VARIATION AND RIGIDITY OF QUASI-LOCAL MASS

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ABSTRACT. Inspired by the work of Chen and Zhang [5], we derive an evolution formula for the Chen-Wang-Wang-Yau quasi-local energy in reference to a static space. If the reference static space represents a mass minimizing, static extension of the initial surface Σ, we observe that the derivative of the Chen-Wang-Wang-Yau quasi-local energy is equal to the derivative of the Bartnik quasi-local mass at Σ.

Combining the evolution formula for the quasi-local energy with a localized Penrose inequality proved in [10], we prove a rigidity theorem for compact 3-manifolds with nonnegative scalar curvature, with boundary. This rigidity theorem in turn gives a characterization of the equality case of the localized Penrose inequality in 3-dimension.

1. INTRODUCTION

The purpose in this paper is twofold. We derive a derivative formula for the integral

$$\int_{\Sigma_t} N(\bar{H} - H)\, d\sigma \tag{1.1}$$

along a family of hypersurfaces $\{\Sigma_t\}$ evolving in a Riemannian manifold (M, g) with an assumption that Σ_t can be isometrically embedded in a static space $(\mathbb{N}, \bar{g})$ as a comparison hypersurface $\bar{\Sigma}_t$. Here H, $\bar{H}$ are the mean curvature of Σ_t, $\bar{\Sigma}_t$ in (M, g), $(\mathbb{N}, \bar{g})$, respectively, and N is the static potential on $(\mathbb{N}, \bar{g})$. When $\{\Sigma_t\}$ is a family of closed 2-surfaces in a 3-manifold (M, g), integral (1.1) represents the Chen-Wang-Wang-Yau quasi-local energy in reference to the static space $(\mathbb{N}, \bar{g})$. In this case, if $(\mathbb{N}, \bar{g})$ represents a mass minimizing, static extension of the initial surface Σ_0, we find that the derivative of the quasi-local energy agrees with the derivative of the Bartnik quasi-local mass at Σ_0 (see (2.7) in Section 2).

We also apply the derivative formula of (1.1) to prove a rigidity theorem for compact 3-manifolds with nonnegative scalar curvature, with boundary. Precisely, we have

Theorem 1.1. *Let $(\Omega, \breve{g})$ be a compact, connected, orientable, 3-dimensional Riemannian manifold with nonnegative scalar curvature, with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of two pieces, Σ_O and Σ_H, where*

- (i) *Σ_O has positive mean curvature H; and*
- (ii) *Σ_H is a minimal hypersurface (with one or more components) and there are no other closed minimal hypersurfaces in $(\Omega, \breve{g})$.*

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Let $\mathbb{M}^3_m$ *be a* 3*-dimensional spatial Schwarzschild manifold with mass* $m > 0$ *outside the horizon. Suppose* $\Sigma_{_O}$ *is isometric to a convex surface* $\Sigma \subset \mathbb{M}^3_m$ *which encloses a domain* Ω_m *with the horizon* $\partial \mathbb{M}^3_m$*. Suppose* $\overline{\mathrm{Ric}}(\nu, \nu) \le 0$ *on* Σ*, where* $\overline{\mathrm{Ric}}$ *is the Ricci curvature of the Schwarzschild metric* $\bar{g}$ *on* $\mathbb{M}^3_m$ *and* ν *is the outward unit normal to* Σ*. Let* H_m *be the mean curvature of* Σ *in* $\mathbb{M}^3_m$ *and* $|\Sigma_{_H}|$ *be the area of* $\Sigma_{_H}$ *in* $(\Omega, \breve{g})$*. If* $H = H_m$ *and* $\sqrt{\frac{|\Sigma_{_H}|}{16\pi}} = m$*, then* $(\Omega, \breve{g})$ *is isometric to* $(\Omega_m, \bar{g})$*.*

Theorem 1.1 gives a characterization of the equality case of a localized Penrose inequality proved in [10].

Theorem 1.2 ([10])**.** *Let* $(\Omega, \breve{g})$ *be a compact, connected, orientable,* 3*-dimensional Riemannian manifold with nonnegative scalar curvature, with boundary* $\partial\Omega$*. Suppose* $\partial\Omega$ *is the disjoint union of two pieces,* $\Sigma_{_O}$ *and* $\Sigma_{_H}$*, where*

(i) $\Sigma_{_O}$ *has positive mean curvature* H*; and*
(ii) $\Sigma_{_H}$*, if nonempty, is a minimal hypersurface (with one or more components) and there are no other closed minimal hypersurfaces in* $(\Omega, \breve{g})$*.*

Let $\mathbb{M}^3_m$ *be a* 3*-dimensional spatial Schwarzschild manifold with mass* $m > 0$ *outside the horizon. Suppose* $\Sigma_{_O}$ *is isometric to a convex surface* $\Sigma \subset \mathbb{M}^3_m$ *which encloses a domain* Ω_m *with the horizon* $\partial \mathbb{M}^3_m$*. Suppose* $\overline{\mathrm{Ric}}(\nu, \nu) \le 0$ *on* Σ*, where* $\overline{\mathrm{Ric}}$ *is the Ricci curvature of the Schwarzschild metric* $\bar{g}$ *on* $\mathbb{M}^3_m$ *and* ν *is the outward unit normal to* Σ*. Then*

$$m + \frac{1}{8\pi} \int_\Sigma N(H_m - H)\, d\sigma \ge \sqrt{\frac{|\Sigma_{_H}|}{16\pi}}. \tag{1.2}$$

