EXISTENCE AND NONEXISTENCE TO EXTERIOR DIRICHLET PROBLEM FOR MONGE-AMPÈRE EQUATION

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Abstract. We consider the exterior Dirichlet problem for Monge-Ampère equation with prescribed asymptotic behavior. Based on earlier work by Caffarelli and the first named author, we complete the characterization of the existence and nonexistence of solutions in terms of their asymptotic behaviors.

1. Introduction

A classic theorem of Jörgens [11], Calabi [7] and Pogorelov [16] states that any classical convex solution of

$$\det(D^2u)=1 \quad in \quad \mathbb{R}^n$$

must be a quadratic polynomial.

A simpler and more analytical proof, along the lines of affine geometry, was later given by Cheng and Yau [8]. The theorem was extended by Caffarelli [3] to viscosity solutions. Another proof of the theorem was given by Jost and Xin [12]. Trudinger and Wang [17] proved that if Ω is an open convex subset of $\mathbb{R}^n$ and u is a convex C^2 solution of $\det(D^2u)=1$ in Ω with $\lim_{x\to\partial\Omega}u(x)=\infty$, then $\Omega=\mathbb{R}^n$.

In 2003, Caffarelli and the first named author [4] extended the Jörgens-Calabi-Pogorelov theorem to exterior domains. In a subsequent paper [5], they also gave another extension: a classical convex solution u of $\det(D^2u)=f$ in $\mathbb{R}^n$ with a periodic positive f must be the sum of a quadratic polynomial and a periodic function. Moreover, their approach in [4] is enough to establish the existence of solutions to the exterior Dirichlet problem for Monge-Ampère equations. More specifically, let

$$\mathcal{A}=\{A|A \text{ is a real } n\times n \text{ symmetric positive definite matrix with} \det A=1\}.$$

Theorem A ([4]). *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n\geq 3$ and $\varphi\in C^2(\partial D)$. Then for any $A\in\mathcal{A}$, $b\in\mathbb{R}^n$, there exists a constant c_1 depending only on n,D,φ,b and A, such that for every $c>c_1$, there exists a unique solution $u\in C^\infty(\mathbb{R}^n\setminus\bar{D})\cap C^0(\overline{\mathbb{R}^n\setminus D})$ satisfying*

$$\begin{cases}\det(D^2u)=1 \quad in \quad \mathbb{R}^n\setminus\bar{D},\\ u=\varphi \quad on \quad \partial D,\\ \lim_{|x|\to\infty}\left|u(x)-\left(\frac{1}{2}x'Ax+bx+c\right)\right|=0.\end{cases}$$

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For $n = 2$, the exterior Dirichlet problem was studied earlier by Ferrer, Martínez and Milán [9, 10] using complex variable methods.

It is desirable to completely understand the relation between the constant c and the exsitence and nonexistence of solutions to the exterior Dirichlet problem. More precisely,

Question 1.1. *Concerning Theorem A, is there a sharp constant C_* such that we have the existence of solutions for $c \geq C_*$ while have the nonexistence for $c < C_*$?*

In [18], Wang and Bao answered the question among radially symmetric solutions. A special case of their theorem is as follows:

Theorem B ([18]). *For $n \geq 3$, let $C_* = -\frac{1}{2} + \int_1^\infty s((1-s^{-n})^{\frac{1}{n}} - 1)ds$. Then for every $c \geq C_*$, there exists a unique radially symmetric solution $u \in C^\infty(\mathbb{R}^n \setminus \overline{B_1(0)}) \cap C^1(\overline{\mathbb{R}^n \setminus B_1(0)})$ satisfying*

$$\begin{cases} \det(D^2 u) = 1 \quad in \quad \mathbb{R}^n \setminus \overline{B_1(0)}, \\ u = 0 \quad on \quad \partial B_1(0), \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}|x|^2 + c\right)| = 0. \end{cases}$$

While for $c < C_*$, there is no radially symmetric classical solution for the above problem.

In this paper, we give an affirmative answer to Question 1.1. To begin with, let us recall the definition of viscosity solutions.

Let Ω be a domain in $\mathbb{R}^n$, $g \in C^0(\Omega)$ be a positive function, and $u \in C^0(\Omega)$ be a locally convex function. We say that u is a viscosity subsolution of

$$\det(D^2 u) = g \quad in \quad \Omega \tag{1.1}$$

if for any $\bar{x} \in \Omega$ and every convex function $\varphi \in C^2(\Omega)$ satisfying

$$\varphi \geq u \quad in \quad \Omega \quad and \quad \varphi(\bar{x}) = u(\bar{x}),$$

we have

$$\det(D^2\varphi(\bar{x})) \geq g(\bar{x}).$$

Similarly, we say u is a viscosity supsolution of (1.1) if for any $\bar{x} \in \Omega$ and every convex function $\varphi \in C^2(\Omega)$ satisfying

$$\varphi \geq u \quad in \quad \Omega \quad and \quad \varphi(\bar{x}) = u(\bar{x}),$$

we have

$$\det(D^2\varphi(\bar{x})) \leq g(\bar{x}).$$

We say u is a viscosity solution of (1.1) if u is both a viscosity subsolution and a viscosity supsolution of (1.1).

Theorem 1.2. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$ and $\varphi \in C^2(\partial D)$. Then for any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, there exists a constant C_* depending only on n, D, φ, b and A, such that for every $c \geq C_*$, there exists a unique solution $u \in C^\infty(\mathbb{R}^n \setminus \bar{D}) \cap C^0(\overline{\mathbb{R}^n \setminus D})$ satisfying*

$$\begin{cases} \det(D^2u) = 1 \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

While for $c < C_*$, there is no viscosity solution for the above problem.

