Ground-state solutions of a two-component elliptic system in $\mathbb{R}^4$ with the Sobolev critical exponent*

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Abstract: In this paper, we continue our study in [17] on the following elliptic system:

$$\begin{cases} -\varepsilon^2\Delta u_1 + \lambda_1 u_1 = \alpha_1 u_1^{p-1} + \mu_1 u_1^3 + \beta u_2^2 u_1 & \text{in } \Omega, \\ -\varepsilon^2\Delta u_2 + \lambda_2 u_2 = \alpha_2 u_2^{p-1} + \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \quad u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ are constants, $\lambda_1, \lambda_2, \varepsilon > 0$ are parameters and $2 < p < 2^* = 4$. By further employing some canonical rescalings, we prove by variational methods that the above elliptic system has a ground-state solution for *all* $\beta \geq -\sqrt{\mu_1\mu_2}$ if ε and $\min\{\lambda_1, \lambda_2\}$ both sufficiently small. We also observe that the ground-state solution will *vanish* with the rate $(\max\{\lambda_1, \lambda_2\})^{\frac{1}{p-2}}$ if both $\varepsilon \to 0$ and $\lambda_1, \lambda_2 \to 0$. These results, together with that in [1, 17], suggest that the subcritical terms $\alpha_1 u_1^{p-1}, \alpha_2 u_2^{p-1}$ have strong effects on the structure of the ground-state solutions of the above system.

Keywords: Elliptic system; Ground-state solution; Variational method; Sobolev critical exponent; Asymptotic property.

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1 Introduction

In this paper, we continue our study in [17] on the following elliptic system:

$$\begin{cases} -\varepsilon^2\Delta u_1 + \lambda_1 u_1 = \alpha_1 u_1^{p-1} + \mu_1 u_1^3 + \beta u_2^2 u_1 & \text{in } \Omega, \\ -\varepsilon^2\Delta u_2 + \lambda_2 u_2 = \alpha_2 u_2^{p-1} + \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \quad u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^4$ is a bounded domain, $\mu_i, \alpha_i > 0$ and $\beta \neq 0$ are constants, $\lambda_i, \varepsilon > 0$ are parameters and $2 < p < 2^* = 4$.

System (1.1) is critical since the cubic terms u_1^3, u_2^3 and the coupling terms $u_2^2 u_1, u_1^2 u_2$ are all of critical growth in the sense of the Sobolev embedding. Such systems have been studied extensively in the literature, see, for example, [3–5, 11–17] and the references therein. By applying the Pohozaev identity, we can see that (1.1) has *no* solution when Ω is star-shaped for $\lambda_1, \lambda_2 > 0$ and $\alpha_1 = \alpha_2 = 0$. In the very recent work [17], by using variational methods, we proved the existence of one spiked solution to (1.1) for $\varepsilon > 0$ sufficiently small, under one of the following three cases:

(1) $\beta > 0$ sufficiently large.

(2) $-\sqrt{\mu_1\mu_2} < \beta < 0$ or $\beta > 0$ sufficiently small.

(3) $\beta \leq -\sqrt{\mu_1\mu_2}$ and $\alpha_1, \alpha_2 > 0$ sufficiently small.

Moreover, this solution is also a ground-state solution that has the same concentration behaviors of the ones in [9] as $\varepsilon \to 0$. Our results suggest that the appearances of the subcritical terms $\alpha_1 u_1^{p-1}, \alpha_2 u_2^{p-1}$ do affect the structure of the solutions of (1.1).

On the other hand, it is well-known that (1.1) with $\alpha_1 = \alpha_2 = 0$ has *no* solutions in one of the following two cases:

(a) $\lambda_1 \leq \lambda_2$ and $\mu_2 \leq \beta \leq \mu_1$ with one strictly inequality holding.

(b) $\lambda_2 \leq \lambda_1$ and $\mu_1 \leq \beta \leq \mu_2$ with one strictly inequality holding.

Based on our very recent work [17], it is natural to ask that *can the subcritical terms* $\alpha_1 u_1^{p-1}, \alpha_2 u_2^{p-1}$ *affect the structure of the solutions of* (1.1) *strongly enough such that* (1.1) *with* $\alpha_1, \alpha_2 > 0$ *has a solution in the above two cases* (a) *and* (b)*?* Since to the best of our knowledge, this question has not been studied yet in the literature, the main purpose of this paper is to investigate this natural question.

Let us give some words about our strategies in studying the above question. We notice that in the recent work [10], the asymptotic properties of the unique ground-state solution of the following equation,

$$\begin{cases} -\Delta u + \varepsilon u = |u|^{p-2}u - |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), \end{cases}$$

as $\varepsilon \to 0^+$, has been studied by variational arguments, where $2 < p < q$. It has been observed by employing some canonical rescalings, which are associated with the lowest order nonlinear term $|u|^{p-2}u$, that the asymptotic properties depends strongly on the relation between the order p and critical Sobolev exponent 2^*. Now, motivated by [10], if we introduce the following canonical rescaling,

$$v_i(x) = \left(\frac{1}{\min\{\lambda_1, \lambda_2\}}\right)^{\frac{1}{p-2}} u_i\left(x_0 + \frac{\varepsilon}{\sqrt{\min\{\lambda_1, \lambda_2\}}} x\right),$$

to (1.1), then we will have the following system:

$$\begin{cases} -\Delta u_1 + \dfrac{\lambda_1}{\min\{\lambda_1,\lambda_2\}} u_1 = \alpha_1 u_1^{p-1} + (\min\{\lambda_1,\lambda_2\})^{\frac{4-p}{p-2}} (\mu_1 u_1^3 + \beta u_2^2 u_1) & \text{in } \breve{\Omega}_\varepsilon, \\ -\Delta u_2 + \dfrac{\lambda_2}{\min\{\lambda_1,\lambda_2\}} u_2 = \alpha_2 u_2^{p-1} + (\min\{\lambda_1,\lambda_2\})^{\frac{4-p}{p-2}} (\mu_2 u_2^3 + \beta u_1^2 u_2) & \text{in } \breve{\Omega}_\varepsilon, \\ u_1, u_2 > 0 \quad \text{in } \breve{\Omega}_\varepsilon, \quad u_1 = u_2 = 0 \quad \text{on } \partial\breve{\Omega}_\varepsilon, \end{cases}$$

where $x_0 \in \Omega$ and

$$\breve{\Omega}_\varepsilon = \{y \in \mathbb{R}^4 \mid \frac{\varepsilon}{\sqrt{\min\{\lambda_1,\lambda_2\}}} y + x_0 \in \Omega\}.$$

