}$, we obtain $\Sigma_t^i = \hat{\Sigma}_t^i \cap (M \times \{0\})$ and $\Sigma_t = \hat{\Sigma}_t \cap (M \times \{0\})$. Since $\hat{\Sigma}_t^i$ solves the smooth IMCF, its mean curvature in $M \times \mathbb{R}$ is

$$H_{\hat{\Sigma}_t^i} = \operatorname{div}_M \left(\frac{\nabla u^{\epsilon_i}}{\sqrt{|\nabla u^{\epsilon_i}|^2 + \epsilon_i^2}} \right) = \sqrt{|\nabla u^{\epsilon_i}|^2 + \epsilon_i^2}. \quad (2.7)$$

The mean curvature of Σ_t^i considered as a hypersurface in M is

$$\begin{aligned} H_{\Sigma_t^i} &= \operatorname{div}_M \left(\frac{\nabla u^{\epsilon_i}}{|\nabla u^{\epsilon_i}|} \right) \\ &= H_{\hat{\Sigma}_t^i} \frac{\sqrt{|\nabla u^{\epsilon_i}|^2 + \epsilon_i^2}}{|\nabla u^{\epsilon_i}|} + \frac{\nabla u^{\epsilon_i}}{\sqrt{|\nabla u^{\epsilon_i}|^2 + \epsilon_i^2}} \cdot \nabla \left(\frac{\sqrt{|\nabla u^{\epsilon_i}|^2 + \epsilon_i^2}}{|\nabla u^{\epsilon_i}|} \right) \end{aligned}$$

$$= H_{\hat{\Sigma}_t^i} \frac{\sqrt{|\nabla u^{\epsilon_i}|^2 + \epsilon_i^2}}{|\nabla u^{\epsilon_i}|} - \epsilon_i^2 \frac{\nabla u^{\epsilon_i} \cdot \nabla H_{\hat{\Sigma}_t^i}}{|\nabla u^{\epsilon_i}|^3 H_{\hat{\Sigma}_t^i}}. \quad (2.8)$$

Since the limit function u of u^{ϵ_i} has $|\nabla u| > 0$ a.e. on Σ_t , using the weak convergence of $\nabla H_{\hat{\Sigma}_t^i}/H_{\hat{\Sigma}_t^i}$ (as in the proof of Lemma 5.2 in [10]), we have that the second term on the right hand side of (2.8) converges to zero locally in L^2 sense as $\epsilon_i \rightarrow 0$. Thus, the mean curvature $H_{\Sigma_t^i}$ of the sliced hypersurface Σ_t^i converges to the weak mean curvature H_{Σ_t} of Σ_t locally in L^2 sense for a.e. $t \geq 0$.

3. THE SMOOTH CASE

As we mentioned in §1, the key step to prove Theorem 1.1 is to show the monotonicity of $Q(t)$ defined in (1.7) along the weak IMCF. In this section, we firstly show how to derive the monotonicity of $Q(t)$ in the smooth case. Let Σ_t be a smooth solution of the IMCF (2.1). It's well known that the following evolution equations for the area form $d\mu$ and mean curvature H of Σ_t in (M^n, g) hold.

Lemma 3.1.

$$\partial_t d\mu_t = d\mu_t, \quad (3.1)$$

$$\partial_t H = -\Delta_{\Sigma} \frac{1}{H} - \frac{1}{H} (|A|^2 + Ric(\nu, \nu)) \quad (3.2)$$

Employing the above two evolution equations, we can derive the monotonicity of $Q(t)$ in the smooth case.

Theorem 3.2. *Let Σ_t be a smooth solution of the IMCF (2.1). For any $0 < t_1 < t_2 < T$, if Σ_t is homologous to the horizon for all $t \in [t_1, t_2]$, then*

$$Q(t_2) \leq Q(t_1)$$

with equality holds if and only if each Σ_t is totally umbilic for $t \in [t_1, t_2]$. If Σ_t is null-homologous for all $t \in [t_1, t_2]$, then

$$\tilde{Q}(t_2) \leq \tilde{Q}(t_1)$$

with equality holds if and only if each Σ_t is totally umbilic for $t \in [t_1, t_2]$, and $Q(t)$ is strictly decreasing in time $t \in [t_1, t_2]$, where

$$\tilde{Q}(t) := |\Sigma_t|^{-\frac{n-2}{n-1}} \int_{\Sigma_t} f H d\mu_t.$$

Proof. The case that Σ_t is homologous to horizon has been treated in [3, 13]. For convenience of readers, we include the proof here. Using the evolution equations (3.1)–(3.2),

$$\begin{aligned} \frac{d}{dt} \int_{\Sigma_t} f H d\mu_t &= \int_{\Sigma_t} (\partial_t f H + f \partial_t H + f H) d\mu_t \\ &= \int_{\Sigma_t} \left(\langle \nabla f, \nu \rangle - f \Delta_{\Sigma} \frac{1}{H} - \frac{f}{H} (|A|^2 + Ric(\nu, \nu)) + f H \right) d\mu_t \\ &\leq \int_{\Sigma_t} \left(\langle \nabla f, \nu \rangle - \frac{1}{H} (\Delta_{\Sigma} f + f Ric(\nu, \nu)) + \frac{n-2}{n-1} f H \right) d\mu_t, \end{aligned} \quad (3.3)$$

where we used $|A|^2 \geq H^2/(n-1)$ in the last inequality. Combining the identity

$$\Delta_{\Sigma} f = \Delta f - \nabla^2 f(\nu, \nu) - H \nu \cdot \nabla f$$

and the static equation (1.4), we have

$$\Delta_{\Sigma} f + f \text{Ric}(\nu, \nu) = -H\nu \cdot \nabla f. \quad (3.4)$$

Substituting (3.4) into (3.3) yields that

$$\frac{d}{dt} \int_{\Sigma_t} f H d\mu_t \leq \int_{\Sigma_t} \left(\frac{n-2}{n-1} f H + 2 \langle \nabla f, \nu \rangle \right) d\mu_t. \quad (3.5)$$

If equality holds in (3.5), then $|A|^2 = H^2/(n-1)$ and Σ_t is totally umbilical.