Apart from the exterior Dirichlet problem with constant 1 on the right hand side, it is natural to consider the problem with a general right hand side. In the case that the right hand side is an appropriate perturbation of 1 near infinity, the existence and uniqueness ware given by Bao, Li and Zhang [2]. More precisely, assume that

$$g \in C^0(\mathbb{R}^n), \quad 0 < \inf_{\mathbb{R}^n} g \leq \sup_{\mathbb{R}^n} g < \infty,$$

and for some integer $m \geq 3$ and some constant $\beta > 2$ such that $D^m g$ exists outside a compact subset of $\mathbb{R}^n$ and

$$\lim_{|x|\to\infty} |x|^{\beta+|\alpha|}|D^\alpha(g(x) - 1)| < \infty, \quad |\alpha| = 0, \cdots, m.$$

Theorem C ([2])**.** *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$, $\varphi \in C^2(\partial D)$ and g satisfy the above assumption. Then for any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, there exists a constant c_1 depending only on n, D, φ, b, g and A, such that for every $c > c_1$, there exists a unique viscosity solution u satisfying*

$$\begin{cases} \det(D^2u) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

Similar to Theorem 1.2, we also give for this problem the characterization of the existence and nonexistence of solutions in terms of their asymptotic behavior for this problem.

Theorem 1.3. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$, $\varphi \in C^2(\partial D)$ and g satisfy the above assumption. Then for any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, there exists a constant C_* depending only on n, D, φ, b, g and A, such that for every $c \geq C_*$, there exists a unique viscosity solution u satisfying*

$$\begin{cases} \det(D^2u) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

While for $c < C_*$, there is no viscosity solution for the above problem.

The main contribution of our results is that we give a complete characterization of the existence and nonexistecne of solutions to the exterior Dirichlet problem in terms of their asymptotic behavior. Note that this is different from the interior Dirichlet problem, for which the existence and uniqueness were established by the seminal work of Caffarelli, Nirenberg and Spruck [6]. The special feather of the exterior Dirichlet problem is that it depends on the asymptotic behavior in a non-trivial way. Our result reveals the intimate relation between the asymptotic behavior and the existence and nonexistence. Moreover, our result is also sharp in the sense that all viscosity solutions to the exterior Dirichlet problem is asymptotic to a quadratic polynomial by the result of Caffarelli and Li [4].

The organization of the paper is as follows: In section 2, we prove Theorem 1.2 and Theorem 1.3. Theorems 1.2 is proved as follows. We first prove in Lemma 2.1 that for any given data D, φ, A and b, there exists c_2 such that for any $c < c_2$ there exists no subsolution u of

$$\begin{cases} \det(D^2u) \geq 1 \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \end{cases} \tag{1.2}$$

having the asymptotic behavior $\frac{1}{2}x'Ax + bx + c$.

Then we prove in Lemma 2.4 that if there is a solution of (1.2) with asymptotic behavior $\frac{1}{2}x'Ax + bx + c_3$, then for every $c > c_3$, there is a solution of (1.2) with asymptotic behavior $\frac{1}{2}x'Ax+bx+c$. Thus, in view of Theorem A, for any given data D, φ, A and b, there exists a C_* such that for every $c > C_*$ there exists a solution of (1.2) with asymptotic behavior $\frac{1}{2}x'Ax + bx + c$, while for every $c < C_*$ there is no such solution. Finally we prove that there is such a solution for $c = C_*$. Theorem 1.3 is proved similarly.

In section 3, we extend some results to more general fully nonlinear elliptic equations.

2. Monge-Ampère equation

In this section, we prove Theorem 1.2 and Theorem 1.3. To begin with, we prove that there is no viscosity subsolution for c very negative.

Lemma 2.1. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$ and $\varphi \in C^2(\partial D)$. Then for any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, there exists a constant c_2 depending only on n, D, φ, b and A, such that for $c < c_2$, there is no viscosity subsolution u satisfying*

$$\begin{cases} \det(D^2u) \geq 1 \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

Proof. Note that by an affine transformation and adding a linear function to u, we only need to consider $A = I$ and $b = 0$. Thus without loss of generality, we may assume that u has the asymptotic $\frac{|x|^2}{2} + c$, $\varphi \geq 0$ and $D \subset B_1(0)$ such that there exists $x_0 \in \partial D \cap \partial B_1(0)$.

Consider

$$v(x) = \sup\{u(Tx) | T : \mathbb{R}^n \to \mathbb{R}^n \text{ is an orthogonal transformation}\}. \tag{2.1}$$

For any compact set $K \subset \mathbb{R}^n \setminus \bar{B}_1$, we have

$$|u| + |\nabla u| \leq C(K) \quad \text{in} \quad K.$$

It is clear from the definition of v that

$$|v| + |\nabla v| \leq C(K) \quad \text{in} \quad K.$$

By a standard argument, v is a locally convex viscosity subsolution satisfying

$$\begin{cases} \det(D^2 v) \geq 1 \quad in \quad \mathbb{R}^n \setminus \overline{B_1(0)}, \\ v \geq 0 \quad on \quad \partial B_1(0), \\ \lim_{|x|\to\infty} |v(x) - \left(\frac{1}{2}|x|^2 + c\right)| = 0. \end{cases}$$

The second line is due to the fact that $\varphi \geq 0$.

Clearly v is radially symmetric.

