Spikes of the two-component elliptic system in $\mathbb{R}^4$ with the critical Sobolev exponent*

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Abstract: Consider the following elliptic system:

$$\begin{cases} -\varepsilon^2\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \Omega, \\ -\varepsilon^2\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \quad u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain, $\lambda_i, \mu_i, \alpha_i > 0$ $(i = 1, 2)$ and $\beta \neq 0$ are constants, $\varepsilon > 0$ is a small parameter and $2 < p < 2^* = 4$. By using variational methods, we study the existence of ground state solutions to this system for sufficiently small $\varepsilon > 0$. The concentration behaviors of least-energy solutions as $\varepsilon \to 0^+$ are also studied. Furthermore, by combining elliptic estimates and local energy estimates, we obtain the locations of these spikes as $\varepsilon \to 0^+$.

Keywords: elliptic system; critical Sobolev exponent; spike; semi-classical solution; variational method.

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1 Introduction

In this paper, we study the following elliptic system:

$$\begin{cases} -\varepsilon^2\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \Omega, \\ -\varepsilon^2\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \quad u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{S}_\varepsilon$}$$

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where $\Omega \subset \mathbb{R}^4$ is a bounded domain, $\lambda_i, \mu_i, \alpha_i > 0$ $(i = 1, 2)$ and $\beta \neq 0$ are constants, $\varepsilon > 0$ is a small parameter and $2 < p < 2^* = 4$.

The solutions of $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) in $\mathbb{R}^N$ with $1 \leq N \leq 3$ for $\alpha_1 = \alpha_2 = 0$ are related to the solitary wave solutions of the following two coupled nonlinear Schrödinger equations, which are also known in the literature as the Gross-Pitaevskii equations (e.g., [21, 40]):

$$\begin{cases} -\iota\dfrac{\partial}{\partial t}\Psi_1 = \Delta\Psi_1 + \mu_1|\Psi_1|^2\Psi_1 + \beta|\Psi_2|^2\Psi_1, \\ -\iota\dfrac{\partial}{\partial t}\Psi_2 = \Delta\Psi_2 + \mu_2|\Psi_2|^2\Psi_2 + \beta|\Psi_1|^2\Psi_2, \\ \Psi_i = \Psi_i(t,x) \in H^1(\mathbb{R}^N;\mathbb{C}), \quad i = 1, 2, \quad N = 1, 2, 3. \end{cases}$$

Here, ι is the imaginary unit. This system appears in many physical problems. For example, in Hartree-Fock theory, the Gross-Pitaevskii equations can be used to describe a binary mixture of Bose-Einstein condensates in two hyperfine states: $|1\rangle$ and $|2\rangle$ (cf. [18]). Solutions $\Psi_j (j = 1, 2)$ are the corresponding condensate amplitudes and μ_i are the intraspecies and interspecies scattering lengths. β is the interaction of the states $|1\rangle$ and $|2\rangle$ and the interaction is attractive if $\beta > 0$ and repulsive if $\beta < 0$. The Gross-Pitaevskii equations also arise in nonlinear optics (cf. [1]). To obtain solitary wave solutions, we set $\Psi_i(t,x) = e^{\iota t\lambda_i}u_i(x)$ for both $i = 1, 2$. Then, u_i satisfy $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) for $\alpha_1 = \alpha_2 = 0$. Due to important applications in physics, system $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) in low dimensions $(1 \leq N \leq 3)$ for $\alpha_1 = \alpha_2 = 0$ has been studied extensively in the last decades. Since it seems almost impossible for us to provide a complete list of references, we refer the readers only to [3, 9, 12, 16, 25, 27, 29, 30, 35, 38–40, 44, 46] and the references therein, where various existence theorems of solitary wave solutions were established.

Recently, $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) for $\alpha_1 = \alpha_2 = 0$ in $\mathbb{R}^N$ with $N \geq 4$ has begun to attract attention (cf. [10, 41]). The cubic nonlinearities and coupled terms all exhibit critical growth for $N = 4$ and even super-critical growth for $N \geq 5$ with respect to the critical Sobolev exponent. Thus, the study of these cases is much more complicated than that in low dimensions from the viewpoint of calculus of variations. By applying a truncation argument, Tavares and Terracini in [41] proved that the k-component system $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) for $\alpha_1 = \alpha_2 = 0$ has infinitely many sign-changing solutions for all $N \geq 2$ and $k \geq 2$ with $\mu_1, \mu_2, \cdots, \mu_k \leq 0$, whereas $\lambda_i > 0$ for all $i = 1, 2, \cdots, k$ appeared as the Lagrange multipliers, which are not specified in advance. In [10], by establishing the threshold for the compactness of the (PS) sequence to the 2-component system $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) for $\alpha_1 = \alpha_2 = 0$ and $N = 4$ and performing a careful and complicated analysis, Chen and Zou proved that the 2-component system $(\mathcal{S}_1)$ (i.e., $\varepsilon = 1$) for $\alpha_1 = \alpha_2 = 0$ has a positive least-energy solution for $N = 4$ and $-\sigma_1 < \lambda_i < 0$ for all $i = 1, 2$, where σ_1 is the first eigenvalue of $-\Delta$ in $L^2(\Omega)$ with Dirichlet boundary conditions. There are other studies on elliptic systems with the critical Sobolev exponent; see, for example, [2, 11–15, 32, 36, 37, 48] and the references therein, which contain many interesting results.

When $\varepsilon > 0$ is a small parameter, the solutions of $(\mathcal{S}_\varepsilon)$ are called semi-classical bound state solutions. Such solutions have been studied extensively for single equations in the past thirty years. Since it seems almost impossible for us to provide a complete list of references and many results are well known, we refer the readers only to [6,8,34,42,49] and the references therein. To the best of our knowledge, the first result for such solutions of the elliptic system is contributed by Lin and Wei in [26], where $(\mathcal{S}_\varepsilon)$ with $\alpha_1 = \alpha_2 = 0$ for $1 \le N \le 3$ is studied. By using variational methods, the authors proved that this system has a least-energy solution for sufficiently small $\varepsilon > 0$. They also studied the concentration behaviors of least-energy solutions and described the locations of spikes as $\varepsilon \to 0^+$. Since then, many works have been devoted to semi-classical solutions of elliptic systems in low dimensions ($N = 1, 2, 3$) and various similar results have been established. We refer the readers to [7, 23, 27, 28, 31, 33, 45, 51] and the references therein for such studies. In particular, in [51], by using the radial symmetry of the annulus $A = \{x \in \mathbb{R}^4 \mid a \le |x| \le b\}$, Zhang and Marcos do O reduced $(\mathcal{S}_\varepsilon)$ in A for $\alpha_1 = \alpha_2 = 0$ to a new system in $\mathbb{R}^3$. Then by using variational methods, they showed that $(\mathcal{S}_\varepsilon)$ in A for $\alpha_1 = \alpha_2 = 0$ has a solution for sufficiently small $\varepsilon > 0$ with $\beta > 0$ being in a range. Moreover, the solutions found by them are spiked solutions will concentrate on the sphere $|x| = a$ as $\varepsilon \to 0$.

Motivated by the above discussion, we pose the following questions: What happens to $(\mathcal{S}_\varepsilon)$ for $N = 4$ in a generally bounded domain Ω? Do similar results hold? For $N = 4$, $(\mathcal{S}_\varepsilon)$ is of critical growth, namely, cubic terms u_1^3, u_2^3 and coupling terms $u_2^2 u_1, u_1^2 u_2$ are all of critical growth in the sense of the Sobolev embedding, and it is well known that, for a Sobolev critical equation or system, the existence of a nontrivial solution is delicate. Indeed, if one wants to take positive λ_1, λ_2 in $(\mathcal{S}_\varepsilon)$ in a generally bounded domain Ω then $\alpha_1, \alpha_2 > 0$ seems to be necessary in the sense that $(\mathcal{S}_\varepsilon)$ has *no* solution when Ω is star-shaped for $\lambda_1, \lambda_2 > 0$ and $\alpha_1 = \alpha_2 = 0$. This nonexistence result can be obtained by applying the Pohozaev identity, as in [10]. On the other hand, $\alpha_1, \alpha_2 > 0$ also seems to be necessary in observing the concentration behavior of solutions of $(\mathcal{S}_\varepsilon)$ as $\varepsilon \to 0^+$. An interesting phenomenon in the semi-classical setting $\varepsilon \to 0^+$ is that solutions will concentrate around their maximum points and convergence strongly to the nontrivial solution of the limit equation or system under a suitable scaling that is centered at the maximum point. Thus, the existence of nontrivial solutions of the limit equation or system is very important for observing this phenomenon. Inspired by [26], it is natural to conjecture that

$$\begin{cases} -\Delta u + \lambda_i u = \mu_i u^3 & \text{in } \mathbb{R}^4, \\ u > 0 \text{ in } \mathbb{R}^4, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 & \text{in } \mathbb{R}^4, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^4, \\ u_1, u_2 > 0 \text{ in } \mathbb{R}^4, \end{cases} \tag{1.2}$$

are the limit equation or system of $(\mathcal{S}_\varepsilon)$ if $\alpha_1 = \alpha_2 = 0$. Applying the Pohozaev identity as in [13] and [50] yields that (1.1)- (1.2) have *no* solution for $\lambda_1, \lambda_2 > 0$. Thus, $\alpha_1, \alpha_2 > 0$ seems also to be necessary for observing the concentration behavior of solutions as $\varepsilon \to 0^+$. Another reason for the appearance of subcritical terms u_1^{p-1}, u_2^{p-1}

in our study on $(\mathcal{S}_\varepsilon)$ comes from the viewpoint of the calculus of variations. Recall that cubic terms u_1^3, u_2^3 and coupling terms $u_2^2u_1, u_1^2u_2$ are all of critical growth in the sense of the Sobolev embedding, thus, a major difficulty in studying $(\mathcal{S}_\varepsilon)$ via variational methods is the lack of compactness. In this scenario, a typical strategy, which is proposed by Brezís and Nirenberg in [5], is to control the energy level to be less than a threshold that is always generated by the energy level of the ground states of the pure critical "limit" functional. In this argument, the negativity of the subcritical terms in the energy functionals plays an important role in controlling the energy value to be less than the threshold. Since we take $\lambda_1, \lambda_2 > 0$ in $(\mathcal{S}_\varepsilon)$, $\alpha_1, \alpha_2 > 0$ seems to be necessary for our study on $(\mathcal{S}_\varepsilon)$. For these reasons, the subcritical terms u_1^{p-1}, u_2^{p-1} seem to play an important role in the study of $(\mathcal{S}_\varepsilon)$ for the case $N = 4$. We have observed that the subcritical terms u_1^{p-1}, u_2^{p-1} have additional strong effects on the structure of the solutions of $(\mathcal{S}_\varepsilon)$, which we will report elsewhere.

Let us present the necessary notations and definitions before we state our existence results for $(\mathcal{S}_\varepsilon)$. Let $\mathcal{H}_{i,\varepsilon,\Omega}$ be the Hilbert space of $H_0^1(\Omega)$ that is equipped with the inner product

$$\langle u, v\rangle_{i,\varepsilon,\Omega} = \int_\Omega \varepsilon^2\nabla u\nabla v + \lambda_i uvdx.$$

For $i = 1, 2$, since $\lambda_i > 0$ and $\varepsilon > 0$, $\mathcal{H}_{i,\varepsilon,\Omega}$ are Hilbert spaces with the corresponding norm $\|u\|_{i,\varepsilon,\Omega} = \langle u, u\rangle_{i,\varepsilon,\Omega}^{\frac{1}{2}}$. Set $\mathcal{H}_{\varepsilon,\Omega} = \mathcal{H}_{1,\varepsilon,\Omega} \times \mathcal{H}_{2,\varepsilon,\Omega}$. Then $\mathcal{H}_{\varepsilon,\Omega}$ is also a Hilbert space with the inner product

$$\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\rangle_{\varepsilon,\Omega} = \sum_{i=1}^{2}\langle u_i, v_i\rangle_{i,\varepsilon,\Omega}$$

and the corresponding norm of $\mathcal{H}_{\varepsilon,\Omega}$ is expressed as $\|\overrightarrow{\mathbf{u}}\|_{\varepsilon,\Omega} = \langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{u}}\rangle_{\varepsilon,\Omega}^{\frac{1}{2}}$. Here, u_i, v_i are the ith components of $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}$, respectively. For simplicity, we denote the subscript ${}_{i,1,\Omega}$ by ${}_{i,\Omega}$. Define

$$\begin{aligned}\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}}) &= \sum_{i=1}^{2}(\frac{1}{2}\|u_i\|_{i,\varepsilon,\Omega}^2 - \frac{\alpha_i}{p}\|u_i\|_{L^p(\Omega)}^p - \frac{\mu_i}{4}\|u_i\|_{L^4(\Omega)}^4)\\ &\quad -\frac{\beta}{2}\|u_1u_2\|_{L^2(\Omega)}^2,\end{aligned}$$

where $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$ is the typical norm in $L^p(\Omega)$. Then, $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ is of class C^2 in $\mathcal{H}_{\varepsilon,\Omega}$.

Definition 1.1. $(u_1, u_2) = \overrightarrow{\mathbf{u}}$ *is called a nontrivial critical point of* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *if* $\mathcal{J}'_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}}) = 0$ *in* $\mathcal{H}_{\varepsilon,\Omega}^{-1}$ *with* $u_1 \not\equiv 0$ *and* $u_2 \not\equiv 0$. $(u_1, u_2) = \overrightarrow{\mathbf{u}}$ *is called a positive critical point of* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *if* $\overrightarrow{\mathbf{u}}$ *is a nontrivial critical point and* $u_i > 0$ *for both* $i = 1, 2$. *Here,* $\mathcal{J}'_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *is the Fréchet derivative of* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *and* $\mathcal{H}_{\varepsilon,\Omega}^{-1}$ *is the dual space of* $\mathcal{H}_{\varepsilon,\Omega}$.

By the above definitions, positive critical points of $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ are equivalent to solutions of $(\mathcal{S}_\varepsilon)$.

Definition 1.2. $(u_1, u_2) = \overrightarrow{\mathbf{u}}$ *is called a least-energy solution of* $(\mathcal{S}_\varepsilon)$ *if* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}}) \leq \mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{v}})$ *for all nontrivial critical points* $\overrightarrow{\mathbf{v}}$.

Now, our existence results for $(\mathcal{S}_\varepsilon)$ in $\mathbb{R}^4$ can be stated as follows.

Theorem 1.1. *Let* $\lambda_i, \mu_i, \alpha_i > 0$ $(i = 1, 2)$ *and* $\beta \neq 0$. *Then there exist three positive constants, namely,* α_0, β_0 *and* β_1 *with* $\beta_0 < \beta_1$ *such that* $(\mathcal{S}_\varepsilon)$ *has a least-energy solution* $\overrightarrow{\mathbf{u}}_\varepsilon$ *when* $\varepsilon > 0$ *is sufficiently small and one of the following three cases holds:*

(1) $\beta > \beta_1$,

(2) $-\sqrt{\mu_1\mu_2} \leq \beta < \beta_0$,

(3) $\beta < -\sqrt{\mu_1\mu_2}$ *and* $|\overrightarrow{\alpha}| < \alpha_0$,

where $\overrightarrow{\alpha} = (\alpha_1, \alpha_2)$.

Remark 1.1. *We mainly use Nehari's manifold approach to prove theorem 1.1. However, due to the appearance of subcritical terms* $\frac{\alpha_i}{p}\|u_i\|^p_{L^p(\Omega)}$, *the related fibering maps of* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *are highly complex. In this scenario, we will apply the implicit function theorem as in [47] and Miranda's theorem as in [12] to prove that the Nehari manifold of* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *is a natural constraint in* $\mathcal{H}_{\varepsilon,\Omega}$, *which is crucial for proving theorem 1.1. The existence of* α_0 *is essential in our argument for this property of the Nehari manifold in the case* $\beta < -\sqrt{\mu_1\mu_2}$. *Indeed, in the case* $\beta < -\sqrt{\mu_1\mu_2}$, *the functional*

$$\mu_1\|u_1\|^4_{L^4(\Omega)} + \mu_2\|u_2\|^4_{L^4(\Omega)} + 2\beta\|u_1u_2\|^2_{L^2(\Omega)} \tag{1.3}$$

is indefinite in the working space $\mathcal{H}_{\varepsilon,\Omega}$. *Since* $2 < p < 4$, *it is difficult for us to show that the Nehari manifold of* $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *is a natural constraint in* $\mathcal{H}_{\varepsilon,\Omega}$ *for all* $\alpha_1, \alpha_2 > 0$ *in the case* $\beta < -\sqrt{\mu_1\mu_2}$. *Hence, we introduce the truncation functional* $\chi_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\right)$ *to deal with the subcritical terms* $\frac{\alpha_i}{p}\|u_i\|^p_{L^p(\Omega)}$ *by regarding them as perturbations of the following functional*

$$\mathcal{J}^*_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}}) = \sum_{i=1}^{2}(\frac{1}{2}\|u_i\|^2_{i,\varepsilon,\Omega} - \frac{\mu_i}{4}\|u_i\|^4_{L^4(\Omega)}) - \frac{\beta}{2}\|u_1u_2\|^2_{L^2(\Omega)}$$

in the case $\beta < -\sqrt{\mu_1\mu_2}$, *since it has been proved in [10] that the Nehari manifold of* $\mathcal{J}^*_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ *is a natural constraint in* $\mathcal{H}_{\varepsilon,\Omega}$ *for all* $\beta < 0$.

To the best of our knowledge, the concentration behaviors of least-energy solutions and the locations of spikes of $(\mathcal{S}_\varepsilon)$ with $N = 4$ as $\varepsilon \to 0^+$ have yet to be studied in the literature. Thus, we shall also explore these two problems in the current paper. Since the concentration behaviors depend on β, we re-denote $\overrightarrow{\mathbf{u}}_\varepsilon$ by $\overrightarrow{\mathbf{u}}_{\varepsilon,\beta}$.

Theorem 1.2. *Let* $p_i^{\varepsilon,\beta}$ *be the maximum point of* $u_i^{\varepsilon,\beta}$, *respectively, for* $i = 1, 2$, *where* $\overrightarrow{\mathbf{u}}_{\varepsilon,\beta} = (u_1^{\varepsilon,\beta}, u_2^{\varepsilon,\beta})$ *is a least-energy solution of* $(\mathcal{S}_\varepsilon)$, *which is found in theorem 1.1. Then, the following hold:*

(a) *If $\beta<0$, then $(u_1^{\varepsilon,\beta}(\varepsilon y+p_1^{\varepsilon,\beta}),u_2^{\varepsilon,\beta}(\varepsilon y+p_2^{\varepsilon,\beta}))\to(v_1^0,v_2^0)$ strongly in $H^1(\mathbb{R}^4)\times H^1(\mathbb{R}^4)$ as $\varepsilon\to 0^+$, where v_i^0 is a least-energy solution of the following equation:*

$$\begin{cases} -\Delta u+\lambda_i u=\mu_i u^3+\alpha_i u^{p-1} & \text{in } \mathbb{R}^4,\\ u>0 \text{ in } \mathbb{R}^4, \quad u\to 0 \text{ as } |x|\to+\infty \end{cases} \tag{1.4}$$

for $i=1,2$. Moreover, $\frac{|p_1^{\varepsilon,\beta}-p_2^{\varepsilon,\beta}|}{\varepsilon}\to+\infty$ as $\varepsilon\to 0^+$.

(b) *If $\beta>0$, then $(u_1^{\varepsilon,\beta}(\varepsilon y+p_1^{\varepsilon,\beta}),u_2^{\varepsilon,\beta}(\varepsilon y+p_2^{\varepsilon,\beta}))\to(v_1^*,v_2^*)$ strongly in $H^1(\mathbb{R}^4)\times H^1(\mathbb{R}^4)$ as $\varepsilon\to 0^+$, where (v_1^*,v_2^*) is the least-energy solution of the following elliptic system:*

$$\begin{cases} -\Delta u_1+\lambda_1 u_1=\mu_1 u_1^3+\alpha_1 u_1^{p-1}+\beta u_2^2 u_1 & \text{in } \mathbb{R}^4,\\ -\Delta u_2+\lambda_2 u_2=\mu_2 u_2^3+\alpha_2 u_2^{p-1}+\beta u_1^2 u_2 & \text{in } \mathbb{R}^4,\\ u_1,u_2>0 \text{ in } \mathbb{R}^4, \quad u_1,u_2\to 0 \text{ as } |x|\to+\infty. \end{cases} \tag{1.5}$$

Moreover, $\frac{|p_1^{\varepsilon,\beta}-p_2^{\varepsilon,\beta}|}{\varepsilon}\to 0$ as $\varepsilon\to 0^+$.

Remark 1.2. *As in [26], by Theorem 1.2, the concentration behaviors of $\vec{\mathbf{u}}_{\varepsilon,\beta}$ as $\varepsilon\to 0^+$ are differ between $\beta<0$ and $\beta>0$, which is caused by the different limits of the coupled term $\int_\Omega (u_1^{\varepsilon,\beta})^2(u_2^{\varepsilon,\beta})^2dx$ as $\varepsilon\to 0^+$. Indeed, based on observations of the limit of energy values as $\varepsilon\to 0^+$ (see Lemma 4.1), we deduce that $\int_\Omega (u_1^{\varepsilon,\beta})^2(u_2^{\varepsilon,\beta})^2dx\to 0$ for $\beta<0$ as $\varepsilon\to 0^+$. This, together with a suitable scaling that is centered at maximum points $p_1^{\varepsilon,\beta},p_2^{\varepsilon,\beta}$ and uniformly elliptic estimates, implies that spikes will repel each other and behave like two separate spikes for $\beta<0$, that is $\frac{|p_1^{\varepsilon,\beta}-p_2^{\varepsilon,\beta}|}{\varepsilon}\to+\infty$ as $\varepsilon\to 0^+$. In the case $\beta>0$, by observations of the limit of energy values as $\varepsilon\to 0^+$ (see also Lemma 4.1), we deduce that $\int_\Omega (u_1^{\varepsilon,\beta})^2(u_2^{\varepsilon,\beta})^2dx\geq C+o(1)$. Together with a suitable scaling that is centered at maximum points $p_1^{\varepsilon,\beta},p_2^{\varepsilon,\beta}$, this implies that the spikes will attract each other and behave like a single spike, that is, $\frac{|p_1^{\varepsilon,\beta}-p_2^{\varepsilon,\beta}|}{\varepsilon}$ is bounded as $\varepsilon\to 0^+$. Combining with the radial symmetry of the solutions of the limit system for $\beta>0$, we obtain that $\frac{|p_1^{\varepsilon,\beta}-p_2^{\varepsilon,\beta}|}{\varepsilon}\to 0$ as $\varepsilon\to 0^+$ since $p_1^{\varepsilon,\beta},p_2^{\varepsilon,\beta}$ are the maximum points of $u_1^{\varepsilon,\beta}$ and $u_2^{\varepsilon,\beta}$, respectively. However, compared with the argument in [26] for such results, an additional difficulty in proving Theorem 1.2 is that the uniform boundedness of $u_i^{\varepsilon,\beta}(\varepsilon y+p_i^{\varepsilon,\beta})$ in $L^\infty(\mathbb{R}^4)$, which is crucial for proving the result, can not be obtained by directly applying the Moser iteration due to the critical growth of cubic nonlinearities and the coupled term in dimension four. By Lemma 4.1, $\{u_i^{\varepsilon,\beta}(\varepsilon y+p_i^{\varepsilon,\beta})\}$ is bounded in $H^1(\mathbb{R}^4)$ for both $i=1,2$. Thus, to show the boundedness of $\{u_i^{\varepsilon,\beta}(\varepsilon y+p_i^{\varepsilon,\beta})\}$ in $L^\infty(\mathbb{R}^N)$ for both $i=1,2$, we prove the strong convergence of $(u_1^{\varepsilon,\beta}(\varepsilon y+p_1^{\varepsilon,\beta}),u_2^{\varepsilon,\beta}(\varepsilon y+p_2^{\varepsilon,\beta}))$ in $H^1(\mathbb{R}^4)\times H^1(\mathbb{R}^4)$ as $\varepsilon\to 0^+$ and apply the Moser iteration as in [8, 10].*

Finally, we study the locations of the spikes p_i^ε as $\varepsilon\to 0^+$.

Theorem 1.3. *Let $p_i^{\varepsilon,\beta}$ be the maximum points of $u_i^{\varepsilon,\beta}$, respectively, for $i = 1, 2$, where $\vec{\mathbf{u}}_{\varepsilon,\beta} = (u_1^{\varepsilon,\beta}, u_2^{\varepsilon,\beta})$ is a least-energy solution of $(\mathcal{S}_\varepsilon)$, which is found in theorem 1.1. Then, the following hold:*

(a) *If $\beta > 0$, then $\lim_{\varepsilon\to 0^+} \mathcal{D}_i^{\varepsilon,\beta} = \mathcal{D}$, where*

$$\mathcal{D} = \max_{p\in\Omega} dist(p, \partial\Omega), \quad \mathcal{D}_i^{\varepsilon,\beta} = dist(p_i^{\varepsilon,\beta}, \partial\Omega), \quad i = 1, 2.$$

(b) *If $\beta < 0$, then $\lim_{\varepsilon\to 0^+} \varphi(p_1^{\varepsilon,\beta}, p_2^{\varepsilon,\beta}) = \varphi(P_1^*, P_2^*)$, where*

$$\varphi(P_1, P_2) = \min_{i=1,2}\{\min\{\sqrt{\lambda_i}|P_1 - P_2|, \sqrt{\lambda_i} dist(P_i, \partial\Omega)\}\}$$

and (P_1^*, P_2^*) satisfies $\varphi(P_1^*, P_2^*) = \max_{(P_1,P_2)\in\Omega\times\Omega} \varphi(P_1, P_2)$.

Remark 1.3. *The argument in [26] for proving a similar result to (b) of Theorem 1.3 is based on the nondegeneracy of v_i^0 $(i = 1, 2)$ in the case $1 \le N \le 3$, where v_i^0 is specified in Theorem 1.2. To the best of our knowledge, it is unknown whether v_i^0 $(i = 1, 2)$ is nondegenerate or not in the case $N = 4$. Thus, the method of [26] is invalid in our case. By Theorem 1.2, the spikes satisfy $\frac{|p_1^{\varepsilon,\beta} - p_2^{\varepsilon,\beta}|}{\varepsilon} \to +\infty$ as $\varepsilon \to 0^+$ for $\beta < 0$, which yields that the two components of the least-energy solution $\vec{\mathbf{u}}_{\varepsilon,\beta}$ will separate little by little as $\varepsilon \to 0^+$. This phenomenon is very similar to the phenomenon of phase separation of elliptic systems as $\beta \to -\infty$, i.e., the two components of solutions to elliptic systems will separate little by little as $\beta \to -\infty$. Since it has been observed in [11, 12, 40, 44, 46, 47] and the references therein that the phenomenon of phase separation in elliptic systems is related to the sign-changing solutions of the single equation, inspired by [17], where a similar scenario was faced in the study of the locations of the spikes in a sign-changing solution of a single equation with jumping nonlinearities, we introduce the function $\varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon)$, which is specified by (5.16) and the set $\Lambda_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon)$, which is specified by (5.19), for our case to prove (b) of Theorem 1.3.*

Remark 1.4. *Via various necessary modifications, Theorems 1.1–1.3 can be generalized slightly to the following system:*

$$\begin{cases} -\varepsilon^2\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p_1-1} + \beta u_2^2 u_1 & \text{in } \Omega, \\ -\varepsilon^2\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p_2-1} + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \quad u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $2 < p_1, p_2 < 4$. However, we do not want to become trapped in unnecessarily complex calculations due to $p_1 \ne p_2$. For this reason, we only present the proofs of Theorems 1.1–1.3 for $p_1 = p_2 = p$.

Notations. Throughout this paper, C and C' are indiscriminately used to denote various absolute positive constants. We also list notations that are used frequently below.

$$\begin{aligned} \|u\|^p_{L^p(\Omega)} &= \int_\Omega |u|^p dx, & \overrightarrow{\mathbf{u}} &= (u_1, u_2, \cdots, u_k), \\ \overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}} &= (t_1u_1, t_2u_2, \cdots, t_ku_k), & \mathbb{B}_r(x) &= \{y \in \mathbb{R}^4 \mid |y-x| < r\}, \\ t\overrightarrow{\mathbf{u}} &= (tu_1, tu_2, \cdots, tu_k), & \overrightarrow{\mathbf{u}}_n &= (u_1^n, u_2^n, \cdots, u_k^n), \\ \mathbb{R}^4_+ &= \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 > 0\}, & \mathbb{R}^+ &= (0, +\infty). \end{aligned}$$

$O(|\overrightarrow{\mathbf{b}}|)$ is used to denote quantities that tend toward zero as $|\overrightarrow{\mathbf{b}}| \to 0$, where $|\overrightarrow{\mathbf{b}}|$ is the typical norm in $\mathbb{R}^k$ of the vector $\overrightarrow{\mathbf{b}}$. Other cited quantities, for example, $d_{i,\varepsilon,\Omega}$ and A_ε, can be found in the appendix.