If Σ_t is homologous to the horizon for all $t \in [t_1, t_2]$, then denote Ω_t denote the region bounded by Σ_t and the horizon ∂M . Applying the divergence theorem and noting that $\Delta f = 0$ on M , we get

$$\int_{\Sigma_t} \langle \nabla f, \nu \rangle d\mu_t = \int_{\Omega_t} \Delta f + \int_{\partial M} \nabla f \cdot \nu_{\partial M} = m(n-2)\omega_{n-1}$$

which is a constant. Thus we obtain

$$\frac{d}{dt} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right) \leq \frac{n-2}{n-1} \left(\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right). \quad (3.6)$$

If $\Sigma_t = \partial\Omega_t$ is null-homologous for all $t \in [t_1, t_2]$, we have

$$\int_{\Sigma_t} \langle \nabla f, \nu \rangle d\mu_t = \int_{\Omega_t} \Delta f = 0.$$

Then

$$\frac{d}{dt} \int_{\Sigma_t} f H d\mu_t \leq \frac{n-2}{n-1} \int_{\Sigma_t} f H d\mu_t. \quad (3.7)$$

Thus the theorem follows directly from (3.6)–(3.7) and the evolution equation of the area $|\Sigma_t|$

$$\frac{d}{dt} |\Sigma_t| = |\Sigma_t|.$$

□

4. THE MONOTONICITY

Firstly, we prove the following lemma which was inspired by Lemma A.1 of [6].

Lemma 4.1. *Suppose that Ω is a smooth bounded domain in (M^n, g) and $\Sigma = \partial\Omega \setminus \partial M$. Let $u : \Omega^c \rightarrow \mathbb{R}_+$ be a smooth proper function with $u|_{\Sigma} = 0$. Let $t > 0$, $\Omega_t = \{u \leq t\}$ and $\Phi : (0, t) \rightarrow \mathbb{R}_+$ be Lipschitz and compactly supported in $(0, t)$. Then $\varphi = \Phi \circ u : \Omega_t \rightarrow \mathbb{R}_+$ satisfies*

$$- \int_{\Omega_t} f \nabla \varphi \cdot \nu H dv_g = \int_{\Omega_t} \varphi (2 \nabla f \cdot \nu H + f H^2 - f |A|^2) dv_g, \quad (4.1)$$

where ν, H, A denote the unit outward normal, mean curvature and second fundamental form of the level sets of u , ∇ be the gradient operator on (M^n, g) and $\nabla \varphi \cdot \nu = g(\nabla \varphi, \nu)$.

Proof. The Sard's theorem implies that the level set $\Sigma_s = \{x \in \Omega^c : u(x) = s\}$ is regular ($\nabla u \neq 0$ on Σ_s) for a.e. $s > 0$. Let $U \subset \Omega^c$ be the open subset where $\nabla u \neq 0$. For any regular level set Σ_s with outward unit normal ν , in $\Sigma_s \cap U$ the variation vector field along Σ_s is $\nabla u / |\nabla u|^2$ and $\nu = \nabla u / |\nabla u|$. By the second variation formula for area, we have

$$- \frac{1}{|\nabla u|} \nu \cdot \nabla H = \Delta_{\Sigma_s} |\nabla u|^{-1} + \frac{1}{|\nabla u|} (|A|^2 + \text{Ric}(\nu, \nu)) \quad (4.2)$$

in $\Sigma_s \cap U$, where Δ_{Σ_s} denotes the Laplacian operator with respect to the induced metric on Σ_s . We multiply (4.2) by f and integrate over Σ_s .

$$\begin{aligned} - \int_{\Sigma_s} \frac{f}{|\nabla u|} \nu \cdot \nabla H d\mu_s &= \int_{\Sigma_s} f \Delta_{\Sigma_s} |\nabla u|^{-1} + \int_{\Sigma_s} \frac{f}{|\nabla u|} (|A|^2 + \text{Ric}(\nu, \nu)) d\mu_s \\ &= \int_{\Sigma_s} \frac{1}{|\nabla u|} (\Delta_{\Sigma_s} f + f|A|^2 + f \text{Ric}(\nu, \nu)) d\mu_s, \end{aligned}$$

where we used the divergence theorem in the second equality. Applying the identity (3.4), we obtain

$$- \int_{\Sigma_s} \frac{f}{|\nabla u|} \nu \cdot \nabla H d\mu_s = \int_{\Sigma_s} \frac{1}{|\nabla u|} (-H\nu \cdot \nabla f + f|A|^2) d\mu_s. \quad (4.3)$$

As Σ_s is regular for a.e. $s > 0$, the coarea formula and (4.3) imply that

$$\begin{aligned} \int_{\Omega_t} f \varphi \nu \cdot \nabla H dv_g &= \int_0^t \Phi(s) \int_{\Sigma_s} \frac{f}{|\nabla u|} \nu \cdot \nabla H d\mu_s ds \\ &= \int_0^t \Phi(s) \int_{\Sigma_s} \frac{1}{|\nabla u|} (H\nu \cdot \nabla f - f|A|^2) d\mu_s ds \\ &= \int_{\Omega_t} \varphi (H\nu \cdot \nabla f - f|A|^2) dv_g. \end{aligned} \quad (4.4)$$