As in [10], by letting $\min\{\lambda_1,\lambda_2\}$ sufficiently small,the coupled terms $\beta u_2^2 u_1, \beta u_2^2 u_1$ and the critical terms $\mu_1 u_1^3, , \mu_2 u_2^3$ can be regard as small perturbations, which means that the lowest order terms $\alpha_1 u_1^{p-1}, \alpha_2 u_2^{p-1}$ are the main terms in the above system. Thus, we can find solutions in the cases (a) and (b) under some further assumptions. If we go further in this direction by employing another canonical rescaling,

$$\tilde{v}_i(x) = \left(\frac{1}{\max\{\lambda_1,\lambda_2\}}\right)^{\frac{1}{p-2}} u_i\left(x_0 + \frac{\varepsilon}{\sqrt{\max\{\lambda_1,\lambda_2\}}} x\right),$$

to (1.1), then we will have the following system:

$$\begin{cases} -\Delta u_1 + \dfrac{\lambda_1}{\max\{\lambda_1,\lambda_2\}} u_1 = \alpha_1 u_1^{p-1} + (\max\{\lambda_1,\lambda_2\})^{\frac{4-p}{p-2}} (\mu_1 u_1^3 + \beta u_2^2 u_1) & \text{in } \tilde{\Omega}_\varepsilon, \\ -\Delta u_2 + \dfrac{\lambda_2}{\max\{\lambda_1,\lambda_2\}} u_2 = \alpha_2 u_2^{p-1} + (\max\{\lambda_1,\lambda_2\})^{\frac{4-p}{p-2}} (\mu_2 u_2^3 + \beta u_1^2 u_2) & \text{in } \tilde{\Omega}_\varepsilon, \\ u_1, u_2 > 0 \quad \text{in } \tilde{\Omega}_\varepsilon, \quad u_1 = u_2 = 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon, \end{cases}$$

where $x_0 \in \Omega$ and

$$\tilde{\Omega}_\varepsilon = \{y \in \mathbb{R}^4 \mid \frac{\varepsilon}{\sqrt{\max\{\lambda_1,\lambda_2\}}} y + x_0 \in \Omega\}.$$

By letting $\varepsilon \to 0$ first and $\max\{\lambda_1,\lambda_2\} \to 0$ next, we can obtain the scalar field equations

$$\begin{cases} -\Delta u_1 + \tilde{\lambda}_1 u_1 = \alpha_1 u_1^{p-1} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \tilde{\lambda}_2 u_2 = \alpha_2 u_2^{p-1} & \text{in } \mathbb{R}^N, \\ u_1, u_2 > 0 \quad \text{in } \mathbb{R}^N, \quad u_1, u_2 \to 0 \quad \text{as } |x| \to +\infty, \end{cases}$$

where $\tilde{\lambda}_i = \lim_{\max\{\lambda_1,\lambda_2\}\to 0} \frac{\lambda_i}{\max\{\lambda_1,\lambda_2\}}$. The above formal analysis yields that $\tilde{v}_i(x)$ will converge to the unique ground-state solution of the above scalar field equation and u_i will vanish in passing to the limit. These properties seems to be quite different from those of $\varepsilon \to 0$ or $\lambda_1, \lambda_2 \to 0$, only (cf. [4, 9, 13, 17]).

Let us give some necessary notations before we state our main results. Let $\mathcal{H}_{\lambda_i,\varepsilon,\Omega}$ be the Hilbert space of $H_0^1(\Omega)$ equipped with the inner product

$$\langle u, v\rangle_{\lambda_i,\varepsilon,\Omega} = \int_\Omega \varepsilon^2 \nabla u \nabla v + \lambda_i uv dx.$$

For $i = 1, 2$, since $\lambda_i > 0$ and $\varepsilon > 0$, $\mathcal{H}_{\lambda_i,\varepsilon,\Omega}$ is a Hilbert space and the corresponding norm is given by $\|u\|_{\lambda_i,\varepsilon,\Omega} = \langle u, u\rangle^{\frac{1}{2}}_{\lambda_i,\varepsilon,\Omega}$. Set $\mathcal{H}_{\varepsilon,\Omega} = \mathcal{H}_{\lambda_1,\varepsilon,\Omega} \times \mathcal{H}_{\lambda_2,\varepsilon,\Omega}$. Then $\mathcal{H}_{\varepsilon,\Omega}$ is a Hilbert space with the inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle_{\varepsilon,\Omega} = \sum_{i=1}^{2} \langle u_i, v_i\rangle_{\lambda_i,\varepsilon,\Omega}.$$

The corresponding norm is given by $\|\mathbf{u}\|_{\varepsilon,\Omega} = \langle \mathbf{u}, \mathbf{u}\rangle^{\frac{1}{2}}_{\varepsilon,\Omega}$. Here, u_i, v_i are the ith component of $\mathbf{u}, \mathbf{v}$, respectively. Define a functional in $\mathcal{H}_{\varepsilon,\Omega}$ as follows:

$$\mathcal{J}_{\varepsilon,\Omega}(\mathbf{u}) = \sum_{i=1}^{2}(\frac{1}{2}\|u_i\|^2_{\lambda_i,\varepsilon,\Omega} - \frac{\alpha_i}{p}\|u_i\|^p_{\Omega,p}) - \frac{1}{4}\sum_{i=1}^{2}\mu_i\|u_i\|^4_{\Omega,4} - \frac{\beta}{2}\|u_1^2u_2^2\|_{\Omega,1}.$$

Here, $\|\cdot\|_{\Omega,p}$ is the standard norm in $L^p(\Omega)$ for all $p \geq 1$ which is given by $\|u\|_{\Omega,p} = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$.

Definition 1.1. *The vector* $(u_1, u_2) = \mathbf{u} \in \mathcal{H}_{\varepsilon,\Omega}$ *is called a nontrivial critical point of* $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{v})$ *if* $\mathcal{J}'_{\varepsilon,\Omega}(\mathbf{u}) = 0$ *in* $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ *with* $u_1 \not\equiv 0$ *and* $u_2 \not\equiv 0$. $\mathbf{u} \in \mathcal{H}_{\varepsilon,\Omega}$ *is called a semi-trivial critical point of* $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{v})$ *if* $\mathcal{J}'_{\varepsilon,\Omega}(\mathbf{u}) = 0$ *in* $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ *with* $u_1 \not\equiv 0$ *or* $u_2 \not\equiv 0$. *Here,* $\mathcal{J}'_{\varepsilon,\Omega}(\mathbf{u})$ *is the Fréchet derivative of* $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{u})$ *and* $\mathcal{H}^{-1}_{\varepsilon,\Omega}$ *is the dual space of* $\mathcal{H}_{\varepsilon,\Omega}$. $\mathbf{u} \in \mathcal{H}_{\varepsilon,\Omega}$ *is called a positive critical point of* $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{v})$ *if* $\mathbf{u}$ *is a nontrivial critical point and* $u_i > 0$ *for both* $i = 1, 2$.

Clearly, $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{u})$ is of class C^2 in $\mathcal{H}_{\varepsilon,\Omega}$ and the positive critical points of $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{u})$ are the solutions of (1.1).

Definition 1.2. $\mathbf{u}$ *is called a ground-state solution of* (1.1) *if* u *is a nontrivial solution of* (1.1) *and* $\mathcal{J}_{\varepsilon,\Omega}(\mathbf{u}) \leq \mathcal{J}_{\varepsilon,\Omega}(\mathbf{v})$ *for any nontrivial solution* $\mathbf{v}$.

Now, our main results can be stated as follows.