2 Existence result

Let $\chi_\beta(s)$ be a smooth function in $[0, +\infty)$ such that $\chi_\beta(s) \equiv 1$ for $\beta > -\sqrt{\mu_1\mu_2}$ and

$$\chi_\beta(s) = \begin{cases} 1, & 0 \le s \le 1, \\ 0, & s \ge 2 \end{cases} \tag{2.1}$$

for $\beta \le -\sqrt{\mu_1\mu_2}$. Moreover, we require $-2 \le \chi'_\beta(s) \le 0$ in $[0, +\infty)$. Let

$$T^2 \ge (16 + \sum_{i=1}^{2} \frac{4p}{\mu_i(p-2)})\mathcal{S}^2 \tag{2.2}$$

be a constant and define

$$\begin{aligned} \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) \quad = \quad & \sum_{i=1}^{2}(\frac{1}{2}\|u_i\|^2_{i,\varepsilon,\Omega} - \frac{\alpha_i}{p}\chi_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\right)\|u_i\|^p_{L^p(\Omega)} - \frac{\mu_i}{4}\|u_i\|^4_{L^4(\Omega)}) \\ & -\frac{\beta}{2}\|u_1u_2\|^2_{L^2(\Omega)}, \end{aligned}$$

Then, $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ is of class C^2 and

$$\begin{aligned} \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{v}} \quad = \quad & \sum_{i=1}^{2}\Bigg((1 - \frac{2\alpha_i}{pT^2\varepsilon^4}\|u_i\|^p_{L^p(\Omega)}\chi'_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\right))\langle u_i, v_i\rangle_{i,\varepsilon,\Omega} \\ & -\alpha_i\chi_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\right)\int_\Omega |u_i|^{p-2}u_iv_idx - \mu_i\int_\Omega u_i^3v_idx\Bigg) \\ & -\beta\int_\Omega (u_2^2u_1v_1 + u_1^2u_2v_2)dx \end{aligned}$$

for all $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \in \mathcal{H}_{\varepsilon,\Omega}$.

Definition 2.1. *The vector $(u_1, u_2) = \overrightarrow{\mathbf{u}} \in \mathcal{H}_{\varepsilon,\Omega}$ is called a nontrivial critical point of $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ if $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) = 0$ in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ with $u_1 \not\equiv 0$ and $u_2 \not\equiv 0$. The vector $(u_1, u_2) =$*

$\overrightarrow{\mathbf{u}} \in \mathcal{H}_{\varepsilon,\Omega}$ is called a semi-trivial critical point of $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ if $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) = 0$ in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ with $u_1 \not\equiv 0$ or $u_2 \not\equiv 0$. The vector $(u_1, u_2) = \overrightarrow{\mathbf{u}} \in \mathcal{H}_{\varepsilon,\Omega}$ is called a positive critical point of $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ if $\overrightarrow{\mathbf{u}}$ is a nontrivial critical point and $u_i > 0$ for both $i = 1, 2$. Here, $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ is the Fréchet derivative of $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ and $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ is the dual space of $\mathcal{H}_{\varepsilon,\Omega}$.

By Definition 2.1 and the construction of χ_β that is specified by (2.1), critical points of $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ are also critical points of $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ for $\beta > -\sqrt{\mu_1\mu_2}$ while critical points of $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ will be critical points of $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}})$ for $\beta \leq -\sqrt{\mu_1\mu_2}$ if $\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega} \leq \varepsilon^4 T^2$.

Without loss of generality, we assume $0 \in \Omega$ and set $\Omega_\varepsilon = \{x \in \mathbb{R}^4 \mid \varepsilon x \in \Omega\}$. Now, let

$$\mathcal{N}_{\varepsilon,\Omega,T} = \{\overrightarrow{\mathbf{u}} = (u_1, u_2) \in \widetilde{\mathcal{H}}_{\varepsilon,\Omega} \mid \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}^1 = \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}^2 = 0\},$$

with $\widetilde{\mathcal{H}}_{\varepsilon,\Omega} = (\mathcal{H}_{1,\varepsilon,\Omega}\backslash\{0\}) \times (\mathcal{H}_{2,\varepsilon,\Omega}\backslash\{0\})$, $\overrightarrow{\mathbf{u}}^1 = (u_1, 0)$ and $\overrightarrow{\mathbf{u}}^2 = (0, u_2)$. It follows that $\mathcal{N}_{\varepsilon,\Omega,T} \neq \emptyset$. We also define

$$\mathcal{N}'_{\varepsilon,\Omega,T} = \{\overrightarrow{\mathbf{u}} = (u_1, u_2) \in \mathcal{H}_{\varepsilon,\Omega}\backslash\{\overrightarrow{\mathbf{0}}\} \mid \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}} = 0\}.$$

Lemma 2.1. *We have $\mathcal{N}_{\varepsilon,\Omega,T} \subset \mathcal{N}'_{\varepsilon,\Omega,T}$ and $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ is bounded from below on $\mathcal{N}'_{\varepsilon,\Omega,T}$.*

Proof. By the definitions of $\mathcal{N}_{\varepsilon,\Omega,T}$ and $\mathcal{N}'_{\varepsilon,\Omega,T}$, $\mathcal{N}_{\varepsilon,\Omega,T} \subset \mathcal{N}'_{\varepsilon,\Omega,T}$. Now, let $\overrightarrow{\mathbf{u}} \in \mathcal{N}'_{\varepsilon,\Omega,T}$. Then, by the construction of χ_β and the Hölder inequality,

$$\begin{aligned}
\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) &= \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) - \frac{1}{p}\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}} \\
&= \frac{p-2}{2p}\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega} + \frac{4-p}{4p}(\sum_{i=1}^{2}\mu_i\|u_i\|^4_{L^4(\Omega)} + 2\beta\|u_1u_2\|^2_{L^2(\Omega)}) \\
&\geq \frac{p-2}{2p}\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega} \qquad (2.3) \\
&\geq 0
\end{aligned}$$

for $\beta > -\sqrt{\mu_1\mu_2}$ since $2 < p < 4$. In the case $\beta \leq -\sqrt{\mu_1\mu_2}$, we have

$$\begin{aligned}
\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) &= \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) - \frac{1}{4}\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}} \\
&= (\frac{1}{4} + \sum_{i=1}^{2}\frac{\alpha_i}{2pT^2\varepsilon^4}\|u_i\|^p_{L^p(\Omega)}\chi'_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\right))\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega} \\
&\quad -\frac{4-p}{4p}\sum_{i=1}^{2}\alpha_i\|u_i\|^p_{L^p(\Omega)}\chi_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\right). \qquad (2.4)
\end{aligned}$$

It follows from the construction of χ_β that $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) \geq 0$ for $\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega} \geq 2\varepsilon^4T^2$. For $\|\overrightarrow{\mathbf{u}}\|^2_{\varepsilon,\Omega} < 2\varepsilon^4T^2$, we have that $\sum_{i=1}^2 \|\overline{u}_i\|^2_{i,\Omega_\varepsilon} < 2T^2$ and

$$\|u_i\|^p_{L^p(\Omega)} = \varepsilon^4\|\overline{u}_i\|^p_{L^p(\mathbb{R}^4)} \leq \varepsilon^4\mathcal{S}_{p,i}^{-\frac{p}{2}}\|\overline{u}_i\|^p_{i,\mathbb{R}^4} \leq \mathcal{S}_{p,i}^{-\frac{p}{2}}(2T^2)^{\frac{p-2}{2}}\|u_i\|^2_{i,\varepsilon,\Omega} \qquad (2.5)$$

for both $i = 1, 2$. Here, $\overline{u}_i(x) = u_i(\varepsilon x)$ and we regard $\overline{u}_i$ as a function in $H^1(\mathbb{R}^4)$ by defining $\overline{u}_i \equiv 0$ in $\mathbb{R}^4 \backslash \Omega_\varepsilon$ and $\mathcal{S}_{p,i}$ is the optimal embedding constant from $\mathcal{H}_{i,\mathbb{R}^4} \to L^p(\mathbb{R}^4)$, which is defined by

$$\mathcal{S}_{p,i} = \inf\{\|u_i\|_{i,\mathbb{R}^4}^2 \mid u \in \mathcal{H}_{i,\mathbb{R}^4}, \|u\|_{L^p(\mathbb{R}^4)} = 1\}.$$

Thus, by (2.4), $\mathcal{J}_{\varepsilon,\Omega}(\overrightarrow{\mathbf{u}}) \geq -\varepsilon^4 C$ for all $\overrightarrow{\mathbf{u}} \in \mathcal{N}'_{\varepsilon,\Omega,T}$ in the case $\beta \leq -\sqrt{\mu_1\mu_2}$. $\square$

Set

$$c_{\varepsilon,\Omega,T} = \inf_{\overrightarrow{\mathbf{u}} \in \mathcal{N}_{\varepsilon,\Omega,T}} \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}) \quad \text{and} \quad c'_{\varepsilon,\Omega,T} = \inf_{\overrightarrow{\mathbf{u}} \in \mathcal{N}'_{\varepsilon,\Omega,T}} \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$$

Then by Lemma 2.1, $c_{\varepsilon,\Omega,T} \geq c'_{\varepsilon,\Omega,T} > -\infty$.

Lemma 2.2. *Assume $\beta \neq 0$ and let $\varepsilon > 0$ be sufficiently small. Then there exists $\alpha_T > 0$, which only depends on T, such that $\varepsilon^4 C \leq c'_{\varepsilon,\Omega,T} \leq c_{\varepsilon,\Omega,T} \leq \varepsilon^4 \sum_{i=1}^2 \frac{1}{4\mu_i}\mathcal{S}^2$ in the following two cases:*

1. *$\beta > -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$,*
2. *$\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$.*

Proof. Let $\mathbb{B}_{r_i}(x_i) \subset \Omega$ for $i = 1, 2$ and $\mathbb{B}_{r_1}(x_1) \cap \mathbb{B}_{r_2}(x_2) = \emptyset$. Then, by Proposition 7.2, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, there exists a solution $U_{i,\varepsilon}$ of equation $(\mathcal{P}_{i,\varepsilon})$ in $\mathbb{B}_{r_i}(x_i)$ such that $\mathcal{E}_{i,\varepsilon,\mathbb{B}_{r_i}(x_i)}(U_{i,\varepsilon}) < \frac{\varepsilon^4}{4\mu_i}\mathcal{S}^2$. It follows from a standard argument that $\|U_{i,\varepsilon}\|^2_{i,\varepsilon,\mathbb{B}_{r_i}(x_i)} \leq \frac{\varepsilon^4 p}{2(p-2)\mu_i}\mathcal{S}^2$, $i = 1, 2$. Now, we regard $U_{i,\varepsilon}$ as a function in $H_0^1(\Omega)$ by defining $U_{i,\varepsilon} \equiv 0$ in $\Omega \backslash \mathbb{B}_{r_i}(x_i)$ and set $\overrightarrow{\mathbf{U}}_\varepsilon = (U_{1,\varepsilon}, U_{2,\varepsilon})$. By $\mathbb{B}_{r_1}(x_1) \cap \mathbb{B}_{r_2}(x_2) = \emptyset$ and the construction of χ_β, it follows from (2.2) that $\overrightarrow{\mathbf{U}}_\varepsilon \in \mathcal{N}_{\varepsilon,\Omega,T}$. Hence,

$$c_{\varepsilon,\Omega,T} \leq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{U}}_\varepsilon) = \sum_{i=1}^2 \mathcal{E}_{i,\varepsilon}(U_{i,\varepsilon}) < \varepsilon^4 \sum_{i=1}^2 \frac{1}{4\mu_i}\mathcal{S}^2 \tag{2.6}$$

for all β. Now, let $\overrightarrow{\mathbf{u}} \in \mathcal{N}'_{\varepsilon,\Omega,T}$. Then, by the Hölder and Young inequalities, it follows that

$$\|u_i\|^p_{L^p(\Omega)} \leq \|u_i\|^{4-p}_{L^2(\Omega)}\|u_i\|^{2(p-2)}_{L^4(\Omega)} \leq \frac{\lambda_i}{2}\|u_i\|^2_{L^2(\Omega)} + C\|u_i\|^4_{L^4(\Omega)}. \tag{2.7}$$

It follows from the construction of χ_β that

$$\begin{aligned}
\sum_{i=1}^2 \|\overline{u}_i\|^2_{i,\Omega_\varepsilon} &\leq \sum_{i=1}^2 \alpha_i \|\overline{u}_i\|^{4-p}_{L^2(\Omega_\varepsilon)}\|\overline{u}_i\|^{2(p-2)}_{L^4(\Omega_\varepsilon)} + 2|\beta|\|\overline{u}_1\|^2_{L^4(\Omega_\varepsilon)}\|\overline{u}_2\|^2_{L^4(\Omega_\varepsilon)} \\
&\leq \sum_{i=1}^2 (\frac{\lambda_i}{2}\|\overline{u}_i\|^2_{L^2(\Omega_\varepsilon)} + C\|\overline{u}_i\|^4_{L^4(\Omega_\varepsilon)}) + 2|\beta|\|\overline{u}_1\|^2_{L^4(\Omega_\varepsilon)}\|\overline{u}_2\|^2_{L^4(\Omega_\varepsilon)},
\end{aligned}$$

where $\overline{u}_i(x) = u_i(\varepsilon x)$. It follows from the Sobolev inequality that

$$\|\overline{u}_1\|^2_{L^4(\Omega_\varepsilon)} + \|\overline{u}_2\|^2_{L^4(\Omega_\varepsilon)} \geq C_\beta > 0, \tag{2.8}$$

Here, C_β only depends on β. Thanks to (2.3), we have $c'_{\varepsilon,\Omega,T} \geq \varepsilon^4 C_\beta$ for all $\beta > -\sqrt{\mu_1\mu_2}$. For $\beta \leq -\sqrt{\mu_1\mu_2}$, by (2.4) and (2.5), there exists $\alpha_T > 0$ such that $c'_{\varepsilon,\Omega,T} \geq \varepsilon^4 C_\beta$ with $|\overrightarrow{\alpha}| < \alpha_T$. Now, the proof is completed by (2.6) and $c_{\varepsilon,\Omega,T} \geq c'_{\varepsilon,\Omega,T}$. □

By (2.8), both $\mathcal{N}_{\varepsilon,\Omega,T}$ and $\mathcal{N}'_{\varepsilon,\Omega,T}$ are bounded away from $\overrightarrow{\mathbf{0}}$ for $\beta > -\sqrt{\mu_1\mu_2}$, which implies that both $\mathcal{N}_{\varepsilon,\Omega,T}$ and $\mathcal{N}'_{\varepsilon,\Omega,T}$ are closed in $\mathcal{H}_{\varepsilon,\Omega}$ for $\beta > -\sqrt{\mu_1\mu_2}$. In the case $\beta \leq -\sqrt{\mu_1\mu_2}$, by taking $\alpha_T > 0$ sufficiently small if necessary, it follows from (2.5) that both $\mathcal{N}_{\varepsilon,\Omega,T}$ and $\mathcal{N}'_{\varepsilon,\Omega,T}$ are also bounded away from $\overrightarrow{\mathbf{0}}$ for $\beta \leq -\sqrt{\mu_1\mu_2}$ with $|\overrightarrow{\alpha}| < \alpha_T$, which implies that both $\mathcal{N}_{\varepsilon,\Omega,T}$ and $\mathcal{N}'_{\varepsilon,\Omega,T}$ are also closed in $\mathcal{H}_{\varepsilon,\Omega}$ for $\beta \leq -\sqrt{\mu_1\mu_2}$ with $|\overrightarrow{\alpha}| < \alpha_T$. Now, by Ekeland's variational principle, there exists $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{N}_{\varepsilon,\Omega,T}$ such that

(1) $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = c_{\varepsilon,\Omega,T} + o_n(1)$,

(2) $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{v}}) \geq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) - \frac{1}{n}\|\overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{u}}_n\|_{\varepsilon,\Omega}$ for all $\overrightarrow{\mathbf{v}} \in \mathcal{N}_{\varepsilon,\Omega,T}$.

There also exists $\{\overrightarrow{\mathbf{u}}'_n\} \subset \mathcal{N}'_{\varepsilon,\Omega}$ such that

(1′) $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_n) = c'_{\varepsilon,\Omega,T} + o_n(1)$,

(2′) $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{v}}) \geq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_n) - \frac{1}{n}\|\overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{u}}'_n\|_{\varepsilon,\Omega}$ for all $\overrightarrow{\mathbf{v}} \in \mathcal{N}'_{\varepsilon,\Omega,T}$.

Define

$$\Phi_n(\overrightarrow{\mathbf{t}}) = \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_n) \quad \text{and} \quad \widetilde{\Phi}_n(t) = \mathcal{J}_{\varepsilon,\Omega,T}(t\overrightarrow{\mathbf{u}}'_n),$$

where $\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_n = (t_1 u_1^n, t_2 u_2^n)$ and $t\overrightarrow{\mathbf{u}}'_n = (t(u_1^n)', t(u_2^n)')$.

Lemma 2.3. *Let $\varepsilon > 0$ be sufficiently small and $\alpha_1, \alpha_2 > 0$. Then,*

(i) $\|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega} < \varepsilon^4 T^2$ *and* $\|\overrightarrow{\mathbf{u}}'_n\|^2_{\varepsilon,\Omega} < \varepsilon^4 T^2$ *in one of the following two cases:*

- *(a)* $\beta > -\sqrt{\mu_1\mu_2}$,
- *(b)* $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $|\overrightarrow{\alpha}| < \alpha_T$, *where* α_T *is specified in Lemma 2.2.*

(ii) *There exists* $\beta_0 \in (0, \min\{\mu_1, \mu_2\})$ *that is independent of* ε *such that* $\{\overrightarrow{\mathbf{u}}_n\}$ *is a* (PS) *sequence of* $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ *at the energy level* $c_{\varepsilon,\Omega,T}$ *in one of the following two cases:*

- *(a)* $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$,
- *(b)* $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $|\overrightarrow{\alpha}| < \alpha_T$, *where* α_T *is specified in Lemma 2.2.*

That is,

$$\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = c_{\varepsilon,\Omega,T} + o_n(1); \quad \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = o_n(1) \text{ strongly in } \mathcal{H}^{-1}_{\varepsilon,\Omega}.$$

Moreover, $\Phi_n(\overrightarrow{\mathbf{1}}) \geq \Phi_n(\overrightarrow{\mathbf{t}})$ *for all* $n \in \mathbb{N}$ *and* $\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^2$.

(iii) $\{\overrightarrow{\mathbf{u}}'_n\}$ *is a* (PS) *sequence of* $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}})$ *at the energy level* $c'_{\varepsilon,\Omega,T}$ *for all* $\beta > 0$. *Moreover,* $\widetilde{\Phi}_n(1) \geq \widetilde{\Phi}_n(t)$ *for all* $n \in \mathbb{N}$ *and* $t > 0$.

Proof. (i) We only present the proof for $\overrightarrow{\mathbf{u}}_n$ since that of $\overrightarrow{\mathbf{u}}'_n$ is similar. By Lemma 2.2, it follows from (2.3) and $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = c_{\varepsilon,\Omega,T} + o_n(1)$ that

$$\|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega} < \varepsilon^4 \frac{2p}{p-2} \sum_{i=1}^{2} \frac{1}{4\mu_i} \mathcal{S}^2 + o_n(1) \tag{2.9}$$

in the case $\beta > -\sqrt{\mu_1\mu_2}$. For $\beta \leq -\sqrt{\mu_1\mu_2}$, by Lemma 2.2 and $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = c_{\varepsilon,\Omega,T} + o_n(1)$, it follows from (2.4) and (2.5) that

$$\begin{aligned} & \varepsilon^4 \sum_{i=1}^{2} \frac{1}{4\mu_i} \mathcal{S}^2 + o_n(1) \\ > \ & (\frac{1}{4} + \frac{\alpha_i}{2pT^2\varepsilon^4} \mathcal{S}_{p,i}^{-\frac{p}{2}} (2T^2)^{\frac{p-2}{2}} \|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega} \chi'_\beta\Big(\frac{\|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega}}{T^2\varepsilon^4}\Big)) \|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega} \\ & - \frac{4-p}{4p} \sum_{i=1}^{2} \alpha_i \mathcal{S}_{p,i}^{-\frac{p}{2}} (2T^2)^{\frac{p-2}{2}} \|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega}. \end{aligned}$$

By choosing α_T sufficiently small if necessary, we have from the construction of χ_β that

$$\varepsilon^4 \sum_{i=1}^{2} \frac{1}{4\mu_i} \mathcal{S}^2 + o_n(1) > \frac{1}{8} \|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega} \tag{2.10}$$

for $|\overrightarrow{\alpha}| < \alpha_T$ in the case $\beta \leq -\sqrt{\mu_1\mu_2}$. Combining (2.9)–(2.10) and (2.2) yields $\|\overrightarrow{\mathbf{u}}_n\|^2_{\varepsilon,\Omega} < \varepsilon^4 T^2$ in the following two cases:

1. $\beta > -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$,
2. $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$.

(ii) Let $\overrightarrow{\mathbf{w}} \in \mathcal{H}_{\varepsilon,\Omega}$. For every $n \in \mathbb{N}$, we consider the system $\overrightarrow{\mathbf{\Psi}}_n(\overrightarrow{\mathbf{t}}, l) = \overrightarrow{\mathbf{0}}$, where $\overrightarrow{\mathbf{\Psi}}_n(\overrightarrow{\mathbf{t}}, l) = (\Psi_1^n(\overrightarrow{\mathbf{t}}, l), \Psi_2^n(\overrightarrow{\mathbf{t}}, l))$ with

$$\begin{aligned} \Psi_i^n(\overrightarrow{\mathbf{t}}, l) \ &= \ \|t_iu_i^n + lw_i\|^2_{i,\varepsilon,\Omega} - \alpha_i\|t_iu_i^n + lw_i\|^p_{L^p(\Omega)} - \mu_i\|t_iu_i^n + lw_i\|^4_{L^4(\Omega)} \\ & \quad -\beta\|(t_1u_1^n + lw_1)(t_2u_2^n + lw_2)\|^2_{L^2(\Omega)}. \end{aligned}$$

$\overrightarrow{\mathbf{\Psi}}_n(\overrightarrow{\mathbf{t}}, l)$ is of class C^1. Moreover, since $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{N}_{\varepsilon,\Omega,T}$, we also have that

$$\overrightarrow{\mathbf{\Psi}}_n(\overrightarrow{\mathbf{1}}, 0) = \overrightarrow{\mathbf{0}}.$$

By a direct calculation, it follows from (i) and the construction of χ_β that

$$\begin{aligned} \frac{\partial \Psi_i^n(\overrightarrow{\mathbf{1}}, 0)}{\partial t_i} \ &= \ 2\|u_i^n\|^2_{i,\varepsilon,\Omega} - p\alpha_i\|u_i^n\|^p_{L^p(\Omega)} - 4\mu_i\|u_i^n\|^4_{L^4(\Omega)} - 2\beta\|u_1^nu_2^n\|^2_{L^2(\Omega)} \\ &= \ -(p-2)\alpha_i\|u_i^n\|^p_{L^p(\Omega)} - 2\mu_i\|u_i^n\|^4_{L^4(\Omega)} \end{aligned} \tag{2.11}$$

for $i = 1, 2$ and

$$\frac{\partial \Psi_1^n(\overrightarrow{\mathbf{1}}, 0)}{\partial t_2} = \frac{\partial \Psi_2^n(\overrightarrow{\mathbf{1}}, 0)}{\partial t_1} = -2\beta \|u_1^n u_2^n\|_{L^2(\Omega)}^2. \tag{2.12}$$

Set

$$\Theta_n = (\theta_{ij}^n)_{i,j=1,2} \tag{2.13}$$

with $\theta_{ij}^n = \frac{\partial \Psi_i^n(\overrightarrow{\mathbf{1}}, 0)}{\partial t_j}$. Then, by (2.11) and (2.12), it follows from $p > 2$ and the Hölder inequality that

$$\det(\Theta_n) > 4(\mu_1\mu_2 - \beta^2)\|u_1^n\|_{L^4(\Omega)}^4 \|u_2^n\|_{L^4(\Omega)}^4. \tag{2.14}$$

If $-\sqrt{\mu_1\mu_2} < \beta < 0$, then by a similar calculation for (2.7) and the Sobolev and Hölder inequalities, it follows from $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{N}_{\varepsilon,\Omega,T}$ and the construction of χ_β that

$$\varepsilon^4 \|\nabla \overline{u}_i^n\|_{L^2(\Omega_\varepsilon)}^2 \leq \|u_i^n\|_{i,\varepsilon,\Omega}^2 \leq C\|u_i^n\|_{L^4(\Omega)}^4 \leq \varepsilon^4 C \|\nabla \overline{u}_i^n\|_{L^2(\Omega_\varepsilon)}^4 \tag{2.15}$$

for both $i = 1, 2$, which implies

$$\|\overline{u}_i^n\|_{L^4(\Omega_\varepsilon)}^4 \geq C \tag{2.16}$$

for both $i = 1, 2$ in the case $-\sqrt{\mu_1\mu_2} < \beta < 0$. Here, $\overline{u}_i(x) = u_i(\varepsilon x)$. On the other hand, when $\beta > 0$, by similar arguments to those that are used in (2.15),

$$\|\overline{u}_1^n\|_{1,\Omega_\varepsilon}^2 \leq C\|\overline{u}_1^n\|_{L^4(\Omega_\varepsilon)}^4 + \beta \|\overline{u}_1^n\|_{L^4(\Omega_\varepsilon)}^2 \|\overline{u}_2^n\|_{L^4(\Omega_\varepsilon)}^2,$$
$$\|\overline{u}_2^n\|_{2,\Omega_\varepsilon}^2 \leq C\|\overline{u}_2^n\|_{L^4(\Omega_\varepsilon)}^4 + \beta \|\overline{u}_1^n\|_{L^4(\Omega_\varepsilon)}^2 \|\overline{u}_2^n\|_{L^4(\Omega_\varepsilon)}^2.$$

It follows from the Sobolev inequality that

$$C \leq \|\overline{u}_1^n\|_{L^4(\Omega_\varepsilon)}^2 + \beta \|\overline{u}_2^n\|_{L^4(\Omega_\varepsilon)}^2, \tag{2.17}$$
$$C \leq \|\overline{u}_2^n\|_{L^4(\Omega_\varepsilon)}^2 + \beta \|\overline{u}_1^n\|_{L^4(\Omega_\varepsilon)}^2. \tag{2.18}$$

By (1) and (2.3), it follows from Lemma 2.2 that

$$C' \geq \|\overline{u}_2^n\|_{L^4(\Omega_\varepsilon)}^2 + \|\overline{u}_1^n\|_{L^4(\Omega_\varepsilon)}^2 \tag{2.19}$$

for $\beta > 0$. Combining (2.17)–(2.19), there exists $\beta_0 \in (0, \sqrt{\mu_1\mu_2})$ that is independent of ε such that

$$\|\overline{u}_i^n\|_{L^4(\Omega_\varepsilon)}^4 \geq C \tag{2.20}$$

for both $i = 1, 2$ in the case $0 < \beta < \beta_0$. From (2.14), (2.16) and (2.20), it follows that $\det(\Theta_n) \geq \varepsilon^4 C > 0$ in the case $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$. For $\beta \leq -\sqrt{\mu_1\mu_2}$, by $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{N}_{\varepsilon,\Omega,T}$ and (2.5), it follows from the construction of χ_β that

$$(1 - \alpha_i \mathcal{S}_{p,i}^{-\frac{p}{2}} (2T^2)^{\frac{p-2}{2}}) \|u_i^n\|_{i,\varepsilon,\Omega}^2 \leq \mu_i \|u_i^n\|_{L^4(\Omega)}^4 + \beta \|u_1^n u_2^n\|_{L^2(\Omega)}^2$$

for both $i = 1, 2$. By taking α_T sufficiently small if necessary, it follows from the Sobolev inequality that

$$\mu_i \|u_i^n\|_{L^4(\Omega)}^4 + \beta \|u_1^n u_2^n\|_{L^2(\Omega)}^2 \geq \varepsilon^4 C \tag{2.21}$$

for $|\overrightarrow{\alpha}| < \alpha_T$ in the case $\beta \leq -\sqrt{\mu_1\mu_2}$, both $i = 1, 2$. Hence,

$$\prod_{i=1}^{2} (\mu_i \|u_i^n\|_{L^4(\Omega)}^4) - \beta^2 \|u_1^n u_2^n\|_{L^2(\Omega)}^4 \geq \varepsilon^8 C$$

for $|\overrightarrow{\alpha}| < \alpha_T$ in the case $\beta \leq -\sqrt{\mu_1\mu_2}$. Thus, by $p > 2$, we also have that $\det(\Theta_n) \geq \varepsilon^8 C > 0$ for $|\overrightarrow{\alpha}| < \alpha_T$ in the case $\beta \leq -\sqrt{\mu_1\mu_2}$. Since $\det(\Theta_n) \geq \varepsilon^8 C > 0$ always holds, by the implicit function theorem, there exist $\sigma_n > 0$ and $\overrightarrow{\mathbf{t}}_n(l) \in C^1([-\sigma_n, \sigma_n], [\frac{1}{2}, \frac{3}{2}]^2)$ such that $\{\overrightarrow{\mathbf{t}}_n(l) \circ \overrightarrow{\mathbf{u}}_n + l\overrightarrow{\mathbf{w}}\} \subset \mathcal{N}_{\varepsilon,\Omega,T}$ in the following two cases:

1. $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ and $\alpha_1, \alpha_2 > 0$,
2. $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$.