We firstly assume that $\Phi \in C^1$. Then in the open subset $U = \{x \in \Omega^c : |\nabla u| \neq 0\}$, we have

$$\begin{aligned} \text{div}(f\varphi H\nu) &= \varphi \nabla f \cdot \nu H + f \nabla \varphi \cdot \nu H + f \varphi \nu \cdot \nabla H + f \varphi H \text{div } \nu \\ &= \varphi \nabla f \cdot \nu H + f \Phi' \circ u |\nabla u| H + f \varphi \nu \cdot \nabla H + f \varphi H \text{div } \nu, \end{aligned} \quad (4.5)$$

where div is the divergence operator on (M^n, g) . Since φ is compactly supported in Ω_t , integrating (4.5) yields that

$$\begin{aligned} - \int_{\Omega_t} f \nabla \varphi \cdot \nu H dv_g &= - \int_{\Omega_t} \text{div}(f\varphi H\nu) dv_g \\ &\quad + \int_{\Omega_t} (\varphi \nabla f \cdot \nu H + f \varphi \nu \cdot \nabla H + f \varphi H \text{div } \nu) dv_g \\ &= \int_{\Omega_t} \varphi (2H \nabla f \cdot \nu - f|A|^2) dv_g + \int_{\Omega_t} f \varphi H \text{div } \nu dv_g, \end{aligned} \quad (4.6)$$

where we used the divergence theorem and (4.4).

We now deal with the last term in (4.6). Since Σ_s is regular for a.e. $s > 0$, the co-area formula and the first variation formula for area imply that

$$\begin{aligned} \int_{\Omega_t} f \varphi H \text{div}(\nu) dv_g &= \int_0^t \int_{\Sigma_s} \frac{f \varphi H}{|\nabla u|} \text{div}(\nu) d\mu_s ds \\ &= \int_0^t \int_{\Sigma_s} \frac{f \varphi H}{|\nabla u|} \text{div}_{\Sigma_s} \nu d\mu_s ds \\ &= \int_0^t \int_{\Sigma_s} \text{div}_{\Sigma_s} \left(\frac{f \varphi H}{|\nabla u|} \nu \right) d\mu_s ds \\ &= \int_0^t \int_{\Sigma_s} \frac{f \varphi H^2}{|\nabla u|} d\mu_s ds \end{aligned}$$

$$= \int_{\Omega_t} f \varphi H^2 dv_g, \quad (4.7)$$

where in the second equality we used the fact that $\operatorname{div}(\nu) = \operatorname{div}_{\Sigma_s} \nu$ on $\Sigma_s \cap U$. Substituting (4.7) into (4.6) yields that

$$- \int_{\Omega_t} f \nabla \varphi \cdot \nu H dv_g = \int_{\Omega_t} \varphi (2 \nabla f \cdot \nu H + f H^2 - f |A|^2) dv_g \quad (4.8)$$

for $\Phi \in C^1$. Since Lipschitz function can be approximated by C^1 function up to a set of measure zero (see [18, p.32]), we conclude that (4.8) also holds for Lipschitz function Φ by approximation. \square

4.1. The case that Σ is homologous to horizon.

Lemma 4.2. *Let Ω be a smooth bounded domain in the Schwarzschild space (M^n, g) . Suppose that the boundary $\partial\Omega = \Sigma \cup \partial M$ and Σ is outward minimizing. Let $\{\Sigma_t\}$ be the weak solution of IMCF in $\Omega^c = M \setminus \Omega$ with initial data Σ . Then for all $0 < \bar{t} < t$,*

$$\int_{\Sigma_t} f H d\mu_t \leq \int_{\Sigma_{\bar{t}}} f H d\mu + \frac{n-2}{n-1} \int_{\bar{t}}^t \left(\int_{\Sigma_s} f H d\mu_s + 2(n-1)m\omega_{n-1} \right) ds \quad (4.9)$$

Proof. As in the discussion in §2, the weak solution of IMCF $u \in C_{\text{loc}}^{0,1}(\Omega^c)$ can be approximated by smooth proper functions u_i locally uniformly in Ω^c , with $C^{1,\alpha}$ convergence of the level sets Σ_s^i away from the singular set Z and L^2 convergence of the weak mean curvature $H_i := H_{\Sigma_s^i}$ of level sets for a.e. $s > 0$. Moreover, we can show that H_i converges to the mean curvature H of the weak solution Σ_s of IMCF in locally L^2 sense in any domain Ω_t . In fact, by the coarea formula and (2.7)–(2.8),

$$\begin{aligned} \int_{\Omega_t^i} \phi H_i &= \int_0^t \int_{\Sigma_s^i} \frac{\phi H_i}{|\nabla u_i|} d\mu_s^i ds \\ &= \int_0^t \int_{\Sigma_s^i} \phi \left(\frac{|\nabla u_i|^2 + \epsilon_i^2}{|\nabla u_i|^2} - \epsilon_i^2 \frac{\nabla u_i \cdot \nabla H_{\Sigma_s^i}}{|\nabla u_i|^4 H_{\Sigma_s^i}} \right) d\mu_s^i ds, \end{aligned}$$

where $\phi \in C_c^0(\Omega_t)$, $\Omega_t^i = \{x \in \Omega^c : u_i(x) < t\}$ and $\Omega_t = \{x \in \Omega^c : u(x) < t\}$. By the fact that $|\nabla u| > 0$ a.e. on Σ_s and the weak convergence of $\nabla H_{\Sigma_s^i}/H_{\Sigma_s^i}$, we have that $\int_{\Omega_t^i} \phi H_i \rightarrow \int_{\Omega_t} \phi H$ as $i \rightarrow \infty$. Similarly we have the convergence of $\int_{\Omega_t^i} \phi H_i^2 \rightarrow \int_{\Omega_t} \phi H^2$.