We first prove that v is monotonically nondecreasing in $[1, \infty)$. Suppose not, then since $v(r) \to \infty$ as $r \to \infty$, there exists $1 < r_1 < r_2 < r_3 < \infty$ such that $v(r_1) = v(r_3) > v(r_2)$, violating the local convexity of v. Indeed, let $h(t) := v(t, r_1, 0, \cdots, 0)$, $-r_2 < t < r_2$. By the local convexity of v, h is locally convex in $(-r_2, r_2)$ and therefore convex in $(-r_2, r_2)$. For $\bar{t} = \sqrt{r_2^2 - r_1^2} \in (0, r_2)$, $h(\bar{t}) = h(-\bar{t}) = v(r_2) < v(r_1) = h(0)$, which violates the convexity of h in $(-r_2, r_2)$. Note that we have slightly abused notation by writing $v(x) = v(|x|)$ for $x \in \mathbb{R}^n$.

We already know that v is locally convex, so v is locally Lipschitz. Thus v' exists almost everywhere and v' is monotonically nondecreasing. Since v is monotonically nondecreasing, $v' \geq 0$ almost everywhere.

On the other hand, we know that monotone functions are differentiable almost everywhere, so v'' exists and $v'' \geq 0$ almost everythere.

To proceed, we first prove that $v'' \in L^1_{loc}[1, \infty)$. Indeed, let $g_m(s) = m(v'(s + \frac{1}{m}) - v'(s))$, then by Fatou's lemma, we have

$$\begin{aligned} \int_t^r v'' ds &\leq \liminf_{m\to\infty} \int_t^r g_m(s) ds = \liminf_{m\to\infty} \left(m \int_r^{r+\frac{1}{m}} v'(s) ds - m \int_t^{t+\frac{1}{m}} v'(s) ds \right) \\ &\leq 2 \sup_{[1,r+1]} v' < \infty \end{aligned}$$

Secondly, we show that for all $r \geq 1$ such that $v'(r)$ and $v''(r)$ are defined, we have

$$\lim_{h\to 0} \frac{v(r+h) - v(r) - v'(r)h}{h^2} = \frac{1}{2} v''(r)$$

Indeed, we have

$$\begin{aligned} v(r+h) = v(r) + \int_0^h v'(r+s) ds &= v(r) + \int_0^h (v'(r) + v''(r)s + o(s)) ds \\ &= v(r) + v'(r)h + \frac{1}{2} v''(r) h^2 + o(h^2). \end{aligned}$$

By taking limit on h, we have the above formula.

For any $\epsilon > 0$, we can choose h_0 small enough such that,

$$v(r+h) \le v(r) + v'(r)h + \frac{1}{2}\left(v''(r)+\epsilon\right)h^2,$$

for any $|h| \le h_0$. Define, for $|h| \le h_0$,

$$\phi(r+h) = v(r) + v'(r)h + \frac{1}{2}\left(v''(r)+\epsilon\right)h^2.$$

Thus $\phi \in C^2(B_{h_0}(r))$ satisfies $\phi \ge v$ in $B_{h_0}(r)$, $\phi(r) = v(r)$, and therefore by the definition of viscosity subsolution,

$$\det(D^2\phi(r)) \ge 1.$$

Let $\epsilon \to 0$, we obtain

$$\det(D^2 v) \ge 1 \quad a.e. \quad in \quad \mathbb{R}^n \setminus \bar{B}_1(0).$$

Note that v is radially symmetric, we have

$$(D^2 v) = diag(v'', \frac{v'}{r}, \cdots, \frac{v'}{r}) \quad a.e. \quad in \quad \mathbb{R}^n \setminus \bar{B}_1(0).$$

Thus we can write the above equation as

$$v''\frac{v'^{n-1}}{r^{n-1}} \ge 1 \quad a.e. \quad in \quad [1,\infty).$$

Integrating the above, for all $1 \le t < r < \infty$, we have

$$\int_t^r v''v'^{n-1}ds \ge \int_t^r s^{n-1}ds. \tag{2.2}$$

Let φ_ϵ be the standard mollifier, then

$$v_\epsilon = v * \varphi_\epsilon$$

is a smooth sequence converging to v in $C^0_{loc}(1,\infty)$.

Since $v'' \in L^1_{loc}[1,\infty)$, we have

$$v'_\epsilon = v' * \varphi_\epsilon, \quad v''_\epsilon = v'' * \varphi_\epsilon,$$

and $v'_\epsilon \to v'$, $v''_\epsilon \to v''$, a.e. in $(1,\infty)$.

By Fatou's lemma, for a.e. $1 < t < r < \infty$, we have

$$\int_t^r v''v'^{n-1}ds \le \liminf_{\epsilon\to 0}\int_t^r v''_\epsilon {v'_\epsilon}^{n-1}ds.$$

i.e.

$$n\int_t^r v''v'^{n-1}ds \le \liminf_{\epsilon\to 0}\left({v'_\epsilon}^n(r) - {v'_\epsilon}^n(t)\right) = v'^n(r) - v'^n(t).$$

Together with (2.2), for a.e. $1 < t < r < \infty$, we have

$$v'^n(r) - v'^n(t) \ge r^n - t^n.$$

Since $v'(t) \ge 0$ and let $t \to 1^+$, we have, for almost all r,

$$v'(r) \ge (r^n - 1)^{\frac{1}{n}}.$$

Now

$$v(r) = v(1) + \int_1^r v' ds \geq \int_1^r (s^n - 1)^{\frac{1}{n}} ds \geq \frac{1}{2} r^2 + C,$$

for r large enough, here C is a constant under control, see in [4]. It follows that $c \geq C$. □

We now prove the nonexistence in the case that g is a perturbation of 1 at infinity.