Since (2) holds and $\det(\Theta_n) \geq \varepsilon^8 C > 0$ for the above two cases, by applying the Taylor expansion to $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}}_n(l) \circ \overrightarrow{\mathbf{u}}_n + l\overrightarrow{\mathbf{w}})$ in a standard way (cf. [10]), we obtain that $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = o_n(1)$ strongly in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$. In what follows, we will show that $\Phi_n(\overrightarrow{\mathbf{1}}) \geq \Phi_n(\overrightarrow{\mathbf{t}})$ for all $n \in \mathbb{N}$ and $\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^2$. Consider the system $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}, \tau) = \overrightarrow{\mathbf{0}}$, where $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}, \tau) = (\Gamma_1^n(\overrightarrow{\mathbf{t}}, \tau), \Gamma_2^n(\overrightarrow{\mathbf{t}}, \tau))$ with

$$\Gamma_i^n(\overrightarrow{\mathbf{t}}, \tau) = \|t_i u_i^n\|_{i,\varepsilon,\Omega}^2 - \alpha_i \|t_i u_i^n\|_{L^p(\Omega)}^p - \mu_i \|t_i u_i^n\|_{L^4(\Omega)}^4 - \tau \|(t_1 u_1^n)(t_2 u_2^n)\|_{L^2(\Omega)}^2.$$

$\overrightarrow{\mathbf{\Gamma}}_n$ is of class C^1 with $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{s}}_n, 0) = \overrightarrow{\mathbf{0}}$ for some $\overrightarrow{\mathbf{s}}_n \in (\mathbb{R}^+)^2$ and $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{1}}, \beta) = \overrightarrow{\mathbf{0}}$. Moreover,

$$s_i^n \frac{\partial \Gamma_i^n(\overrightarrow{\mathbf{s}}_n, 0)}{\partial t_i} = -(p-2)\alpha_i \|s_i^n u_i^n\|_{L^p(\Omega)}^p - 2\mu_i \|s_i^n u_i^n\|_{L^4(\Omega)}^4 \tag{2.22}$$

for both $i = 1, 2$ and

$$s_1^n \frac{\partial \Gamma_1^n(\overrightarrow{\mathbf{s}}_n, 0)}{\partial t_2} = s_2^n \frac{\partial \Gamma_2^n(\overrightarrow{\mathbf{s}}_n, 0)}{\partial t_1} = 0. \tag{2.23}$$

$\det(\widetilde{\Theta}_n) > 0$, where $\widetilde{\Theta}_n = (\widetilde{\theta}_{ij}^n)_{i,j=1,2}$ with $\widetilde{\theta}_{ij}^n = \frac{\partial \Gamma_i^n(\overrightarrow{\mathbf{s}}_n, 0)}{\partial t_j}$. By the implicit function theorem, there exists a C^1 vector-valued function $\overrightarrow{\mathbf{t}}_n(\tau)$ with $\overrightarrow{\mathbf{t}}_n(0) = \overrightarrow{\mathbf{s}}_n$ and $t_i^n(\tau) > 0$ such that $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}(\tau), \tau) = \overrightarrow{\mathbf{0}}$ for sufficiently small τ. Moreover, by (2.20), τ is uniform in n. Thus, taking β_0 sufficiently small if necessary, it follows that $\overrightarrow{\mathbf{1}}$ is the unique solution of $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}, \beta) = \overrightarrow{\mathbf{0}}$ for $0 < \beta < \beta_0$. For $\beta < 0$, we follow the strategy in the proof of [47, Lemma 2.2] by considering the following set

$$\mathcal{Z}_n = \{\tau \in [0, \beta] \mid \overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}, \tau) = \overrightarrow{\mathbf{0}} \text{ is uniquely solvable in } (\mathbb{R}^+)^2\}.$$

It follows that $0 \in \mathcal{Z}_n$. Moreover, we claim that $\{\overrightarrow{\mathbf{t}}_n(\tau)\}_{\tau\in\mathcal{Z}_n}$ is both bounded from above and below away from 0, where $\overrightarrow{\mathbf{t}}_n(\tau)$ is the unique solution of $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}, \tau) = \overrightarrow{\mathbf{0}}$. Indeed, by (i) and (2.16), it follows from $\beta < 0$ and $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}_n(\tau), \tau) = \overrightarrow{\mathbf{0}}$ that $t_i^n(\tau) \geq C$ with some $C > 0$ uniformly for n and τ. On the other hand, if $t_1^n(\tau) \to +\infty$ and $t_2^n(\tau) \leq C$, then from (i) and (2.16), it follows that $\Gamma_1^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau) \to -\infty$, which contradicts $\Gamma_1^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau) = 0$. Similarly, it is also impossible if $t_1^n(\tau) \leq C$ and $t_2^n(\tau) \to +\infty$. The case of $t_i^n(\tau) \to +\infty$ remains to be excluded. From $\det(\Theta_n) \geq \varepsilon^8 C > 0$, it follows that Θ_n, which is specified by (2.13), is strictly diagonally dominant for $\beta < 0$ and the first eigenvalue of Θ_n is bounded below away from 0 uniformly for n. Thus, $\sum_{i=1}^2 \Gamma_i^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau) \to -\infty$ in this case, which is also impossible because $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}_n(\tau), \tau) = \overrightarrow{\mathbf{0}}$. Now, for every $\tau \in \mathcal{Z}_n$, by similar calculations for Θ_n, we obtain

$$t_i^n(\tau)\frac{\partial \Gamma_i^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau)}{\partial t_i} = -(p-2)\alpha_i \|t_i^n(\tau)u_i^n\|_{L^p(\Omega)}^p - 2\mu_i \|t_i^n(\tau)u_i^n\|_{L^4(\Omega)}^4 \quad (2.24)$$

and

$$\begin{aligned} t_2^n(\tau)\frac{\partial \Gamma_1^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau)}{\partial t_2} &= t_1^n(\tau)\frac{\partial \Gamma_2^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau)}{\partial t_1} \\ &= -2\beta \|(t_1^n(\tau)u_1^n)(t_2^n(\tau)u_2^n)\|_{L^2(\Omega)}^2. \end{aligned} \quad (2.25)$$

By taking α_T sufficiently small if necessary for $\beta \leq -\sqrt{\mu_1\mu_2}$, we obtain via the similar calculations to those for Θ_n that $\det(\widehat{\Theta}_n(\tau)) > 0$ for all n and τ, where $\widehat{\Theta}_n(\tau) = (\widehat{\theta}_{ij}^n(\tau))_{i,j=1,2}$ with $\widehat{\theta}_{ij}^n(\tau) = \frac{\partial \Gamma_i^n(\overrightarrow{\mathbf{t}}_n(\tau), \tau)}{\partial t_j}$. Now, by applying the implicit function theorem, we can extend $\overrightarrow{\mathbf{t}}_n(\tau)$ to $\tau = 0$ for all n, since $\{\overrightarrow{\mathbf{t}}_n(\tau)\}_{\tau\in\mathcal{Z}_n}$ is bounded from both above and below away from 0. Since $\overrightarrow{\mathbf{t}}_n(0) = \overrightarrow{\mathbf{s}}_n$ is unique, by similar arguments to those that are used in the proof of [47, Lemma 2.2], we obtain that $\overrightarrow{\mathbf{1}}$ is the unique solution of $\overrightarrow{\mathbf{\Gamma}}_n(\overrightarrow{\mathbf{t}}, \beta) = \overrightarrow{\mathbf{0}}$, which implies that $\overrightarrow{\mathbf{1}}$ is the unique critical point of $\Phi_n(\overrightarrow{\mathbf{t}})$ for all $n \in \mathbb{N}$. Recall that

$$\prod_{i=1}^{2}(\mu_i \|u_i^n\|_{L^4(\Omega)}^4) - \beta^2 \|u_1^n u_2^n\|_{L^2(\Omega)}^4 \geq \varepsilon^8 C \quad \text{and} \quad \|u_i^n\|_{L^4(\Omega)}^4 \geq \varepsilon^4 C$$

for both $i = 1, 2$. We observe that $\Phi_n(\overrightarrow{\mathbf{t}}) \to -\infty$ as $|\overrightarrow{\mathbf{t}}| \to +\infty$. It follows that $\Phi_n(\overrightarrow{\mathbf{t}})$ has a maximum point in $\overline{(\mathbb{R}^+)^2}$ for all $n \in \mathbb{N}$. Since $p > 2$, by a standard argument, either $\frac{\partial \Phi_n(\overrightarrow{\mathbf{t}})}{\partial t_1} > 0$ or $\frac{\partial \Phi_n(\overrightarrow{\mathbf{t}})}{\partial t_2} > 0$ near $\partial\overline{(\mathbb{R}^+)^2}$. Hence, $\overrightarrow{\mathbf{1}}$ is the unique maximum point of $\Phi_n(\overrightarrow{\mathbf{t}})$ in $\overline{(\mathbb{R}^+)^2}$ and $\Phi_n(\overrightarrow{\mathbf{1}}) \geq \Phi_n(\overrightarrow{\mathbf{t}})$ for all $n \in \mathbb{N}$ and $\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^2$.

(iii) Let $\overrightarrow{\mathbf{w}} \in \mathcal{H}_{\varepsilon,\Omega}$. For every $n \in \mathbb{N}$, we consider the equation

$$
\begin{aligned}
\Upsilon_n(t,l) &= \sum_{i=1}^{2}(\|tu_i^n + lw_i\|_{i,\varepsilon,\Omega}^2 - \alpha_i\|tu_i^n + lw_i\|_{L^p(\Omega)}^p - \mu_i\|tu_i^n + lw_i\|_{L^4(\Omega)}^4) \\
&\quad -2\beta\|(tu_1^n + lw_1)(tu_2^n + lw_2)\|_{L^2(\Omega)}^2.
\end{aligned}
$$

$\Upsilon_n(t,l)$ is of class C^1. Moreover, since $\{\overrightarrow{\mathbf{u}}'_n\} \subset \mathcal{N}'_{\varepsilon,\Omega,T}$, we also have from (i) and the construction of χ_β that is specified by (2.1) that $\Upsilon_n(1,0) = 0$. By a direct calculation, we obtain that

$$
\frac{\partial \Upsilon_n(1,0)}{\partial t} = -\sum_{i=1}^{2}((p-2)\alpha_i\|u_i^n\|_{L^p(\Omega)}^p + 2\mu_i\|u_i^n\|_{L^4(\Omega)}^4) - 4\beta\|u_1^n u_2^n\|_{L^2(\Omega)}^2.
$$

Since $p > 2$, by a similar argument to that used for (2.8), $\frac{\partial \Upsilon_n(1,0)}{\partial t} \leq -C < 0$ for all $\beta > 0$. Now, by the implicit function theorem, there exist $\sigma_n > 0$ and $t_n(l) \in C^1([-\sigma_n, \sigma_n], [\frac{1}{2}, \frac{3}{2}])$ such that $\{t_n(l)\overrightarrow{\mathbf{u}}'_n + l\overrightarrow{\mathbf{w}}\} \subset \mathcal{N}'_{\varepsilon,\Omega,T}$ for all $\beta > 0$. By applying the Taylor expansion to $\mathcal{J}_{\varepsilon,\Omega,T}(t_n(l)\overrightarrow{\mathbf{u}}'_n + l\overrightarrow{\mathbf{w}})$, we can show that $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_n) = o_n(1)$ strongly in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$. Since $p > 2$ and $\beta > 0$, we can prove $\widetilde{\Phi}_n(1) \geq \widetilde{\Phi}_n(t)$ for all $n \in \mathbb{N}$ and $t > 0$ in a standard way. □

We also require the following important energy estimates:

Lemma 2.4. *Let α_T and β_0 be defined by Lemmas 2.2 and 2.3, respectively. Then, for sufficiently small $\varepsilon > 0$ and $\alpha_1, \alpha_2 > 0$,*

(i) *we have*

$$
c_{\varepsilon,\Omega,T} < \min\{d_{1,\varepsilon,\Omega} + \frac{\varepsilon^4}{4\mu_2}\mathcal{S}^2, d_{2,\varepsilon,\Omega} + \frac{\varepsilon^4}{4\mu_1}\mathcal{S}^2, A_\varepsilon\} - \varepsilon^4 C_\beta
$$

under one of the following two cases:

(a) $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$,

(b) $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $|\overrightarrow{\alpha}| < \alpha_T$,

where $d_{i,\varepsilon,\Omega}$ and A_ε are specified by Propositions 7.2 and 7.3, respectively.

(ii) *there exists $\beta_1 > \max\{\mu_1, \mu_2\}$ that is independent of ε such that*

$$
c'_{\varepsilon,\Omega,T} < \min\{d_{1,\varepsilon,\Omega}, d_{2,\varepsilon,\Omega}, A_\varepsilon\} - \varepsilon^4 C_\beta
$$

for $\beta > \beta_1$.

Here, C_β only depends on β.

Proof. (i) By Proposition 7.2, $\widetilde{U}_{i,\varepsilon}$ is a least-energy solution of the following equation:

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_i u = \mu_i u^3 + \alpha_i u^{p-1} & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, & u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then $\mathcal{E}_{i,\varepsilon,\Omega}(\widetilde{U}_{i,\varepsilon}) = d_{i,\varepsilon,\Omega}, i = 1,2$. Since $d_{i,\varepsilon,\Omega} < \frac{\varepsilon^4}{4\mu_i}\mathcal{S}^2$ and $2 < p < 4$, by a standard argument, we have

$$\begin{aligned} d_{i,\varepsilon,\Omega} &= \mathcal{E}_{i,\varepsilon,\Omega}(\widetilde{U}_{i,\varepsilon}) - \frac{1}{p}\mathcal{E}'_{i,\varepsilon,\Omega}(\widetilde{U}_{i,\varepsilon})\widetilde{U}_{i,\varepsilon} \\ &= \frac{p-2}{2p}\|\widetilde{U}_{i,\varepsilon}\|^2_{i,\varepsilon,\Omega} + \frac{4-p}{4p}\|\widetilde{U}_{i,\varepsilon}\|^4_{L^4(\Omega)}. \end{aligned}$$

It follows that

$$\|\widetilde{U}_{i,\varepsilon}\|^4_{L^4(\Omega)} < \frac{\varepsilon^4 p}{(4-p)\mu_i}\mathcal{S}^2 \quad \text{and} \quad \|\widetilde{U}_{i,\varepsilon}\|^2_{i,\varepsilon,\Omega} < \frac{\varepsilon^4 p}{2(p-2)\mu_i}\mathcal{S}^2, \quad i = 1,2. \tag{2.26}$$

On the other hand, by a similar argument to that used for (2.7), we also have that

$$\|\widetilde{U}_{i,\varepsilon}\|^2_{i,\varepsilon,\Omega} \geq \varepsilon^4 C \quad \text{and} \quad \|\widetilde{U}_{i,\varepsilon}\|^4_{L^4(\Omega)} \geq \varepsilon^4 C. \tag{2.27}$$

Let $x_R \in \Omega_\varepsilon$ and $R > 0$ satisfy $\mathbb{B}_{3R}(x_R) \subset \Omega_\varepsilon$. Take $\Psi \in C_0^2(\mathbb{B}_2(0))$ that satisfies $0 \leq \Psi(x) \leq 1$ and $\Psi(x) \equiv 1$ in $\mathbb{B}_1(0)$. Set $\varphi_R^*(x) = \Psi\left(\frac{x-x_R}{R}\right)$ for $x \in \mathbb{B}_{2R}(x_R)$ and $\varphi_R^*(x) = 0$ for $x \in \Omega_\varepsilon \backslash \mathbb{B}_{2R}(x_R)$. Then $\varphi_R^*(x) \in C_0^2(\Omega_\varepsilon)$ and

$$\varphi_R^*(x) = \begin{cases} 1, & x \in \mathbb{B}_R(x_R); \\ 0, & x \in \Omega_\varepsilon \backslash \mathbb{B}_{2R}(x_R), \end{cases}$$

and $|\nabla\varphi_R^*(x)| \leq \frac{C}{R}$. Let

$$V_\sigma(x) = \frac{2\sqrt{2}\sigma}{(\sigma^2 + |x|^2)}, \quad v_\sigma(x) = \varphi_R^*(x)V_\sigma(\frac{x - x_R}{R}). \tag{2.28}$$

Here, $V_\sigma(x)$ is the well-known Talanti function, which is also a bubble. Then, it is well known that $v_\sigma \rightharpoonup 0$ weakly in $H_0^1(\Omega_\varepsilon) \cap L^4(\Omega_\varepsilon)$ and $v_\sigma \to 0$ strongly in $L^r(\Omega_\varepsilon)$ for all $1 \leq r < 4$ as $\sigma \to 0$. Moreover, by a well-known calculation (cf. [43]), we also have that

$$\|\nabla v_\sigma\|^2_{L^2(\Omega_\varepsilon)} = \mathcal{S}^2 + O(\sigma^2), \quad \|v_\sigma\|^4_{L^4(\Omega_\varepsilon)} = \mathcal{S}^2 + O(\sigma^4) \tag{2.29}$$

and

$$\|v_\sigma\|^p_{L^p(\Omega_\varepsilon)} \geq C(\sigma^{4-p} + \sigma^p), \quad \|v_\sigma\|^2_{L^2(\Omega_\varepsilon)} \leq C'\sigma^2|\ln\sigma| + O(\sigma^2). \tag{2.30}$$

Since $0 \in \Omega$, we must have $\Omega_{\varepsilon_1} \subset \Omega_{\varepsilon_2}$ with $\varepsilon_1 > \varepsilon_2$, which implies that $O(\sigma^2)$, $O(\sigma^4)$ and the constants C, C' in (2.29) and (2.30) can be chosen such that they are independent of $\varepsilon < 1$. Let us consider the following system

$$\begin{cases} \eta_1(t_1, t_2) = \|t_1 \widetilde{U}_{1,\varepsilon}\|^2_{1,\varepsilon,\Omega} - \alpha_1 \|t_1 \widetilde{U}_{1,\varepsilon}\|^p_{L^p(\Omega)} - \mu_1 \|t_1 \widetilde{U}_{1,\varepsilon}\|^4_{L^4(\Omega)} \\ \qquad\qquad - \beta \|(t_1 \widetilde{U}_{1,\varepsilon})(t_2 u_\sigma)\|^2_{L^2(\Omega)}, \\ \eta_2(t_1, t_2) = \|t_2 u_\sigma\|^2_{2,\varepsilon,\Omega} - \alpha_2 \|t_2 u_\sigma\|^p_{L^p(\Omega)} - \mu_2 \|t_2 u_\sigma\|^4_{L^4(\Omega)} \\ \qquad\qquad - \beta \|(t_1 \widetilde{U}_{1,\varepsilon})(t_2 u_\sigma)\|^2_{L^2(\Omega)}, \end{cases}$$

where $u_\sigma(x) = v_\sigma(\varepsilon^{-1} x)$. Set

$$t_1^* = \left(\frac{\|\widetilde{U}_{1,\varepsilon}\|^2_{1,\varepsilon,\Omega}}{\mu_1 \|\widetilde{U}_{1,\varepsilon}\|^4_{L^4(\Omega)}} \right)^{\frac{1}{2}} \quad \text{and} \quad t_2^* = \left(\frac{\|u_\sigma\|^2_{2,\varepsilon,\Omega}}{\mu_2 \|u_\sigma\|^4_{L^4(\Omega)}} \right)^{\frac{1}{2}}.$$

By (2.26) and (2.27), we have $t_1^* \leq C$. Moreover, by (2.29), we also have $t_2^* \leq C$ for sufficiently small σ. By the classical elliptic regularity theorem, we have $\widetilde{U}_{1,\varepsilon} \in C^2$. Thus, from (2.30), $2 < p < 4$ and Proposition 7.2, it follows that

$$\begin{aligned} &|\beta| \|(t_1^* \widetilde{U}_{1,\varepsilon})(t_2 u_\sigma)\|^2_{L^2(\Omega)} - \alpha_1 \|t_1^* \widetilde{U}_{1,\varepsilon}\|^p_{L^p(\Omega)} \\ &\leq \varepsilon^4 \|\widetilde{U}_{1,\varepsilon}\|^2_{L^\infty(\Omega)} C |\beta| (\sigma^2 |\ln \sigma| + O(\sigma^2)) - \alpha_1 \|t_1^* \widetilde{U}_{1,\varepsilon}\|^p_{L^p(\Omega)} \\ &< 0 \end{aligned}$$

for all $t_2 \in [0, t_2^*]$ and

$$\begin{aligned} &|\beta| \|(t_1 \widetilde{U}_{1,\varepsilon})(t_2^* u_\sigma)\|^2_{L^2(\Omega)} - \alpha_2 \|t_2^* u_\sigma\|^p_{L^p(\Omega)} \\ &\leq \varepsilon^4 (\|\widetilde{U}_{1,\varepsilon}\|^2_{L^\infty(\Omega)} C |\beta| (\sigma^2 |\ln \sigma| + O(\sigma^2)) - C'(\sigma^{4-p} + \sigma^p)) \\ &< 0 \end{aligned}$$

for all $t_1 \in [0, t_1^*]$ with sufficiently small σ. It follows that $\eta_1(t_1^*, t_2) < 0$ for all $t_2 \in [0, t_2^*]$ and $\eta_2(t_1, t_2^*) < 0$ for all $t_1 \in [0, t_1^*]$ with sufficiently small σ. On the other hand, by similar arguments to those that were used in (2.15), we have

$$\eta_1(t_1, t_2) \geq t_1^2 (\|\widetilde{U}_{1,\varepsilon}\|^2_{1,\varepsilon,\Omega} - C t_1^2 \|\widetilde{U}_{1,\varepsilon}\|^4_{L^4(\Omega)} - \beta \|(\widetilde{U}_{1,\varepsilon})(t_2^* u_\sigma)\|^2_{L^2(\Omega)})$$

for all $t_2 \in [0, t_2^*]$ and

$$\eta_2(t_1, t_2) \geq t_2^2 (\|u_\sigma\|^2_{2,\varepsilon,\Omega} - C t_2^2 \|u_\sigma\|^4_{L^4(\Omega)} - \beta \|(t_1^* \widetilde{U}_{1,\varepsilon})(u_\sigma)\|^2_{L^2(\Omega)})$$

for all $t_1 \in [0, t_1^*]$ with $\beta > 0$ while

$$\eta_1(t_1, t_2) \geq t_1^2 (\|\widetilde{U}_{1,\varepsilon}\|^2_{1,\varepsilon,\Omega} - C t_1^2 \|\widetilde{U}_{1,\varepsilon}\|^4_{L^4(\Omega)})$$

for all $t_2 \in [0, t_2^*]$ and

$$\eta_2(t_1, t_2) \geq t_2^2 (\|u_\sigma\|^2_{2,\varepsilon,\Omega} - C t_2^2 \|u_\sigma\|^4_{L^4(\Omega)})$$

for all $t_1 \in [0, t_1^*]$ with $\beta < 0$. Since (2.26)–(2.29) hold, by taking sufficiently small β_0 if necessary and using a standard argument, we obtain that there exist $t_1^{**}, t_2^{**} \geq C'$

such that $\eta_1(t_1^{**}, t_2) > 0$ for all $t_2 \in [t_2^{**}, t_2^*]$ and $\eta_2(t_1, t_2^{**}) > 0$ for all $t_1 \in [t_1^{**}, t_1^*]$ with $\beta < \beta_0$ and σ sufficiently small. Now, applying Miranda's theorem (cf. [12, Lemma 3.1]), we obtain that there exists $(\widetilde{t}_{1,\varepsilon}, \widetilde{t}_{2,\varepsilon}) \in [t_1^{**}, t_1^*] \times [t_2^{**}, t_2^*]$ such that $\eta_1(\widetilde{t}_{1,\varepsilon}, \widetilde{t}_{2,\varepsilon}) = \eta_2(\widetilde{t}_{1,\varepsilon}, \widetilde{t}_{2,\varepsilon}) = 0$. It follows from (2.2), (2.26) and (2.27) that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\widetilde{\mathbf{U}}}_\varepsilon \in \mathcal{N}_{\varepsilon,\Omega,T}$ for σ sufficiently small, where $\overrightarrow{\mathbf{t}}_\varepsilon = (\widetilde{t}_{1,\varepsilon}, \widetilde{t}_{2,\varepsilon})$ and $\overrightarrow{\widetilde{\mathbf{U}}}_\varepsilon = (\widetilde{U}_{1,\varepsilon}, u_\sigma)$. By the choice of u_σ and Proposition 7.2 in the Appendix, we have

$$|\beta| \|(\widetilde{t}_{1,\varepsilon}\widetilde{U}_{1,\varepsilon})(\widetilde{t}_{2,\varepsilon}u_\sigma)\|^2_{L^2(\Omega)} \le |\beta| C \|u_\sigma\|^2_{L^2(\Omega)}.$$

Since $2 < p < 4$, via a standard argument we obtain that

$$\begin{aligned} c_{\varepsilon,\Omega,T} &\le \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\widetilde{\mathbf{U}}}_\varepsilon) \\ &\le \mathcal{E}_{1,\varepsilon}(\widetilde{U}_{1,\varepsilon}) + \frac{\varepsilon^4}{4\mu_2}\mathcal{S}^2 + \varepsilon^4(C|\beta|\sigma^2|\ln\sigma| + |\beta|O(\sigma^2) - C'\sigma^{4-p}) \\ &< d_{1,\varepsilon,\Omega} + \frac{\varepsilon^4}{4\mu_2}\mathcal{S}^2 - \varepsilon^4 C_\beta \end{aligned}$$

with σ sufficiently small, where C_β depends only on β. Similarly, we show that

$$c_{\varepsilon,\Omega,T} < d_{2,\varepsilon,\Omega} + \frac{\varepsilon^4}{4\mu_1}\mathcal{S}^2 - \varepsilon^4 C_\beta.$$

It remains to show that $c_{\varepsilon,\Omega,T} < A_\varepsilon - \varepsilon^4 C$. For $\beta < 0$, by Proposition 7.3, we have $A_\varepsilon = \frac{\varepsilon^4}{4\mu_1}\mathcal{S}^2 + \frac{\varepsilon^4}{4\mu_2}\mathcal{S}^2$. Since $d_{i,\varepsilon,\Omega} < \frac{\varepsilon^4}{4\mu_i}\mathcal{S}^2 - \varepsilon^4 C$, we must have $c_{\varepsilon,\Omega,T} < A_\varepsilon - \varepsilon^4 C$ for $\beta < 0$. For $\beta > 0$, let

$$\overrightarrow{\mathbf{U}}_\sigma = (\sqrt{k_1}u_\sigma, \sqrt{k_2}u_\sigma)$$

and consider the following system

$$\begin{aligned} \eta_i(t_1, t_2) &= \|t_i\sqrt{k_i}u_\sigma\|^2_{i,\varepsilon,\Omega} - \alpha_i\|t_i\sqrt{k_i}u_\sigma\|^p_{L^p(\Omega)} - \mu_i\|t_i\sqrt{k_i}u_\sigma\|^4_{L^4(\Omega)} \\ &\quad -\beta\|(t_1\sqrt{k_1}u_\sigma)(t_2\sqrt{k_2}u_\sigma)\|^2_{L^2(\Omega)}, \end{aligned}$$

where k_1, k_2 satisfy (7.9). Since (2.2), (2.29)–(2.30) hold, taking into account $k_i \to \frac{1}{\mu_i}$ as $\beta \to 0$, we can apply Miranda's theorem as for $\eta_i(t_1, t_2)$ to show that there exists $(\bar{t}_{1,\varepsilon}, \bar{t}_{2,\varepsilon}) \in [C', C] \times [C', C]$ such that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{U}}_\sigma \in \mathcal{N}_{\varepsilon,\Omega,T}$ for $\beta > 0$ sufficiently small. By taking β_0 sufficiently small if necessary, we obtain that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{U}}_\sigma \in \mathcal{N}_{\varepsilon,\Omega,T}$ for $0 < \beta < \beta_0$. Thus, it follows from (2.29) and (2.30) that

$$\begin{aligned} c_{\varepsilon,\Omega,T} &\le \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{U}}_\sigma) \\ &= \varepsilon^4\mathcal{S}^2\bigg(\sum_{i=1}^2\Big(\frac{(\bar{t}_{i,\varepsilon}\sqrt{k_i})^2}{2} - \frac{\mu_i(\bar{t}_{i,\varepsilon}\sqrt{k_i})^4}{4}\Big) - \frac{\beta(\bar{t}_{1,\varepsilon}\sqrt{k_1})^2(\bar{t}_{2,\varepsilon}\sqrt{k_2})^2}{2}\bigg) \\ &\quad +\varepsilon^4(C\sigma^2|\ln\sigma| + O(\sigma^2) - C(\sigma^{4-p} + \sigma^p)). \end{aligned} \tag{2.31}$$

Since $\beta_0 < \sqrt{\mu_1\mu_2}$, k_1, k_2 uniquely satisfy (7.9) for $0 < \beta < \beta_0$. Since $2 < p < 4$ and by a similar argument to that used in the proof of [46, Lemma 3.1], we obtain from (2.31) and (7.9) once more that

$$\begin{aligned} c_{\varepsilon,\Omega,T} &\leq \varepsilon^4\mathcal{S}^2\bigg(\sum_{i=1}^{2}(\frac{k_i}{2} - \frac{\mu_i k_i^2}{4}) - \frac{\beta k_1 k_2}{2}\bigg) \\ &\quad + \varepsilon^4(C\sigma^2|\ln\sigma| + O(\sigma^2) - C(\sigma^{4-p} + \sigma^p)) \\ &< \varepsilon^4\frac{k_1+k_2}{4}\mathcal{S}^2 - \varepsilon^4 C \end{aligned}$$

for σ sufficiently small. It follows from Proposition 7.3 that $c_{\varepsilon,\Omega} < A_\varepsilon - \varepsilon^4 C$ for $0 < \beta < \beta_0$. In summary, we finally have that

$$c_{\varepsilon,\Omega,T} < \min\{d_{1,\varepsilon,\Omega} + \frac{\varepsilon^4}{4\mu_2}\mathcal{S}^2, d_{2,\varepsilon,\Omega} + \frac{\varepsilon^4}{4\mu_1}\mathcal{S}^2, A_\varepsilon\} - \varepsilon^4 C_\beta$$

in the following two cases:

1. $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ and $\alpha_1, \alpha_2 > 0$,
2. $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$,

where C_β depends only on β.