For any nonnegative Lipschitz function $\Phi \in \operatorname{Lip}(0, t)$ with compact support in $(0, t)$ and $\varphi_i = \Phi \circ u_i$, by (4.1) we have

$$\begin{aligned} - \int_{\Omega_t} f \nabla \varphi_i \cdot \nu_i H_i dv_g &= \int_{\Omega_t} \varphi_i (2 \nabla f \cdot \nu_i H_i + f H_i^2 - f |A_i|^2) dv_g \\ &\leq \int_{\Omega_t} \varphi_i \left(2 \nabla f \cdot \nu_i H_i + \frac{n-2}{n-1} f H_i^2 \right) dv_g, \end{aligned}$$

where we used $|A_i|^2 \geq H_i^2/(n-1)$ in the last inequality. Taking the limit of $i \rightarrow \infty$, and using the convergence of u_i and H_i , we obtain that

$$- \int_{\Omega_t} f \nabla \varphi \cdot \nu H dv_g \leq \int_{\Omega_t} \varphi \left(2 \nabla f \cdot \nu H + \frac{n-2}{n-1} f H^2 \right) dv_g \quad (4.10)$$

As u is the weak solution of IMCF, we have $H = |\nabla u|$ a.e. in Ω_t . Also note that by Rademacher's theorem, the Lipschitz function Φ is differentiable a.e. in $(0, t)$. From the coarea formula and (4.10), we have

$$\begin{aligned}
-\int_0^t \Phi'(s) \int_{\Sigma_s} f H d\mu_s ds &= -\int_{\Omega_t} \Phi'(s) \nabla u \cdot \nu f H dv_g = -\int_{\Omega_t} f \nabla \varphi \cdot \nu H dv_g \\
&\leq \int_{\Omega_t} \varphi \left(2\nabla f \cdot \nu H + \frac{n-2}{n-1} f H^2 \right) dv_g \\
&= \int_{\Omega_t} \varphi \left(2\nabla f \cdot \nu + \frac{n-2}{n-1} f H \right) |\nabla u| dv_g \\
&= \int_0^t \Phi(s) \int_{\Sigma_s} \left(2\nabla f \cdot \nu + \frac{n-2}{n-1} f H \right) d\mu_s ds. \tag{4.11}
\end{aligned}$$

For any $0 < \bar{t} < t$ and $0 < \delta < (t - \bar{t})/2$, define Φ by

$$\Phi(s) = \begin{cases} 0, & \text{on } [0, \bar{t}] \\ (s - \bar{t})/\delta, & \text{on } [\bar{t}, \bar{t} + \delta] \\ 1, & \text{on } [\bar{t} + \delta, t - \delta] \\ (t - s)/\delta, & \text{on } [t - \delta, t] \end{cases}$$

The left hand side of (4.11) is equal to

$$\frac{1}{\delta} \int_{t-\delta}^t \int_{\Sigma_s} f H d\mu_s ds - \frac{1}{\delta} \int_{\bar{t}}^{\bar{t}+\delta} \int_{\Sigma_s} f H d\mu_s ds$$

Since for a.e. $s > 0$, the level set Σ_s^i of u_i converges to Σ_s in $C^{1,\alpha}$ away from the singular set Z of Hausdorff dimension at most $n - 8$ with L^2 convergence of the weak mean curvature, taking the limits $\delta \rightarrow 0$ in (4.11), we find that for a.e. $0 < \bar{t} < t$

$$\int_{\Sigma_t} f H d\mu_t - \int_{\Sigma_{\bar{t}}} f H d\mu_{\bar{t}} \leq \int_{\bar{t}}^t \int_{\Sigma_s} \left(2\nabla f \cdot \nu + \frac{n-2}{n-1} f H \right) d\mu_s ds. \tag{4.12}$$

To show (4.12) holds for all pair of $0 < \bar{t} < t$, we use the $C^{1,\beta}$ convergence (2.3) and the weak convergence of mean curvature. For any $t > 0$, we can find a sequence of time $t_i \nearrow t$ such that $0 < \bar{t} < t_i$ satisfies (4.12), then $\Sigma_{t_i} \rightarrow \Sigma_t$ in $C^{1,\beta}$ away from the singular set Z as $i \rightarrow \infty$ by (2.3). As the weak mean curvature of Σ_{t_i} equals to $|\nabla u|$ a.e. and is uniformly bounded for a.e. $x \in \Sigma_{t_i}$, it follows from the Riesz Representation theorem that (see (1.13) in [10])

$$\int_{\Sigma_{t_i}} H_{\Sigma_{t_i}} \nu_{\Sigma_{t_i}} \cdot X \rightarrow \int_{\Sigma_t} H_{\Sigma_t} \nu_{\Sigma_t} \cdot X, \quad X \in C_c^0(TM). \tag{4.13}$$

Then

$$\int_{\Sigma_t} f H d\mu_t = \lim_{i \rightarrow \infty} \int_{\Sigma_{t_i}} f H_{t_i} d\mu_{t_i} \tag{4.14}$$

and (4.12) holds for all $t > 0$ and a.e. $\bar{t} > 0$ with $\bar{t} < t$. Similarly for any \bar{t} with $0 < \bar{t} < t$, we can find a sequence of time $\bar{t}_i \searrow \bar{t}$ such that $0 < \bar{t}_i < t$ satisfies (4.12). By the convergence (2.3) and (4.13), we have

$$\int_{\Sigma_{\bar{t}}} f H d\mu_{\bar{t}} = \lim_{i \rightarrow \infty} \int_{\Sigma_{\bar{t}_i}} f H d\mu_{\bar{t}_i}. \tag{4.15}$$