Here N *is the static potential on* $\mathbb{M}^3_m$*,* H_m *is the mean curvature of* Σ *in* $\mathbb{M}^3_m$*, and* $|\Sigma_{_H}|$ *is the area of* $\Sigma_{_H}$ *in* $(\Omega, \breve{g})$*. Furthermore, equality in* (1.2) *holds if and only if*

$$H = H_m, \ \sqrt{\frac{|\Sigma_{_H}|}{16\pi}} = m. \tag{1.3}$$

By Theorems 1.1 and (1.3), we have the following rigidity statement concerning the equality case of (1.2).

Theorem 1.3. *Equality in* (1.2) *in Theorem 1.2 holds if and only if* $(\Omega, \breve{g})$ *is isometric to* $(\Omega_m, \bar{g})$*.*

Our motivation to consider the evolution of (1.1) and the proof of Theorem 1.1 are inspired by a recent paper of Chen and Zhang [5]. In [5], Chen-Zhang proved the global rigidity of a convex surface Σ with $\overline{\mathrm{Ric}}(\nu, \nu) \le 0$ among all isometric surfaces Σ' in $\mathbb{M}^3_m$ having the same mean curvature and enclosing the horizon. As a key step in their proof, they computed the first variation of the quasi-local energy of Σ' in reference to $\mathbb{M}^3_m$. Such a variational consideration is made possible by the openness result of solutions to the isometric embedding problem into warped product space, which is due to Li and Wang [8]. Combining the variation formula with inequality (1.2), Chen-Zhang established the rigidity of Σ in $\mathbb{M}^3_m$.

This paper may be viewed as a further application of the method of Chen-Zhang. In Section 2, we compute the derivative of (1.1) (see Formula 2.1) and relate it to the derivative of the Bartnik quasi-local mass. In Section 3, we prove Theorem 1.1 by applying Formula 2.1 and Theorem 1.2. In Section 4, we give a proof of the derivative formula of Bartnik mass for a specific class of Bartnik data, by applying Formula 2.1 and results in [10].

2. Evolution of quasi-local mass

In this section we derive a formula that is inspired by [5, Lemma 2]. First we fix some notations. Let (M, g) be an $(n+1)$-dimensional Riemannian manifold and Σ be an n-dimensional closed manifold. Consider a family of immersed hypersurfaces $\{\Sigma_t\}$ evolving in (M, g) according to

$$F : \Sigma \times I \longrightarrow M, \quad \frac{\partial F}{\partial t} = \eta\nu.$$

Here F is a smooth map, I is some open interval containing 0, $\Sigma_t = F_t(\Sigma)$ with $F_t(\cdot) = F(\cdot, t)$, ν is a chosen unit normal to $\Sigma_t = F_t(\Sigma)$, and η denotes the speed of the evolution of $\{\Sigma_t\}$.

Let $(\mathbb{N}, \bar{g})$ denote an $(n+1)$-dimensional *static* Riemannian manifold. Here $(\mathbb{N}, \bar{g})$ is called static (cf. [6]) if there exists a nontrivial function N such that

$$(\bar{\Delta} N)\bar{g} - \bar{D}^2 N + N\bar{Ric} = 0, \tag{2.1}$$

where $\bar{Ric}$ is the Ricci curvature of $(\mathbb{N}, \bar{g})$, $\bar{D}^2 N$ is the Hessian of N and $\bar{\Delta}$ is the Laplacian of N. The function N is called a static potential on $(\mathbb{N}, \bar{g})$.

In what follows, we consider another family of immersed hypersurfaces $\{\bar{\Sigma}_t\}$ evolving in $(\mathbb{N}, \bar{g})$ according to

$$\bar{F} : \Sigma \times I \longrightarrow \mathbb{N}$$

with $\bar{\Sigma}_t = \bar{F}_t(\Sigma)$ and $\bar{F}_t(\cdot) = \bar{F}(\cdot, t)$. We will make an important assumption:

$$\bar{F}_t^*(\bar{g}) = F_t^*(g), \ \forall\, t \in I. \tag{2.2}$$

In particular, this means that $\bar{\Sigma}_t$ is assumed to be isometric to Σ_t for each t.

Remark 2.1. We emphasize that, when $n = 2$, given any $\{\Sigma_t\}$ in (M, g), there exists such a family of $\{\bar{\Sigma}_t\}$ in $(\mathbb{N}, \bar{g})$ satisfying condition (2.2). This is guaranteed by the openness result of solutions to the isometric embedding problem, which is due to Li and Wang [8, 9].