Lemma 2.2. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$, $\varphi \in C^2(\partial D)$ and g satisfy the assumption in Theorem 1.3. Then for any $A \in \mathcal{A}$, $b \in \mathbb{R}^n$, there exists a constant c_2 dpending only on n, D, φ, b, g and A, such that for $c < c_2$, there is no viscosity subsolution u satisfying*

$$\begin{cases} \det(D^2 u) \geq g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x| \to \infty} |u(x) - \left(\frac{1}{2} x' A x + bx + c\right)| = 0. \end{cases}$$

Proof. Note that we can run the same argument as in Lemma 2.1. Let

$$g_1(r) = \inf_{|x|=r} g(x).$$

Then for a.e. $1 < t < r < \infty$,

$$v'^n(r) - v'^n(t) \geq \int_t^r n s^{n-1} g_1(s) ds.$$

Let $t \to 1^+$, we have for a.e. $r > 1$,

$$v'(r) \geq \left(\int_1^r n s^{n-1} g_1(s) ds \right)^{\frac{1}{n}}.$$

Thus for all $r > 1$,

$$v(r) \geq v(1) + \int_1^r \left(\int_1^l n s^{n-1} g_1(s) ds \right)^{\frac{1}{n}} dl.$$

Now by Lemma 3.1 in [2], we have

$$\int_1^r \left(\int_1^l n s^{n-1} g_1(s) ds \right)^{\frac{1}{n}} dl \geq \frac{1}{2} r^2 + C,$$

where C is under control. It follows that that $c \geq C$. □

As mentioned in the introduction, we have

Corollary 2.3. *For $n \geq 3$, let u be a viscosity solution satisfying*

$$\begin{cases} \det(D^2 u) = 1 \quad in \quad \mathbb{R}^n \setminus \overline{B_1(0)}, \\ u = 0 \quad on \quad \partial B_1(0), \\ \lim_{|x| \to \infty} |u(x) - \left(\frac{1}{2} |x|^2 + c\right)| = 0. \end{cases}$$

Then u is radially symmetric.

Proof. Let u be a viscosity solution for the above problem and let v be defined as in (2.1). Then v is a locally convex viscosity subsolution to the exterior Dirichlet problem

$$\begin{cases} \det(D^2 v) \geq 1 \quad in \quad \mathbb{R}^n \setminus \overline{B_1(0)}, \\ v = 0 \quad on \quad \partial B_1(0), \\ \lim_{|x|\to\infty} |v(x) - \left(\frac{1}{2}|x|^2 + c\right)| = 0. \end{cases}$$

Clearly v is radially symmetric.

$\forall \epsilon > 0$, there exists $R > 1$ such that

$$u(x) + \epsilon \geq v(x), \quad |x| \geq R.$$

Applying the comparison principle, see e.g. Proposition 2.1 in [4], we have $u(x) + \epsilon \geq v(x)$ for $1 \leq |x| \leq R$. Thus $u + \epsilon \geq v$ in $\mathbb{R}^n \setminus B_1(0)$. Sending ϵ to 0, it gives $u \geq v$ in $\mathbb{R}^n \setminus B_1(0)$.

On the other hand, by the definition of v, we have $v \geq u$. Thus $u = v$ is radially symmetric.

□

We now prove that if we have a viscosity solution with c_3, then we have a viscosity solution with all $c \geq c_3$.

Lemma 2.4. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$, $\varphi \in C^2(\partial D)$, g satisfy the assumption in Theorem 1.3, $A \in \mathcal{A}$ and $b \in \mathbb{R}^n$. Suppose that there exists a viscosity solution u_3 satisfying*

$$\begin{cases} \det(D^2 u) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c_3\right)| = 0. \end{cases}$$

Then for all $c \geq c_3$, there exists a viscosity solution u satisfying

$$\begin{cases} \det(D^2 u) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

Proof. By Theorem C, there exists a viscosity solution u satisfying

$$\begin{cases} \det(D^2 u) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

for all $c \geq c_1 > c_3$.

Thus we only need to prove that there exists a viscosity solution u for all $c_3 < c < c_1$.

As before, we may assume without loss of generality that $A = I$ and $b = 0$.

We consider viscosity subsolutions v to the exterior Dirichlet problem

$$\begin{cases} \det(D^2 v) \geq g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ v = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |v(x) - \left(\frac{1}{2}|x|^2 + c\right)| = 0. \end{cases} \tag{2.3}$$

First of all, we show that there exists at least one viscosity subsolution.

Let u_3 and u_1 be solutions with asymptotic behavior $\frac{1}{2}|x|^2 + c_3$ and $\frac{1}{2}|x|^2 + c_1$ respectively.

Note that $c = \alpha c_3 + (1-\alpha)c_1$ for some $0 < \alpha < 1$ and let

$$u_c := \alpha u_3 + (1-\alpha)u_1.$$

By Alexandrov's theorem [1], locally convex functions are second order differentiable almost everywhere. Thus both u_1 and u_3 are second order differentiable almost everywhere and $D^2u_1 \geq 0, D^2u_3 \geq 0$ a.e.

We first prove that $\det(D^2u_1) = g$ a.e. in $\mathbb{R}^n \setminus \bar{D}$. Let $\bar{x} \in \mathbb{R}^n \setminus \bar{D}$ be a point such that u_1 is second order differentiable, without loss of generality, let $\bar{x} = 0$. Define

$$\phi_\delta(x) = u_1(0) + Du_1(0)\cdot x + \frac{1}{2}x'(D^2u_1(0) + \delta)x.$$

Clearly $\phi_\delta(0) = u_1(0)$. Choose δ so small such that $\phi_\delta(x) \geq u_1(x)$ near 0, since u_1 is a viscosity subsolution, we have

$$\det(D^2\phi_\delta(0)) \geq g(0).$$

Sending δ to 0, we have $\det(D^2u_1(0)) \geq g(0)$. In particular, $D^2u_1(0) > 0$.