Recall that Heidusch [8] proved the optimal local $C_{\text{loc}}^{1,1}$ regularity for the level sets $\Sigma_{\bar{t}}$ and $\Sigma'_{\bar{t}}$ away from the singular set Z . By [10, (1.15)] the weak mean curvature H of $\Sigma_{\bar{t}}$ and $\Sigma'_{\bar{t}}$ satisfy

$$H = 0 \text{ on } \Sigma'_{\bar{t}} \setminus (\Sigma_{\bar{t}} \cup Z), \quad H_{\Sigma'_{\bar{t}}} = H_{\Sigma_{\bar{t}}} \geq 0, \text{ a.e. on } \Sigma'_{\bar{t}} \cap \Sigma_{\bar{t}}.$$

As the weak mean curvature H is nonnegative on $\Sigma_{\bar{t}}$, we deduce that

$$\int_{\Sigma'_{\bar{t}}} f H d\mu_{\bar{t}} \leq \int_{\Sigma_{\bar{t}}} f H d\mu_{\bar{t}}. \quad (4.16)$$

Thus by (4.14)–(4.16), we conclude that (4.12) holds for all pair of $0 < \bar{t} < t$. Since Σ is assumed to be outward minimizing, (4.12) is also true for $\bar{t} = 0$.

Finally, for the first integral on the right hand side of (4.12), using the divergence theorem and noting that $\Delta f = 0$ on M , we get

$$\int_{\Sigma_s} \nabla f \cdot \nu = \int_{\Omega_s \cup \Omega} \Delta f + \int_{\partial M} \nabla f \cdot \nu_{\partial M} = m(n-2)\omega_{n-1}, \quad (4.17)$$

by first computing on Σ_s^i and then passing to limits. Inserting (4.17) into (4.12), we obtain the inequality (4.9) for all pair of $0 < \bar{t} < t$. \square

Proposition 4.3. *Under the assumption of Lemma 4.2, the quantity $Q(t)$ is monotone non-increasing for all t . Moreover, if $Q(t) = Q(\bar{t})$ for some pair $0 < \bar{t} < t$, we have that Σ_s is umbilic a.e. for a.e. $s \in [\bar{t}, t]$.*

Proof. By Gronwall's lemma, (4.9) implies that

$$\int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \leq \left(\int_{\Sigma_{\bar{t}}} f H d\mu + 2(n-1)m\omega_{n-1} \right) e^{\frac{n-2}{n-1}(t-\bar{t})}$$

for all $0 < \bar{t} < t$. Since Σ is outward minimizing, Theorem 2.1 implies that $|\Sigma_t| = e^t |\Sigma|$ for all $t \geq 0$. Then the quantity $Q(t)$ is monotone non-increasing for all $t \geq 0$. If $Q(t) = Q(\bar{t})$ for some pair $0 < \bar{t} < t$, from the proof of Lemma 4.2, we have that $H^2 = (n-1)|A|^2$ a.e. on Σ_s for a.e. $s \in [\bar{t}, t]$. \square

4.2. The case that Σ is null homologous. Now we consider the case that the bounded domain Ω has boundary $\partial\Omega = \Sigma$, which is null-homologous and outward minimizing. By the argument in [10, §6], we fill-in the region W bounded by the horizon ∂M to obtain a new space \tilde{M} , and then run the weak IMCF in \tilde{M} with initial condition Σ , except that when the flow Σ_t is nearly entering the filled-in region W , we jump to a strictly minimizing hull F enclosing $\Omega_t \cup W$. Then we restart the flow from ∂F .

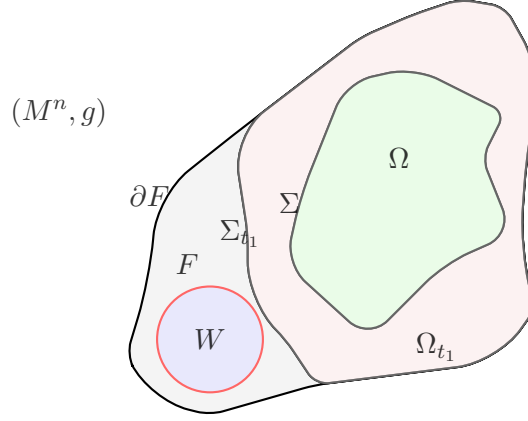


Figure 1: F is strictly minimizing hull of Ω_{t_1} and W

Suppose that t_1 is the jump time. Then before t_1 , each Σ_t is null homologous. Using the divergence theorem as in (4.17), we have that $\int_{\Sigma_t} \nabla f \cdot \nu = 0$ if $t \leq t_1$. The similar argument as in Lemma 4.2 and Proposition (4.5) implies that

$$|\Sigma_t|^{-\frac{n-2}{n-1}} \int_{\Sigma_t} f H d\mu_t$$

is monotone non-increasing in time t if $t \leq t_1$. As $|\Sigma_t|$ is increasing and the mass $m > 0$, we have that $Q(t)$ is strictly decreasing for $t \leq t_1$.

Lemma 4.4. ∂F is $C^{1,\alpha}$ away from a singular set of Hausdorff dimension at most $n - 8$.