(ii) Let $\overrightarrow{\mathbf{U}^*}_\varepsilon = (\widetilde{U}_{1,\varepsilon}, \widetilde{U}_{1,\varepsilon})$. Since $p > 2$, by a standard argument, we obtain from (2.2) and (2.26) that there exists $t_\beta > 0$ such that $t_\beta\overrightarrow{\mathbf{U}^*}_\varepsilon \in \mathcal{N}'_{\varepsilon,\Omega,T}$. We claim that $t_\beta \to 0$ as $\beta \to +\infty$. Indeed, suppose the contrary. Then, without loss of generality, we may assume that $t_\beta \geq C$ for β sufficiently large. It follows from (2.26)–(2.27) that

$$c'_{\varepsilon,\Omega,T} \leq \mathcal{J}_{\varepsilon,\Omega,T}(t_\beta\overrightarrow{\mathbf{U}^*}_\varepsilon) \leq \varepsilon^4 t_\beta^2(C - \beta C') \to -\infty$$

as $\beta \to +\infty$, which contradicts Lemma 2.2. Now, since $t_\beta \to 0$ as $\beta \to +\infty$ and $p > 2$, it follows from $t_\beta\overrightarrow{\mathbf{U}^*}_\varepsilon \in \mathcal{N}'_{\varepsilon,\Omega,T}$ and (2.26) and the Hölder inequality that

$$\sum_{i=1}^{2}\|\widetilde{U}_{1,\varepsilon}\|^2_{i,\varepsilon,\Omega} = 2t_\beta^2\beta\|\widetilde{U}_{1,\varepsilon}\|^4_{L^4(\Omega)} + o(1)\varepsilon^4,$$

where $o(1) \to 0$ uniformly for $\varepsilon < 1$ as $\beta \to +\infty$. This implies

$$c'_{\varepsilon,\Omega,T} \leq \mathcal{J}_{\varepsilon,\Omega,T}(t_\beta\overrightarrow{\mathbf{U}^*}_\varepsilon) \leq \varepsilon^4 t_\beta^2(\frac{1}{4}\sum_{i=1}^{2}\|\overline{U}_{1,\varepsilon}\|^2_{i,\Omega_\varepsilon} + o(1)),$$

where $\overline{U}_{1,\varepsilon}(x) = \widetilde{U}_{1,\varepsilon}(\varepsilon x)$. Thus, by (2.26) and Propositions 7.2–7.3, there exists $\beta_1 > \beta_0$ that is independent of ε such that

$$c'_{\varepsilon,\Omega,T} < \min\{d_{1,\varepsilon,\Omega}, d_{2,\varepsilon,\Omega}, A_\varepsilon\} - \varepsilon^4 C_\beta$$

for all $\beta > \beta_1$. $\square$

By Lemma 2.3 and the construction of χ_β, we obtain $\overrightarrow{\mathbf{u}}_n \rightharpoonup \overrightarrow{\mathbf{u}}_0$ and $\overrightarrow{\mathbf{u}}'_n \rightharpoonup \overrightarrow{\mathbf{u}}'_0$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n\to\infty$ and $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_0)=\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_0)=0$ in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$.

Proposition 2.1. *Assume $\alpha_1,\alpha_2>0$. Let α_T, β_0 and β_1 be defined by Lemmas 2.2-2.4, respectively. Then, $(\mathcal{S}_\varepsilon)$ has a least-energy solution $\overrightarrow{\mathbf{u}}_\varepsilon$ with sufficiently small $\varepsilon>0$ in the following two cases:*

1. *either* $-\sqrt{\mu_1\mu_2}<\beta<\beta_0$ *or* $\beta>\beta_1$,
2. $\beta\le-\sqrt{\mu_1\mu_2}$ *with* $|\overrightarrow{\alpha}|<\alpha_T$.

Moreover, $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon)=c_{\varepsilon,\Omega,T}$ for $\beta<\beta_0$ and $\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon)=c'_{\varepsilon,\Omega,T}$ for $\beta>\beta_1$.

Proof. Let $\overrightarrow{\mathbf{v}}_n=\overrightarrow{\mathbf{u}}_n-\overrightarrow{\mathbf{u}}_0$ and $\overrightarrow{\mathbf{v}}'_n=\overrightarrow{\mathbf{u}}'_n-\overrightarrow{\mathbf{u}}'_0$. Since $\overrightarrow{\mathbf{u}}_n \rightharpoonup \overrightarrow{\mathbf{u}}_0$ and $\overrightarrow{\mathbf{u}}'_n \rightharpoonup \overrightarrow{\mathbf{u}}'_0$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n\to\infty$, we have $\overrightarrow{\mathbf{v}}_n, \overrightarrow{\mathbf{v}}'_n\rightharpoonup 0$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n\to\infty$.

Claim. 1 $\overrightarrow{\mathbf{u}}'_0$ is nontrivial for $\beta>\beta_1$.

Indeed, suppose $\overrightarrow{\mathbf{u}}'_0=\overrightarrow{\mathbf{0}}$. Then, $\overrightarrow{\mathbf{u}}'_n\rightharpoonup\overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n\to\infty$. It follows from $\overrightarrow{\mathbf{u}}'_n\in\mathcal{N}'_{\varepsilon,\Omega,T}$ and the Sobolev embedding theorem that

$$\sum_{i=1}^2\|(u_i^n)'\|^2_{i,\varepsilon,\Omega}=\sum_{i=1}^2\mu_i\|(u_i^n)'\|^4_{L^4(\Omega)}+2\beta\|(u_1^n)'(u_2^n)'\|^2_{L^2(\Omega)}+o_n(1). \quad (2.32)$$

Let $(u_i^n)'$ be a function in $D^{1,2}(\mathbb{R}^4)$ by setting $(u_i^n)'\equiv0$ outside Ω. Then, by a standard argument and (2.32), we obtain that

$$\begin{aligned} c'_{\varepsilon,\Omega,T} &= \frac14\sum_{i=1}^2\|(u_i^n)'\|^2_{i,\varepsilon,\Omega}+o_n(1)\\ &\ge \min_{\overrightarrow{\mathbf{u}}\in\widetilde{\mathcal{D}}}\frac14\left(\frac{\sum_{i=1}^2\|u_i\|^2_{i,\varepsilon,\mathbb{R}^4}}{(\sum_{i=1}^2\mu_i\|u_i\|^4_{L^4(\mathbb{R}^4)}+2\beta\|u_1u_2\|^2_{L^2(\mathbb{R}^4)})^{\frac12}}\right)^2+o_n(1). \end{aligned}$$

Let

$$c_{\varepsilon,\beta}=\min_{\overrightarrow{\mathbf{u}}\in\widetilde{\mathcal{D}}}\frac14\left(\frac{\sum_{i=1}^2\|u_i\|^2_{i,\varepsilon,\mathbb{R}^4}}{(\sum_{i=1}^2\mu_i\|u_i\|^4_{L^4(\mathbb{R}^4)}+2\beta\|u_1u_2\|^2_{L^2(\mathbb{R}^4)})^{\frac12}}\right)^2.$$

Then, by testing $c_{\varepsilon,\beta}$ with $(\overline{V}_\sigma,\overline{V}_\sigma)$, we observe that $c_{\varepsilon,\beta}<\varepsilon^4\min\{\frac{1}{4\mu_1}\mathcal{S}^2,\frac{1}{4\mu_2}\mathcal{S}^2\}$ for β sufficiently large. Here, $\overline{V}_\sigma(x)=V_\sigma(\varepsilon x)$ and V_σ is the well-known Talanti function, which is specified by (2.28). By taking β_1 sufficiently large if necessary, it follows from a standard argument (cf. [10, (5.43)-(5.44) and (5.46)]) that $c_{\varepsilon,\beta}=A_\varepsilon$ for $\beta>\beta_1$. This implies that $c'_{\varepsilon,\Omega,T}\ge A_\varepsilon$ in the case $\beta>\beta_1$, which contradicts Lemma 2.4. Thus, $\overrightarrow{\mathbf{u}}'_0\ne\overrightarrow{\mathbf{0}}$ for $\beta>\beta_1$. Applying Lemma 2.4 once more yields that $\overrightarrow{\mathbf{u}}'_0$ is also not semi-trivial. Hence, $\overrightarrow{\mathbf{u}}'_0$ must be nontrivial.

Claim. 2 $\overrightarrow{\mathbf{u}}_0$ is nontrivial for $\beta<\beta_0$.

First, we prove that $\overrightarrow{\mathbf{u}}_0\ne\overrightarrow{\mathbf{0}}$ for $\beta<\beta_0$. Indeed, suppose the contrary. Then, $\overrightarrow{\mathbf{u}}_n\rightharpoonup\overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n\to\infty$. It follows from $\overrightarrow{\mathbf{u}}_n\in\mathcal{N}_{\varepsilon,\Omega,T}$ and the Sobolev embedding theorem that

$$\|u_i^n\|^2_{i,\varepsilon,\Omega}=\mu_i\|u_i^n\|^4_{L^4(\Omega)}+\beta\|u_1^nu_2^n\|^2_{L^2(\Omega)}+o_n(1),\ \ i=1,2. \quad (2.33)$$

By setting $u_i^n \equiv 0$ outside Ω, we also regard u_i^n as a function in $D^{1,2}(\mathbb{R}^4)$. From (2.33), it follows that there exists $\overrightarrow{\mathbf{t}}_n \to \overrightarrow{\mathbf{1}}$ as $n \to \infty$ such that $\overrightarrow{\mathbf{t}}_n \circ \overrightarrow{\mathbf{u}}_n \in \mathcal{V}_\varepsilon$ (cf. [46]), where $\mathcal{V}_\varepsilon$ is specified by (7.8). It follows from Lemma 2.2 and the Sobolev embedding theorem that

$$\begin{aligned} c_{\varepsilon,\Omega,T} &= \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) + o_n(1) \\ &\geq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}}_n \circ \overrightarrow{\mathbf{u}}_n) + o_n(1) \\ &= \mathcal{I}_\varepsilon(\overrightarrow{\mathbf{t}}_n \circ \overrightarrow{\mathbf{u}}_n) + o_n(1) \\ &\geq A_\varepsilon + o_n(1) \end{aligned}$$

for $\beta < \beta_0$, which contradicts Lemma 2.4. Here, $\mathcal{I}_\varepsilon(\overrightarrow{\mathbf{u}})$ is specified by (7.7). Next, we prove that $\overrightarrow{\mathbf{u}}_0$ is also not semi-trivial for $\beta < \beta_0$. Suppose otherwise. Then, without loss of generality, we may assume $\overrightarrow{\mathbf{u}}_0 = (u_1^0, 0)$ with $u_1^0 \neq 0$. Let $v_n = u_1^n - u_1^0$. Then, $\overrightarrow{\mathbf{v}}_n = (v_n, u_2^n) \rightharpoonup \overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n \to \infty$. By the Brezís-Lieb lemma and [13, Lemma 2.3], we obtain that

$$\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_n) = \mathcal{E}_{1,\varepsilon,\Omega}(u_1^0) + \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{v}}_n) + o_n(1), \tag{2.34}$$

where $\mathcal{E}_{1,\varepsilon,\Omega}(u)$ is specified by (7.4). Since $\overrightarrow{\mathbf{u}}_n \rightharpoonup \overrightarrow{\mathbf{u}}_0$ weakly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n \to \infty$ and $u_2^0 \equiv 0$, it is easy to show that $\mathcal{E}'_{1,\varepsilon,\Omega}(u_1^0) = 0$ in $\mathcal{H}^{-1}_{1,\varepsilon,\Omega}$, where $\mathcal{H}^{-1}_{1,\varepsilon,\Omega}$ is the dual space of $\mathcal{H}_{1,\varepsilon,\Omega}$. Now, by similar arguments to those that are used above, it follows from (2.34) that

$$c_{\varepsilon,\Omega,T} \geq d_{1,\varepsilon,\Omega} + A_\varepsilon,$$

which also contradicts Lemma 2.4.

Now, since $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_0) = \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_0) = 0$ in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ and both $\overrightarrow{\mathbf{u}}_0$ and $\overrightarrow{\mathbf{u}}'_0$ are nontrivial, $c_{\varepsilon,\Omega,T}$ and $c'_{\varepsilon,\Omega,T}$ are attained by $\overrightarrow{\mathbf{u}}_\varepsilon$ and $\overrightarrow{\mathbf{u}}'_\varepsilon$, respectively, for $\beta < \beta_0$ and $\beta > \beta_1$. Here, $\overrightarrow{\mathbf{u}}_\varepsilon = (|u_1^0|, |u_2^0|)$ and $\overrightarrow{\mathbf{u}}'_\varepsilon = (|(u_1^0)'|, |(u_2^0)'|)$. Moreover, $\overrightarrow{\mathbf{u}}_n \to \overrightarrow{\mathbf{u}}_0$ and $\overrightarrow{\mathbf{u}}'_n \to \overrightarrow{\mathbf{u}}'_0$ strongly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n \to \infty$. By the method of Lagrange multipliers, there exist ρ_1, ρ_2 and ρ_3 such that

$$\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) - \rho_1(\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon)\overrightarrow{\mathbf{u}}^1_\varepsilon)' - \rho_2(\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon)\overrightarrow{\mathbf{u}}^2_\varepsilon)' = 0 \tag{2.35}$$

and

$$\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_\varepsilon) - \rho_3(\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_\varepsilon)\overrightarrow{\mathbf{u}}'_\varepsilon)' = 0 \tag{2.36}$$

in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$. Multiplying (2.35) and (2.36) with $\overrightarrow{\mathbf{u}}^i_\varepsilon$ and $\overrightarrow{\mathbf{u}}'_\varepsilon$, respectively, yields

$$\rho_1(\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon)\overrightarrow{\mathbf{u}}^1_\varepsilon)'\overrightarrow{\mathbf{u}}^i_\varepsilon + \rho_2(\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon)\overrightarrow{\mathbf{u}}^2_\varepsilon)'\overrightarrow{\mathbf{u}}^i_\varepsilon = 0, i = 1, 2 \tag{2.37}$$

and

$$\rho_3(\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_\varepsilon)\overrightarrow{\mathbf{u}}'_\varepsilon)'\overrightarrow{\mathbf{u}}'_\varepsilon = 0. \tag{2.38}$$

Since $\det(\Theta_n) \geq C$, which is specified by (2.13), and $\overrightarrow{\mathbf{u}}_n \to \overrightarrow{\mathbf{u}}_0$ strongly in $\mathcal{H}_{\varepsilon,\Omega}$ as $n \to \infty$, we obtain that $\overrightarrow{\mathbf{0}}$ is the unique solution of the System (2.37). However, since $p > 2$, 0 is only solution of Equation (2.38). Thus, by (2.35) and (2.36), we have $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) = 0$ in the following two cases:

1. $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ and $\alpha_1, \alpha_2 > 0$,

2. $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$,

and $\mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}'_\varepsilon) = 0$ for $\beta > \beta_1$ and $\alpha_1, \alpha_2 > 0$ in $\mathcal{H}^{-1}_{\varepsilon,\Omega}$. Since Lemma 2.3 holds, by standard elliptic estimates and the maximum principle, $\overrightarrow{\mathbf{u}}_\varepsilon$ and $\overrightarrow{\mathbf{u}}'_\varepsilon$ are also solutions of $(\mathcal{S}_\varepsilon)$. Recall the construction of χ_β. We also know that $\overrightarrow{\mathbf{u}}_\varepsilon$ and $\overrightarrow{\mathbf{u}}'_\varepsilon$ are both least-energy solutions. □

Proof of Theorem 1.1: It follows immediately from (2.2), Lemma 2.3 and Proposition 2.1. □

3 Limiting problem in $\mathbb{R}^4$

In this section, we mainly consider the following system

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \mathbb{R}^4, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \mathbb{R}^4, \\ u_1, u_2 > 0 \text{ in } \mathbb{R}^4, \quad u_1, u_2 \to 0 \text{ as } |x| \to +\infty. \end{cases} \qquad (\mathcal{S}_*)$$

Let $\mathcal{H}_{\mathbb{R}^4} = \mathcal{H}_{1,\mathbb{R}^4} \times \mathcal{H}_{2,\mathbb{R}^4}$, where $\mathcal{H}_{i,\mathbb{R}^4}$ are specified in the Appendix. Then $\mathcal{H}_{\mathbb{R}^4}$ is a Hilbert space that is equipped with the inner product

$$\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \rangle_{\mathbb{R}^4} = \sum_{i=1}^{2} \langle u_i, v_i \rangle_{i,\mathbb{R}^4}.$$

The corresponding norm is specified by $\|\overrightarrow{\mathbf{u}}\|_{\mathbb{R}^4} = \langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{u}} \rangle^{\frac{1}{2}}_{\mathbb{R}^4}$. Here, u_i, v_i are the ith components of $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}$, respectively. Set

$$\begin{aligned} \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}}) &= \sum_{i=1}^{2} (\frac{1}{2}\|u_i\|^2_{i,\mathbb{R}^4} - \frac{\alpha_i}{p}\|u_i\|^p_{L^p(\mathbb{R}^4)} \chi_\beta\left(\frac{\|\overrightarrow{\mathbf{u}}\|_{\mathbb{R}^4}}{T}\right) - \frac{\mu_i}{4}\|u_i\|^4_{L^4(\mathbb{R}^4)} \\ &\quad - \frac{\beta}{2}\|u_1 u_2\|^2_{L^2(\mathbb{R}^4)}, \end{aligned} \qquad (3.1)$$

where $\chi_\beta(s)$ is specified by (2.1). $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}})$ is of class C^2. Define

$$B = \inf_{\mathcal{N}_{\mathbb{R}^4,T}} \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}}) \quad \text{and} \quad B' = \inf_{\mathcal{N}'_{\mathbb{R}^4,T}} \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}}),$$

where

$$\mathcal{N}_{\mathbb{R}^4,T} = \{\overrightarrow{\mathbf{u}} = (u_1, u_2) \in \widetilde{\mathcal{H}}_{\mathbb{R}^4} \mid \mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}^1 = \mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}^2 = 0\}, \qquad (3.2)$$

with $\widetilde{\mathcal{H}}_{\mathbb{R}^4} = (\mathcal{H}_{1,\mathbb{R}^4}\backslash\{0\}) \times (\mathcal{H}_{2,\mathbb{R}^4}\backslash\{0\})$, $\overrightarrow{\mathbf{u}}^1 = (u_1, 0)$ and $\overrightarrow{\mathbf{u}}^2 = (0, u_2)$ and

$$\mathcal{N}'_{\mathbb{R}^4,T} = \{\overrightarrow{\mathbf{u}} = (u_1, u_2) \in \mathcal{H}_{\mathbb{R}^4}\backslash\{\overrightarrow{\mathbf{0}}\} \mid \mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}} = 0\}. \qquad (3.3)$$

Now, our results for $(\mathcal{S}_*)$ can be stated as follows.

Proposition 3.1. *Let α_T, β_0 and β_1 be as specified by Lemmas 2.2–2.4, respectively. Then,*

(i) $B = \sum_{i=1}^{2} d_{i,\mathbb{R}^4}$ *and* B *can not be attained in one of the following two cases*

(a) $-\sqrt{\mu_1\mu_2} < \beta < 0$*;*

(b) $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $|\overrightarrow{\alpha}| < \alpha_T$*,*

where $d_{i,\mathbb{R}^4}$ *are specified by* (7.3)*.*

(ii) *There exists* $\overrightarrow{\mathbf{U}}_*$ *with* U_i^* *radial symmetry such that* $\overrightarrow{\mathbf{U}}_*$ *is a solution of* $(\mathcal{S}_*)$ *and* $\overrightarrow{\mathbf{U}}_*$ *attains* B *for* $0 < \beta < \beta_0$ *or* $\beta > \beta_1$*. Moreover,* $B < \sum_{i=1}^{2} d_{i,\mathbb{R}^4}$ *for* $0 < \beta < \beta_0$ *and* $B = B'$ *for* $\beta > \beta_1$*.*

Proof. (i) Let $U_{2,\mathbb{R}^4}^R(x) = U_{2,\mathbb{R}^4}^R(x - Re_1)$, where $e_1 = (1, 0, 0, 0)$. Then,

$$\|U_{1,\mathbb{R}^4}U_{2,\mathbb{R}^4}^R\|_{L^2(\mathbb{R}^4)}^2 \to 0$$

as $R \to +\infty$. It follows that

$$\mu_1\mu_2\|U_{1,\mathbb{R}^4}\|_{L^4(\mathbb{R}^4)}^4\|U_{2,\mathbb{R}^4}^R\|_{L^4(\mathbb{R}^4)}^4 - \beta^2\|U_{1,\mathbb{R}^4}U_{2,\mathbb{R}^4}^R\|_{L^2(\mathbb{R}^4)}^2 \geq C$$

and

$$\|U_{1,\mathbb{R}^4}\|_{L^4(\mathbb{R}^4)}^4 \geq C, \quad \|U_{2,\mathbb{R}^4}^R\|_{L^4(\mathbb{R}^4)}^4 \geq C$$

for sufficiently large R. Since (2.2) holds and $d_{i,\mathbb{R}^4} \in (0, \frac{1}{4\mu_i}\mathcal{S}^2)$, we can apply Miranda's theorem as for $\eta_i(t_1, t_2)$ in the proof of Lemma 2.4 to show that there exists $\overrightarrow{\mathbf{t}}_R \in (\mathbb{R}^+)^2$ with $\overrightarrow{\mathbf{t}}_R \to \overrightarrow{\mathbf{1}}$ as $R \to +\infty$ such that $\overrightarrow{\mathbf{t}}_R \circ \overrightarrow{\mathbf{U}}_R \in \mathcal{N}_{\mathbb{R}^4,T}$ and $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{t}}_R \circ \overrightarrow{\mathbf{U}}_R) = \max_{\overrightarrow{\mathbf{t}} \in \mathbb{R}_+^2} \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{U}}_R)$ for sufficiently large R, where $\overrightarrow{\mathbf{U}}_R = (U_{1,\mathbb{R}^4}, U_{2,\mathbb{R}^4}^R)$. Moreover, taking into account (2.2), we can use similar arguments to those that are used for $\Phi_n(\overrightarrow{\mathbf{t}})$ in the proof of Lemma 2.3 to show that

$$\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}}) = \max_{\overrightarrow{\mathbf{s}} \in \mathbb{R}_+^2} \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{s}} \circ \overrightarrow{\mathbf{u}})$$

for all $\overrightarrow{\mathbf{u}} \in \mathcal{N}_{\mathbb{R}^4,T}$ with $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{u}}) < \frac{1}{4\mu_1}\mathcal{S}^2 + \frac{1}{4\mu_2}\mathcal{S}^2$. Now, since $d_{i,\mathbb{R}^4}$ is attained by $U_{i,\mathbb{R}^4}$, $i = 1, 2$, and $\beta < 0$, by a similar argument to that adopted for [10, Theorem 1.5], we can show that $B = \sum_{i=1}^{2} d_{i,\mathbb{R}^4}$ and this can not be attained for $\beta < 0$.