Similary, define

$$\phi_\delta(x) = u_1(0) + Du_1(0)\cdot x + \frac{1}{2}x'(D^2u_1(0) - \delta)x.$$

Clearly $\phi_\delta(0) = u_1(0)$. Choose δ so small such that $\phi_\delta(x) \leq u_1(x)$ near 0 and that ϕ_δ is convex, since u_1 is a viscosity supsolution, we have

$$\det(D^2\phi_\delta(0)) \leq g(0).$$

Sending δ to 0, we have $\det(D^2u_1)(0) \leq g(0)$.

Thus we have

$$\det(D^2u_1) = g, \quad a.e.$$

Similarly

$$\det(D^2u_3) = g, \quad a.e.$$

By the concavity of $\det^{\frac{1}{n}}$, we have

$$\det(D^2u_c)^{\frac{1}{n}} \geq \alpha \det(D^2u_3)^{\frac{1}{n}} + (1-\alpha)\det(D^2u_1)^{\frac{1}{n}} = g^{\frac{1}{n}}, \quad a.e.$$

We now prove that u_c is a viscosity subsolution.

Let $\bar{x} \in \Omega$ be an arbitrary point, say $\bar{x} = 0$. For any $\phi \in C^2$, such that $\phi(0) = u_c(0)$ and $\phi(x) \geq u_c(x)$ for x near 0.

For $\delta > 0$ small, define $\phi_\delta(x) = \phi(x) + \delta|x|^2$, then

$$\phi_\delta(x) \geq u_c(x) + \delta^3, \quad for \quad |x| = \delta.$$

Consider

$$\xi(x) = \phi_\delta(x) - u_c(x) - \delta^4.$$

Then $\xi(0) = -\delta^4$ and $\xi(x) > 0$ for $|x| = \delta$. Since $D^2 u_c \geq 0$ almost everythere, we have $D^2\xi \leq C$ almost everywhere. It follows from the Alexandrov-Bakelman-Pucci inequality that

$$\delta^4 \leq C \left(\int_{\{\xi = \Gamma_\xi\}} \det(D^2\Gamma_\xi) \right)^{\frac{1}{n}},$$

where Γ_ξ is the convex envelope of ξ.

Thus

$$\int_{\{\xi = \Gamma_\xi\}} \det(D^2\Gamma_\xi) > 0.$$

In particular $\{\xi = \Gamma_\xi\}$ has positive Lebesgue measure. Let $x \in \{\xi = \Gamma_\xi\} \cap B_\delta(0)$ where u_c is second order differentiable and $\det(D^2 u_c(x)) \geq g(x)$. Thus

$$\xi(y) \geq \Gamma_\xi(x) + \nabla\Gamma_\xi(x) \cdot (y - x),$$

and $D^2\xi(x) \geq 0$, i.e.

$$D^2\phi_\delta(x) \geq D^2 u_c(x).$$

It follows that $\det(D^2\phi_\delta(x)) \geq \det(D^2 u_c(x)) \geq g(x)$. Sending δ to 0, we have $\det(D^2\phi(0)) \geq g(0)$. Thus u_c is a viscosity subsolution.

On the other hand,

$$u_c = \varphi \quad on \quad \partial D,$$

and

$$\lim_{x\to\infty} (u_c - \frac{1}{2}|x|^2) = c.$$

Thus u_c is a viscosity subsolution of (2.3).

Now for every viscosity subsolution v of (2.3), we have

$$v = u_3 = \varphi \quad on \quad \partial D,$$

and

$$\lim_{|x|\to\infty} (v(x) - u_3(x)) = c - c_3 > 0.$$

We deduce from the comparison principle, applied to v and $u_3 + c - c_3$, as before that

$$v - u_3 \leq c - c_3 \quad on \quad \mathbb{R}^n \setminus \bar{D}.$$

Let

$$u(x) = \sup\{v(x) | v \text{ is a viscosity subsolution of (2.3)}\}.$$

Now by the comparison principle, $v \leq u_1$ for all such v, thus $u \leq u_1$. On the other hand, u_c is a viscosity subsolution, thus $u \geq u_c$. Then a standard argument shows that u is a viscosity solution of

$$\begin{cases} \det(D^2 u) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D. \end{cases}$$

Now we only need to consider the asymptotic behavior. Since for all such v, we have

$$\lim_{|x|\to\infty} (v(x) - \frac{1}{2}|x|^2) = c.$$

Thus

$$\liminf_{|x|\to\infty} (u(x) - \frac{1}{2}|x|^2) \geq c.$$

On the other hand, as $v - u_3 \leq c - c_3$ for all such v, we have $u - u_3 \leq c - c_3$ in $\mathbb{R}^n \setminus D$, and therefore

$$\limsup_{|x|\to\infty} (u(x) - \frac{1}{2}|x|^2) \leq \lim_{|x|\to\infty} (u_3 + c - c_3 - \frac{1}{2}|x|^2) = c.$$

Thus u has the asymptotic behavior $\frac{1}{2}|x|^2 + c$.

□

Proof of the Theorem 1.2 and Theorem 1.3:

Proof. By Theorem A and Theorem C, there exists a unique viscosity solution for $c > c_1$, without loss of generality, we assume that c_1 is the infimum of all such c_1. On the other hand, by Lemma 2.1 and Lemma 2.2, there exists c_2 such that there is no viscosity solution for $c < c_2$, without loss of generality, we assume that c_2 is the supremum of all such c_2. Now we prove that $c_1 = c_2$. Indeed, if this is not true, then there exists $c_2 \leq c_3 < c_1$ such that we have a viscosity solution with asymptotic behavior $\frac{1}{2}x'Ax + bx + c_3$, for otherwise it violates the fact that c_2 is the supereme of all such c_2. Now by Lemma 2.4, we know that for all $c \geq c_3$, we have viscosity solution with asymptotic behavior $\frac{1}{2}x'Ax + bx + c$. This violates the fact that c_1 is the infimum of all such c_1.