Theorem 1.1. *Let* $\beta \geq -\sqrt{\mu_1\mu_2}$ *and* $\lambda_i, \alpha_i, \mu_i > 0$. *Let* $D_0 > 0$ *be an absolute constant. Then we have the following.*

(1) *There exists* $\lambda_* > 0$ *only dependent on* D_0, μ_1, μ_2 *and* β *such that* (1.1) *has a ground-state solution* $(u_{1,\varepsilon}, u_{2,\varepsilon})$ *for* $0 < \min\{\lambda_1, \lambda_2\} < \lambda_*$ *and* $0 < \varepsilon < D_0\sqrt{\min\{\lambda_1, \lambda_2\}}$.

(2) *Let* $\varepsilon_n, \lambda_1^n, \lambda_2^n \to 0^+$ *and* (u_1^n, u_2^n) *be the related solution obtained in* (1). *Suppose* $\mathcal{A} = \lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}} < +\infty$, *then*

$$u_i^n(x_i^n) = O((\max\{\lambda_1^n, \lambda_2^n\})^{\frac{1}{p-2}})$$

and

$$v_i^n(x) = \left(\frac{1}{\max\{\lambda_1^n, \lambda_2^n\}}\right)^{\frac{1}{p-2}} u_i^n\left(x_i^n + \frac{\varepsilon_n}{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}}x\right)$$

converges to some v_i strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$, where v_i is the solution of the following equation

$$\begin{cases} -\Delta u + \lambda_i^* u = \alpha_i |u|^{p-2}u & \text{in } \Omega_i^*, \\ u \in H_0^1(\Omega_i^*). \end{cases} \tag{1.2}$$

Here x_i^n is, respectively, the maximum point of u_i^n, Ω_i^* is the limit of

$$\Omega_{i,n} = \{y \in \mathbb{R}^4 \mid \frac{\varepsilon_n}{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}}y + x_i^n \in \Omega\}$$

as $n \to \infty$ and $\lambda_i^* = \lim_{n\to\infty} \frac{\lambda_i^n}{\max\{\lambda_1^n, \lambda_2^n\}}$. Moreover,

- (i) *If $\mathcal{A} = \lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}} > 0$, then Ω_i^* are bounded and v_i are a ground-state solution of* (1.2)*, respectively.*
- (ii) *If $\mathcal{A} = \lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}} = 0$, then $\Omega_i^* = \mathbb{R}^4$ and v_i is the unique radially ground-state solution of* (1.2) *for $\lambda_i^* > 0$ while $v_i \equiv 0$ for $\lambda_i^* = 0$. Furthermore, $\frac{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}|x_1^n - x_2^n|}{\varepsilon_n} \to 0$ as $n \to \infty$.*

Remark 1.1. (a) *By introducing the canonical re-scaling given by* (2.2)*, we can re-scale system* (1.1) *into* (2.3)*. In such system, the cubic nonlinearities and the coupled term can be treated as the perturbation terms for λ_1 small enough and $\varepsilon \ll \sqrt{\lambda_1}$. Here, without loss of generality, we assume $\lambda_1 = \min\{\lambda_1, \lambda_2\}$. Thus, under this idea, we can prove by the method of the Nehari manifold that system* (1.1) *has a ground-state solution for all $\beta \geq -\sqrt{\mu_1\mu_2}$ under the conditions $0 < \lambda_1 < \lambda_*$ and $0 < \varepsilon < D_0\sqrt{\lambda_1}$, which is stated by* (1) *of Theorem 1.1.*

(b) (2) *of Theorem 1.1 yields that the scalar field equation* (1.2) *is the limit equation of system* (1.1) *if both $\varepsilon_n \to 0$ and $max\{\lambda_1^n, \lambda_2^n\} \to 0$ with*

$$\mathcal{A} = \lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\max\{\lambda_1^n, \lambda_2^n\}}} < +\infty.$$

Moreover, the ground-state solutions will vanish at the rate of $(\max\{\lambda_1^n, \lambda_2^n\})^{\frac{1}{p-2}}$. To the best of our knowledge, this is the first result about such asymptotic property of system (1.1)*.*

This paper is organized as follows. In Section 2, we obtain an existence result of (1.1) by applying the method of the Nehari manifold. In Section 3, we study the concentration behavior of the ground state solution as $\varepsilon \to 0^+$ and $\lambda_1, \lambda_2 \to 0^+$.

2 The existence results

2.1 Some preliminaries

In this section,we are interesting in finding a ground state solution of (1.1) for all $\beta \geq -\sqrt{\mu_1\mu_2}$. Without loss of generality, we assume that $0 \in \Omega$, $\lambda_1 \leq \lambda_2$ and $D_0 = 1$. Let

$$\Omega_\varepsilon = \{x \in \mathbb{R}^4 \mid \varepsilon x \in \Omega\}.$$

Then it is easy to see that $\Omega_\varepsilon \to \mathbb{R}^4$ as $\varepsilon \to 0^+$. Moreover, it is easy to see that $\mathbf{u} = (u_1, u_2)$ is a solution of (1.1) if and only if $\widehat{\mathbf{u}} = (\widehat{u}_1, \widehat{u}_2)$ with $\widehat{u}_i(x) = u_i(\varepsilon x)$ is a solution of the following system

$$\begin{cases} -\Delta\widehat{u}_1 + \lambda_1\widehat{u}_1 = \mu_1\widehat{u}_1^3 + \alpha_1\widehat{u}_1^{p-1} + \beta\widehat{u}_2^2\widehat{u}_1 & \text{in } \Omega_\varepsilon, \\ -\Delta\widehat{u}_2 + \lambda_2\widehat{u}_2 = \mu_2\widehat{u}_2^3 + \alpha_2\widehat{u}_2^{p-1} + \beta\widehat{u}_1^2\widehat{u}_2 & \text{in } \Omega_\varepsilon, \\ \widehat{u}_1, \widehat{u}_2 > 0 \quad \text{in } \Omega, \quad \widehat{u}_1 = \widehat{u}_2 = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.1}$$

We also define a canonical rescaling of $\widehat{\mathbf{u}}$ and denote it by $\widetilde{\mathbf{u}}$. That is,

$$\widetilde{u}_i(x) = \left(\frac{1}{\lambda_1}\right)^{\frac{1}{p-2}} \widehat{u}_i\left(\frac{x}{\sqrt{\lambda_1}}\right). \tag{2.2}$$

Clearly, for every $u_i \in \mathcal{H}_{\lambda_i,\varepsilon,\Omega}$, we have $\widetilde{u}_i \in \mathcal{H}_{\frac{\lambda_i}{\lambda_1},1,\widetilde{\Omega}_\varepsilon}$, where

$$\widetilde{\Omega}_\varepsilon = \{x \in \mathbb{R}^4 \mid \frac{\varepsilon}{\sqrt{\lambda_1}}x \in \Omega\}.$$