We will compute

$$\frac{d}{dt}\int_\Sigma N_t(\bar{H}_t - H_t)\, d\sigma_t,$$

where $N_t = \bar{F}_t^*(N)$ is the pull back of the static potential N on $(\mathbb{N}, \bar{g})$; H_t, $\bar{H}_t$ are the mean curvature of Σ_t, $\bar{\Sigma}_t$ in (M, g), $(\mathbb{N}, \bar{g})$, respectively; and $d\sigma_t$ is the area element of the pull back metric $\gamma_t = \bar{F}_t^*(\bar{g}) = F_t^*(g)$. For simplicity, the lower index t is omitted below.

Formula 2.1. *Given* $\{\Sigma_t\}$, $\{\bar\Sigma_t\}$ *evolving in* (M,g), $(\mathbb{N},\bar g)$ *as specified above,*

$$\begin{aligned} &\frac{d}{dt}\int_\Sigma N(\bar H - H)\,d\sigma \\ &= \int_\Sigma N\left[\frac12|A-\bar A|^2 - \frac12|H-\bar H|^2 + \frac12(R-\bar R)\right]\eta\,d\sigma \\ &\quad + \int_\Sigma \left[(f-\eta)\frac{\partial N}{\partial\bar\nu} + \langle\nabla N, X\rangle\right](\bar H - H)\,d\sigma. \end{aligned} \tag{2.3}$$

Here A, $\bar A$ *are the second fundamental forms of* Σ_t, $\bar\Sigma_t$ *in* (M,g), $(\mathbb{N},\bar g)$, *respectively;* R, $\bar R$ *are the scalar curvature of* (M,g), $(\mathbb{N},\bar g)$, *respectively;* f *and* X *are the lapse and the shift associated to* $\frac{\partial\bar F}{\partial t}$, *i.e.* $\frac{\partial\bar F}{\partial t} = f\bar\nu + X$, *where* f *is a function and* X *is tangential to* $\bar\Sigma_t$*; and* ∇ *denotes the gradient on* $(\bar\Sigma_t,\gamma)$.

Remark 2.2. Suppose (M,g) and $(\mathbb{N},\bar g)$ both are $\mathbb{M}^3_m$ and suppose $H=\bar H$ at $t=0$, (2.3) becomes

$$\frac{d}{dt}\Big|_{t=0}\int_\Sigma N(\bar H - H)\,d\sigma = \frac12\int_\Sigma N|A-\bar A|^2\eta\,d\sigma. \tag{2.4}$$

This is the formula in [5, Lemma 2].

Remark 2.3. If $X=0$ and $R=\bar R$, (2.3) reduces to

$$\begin{aligned} &\frac{d}{dt}\int_\Sigma N(\bar H - H)\,d\sigma \\ &= \int_\Sigma N\left[\frac12|A-\bar A|^2 - \frac12|H-\bar H|^2\right]\eta\,d\sigma + \int_\Sigma (f-\eta)\frac{\partial N}{\partial\bar\nu}(\bar H - H)\,d\sigma \\ &= \int_\Sigma \eta^{-1}(f-\eta)^2\left(-N\sigma_2 - \bar H\frac{\partial N}{\partial\bar\nu}\right)d\sigma. \end{aligned} \tag{2.5}$$

This is the formula in [10, Proposition 2.2].

We comment on the physical meaning of (2.3). Suppose $n=2$. In [4], Chen, Wang, Wang and Yau introduced a notion of quasi-local energy in reference to a static spacetime. (The notion is a generalization of the Wang-Yau quasi-local energy [13, 14] for which the reference spacetime is $\mathbb{R}^{3,1}$.) Setting $\tau=0$ in equation (2.10) in [4], one sees that the quasi-local energy of a 2-surface $\Sigma\subset(M^3,g)$ defined in [4] with respect to an isometric embedding of Σ into a constant t-slice of the static spacetime $(\mathbb{R}^1\times\mathbb{N}, -N^2dt^2+\bar g)$ is the integral

$$\frac{1}{8\pi}\int_\Sigma N(\bar H - H)\,d\sigma.$$

Therefore, up to a multiplicative constant, (2.3) gives the evolution formula of the Chen-Wang-Wang-Yau quasi-local energy of the flowing surfaces $\{\Sigma_t\}$ in (M,g).

Next, we tie (2.3) with the evolution formula of the Bartnik quasi-local mass $\mathfrak{m}_{_B}(\cdot)$. We defer the detailed definition of the Bartnik mass $\mathfrak{m}_{_B}(\cdot)$ to Section 4. For the moment, we recall the following evolution formula of $\mathfrak{m}_{_B}(\cdot)$ derived in [12, Theorem 3.1] under a stringent condition.