Now we may denote the unique constant $C_* = c_1 = c_2$. By the discussion above, we have viscosity solution for all $c > C_*$ and we have no viscosity solution for all $c < C_*$. We only need to prove that we have viscosity solution for C_*.

By the above, we have a sequence of viscosity solutions u_j satisfying

$$\begin{cases} \det(D^2 u_j) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u_j = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u_j(x) - \left(\frac{1}{2}x'Ax + bx + C_* + \frac{1}{j}\right)| = 0. \end{cases}$$

By comparison principle, we know that $u_i \geq u_j$, $\forall i \leq j$. Thus u_j is monotonically nonincreasing as $j \to \infty$.

Define

$$u_\infty(x) = \lim_{j\to\infty} u_j(x), \quad x \in \mathbb{R}^n \setminus D.$$

We claim that u_∞ is a viscosity solution of

$$\begin{cases} \det(D^2 v) = g \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ v = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |v(x) - \left(\frac{1}{2}x'Ax + bx + C_*\right)| = 0. \end{cases} \tag{2.4}$$

We note that by the comparison principle, $u_i \geq u_j - \frac{1}{j} + \frac{1}{i}$ for all $i \geq j$. Sending i to ∞, we have $u_j \geq u_\infty \geq u_j - \frac{1}{j}$ for all j. Thus

$$\lim_{j\to\infty} \|u_j - u_\infty\|_{L^\infty(\mathbb{R}^n\setminus D)} = 0.$$

It follows that $\lim_{|x|\to\infty} |u_\infty(x) - \left(\frac{1}{2}x'Ax + bx + C_*\right)| = 0$.

Since $\{u_j\}$ satisfies the first two lines in (2.4), and $u_j \to u_\infty$ uniformly, it is standard that u_∞ also satisfies the first two lines of (2.4).

□

3. Further discussions

In this section, we generalize some results above to more general fully nonlinear elliptic equations. To begin with, let us recall some definitions. We have the following two equivalent definitions for the equations, see [14] for more detail.

Definition 1 : Let $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1 := \{\lambda \in \mathbb{R}^n | \sum_{i=1}^n \lambda_i > 0\},$$

where $\Gamma_n := \{\lambda \in \mathbb{R}^n | \lambda_i > 0, i = 1, \cdots, n\}$ is the positive cone.

Here Γ being symmetric means $(\lambda_1, \cdots, \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, \cdots, \lambda_{\lambda_n}) \in \Gamma$ for any perturbation $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$.

Assume that $f \in C^1(\Gamma) \cap C^0(\bar{\Gamma})$ is concave and symmetric in λ_i satisfying

$$f|_{\partial\Gamma} = 0, \quad \nabla f \in \Gamma_n \quad on \quad \Gamma,$$

and

$$\lim_{s\to\infty} f(s\lambda) = \infty, \quad \lambda \in \Gamma.$$

Then the equation is

$$f(\lambda(D^2 u)) = 1, \quad \lambda(D^2 u) \in \Gamma.$$

Definition 2: Let V be an open symmetric convex subset of $\mathbb{R}^n$ with $\partial V \neq \emptyset$ and $\partial V \in C^1$. Assume that

$$\nu(\lambda) \in \Gamma_n, \quad \forall \lambda \in \partial V,$$

and

$$\nu(\lambda)\cdot\lambda > 0, \quad \forall \lambda \in \partial V,$$

where $\nu(\lambda)$ is the inner unit normal of ∂V at λ.

Then the equation is

$$\lambda(D^2 u) \in \partial V. \tag{3.1}$$

Remark 3.1. *In particular, if $f = \det^{\frac{1}{n}}$, then $\Gamma = \Gamma_n$. If $f = \sigma_k^{\frac{1}{k}}$, the k-th elementary symmetric function, then $\Gamma = \Gamma_k := \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, j = 1, \cdots, k\}$. If $f = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}}$ for $k > l$, then $\Gamma = \Gamma_k$.*

To proceed, let us recall the definition of viscosity solutions, see [13, 15] in this context. We first define the upper semi-continous and lower semi-continuous functions. Let $S \subset \mathbb{R}^n$, we denote $USC(S)$ the set of functions $\phi : S \to \mathbb{R} \cup \{-\infty\}$, $\phi \not\equiv -\infty$ in S, satisfying

$$\limsup_{x\to\bar{x}} \phi(x) \le \phi(\bar{x}), \quad \forall \bar{x} \in S.$$

Similarly, we denote $LSC(S)$ the set of functions $\phi : S \to \mathbb{R} \cup \{+\infty\}$, $\phi \not\equiv +\infty$ in S, satisfying

$$\liminf_{x\to\bar{x}} \phi(x) \ge \phi(\bar{x}), \quad \forall \bar{x} \in S.$$

Let Ω be a domain in $\mathbb{R}^n$. For a function u in $USC(\Omega)$, we say u is a viscosity subsolution in Ω if for any $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ with

$$(u-\varphi)(x_0) = 0, \quad u - \varphi \le 0 \quad in \quad \Omega.$$

there holds

$$\lambda(D^2\varphi) \in \bar{V}.$$

For a function u in $LSC(\Omega)$, we say u is a viscosity supsolution in Ω if for any $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ with

$$(u-\varphi)(x_0) = 0, \quad u - \varphi \ge 0 \quad in \quad \Omega.$$

there holds

$$\lambda(D^2\varphi) \in \mathbb{R}^n \setminus V.$$

We say that a function $u \in C^0(\Omega)$ is a viscosity solution of

$$\lambda(D^2 u) \in \partial V,$$

in Ω if it is both a viscosity subsolution and a viscosity supsolution.