Moreover, $\widehat{\mathbf{u}}$ is a solution of (2.1) if and only if $\widetilde{\mathbf{u}}$ is a solution of the following system

$$\begin{cases} -\Delta\widetilde{u}_1 + \widetilde{u}_1 = \alpha_1\widetilde{u}_1^{p-1} + (\lambda_1)^{\frac{4-p}{p-2}}(\mu_1\widetilde{u}_1^3 + \beta\widetilde{u}_2^2\widetilde{u}_1) & \text{in } \widetilde{\Omega}_\varepsilon, \\ -\Delta\widetilde{u}_2 + \dfrac{\lambda_2}{\lambda_1}\widetilde{u}_2 = \alpha_2\widetilde{u}_2^{p-1} + (\lambda_1)^{\frac{4-p}{p-2}}(\mu_2\widetilde{u}_2^3 + \beta\widetilde{u}_1^2\widetilde{u}_2) & \text{in } \widetilde{\Omega}_\varepsilon, \\ \widetilde{u}_1, \widetilde{u}_2 > 0 \quad \text{in } \widetilde{\Omega}_\varepsilon, \quad \widetilde{u}_1 = \widetilde{u}_2 = 0 & \text{on } \widetilde{\Omega}_\varepsilon, \end{cases} \tag{2.3}$$

By a direct calculation, we have

$$\int_{\widetilde{\Omega}_\varepsilon} |\nabla\widetilde{u}_i|^2 dx = (\lambda_1)^{1-\frac{2}{p-2}} \int_{\Omega_\varepsilon} |\nabla\widehat{u}_i|^2 dx = (\lambda_1)^{1-\frac{2}{p-2}} \varepsilon^{-2} \int_\Omega |\nabla u_i|^2 dx \tag{2.4}$$

and

$$\int_{\widetilde{\Omega}_\varepsilon} |\widetilde{u}_i|^r dx = (\lambda_1)^{2-\frac{r}{p-2}} \int_{\Omega_\varepsilon} |\widehat{u}_i|^r dx = (\lambda_1)^{2-\frac{r}{p-2}} \varepsilon^{-4} \int_\Omega |u_i|^r dx \tag{2.5}$$

for both $i = 1, 2$ and all $2 \leq r \leq 4$. Let

$$\begin{aligned} \mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}) \quad &= \quad \sum_{i=1}^2 \left(\frac{1}{2}(\|\nabla\widetilde{u}_i\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i\|_{\widetilde{\Omega}_\varepsilon,2}^2) - \frac{\alpha_i}{p}\|\widetilde{u}_i\|_{\widetilde{\Omega}_\varepsilon,p}^p\right) \\ &\quad -(\lambda_1)^{\frac{4-p}{p-2}}\left(\frac{1}{4}\sum_{i=1}^2 \mu_i\|\widetilde{u}_i\|_{\widetilde{\Omega}_\varepsilon,4}^4 + \frac{\beta}{2}\|\widetilde{u}_1^2\widetilde{u}_2^2\|_{\widetilde{\Omega}_\varepsilon,1}\right). \end{aligned}$$

Then it is easy to see that $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ is of class C^2 in $\mathcal{H}_{*,\widetilde{\Omega}_\varepsilon} = \mathcal{H}_{\frac{\lambda_1}{\lambda_1},1,\widetilde{\Omega}_\varepsilon} \times \mathcal{H}_{\frac{\lambda_2}{\lambda_1},1,\widetilde{\Omega}_\varepsilon}$ and positive critical points of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ are equivalent to solutions of (2.3). Moreover, by (2.4) and (2.5), we have

$$\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}) = (\lambda_1)^{1-\frac{2}{p-2}}\mathcal{J}_{1,\Omega_\varepsilon}(\widehat{\mathbf{u}}) = (\lambda_1)^{1-\frac{2}{p-2}}\varepsilon^{-4}\mathcal{J}_{\varepsilon,\Omega}(\mathbf{u}). \tag{2.6}$$

Definition 2.1. $\widetilde{\mathbf{u}}$ *is called a ground state solution of* (2.3) *if* $\widetilde{\mathbf{u}}$ *is a nontrivial solution of* (2.3) *and* $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}) \leq \mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{v}})$ *for any nontrivial solution* $\widetilde{\mathbf{v}}$.

By (2.6) and Definition 1.2, if $\widetilde{\mathbf{w}}$ is a ground state solution of (2.3), then $\mathbf{w}$ is a ground state solution of (1.1), where $\mathbf{w} = (w_1, w_2)$ with $w_i(x) = \lambda_1^{\frac{1}{p-2}}\widetilde{w}_i(\frac{\sqrt{\lambda_1}}{\varepsilon}x)$.

2.2 The Nehari manifold

Define the Nehari manifold of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ as follows:

$$\mathcal{N}_{\widetilde{\Omega}_\varepsilon} = \{\widetilde{\mathbf{u}} \in \widetilde{\mathcal{H}}_{*,\widetilde{\Omega}_\varepsilon} \mid \mathcal{E}'_{\Omega^*_{j,n}}(\mathbf{u})\mathbf{u}^k = 0, k = 1, 2\},$$

where $\widetilde{\mathcal{H}}_{*,\widetilde{\Omega}_\varepsilon} = (\mathcal{H}_{\frac{\lambda_1}{\lambda_1},1,\widetilde{\Omega}_\varepsilon}\backslash\{0\}) \times (\mathcal{H}_{\frac{\lambda_2}{\lambda_1},1,\widetilde{\Omega}_\varepsilon}\backslash\{0\})$, $\mathbf{u}^1 = (u_1, 0)$ and $\mathbf{u}^2 = (0, u_2)$. Clearly, all nontrivial solutions of (2.3) are contained in $\mathcal{N}_{\widetilde{\Omega}_\varepsilon}$. Thus, if $\widetilde{\mathbf{w}} \in \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ attains the minimum of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ in $\mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ and $\widetilde{\mathbf{w}}$ is also a positive critical point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$, then $\widetilde{\mathbf{w}}$ is a ground state solution of (2.3). Here, we say $\widetilde{\mathbf{w}}$ is a positive critical point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$, if $\widetilde{\mathbf{w}}$ is a critical point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ and $\widetilde{\mathbf{w}} = (\widetilde{w}_1, \widetilde{w}_2)$ with $\widetilde{w}_i > 0$.

Lemma 2.1. *Let* $\beta \geq -\sqrt{\mu_1\mu_2}$. *Then* $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}) \geq 0$ *on* $\mathcal{N}_{\widetilde{\Omega}_\varepsilon}$.

Proof. Since $\beta \geq -\sqrt{\mu_1\mu_2}$, the functional

$$\sum_{i=1}^{2}\mu_i\|\widetilde{u}_i^n\|^4_{\widetilde{\Omega}_\varepsilon,4} + 2\beta\|(\widetilde{u}_i^n)^2(\widetilde{u}_i^n)^2\|_{\widetilde{\Omega}_\varepsilon,1}$$

is nonnegative definite in $\mathcal{H}_{*,\widetilde{\Omega}_\varepsilon}$ by the Hölder inequality. Thus, the conclusion follows immediately from $2 < p < 4$ and a standard calculation. □

By Lemma 2.1,

$$c_\varepsilon = \inf_{\mathcal{N}_{\widetilde{\Omega}_\varepsilon}}\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}) \tag{2.7}$$

is well defined.