Formula 2.2 ([12])**.** *Suppose Σ_t has a mass minimizing, static extension (M^s_t, g^s_t) such that $\{(M^s_t, g^s_t)\}$ depends smoothly on t. One has*

$$\frac{d}{dt}|_{t=0}\mathfrak{m}_{_B}(\Sigma_t) = \frac{1}{16\pi}\int_\Sigma N\left(|A-\bar{A}|^2 + R\right)\eta\, d\sigma. \tag{2.6}$$

To relate (2.3) to (2.6), we assume that $(\mathbb{N}, \bar{g})$ represents a mass minimizing, static extension of the surface $\Sigma_0 \subset (M, g)$. Then, by assumption, $H = \bar{H}$ at $t = 0$. It follows from (2.3) and (2.6) that

$$\begin{aligned} &\frac{d}{dt}|_{t=0}\frac{1}{8\pi}\int_\Sigma N(\bar{H} - H)\, d\sigma \\ =&\frac{1}{16\pi}\int_\Sigma N\left[|A-\bar{A}|^2 + (R-\bar{R})\right]\eta\, d\sigma \\ =&\frac{d}{dt}|_{t=0}\mathfrak{m}_{_B}(\Sigma_t). \end{aligned}\tag{2.7}$$

We will reflect more on this relation in Section 4.

In the remainder of this section, we give a proof of Formula 2.1.

Proof of Formula 2.1. To verify (2.3), we can pull the metrics g, $\bar{g}$ back to $\Sigma\times I$ via F, $\bar{F}$, respectively, and write them as

$$g = \eta^2 dt^2 + \gamma \tag{2.8}$$

and

$$\bar{g} = f^2dt^2 + \gamma_{\alpha\beta}(dx^\alpha + X^\alpha dt)(dx^\beta + X^\beta dt). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\gamma' = 2\eta A,\ \partial_t d\sigma = \eta H\, d\sigma \tag{2.10}$$

and

$$\gamma' = 2f\bar{A} + L_X\gamma,\ \partial_t d\sigma = (f\bar{H} + \mathrm{div}X)\, d\sigma, \tag{2.11}$$

where $\mathrm{div}X$ is the divergence of X on (Σ, γ). Thus, we have

$$2\eta A = 2f\bar{A} + L_X\gamma, \quad \eta H = f\bar{H} + \mathrm{div}X\, . \tag{2.12}$$

We first compute

$$\frac{d}{dt}\int_\Sigma N\bar{H}\, d\sigma = \int_\Sigma (N'\bar{H} + N\bar{H}')\, d\sigma + N\bar{H}\,\partial_t d\sigma. \tag{2.13}$$

Let $\bar{\nabla}$ denote the gradient on $(\mathbb{N}, \bar{g})$. We have

$$N' = \langle\bar{\nabla}N, \frac{\partial\bar{F}}{\partial t}\rangle = \langle\bar{\nabla}N, f\bar{\nu} + X\rangle = f\frac{\partial N}{\partial\bar{\nu}} + \langle\nabla N, X\rangle. \tag{2.14}$$

Hence,

$$\int_\Sigma N'\bar{H}\,d\sigma = \int_\Sigma \left(f\frac{\partial N}{\partial \bar{\nu}} + \langle \nabla N, X\rangle \right) \bar{H}\,d\sigma. \tag{2.15}$$

Recall that

$$\bar{A}'_{\alpha\beta} = f\bar{A}_{\alpha\delta}\bar{A}^{\delta}_{\beta} + (L_X\bar{A})_{\alpha\beta} - (\nabla^2 f)_{\alpha\beta} + f\langle \bar{R}(\bar{\nu},\partial_\alpha)\bar{\nu},\partial_\beta\rangle, \tag{2.16}$$

where ∇^2 denotes the Hessian on (Σ,γ). Hence,

$$\begin{aligned} \bar{H}' &= (\gamma^{\alpha\beta})'\bar{A}_{\alpha\beta} + \gamma^{\alpha\beta}\bar{A}'_{\alpha\beta} \\ &= -\langle \gamma', \bar{A}\rangle + f|\bar{A}|^2 + \langle \gamma, L_X\bar{A}\rangle - \Delta f - \bar{R}ic(\bar{\nu},\bar{\nu})f. \end{aligned} \tag{2.17}$$

By (2.11),

$$\langle \gamma', \bar{A}\rangle = \langle 2f\bar{A} + L_X\gamma, \bar{A}\rangle = 2f|\bar{A}|^2 + \langle L_X\gamma, \bar{A}\rangle.$$

Thus,

$$\bar{H}' = -\langle L_X\gamma, \bar{A}\rangle + \langle \gamma, L_X\bar{A}\rangle - \Delta f - f|\bar{A}|^2 - \bar{R}ic(\bar{\nu},\bar{\nu})f. \tag{2.18}$$

One checks that

$$-\langle L_X\gamma, \bar{A}\rangle + \langle \gamma, L_X\bar{A}\rangle = \langle X, \nabla\bar{H}\rangle. \tag{2.19}$$

Hence,

$$\bar{H}' = -\Delta f - f|\bar{A}|^2 - \bar{R}ic(\bar{\nu},\bar{\nu})f + \langle X, \nabla\bar{H}\rangle. \tag{2.20}$$

Thus,

$$\begin{aligned} \int_\Sigma N\bar{H}'\,d\sigma &= \int_\Sigma (-\Delta N - \bar{R}ic(\bar{\nu},\bar{\nu})N)f + N\left[-f|\bar{A}|^2 + \langle X, \nabla\bar{H}\rangle\right]\,d\sigma \\ &= \int_\Sigma \bar{H}\frac{\partial N}{\partial\bar{\nu}}f - Nf|\bar{A}|^2 + N\langle X,\nabla\bar{H}\rangle\,d\sigma. \end{aligned} \tag{2.21}$$

Here we have used

$$\Delta N + \bar{R}ic(\bar{\nu},\bar{\nu})N = -\bar{H}\frac{\partial N}{\partial\bar{\nu}},$$

which follows from the static equation (2.1).