We now state the comparison principle, see Theroem 1.7 in [15].

Lemma 3.2. *Let Ω be a domain in $\mathbb{R}^n$, assume that $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ are respectively a viscosity subsolution and a viscosity supsolution of (3.1) in Ω satisfying $u \ge v$ on $\partial\Omega$. Then $u \ge v$ in Ω.*

We now state the first result in this section.

Lemma 3.3. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \geq 3$, let $\varphi \in C^2(\partial D)$, let A be a real positive definite $n \times n$ symmetric matrix such that $\lambda(A) \in \partial V$, $b \in \mathbb{R}^n$. Suppose there exist a two constants $c_3 < c_1$ such that there exist viscosity solutions u_3 and u_1 satisfying*

$$\begin{cases} \lambda(D^2 u) \in \partial V \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D. \end{cases}$$

with asymptotic behavior $\frac{1}{2}x'Ax + bx + c_3$ and $\frac{1}{2}x'Ax + bx + c_1$ respectively as $|x| \to \infty$.

Then for any $c_3 < c < c_1$, there exists a viscosity solution u satisfying

$$\begin{cases} \lambda(D^2 u) \in \partial V \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x| \to \infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases}$$

Proof. We note that by an orthogonal transformation and adding a linear function to u, we only need to consider case $b = 0$ and $A = \Lambda$ is a diagonal matrix.

Now we show that for any $c_3 < c < c_1$, there is a viscosity solution u with asymptotic behavior $\frac{1}{2}x'\Lambda x + c$.

To show this, we consider viscosity subsolutions v to the exterior Dirichlet problem

$$\begin{cases} \lambda(D^2 v) \in \bar{V} \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ v = \varphi \quad on \quad \partial D, \\ \lim_{|x| \to \infty} |v(x) - \left(\frac{1}{2}x'\Lambda x + c\right)| = 0. \end{cases} \tag{3.2}$$

First of all, we show that there exists at least one viscosity subsolution.

Let u_3 and u_1 be solutions with asymptotic behavior $\frac{1}{2}x'\Lambda x + c_3$ and $\frac{1}{2}x'\Lambda x + c_1$ respectively.

Note that $c = \alpha c_3 + (1 - \alpha)c_1$ for some $0 < \alpha < 1$. Let $u_c := \alpha u_3 + (1 - \alpha)u_1$, by Lemma 4.1

$$\lambda(D^2 u_c) \in \bar{V}.$$

On the other hand,

$$u_c = \varphi \quad on \quad \partial D,$$

and

$$\lim_{x \to \infty} (u_c - \frac{1}{2}x'\Lambda x) = c.$$

Thus u_c is a viscosity subsolution of (3.2).

Now for every viscosity subsolution v of (3.2), we have

$$v = u_3 = u_1 \quad on \quad \partial D,$$

$$\lim_{|x| \to \infty} (v(x) - u_1(x)) = c - c_1,$$

$$\lim_{|x| \to \infty} (v(x) - u_3(x)) = c - c_3.$$

We deduce from Lemma 3.2, applied to v and u_1, and then to v and $u_3 + c - c_3$, as before that

$$v \le u_1 \quad in \quad \mathbb{R}^n \setminus D,$$
$$v - u_3 \le c - c_3 \quad in \quad \mathbb{R}^n \setminus D,$$

for all such subsolution v.

Let

$$u(x) = \sup\{v(x) | v \text{ is a viscosity subsolution of (3.2)}\}.$$

Since u_c is a viscosity subsolution of (3.2), we have

$$u \ge u_c \quad in \quad \mathbb{R}^n \setminus D.$$

By the above, every viscosity subsolution of (3.2) satisfies $v \le u_1$, we have

$$u \le u_1 \quad in \quad \mathbb{R}^n \setminus D.$$

Since $u_c = u_1 = \varphi$ on ∂D and $u_c, u_1 \in C^0(\mathbb{R}^n \setminus D)$, we have $u = \varphi$ on ∂D.

By the proof of Theorem 1.5 in [15], u is a viscosity solution of

$$\begin{cases} \lambda(D^2 u) \in \partial V \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D. \end{cases}$$

Now we only need to consider the asymptotic behavior. Since for all such v, we have

$$\lim_{|x|\to\infty} (v - \frac{1}{2} x' \Lambda x) = c.$$

It follows that

$$\liminf_{|x|\to\infty} (u - \frac{1}{2} x' \Lambda x) \ge c.$$

On the other hand, as $v - u_3 \le c - c_3$ for all such v, we have $u - u_3 \le c - c_3$ on $\mathbb{R}^n \setminus D$ and therefore

$$\limsup_{|x|\to\infty} (u - \frac{1}{2} x' \Lambda x) \le \lim_{|x|\to\infty} (u_3 + c - c_3 - \frac{1}{2} x' \Lambda x) = c$$

Thus u has asymptotic behavior $\frac{1}{2} x' \Lambda x + c$.

□

We now consider the limiting case.

Lemma 3.4. *Let D be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$ for $n \ge 3$, let $\varphi \in C^2(\partial D)$, let A be a real positive definite $n \times n$ symmetric matrix such that $\lambda(A) \in \partial V$, $b \in \mathbb{R}^n$. Suppose there exists a sequence $c_j \to c$ such that for each c_j, there exists a viscosity solution u_j satisfying*

$$\begin{cases} \lambda(D^2 u) \in \partial V \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u_j(x) - \left(\frac{1}{2} x' A x + bx + c_j\right)| = 0. \end{cases}$$

Then there exists a viscosity solution u satisfying

$$\begin{cases} \lambda(D^2u) \in \partial V \quad in \quad \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi \quad on \quad \partial D, \\ \lim_{|x|\to\infty} |u(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0. \end{cases} \tag{3.3}$$

Proof. We first note that by passing to a subsequence, we may assume that c_j is monotonically increasing or decreasing. We only need to prove the case that c_j is monotonically decreasing. The other case follows in the same way.