Lemma 2.2. *Let* $\beta \geq -\sqrt{\mu_1\mu_2}$. *Then for* $0 < \varepsilon < \sqrt{\lambda_1}$, *there exists* $C_0 > 0$ *independent of* ε *and* λ_1 *such that* $c_\varepsilon \leq C_0$.

Proof. Since $\frac{\varepsilon}{\sqrt{\lambda_1}} \leq 1$, we can see that $\mathbb{B}_{\frac{C^*}{2}}(0) \subset \widetilde{\Omega}_\varepsilon$, where $C^* = \text{dist}(0, \partial\Omega)$. Let Ω_1 and Ω_2 be two bounded domains with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_i \subset \mathbb{B}_{\frac{C^*}{2}}(0)$ for both $i = 1, 2$. Then it is well known that the following equation

$$\begin{cases} -\Delta u + \lambda_2 u = \alpha_i|u|^{p-2}u & \text{in } \Omega_i, \\ u = 0 & \text{on } \partial\Omega_i \end{cases}$$

has a positive ground state solution u_i. Denote the energy level of u_i by C_i, then C_i is independent of ε and λ_1. Let $C_0 = C_1 + C_2$, then by $\Omega_1 \cap \Omega_2 = \emptyset$ and $\lambda_1 \le \lambda_2$, the conclusion immediately follows from (2.7). □

By the Ekeland's variational principle, there exists $\{\widetilde{\mathbf{u}}_n\} \subset \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ such that

(1) $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n) = c_\varepsilon + o_n(1)$,

(2) $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{v}}) \ge \mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n) - \frac{1}{n}\|\widetilde{\mathbf{v}} - \widetilde{\mathbf{u}}_n\|_{\varepsilon,\Omega}$ for all $\widetilde{\mathbf{v}} \in \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$.

Moreover, by setting $\widetilde{u}_i^n \equiv 0$ outside $\widetilde{\Omega}_\varepsilon$, we can regard $\widetilde{\mathbf{u}}_n = (\widetilde{u}_1^n, \widetilde{u}_2^n) \in D^{1,2}(\mathbb{R}^4) \times D^{1,2}(\mathbb{R}^4)$.

Lemma 2.3. *Let $\beta \ge -\sqrt{\mu_1\mu_2}$. Then there exists $\lambda_* > 0$ such that*

$$\|\nabla \widetilde{u}_i^n\|_{\mathbb{R}^4,2}^2 + \|\widetilde{u}_i^n\|_{\mathbb{R}^4,2}^2 \le C$$

with n large enough for $0 < \lambda_1 < \lambda_*$ and $0 < \varepsilon < \sqrt{\lambda_1}$, where $C > 0$ is a constant independent of ε and λ_1. Moreover, $\{\widetilde{\mathbf{u}}_n\} \subset \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ is also a (PS) sequence of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n)$ at the energy level c_ε.

Proof. By Lemma 2.2, we can see from $\beta \ge -\sqrt{\mu_1\mu_2}$ and $2 < p < 4$ that

$$\begin{aligned} C_0 + o_n(1) &\ge \mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n) - \frac{1}{p}\mathcal{E}'_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n)\widetilde{\mathbf{u}}_n \\ &\ge \frac{p-2}{2p}\sum_{i=1}^{2}(\|\nabla \widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2). \end{aligned}$$

Thus, we have $\{\widetilde{u}_i^n\}$ is bounded in $H^1(\mathbb{R}^4)$ for n. Moreover, by $\lambda_1 \le \lambda_2$, we also have

$$\|\nabla \widetilde{u}_i^n\|_{\mathbb{R}^4,2}^2 + \|\widetilde{u}_i^n\|_{\mathbb{R}^4,2}^2 \le \frac{2p(C_0+1)}{p-2} \tag{2.8}$$

for n large enough, where $\frac{2p(C_0+1)}{p-2} > 0$ is a constant independent of ε and λ_1. It remains to show that $\{\widetilde{\mathbf{u}}_n\} \subset \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ is also a (PS) sequence of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n)$ at the energy level c_ε. Let $\widetilde{\mathbf{w}} \in \mathcal{H}_{*,\widetilde{\Omega}_\varepsilon}$. For every $n \in \mathbb{N}$, we consider the system $\mathbf{\Psi}_n(\mathbf{t}, l) = \mathbf{0}$, where $\mathbf{\Psi}_n(\mathbf{t}, l) = (\Psi_1^n(\mathbf{t}, l), \Psi_2^n(\mathbf{t}, l))$ with

$$\begin{aligned} \Psi_i^n(\mathbf{t}, l) &= \|\nabla(t_i\widetilde{u}_i^n + l\widetilde{w}_i)\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|t_i\widetilde{u}_i^n + l\widetilde{w}_i\|_{\widetilde{\Omega}_\varepsilon,2}^2 - \alpha_i\|t_i\widetilde{u}_i^n + l\widetilde{w}_i\|_{\widetilde{\Omega}_\varepsilon,p}^p \\ &\quad -\lambda_1^{\frac{4-p}{p-2}}(\mu_i\|t_i\widetilde{u}_i^n + l\widetilde{w}_i\|_{\widetilde{\Omega}_\varepsilon,4}^4 + \beta\|(t_1\widetilde{u}_1^n + l\widetilde{w}_1)^2(t_2\widetilde{u}_2^n + l\widetilde{w}_2)^2\|_{\widetilde{\Omega}_\varepsilon,1}). \end{aligned}$$

Clearly, $\mathbf{\Psi}(\mathbf{t}, l)$ is of C^1. Moreover, since $\{\widetilde{\mathbf{u}}_n\} \subset \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$, we also have that $\mathbf{\Psi}_n(\mathbf{1}, 0) = \mathbf{0}$. By a direct calculation, we have

$$\begin{aligned} \frac{\partial \Psi_i^n(\mathbf{1}, 0)}{\partial t_i} &= 2(\|\nabla \widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2) - p\alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p \\ &\quad -\lambda_1^{\frac{4-p}{p-2}}(4\mu_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,4}^4 + 2\beta\|(\widetilde{u}_1^n)^2(\widetilde{u}_2^n)^2\|_{\widetilde{\Omega}_\varepsilon,1}) \\ &= -(p-2)\alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p - 2\mu_i\lambda_1^{\frac{4-p}{p-2}}\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,4}^4 \end{aligned} \tag{2.9}$$

respectively for $i = 1, 2$ and

$$\begin{aligned}\frac{\partial \Psi_2^n(\mathbf{1}, 0)}{\partial t_1} &= \frac{\partial \Psi_1^n(\mathbf{1}, 0)}{\partial t_2} \\ &= -2\beta\lambda_1^{\frac{4-p}{p-2}} \|(\widetilde{u}_i^n)^2(\widetilde{u}_i^n)^2\|_{\widetilde{\Omega}_\varepsilon,1} \end{aligned} \tag{2.10}$$