By (2.15) and (2.21),

$$\begin{aligned} &\int_\Sigma N'\bar{H} + N\bar{H}'\,d\sigma \\ =& \int_\Sigma \left(f\frac{\partial N}{\partial\bar{\nu}} + \langle\nabla N, X\rangle\right)\bar{H} + \bar{H}\frac{\partial N}{\partial\bar{\nu}}f - Nf|\bar{A}|^2 + N\langle X,\nabla\bar{H}\rangle\,d\sigma \\ =& \int_\Sigma 2f\frac{\partial N}{\partial\bar{\nu}}\bar{H} - Nf|\bar{A}|^2 + \langle X, \nabla(N\bar{H})\rangle\,d\sigma. \end{aligned} \tag{2.22}$$

On the other hand, by (2.11),

$$\int_\Sigma N\bar{H}\,\partial_t d\sigma = \int_\Sigma N\bar{H}(f\bar{H} + \mathrm{div}X)\,d\sigma. \tag{2.23}$$

Therefore, it follows from (2.22) and (2.23) that

$$\frac{d}{dt}\int_\Sigma N\bar{H}\,d\sigma = \int_\Sigma 2f\frac{\partial N}{\partial\bar{\nu}}\bar{H} + Nf(\bar{H}^2 - |\bar{A}|^2)\,d\sigma\,. \tag{2.24}$$

To proceed, we note that by (2.11),

$$2f(\bar{H}^2 - |\bar{A}|^2) = \langle \bar{H}\gamma - \bar{A}, 2f\bar{A}\rangle = \langle \bar{H}\gamma - \bar{A}, \gamma'\rangle - \langle \bar{H}\gamma - \bar{A}, L_X\gamma\rangle\,. \tag{2.25}$$

Thus,

$$2\int_\Sigma Nf(\bar{H}^2 - |\bar{A}|^2)\,d\sigma = \int_\Sigma N\langle \bar{H}\gamma - \bar{A}, \gamma'\rangle - N\langle \bar{H}\gamma - \bar{A}, L_X\gamma\rangle\,d\sigma. \tag{2.26}$$

Integrating by parts, we have

$$\begin{aligned}&\int_\Sigma N\langle \bar{H}\gamma - \bar{A}, L_X\gamma\rangle\,d\sigma\\ =&\ -2\int_\Sigma(\bar{H}\gamma - \bar{A})(\nabla N, X) - 2\int_\Sigma N(d\bar{H} - \mathrm{div}\bar{A})(X)\,d\sigma.\end{aligned} \tag{2.27}$$

By the Codazzi equation and the static equation,

$$N(\mathrm{div}\bar{A} - d\bar{H})(X) = N\bar{R}ic(X,\bar{\nu}) = \bar{D}^2N(X,\bar{\nu}). \tag{2.28}$$

Here

$$\bar{D}^2N(X,\nu) = -\bar{A}(\nabla N, X) + X\left(\frac{\partial N}{\partial\bar{\nu}}\right).$$

Hence,

$$\int_\Sigma N\langle \bar{H}\gamma - \bar{A}, L_X\gamma\rangle\,d\sigma = \int_\Sigma -2\bar{H}\langle\nabla N, X\rangle + 2X\left(\frac{\partial N}{\partial\bar{\nu}}\right)\,d\sigma. \tag{2.29}$$

Therefore, (2.24) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\int_\Sigma N\bar{H}\,d\sigma\\ =&\int_\Sigma 2f\frac{\partial N}{\partial\bar{\nu}}\bar{H} + \bar{H}\langle\nabla N, X\rangle - X\left(\frac{\partial N}{\partial\bar{\nu}}\right) + \frac{1}{2}N\langle \bar{H}\gamma - \bar{A}, \gamma'\rangle\,d\sigma.\end{aligned} \tag{2.30}$$

We now turn to the term $\int_\Sigma NH\,d\sigma$. We have

$$\begin{aligned}\frac{d}{dt}\int_\Sigma NH\,d\sigma =& \int_\Sigma N'H + NH' + NH\eta H\,d\sigma\\ =& \int_\Sigma\left(f\frac{\partial N}{\partial\bar{\nu}} + \langle\nabla N, X\rangle\right)H\\ &+ N\left[-\Delta\eta - (|A|^2 + \mathrm{Ric}(\nu,\nu))\eta\right] + NH^2\eta\,d\sigma.\end{aligned} \tag{2.31}$$