Proof. If $n < 8$, then $\Sigma_{t_1} = \partial\Omega_{t_1}$ is $C^{1,\alpha}$ and ∂M is smooth, which combined with the Regularity Theorem 1.3 (ii) of [10] implies that ∂F is $C^{1,\alpha}$. If $n \geq 8$, then Σ_{t_1} has singular set of Hausdorff dimension at most $n - 8$. By the variational formulation of the weak solution of IMCF described in [10, §1], $\Omega_{t_1} = \{u < t_1\}$ minimizes J_u in Ω^c among sets of locally finite perimeter F with $F\Delta E \subset\subset \Omega^c$, where

$$J_u(F) = J_u^K(F) = |\partial^* F \cap K| - \int_{F \cap K} |\nabla u|.$$

and K is any compact set containing $F\Delta E$. Since $|\nabla u|$ is bounded above locally uniformly by Theorem 3.1 of [10], the obstacle $\Omega_{t_1} \cup W$ satisfies the assumption of the main theorem in [4] (see also Proposition 2 of [19]). Therefore, as the strictly minimizing hull of $\Omega_{t_1} \cup W$, F has boundary ∂F which is $C^{1,\alpha}$ away from a singular set of Hausdorff dimension at most $n - 8$. \square

Since Σ_{t_1} is outward minimizing and $F \supset \Omega_{t_1}$, we have

$$|\partial F| \geq |\Sigma_{t_1}|. \quad (4.18)$$

Now we assume that $n < 8$. As Σ_{t_1} is $C^{1,\alpha}$ with nonnegative bounded weak mean curvature, the standard Calderon-Zygmund estimate implies that Σ_{t_1} is of class $W^{2,p}$ for all $1 \leq p < \infty$. We can choose a sequences of sets E_i containing Ω_{t_1} by mollification such that ∂E_i is smooth and converges to $\Sigma_{t_1} = \partial\Omega_{t_1}$ in $C^{1,\alpha} \cap W^{2,p}$. The Regularity Theorem 1.3 and (1.15) of [10] imply that $\partial(E_i \cup W)'$ is $C^{1,1}$ and $H = 0$ on $\partial(E_i \cup W)' \setminus \partial(E_i \cup W)$. Thus

$$\int_{\partial(E_i \cup W)'} f H = \int_{\partial(E_i \cup W)' \cap \partial E_i} f H = \int_{\partial E_i} f H - \int_{\partial E_i \setminus \partial(E_i \cup W)'} f H$$

It can be seen that $(E_i \cup W)' \rightarrow F$ and $\partial(E_i \cup W)' \rightarrow \partial F$ in $C^{1,\alpha}$. Passing to limits and using the nonnegativity of the weak mean curvature on Σ_{t_1} , we obtain

$$\int_{\partial F} fH \leq \int_{\Sigma_{t_1}} fH. \quad (4.19)$$

The estimates (4.18)–(4.19) imply that the quantity $Q(t)$ defined in (1.7) does not increase during the jump. Similar as in [10, §6], we can show that F is a suitable initial condition to restart the flow. We approximate Σ_{t_1} in C^1 by a sequence of smooth hypersurfaces ∂U_i with uniformly bounded mean curvature and $U_i \supset \Omega_{t_1}$, using Lemma 6.2 of [10]. Then by Regularity Theorem 1.3 and (1.15) of [10], $\partial(U_i \cup W)'$ is $C^{1,1}$ and has uniformly bounded mean curvature as well. It can be checked that $\partial(U_i \cup W)'$ converges to ∂F in C^1 . Then slightly mollifying $\partial(U_i \cup W)'$ shows that ∂F is approximated in C^1 by smooth hypersurfaces ∂F^i with uniformly bounded mean curvature. The Existence Theorem 3.1 of [10] gives a solution u^i of weak IMCF $F_t^i, t > 0$ with initial condition F^i , and uniform bounds on the gradient $|\nabla u^i|$. Passing to the limits and applying the Compactness Theorem 2.1 of [10], we obtain the solution u of the weak IMCF with initial condition F . The proof of Proposition 4.5 and the argument in [10, p.407] imply that $Q(t)$ is monotone non-increasing in t for $t \geq 0$ along the weak IMCF F_t with the initial condition F . Thus we conclude that $Q(t)$ is monotone non-increasing for all time t .

Proposition 4.5. *Let $n < 8$ and Ω be a bounded domain with smooth boundary $\partial\Omega = \Sigma$ in the Schwarzschild space (M^n, g) . Assume that Σ is outward minimizing. Then $Q(t)$ is monotone non-increasing for all time along the weak IMCF.*

5. PROOF OF THE MAIN THEOREM

In §4, we proved that the quantity $Q(t)$ is monotone non-increasing along the weak IMCF. In this section, we first estimate the limit of $Q(t)$ as $t \rightarrow \infty$.

Proposition 5.1. *We have*

$$\lim_{t \rightarrow \infty} Q(t) = (n-1)\omega_{n-1}^{\frac{1}{n-1}}. \quad (5.1)$$

Proof. Denote by $U = \mathbb{R}^n \setminus D_{r_0}$ the asymptotic flat end of M . The Schwarzschild metric on U is

$$g = \left(1 + \frac{m}{2}r^{2-n}\right)^{\frac{4}{n-2}} (dr^2 + r^2 g_{\mathbb{S}^{n-1}}),$$

For any $\lambda > 0$, define the blow down object by

$$\Sigma_t^\lambda := \lambda \Sigma_t = \{\lambda x : x \in \Sigma_t\}, \quad g^\lambda(x) := \lambda^2 g(x/\lambda).$$