Set

$$\Theta_n = (\theta_{ij}^n)_{i,j=1,2}$$

with $\theta_{ij}^n = \frac{\partial \Psi_i^n(\mathbf{1},0)}{\partial t_j}$. By (2.8), we can see from the Sobolev inequality that

$$\|\widetilde{u}_i^n\|_{\mathbb{R}^4,4}^2 \leq \frac{2p(C_0+1)}{(p-2)\mathcal{S}} \tag{2.11}$$

for n large enough, where $\mathcal{S}$ is best embedding constant from $H^1(\mathbb{R}^4) \to L^4(\mathbb{R}^4)$ defined by

$$\mathcal{S} = \inf\{\|\nabla u\|_{\mathbb{R}^4,2}^2 \mid u \in H^1(\mathbb{R}^4), \|u\|_{\mathbb{R}^4,4}^2 = 1\}.$$

(2.11) together with $\{\mathbf{u}_n\} \subset \mathcal{N}_{\varepsilon,\Omega}$ and the Hölder inequality, implies

$$\begin{aligned} &\|\nabla \widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 \\ = \ & \alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p + \lambda_1^{\frac{4-p}{p-2}}(\mu_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,4}^4 + \beta\|(\widetilde{u}_i^n)^2(\widetilde{u}_i^n)^2\|_{\widetilde{\Omega}_\varepsilon,1}) \\ \leq \ & \alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p + \lambda_1^{\frac{4-p}{p-2}}\frac{2p(C_0+1)}{(p-2)\mathcal{S}}(\mu_i + |\beta|)\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,4}^2 \end{aligned} \tag{2.12}$$

for n large enough. Let

$$\lambda_* = \left(\frac{p-2}{4p(C_0+1)(\mu_1+\mu_2+|\beta|)}\right)^{\frac{p-2}{4-p}}.$$

Then for $0 < \lambda_1 < \lambda_*$, we have from (2.12) that

$$\|\nabla \widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 \leq 2\alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p$$

for n large enough, which together with $\lambda_1 \leq \lambda_2$, implies

$$\alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p \geq \left(\frac{\mathcal{S}_p}{2}\right)^{\frac{p}{p-2}} \tag{2.13}$$

for n large enough with $0 < \lambda_1 < \lambda_*$. Here, $\mathcal{S}_p$ is best embedding constant from $H^1(\mathbb{R}^4) \to L^p(\mathbb{R}^4)$ defined by

$$\mathcal{S}_p = \inf\{\|\nabla u\|_{\mathbb{R}^4,2}^2 + \|u\|_{\mathbb{R}^4,2}^2 \mid u \in H^1(\mathbb{R}^4), \|u\|_{\mathbb{R}^4,p}^p = 1\}.$$

By $2 < p < 4$, we have from (2.9)–(2.10) that

$$\det(\Theta_n) \geq (p-2)^2\alpha_1\alpha_2\left(\frac{\mathcal{S}_p}{2}\right)^{\frac{2p}{p-2}} > 0 \tag{2.14}$$

for n large enough. Now, we can applying the implicit function theorem and the Taylor's expansion in a standard way (cf. [2]) to show that $\{\widetilde{\mathbf{u}}_n\} \subset \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ is also a (PS) sequence of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n)$ at the energy level c_ε for $0 < \lambda_1 < \lambda_*$. □

By Lemma 2.3, $\{\widetilde{u}_i^n\}$ is bounded in $H^1(\mathbb{R}^4)$ for n. Without loss of generality, we assume that $\widetilde{u}_i^n \rightharpoonup \widetilde{u}_i^*$ weakly in $H^1(\mathbb{R}^4)$ as $n \to \infty$.

Proposition 2.1. *Let $\beta \geq -\sqrt{\mu_1\mu_2}$, $0 < \lambda_1 < \lambda_*$ and $0 < \varepsilon < \sqrt{\lambda_1}$. Then there exists $\widetilde{\mathbf{U}} = (\widetilde{U}_1, \widetilde{U}_2)$ with $\widetilde{U}_i > 0$ in $\widetilde{\Omega}_\varepsilon$ for both $i = 1, 2$ such that $\widetilde{\mathbf{U}}$ is a critical point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ at the energy level c_ε.*

Proof. Clearly, $\widetilde{\mathbf{u}}_* = (\widetilde{u}_1^*, \widetilde{u}_2^*)$ is a critical point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$. Moreover, since $\widetilde{\Omega}_\varepsilon$ is bounded, we can see from the Sobolev embedding theorem and (2.13) that $\widetilde{u}_i^n \to \widetilde{u}_i^*$ strongly in $L^r(\widetilde{\Omega}_\varepsilon)$ for $1 \leq r < 4$ as $n \to \infty$ and $\widetilde{u}_i^* \not\equiv 0$ for both $i = 1, 2$. Thus, we must have $\widetilde{\mathbf{u}}_* \in \mathcal{N}_{\widetilde{\Omega}_\varepsilon}$. It follows that

$$\begin{aligned}
c_\varepsilon + o_n(1) &= \mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n) - \frac{1}{4}\mathcal{E}'_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_n)\widetilde{\mathbf{u}}_n \\
&= \frac{1}{4}\sum_{i=1}^{2}(\|\nabla\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,2}^2) - \frac{4-p}{4p}\sum_{i=1}^{2}\alpha_i\|\widetilde{u}_i^n\|_{\widetilde{\Omega}_\varepsilon,p}^p \\
&\geq \frac{1}{4}\sum_{i=1}^{2}\|\nabla\widetilde{u}_i^*\|_{\widetilde{\Omega}_\varepsilon,2}^2 + \frac{\lambda_i}{\lambda_1}\|\widetilde{u}_i^*\|_{\widetilde{\Omega}_\varepsilon,2}^2 - \frac{4-p}{4p}\sum_{i=1}^{2}\alpha_i\|\widetilde{u}_i^*\|_{\widetilde{\Omega}_\varepsilon,p}^p + o_n(1) \\
&= \mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_*) - \frac{1}{4}\mathcal{E}'_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_*)\widetilde{\mathbf{u}}_* + o_n(1) \\
&\geq c_\varepsilon + o_n(1).
\end{aligned}$$

Hence, $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}}_*) = c_\varepsilon$. That is, $\widetilde{\mathbf{u}}_*$ is a minimum point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ on $\mathcal{N}_{\widetilde{\Omega}_\varepsilon}$. Note that $\widetilde{\mathbf{U}} = (\widetilde{U}_1, \widetilde{U}_2)$ with $\widetilde{U}_i = |\widetilde{u}_i^*|$ is also a minimum point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ on $\mathcal{N}_{\widetilde{\Omega}_\varepsilon}$. Now, by a similar argument as used for (2.14), we can apply the method of Lagrange multipliers in a standard way (cf. [2]) to show that $\mathcal{N}_{\widetilde{\Omega}_\varepsilon}$ is a natural constraint in $\mathcal{H}_{*,\widetilde{\Omega}_\varepsilon}$. Therefore, by the maximum principle, $\widetilde{\mathbf{U}}$ is a critical point of $\mathcal{E}_{\widetilde{\Omega}_\varepsilon}(\widetilde{\mathbf{u}})$ at the energy level c_ε with $\widetilde{U}_i > 0$ for both $i = 1, 2$. □