Here

$$-\int_\Sigma N\Delta\eta\,d\sigma = -\int_\Sigma(\Delta N)\eta\,d\sigma = \int_\Sigma\left(\bar{H}\frac{\partial N}{\partial\bar{\nu}} + \bar{R}ic(\bar{\nu},\bar{\nu})N\right)\eta. \tag{2.32}$$

Therefore,

$$\frac{d}{dt}\int_\Sigma NH\,d\sigma = \int_\Sigma f\frac{\partial N}{\partial\bar\nu}H + \langle\nabla N, X\rangle H + \bar H\frac{\partial N}{\partial \bar\nu}\eta + N\left[\bar Ric(\bar\nu,\bar\nu) - (|A|^2 + \mathrm{Ric}(\nu,\nu)) + H^2\right]\eta\,d\sigma \tag{2.33}$$

We group the zero order terms of N in $\frac{d}{dt}\int_\Sigma N(\bar H - H)\,d\sigma$ first. Using $\gamma' = 2\eta A$, we have

$$\frac{1}{2}N\langle\bar H\gamma - \bar A, \gamma'\rangle = N\langle \bar H\gamma - \bar A, A\rangle\eta. \tag{2.34}$$

Thus, omitting the terms η and N, using the Gauss equation, we have

$$\begin{aligned}&\langle\bar H\gamma - \bar A, A\rangle - \bar Ric(\bar\nu,\bar\nu) + \mathrm{Ric}(\nu,\nu)) + |A|^2 - H^2\\ =\ &\langle\bar H\gamma - \bar A, A\rangle + \frac{1}{2}(R-\bar R) - \frac{1}{2}(H^2 - |A|^2) - \frac{1}{2}(\bar H^2 - |\bar A|^2)\\ =\ &\frac{1}{2}|A-\bar A|^2 - \frac{1}{2}|H - \bar H|^2 + \frac{1}{2}(R - \bar R).\end{aligned} \tag{2.35}$$

Integrating by part and using the fact $\eta H = f\bar H + \mathrm{div}X$, we conclude

$$\begin{aligned}&\frac{d}{dt}\int_\Sigma N(\bar H - H)\,d\sigma\\ =\ &\int_\Sigma N\left[\frac{1}{2}|A-\bar A|^2 - \frac{1}{2}|H-\bar H|^2 + \frac{1}{2}(R-\bar R)\right]\eta\,d\sigma\\ &+\int_\Sigma (2f\bar H - fH - \eta\bar H + \mathrm{div}X)\frac{\partial N}{\partial\bar\nu} + (\bar H - H)\langle\nabla N, X\rangle\,d\sigma\\ =\ &\int_\Sigma N\left[\frac{1}{2}|A-\bar A|^2 - \frac{1}{2}|H-\bar H|^2 + \frac{1}{2}(R-\bar R)\right]\eta\,d\sigma\\ &+\int_\Sigma\left[(f-\eta)\frac{\partial N}{\partial\bar\nu} + \langle\nabla N, X\rangle\right](\bar H - H)\,d\sigma.\end{aligned} \tag{2.36}$$

□

3. Equality case of the localized Penrose inequality

In this section, we apply Formula 2.1, the openness result of the isometric embedding problem [8], and Theorem 1.2 to prove Theorem 1.1.

Proof of Theorem 1.1. Let A, $\bar A$ be the second fundamental form of Σ_o, Σ in $(\Omega, \breve g)$, $\mathbb{M}^3_m$, respectively. Viewing $\bar A$ as a tensor on Σ_o via the surface isometry, we want to show $A = \bar A$.

In $(\Omega,\breve g)$, consider a smooth family of 2-surfaces $\{\Sigma_t\}_{-\epsilon<t\le 0}$ such that $\Sigma_0 = \Sigma_o$ and Σ_t is $|t|$-distance away from Σ_o. We can parametrize $\{\Sigma_t\}$ so that, as t increases, Σ_t evolves in a direction normal to Σ_t and has constant unit speed. Applying the openness result of the isometric embedding problem in [8], we obtain a smooth family

of 2-surfaces $\{\bar{\Sigma}_t\}_{-\epsilon<t\leq 0}$ in $\mathbb{M}^3_m$ so that $\bar{\Sigma}_0 = \Sigma$ and condition (2.2) is satisfied by $\{\Sigma_t\}$ and $\{\bar{\Sigma}_t\}$. By (2.3) and the assumption $H = H_m$, we have

$$\frac{d}{dt}|_{t=0} \int_{\Sigma_t} N(\bar{H} - H)\, d\sigma = \frac{1}{2} \int_{\Sigma_O} N(|A - \bar{A}|^2 + R)\, d\sigma. \tag{3.1}$$

Here N is the static potential on $\mathbb{M}^3_m$, which is positive away from the horizon, and R is the scalar curvature of $(\Omega, \breve{g})$.