Let $r(t)$ be such that $|\Sigma_t| = \omega_{n-1} r(t)^{n-1}$. Then $|\Sigma_t^{1/r(t)}|_{g^{1/r(t)}} = \omega_{n-1}$ and the blow down Lemma 7.1 of [10] implies that

$$\Sigma_t^{1/r(t)} \rightarrow \partial D_1 \quad (5.2)$$

in $C^{1,\alpha}$ as $t \rightarrow \infty$. As in the proof of Lemma 7.1 of [10], there exist constants $C, R_0 > 0$ depending only on the dimension n such that

$$|\nabla u(x)| \leq \frac{C}{|x|}, \quad \text{for all } |x| \geq R_0.$$

By the property (d) of the weak solution of IMCF, we have

$$|H| = |\nabla u| \leq \frac{C}{|x|} \leq \frac{C}{r(t)}, \quad \text{a.e., on } \Sigma_t \quad (5.3)$$

for a.e. sufficiently large t , where we have used (5.2) to relate $|x|$ to $r(t)$. The mean curvature of $\Sigma_t^{1/r(t)}$ with respect to the metric $g^{1/r(t)}$ satisfies

$$H^{1/r(t)}(x) = r(t)H(r(t)x) \leq C, \quad \text{a.e. } x \in \Sigma_t^{1/r(t)}. \quad (5.4)$$

for a.e. sufficiently large t . Write $\Sigma_t^{1/r(t)}$ as graphs of $C^{1,\alpha}$ functions over ∂D_1 . By (5.2) and (5.4), for any sequence of time $t_i \rightarrow \infty$ such that (5.4) holds for time t_i , we have the weak convergence of the mean curvature

$$\int_{\Sigma_{t_i}^{1/r(t_i)}} H_{\Sigma_{t_i}^{1/r(t_i)}} \nu_{\Sigma_{t_i}^{1/r(t_i)}} \cdot X \rightarrow \int_{\partial D_1} H_{\partial D_1} \nu_{\partial D_1} \cdot X, \quad X \in C_c^0(TM), \quad (5.5)$$

Recall that

$$f(x) = \sqrt{1 - 2ms(x)^{2-n}} = 1 - m|x|^{2-n} + O(|x|^{4-2n}), \quad (5.6)$$

where

$$s(x) = |x| \left(1 + \frac{m}{2}|x|^{2-n}\right)^{\frac{2}{n-2}}, \quad \forall x \in U.$$

Then by (5.2), and (5.4) – (5.6), we have that

$$\begin{aligned} \lim_{t_i \rightarrow \infty} |\Sigma_{t_i}|^{-\frac{n-2}{n-1}} \int_{\Sigma_{t_i}} f H d\mu_{t_i} &= \omega_{n-1}^{-\frac{n-2}{n-1}} \lim_{t_i \rightarrow \infty} r(t_i)^{-(n-2)} \int_{\Sigma_{t_i}} f H d\mu_{t_i} \\ &= \omega_{n-1}^{-\frac{n-2}{n-1}} \lim_{t_i \rightarrow \infty} \int_{\Sigma_{t_i}^{1/r(t_i)}} f(r(t_i)x) H^{1/r(t_i)}(x) d\mu_{\Sigma_{t_i}^{1/r(t_i)}} \\ &= (n-1) \omega_{n-1}^{\frac{1}{n-1}}. \end{aligned}$$

Observe that $2(n-1)m\omega_{n-1}$ is a fixed constant and $|\Sigma_{t_i}|$ goes to infinity as $t_i \rightarrow \infty$. Therefore

$$\lim_{t_i \rightarrow \infty} Q(t_i) = (n-1) \omega_{n-1}^{\frac{1}{n-1}},$$

which combined with the monotonicity of $Q(t)$ yields the estimate (5.1). \square

We now complete the proof of our main theorem.

Proof of Theorem 1.1. Proposition 5.1 together with the monotonicity of $Q(t)$ yields the inequality (1.5) in Theorem 1.1 immediately. To complete the proof of Theorem 1.1, it remains to prove the rigidity of the inequality (1.5).

If equality holds in (1.5) for Σ , then $Q(0) = Q(t)$ for all $t > 0$. Then the initial hypersurface Σ must be homologous to the horizon, because if not, $Q(t)$ should be strictly decreasing during the jump as described in §4.2. From the proof of Lemma 4.2, the fact that $Q(0) = Q(t)$ for all $t > 0$ also implies $H^2 = (n-1)|A|^2$ a.e. on Σ_t for almost all time $t \geq 0$. Since Σ is smooth and outward minimizing, Theorem 2.1 implies that $\Sigma_t \rightarrow \Sigma$ locally in $C^{1,\beta}$ as $t \rightarrow 0+$. We can choose a sequence of time $t_i \rightarrow 0+$ with $\int_{\Sigma_{t_i}} |\mathring{A}|^2 = 0$ and Σ_{t_i} converges to Σ locally in $C^{1,\beta}$ as $t_i \rightarrow 0+$. The lower semicontinuity implies

$$\int_{\Sigma} |\mathring{A}|^2 \leq \liminf_{t_i \rightarrow 0+} \int_{\Sigma_{t_i}} |\mathring{A}|^2 = 0.$$

and then Σ is totally umbilic in the Schwarzschild space $(M^n, g) = (\mathbb{R}^n \setminus D_{r_0}, g_{ij})$. Denote the Schwarzschild metric $g = e^{2\psi} \delta_{ij}$ on $\mathbb{R}^n \setminus D_{r_0}$, where

$$\psi = \frac{2}{n-2} \ln \left(1 + \frac{m}{2} r^{2-n}\right).$$

Let $\bar{\nu}$, \bar{H} and \bar{h}_i^j be the unit outward normal, mean curvature and shape operator of Σ with respect to $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$. Then they satisfies the following transformation formula under the conformal change of the ambient metric:

$$e^\psi H = \bar{H} + d\psi \cdot \bar{\nu}, \quad \nu = e^{-\psi} \bar{\nu}, \quad (5.7)$$

$$\mathring{h}_i^j = h_i^j - \frac{H}{n-1} \delta_i^j = e^{-\psi} \left(\bar{h}_i^j - \frac{\bar{H}}{n-1} \delta_i^j \right) = e^{-\psi} \mathring{\bar{h}}_i^j, \quad (5.8)$$

where \mathring{h}_i^j and $\mathring{\bar{h}}_i^j$ are the trace-less second fundamental forms of Σ in $\mathbb{R}^n \setminus D_{r_0}$ with respect to the metrics δ_{ij} and g_{ij} respectively. Equation (5.8) says that the totally umbilicity of a hypersurface is invariant under the conformal change of the ambient metric. Then Σ is totally umbilic in $\mathbb{R}^n \setminus D_{r_0}$ with respect to the Euclidean metric δ_{ij} and therefore is a sphere in $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$.