3 Concentration behaviors

In this section, we are interesting in studying the concentration behavior of the ground state solution of (1.1) as $\varepsilon \to 0^+$ and $\lambda = (\lambda_1, \lambda_2) \to \mathbf{0}$. For this purpose, we take $\varepsilon_n \to 0^+$ and $\lambda_n = (\lambda_1^n, \lambda_2^n) \to \mathbf{0}$ as $n \to \infty$. Without loss of generality, we assume that $\varepsilon_n < 1$ and $\lambda_1^n \leq \lambda_2^n < 1$ are all decreasing for n. We also assume that $\lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\lambda_i^n}} = \mathcal{A}_i \in [0, +\infty]$. By the assumption $\lambda_1^n \leq \lambda_2^n$, we have $\mathcal{A}_2 \leq \mathcal{A}_1$. We are also interesting in studying the concentration behavior of the ground state solution of (1.1) as $\varepsilon \to 0^+$ and $\lambda_1 \to 0$. However, this case is similar to the situation of $\mathcal{A}_2 = 0$ if $\varepsilon \to 0^+$ and $\lambda = (\lambda_1, \lambda_2) \to \mathbf{0}$. Due to this reason, we only study the case $\varepsilon \to 0^+$ and $\lambda = (\lambda_1, \lambda_2) \to \mathbf{0}$ in what follows.

Proposition 3.1. *Let $\beta \geq -\sqrt{\mu_1\mu_2}$ and $\varepsilon_n \to 0^+$ and $\lambda_n = (\lambda_1^n, \lambda_2^n) \to \mathbf{0}$ as $n \to \infty$. If $\mathcal{A}_2 < +\infty$ then* (1.1) *has a ground state solution $\mathbf{u}_n$ for n large enough.*

Proof. For every n, we consider the following canonical rescaling of $\widehat{\mathbf{u}}$:

$$\overline{u}_i(x) = \left(\frac{1}{\lambda_2^n}\right)^{\frac{1}{p-2}} \widehat{u}_i\left(\frac{x}{\sqrt{\lambda_2^n}}\right), \tag{3.1}$$

where $\widehat{u}_i$ is given by (2.2). Define

$$\begin{aligned}\overline{\mathcal{E}}_{\widehat{\Omega}_n}(\overline{\mathbf{u}}) \quad = \quad & \sum_{i=1}^{2}\Big(\frac{1}{2}(\|\nabla\overline{u}_i\|_{\widehat{\Omega}_n,2}^2 + \frac{\lambda_i^n}{\lambda_2^n}\|\overline{u}_i\|_{\widehat{\Omega}_n,2}^2) - \frac{\alpha_i}{p}\|\overline{u}_i\|_{\widehat{\Omega}_n,p}^p)\Big) \\ & -(\lambda_2^n)^{\frac{4-p}{p-2}}\Big(\frac{1}{4}\sum_{i=1}^{2}\mu_i\|\overline{u}_i\|_{\widehat{\Omega}_n,4}^4 + \frac{\beta}{2}\|\overline{u}_1^2\overline{u}_2^2\|_{\widehat{\Omega}_n,1}\Big),\end{aligned}$$

where

$$\widehat{\Omega}_n = \{x \in \mathbb{R}^4 \mid \frac{\varepsilon_n}{\sqrt{\lambda_2^n}}x \in \Omega\}.$$

Then it is easy to see that $\overline{\mathcal{E}}_{\widehat{\Omega}_n}(\overline{\mathbf{u}})$ is of class C^2 in $\mathcal{H}_{**,\widehat{\Omega}_n} = \mathcal{H}_{\frac{\lambda_1^n}{\lambda_2^n},1,\widehat{\Omega}_n} \times \mathcal{H}_{\frac{\lambda_2^n}{\lambda_2^n},1,\widehat{\Omega}_n}$. Since $\lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\lambda_2^n}} = \mathcal{A}_2 < +\infty$, by using the canonical rescaling (3.1), we can see that $\mathbb{B}_{\frac{C^*}{2}}(0) \subset \widehat{\Omega}_n$ for all n large enough, where $C^* = \frac{\text{dist}(0,\partial\Omega)}{\mathcal{A}_2+1}$. Now, by a similar argument as used for Lemma 2.2, we can see that $\overline{c}_n \leq C_0$, where the up-bound C_0 is independent of n large enough. Here,

$$\overline{c}_n = \inf_{\overline{\mathcal{N}}_{\widehat{\Omega}_n}} \overline{\mathcal{E}}_{\widehat{\Omega}_n}(\overline{\mathbf{u}})$$

with

$$\overline{\mathcal{N}}_{\widehat{\Omega}_n} = \{\overline{\mathbf{u}} \in \overline{\mathcal{H}}_{**,\widehat{\Omega}_n} \mid \overline{\mathcal{E}}'_{\widehat{\Omega}_n}(\overline{\mathbf{u}})\overline{\mathbf{u}}^k = 0, k = 1, 2\}$$

and $\overline{\mathcal{H}}_{**,\widehat{\Omega}_n} = (\mathcal{H}_{\frac{\lambda_1^n}{\lambda_2^n},1,\widehat{\Omega}_n}\backslash\{0\}) \times (\mathcal{H}_{\frac{\lambda_2^n}{\lambda_2^n},1,\widehat{\Omega}_n}\backslash\{0\})$, $\overline{\mathbf{u}}^1 = (\overline{u}_1, 0)$ and $\overline{\mathbf{u}}^2 = (0, \overline{u}_2)$. Thus, by a similar argument as used for Lemma 2.3 and Proposition 2.1, $\overline{\mathcal{E}}_{\widehat{\Omega}_n}(\overline{\mathbf{u}})$ has a positive critical point at the energy level $\overline{c}_n$ for n large enough. It follows from (3.1) that (1.1) has a ground state solution $\mathbf{u}_n$ for n large enough. □

By the classical elliptic estimates (cf. [6]), we can see that $u_i^n \in C^2(\Omega)\cap C(\overline{\Omega})$ for $i=1,2$. Let x_i^n be the maximum points of u_i^n, respectively for $i=1,2$. Define

$$v_i^{1,n}(y) = u_i^n(\varepsilon_n y + x_1^n) \quad \text{and} \quad v_i^{2,n}(y) = u_i^n(\varepsilon_n y + x_2^n) \tag{3.2}$$

for $i=1,2$. Then $v_i^{1,n} \in H_0^1(\Omega_{1,n})$ and $v_i^{2,n} \in H_0^1(\Omega_{2,n})$ respectively satisfy

$$\begin{cases} -\Delta v_1^{j,n} + \lambda_1^n v_1^{j,n} = \mu_1 (v_1^{j,n})^3 + \alpha_1 (v_1^{j,n})^{p-1} + \beta (v_2^{j,n})^2 v_1^{j,n} & \text{in } \Omega_{j,n}, \\ -\Delta v_2^{j,n} + \lambda_2^n v_2^{j,n} = \mu_2 (v_2^{j,n})^3 + \alpha_2 (v_2^{j,n})^{p-1} + \beta (v_1^{j,n})^2 v_2^{j,n} & \text{in } \Omega_{j,n}, \\ v_1^{j,n}, v_2^{j,n} > 0 \quad \text{in } \Omega_{j,n}, \quad v_1^{j,n} = v_2^{j,n} = 0 \quad \text{on } \partial\Omega_{j,n}, \end{cases} \tag{3.3}$$

where

$$\Omega_{j,n} = \{y \in \mathbb{R}^4 \mid \varepsilon_n y + x_j^n \in \Omega\} \quad \text{for } j=1,2.$$