(ii) By setting $u \equiv 0$ outside Ω, we can regard any $u \in H_0^1(\Omega)$ as a function in $H^1(\mathbb{R}^4)$. Now, applying a similar argument for $\overrightarrow{\widetilde{\mathbf{U}}}_\varepsilon = (\widetilde{U}_{1,\varepsilon}, u_\sigma)$ to that in the proof of Lemma 2.4 to $(U_{1,\mathbb{R}^4}, v_\sigma)$ yields $B < d_{1,\mathbb{R}^4} + \frac{1}{4\mu_2}\mathcal{S}^2$ for all $0 < \beta < \beta_0$. Similarly, we can also show that $B < d_{2,\mathbb{R}^4} + \frac{1}{4\mu_1}\mathcal{S}^2$ for all $0 < \beta < \beta_0$. By following a similar argument to that used for $\overrightarrow{\widetilde{\mathbf{U}}}_\sigma = (\sqrt{k_1}u_\sigma, \sqrt{k_2}u_\sigma)$ in the proof of Lemma 2.4, we can prove that $B < A_1$ for all $0 < \beta < \beta_0$. Thus,

$$B < \min\{d_{1,\mathbb{R}^4} + \frac{1}{4\mu_2}\mathcal{S}^2, d_{2,\mathbb{R}^4} + \frac{1}{4\mu_1}\mathcal{S}^2, A_1\} \tag{3.4}$$

for all $0 < \beta < \beta_0$. Moreover, applying the argument that was used for $\overrightarrow{\mathbf{U}^*}_\varepsilon = (\widetilde{U}_{1,\varepsilon}, \widetilde{U}_{1,\varepsilon})$ in the proof of Lemma 2.4 to $(U_{1,\mathbb{R}^4}, U_{1,\mathbb{R}^4})$ yields

$$B' < \min\{d_{1,\mathbb{R}^4}, d_{2,\mathbb{R}^4}, A_1\} \tag{3.5}$$

for all $\beta > \beta_1$. On the other hand, let $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{N}_{\mathbb{R}^4,T}$ and $\{\overrightarrow{\mathbf{u}}'_n\} \subset \mathcal{N}'_{\mathbb{R}^4,T}$ be the sequences that are obtained by Ekeland's principle. Then, by $\beta > 0$ and Schwartz's symmetrization (cf. [26]), we obtain that $\overrightarrow{\mathbf{u}}_n$ and $\overrightarrow{\mathbf{u}}'_n$ can be chosen to be radially symmetric. Due to the compactness of the embedding map $\mathcal{H}_{i,\mathbb{R}^4} \to L^r_{Rad}(\mathbb{R}^4)$ for all $2 < r < 4$, by (3.4) and (3.5), we follow similar arguments to those that are used in the proof of Proposition 2.1 to prove that there exists $\overrightarrow{\mathbf{U}}_*$ with U_i^* radially symmetric such that $\overrightarrow{\mathbf{U}}_*$ is a solution of $(\mathcal{S}_*)$ and $\overrightarrow{\mathbf{U}}_*$ attains B for $0 < \beta < \beta_0$ or $\beta > \beta_1$ with $B = B'$ for $\beta > \beta_1$, where $L^r_{Rad}(\mathbb{R}^4) = \{u \in L^r(\mathbb{R}^4) \mid u \text{ is radially symmetric}\}$. It remains to show that $B < \sum_{i=1}^2 d_{i,\mathbb{R}^4}$ for $0 < \beta < \beta_0$. Indeed, by considering $\overrightarrow{\mathbf{U}}_{\mathbb{R}^4} = (U_{1,\mathbb{R}^4}, U_{2,\mathbb{R}^4})$ and taking into account (2.2), we can apply Miranda's theorem as for $\eta_i(t_1, t_2)$ in the proof of Lemma 2.4 to show that $\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{U}}_{\mathbb{R}^4} \in \mathcal{N}_{\mathbb{R}^4,T}$ for $\beta > 0$ sufficiently small with some $\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^2$. By taking β_0 sufficiently small if necessary, we obtain that $\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{U}}_{\mathbb{R}^4} \in \mathcal{N}_{\mathbb{R}^4,T}$ for $0 < \beta < \beta_0$ with some $\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^2$. Since $p > 2$, by a standard argument, it follows from $\beta > 0$ that

$$\sum_{i=1}^2 d_{i,\mathbb{R}^4} = \sum_{i=1}^2 \mathcal{E}_{i,\mathbb{R}^4}(U_{i,\mathbb{R}^4}) \geq \sum_{i=1}^2 \mathcal{E}_{i,\mathbb{R}^4}(t_i U_{i,\mathbb{R}^4}) > \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{U}}_{\mathbb{R}^4}) \geq B,$$

where $\mathcal{E}_{i,\mathbb{R}^4}(u)$ are specified by (7.1). □

Next, we establish a Liouville-type result for the following system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \mathbb{R}^4_+, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \mathbb{R}^4_+, \\ u_1, u_2 \geq 0 \text{ in } \mathbb{R}^4_+, \quad u_1, u_2 \to 0 \text{ as } |x| \to +\infty, \\ u_1 = u_2 = 0 \text{ on } \partial\mathbb{R}^4_+, \end{cases} \tag{$\mathcal{S}_{**}$}$$

where $\mathbb{R}^4_+ = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 > 0\}$. Our result can be expressed as follows:

Proposition 3.2. $(\mathcal{S}_{**})$ *has no solution unless* $u_1 \equiv 0$ *and* $u_2 \equiv 0$.

Proof. Suppose the contrary and let $\overrightarrow{\mathbf{u}} = (u_1, u_2)$ be a solution of $(\mathcal{S}_{**})$ such that either $u_1 \not\equiv 0$ or $u_2 \not\equiv 0$. Then, by the classical elliptic regularity theorem, u_i is of class C^2. Thus, by the strong maximum principle, $\frac{\partial u_i}{\partial x_4} > 0$ on $\partial\mathbb{R}^4_+$ for either $i = 1$ or $i = 2$. Now, multiplying $(\mathcal{S}_{**})$ with $(\frac{\partial u_1}{\partial x_4}, \frac{\partial u_2}{\partial x_4})$ and integrating by parts yields

$$0 = \int_{\mathbb{R}^4_+} \sum_{i=1}^2 \Delta u_i \frac{\partial u_i}{\partial x_4} dx = \frac{1}{2} \int_{\partial\mathbb{R}^4_+} \sum_{i=1}^2 |\frac{\partial u_1}{\partial x_4}|^2 ds > 0,$$

which is a contradiction. □

4 Concentration behavior as $\varepsilon \to 0^+$

By Proposition 2.1, $(\mathcal{S}_\varepsilon)$ has a least-energy solution $\overrightarrow{\mathbf{u}}_\varepsilon$ with ε sufficiently small and $\alpha_1, \alpha_2 > 0$ in the following two cases:

1. either $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ or $\beta > \beta_1$,
2. $\beta \leq -\sqrt{\mu_1\mu_2}$ with $|\overrightarrow{\alpha}| < \alpha_T$.

In this section, we will obtain results for $\overrightarrow{\mathbf{u}}_\varepsilon$ as $\varepsilon \to 0^+$. First, we present an estimate for $\varepsilon^{-4}c_{\varepsilon,\Omega,T}$ as $\varepsilon \to 0^+$.

Lemma 4.1. *Let α_T, β_0 and β_1 be as specified by Lemmas 2.2–2.4, respectively. Then, the following hold:*

(i) $\varepsilon^{-4}c_{\varepsilon,\Omega,T} = B + o(1)$ *in the following two cases:*

(a) $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ *and* $\alpha_1, \alpha_2 > 0$*,*

(b) $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $\alpha_1, \alpha_2 > 0$ *with* $|\overrightarrow{\alpha}| < \alpha_T$*.*

(ii) $\varepsilon^{-4}c'_{\varepsilon,\Omega,T} = B' + o(1)$ *for* $\beta > \beta_1$ *and* $\alpha_1, \alpha_2 > 0$*.*

Proof. (1) Let $\{\overrightarrow{\mathbf{U}}_n\} \in \mathcal{N}_{\mathbb{R}^4,T}$ and $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{U}}_n) < B + \frac{1}{n}$, where $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{U}})$ is specified by (3.1). Since $\Omega_\varepsilon \to \mathbb{R}^4$ as $\varepsilon \to 0^+$, where $\Omega_\varepsilon = \{y \in \mathbb{R}^4 \mid \varepsilon y \in \Omega\}$, there exists $\{\overrightarrow{\mathbf{U}}_{n,\varepsilon}\} \subset \mathcal{H}_{\Omega_\varepsilon}$ such that $\overrightarrow{\mathbf{U}}_{n,\varepsilon} \to \overrightarrow{\mathbf{U}}_n$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. It follows from (2.2), the construction of χ_β that is specified by (2.1) and Propositions 3.1 and 7.2 that

$$\begin{aligned} \|U_1^{n,\varepsilon}\|^2_{1,\varepsilon,\Omega} - \alpha_1\|U_1^{n,\varepsilon}\|^p_{L^p(\Omega)} - \mu_1\|U_1^{n,\varepsilon}\|^4_{L^4(\Omega)} - \beta\|U_1^{n,\varepsilon}U_2^{n,\varepsilon}\|^2_{L^2(\Omega)} &= o(1), \\ \|U_2^{n,\varepsilon}\|^2_{1,\varepsilon,\Omega} - \alpha_2\|U_2^{n,\varepsilon}\|^p_{L^p(\Omega)} - \mu_2\|U_2^{n,\varepsilon}\|^4_{L^4(\Omega)} - \beta\|U_1^{n,\varepsilon}U_2^{n,\varepsilon}\|^2_{L^2(\Omega)} &= o(1). \end{aligned}$$

Since $\{\overrightarrow{\mathbf{U}}_n\}$ is bounded in $\mathcal{H}$, we can apply Miranda's theorem as for $\eta_i(t_1, t_2)$ in the proof of Lemma 2.4 to show that there exists $\overrightarrow{\mathbf{t}}_{n,\varepsilon}$ with $\overrightarrow{\mathbf{t}}_{n,\varepsilon} \to \overrightarrow{\mathbf{1}}$ as $\varepsilon \to 0^+$ such that $\overrightarrow{\mathbf{t}}_{n,\varepsilon} \circ \overrightarrow{\mathbf{U}}_{n,\varepsilon} \in \mathcal{N}_{\varepsilon,\Omega,T}$ for $\beta > 0$ sufficiently small. By taking β_0 sufficiently small if necessary, we obtain that $\overrightarrow{\mathbf{t}}_{n,\varepsilon} \circ \overrightarrow{\mathbf{U}}_{n,\varepsilon} \in \mathcal{N}_{\varepsilon,\Omega,T}$ for $0 < \beta < \beta_0$. Taking into account (2.2), we can use the implicit function theorem as for $\Gamma_i^n(\overrightarrow{\mathbf{t}}, \tau)$ in the proof of Lemma 2.3 to show that there exists $\overrightarrow{\mathbf{t}}_{n,\varepsilon}$ with $\overrightarrow{\mathbf{t}}_{n,\varepsilon} \to \overrightarrow{\mathbf{1}}$ as $\varepsilon \to 0^+$ such that $\overrightarrow{\mathbf{t}}_{n,\varepsilon} \circ \overrightarrow{\mathbf{U}}_{n,\varepsilon} \in \mathcal{N}_{\varepsilon,\Omega,T}$ for $\beta < 0$. Since $\overrightarrow{\mathbf{U}}_n \in \mathcal{N}_{\mathbb{R}^4,T}$ with $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{U}}_n) < B + \frac{1}{n}$, we can use similar arguments to those used for $\Phi_n(\overrightarrow{\mathbf{t}})$ in the proof of Lemma 2.3 to show that

$$\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{U}}_n) \geq \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{U}}_n) \quad \text{for all } \overrightarrow{\mathbf{t}} \in (\mathbb{R}_+)^2.$$

Thus, by a standard argument, $\lim_{\varepsilon\to 0^+} \varepsilon^{-4}c_{\varepsilon,\Omega,T} \leq B + \frac{1}{n}$ under condition $(a1)$ or $(a2)$. Let $n \to \infty$. We have $\lim_{\varepsilon\to 0^+} \varepsilon^{-4}c_{\varepsilon,\Omega,T} \leq B$. In contrast, by setting $u \equiv 0$ outside Ω_ε, we can regard $\overrightarrow{\mathbf{u}} \in \mathcal{H}_{\Omega_\varepsilon}$ as in $\mathcal{H}_{\mathbb{R}^4}$. It follows that $\mathcal{N}_{\Omega_\varepsilon,T} \subset \mathcal{N}_{\mathbb{R}^4,T}$, which, together with the standard scaling technique, implies $\lim_{\varepsilon\to 0^+} \varepsilon^{-4}c_{\varepsilon,\Omega,T} \geq B$.

(ii) The proof is similar to but simpler than that of (i). □

Let p_i^ε be the maximum point of $u_i^\varepsilon (i = 1, 2)$ and define

$$\Omega_{i,\varepsilon} = \{x \in \mathbb{R}^4 \mid \varepsilon x + p_i^\varepsilon \in \Omega\}.$$

Lemma 4.2. *Assume $\alpha_1, \alpha_2 > 0$. Let α_T, β_0 and β_1 be as specified by Lemmas 2.2-2.4, respectively. Then, we have $\Omega_{i,\varepsilon} \to \mathbb{R}^4$ as $\varepsilon \to 0^+ (i = 1, 2)$ under one of the following two cases:*

1. *either* $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ *or* $\beta > \beta_1$,
2. $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $|\overrightarrow{\alpha}| < \alpha_T$.

Proof. We only present the proof of $\Omega_{1,\varepsilon}$ since that of $\Omega_{2,\varepsilon}$ is similar. Suppose the contrary. Then, as in [26], we may assume $\Omega_{1,\varepsilon} \to \mathbb{R}^4_+$ as $\varepsilon \to 0^+$ without loss of generality, where $\mathbb{R}^4_+ = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 > 0\}$. However, since $\overrightarrow{\mathbf{u}}_\varepsilon$ is a solution of $(\mathcal{S}_\varepsilon)$, by a standard scaling technique, we obtain that $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies

$$\begin{cases} -\Delta v_1^\varepsilon + \lambda_1 v_1^\varepsilon = \mu_1 (v_1^\varepsilon)^3 + \alpha_1 (v_1^\varepsilon)^{p-1} + \beta (v_2^\varepsilon)^2 v_1^\varepsilon & \text{in } \Omega_{1,\varepsilon}, \\ -\Delta v_2^\varepsilon + \lambda_2 v_2^\varepsilon = \mu_2 (v_2^\varepsilon)^3 + \alpha_2 (v_2^\varepsilon)^{p-1} + \beta (v_1^\varepsilon)^2 v_2^\varepsilon & \text{in } \Omega_{1,\varepsilon}, \\ v_1^\varepsilon, v_2^\varepsilon > 0 \quad \text{in } \Omega_{1,\varepsilon}, \quad v_1^\varepsilon = v_2^\varepsilon = 0 \quad \text{on } \partial\Omega_{1,\varepsilon}, \end{cases} \qquad (\mathcal{S}_\varepsilon^*)$$

where

$$v_i^\varepsilon(x) = u_i^\varepsilon(p_1^\varepsilon + \varepsilon x), \ \ i = 1, 2.$$

Moreover, since $|\overrightarrow{\alpha}| < \alpha_T$ for $\beta < -\sqrt{\mu_1\mu_2}$, by Lemma 2.4, Proposition 2.1 and Proposition 7.2, it follows from a standard scaling technique and a similar argument to that used in the proof of Lemma 2.1 that $\{v_i^\varepsilon\}$ are bounded in $\mathcal{H}_{i,\mathbb{R}^4}$. Here, we regard v_i^ε as a function in $\mathcal{H}_{i,\mathbb{R}^4}$ by setting $v_i^\varepsilon \equiv 0$ outside $\Omega_{1,\varepsilon}$. Without loss of generality, we assume that $\overrightarrow{\mathbf{v}}_\varepsilon \rightharpoonup \overrightarrow{\mathbf{v}}_0$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. Since $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies $(\mathcal{S}_\varepsilon^*)$, $\overrightarrow{\mathbf{v}}_0$ is a solution of $(\mathcal{S}_{**})$ because $\Omega_{1,\varepsilon} \to \mathbb{R}^4_+$ as $\varepsilon \to 0^+$. By Proposition 3.2, $\overrightarrow{\mathbf{v}}_0 = \overrightarrow{\mathbf{0}}$. Thus, $\overrightarrow{\mathbf{v}}_\varepsilon \rightharpoonup \overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. If

$$\lim_{\varepsilon \to 0^+} \sup_{y \in \mathbb{R}^4} \int_{\mathbb{B}_\rho(y)} \sum_{i=1}^{2} (v_i^\varepsilon)^2 dx = 0$$

for all $\rho > 0$, then by Lions' lemma, we have $\overrightarrow{\mathbf{v}}_\varepsilon \to \overrightarrow{\mathbf{0}}$ strongly in $\mathcal{L}^r(\mathbb{R}^4)$ for all $2 < r < 4$ as $\varepsilon \to 0^+$, where $\mathcal{L}^r(\mathbb{R}^4) = (L^r(\mathbb{R}^4))^2$. It follows from $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies $(\mathcal{S}_\varepsilon^*)$ that

$$\|\nabla v_1^\varepsilon\|^2_{L^2(\mathbb{R}^4)} + \lambda_1 \|v_1^\varepsilon\|^2_{L^2(\mathbb{R}^4)} - \mu_1 \|v_1^\varepsilon\|^4_{L^4(\mathbb{R}^4)} - \beta \|v_1^\varepsilon v_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} = o(1), \quad (4.1)$$

$$\|\nabla v_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} + \lambda_2 \|v_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} - \mu_2 \|v_2^\varepsilon\|^4_{L^4(\mathbb{R}^4)} - \beta \|v_1^\varepsilon v_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} = o(1). \quad (4.2)$$

Since $\lambda_1, \lambda_2 > 0$, by similar arguments to those used in the proofs of [46, Lemma 3.1] and [22, Lemma 5.1], it follows from (4.1) and (4.2) that $\lim_{\varepsilon \to 0^+} (c_{\varepsilon,\Omega,T} - \varepsilon^{-4} A_\varepsilon) \geq 0$ in the following two cases:

1. $-\sqrt{\mu_1\mu_2} < \beta < \beta_0$ and $\alpha_1, \alpha_2 > 0$,

2. $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$.

In the case $\beta > \beta_1$, since $p > 2$, it also follows from a standard argument that $\lim_{\varepsilon\to 0^+}(c'_{\varepsilon,\Omega,T} - \varepsilon^{-4}A_\varepsilon) \geq 0$. However, this is impossible since Lemma 2.4 holds. Therefore, there exist $\rho > 0$ and $y^\varepsilon \in \mathbb{R}^4$ such that

$$\int_{\mathbb{B}_\rho(y^\varepsilon)} \sum_{i=1}^{2} (v_i^\varepsilon)^2 dx \geq C > 0. \tag{4.3}$$

Without loss of generality, we assume that $\int_{\mathbb{B}_\rho(y^\varepsilon)} (v_1^\varepsilon)^2 dx \geq C > 0$. Since $\overrightarrow{\mathbf{v}}_\varepsilon \rightharpoonup \overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$, by the Sobolev embedding theorem, $|y^\varepsilon| \to +\infty$. Let $v_{i,1}^\varepsilon(x) = v_i^\varepsilon(x + y^\varepsilon)$. Then $\overrightarrow{\overrightarrow{\mathbf{v}}}_\varepsilon = (v_{1,1}^\varepsilon, v_{2,1}^\varepsilon) \rightharpoonup \overrightarrow{\overrightarrow{\mathbf{v}}}_0 = (v_{1,1}^0, v_{2,1}^0)$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$ with $v_{i,1}^0 \geq 0$ and $v_{1,1}^0 \not\equiv 0$. Let $\Omega_{1,\varepsilon}^* = \Omega_{1,\varepsilon} - y^\varepsilon$ and assume $\Omega_{1,\varepsilon}^* \to \Omega^*$ as $\varepsilon \to 0^+$, where Ω^* is either the whole space $\mathbb{R}^4$ or the half-space $\mathbb{R}_+^4$ under rotations and translations. Since $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies $(\mathcal{S}_\varepsilon^*)$, $\overrightarrow{\overrightarrow{\mathbf{v}}}_0$ is a solution of the following system

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \Omega^*, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \Omega^*, \\ u_1, u_2 \geq 0 \text{ in } \Omega^*, \quad u_1, u_2 \to 0 \text{ as } |x| \to +\infty, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega^*, \end{cases} \tag{$\overline{\mathcal{S}}_{**}$}$$

Since $v_{1,1}^0 \not\equiv 0$, by Proposition 3.2, $\Omega^* = \mathbb{R}^4$.

Case. 1 $\beta < 0$

If $v_{2,1}^0 \not\equiv 0$, then by Proposition 3.1,

$$\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\overrightarrow{\mathbf{v}}}_0) > B = \sum_{i=1}^{2} d_{i,\mathbb{R}^4}.$$

Let $\overrightarrow{\mathbf{w}}_\varepsilon = \overrightarrow{\overrightarrow{\mathbf{v}}}_\varepsilon - \overrightarrow{\overrightarrow{\mathbf{v}}}_0$. Then, $\overrightarrow{\mathbf{w}}_\varepsilon \rightharpoonup \overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. By applying the Brezís-Lieb lemma and [13, Lemma 2.3], we also obtain that

$$\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) = \mathcal{J}_{\Omega_{1,\varepsilon}^*,T}(\overrightarrow{\overrightarrow{\mathbf{v}}}_\varepsilon) = \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\overrightarrow{\mathbf{v}}}_0) + \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon) + o(1) \tag{4.4}$$

and

$$\mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon)\overrightarrow{\mathbf{w}}_\varepsilon^1 = \mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon)\overrightarrow{\mathbf{w}}_\varepsilon^1 = o(1), \tag{4.5}$$

where $\overrightarrow{\mathbf{w}}_\varepsilon^1 = (w_1^\varepsilon, 0)$ and $\overrightarrow{\mathbf{w}}_\varepsilon^2 = (0, w_2^\varepsilon)$. Since $|\overrightarrow{\alpha}| < \alpha_T$ for $\beta < -\sqrt{\mu_1\mu_2}$, by a similar argument to that used for (2.3) and (2.4) in the proof of Lemma 2.2 for $-\sqrt{\mu_1\mu_2} < \beta < 0$ and $\beta \leq -\sqrt{\mu_1\mu_2}$, respectively, $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon) \geq 0$, which contradicts Lemma 4.1. Thus, $v_{2,1}^0 = 0$. Then, similar to (4.4) and (4.5),

$$\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) = \mathcal{J}_{\Omega_{1,\varepsilon}^*,T}(\overrightarrow{\overrightarrow{\mathbf{v}}}_\varepsilon) = \mathcal{E}_{1,\mathbb{R}^4}(v_{1,1}^0) + \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon) + o(1) \tag{4.6}$$

and

$$\mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon)\overrightarrow{\mathbf{w}}^1_\varepsilon = \mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon)\overrightarrow{\mathbf{w}}^1_\varepsilon = o(1), \tag{4.7}$$

where $\mathcal{E}_{1,\mathbb{R}^4}(u)$ is specified by (7.1). It follows that $\mathcal{E}'_{1,\mathbb{R}^4}(v^0_{1,1}) = 0$. Moreover, in this scenario, $w^\varepsilon_2 = v^\varepsilon_{2,1}$. Thus, by a similar argument to that use for (2.15),

$$\|\nabla w^\varepsilon_2\|^2_{L^2(\mathbb{R}^4)} + \lambda_2\|w^\varepsilon_2\|^2_{L^2(\mathbb{R}^4)} \geq C + o(1).$$

It follows from $\beta < 0$ and (4.7) that there exists $0 < t_\varepsilon \leq 1 + o(1)$ such that $t_\varepsilon w^\varepsilon_2 \in \mathcal{M}_{2,\mathbb{R}^4}$, which is specified by (7.2). Recall that $|\overrightarrow{\alpha}| < \alpha_T$ for $\beta < -\sqrt{\mu_1\mu_2}$. Thus, by a similar argument to that used for (2.3) and (2.4) in the proof of Lemma 2.2 for $-\sqrt{\mu_1\mu_2} < \beta < 0$ and $\beta \leq -\sqrt{\mu_1\mu_2}$, respectively, it follows from (2.2), (4.6), Lemma 4.1 and Proposition 3.1 that $w^\varepsilon_1 \to 0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$ as $\varepsilon \to 0^+$, which implies $v^\varepsilon_{1,1} \to v^0_{1,1}$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$ as $\varepsilon \to 0^+$. Since $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies the system $(\mathcal{S}_*)$, for $\beta < 0$, we can apply Moser's iteration as in [8] to show that $v^\varepsilon_{1,1}$ is uniformly bounded in $L^q(\mathbb{R}^4)$ for all $q \geq 2$. Since $\overrightarrow{\mathbf{v}}_0$ is a nontrivial solution of $(\mathcal{S}_*)$, by the classical elliptic regularity, $v^0_{1,1} \in L^\infty(\mathbb{R}^4)$. It follows from the Taylor expansion that $w^\varepsilon_1 = v^\varepsilon_{1,1} - v^0_{1,1}$ satisfies the following equation

$$-\Delta w^\varepsilon_1 + \lambda_1 w^\varepsilon_1 \leq \alpha_1(p-1)(v^\varepsilon_{1,1} + v^0_{1,1})^{p-2}w^\varepsilon_1 + 3\mu_1(v^\varepsilon_{1,1} + v^0_{1,1})^2 w^\varepsilon_1$$

in $\mathbb{R}^4$. Since $w^\varepsilon_1 \to 0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$, by applying Moser's iteration as in [8], we obtain that $w^\varepsilon_1 \to 0$ strongly in $L^q(\mathbb{R}^4)$ for all $q \geq 2$. It follows from [19, Theorem 8.17] (see also [8, Lemma 4.3]) that $v^\varepsilon_{1,1} \to 0$ as $|x| \to +\infty$ uniformly for ε, which contradicts $v^\varepsilon_{1,1}(-y_\varepsilon) = v^\varepsilon_1(0)$ being the maximum value and $|y^\varepsilon| \to +\infty$. Thus, we must have that $\Omega_{1,\varepsilon} \to \mathbb{R}^4$ as $\varepsilon \to 0^+$ for $\beta < 0$.

Case. 2 $\beta > 0$

If $v^0_{2,1} = 0$, then similar to the case $\beta < 0$, we have that (4.6)–(4.7) hold and

$$\|\nabla w^\varepsilon_2\|^2_{L^2(\mathbb{R}^4)} + \lambda_2\|w^\varepsilon_2\|^2_{L^2(\mathbb{R}^4)} \geq C + o(1).$$

If we also have

$$\|\nabla w^\varepsilon_1\|^2_{L^2(\mathbb{R}^4)} + \lambda_1\|w^\varepsilon_1\|^2_{L^2(\mathbb{R}^4)} \geq C + o(1)$$

in this scenario, then by (4.7), we can apply Miranda's theorem as for $\eta_i(t_1, t_2)$ in the proof of Lemma 2.4 to show that there exists $\overrightarrow{\mathbf{t}}_\varepsilon \in (\mathbb{R}^+)^2$ with $\overrightarrow{\mathbf{t}}_\varepsilon \to \overrightarrow{\mathbf{1}}$ as $\varepsilon \to 0$ such that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{w}}_\varepsilon \in \mathcal{N}_{\mathbb{R}^4,T}$ for $\beta > 0$ sufficiently small. Here, $\mathcal{N}_{\mathbb{R}^4,T}$ is specified by (3.2). By taking β_0 sufficiently small if necessary, we obtain that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{w}}_\varepsilon \in \mathcal{N}_{\mathbb{R}^4,T}$ for $0 < \beta < \beta_0$. It follows that (4.6) contradicts Lemma 4.1 since $\{\overrightarrow{\mathbf{w}}_\varepsilon\}$ is bounded in $\mathcal{H}_{\mathbb{R}^4}$ and $v^0_{1,1} \neq 0$ satisfies $\mathcal{E}'_{1,\mathbb{R}^4}(v^0_{1,1}) = 0$. Thus, $w^\varepsilon_1 \to 0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$ as $\varepsilon \to 0^+$. It follows from (4.7), the Hölder inequality and $\|\nabla w^\varepsilon_2\|^2_{L^2(\mathbb{R}^4)} + \lambda_2\|w^\varepsilon_2\|^2_{L^2(\mathbb{R}^4)} \geq C + o(1)$ that there exists $t_\varepsilon \to 1$ as $\varepsilon \to 0$ such that $t_\varepsilon w^\varepsilon_2 \in \mathcal{M}_{2,\mathbb{R}^4}$. Thus, by (4.6),

$$\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) \geq \sum_{i=1}^{2} d_{i,\mathbb{R}^4} + o(1),$$

which contradicts Proposition 3.1 and Lemma 4.1 for $0 < \beta < \beta_0$. For $\beta > \beta_1$, it follows from

$$\|\nabla w_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} + \lambda_2 \|w_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} \geq C + o(1)$$

that there exists $s_\varepsilon > 0$ such that $s_\varepsilon \overrightarrow{\mathbf{w}}_\varepsilon \in \mathcal{N}'_{\mathbb{R}^4,T}$, which is specified by (3.3). It follows that (4.6) contradicts Lemma 4.1 since $v^0_{1,1} \neq 0$ satisfies $\mathcal{E}'_{1,\mathbb{R}^4}(v^0_{1,1}) = 0$. Thus, we must have that $v^0_{2,1} \neq 0$. It follows from Proposition 3.1 and Lemma 4.1 that $\overrightarrow{\mathbf{v}}_0$ is a least-energy solution of $(\mathcal{S}_*)$ and $\overrightarrow{\mathbf{v}}_\varepsilon \to \overrightarrow{\mathbf{v}}_0$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. For $\beta > 0$, $w_1^\varepsilon = v^\varepsilon_{1,1} - v^0_{1,1}$ satisfies the following equation by the Taylor expansion

$$-\Delta w_1^\varepsilon + \lambda_1 w_1^\varepsilon \leq \alpha_1 (v^\varepsilon_{1,1} + v^0_{1,1})^{p-2} w_1^\varepsilon + \mu_1 (v^\varepsilon_{1,1} + v^0_{1,1})^2 w_1^\varepsilon + \beta (v^\varepsilon_{1,2}) w_1^\varepsilon + o(1)$$

in $\mathbb{R}^4$. Thus, we obtain a contradiction similar to the case $\beta < 0$. Hence, $\Omega_{1,\varepsilon} \to \mathbb{R}^4$ as $\varepsilon \to 0^+$ for $0 < \beta < \beta_0$ or $\beta > \beta_1$. □

Let $v_i^\varepsilon(x) = u_i^\varepsilon(p_1^\varepsilon + \varepsilon x)$ and $\widetilde{v}_i^\varepsilon(x) = u_i^\varepsilon(p_2^\varepsilon + \varepsilon x)$ for $i = 1, 2$, respectively. Then, the following proposition holds:

Proposition 4.1. *Let α_T, β_0 and β_1 be as specified by Lemmas 2.2–2.4, respectively.*

- (i) *$\overrightarrow{\mathbf{V}}^*_\varepsilon = (v_1^\varepsilon, \widetilde{v}_2^\varepsilon) \to \overrightarrow{\mathbf{v}}_0$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$ in the following two cases:*
  - *(a) $-\sqrt{\mu_1\mu_2} < \beta < 0$ and $\alpha_1, \alpha_2 > 0$,*
  - *(b) $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ with $|\overrightarrow{\alpha}| < \alpha_T$,*

  where v_i^0 is a least-energy solution of $(\mathcal{P}_i)$. Moreover, $\frac{|p_1^\varepsilon - p_2^\varepsilon|}{\varepsilon} \to +\infty$ as $\varepsilon \to 0^+$.
- (ii) *For $\beta \in (0, \beta_0) \cup (\beta_1, +\infty)$, $\overrightarrow{\mathbf{V}}^*_\varepsilon \to \overrightarrow{\mathbf{v}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$, where $\overrightarrow{\mathbf{v}}_*$ is the least-energy solution of $(\mathcal{S}_*)$. Moreover, $\frac{|p_1^\varepsilon - p_2^\varepsilon|}{\varepsilon} \to 0$ as $\varepsilon \to 0^+$.*

Proof. (i) By a similar argument to that used in the proof of Lemma 4.2 for $\beta < 0$, we can show that either $v_1^0 \equiv 0$ or $v_2^0 \equiv 0$. By the scaling technique, it follows from Lemma 2.4 and a similar argument to that used for (2.16) that $\|v_i^\varepsilon\|^4_{L^4(\mathbb{R}^4)} \geq C$ both for $i = 1, 2$. If $\|v_i^\varepsilon\|^p_{L^p(\mathbb{R}^4)} = o(1)$, then by the Sobolev inequality, $\beta < 0$ and $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfying the system $(\mathcal{S}_*)$, we obtain that

$$\mathcal{J}_{\Omega_{1,\varepsilon},T}(\overrightarrow{\mathbf{v}}_\varepsilon) \geq \sum_{i=1}^{2} \frac{1}{4\mu_i}\mathcal{S}^2 + o(1),$$

which contradicts Lemma 2.4 and Proposition 7.2. Thus, we must have

$$\sum_{i=1}^{2} \|v_i^\varepsilon\|^p_{L^p(\mathbb{R}^4)} \geq C.$$

If $\|v_1^\varepsilon\|^p_{L^p(\mathbb{R}^4)} = o(1)$, then $\|v_2^\varepsilon\|^p_{L^p(\mathbb{R}^4)} \geq C$. By the Hölder and Sobolev inequalities, $\|v_2^\varepsilon\|^2_{L^2(\mathbb{R}^4)} \geq C$. Since $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies the system $(\mathcal{S}_*)$ and $\beta < 0$,

$$\|v_1^\varepsilon\|^2_{1,\mathbb{R}^4} \leq \mu_1 \|v_1^\varepsilon\|^4_{L^4(\mathbb{R}^4)} + o(1)$$

and

$$\|v_2^\varepsilon\|_{2,\mathbb{R}^4}^2 \leq \alpha_2\|v_2^\varepsilon\|_{L^p(\mathbb{R}^4)}^p + \mu_2\|v_2^\varepsilon\|_{L^4(\mathbb{R}^4)}^4 + o(1).$$

By a standard argument, we deduce that either $v_1^\varepsilon \to 0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$ as $\varepsilon \to 0^+$ or there exists $0 < s_i^\varepsilon \leq 1 + o(1)$ such that

$$\frac{1}{2}\|s_1^\varepsilon v_1^\varepsilon\|_{1,\mathbb{R}^4}^2 - \frac{\mu_1}{4}\|s_1^\varepsilon v_1^\varepsilon\|_{L^4(\mathbb{R}^4)}^4 \geq \frac{1}{4\mu_1}\mathcal{S}^2; \quad s_2^\varepsilon v_2^\varepsilon \in \mathcal{M}_{2,\mathbb{R}^4}.$$

If $0 < s_i^\varepsilon \leq 1 + o(1)$ such that $\frac{1}{2}\|s_1^\varepsilon v_1^\varepsilon\|_{1,\mathbb{R}^4}^2 - \frac{\mu_1}{4}\|s_1^\varepsilon v_1^\varepsilon\|_{L^4(\mathbb{R}^4)}^4 \geq \frac{1}{4\mu_1}\mathcal{S}^2$ and $s_2^\varepsilon v_2^\varepsilon \in \mathcal{M}_{2,\mathbb{R}^4}$, then by a similar argument to that used in Lemma 2.3 and $\beta < 0$,

$$\begin{aligned}
\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) &\geq \varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{s}}_\varepsilon \circ \overrightarrow{\mathbf{u}}_\varepsilon) \\
&= \mathcal{J}_{\Omega_{1,\varepsilon},T}(\overrightarrow{\mathbf{s}}_\varepsilon \circ \overrightarrow{\mathbf{v}}_\varepsilon) \\
&\geq \frac{1}{2}\|s_1^\varepsilon v_1^\varepsilon\|_{1,\mathbb{R}^4}^2 - \frac{\mu_1}{4}\|s_1^\varepsilon v_1^\varepsilon\|_{L^4(\mathbb{R}^4)}^4 + \mathcal{E}_{2,\mathbb{R}^4}(s_2^\varepsilon v_2^\varepsilon) \\
&\geq \frac{1}{4\mu_1}\mathcal{S}^2 + d_{2,\mathbb{R}^4},
\end{aligned}$$

which also contradicts Lemma 2.4. Thus, $v_1^\varepsilon \to 0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$ as $\varepsilon \to 0^+$ in this case. Since $\beta < 0$ and $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies the system $(\mathcal{S}_*)$, by applying Moser's iteration as in [8], we can show that $v_1^\varepsilon \to 0$ strongly in $L^q(\mathbb{R}^4)$ as $\varepsilon \to 0^+$ for all $q \geq 2$. By the classical L^p estimate for elliptic equations and the Sobolev embedding theorem, we also have that $v_1^\varepsilon \to 0$ strongly in $C^1(\mathbb{R}^4)$. Therefore, $v_1^\varepsilon(0) \to 0$ as $\varepsilon \to 0^+$. However, since p_1^ε is the maximum point of u_1^ε, 0 is the maximum point of v_1^0. By $p > 2$, it follows from $\beta < 0$ that

$$\lambda_1 v_1^\varepsilon(0) \leq \alpha_1(v_1^\varepsilon(0))^{p-1} + \mu_1(v_1^\varepsilon(0))^2, \tag{4.8}$$

which implies $v_1^\varepsilon(0) \geq C$. This is impossible. Thus, we must have $\|v_1^\varepsilon\|_{L^p(\mathbb{R}^4)}^p \geq C$. If we also have $\|v_2^\varepsilon\|_{L^p(\mathbb{R}^4)}^p \geq C$, then by the Hölder and Sobolev inequalities, $\|v_i^\varepsilon\|_{L^2(\mathbb{R}^4)}^2 \geq C$ for both $i = 1, 2$. Thus, since $\beta < 0$, there exists $0 < s_i^\varepsilon \leq 1$ such that $s_i^\varepsilon v_i^\varepsilon \in \mathcal{M}_{i,\mathbb{R}^4}$ both for $i = 1, 2$. By a similar argument to that used in Lemma 2.3 and since $\beta < 0$, we obtain

$$\begin{aligned}
\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) &\geq \varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{s}}_\varepsilon \circ \overrightarrow{\mathbf{u}}_\varepsilon) \\
&= \mathcal{J}_{\Omega_{1,\varepsilon},T}(\overrightarrow{\mathbf{s}}_\varepsilon \circ \overrightarrow{\mathbf{v}}_\varepsilon) \\
&\geq \sum_{i=1}^{2}\mathcal{E}_{i,\mathbb{R}^4}(s_i^\varepsilon v_i^\varepsilon) \\
&\geq \sum_{i=1}^{2} d_{i,\mathbb{R}^4}.
\end{aligned} \tag{4.9}$$

Combined with Lemma 4.1, this implies that $\overrightarrow{\mathbf{s}}_\varepsilon = \overrightarrow{\mathbf{1}} + o(1)$ and $\|v_1^\varepsilon v_2^\varepsilon\|_{L^2(\mathbb{R}^4)}^2 = o(1)$. However, since $\|v_i^\varepsilon\|_{L^p(\mathbb{R}^4)}^p \geq C$ in this scenario, by Lions' Lemma, there exist

$\{y_j^\varepsilon\} \subset \mathbb{R}^4$ such that $\overrightarrow{\mathbf{v}}_\varepsilon = (v_{1,j}^\varepsilon, v_{2,j}^\varepsilon) \rightharpoonup \overrightarrow{\mathbf{v}}_0 = (v_{1,j}^0, v_{2,j}^0)$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$, where $v_{i,j}^\varepsilon(x) = v_i(x + y_j^\varepsilon)$ for $i, j = 1, 2$. Moreover, $v_{i,i}^0 \not\equiv 0$ for $i = 1, 2$. Let $\overrightarrow{\mathbf{w}}_\varepsilon = \overrightarrow{\mathbf{v}}_\varepsilon - \overrightarrow{\mathbf{v}}_0$. Then, $\overrightarrow{\mathbf{w}}_\varepsilon \rightharpoonup \overrightarrow{\mathbf{0}}$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. Since $v_{1,1}^0 \neq 0$, by Proposition 3.2, $\overrightarrow{\mathbf{v}}_0$ is a solution of $(\mathcal{S}_*)$. Thus, by the Brezís-Lieb lemma and [13, Lemma 2.3],

$$\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) = \mathcal{J}_{\Omega_{1,\varepsilon},T}(\overrightarrow{\mathbf{v}}_\varepsilon) = \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{v}}_0) + \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon) + o(1) \tag{4.10}$$

and

$$\mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon)\overrightarrow{\mathbf{w}}_\varepsilon^1 = \mathcal{J}'_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon)\overrightarrow{\mathbf{w}}_\varepsilon^2 = o(1),$$

where $\overrightarrow{\mathbf{w}}_\varepsilon^1 = (w_1^\varepsilon, 0)$ and $\overrightarrow{\mathbf{w}}_\varepsilon^2 = (0, w_2^\varepsilon)$. As above, we also have $\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{w}}_\varepsilon) \geq 0$. Hence, by Proposition 3.1 and Lemma 4.2, it follows from $\beta < 0$ and Lemma 4.1 that $v_{2,1}^0 \equiv 0$. Hence, $v_{1,1}^0$ is a solution of $(\mathcal{P}_1)$. Since $v_{1,1}^0 \geq 0$ and $v_{1,1}^0 \not\equiv 0$, by the maximum principle and a standard argument, we must have $\mathcal{E}_{1,\mathbb{R}^4}(v_{1,1}^0) \geq d_{1,\mathbb{R}^4}$. Similarly, $v_{1,2}^0 \equiv 0$ and $v_{2,2}^0 \not\equiv 0$ with $\mathcal{E}_{2,\mathbb{R}^4}(v_{2,2}^0) \geq d_{2,\mathbb{R}^4}$. Since $\|v_1^\varepsilon v_2^\varepsilon\|_{L^2(\mathbb{R}^4)}^2 = o(1)$, by translation and a similar calculation in (4.9), we obtain that $v_{i,i}^\varepsilon \to v_{i,i}^0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$ as $\varepsilon \to 0^+$ for $i = 1, 2$. Since $\beta < 0$ and $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies the system $(\mathcal{S}_*)$, by applying a modified Moser's iteration (cf. [8]), we can show that $v_{1,1}^\varepsilon$ is uniformly bounded in $L^q(\mathbb{R}^4)$ for all $q \geq 2$. Since $v_{2,1}^0 \equiv 0$, $v_{1,1}^0$ is also a nontrivial solution of $(\mathcal{P}_1)$. By the classical elliptic regularity, $v_{1,1}^0 \in L^\infty(\mathbb{R}^4)$. By Taylor expansion and since $\beta < 0$, we obtain that $w_1^\varepsilon = v_{1,1}^\varepsilon - v_{1,1}^0$ satisfies the following equation:

$$-\Delta w_1^\varepsilon + \lambda_1 w_1^\varepsilon \leq \alpha_1(p-1)(v_{1,1}^\varepsilon + v_{1,1}^0)^{p-2} w_1^\varepsilon + 3\mu_1(v_{1,1}^\varepsilon + v_{1,1}^0)^2 w_1^\varepsilon + o(1)$$

in $\mathbb{R}^4$. Since $w_1^\varepsilon \to 0$ strongly in $\mathcal{H}_{1,\mathbb{R}^4}$, by applying the Moser's iteration as in [8], we obtain that $w_1^\varepsilon \to 0$ strongly in $L^q(\mathbb{R}^4)$ for all $q \geq 2$. It follows from [19, Theorem 8.17] (see also [8, Lemma 4.3]) that $v_{1,1}^\varepsilon \to 0$ as $|x| \to +\infty$ uniformly for ε. Since $v_{1,1}^\varepsilon(-y_\varepsilon) = v_1^\varepsilon(0)$, we obtain a contradiction. Thus, we must have that $\|v_2^\varepsilon\|_{L^p(\mathbb{R}^4)}^p = o(1)$, which implies $v_1^0 \not\equiv 0$ and $v_2^0 \equiv 0$. Similarly, we can also show that $\widetilde{v}_1^0 \equiv 0$ and $\widetilde{v}_2^0 \not\equiv 0$, where $\overrightarrow{\mathbf{v}}_\varepsilon \rightharpoonup \overrightarrow{\mathbf{v}}_0$ weakly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$. Now, by Proposition 3.1 and Lemma 4.1, we can easy to show that $\overrightarrow{\mathbf{V}}_\varepsilon^* = (v_1^\varepsilon, \widetilde{v}_2^\varepsilon) \to \overrightarrow{\mathbf{v}}_0$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$, where v_i^0 is a least-energy solution of $(\mathcal{P}_i)$, $i = 1, 2$. However, since $\beta < 0$ and $\overrightarrow{\mathbf{v}}_\varepsilon$ satisfies the system $(\mathcal{S}_*)$, by applying Moser's iteration as in [8], we can show that $\widetilde{v}_2^\varepsilon \to 0$ strongly in $L^q(\mathbb{R}^4)$ as $\varepsilon \to 0$ for all $q \geq 2$. By the classical elliptic regularity, $\widetilde{v}_2^\varepsilon \to 0$ strongly in $C^1(\mathbb{R}^4)$. It follows from $\widetilde{v}_2^\varepsilon(x) = v_2^\varepsilon(x + \frac{p_\varepsilon^2 - p_1^\varepsilon}{\varepsilon})$ that $\frac{|p_1^\varepsilon - p_2^\varepsilon|}{\varepsilon} \to +\infty$ since $v_2^\varepsilon \rightharpoonup 0$ in $\mathcal{H}_{2,\mathbb{R}^4}$ as $\varepsilon \to 0^+$.

(ii) The proof is similar to that of (i); hence, we only identify the differences. Suppose that $\|v_1^\varepsilon v_2^\varepsilon\|_{L^2(\mathbb{R}^4)}^2 = o(1)$. Then, by a similar argument to that used in (i) for (4.9), it follows from Lemma 4.1 that $B \geq \sum_{i=1}^2 d_{i,\mathbb{R}^4}$, which contradicts Proposition 3.1. Thus, we must have $\|v_1^\varepsilon v_2^\varepsilon\|_{L^2(\mathbb{R}^4)}^2 \geq C$ for $\beta \in (0, \beta_0) \cup (\beta_1, +\infty)$. By applying Lion's lemma as for (4.3), we can show that there exists $\{y^\varepsilon\} \subset \mathbb{R}^4$ such that

$v_{i,1}^{\varepsilon}(x) = v_i^{\varepsilon}(x + y^{\varepsilon}) \rightharpoonup v_{i,1}^0$ weakly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$ and $v_{1,1}^0 \not\equiv 0$. Now, by a similar argument to that used in the proof of Lemma 4.2 for $\beta > 0$, we obtain that $v_{2,1}^0 \neq 0$ and $v_{i,1}^{\varepsilon} \to v_{i,1}^0$ strongly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$. By applying a similar regularity argument to that used in (i), we obtain that $v_{i,1}^{\varepsilon} \to v_{i,1}^0$ strongly $\mathcal{L}^{\infty}(\mathbb{R}^4) \cap \mathcal{C}^1(\mathbb{R}^4)$ as $\varepsilon \to 0^+$, where $\mathcal{L}^{\infty}(\mathbb{R}^4) = L^{\infty}(\mathbb{R}^4) \times L^{\infty}(\mathbb{R}^4)$ and $\mathcal{C}^1(\mathbb{R}^4) = (C^1(\mathbb{R}^4))^2$. Since 0 is the maximum point of v_1^{ε}, we must have that $v_1^0 \not\equiv 0$, which implies $|y^{\varepsilon}| \leq C$ and $v_2^0 \not\equiv 0$. By a similar argument to that used for (4.10), $v_i^{\varepsilon} \to v_i^0$ strongly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$. Also, by applying a similar regularity argument to that used in (i), we obtain that $v_i^{\varepsilon} \to v_i^0$ strongly in $\mathcal{L}^{\infty}(\mathbb{R}^4) \cap \mathcal{C}^1(\mathbb{R}^4)$ as $\varepsilon \to 0^+$. Similarly, we also obtain that $\overrightarrow{\widetilde{\mathbf{v}}}_{\varepsilon} \to \overrightarrow{\widetilde{\mathbf{v}}}_0$ strongly in $\mathcal{L}^{\infty}(\mathbb{R}^4) \cap \mathcal{C}^1(\mathbb{R}^4)$ as $\varepsilon \to 0^+$, where both $\overrightarrow{\mathbf{v}}_0$ and $\overrightarrow{\widetilde{\mathbf{v}}}_0$ are the least-energy solution of $(\mathcal{S}_*)$. By the results in [4], it follows from $\beta > 0$ that $\overrightarrow{\mathbf{v}}_0$ and $\overrightarrow{\widetilde{\mathbf{v}}}_0$ are both radially symmetric and monotonic. Recall that p_i^{ε} is the maximum point of u_i^{ε},$i = 1, 2$, Thus, since $\overrightarrow{\mathbf{v}}_{\varepsilon} \to \overrightarrow{\mathbf{v}}_0$ and $\overrightarrow{\widetilde{\mathbf{v}}}_{\varepsilon} \to \overrightarrow{\widetilde{\mathbf{v}}}_0$ strongly in $\mathcal{C}^1(\mathbb{R}^4)$ as $\varepsilon \to 0^+$, we must have that 0 is the maximum point of v_i^0 and $\widetilde{v}_i^0$ for all $i = 1, 2$. Suppose that $\frac{|p_1^{\varepsilon} - p_2^{\varepsilon}|}{\varepsilon} \to +\infty$. Then, since $\widetilde{v}_2^{\varepsilon}(x) = v_2^{\varepsilon}(x + \frac{p_1^{\varepsilon} - p_2^{\varepsilon}}{\varepsilon})$, it follows from $\overrightarrow{\mathbf{v}}_{\varepsilon} \to \overrightarrow{\mathbf{v}}_0$ and $\overrightarrow{\widetilde{\mathbf{v}}}_{\varepsilon} \to \overrightarrow{\widetilde{\mathbf{v}}}_0$ strongly in $\mathcal{L}^{\infty}(\mathbb{R}^4)$ as $\varepsilon \to 0^+$ that $v_2^0(0) = 0$, which is impossible since $v_2^0 > 0$ in $\mathbb{R}^4$ by the maximum principle. Thus, we must have $|\frac{p_1^{\varepsilon} - p_2^{\varepsilon}}{\varepsilon}| \leq C$. Without loss of generality, we assume that $\frac{|p_1^{\varepsilon} - p_2^{\varepsilon}|}{\varepsilon} \to p_0$ as $\varepsilon \to 0^+$. We claim that $p_0 = 0$. Indeed, since $\widetilde{v}_2^{\varepsilon}(x) = v_2^{\varepsilon}(x + \frac{p_1^{\varepsilon} - p_2^{\varepsilon}}{\varepsilon})$, $v_2^0(x + p_0) = \widetilde{v}_2^0(x)$ for all $x \in \mathbb{R}^4$; in particular, $v_2^0(p_0) = \widetilde{v}_2^0(0)$ and $v_2^0(0) = \widetilde{v}_2^0(-p_0)$. Since 0 is the maximum point of v_2^0 and $\widetilde{v}_2^0$, we must have $v_2^0(0) = \widetilde{v}_2^0(0)$. It follows that $v_2^0(p_0) = v_2^0(0)$ and $\widetilde{v}_2^0(-p_0) = \widetilde{v}_2^0(0)$. Since $\widetilde{v}_2^0$ is radially symmetric, we also have that $\widetilde{v}_2^0(p_0) = \widetilde{v}_2^0(0)$. Now, by iteration, we obtain that $v_2^0(0) = v_2^0(kp_0)$ for all $k \in \mathbb{N}$. This is impossible if $p_0 \neq 0$ since $v_2^0(kp_0) \to 0$ as $k \to \infty$ in this scenario. Now, set $\overrightarrow{\mathbf{v}}_* = (v_1^0, \widetilde{v}_2^0)$. Since $\frac{|p_1^{\varepsilon} - p_2^{\varepsilon}|}{\varepsilon} \to 0$ as $\varepsilon \to 0^+$, $\overrightarrow{\mathbf{v}}_{\varepsilon} \to \overrightarrow{\mathbf{v}}_0$ and $\overrightarrow{\widetilde{\mathbf{v}}}_{\varepsilon} \to \overrightarrow{\widetilde{\mathbf{v}}}_0$ strongly in $\mathcal{H}_{\mathbb{R}^4} \cap \mathcal{C}^1(\mathbb{R}^4)$ as $\varepsilon \to 0^+$, we have that $\overrightarrow{\mathbf{V}}_{\varepsilon}^* \to \overrightarrow{\mathbf{v}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4} \cap \mathcal{C}^1(\mathbb{R}^4)$ as $\varepsilon \to 0^+$. Moreover, by $\frac{|p_1^{\varepsilon} - p_2^{\varepsilon}|}{\varepsilon} \to 0$ and Proposition 3.1, $\overrightarrow{\mathbf{v}}_*$ is the least-energy solution of $(\mathcal{S}_*)$. □

Proof of Theorem 1.2: It follows immediately from Proposition 4.1. □

5 Locations of the spikes as $\varepsilon \to 0^+$

First, we study locations of the spikes for $\beta > 0$. Without loss of generality, we assume that

$$\text{dist}(0, \partial\Omega) = \mathcal{D} = \max_{p \in \Omega} \text{dist}(p, \partial\Omega).$$

Then, $\mathcal{D} > 0$ and $\mathbb{B}_{\mathcal{D}} \subset \Omega$. Let us consider the following system:

$$\begin{cases} -\varepsilon^2 \Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \mathbb{B}_{\mathcal{D}}, \\ -\varepsilon^2 \Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \mathbb{B}_{\mathcal{D}}, \\ u_1, u_2 > 0 \quad \text{in } \mathbb{B}_{\mathcal{D}}, \quad u_1 = u_2 = 0 \quad \text{on } \partial\mathbb{B}_{\mathcal{D}}. \end{cases} \qquad (\mathcal{S}_{\varepsilon}^0)$$

Then, by a similar argument to that used for Proposition 2.1, we can show that $(\mathcal{S}_\varepsilon^0)$ has a least-energy solution $\overrightarrow{\mathbf{u}}_\varepsilon$ for $0 < \beta < \beta_0$ or $\beta > \beta_1$ with $\mathcal{J}_{\varepsilon,\mathbb{B}_\mathcal{D},T}(\overrightarrow{\mathbf{u}}_\varepsilon) = c_{\varepsilon,\mathbb{B}_\mathcal{D},T}$. Since $\mathbb{B}_\mathcal{D}$ is radially symmetric and $\beta > 0$, by Schwartz's symmetrization, $\overrightarrow{\mathbf{u}}_\varepsilon$ is also radially symmetric.

Lemma 5.1. *For every sufficiently small $\sigma > 0$, we have $\widetilde{u}_i^\varepsilon + \varepsilon|\nabla\widetilde{u}_i^\varepsilon| \leq Ce^{-\frac{(1-\sigma)\sqrt{\lambda_i}|x|}{\varepsilon}}$ with sufficiently small $\varepsilon > 0$.*

Proof. Since $\overrightarrow{\mathbf{u}}_\varepsilon$ is a solution of $(\mathcal{S}_\varepsilon^0)$, by the classical elliptic regularity theory, $\overrightarrow{\mathbf{u}}_\varepsilon \in \mathcal{C}^2(\mathbb{B}_\mathcal{D}) = (C^2(\mathbb{B}_\mathcal{D}))^2$. It follows that $\overrightarrow{\mathbf{u}}_\varepsilon = (\overline{u}_1^\varepsilon, \overline{u}_2^\varepsilon) \in \mathcal{C}^2(\mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}})$ with $\overline{u}_i^\varepsilon(x) = \widetilde{u}_i^\varepsilon(\varepsilon x)$. Moreover, $\overrightarrow{\mathbf{u}}_\varepsilon$ also satisfies the following system

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \alpha_1 u_1^{p-1} + \beta u_2^2 u_1 & \text{in } \mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}}, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \alpha_2 u_2^{p-1} + \beta u_1^2 u_2 & \text{in } \mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}}, \\ u_1, u_2 > 0 \quad \text{in } \mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}}, \quad u_1 = u_2 = 0 \quad \text{on } \partial\mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}}. \end{cases} \qquad (\mathcal{S}_\varepsilon^{00})$$

By a similar argument to that used for Proposition 4.1, $\overrightarrow{\mathbf{u}}_\varepsilon \to \overrightarrow{\mathbf{v}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4} \cap \mathcal{L}^\infty(\mathbb{R}^4)$ as $\varepsilon \to 0^+$, where $\overrightarrow{\mathbf{v}}_*$ is a least-energy solution of $(\mathcal{S}_*)$. It follows from Moser's iteration as used in [8] and [19, Theorem 8.17] (see also [8, Lemma 4.3]) that

$$\lim_{|x|\to+\infty} \overline{u}_i^\varepsilon = 0 \quad \text{uniformly for } \varepsilon > 0 \text{ sufficiently small.}$$

Thus, since $\lambda_i > 0$, by using the maximum principle in a standard way, we obtain that $\overline{u}_i^\varepsilon(x) \leq Ce^{-(1-\sigma)\sqrt{\lambda_i}|x|}$ with $\varepsilon > 0$ sufficiently small for every sufficiently small $\sigma > 0$. Therefore, $\widetilde{u}_i^\varepsilon \leq Ce^{-\frac{(1-\sigma)\sqrt{\lambda_i}|x|}{\varepsilon}}$. However, since $\overrightarrow{\mathbf{u}}_\varepsilon$ is radially symmetric and $\overrightarrow{\mathbf{u}}_\varepsilon$ satisfies $(\mathcal{S}_\varepsilon^{00})$,

$$\frac{\lambda_i}{2}\overline{u}_i^\varepsilon(r) \leq (\overline{u}_i^\varepsilon(r))'' + \frac{3}{r}(\overline{u}_i^\varepsilon(r))' \leq \lambda_i \overline{u}_i^\varepsilon(r)$$

for $\frac{\mathcal{D}}{\varepsilon} - 1 < r < \frac{\mathcal{D}}{\varepsilon}$ with $\varepsilon > 0$ sufficiently small. By the results in [4], $(\overline{u}_i^\varepsilon(r))' \leq 0$. Thus, $(\overline{u}_i^\varepsilon(r))'' > 0$. If $(\overline{u}_i^\varepsilon(\frac{\mathcal{D}}{\varepsilon}))' \leq -C$, then by the Harnack inequality, $(\overline{u}_i^\varepsilon(r))' \leq C\overline{u}_i^\varepsilon(r)$ for $\frac{\mathcal{D}}{\varepsilon} - 1 < r < \frac{\mathcal{D}}{\varepsilon}$. Otherwise, by integrating the above inequality over the interval $[r, \frac{\mathcal{D}}{\varepsilon}]$, we obtain $(\overline{u}_i^\varepsilon(r))' \leq C\overline{u}_i^\varepsilon(r) + Ce^{-(1-\sigma)\sqrt{\lambda_i}r}$. It follows that $|\nabla\overline{u}_i^\varepsilon| \leq Ce^{-(1-\sigma)\sqrt{\lambda_i}|x|}$ with sufficiently small $\varepsilon > 0$ for every $\sigma > 0$ sufficiently small, which implies $\varepsilon|\nabla\widetilde{u}_i^\varepsilon| \leq Ce^{-\frac{(1-\sigma)\sqrt{\lambda_i}|x|}{\varepsilon}}$. □

Recall that $\text{dist}(0, \partial\Omega) = \mathcal{D} = \max_{p\in\Omega} \text{dist}(p, \partial\Omega)$. Then, we have the following upper bound of $c_{\varepsilon,\Omega,T}$ with $\beta > 0$.