Suppose $A \neq \bar{A}$. Then, by (3.1) and the assumption $R \geq 0$,

$$\frac{d}{dt}|_{t=0} \int_{\Sigma_t} N(\bar{H} - H)\, d\sigma > 0. \tag{3.2}$$

Thus, for small $t < 0$,

$$\int_{\Sigma_t} N(\bar{H} - H)\, d\sigma < 0. \tag{3.3}$$

We claim (3.3) contradicts Theorem 1.2. To see this, we can first consider the case $\overline{\mathrm{Ric}}(\nu, \nu) < 0$ on Σ. By choosing ϵ small, we may assume $\overline{\mathrm{Ric}}(\nu, \nu) < 0$ on each $\bar{\Sigma}_t$. Hence, we can apply Theorem 1.2 to the region in Ω enclosed by Σ_t and Σ_H. It follows from (1.2) and the assumption $m = \sqrt{\frac{|\Sigma_H|}{16\pi}}$ that

$$\int_{\Sigma_t} N(\bar{H} - H)\, d\sigma \geq 0. \tag{3.4}$$

This is a contradiction to (3.3).

To include the case $\overline{\mathrm{Ric}}(\nu, \nu) \leq 0$ on Σ, we point out that this assumption was imposed in [10] only to guarantee that the flow in $\mathbb{M}^3_m$, which starts from Σ and satisfies equation (4.2) in [10], has the property that its leaves have positive scalar curvature (see Lemma 3.8 in [10]). Now, if Σ is slightly perturbed to a nearby surface Σ' in $\mathbb{M}^3_m$, though Σ' may not satisfy $\overline{\mathrm{Ric}}(\nu, \nu) \leq 0$, the flow to (4.2) in [10] starting from Σ' remains to have such a property. (More precisely, this follows from estimates in Lemmas 3.6, 3.7 and 3.11 of [10].) Therefore, for small $t < 0$, we can still apply Theorem 1.2 to conclude (3.4), which contradicts (3.3).

Thus we have $A = \bar{A}$. For the same reason, we also know $R = 0$ along Σ_O in $(\Omega, \breve{g})$. Next, we consider the manifold $(\hat{M}, \hat{g})$ obtained by gluing $(\Omega, \breve{g})$ and $(\mathbb{M}^3_m \setminus \Omega_m, \bar{g})$ along Σ_O that is identified with Σ. Since $A = \bar{A}$, the metric $\hat{g}$ on $\hat{M}$ is $C^{1,1}$ across Σ_O and is smooth up to Σ_O from its both sides in $\hat{M}$. To finish the proof, we check that the rigidity statement of the Riemannian Penrose inequality holds on this $(\hat{M}, \hat{g})$.

We apply the conformal flow used by Bray [2] in his proof of the Riemannian Penrose inequality. Since $\hat{g}$ is $C^{1,1}$, equations (13) - (16) in [2] which define the flow hold in the classical sense when g_0 is replaced by $\hat{g}$. Existence of this flow with initial condition $\hat{g}$ follows from Section 4 in [2]. The difference is that, along the flow which we denote by $\{\hat{g}(t)\}$, the outer minimizing horizon $\Sigma(t)$ is $C^{2,\alpha}$ and the green function in Theorems 8 and 9 in [2] is $C^{2,\alpha}$, for any $0 < \alpha < 1$. These regularities are sufficient to show Theorem 6 in [2] holds, i.e. the area of $\Sigma(t)$ stays the same; and the results on the mass and the capacity in Theorems 8 and 9 in [2] remain valid. Moreover, at $t=0$, by the proof of Theorem 10 in [2], i.e. equation (113), we have

$$\frac{d}{dt^+} m(t)|_{t=0} = \mathcal{E}(\Sigma_{_H}, \hat{g}) - 2m \le 0, \tag{3.5}$$

where $\mathcal{E}(\Sigma_{_H}, \hat{g})$ is the capacity of $\Sigma_{_H}$ in $(\hat{M}, \hat{g})$ and the inequality in (3.5) is given by Theorem 9 in [2].

Now, if $\frac{d}{dt^+} m(t)|_{t=0} < 0$, then for t small, we would have

$$m(t) < m = \sqrt{\frac{|\Sigma_{_H}|}{16\pi}} = \sqrt{\frac{|\Sigma(t)|}{16\pi}}, \tag{3.6}$$

where $m(t)$ is the mass of $\hat{g}(t)$. But (3.6) violates the Riemannian Penrose inequality (for metrics possibly with corner along a hypersurface, cf. [11]). Thus, we must have

$$\frac{d}{dt^+} m(t)|_{t=0} = \mathcal{E}(\Sigma_{_H}, \hat{g}) - 2m = 0.$$

Since Theorem 9 in [2] holds on $(\hat{M}, \hat{g})$, by its rigidity statement we conclude that $(\hat{M}, \hat{g})$ is isometric to $\mathbb{M}^3_m$. □

4. Implication on Bartnik mass

In (2.7) of Section 2, we have observed that, if $(\mathbb{N}, \bar{g})$ represents a mass minimizing, static extension of $\Sigma_0 \subset (M, g)$, then

$$\frac{d}{dt}|_{t=0} \mathfrak{m}_{_B}(\Sigma_t) = \frac{d}{dt}|_{t=0} \frac{1}{8\pi} \int_\Sigma N(\bar{H} - H)\, d\sigma. \tag{4.1}$$

However, this observation was based on (2.6) in Formula 2.2, which requires a rather stringent assumption that mass minimizing, static extensions of $\{\Sigma_t\}$ exist and depend smoothly on t.