We next show that Σ is a sphere centered at the origin in $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$, and is a slice $\{s\} \times \mathbb{S}^{n-1}$ if considered as a hypersurface in the Schwarzschild space. Suppose that the radius of the sphere Σ is r . Then the mean curvature of Σ in $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$ is $\bar{H} = (n-1)/r$ and

$$d\psi \cdot \bar{\nu} = -\frac{mr^{1-n}}{1 + \frac{m}{2}r^{2-n}} \partial_r \cdot \bar{\nu} \geq -\frac{mr^{1-n}}{1 + \frac{m}{2}r^{2-n}}. \quad (5.9)$$

This implies that the mean curvature of Σ in the $(\mathbb{R}^n \setminus D_{r_0}, g_{ij})$ satisfies $H = e^{-\psi}(\bar{H} + d\psi \cdot \bar{\nu}) > 0$. Since Σ is strictly mean convex, starting from Σ there exists a unique smooth solution to the IMCF (2.1) in Schwarzschild space, which coincides with the weak solution for a short time $t \in [0, \delta)$ by the Smooth Start Lemma 2.4 of [10]. Arguing similarly as before, each Σ_t , $t \in [0, \delta)$, is a sphere in $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$. By the conformal transformation formulas (5.7), Σ_t solves the following corresponding flow in Euclidean space $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$

$$\frac{\partial}{\partial t} X(x, t) = \frac{1}{\bar{H} + d\psi \cdot \bar{\nu}} \bar{\nu}(x, t), \quad t \in [0, \delta). \quad (5.10)$$

Under the flow (5.10), the shape operator \bar{h}_i^j of Σ_t in $(\mathbb{R}^n \setminus D_{r_0}, \delta_{ij})$ evolves by

$$\begin{aligned} \frac{\partial}{\partial t} \bar{h}_i^j &= -\nabla^j \nabla_i \left(\frac{1}{\bar{H} + d\psi \cdot \bar{\nu}} \right) - \frac{\bar{h}_i^k \bar{h}_k^j}{\bar{H} + d\psi \cdot \bar{\nu}} \\ &= \frac{\nabla^j \nabla_i (d\psi \cdot \bar{\nu})}{(\bar{H} + d\psi \cdot \bar{\nu})^2} - \frac{2\nabla^j (d\psi \cdot \bar{\nu}) \nabla_i (d\psi \cdot \bar{\nu})}{(\bar{H} + d\psi \cdot \bar{\nu})^3} - \frac{\bar{H}^2 \delta_i^j}{(n-1)^2 (\bar{H} + d\psi \cdot \bar{\nu})} \end{aligned}$$

As in (5.9),

$$d\psi \cdot \bar{\nu} = -\frac{mr^{1-n}}{1 + \frac{m}{2}r^{2-n}} \partial_r \cdot \bar{\nu} = v(|X|^2) X \cdot \bar{\nu},$$

where $|X|^2 = r^2$, $X \cdot \bar{\nu} = r \partial_r \cdot \bar{\nu}$ and $v(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function given by

$$v(x) := -\frac{mx^{-n/2}}{1 + \frac{m}{2}x^{1-\frac{n}{2}}}.$$

Then

$$\nabla_i (d\psi \cdot \bar{\nu}) = \left(2v' \langle X, \bar{\nu} \rangle + \frac{\bar{H}}{n-1} v \right) \langle X, e_i \rangle$$

and

$$\begin{aligned} \nabla^j \nabla_i (d\psi \cdot \bar{\nu}) &= 4 \left(v'' \langle X, \bar{\nu} \rangle + \frac{\bar{H}}{n-1} v' \right) \langle X, e_i \rangle \langle X, e_j \rangle \\ &\quad + \left(2v' \langle X, \bar{\nu} \rangle + \frac{\bar{H}}{n-1} v \right) \left(1 - \frac{\bar{H}}{n-1} \langle X, \bar{\nu} \rangle \right) \delta_i^j. \end{aligned}$$

Since Σ_t is totally umbilic for $t \in [0, \delta)$, the trace-less second fundamental form \bar{h}_i^j is zero for all time $t \in [0, \delta)$. Then

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \bar{h}_i^j = \frac{\partial}{\partial t} \left(\bar{h}_i^j - \frac{\bar{H}}{n-1} \delta_i^j \right) \\ &= \left(\frac{4 \left(v'' \langle X, \bar{\nu} \rangle + \frac{\bar{H}}{n-1} v' \right)}{(\bar{H} + d\psi \cdot \bar{\nu})^2} - \frac{2 \left(2v' \langle X, \bar{\nu} \rangle + \frac{\bar{H}}{n-1} v \right)^2}{(\bar{H} + d\psi \cdot \bar{\nu})^3} \right) \\ &\quad \times \left(\langle X, e_i \rangle \langle X, e_j \rangle - \frac{|X^\top|^2}{n-1} \delta_i^j \right), \end{aligned}$$

where X^\top denotes the tangential part of the position vector. It follows that $\langle X, e_i \rangle^2 = |X^\top|^2 / (n-1)$ on Σ_t and is independent of the direction e_i . This can occur only if position vector X is parallel to the normal vector at X and each $\Sigma_t, t \in [0, \delta)$, is a sphere centered at the origin. Therefore, each $\Sigma_t, t \in [0, \delta)$, is a slice $\{s\} \times \mathbb{S}^{n-1}$ in the Schwarzschild space. This completes the proof of Theorem 1.1. \square

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