Lemma 5.2. *Let $\beta > 0$ and $\varepsilon > 0$ be sufficiently small. Then, for every sufficiently small $\sigma > 0$,*

$$c_{\varepsilon,\Omega,T} \leq \varepsilon^4\Big(B + C\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}\mathcal{D}}{\varepsilon}}\Big)$$

with $0 < \beta < \beta_0$ *while*

$$c_{\varepsilon,\Omega,T} \leq \varepsilon^4(B' + C\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}\mathcal{D}}{\varepsilon}})$$

with $\beta > \beta_1$.

Proof. Let ϕ_ε be a smooth radially symmetric function such that $0 \leq \phi_\varepsilon \leq 1$ and

$$\phi_\varepsilon(x) = \begin{cases} 1, & x \in \mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}-1}; \\ 0, & x \in \mathbb{R}^4\backslash\mathbb{B}_{\frac{\mathcal{D}}{\varepsilon}}. \end{cases}$$

Set $\overrightarrow{\overline{\mathbf{U}}}_{*,\varepsilon} = (U_1^*\phi_\varepsilon, U_2^*\phi_\varepsilon)$, where $\overrightarrow{\mathbf{U}}_*$ is a least-energy solution of $(\mathcal{S}_*)$, which is specified by Proposition 3.1. Then $\overrightarrow{\overline{\mathbf{U}}}_{*,\varepsilon} \in \mathcal{H}_{\Omega_\varepsilon}$. Since $\overrightarrow{\overline{\mathbf{U}}}_{*,\varepsilon} \to \overrightarrow{\mathbf{U}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0$, by a similar argument to that used in the proof of Lemma 4.1, we can show that there exits $\overrightarrow{\mathbf{t}}_\varepsilon$ with $\overrightarrow{\mathbf{t}}_\varepsilon \to \overrightarrow{\mathbf{1}}$ as $\varepsilon \to 0^+$ such that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\overline{\mathbf{U}}}_{*,\varepsilon} \in \mathcal{N}_{\Omega_\varepsilon,T}$. It follows from Lemma 5.1 and a similar argument to that used for Lemma 2.3 that

$$\begin{aligned} B &= \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{U}}_*) \\ &\geq \mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{U}}_*) \\ &\geq \mathcal{J}_{\Omega_\varepsilon,T}(\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\overline{\mathbf{U}}}_*) - C\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}\mathcal{D}}{\varepsilon}}. \end{aligned}$$

By the standard scaling technique, we obtain

$$c_{\varepsilon,\Omega,T} \leq \varepsilon^4(B + C\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}\mathcal{D}}{\varepsilon}})$$

for $0 < \beta < \beta_0$. Similarly,

$$c_{\varepsilon,\Omega,T} \leq \varepsilon^4(B' + C\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}\mathcal{D}}{\varepsilon}})$$

for $\beta > \beta_1$. □

Recall that $\overrightarrow{\mathbf{u}}_\varepsilon$ is a least-energy solution of $(\mathcal{S}_\varepsilon)$ for $0 < \beta < \beta_0$ or $\beta > \beta_1$.

Lemma 5.3. *Let* $0 < \beta < \beta_0$ *or* $\beta > \beta_1$. *Then, for every sufficiently small* $\sigma > 0$, *we have* $u_i^\varepsilon + \varepsilon|\nabla u_i^\varepsilon| \leq Ce^{-\frac{(1-\sigma)\sqrt{\lambda_i}|x-p_i^\varepsilon|}{\varepsilon}}$ *with sufficiently small* $\varepsilon > 0$ *for* $i = 1, 2$.

Proof. Recall that $v_1^\varepsilon(x) = u_1^\varepsilon(p_1^\varepsilon + \varepsilon x)$ and $\widetilde{v}_2^\varepsilon(x) = u_2^\varepsilon(p_2^\varepsilon + \varepsilon x)$. By Proposition 4.1, $(v_1^\varepsilon, \widetilde{v}_2^\varepsilon) \to \overrightarrow{\mathbf{v}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4} \cap \mathcal{L}^\infty(\mathbb{R}^4)$ as $\varepsilon \to 0^+$. It follows from Moser's iteration as used in [8] and [19, Theorem 8.17] (see also [8, Lemma 4.3]) that

$$\lim_{|x|\to+\infty} v_1^\varepsilon = \lim_{|x|\to+\infty} \widetilde{v}_2^\varepsilon = 0 \quad \text{uniformly for } \varepsilon > 0 \text{ sufficiently small.}$$

Since $\lambda_i > 0$, by applying the maximum principle in a standard way, we can show that

$$v_1^\varepsilon \leq Ce^{-(1-\sigma)\sqrt{\lambda_1}|x|} \quad \text{and} \quad \widetilde{v}_2^\varepsilon \leq Ce^{-(1-\sigma)\sqrt{\lambda_2}|x|}$$

for all sufficiently small $\sigma > 0$ with sufficiently small $\varepsilon > 0$. Hence, by the standard scaling technique, $u_i^\varepsilon \leq Ce^{-\frac{(1-\sigma)\sqrt{\lambda_i}|x-p_i^\varepsilon|}{\varepsilon}}$ for $i = 1, 2$. By Harnack's inequality, we also have $u_i^\varepsilon + \varepsilon|\nabla u_i^\varepsilon| \leq Ce^{-\frac{(1-\sigma)\sqrt{\lambda_i}|x-p_i^\varepsilon|}{\varepsilon}}$ with sufficiently small $\varepsilon > 0$ for $i = 1, 2$. □

Set $\mathcal{D}_i^\varepsilon = \text{dist}(p_i^\varepsilon, \partial\Omega)$ for $i = 1, 2$. Then, $\mathbb{B}_{\mathcal{D}_i^\varepsilon}(p_i^\varepsilon) \subset \Omega$ for both $i = 1, 2$. Without loss of generality, by Proposition 4.1, we assume that $\mathcal{D}_i^\varepsilon \to \mathcal{D}_0 = \text{dist}(p_0, \partial\Omega)$ as $\varepsilon \to 0^+$ with $p_i^\varepsilon \to p_0$ as $\varepsilon \to 0^+$ for $0 < \beta < \beta_0$ or $\beta > \beta_1$. As in [26], for $\delta > 0$, we choose $\mathcal{D}_0' < \mathcal{D}_0 + \delta$ such that

$$\text{meas}(\mathbb{B}_{\mathcal{D}_0'}(p_0)) = \text{meas}(\mathbb{B}_{\mathcal{D}_0+\delta}(p_0) \cap \Omega).$$

We also choose $\delta' < \delta$ such that $\mathcal{D}_0' < \mathcal{D}_0 + \delta'$. Now, consider the following smooth cut-off function

$$\eta_i^\varepsilon(s) = \begin{cases} 1, & \text{for } 0 \leq s \leq \mathcal{D}_i^\varepsilon + \delta', \\ 0, & \text{for } s > \mathcal{D}_i^\varepsilon + \delta \end{cases}$$

with $0 \leq \eta_i^\varepsilon \leq 1$ and $|(\eta_i^\varepsilon)'| \leq C$. Let $\overrightarrow{\widehat{\mathbf{u}}}_\varepsilon = (\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon)$ with $\widehat{u}_i^\varepsilon = u_i^\varepsilon(x)\eta_i^\varepsilon(|x - p_i^\varepsilon|)$. Then by Lemma 2.3, Proposition 4.1 and Lemma 5.3, we have

$$\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{u}}_\varepsilon) \geq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\widehat{\mathbf{u}}}_\varepsilon) - C(\overrightarrow{\mathbf{t}})\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}(\mathcal{D}_i^\varepsilon+\delta')}{\varepsilon}} \tag{5.1}$$

for $0 < \beta < \beta_0$ or $\beta > \beta_1$. Here, $C(\overrightarrow{\mathbf{t}})$ is bounded from above if $\overrightarrow{\mathbf{t}}$ is bounded from above. Let $R_\varepsilon = \frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon}$ where $\widetilde{\mathcal{D}}_1^\varepsilon$ is chosen such that

$$\text{meas}(\mathbb{B}_{\widetilde{\mathcal{D}}_1^\varepsilon}(p_1^\varepsilon)) = \text{meas}(\mathbb{B}_{\mathcal{D}_1^\varepsilon+\delta}(p_1^\varepsilon) \cap \Omega) \quad \text{and} \quad \widetilde{\mathcal{D}}_1^\varepsilon > \mathcal{D}_1^\varepsilon + \frac{1}{2}\delta.$$

Moreover, by Schwartz's symmetrization and since $\beta > 0$,

$$\mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\widehat{\mathbf{u}}}_\varepsilon) \geq \varepsilon^4 \mathcal{J}_{\mathbb{B}_{R_\varepsilon},T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\widehat{\mathbf{u}}}_{\varepsilon,*}). \tag{5.2}$$

Here, $\widehat{u}_i^{\varepsilon,*}$ is Schwartz's symmetrization of $\widehat{u}_i^\varepsilon$ in $\mathbb{B}_{\widetilde{\mathcal{D}}_1^\varepsilon}$ and $\mathcal{J}_{\mathbb{B}_{R_\varepsilon},T}(\overrightarrow{\mathbf{u}}) = \mathcal{J}_{1,\mathbb{B}_{R_\varepsilon},T}(\overrightarrow{\mathbf{u}})$.

Lemma 5.4. *Let $\beta > 0$. Then, for every sufficiently small $\delta > 0$, we have*

$$c_{\varepsilon,\Omega,T} \geq \varepsilon^4(B + C\sum_{i=1}^{2} e^{-\frac{2(1+\sigma)\sqrt{\lambda_i}(\mathcal{D}_1^\varepsilon+\frac{1}{2}\delta)}{\varepsilon}})$$

with $0 < \beta < \beta_0$ and sufficiently small $\varepsilon > 0$ and

$$c_{\varepsilon,\Omega,T} \geq \varepsilon^4(B' + C\sum_{i=1}^{2} e^{-\frac{2(1+\sigma)\sqrt{\lambda_i}(\mathcal{D}_1^\varepsilon+\frac{1}{2}\delta)}{\varepsilon}})$$

with $\beta > \beta_1$ and sufficiently small $\varepsilon > 0$.

Proof. The proof is similar to that of [26, Theorem 4.1]; therefore, we only sketch it and identify the differences. By a similar argument to that used for Lemma 5.1, the system $(\mathcal{S}_\varepsilon^{00})$ has a radial least-energy solution $\overrightarrow{\mathbf{v}}_{\varepsilon,*}$ in $\mathbb{B}_{R_\varepsilon}$ for $\beta \in (0, \beta_0) \cup (\beta_1, +\infty)$ with $\varepsilon > 0$ sufficiently small. By Lemma 4.2, $\mathbb{B}_{R_\varepsilon}(p_1^\varepsilon) \to \mathbb{R}^4$ as $\varepsilon \to 0^+$. Thus, by a similar argument to that used for Proposition 4.1, $\overrightarrow{\mathbf{v}}_{\varepsilon,*} \to \overrightarrow{\mathbf{v}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4} \cap \mathcal{L}^\infty(\mathbb{R}^4)$ as $\varepsilon \to 0^+$, where $\overrightarrow{\mathbf{v}}_*$ is a least energy solution of $(\mathcal{S}_*)$. It follows from Moser's iteration as used in [8] and [19, Theorem 8.17] (see also [8, Lemma 4.3]) that

$$\lim_{|x|\to+\infty} \widehat{v}_i^{\varepsilon,*} = 0 \quad \text{uniformly for } \varepsilon > 0 \text{ sufficiently small.}$$

Now, since $\lambda_i > 0$, by using the maximum principle in a standard way, for every $\delta > 0$ sufficiently small, we have

$$Ce^{-(1+\sigma)\sqrt{\lambda_i}(\frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon}-1)} \leq \widehat{v}_i^{\varepsilon,*}\left(\frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon} - 1\right) \leq Ce^{-(1-\sigma)\sqrt{\lambda_i}(\frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon}-1)} \tag{5.3}$$

with $\varepsilon > 0$ sufficiently small. Let us extend $\overrightarrow{\mathbf{v}}_{\varepsilon,*}$ to the whole space $\mathbb{R}^4$ as in the proof of [26, Theorem 4.1] and denote it by $\overrightarrow{\mathbf{v}}_{\varepsilon,**}$. Then, $\overrightarrow{\mathbf{v}}_{\varepsilon,**} \in \mathcal{H}_{\mathbb{R}^4}$. Since $p > 2$, by a similar argument to that used in the proof of [26, Theorem 4.1],

$$\mathcal{J}_{\mathbb{R}^4,T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{v}}_{\varepsilon,**}) \leq \mathcal{J}_{\mathbb{B}_{\widetilde{\mathcal{D}}_1^\varepsilon},T}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{v}}_{\varepsilon,*}) - C(\overrightarrow{\mathbf{t}})\sum_{i=1}^{2} \widehat{v}_i^{\varepsilon,*}\left(\frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon} - 1\right) \tag{5.4}$$

for all $\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^2$ and $C(\overrightarrow{\mathbf{t}})$ is bounded away from 0 if t_i bounded away from 0 for $i = 1, 2$. Since $\overrightarrow{\mathbf{v}}_{\varepsilon,*} \to \overrightarrow{\mathbf{v}}_*$ strongly in $\mathcal{H}_{\mathbb{R}^4}$ as $\varepsilon \to 0^+$ and $\overrightarrow{\mathbf{v}}_*$ is a least-energy solution of $(\mathcal{S}_*)$, by a similar argument to that used in the proof of Lemma 4.1, there exists $\overrightarrow{\mathbf{t}}_\varepsilon \in (\mathbb{R}^+)^2$ with $\overrightarrow{\mathbf{t}}_\varepsilon \to \overrightarrow{\mathbf{1}}$ as $\varepsilon \to 0^+$ such that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{v}}_{\varepsilon,**} \in \mathcal{N}_{\mathbb{R}^4,T}$ for $0 < \beta < \beta_0$. In the case $\beta > \beta_1$, we also have that $t_1^\varepsilon = t_2^\varepsilon$ and $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{v}}_{\varepsilon,**} \in \mathcal{N}'_{\mathbb{R}^4,T}$. Therefore, by (5.4) and a similar argument to that used in the proof of Lemma 2.3,

$$B \leq \mathcal{J}_{\mathbb{B}_{\widetilde{\mathcal{D}}_1^\varepsilon},T}(\overrightarrow{\mathbf{v}}_{\varepsilon,*}) - C\sum_{i=1}^{2} \widehat{v}_i^{\varepsilon,*}\left(\frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon} - 1\right)$$

for $0 < \beta < \beta_0$ and

$$B' \leq \mathcal{J}_{\mathbb{B}_{\widetilde{\mathcal{D}}_1^\varepsilon},T}(\overrightarrow{\mathbf{v}}_{\varepsilon,*}) - C\sum_{i=1}^{2} \widehat{v}_i^{\varepsilon,*}\left(\frac{\widetilde{\mathcal{D}}_1^\varepsilon}{\varepsilon} - 1\right)$$

for $\beta > \beta_1$. However, also by a similar argument to that used in the proof of Lemma 2.3, there exists $\overrightarrow{\mathbf{t}}'_\varepsilon \in (\mathbb{R}^+)^2$ such that $\overrightarrow{\mathbf{t}}'_\varepsilon \circ \overrightarrow{\widehat{\mathbf{u}}}_{\varepsilon,*} \in \mathcal{N}_{\mathbb{B}_{R_\varepsilon},T}$ for $0 < \beta < \beta_0$. In the case $\beta > \beta_1$, we also have that $(t_1^\varepsilon)' = (t_2^\varepsilon)'$ and $\overrightarrow{\mathbf{t}}'_\varepsilon \circ \overrightarrow{\widehat{\mathbf{u}}}_{\varepsilon,*} \in \mathcal{N}'_{\mathbb{B}_{R_\varepsilon},T}$. By Proposition 4.1, it follows from the definition of $\overrightarrow{\widehat{\mathbf{u}}}_{\varepsilon,*}$ that $\overrightarrow{\mathbf{t}}'_\varepsilon \to \overrightarrow{\mathbf{t}}'_0$ as $\varepsilon \to 0^+$ with $(t_i^0)' \leq 1$ for both $i = 1, 2$. Therefore, the conclusion follows from , (5.1)–(5.3). □

Now, we obtain the following:

Proposition 5.1. *Let* $\beta \in (0,\beta_0)\cup(\beta_1,+\infty)$*. Then,* $\mathcal{D}_i^\varepsilon \to \mathcal{D}$ *as* $\varepsilon\to 0^+$ *for all* $i=1,2$.

Proof. By Lemmas 5.2 and 5.4, it follows from Proposition 4.1 that

$$C'\sum_{i=1}^{2} e^{-\frac{2(1+\sigma')\sqrt{\lambda_i}(\mathcal{D}_1^\varepsilon+\frac{1}{2}\delta)}{\varepsilon}} \leq C\sum_{i=1}^{2} e^{-\frac{2(1-\sigma)\sqrt{\lambda_i}\mathcal{D}}{\varepsilon}}.$$

Since σ,σ' and δ are arbitrary, $\liminf_{\varepsilon\to 0^+}\mathcal{D}_1^\varepsilon \geq \mathcal{D}$, which implies that $\lim_{\varepsilon\to 0^+}\mathcal{D}_1^\varepsilon = \mathcal{D}$. By Proposition 4.1, $\lim_{\varepsilon\to 0^+}\mathcal{D}_2^\varepsilon=\mathcal{D}$. □

In what follows, we study the locations of the spikes for $\beta<0$. First, we follow the ideas in [34] to establish an upper-bound of $c_{\varepsilon,\Omega,T}$ for sufficiently small $\varepsilon>0$. Fix $P\in\Omega$ and let $u_i^{\varepsilon,P}$ be the unique solution of the following equation

$$\begin{cases} -\Delta u+\lambda_i u=\mu_i(v_i^0)^3+\alpha_i(v_i^0)^{p-1} & \text{in } \Omega_{\varepsilon,P}, \\ u>0 \quad \text{in } \Omega_{\varepsilon,P}, \quad u=0 \quad \text{on } \partial\Omega_{\varepsilon,P}. \end{cases} \tag{$\mathcal{P}_i^{\varepsilon,P}$}$$

where $\Omega_{\varepsilon,P}=\{y\in\mathbb{R}^4 \mid \varepsilon y+P\in\Omega\}$ and v_i^0 is a least-energy solution of $(\mathcal{P}_i)$, $i=1,2$, which is specified by Proposition 4.1. Since $\Omega_{\varepsilon,P}\to\mathbb{R}^4$ as $\varepsilon\to 0^+$, $u_i^{\varepsilon,P}\to v_i^0$ strongly in $\mathcal{H}_i$ as $\varepsilon\to 0^+$. Let

$$\psi_i^{\varepsilon,P}(x)=-\varepsilon\log\left(\varphi_i^{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)\right) \quad \text{and} \quad V_i^{\varepsilon,P}=e^{\frac{\psi_i^\varepsilon(P)}{\varepsilon}}\varphi_i^{\varepsilon,P}, \quad i=1,2 \tag{5.5}$$

where $\varphi_i^{\varepsilon,P}=v_i^0-u_i^{\varepsilon,P}$. Then, as in [34], since $u_i^{\varepsilon,P}$ and v_i^0 satisfy $(\mathcal{P}_i^{\varepsilon,P})$ and $(\mathcal{P}_i)$ for $i=1,2$, respectively, $\psi_i^{\varepsilon,P}$ and $V_i^{\varepsilon,P}$ respectively satisfy

$$\begin{cases} \varepsilon\Delta\psi_i^{\varepsilon,P}-|\nabla\psi_i^{\varepsilon,P}|^2+\lambda_i=0 & \text{in } \Omega, \\ \psi_i^{\varepsilon,P}=-\varepsilon\log\left(v_i^0\left(\frac{x-P}{\varepsilon}\right)\right) & \text{on } \partial\Omega \end{cases} \tag{$\widetilde{\mathcal{P}}_i^{\varepsilon,P}$}$$

and

$$\begin{cases} \Delta V_i^{\varepsilon,P}-\lambda_i V_i^{\varepsilon,P}=0 & \text{in } \Omega_{\varepsilon,P}, \\ V_i^{\varepsilon,P}(0)=1. \end{cases} \tag{$\widehat{\mathcal{P}}_i^{\varepsilon,P}$}$$

Set $\widetilde{u}_i^{\varepsilon,P}(x)=u_i^{\varepsilon,P}\left(\frac{x-P}{\varepsilon}\right)$. Then, by $(\mathcal{P}_i^{\varepsilon,P})$, $\widetilde{u}_i^{\varepsilon,P}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta\widetilde{u}_i^{\varepsilon,P}+\lambda_i\widetilde{u}_i^{\varepsilon,P}=\mu_i\left(v_{i,P}^{0,\varepsilon}\right)^3+\alpha_i\left(v_{i,P}^{0,\varepsilon}\right)^{p-1} & \text{in } \Omega, \\ \widetilde{u}_i^{\varepsilon,P}>0 \quad \text{in } \Omega, \quad \widetilde{u}_i^{\varepsilon,P}=0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\overline{\mathcal{P}}_i^{\varepsilon,P}$}$$

where $v_{i,P}^{0,\varepsilon}=v_i^0\left(\frac{x-P}{\varepsilon}\right)$, $i=1,2$. We also recall that

$$\text{dist}(0,\partial\Omega)=\mathcal{D}=\max_{p\in\Omega}\text{dist}(p,\partial\Omega).$$

Lemma 5.5. *For ε sufficiently small, we have the following*

(1) $\|\widetilde{u}_i^{\varepsilon,P}\|_{i,\varepsilon,\Omega}^2 = \varepsilon^4\bigg(\|v_i^0\|_{i,\mathbb{R}^4}^2 - (\gamma_i + o(1))\varphi_i^{\varepsilon,P}(0)\bigg)$,

(2) $\alpha_i\|\widetilde{u}_i^{\varepsilon,P}\|_{L^p(\Omega)}^p = \varepsilon^4\bigg(\alpha_i\|v_i^0\|_{L^p(\Omega)}^p - (\gamma_{i,p} + o(1))\varphi_i^{\varepsilon,P}(0)\bigg)$,

(3) $\mu_i\|\widetilde{u}_i^{\varepsilon,P}\|_{L^4(\Omega)}^4 = \varepsilon^4\bigg(\mu_i\|v_i^0\|_{L^4(\Omega)}^4 - (\gamma_{i,4} + o(1))\varphi_i^{\varepsilon,P}(0)\bigg)$,

where $\gamma_i, \gamma_{i,p}$ and $\gamma_{i,4}$ are positive constants that are independent of ε. Moreover,

$$d_{i,\varepsilon,\Omega} \leq \varepsilon^4(d_{i,\mathbb{R}^4} + (a_0 + o(1))\varphi_i^{\varepsilon,P}(0)),$$

where $a_0 > 0$ is a constant that is independent of ε.

Proof. Since the proof is similar to that of [34, Lemma 5.2 and Proposition 5.1], we only sketch it and identify the differences. By the well-known Gidas-Ni-Nirenberg's Theorem [20] (see also in [24, Theorem A]), v_i^0 are radially symmetric for $i = 1, 2$. It follows from $p > 2$ that

$$\begin{aligned}
&\int_{\mathbb{R}^4}(\alpha_i(v_i^0)^{p-1} + \mu_i(v_i^0)^3)V_i^{*,0} = \gamma_i > 0,\\
&\int_{\mathbb{R}^4}\alpha_i p(v_i^0)^{p-1}V_i^{*,0} = \gamma_{i,p} > 0,\\
&\int_{\mathbb{R}^4}4\mu_i(v_i^0)^3V_i^{*,0} = \gamma_{i,4} > 0,
\end{aligned} \tag{5.6}$$

where $V_i^{*,0}$ is the unique positive radial solution of the following equation

$$\begin{cases}\Delta V - \lambda_i V = 0 & \text{in } \mathbb{R}^4,\\ V(0) = 1.\end{cases} \tag{$\widehat{\mathcal{P}}_i^{*,0}$}$$

Since all the integrals in the proof of [34, Lemma 4.7] also make sense in our case, by s similar argument with trivial modifications,

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^4}(\alpha_i(v_i^0)^{p-1} + \mu_i(v_i^0)^3)\widetilde{V}_i = \gamma_i,\\
&\textstyle\int_{\mathbb{R}^4}\alpha_i(v_i^0)^{p-1}\widetilde{V}_i = \gamma_{i,p},\\
&\textstyle\int_{\mathbb{R}^4}\mu_i(v_i^0)^3\widetilde{V}_i = \gamma_{i,4},
\end{aligned}$$

where $\widetilde{V}_i$ is an arbitrary solution of $(\widehat{\mathcal{P}}_i^{*,0})$. However, by [26, Lemma 5.1], $V_i^{\varepsilon,0} \to V_i^0$ strongly in $\mathcal{H}_i$ as $\varepsilon \to 0^+$ for $i = 1, 2$ and

$$\sup_{y\in\Omega_\varepsilon}|e^{-\sqrt{\lambda_i}(1+\sigma)|y|}V_i^{\varepsilon,0}(y)| \leq C$$

for any $0 < \sigma < 1$ and uniformly for sufficiently small ε. Since all the integrals in the proofs of [34, Lemmas 5.2, 5.3 and Proposition 5.1] also make sense in our case, by similar arguments with trivial modifications and from $(2-p)\alpha_i(v_i^0)^p - 2\mu_i(v_i^0)^3 < 0$ in $\mathbb{R}^4$ (since $p > 2$), we obtain the conclusion. □

We make the following observation:

Lemma 5.6. *For sufficiently small* $\varepsilon > 0$*,*

$$\int_\Omega |\widetilde{u}_1^{\varepsilon,P_1}|^2|\widetilde{u}_2^{\varepsilon,P_2}|^2dx = \varepsilon^4(1+o(1))\int_{\mathbb{R}^4}\left|v_1^0\left(x-\frac{P_1}{\varepsilon}\right)\right|^2\left|v_2^0\left(x-\frac{P_2}{\varepsilon}\right)\right|^2dx.$$

Proof. With trivial modifications, the conclusion follows from a similar argument to that used in the proof of (2) of [26, Lemma 5.2]. □

Now, as in [26], for every $(P_1, P_2) \in \Omega^2$, we denote

$$I_\varepsilon[P_1,P_2] = \int_{\mathbb{R}^4}\left|v_1^0\left(x-\frac{P_1}{\varepsilon}\right)\right|^2\left|v_2^0\left(x-\frac{P_2}{\varepsilon}\right)\right|^2dx$$

and

$$\delta_\varepsilon[P_1,P_2] = \sum_{i=1}^{2}\varphi_i^{\varepsilon,P_i}(0) + I_\varepsilon[P_1,P_2].$$

Lemma 5.7. *For sufficiently small* $\varepsilon > 0$*,*

$$C'e^{-\frac{2(1+\sigma)\varphi(P_1,P_2)}{\varepsilon}} \le \delta_\varepsilon[P_1,P_2] \le Ce^{-\frac{2(1-\sigma)\varphi(P_1,P_2)}{\varepsilon}}$$

for sufficiently small $\sigma > 0$*, where*

$$\varphi(P_1,P_2) = \min_{i=1,2}\{\min\{\sqrt{\lambda_i}|P_1-P_2|, \sqrt{\lambda_i}dist(P_i,\partial\Omega)\}\}. \tag{5.7}$$

Proof. By [34, Lemma 4.4],

$$C'e^{-\frac{2(1+\sigma)\sqrt{\lambda_i}dist(P_i,\partial\Omega)}{\varepsilon}} \le \varphi_i^{\varepsilon,P_i}(0) \le Ce^{-\frac{2(1-\sigma)\sqrt{\lambda_i}dist(P_i,\partial\Omega)}{\varepsilon}} \tag{5.8}$$

for any $0 < \sigma < 1$. However, by a scaling technique and [8, Proposition 2.1], for any $0 < \delta < 1$, there exists $C_\delta > 0$ such that

$$v_i^0 \le C_\delta e^{-(1-\delta)\sqrt{\lambda_i}|x|}. \tag{5.9}$$

Moreover, since v_i^0 satisfies $(\mathcal{P}_i)$, by (5.9), we can apply the maximum principle in a standard way to show that for any $0 < \delta < 1$, there exists $C'_\delta > 0$ such that

$$v_i^0 \ge C'_\delta e^{-(1+\delta)\sqrt{\lambda_i}|x|} \tag{5.10}$$

for sufficiently large $|x|$. By translation, (5.9) and (5.10),

$$C'e^{-\frac{2(1+\delta)\min_{i=1,2}\{\sqrt{\lambda_i}|P_1-P_2|\}}{\varepsilon}} \le I_\varepsilon[P_1,P_2] \le Ce^{-\frac{2(1-\delta)\min_{i=1,2}\{\sqrt{\lambda_i}|P_1-P_2|\}}{\varepsilon}}. \tag{5.11}$$

The conclusion follows immediately from (5.8) and (5.11). □

Now, we can obtain an upper bound for $c_{\varepsilon,\Omega,T}$ in the case $\beta < 0$.