In this section, assuming that $\mathfrak{m}_{_B}(\cdot)$ is differentiable, we prove that (2.6) is true whenever the Bartnik data of Σ_0 corresponds to a surface in a Schwarzschild manifold. We recall the definition of $\mathfrak{m}_{_B}(\cdot)$ as follows. Given a closed 2-surface Σ in a 3-manifold (M, g) with nonnegative scalar curvature, $\mathfrak{m}_{_B}(\Sigma)$ is given by

$$\mathfrak{m}_{_B}(\Sigma) = \inf \left\{ \mathfrak{m}(\tilde{g}) \,|\, (\tilde{M}, \tilde{g}) \text{ is an admissible extension of } \Sigma \right\}. \tag{4.2}$$

Here $\mathfrak{m}(\tilde{g})$ is the mass of $(\tilde{M}, \tilde{g})$, which is an asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary $\partial \tilde{M}$. $(\tilde{M}, \tilde{g})$ is called an admissible extension of Σ if $\partial \tilde{M}$ is isometric to Σ and the mean curvature of $\partial \tilde{M}$ equals the mean curvature H of Σ. Moreover, it is assumed that $(\tilde{M}, \tilde{g})$ satisfies certain non-degeneracy condition that prevents $\mathfrak{m}(\tilde{g})$ from becoming trivially small. For instance, one often assumes that $(\tilde{M}, \tilde{g})$ contains no closed minimal surfaces or $\partial \tilde{M}$ is outer minimizing in $(\tilde{M}, \tilde{g})$ (cf. [1, 2, 3, 7]).

With the above definition of $\mathfrak{m}_{_B}(\cdot)$, we have

Theorem 4.1. *Let (M,g) be a 3-manifold of nonnegative scalar curvature. Suppose $\Sigma_0 \subset (M,g)$ is isometric to a convex surface $\bar\Sigma_0$ with $\overline{\mathrm{Ric}}(\nu,\nu) \le 0$ in a spatial Schwarzschild manifold $(\mathbb{M}^3_m,\bar g)$. Suppose $\bar\Sigma_0$ encloses a domain Ω_m with the horizon and Σ_0 and $\bar\Sigma_0$ have the same mean curvature. Let $\{\Sigma_t\}_{|t|<\epsilon}$ be a smooth family of 2-surfaces evolving in (M,g) according to $\frac{\partial F}{\partial t} = \eta\nu$. If $\mathfrak{m}_{_B}(\Sigma_t)$ is differentiable at $t=0$, then*

$$\frac{d}{dt}\Big|_{t=0}\mathfrak{m}_{_B}(\Sigma_t) = \frac{1}{16\pi}\int_{\Sigma_0} N(|A-\bar A|^2 + R)\eta\, d\sigma. \tag{4.3}$$

Here A, $\bar A$ are the second fundamental form of Σ_0, $\bar\Sigma_0$ in (M,g), $\mathbb{M}^3_m$, respectively, and R is the scalar curvature of (M,g).

Proof. First we note that the assumptions on Σ_0 shows $\mathfrak{m}_{_B}(\Sigma_0) = m$. This is because, $\mathbb{M}^3_m \setminus \Omega_m$ is an admissible extension of Σ_0, thus $\mathfrak{m}_{_B}(\Sigma_0) \le m$ by definition. On the other hand, if $(\tilde M, \tilde g)$ is any other admissible extension of Σ_0, by gluing $(\tilde M,\tilde g)$ with Ω_m along $\bar\Sigma_0$ and applying the Riemannian Penrose inequality, we have $\mathfrak{m}(\tilde g) \ge m$. Hence, $\mathfrak{m}_{_B}(\Sigma_0) = m$.

Next, we proceed as in the proof of Theorem 1.1. By the result of Li-Wang [8], there exists a smooth family of surfaces $\{\bar\Sigma_t\}$ in $\mathbb{M}^3_m$ such that $\bar\Sigma_t$ is isometric to Σ_t for each t. By [10, Theorem 5.1], we have

$$\mathfrak{m}_{_B}(\Sigma_t) \le m + \frac{1}{8\pi}\int_{\Sigma_t} N(\bar H - H)\, d\sigma \tag{4.4}$$

for small t. Note that the right side of (4.4) equals m when $t=0$. Therefore, it follows from (4.4) and the fact $\mathfrak{m}_{_B}(\Sigma_0) = m$ that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathfrak{m}_{_B}(\Sigma_t) &= \frac{d}{dt}\Big|_{t=0}\frac{1}{8\pi}\int_{\Sigma_t} N(\bar H - H)\, d\sigma \\ &= \frac{1}{16\pi}\int_{\Sigma_0} N(|A-\bar A|^2 + R)\eta\, d\sigma.\end{aligned} \tag{4.5}$$

Here in the last step we have used Formula 2.1. □

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