Applying Lemma 3.2, we have $u_i - c_i + c_j \le u_j \le u_i$ in $\mathbb{R}^n \setminus D$ for $i \le j$. Thus $\{u_j\}$ is monotonically nonincreasing and $\|u_i - u_j\|_{L^\infty(\mathbb{R}^n\setminus D)} \to 0$ as $i, j \to \infty$. Consequently, for some $u_\infty \in C^0(\mathbb{R}^n \setminus D)$,

$$\lim_{j\to\infty} \|u_j - u_\infty\|_{L^\infty(\mathbb{R}^n\setminus D)} = 0.$$

It follows that

$$\lim_{|x|\to\infty} |u_\infty(x) - \left(\frac{1}{2}x'Ax + bx + c\right)| = 0.$$

Since $\{u_j\}$ satisfies the first two lines in (3.3), and $u_j \to u_\infty$ uniformly, it is standard that u_∞ also satisfies the first two lines of (3.3).

□

4. Appendix

In this appendix, we prove the following lemma.

Lemma 4.1. *Let u, v be two viscosity solutions of (3.1), then for any $0 < \alpha < 1$, $\alpha u + (1-\alpha)v$ is a viscosity subsolution of (3.1).*

Proof. To begin with, let us recall the definition of ϵ-upper envelope of u, see e.g. [15],

$$u^\epsilon(x) := \max_{y\in\bar{\Omega}}\{u(y) - \frac{1}{\epsilon}|y-x|^2\}, \quad x \in \bar{\Omega}.$$

Then

$$u^\epsilon \to u, \quad \epsilon \to 0^+,$$

in $C^0_{loc}(\Omega)$.

And u^ϵ is a viscosity subsolution of (3.1).

Moreover u^ϵ is second order differentiable almost everythere and

$$D^2u^\epsilon \ge -\frac{2}{\epsilon}I, \quad a.e. \quad in \quad \Omega.$$

Let $\bar{x} \in \Omega$ be a point where u^ϵ is second order differentiable, say $\bar{x} = 0$. For $\delta > 0$ small, define

$$\phi(x) = u^\epsilon(0) + \nabla u^\epsilon(0)x + \frac{1}{2}x'(D^2u^\epsilon(0) + \delta)x.$$

Then $\phi(x) \ge u^\epsilon(x)$ near 0 and $\phi(0) = u^\epsilon(0)$.

Since u^ϵ is a viscosity subsolution, we have $\lambda(D^2\varphi)(0) \in \bar{V}$. Sending δ to 0, we have $\lambda(D^2u^\epsilon)(0) \in \bar{V}$. Thus $\lambda(D^2u^\epsilon) \in \bar{V}$ a.e.

Similarly, $\lambda(D^2v^\epsilon) \in \bar{V}$ a.e., where v^ϵ is the ϵ-upper envelope of v.

Define

$$w_\alpha^\epsilon = \alpha u^\epsilon + (1-\alpha)v^\epsilon.$$

Since $\bar{V}$ is convex, it follows that $\lambda(D^2w_\alpha^\epsilon) \in \bar{V}$ a.e.

We now prove that w_α^ϵ is a viscosity subsolution in Ω.

Let $\bar{x} \in \Omega$ be an arbitrary point, say $\bar{x} = 0$. For any $\phi \in C^2$, such that $\phi(0) = w_\alpha^\epsilon(0)$ and $\phi(x) \geq w_\alpha^\epsilon(x)$ for x near 0.

For $\delta > 0$ small, define $\phi_\delta(x) = \phi(x) + \delta|x|^2$, then

$$\phi_\delta(x) \geq w_\alpha^\epsilon(x) + \delta^3, \quad for \quad |x| = \delta.$$

Consider

$$\xi(x) = \phi_\delta(x) - w_\alpha^\epsilon(x) - \delta^4.$$

Then $\xi(0) = -\delta^4$ and $\xi(x) > 0$ for $|x| = \delta$. Since $D^2u^\epsilon \geq -\frac{2}{\epsilon}I$, $D^2v^\epsilon \geq -\frac{2}{\epsilon}I$ almost everywhere, we have $D^2\xi \leq C(\epsilon)I$ almost everywhere. It follows from the Alexandrov-Bakelman-Pucci inequality that

$$\delta^4 \leq C\left(\int_{\{\xi=\Gamma_\xi\}} \det(D^2\Gamma_\xi)\right)^{\frac{1}{n}},$$

where Γ_ξ is the convex envelope of ξ.

As in the previous section, there exists some $x \in \{\xi = \Gamma_\xi\} \cap B_\delta(0)$, where $w_\alpha^\epsilon(x)$ is second order differentiable, $\lambda(D^2w_\alpha^\epsilon)(x) \in \bar{V}$, and $D^2\xi(x) \geq 0$, i.e.

$$D^2\phi_\delta(x) \geq D^2w_\alpha^\epsilon(x).$$

It follows that $\lambda(D^2\phi_\delta)(x) \in \bar{V}$. Sending δ to 0, we have $\lambda(D^2\varphi)(0) \in \bar{V}$. Thus w_α^ϵ is a viscosity subsolution. Since $w_\alpha^\epsilon \to \alpha u + (1-\alpha)v$ in $C^0_{loc}(\Omega)$, sending ϵ to 0, it follows that $\alpha u + (1-\alpha)v$ is a viscosity subsolution.

□

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