Lemma 5.8. *Let α_T be as specified in Lemma 2.2 and $\varepsilon > 0$ be sufficiently small. Then, for*

1. $-\sqrt{\mu_1\mu_2} < \beta < 0$ *and* $\alpha_1, \alpha_2 > 0$,
2. $\beta \leq -\sqrt{\mu_1\mu_2}$ *and* $\alpha_1, \alpha_2 > 0$ *with* $|\overrightarrow{\alpha}| < \alpha_T$,

we have

$$c_{\varepsilon,\Omega,T} \leq \varepsilon^4(\sum_{i=1}^{2} d_{i,\mathbb{R}^4} + a_1 e^{-\frac{2(1-\sigma)\varphi(P_1^*,P_2^*)}{\varepsilon}}),$$

where P_i^* satisfy $\varphi(P_1^*, P_2^*) = \max_{(P_1,P_2)\in\Omega^2} \varphi(P_1, P_2)$.

Proof. Clearly, $\varphi(P_1^*, P_2^*) > 0$ and $P_1^* \neq P_2^*$. Now, consider $\overrightarrow{\mathbf{u}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*} = (\widetilde{u}_1^{\varepsilon,P_1^*}, \widetilde{u}_2^{\varepsilon,P_2^*})$. Then, we must have $\overrightarrow{\mathbf{u}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*} \in \widetilde{\mathcal{H}}_{\varepsilon,\Omega}$. By Lemma 5.6 and (5.11), we also have that

$$\|\widetilde{u}_1^{\varepsilon,P_1^*}\widetilde{u}_2^{\varepsilon,P_2^*}\|_{L^2(\Omega)}^2 = o(1). \tag{5.12}$$

By (2.2) and since $u_i^{\varepsilon,P_i^*} \to v_i^0$ strongly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$, we can apply Miranda's theorem as for $\eta_i(t_1, t_2)$ in the proof of Lemma 2.4 to show that there exists $\overrightarrow{\mathbf{t}}_\varepsilon \in (\mathbb{R}^+)^2$ with $0 < \widetilde{t}_i^\varepsilon < t_i^{\varepsilon,*}$ such that $\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{u}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*} \in \mathcal{N}_{\varepsilon,\Omega,T}$, where

$$t_i^{\varepsilon,*} = \left(\frac{\|\widetilde{u}_i^{\varepsilon,P_i^*}\|_{i,\varepsilon,\Omega}^2}{\mu_i\|\widetilde{u}_i^{\varepsilon,P_i^*}\|_{L^4(\Omega)}^4}\right)^{\frac{1}{2}}$$

are bounded from above by a constant C that is independent of ε. By (5.12) and since $u_i^{\varepsilon,P_i^*} \to v_i^0$ strongly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$, we also have that

$$\prod_{i=1}^{2} \mu_i\|\widetilde{u}_i^{\varepsilon,P_i^*}\|_{L^4(\Omega)}^4 - \beta^2\|\widetilde{u}_1^{\varepsilon,P_1^*}\widetilde{u}_2^{\varepsilon,P_2^*}\|_{L^2(\Omega)}^4 \geq \varepsilon^4 C; \quad \|\widetilde{u}_i^{\varepsilon,P_i^*}\|_{L^4(\Omega)}^4 \geq \varepsilon^4 C.$$

Thus, by applying the implicit function theorem as used for $\Gamma_i^n(\overrightarrow{\mathbf{t}}, \tau)$ with $\beta < 0$ in the proof of Lemma 2.3, we obtain that $\overrightarrow{\mathbf{t}}_\varepsilon$ is unique. Now, let us consider the system $\overrightarrow{\mathbf{\Gamma}}(\overrightarrow{\mathbf{t}}, \tau) = \overrightarrow{\mathbf{0}}$, where $\overrightarrow{\mathbf{\Gamma}}(\overrightarrow{\mathbf{t}}, \tau) = (\Gamma_1(\overrightarrow{\mathbf{t}}, \tau), \Gamma_2(\overrightarrow{\mathbf{t}}, \tau))$ with

$$\begin{aligned}\Gamma_i(\overrightarrow{\mathbf{t}}, \tau) &= \tau^{-4}(\|t_i\widetilde{u}_i^{\tau,P_i^*}\|_{i,\tau,\Omega}^2 - \alpha_i\|t_i\widetilde{u}_i^{\varepsilon,P_i^*}\|_{L^p(\Omega)}^p - \mu_i\|t_i\widetilde{u}_i^{\varepsilon,P_i^*}\|_{L^4(\Omega)}^4 \\ &\quad -\beta\|(t_1\widetilde{u}_1^{\varepsilon,P_1^*})(t_2\widetilde{u}_2^{\varepsilon,P_2^*})\|_{L^2(\Omega)}^2).\end{aligned}$$

$\overrightarrow{\mathbf{\Gamma}}$ is of class C^1 and $\overrightarrow{\mathbf{\Gamma}}(\overrightarrow{\mathbf{t}}_\varepsilon, \varepsilon) = \overrightarrow{\mathbf{0}}$. Moreover, as stated in the proof of Lemma 5.5, all the integrals in the proof of [34, Lemmas 5.2] are well-defined in our case; thus, by similar arguments with trivial modifications, it follows from Lemma 5.6 that

$$\begin{aligned}\Gamma_i(\overrightarrow{\mathbf{t}}, \tau) &= \|t_iv_i^0\|_{i,\mathbb{R}^4}^2 - \alpha_i\|t_iv_i^0\|_{L^p(\Omega)}^p - \mu_i\|t_iv_i^0\|_{L^4(\Omega)}^4 \\ &\quad -(\gamma_it_i^2 - t_i\gamma_i(t_i) + o(1))\varphi_i^{\tau,P_i^*}(0)) - \beta t_1^2t_2^2(1 + o(1))I_\tau[P_1^*, P_2^*],\end{aligned}$$

where $o(1)$ is uniformly bounded and t_i, γ_i are specified by (5.6) and

$$\gamma_i(t_i) \quad = \quad \int_{\mathbb{R}^4} (\alpha_i p(t_i v_i^0)^{p-1} + 4\mu_i(t_i v_i^0)^3) V_i^{*,0}.$$

Since $p > 2$, $\gamma_i t_i^2 - t_i\gamma_i(t_i) < 0$ for $t_i \geq 1$. Thus, by $\beta < 0$, we can extend the functions $\Gamma_i(\overrightarrow{\mathbf{t}}, \tau)$ to $\tau = 0$ as continuously differentiable maps for $t_i \geq 1$. Since $p > 2$, by applying the implicit function theorem, we obtain that

$$t_i^\varepsilon = 1 + O(\delta_\varepsilon[P_1^*, P_2^*]), \quad i = 1, 2. \tag{5.13}$$

Now, by similar computations as in [26, (5.15)] and Taylor's expansion, we obtain that

$$\begin{aligned} c_{\varepsilon,\Omega,T} &\leq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\widetilde{\mathbf{t}}}_\varepsilon \circ \overrightarrow{\widetilde{\mathbf{u}}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*}) \\ &= \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\widetilde{\mathbf{u}}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*}) + \mathcal{J}'_{\varepsilon,\Omega,T}(\overrightarrow{\widetilde{\mathbf{u}}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*})\Big((\overrightarrow{\widetilde{\mathbf{t}}}_\varepsilon - \overrightarrow{\mathbf{1}}) \circ \overrightarrow{\widetilde{\mathbf{u}}}_{\varepsilon,\overrightarrow{\mathbf{P}}_*}\Big) \\ &\quad + O(|\overrightarrow{\widetilde{\mathbf{t}}}_\varepsilon - \overrightarrow{\mathbf{1}}|^2). \end{aligned} \tag{5.14}$$

By Lemmas 5.5-5.7, (5.13) and (5.14), we obtain the conclusion. □

Due to the criticality, we do not know whether the solution of $(\mathcal{P}_i)$ in $\mathbb{R}^4$ is non-degenerate, we must establish the lower-bound of $c_{\varepsilon,T,\Omega}$ for $\beta < 0$ using a different approach from that in [26], which is inspired by [17]. Consider the following equations:

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_i u = \mu_i u^3 + \alpha_i u^{p-1} & \text{in } \mathbb{B}_{r_i}(P_i), \\ u > 0 \quad \text{in } \mathbb{B}_{r_i}(P_i), & u = 0 \quad \text{on } \partial\mathbb{B}_{r_i}(P_i), \end{cases} \tag{$\mathcal{P}_i^{\varepsilon,P_i}$}$$

where $P_i \in \Omega, i = 1, 2$, are two distinct points. By Proposition 7.2, $(\mathcal{P}_i^{\varepsilon,P_i})$ has a solution u_{i,P_i}^ε that satisfies the following:

$$\mathcal{E}_{i,\varepsilon,\mathbb{B}_{r_i}(P_i)}(u_{i,P_i}^\varepsilon) = d_{i,\varepsilon,\mathbb{B}_{r_i}(P_i)}$$

and

$$u_{i,P_i}^\varepsilon(\varepsilon y + P_i) \to v_i^0 \text{ strongly in } \mathcal{H}_{i,\mathbb{R}^4}, \tag{5.15}$$

where v_i^0 is a least-energy solution of $(\mathcal{P}_i)$ and $\mathcal{E}_{i,\varepsilon,\mathbb{B}_{r_i}(P_i)}(u), d_{i,\varepsilon,\mathbb{B}_{r_i}(P_i)}$ are specified by (7.6). Moreover, similar to the proofs of Lemmas 5.2-5.4, we also have

$$d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)\sqrt{\lambda_i}r_i}{\varepsilon}} \leq \frac{d_{i,\varepsilon,\mathbb{B}_{r_i}(P_i)}}{\varepsilon^4} \leq d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1-\sigma)\sqrt{\lambda_i}r_i}{\varepsilon}}.$$

where $d_{i,\mathbb{R}^4}$ is specified by (7.3) and $\sigma \in (0, 1)$. For every $b_1, b_2 > 0$, we also define

$$\begin{aligned} &\varphi_{b_1,b_2}^*(p_1^\varepsilon, p_2^\varepsilon) \\ &= \min\left\{\frac{\sqrt{\lambda_1}b_1 + \sqrt{\lambda_2}b_2}{b_1 + b_2}|p_1^\varepsilon - p_2^\varepsilon|, \sqrt{\lambda_1}dist(p_1^\varepsilon, \partial\Omega), \sqrt{\lambda_2}dist(p_2^\varepsilon, \partial\Omega)\right\}, \end{aligned} \tag{5.16}$$

where p_i^ε is the maximum point of u_i^ε and $\overrightarrow{\mathbf{u}}_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the positive solution that is obtained by Proposition 2.1.

Lemma 5.9. *Let α_T be as specified in Lemma 2.2 and $\varepsilon > 0$ be sufficiently small. Then, for $\beta < 0$ and $|\overrightarrow{\alpha}| < \alpha_T$,*

$$c_{\varepsilon,\Omega,T} \geq \varepsilon^4 (\sum_{i=1}^{2} d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)(\varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon)-\delta)}{\varepsilon}}),$$

where $O(\delta) \to 0$ as $\delta \to 0$ uniformly for ε.

Proof. Since $p > 2$, it is standard to show that there exists $t_i^\varepsilon > 0$ such that $t_i^\varepsilon u_i^\varepsilon \in \mathcal{M}_{i,\varepsilon,\Omega}$, where $\mathcal{M}_{i,\varepsilon,\Omega}$ is specified by (7.5). Moreover, by Proposition 4.1, we also have $t_i^\varepsilon = 1 + o(1)$. Thus, since $|\overrightarrow{\alpha}| < \alpha_T$ and $\beta < 0$, by Lemma 2.3, we obtain that

$$c_{\varepsilon,\Omega,T} \geq \mathcal{J}_{\varepsilon,\Omega,T}(\overrightarrow{\mathbf{t}}_\varepsilon \circ \overrightarrow{\mathbf{u}}_\varepsilon) \geq \sum_{i=1}^{2} \mathcal{E}_{i,\varepsilon,\Omega}(t_i^\varepsilon u_i^\varepsilon) - \frac{\beta}{4} \|(t_1^\varepsilon u_1^\varepsilon)(t_2^\varepsilon u_2^\varepsilon)\|^2_{L^2(\Omega)} \quad (5.17)$$

for sufficiently small $\varepsilon > 0$. By Lemma 5.3, similar to the proof of Lemma 5.4,

$$d_{i,\varepsilon,\Omega} \geq \varepsilon^4 (d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)\sqrt{\lambda_i} dist(p_i^\varepsilon, \partial\Omega)}{\varepsilon}}), \quad i = 1, 2. \quad (5.18)$$

On the other hand, let p^ε be the point in $\Lambda_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon)$ such that

$$|p_1^\varepsilon - p^\varepsilon| + |p_2^\varepsilon - p^\varepsilon| = |p_1^\varepsilon - p_2^\varepsilon|,$$

where $\Lambda_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon)$ is defined by

$$\Lambda_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon) = \{x \in \Omega \mid b_1|x - p_1^\varepsilon| = b_2|x - p_2^\varepsilon|\}. \quad (5.19)$$

We re-denote $|p_i^\varepsilon - p^\varepsilon|$ by $D_i^{\varepsilon,*}$. For clarity, we divide the following proof into two cases.

Case. 1 Either

$$\varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon) = \sqrt{\lambda_1} dist(p_1^\varepsilon, \partial\Omega)$$

or

$$\varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon) = \sqrt{\lambda_2} dist(p_2^\varepsilon, \partial\Omega).$$

Indeed, since $\beta < 0$, by (5.17) and (5.18),

$$\begin{aligned} c_{\varepsilon,\Omega,T} &\geq \varepsilon^4 (\sum_{i=1}^{2} (d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)\sqrt{\lambda_1} dist(p_1^\varepsilon, \partial\Omega)}{\varepsilon}})) \\ &= \varepsilon^4 (\sum_{i=1}^{2} d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)\varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon)}{\varepsilon}}). \end{aligned}$$

Case. 2 $\varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon) = \frac{\sqrt{\lambda_1} b_1 + \sqrt{\lambda_2} b_2}{b_1 + b_2} |p_1^\varepsilon - p_2^\varepsilon|$.

By Proposition 4.1, for every $\delta > 0$, $\mathbb{B}_{D_i^{\varepsilon,*}+\varepsilon\delta}(p_i^\varepsilon) \subset \Omega$ for $\varepsilon > 0$ sufficiently small. Now, let us consider the intersection $\Gamma_\varepsilon = \mathbb{B}_{D_1^{\varepsilon,*}+\varepsilon\delta}(p_1^\varepsilon) \cap \mathbb{B}_{D_2^{\varepsilon,*}+\varepsilon\delta}(p_2^\varepsilon)$. Then, $\Gamma_\varepsilon \subset \mathbb{B}_{D_i^{\varepsilon,*}+\varepsilon\delta}(p_i^\varepsilon) \backslash \mathbb{B}_{D_i^{\varepsilon,*}-\varepsilon\delta}(p_i^\varepsilon)$. Since $D_i^{\varepsilon,*} = \frac{b_i}{b_1+b_2}|p_1^\varepsilon - p_2^\varepsilon|$, by Proposition 4.1 and a standard comparison argument, we obtain that $u_i^\varepsilon \geq Ce^{-\frac{(1+\sigma)\sqrt{\lambda_i}|x-p_i^\varepsilon|}{\varepsilon}}$ in Γ_ε with sufficiently small $\varepsilon > 0$. It follows that

$$\|u_1^\varepsilon u_2^\varepsilon\|_{L^2(\Omega)}^2 \geq Ce^{-\frac{2(1+\sigma)}{\varepsilon}(\sqrt{\lambda_1}(D_1^{\varepsilon,*}-\delta)+\sqrt{\lambda_2}(D_2^{\varepsilon,*}-\delta))}.$$

This, together with (5.17)–(5.18) and $\beta < 0$, implies

$$\begin{aligned} c_{\varepsilon,\Omega,T} &\geq \varepsilon^4\Big(\sum_{i=1}^2 (d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)\sqrt{\lambda_1}dist(p_1^\varepsilon,\partial\Omega)}{\varepsilon}}) \\ &\quad + Ce^{-\frac{2(1+\sigma)}{\varepsilon}(\sqrt{\lambda_1}(D_1^{\varepsilon,*}-\delta)+\sqrt{\lambda_2}(D_2^{\varepsilon,*}-\delta))}\Big) \\ &= \varepsilon^4\Big(\sum_{i=1}^2 d_{i,\mathbb{R}^4} + Ce^{-\frac{2(1+\sigma)(\varphi^*_{b_1,b_2}(p_1^\varepsilon,p_2^\varepsilon)-\delta)}{\varepsilon}}\Big). \end{aligned}$$

This completes the proof. □

Now, we can obtain the following:

Proposition 5.2. *Let* $\beta < 0$. *Then,* $\varphi(p_1^\varepsilon, p_2^\varepsilon) \to \varphi(P_1^*, P_2^*)$ *as* $\varepsilon \to 0^+$, *where* $\varphi(P_1, P_2)$ *is specified by* (5.7).

Proof. By Lemmas 5.8 and 5.9, it follows from Proposition 4.1 that

$$\lim_{\varepsilon\to 0^+} \varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon) \geq \frac{(1-\sigma)(\varphi(P_1^*, P_2^*) - \delta)}{1+\sigma'}.$$

Since σ, σ' and δ are all arbitrary,

$$\lim_{\varepsilon\to 0^+} \varphi^*_{b_1,b_2}(p_1^\varepsilon, p_2^\varepsilon) \geq \varphi(P_1^*, P_2^*).$$

The conclusion is obtained by letting $b_1 \to 0^+$ or $b_2 \to 0^+$. □

Proof of Theorem 1.3: It follows immediately from Propositions 5.1 and 5.2. □

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7 Appendix

In this section, we will list known results that will be used frequently in this paper. Let $\mathcal{H}_{i,\mathbb{R}^4}$ be the Hilbert space $H^1(\mathbb{R}^4)$ that is equipped with the inner product

$$\langle u, v\rangle_{i,\mathbb{R}^4} = \int_{\mathbb{R}^4} \nabla u \nabla v + \lambda_i uv dx.$$

The corresponding norm is specified by $\|u\|_{i,\mathbb{R}^4} = \langle u, u\rangle_{i,\mathbb{R}^4}^{\frac{1}{2}}$. Define

$$\mathcal{E}_{i,\mathbb{R}^4}(u) = \frac{1}{2}\|u\|_{i,\mathbb{R}^4}^2 - \frac{\alpha_i}{p}\|u\|_{L^p(\mathbb{R}^4)}^p - \frac{\mu_i}{4}\|u\|_{L^4(\mathbb{R}^4)}^4. \tag{7.1}$$

Then it is well known that $\mathcal{E}_{i,\mathbb{R}^4}(u)$ is of class C^2 in $\mathcal{H}_{i,\mathbb{R}^4}$. Set

$$\mathcal{M}_{i,\mathbb{R}^4} = \{u \in \mathcal{H}_{i,\mathbb{R}^4}\backslash\{0\} \mid \mathcal{E}'_{i,\mathbb{R}^4}(u)u = 0\}. \tag{7.2}$$

and define

$$d_{i,\mathbb{R}^4} = \inf_{\mathcal{M}_{i,\mathbb{R}^4}} \mathcal{E}_{i,\mathbb{R}^4}(u). \tag{7.3}$$

Proposition 7.1. $0 < d_{i,\mathbb{R}^4} < \frac{1}{4\mu_i}\mathcal{S}^2$ *holds for both* $i = 1, 2$*, where* $\mathcal{S}$ *is the optimal embedding constant from* $H^1(\mathbb{R}^4) \to L^4(\mathbb{R}^4)$*, which is defined by*

$$\mathcal{S} = \inf\{\|\nabla u\|_{L^2(\mathbb{R}^4)}^2 \mid u \in H^1(\mathbb{R}^4), \|u\|_{L^4(\mathbb{R}^4)}^2 = 1\}.$$

Moreover, $d_{i,\mathbb{R}^4}$ *is attained by some* $U_{i,\mathbb{R}^4} \in \mathcal{M}_{i,\mathbb{R}^4}$*, which is also a solution of the following equation*

$$\begin{cases} -\Delta u + \lambda_i u = \mu_i u^3 + \alpha_i u^{p-1} & \text{in } \mathbb{R}^4, \\ u > 0 \text{ in } \mathbb{R}^4, & u \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{$\mathcal{P}_i$}$$

Proof. See the results in [50]. □

Let $\Omega \subset \mathbb{R}^4$ be a bounded domain, $\lambda_i, \mu_i, \alpha_i > 0 (i = 1, 2)$ constants, $\varepsilon > 0$ a small parameter and $2 < p < 2^* = 4$. Let

$$\mathcal{E}_{i,\varepsilon,\Omega}(u) = \frac{1}{2}\|u\|_{i,\varepsilon,\Omega}^2 - \frac{\alpha_i}{p}\|u\|_{L^p(\Omega)}^p - \frac{\mu_i}{4}\|u\|_{L^4(\Omega)}^4. \tag{7.4}$$

Then it is well known that $\mathcal{E}_{i,\varepsilon,\Omega}(u)$ is of class C^2 in $\mathcal{H}_{i,\varepsilon,\Omega}$. Set

$$\mathcal{M}_{i,\varepsilon,\Omega} = \{u \in \mathcal{H}_{i,\varepsilon,\Omega}\backslash\{0\} \mid \mathcal{E}'_{i,\varepsilon,\Omega}(u)u = 0\} \tag{7.5}$$

and define

$$d_{i,\varepsilon,\Omega} = \inf_{\mathcal{M}_{i,\varepsilon,\Omega}} \mathcal{E}_{i,\varepsilon,\Omega}(u), \quad i = 1, 2. \tag{7.6}$$

Proposition 7.2. *Let $\varepsilon > 0$ be sufficiently small. Then, $\varepsilon^4 C' \leq d_{i,\varepsilon,\Omega} \leq \frac{\varepsilon^4}{4\mu_i}\mathcal{S}^2 - \varepsilon^4 C$ for both $i = 1, 2$. Moreover, there exists $\widetilde{U}_{i,\varepsilon} \in \mathcal{M}_{i,\varepsilon,\Omega}$ such that $\mathcal{E}_{i,\varepsilon,\Omega}(\widetilde{U}_{i,\varepsilon}) = d_{i,\varepsilon,\Omega}$, which is also a solution of the following equation:*

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_i u = \mu_i u^3 + \alpha_i u^{p-1} & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, & u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_{i,\varepsilon}$}$$

$i = 1, 2$, Moreover, $\widetilde{U}_{i,\varepsilon}(\varepsilon y + q_i^\varepsilon) \to v_i^0$ strongly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$ and $\{\widetilde{U}_{i,\varepsilon}\}$ is uniformly bounded in $L^\infty(\Omega)$, where q_i^ε is the maximum point of $\widetilde{U}_{i,\varepsilon}$ and v_i^0 is a least-energy solution of $(\mathcal{P}_i)$.

Proof. By the results in [8], $\varepsilon^4 C' \leq d_{i,\varepsilon,\Omega} \leq \frac{\varepsilon^4}{4\mu_i}\mathcal{S}^2 - \varepsilon^4 C$ for both $i = 1, 2$. Regarding the remaining results, we believe that they exist but we can not find the references; thus, we will sketch their proofs here. We only present the proof for $(\mathcal{P}_{1,\varepsilon})$ since that of $(\mathcal{P}_{2,\varepsilon})$ is similar. Once again, by the result in [8], $d_{1,\varepsilon,\Omega} < \frac{\varepsilon^4}{4\mu_1}\mathcal{S}^2 - \varepsilon^4 C$ for sufficiently small $\varepsilon > 0$. Since $p > 2$, $\mathcal{M}_{i,\varepsilon,\Omega}$ is a natural constraint. Hence, by applying the concentration-compactness principle in a standard way, we can show that $(\mathcal{P}_{i,\varepsilon})$ has a least-energy solution $\widetilde{U}_{i,\varepsilon}$ that satisfies $\widetilde{U}_{i,\varepsilon} \in \mathcal{M}_{i,\varepsilon,\Omega}$ and $\mathcal{E}_{i,\varepsilon,\Omega}(\widetilde{U}_{i,\varepsilon}) = d_{i,\varepsilon,\Omega}$. Now, let us consider the functions $\widetilde{U}_{i,\varepsilon}(\varepsilon y + q_i^\varepsilon)$. By a similar argument to that used for Proposition 3.2, we can show that the only solution of the following equation is $u \equiv 0$:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \alpha_1 u^{p-1} & \text{in } \mathbb{R}^4_+, \\ u \geq 0 \quad \text{in } \mathbb{R}^4_+, & u = 0 \quad \text{on } \partial\mathbb{R}^4_+. \end{cases}$$

Thus, if $\Omega^*_{i,\varepsilon} \to \mathbb{R}^4_+$ as $\varepsilon \to 0^+$, then by a similar argument to that used for Lemma 4.2, we can obtain a contradiction since $d_{1,\varepsilon,\Omega} < \frac{\varepsilon^4}{4\mu_1}\mathcal{S}^2 - \varepsilon^4 C$ for $\varepsilon > 0$ sufficiently small. Hence, $\Omega^*_{i,\varepsilon} \to \mathbb{R}^4$ as $\varepsilon \to 0^+$. Now, by the result in [8] and a similar argument to that used in Case. 1 of the proof to Proposition 4.1, we can show that $\widetilde{U}_{i,\varepsilon}(\varepsilon y + q_i^\varepsilon) \to v_i^0$ strongly in $\mathcal{H}_{i,\mathbb{R}^4}$ as $\varepsilon \to 0^+$, where v_i^0 is a least-energy solution of $(\mathcal{P}_i)$. The uniform boundedness of $\{\widetilde{U}_{i,\varepsilon}\}$ in $L^\infty(\Omega)$ can be obtained via standard elliptic estimates (cf. [8, 10]). □

Let

$$\mathcal{I}_\varepsilon(\vec{\mathbf{u}}) = \sum_{i=1}^{2}\left(\frac{\varepsilon^2}{2}\|\nabla u_i\|^2_{L^2(\mathbb{R}^4)} - \frac{\mu_i}{4}\|u_i\|^4_{L^4(\mathbb{R}^4)}\right) - \frac{\beta}{2}\|u_1u_2\|^2_{L^2(\mathbb{R}^4)}. \tag{7.7}$$

Then $\mathcal{I}_\varepsilon$ is of class C^2 in $\mathcal{D} = D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4)$. Set $A_\varepsilon = \inf_{\mathcal{V}_\varepsilon} \mathcal{I}_\varepsilon(\vec{\mathbf{u}})$, where

$$\mathcal{V}_\varepsilon = \{\vec{\mathbf{u}} \in \widetilde{\mathcal{D}} \mid \mathcal{I}'_\varepsilon(\vec{\mathbf{u}})\vec{\mathbf{u}}_1 = \mathcal{I}'_\varepsilon(\vec{\mathbf{u}})\vec{\mathbf{u}}_2 = 0\} \tag{7.8}$$

with $\widetilde{\mathcal{D}} = (D^{1,2}(\mathbb{R}^4)\backslash\{0\}) \times (D^{1,2}(\mathbb{R}^4)\backslash\{0\})$.

Proposition 7.3. *$A_\varepsilon = \varepsilon^4 A_1$ holds. Moreover, $A_1 = \frac{1}{4\mu_1}\mathcal{S}^2 + \frac{1}{4\mu_2}\mathcal{S}^2$ for $\beta < 0$ and $A_1 = \frac{k_1+k_2}{4}\mathcal{S}^2$ for $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$ with k_1, k_2 satisfying*

$$\begin{cases} \mu_1 k_1 + \beta k_2 = 1, \\ \mu_2 k_2 + \beta k_1 = 1. \end{cases} \tag{7.9}$$

Proof. See the results in [10]. □

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