It is well known that $\mathbf{v}_{1,n} = (v_1^{1,n}, v_2^{1,n})$ and $\mathbf{v}_{2,n} = (v_1^{2,n}, v_1^{2,n})$ are two standard scaling vector functions, which is used to analyze the asymptotic behaviors of the solution $\mathbf{u}_n = (u_{1,n}, u_{2,n})$ and the locations of the spikes x_i^n as $n \to \infty$, usually. However, the system (3.3) which is solved by $\mathbf{v}_{j,n}$ is not good enough to understand such properties. Indeed, the formal limit system of (3.3) is the following one

$$\begin{cases} -\Delta v_1^j = \mu_1 (v_1^j)^3 + \alpha_1 (v_1^j)^{p-1} + \beta (v_2^j)^2 v_1^j & \text{in } \Omega_j, \\ -\Delta v_2^j = \mu_2 (v_2^j)^3 + \alpha_2 (v_2^j)^{p-1} + \beta (v_1^j)^2 v_2^j & \text{in } \Omega_j, \\ v_1^j, v_2^j > 0 \quad \text{in } \Omega_j, \quad v_1^j = v_2^j = 0 \quad \text{on } \partial\Omega_j, \end{cases}$$

where Ω_j is the limit of $\Omega_{j,n}$ as $n \to \infty$. Since Ω_j may be unbounded, subcritical terms $\alpha_1 \int_{\Omega_j} (v_1^j)^p$, $\alpha_2 \int_{\Omega_j} (v_2^j)^p$ maybe have no sense. In order to overcome this difficulty, the idea is to re-scale $\mathbf{v}_{j,n}$ such that the subcritical term can make sense after passing to the limit. Now, we re-scale $\mathbf{v}_{j,n}$ by the canonical rescaling:

$$\widetilde{v}_i^{j,n}(y) = \left(\frac{1}{\lambda_2^n}\right)^{\frac{1}{p-2}} v_i^{j,n}\left(\frac{y}{\sqrt{\lambda_2^n}}\right), \quad i,j=1,2. \tag{3.4}$$

Then the vector function $\widetilde{\mathbf{v}}_{j,n} = (\widetilde{v}_1^{j,n}, \widetilde{v}_2^{j,n})$ satisfies

$$\begin{cases} -\Delta \widetilde{v}_1^{j,n} + \dfrac{\lambda_1^n}{\lambda_2^n} \widetilde{v}_1^{j,n} = \alpha_1 (\widetilde{v}_1^{j,n})^{p-1} + (\lambda_2^n)^{\frac{4-p}{p-2}} (\mu_1 (\widetilde{v}_1^{j,n})^3 + \beta (\widetilde{v}_2^{j,n})^2 \widetilde{v}_1^{j,n}) \text{ in } \Omega_{j,n}^*, \\ -\Delta \widetilde{v}_2^{j,n} + \widetilde{v}_2^{j,n} = \alpha_2 (\widetilde{v}_2^{j,n})^{p-1} + (\lambda_2^n)^{\frac{4-p}{p-2}} (\mu_2 (\widetilde{v}_2^{j,n})^3 + \beta (\widetilde{v}_1^{j,n})^2 \widetilde{v}_2^{j,n}) \text{ in } \Omega_{j,n}^*, \\ \widetilde{v}_1^{j,n}, \widetilde{v}_2^{j,n} > 0 \quad \text{in } \Omega_{j,n}^*, \quad \widetilde{v}_1^{j,n} = \widetilde{v}_2^{j,n} = 0 \quad \text{on } \Omega_{j,n}^*, \end{cases} \tag{3.5}$$

where

$$\Omega_{j,n}^* = \{y \in \mathbb{R}^4 \mid \frac{\varepsilon_n}{\sqrt{\lambda_2^n}} y + x_j^n \in \Omega\} \quad \text{for } j=1,2. \tag{3.6}$$

Define

$$\overline{\mathcal{E}}_{\Omega^*_{j,n}}(\overline{\mathbf{u}}) = \sum_{i=1}^{2}(\frac{1}{2}(\|\nabla\overline{u}_i\|^2_{\Omega^*_{j,n},2} + \frac{\lambda_i^n}{\lambda_2^n}\|\overline{u}_i\|^2_{\Omega^*_{j,n},2}) - \frac{\alpha_i}{p}\|\overline{u}_i\|^p_{\Omega^*_{j,n},p})) -(\lambda_2^n)^{\frac{4-p}{p-2}}(\frac{1}{4}\sum_{i=1}^{2}\mu_i\|\overline{u}_i\|^4_{\Omega^*_{j,n},4} + \frac{\beta}{2}\|\overline{u}_1^2\overline{u}_2^2\|_{\Omega^*_{j,n},1}).$$

Then it is easy to see that $\overline{\mathcal{E}}_{\Omega^*_{j,n}}(\overline{\mathbf{u}})$ is of class C^2 in $\mathcal{H}_{**,\Omega^*_{j,n}} = \mathcal{H}_{\frac{\lambda_1^n}{\lambda_2^n},1,\Omega^*_{j,n}} \times \mathcal{H}_{\frac{\lambda_2^n}{\lambda_2^n},1,\Omega^*_{j,n}}$. Let

$$\overline{\mathcal{N}}_{\Omega^*_{j,n}} = \{\overline{\mathbf{u}} \in \overline{\mathcal{H}}_{**,\Omega^*_{j,n}} \mid \overline{\mathcal{E}}'_{\Omega^*_{j,n}}(\overline{\mathbf{u}})\overline{\mathbf{u}}^k = 0, k = 1, 2\} \tag{3.7}$$

where $\overline{\mathcal{H}}_{**,\Omega^*_{j,n}} = (\mathcal{H}_{\frac{\lambda_1^n}{\lambda_2^n},1,\Omega^*_{j,n}}\backslash\{0\}) \times (\mathcal{H}_{\frac{\lambda_2^n}{\lambda_2^n},1,\Omega^*_{j,n}}\backslash\{0\})$, $\overline{\mathbf{u}}^1 = (\overline{u}_1, 0)$ and $\overline{\mathbf{u}}^2 = (0, \overline{u}_2)$. Then by similar calculations of (2.4) and (2.5), we can see from (3.1), (3.2) and (3.4) and Proposition 3.1 that

$$\overline{\mathcal{E}}_{\Omega^*_{j,n}}(\widetilde{\mathbf{v}}_{j,n}) = \overline{c}_{j,n} = \inf_{\overline{\mathcal{N}}_{\Omega^*_{j,n}}} \overline{\mathcal{E}}_{\Omega^*_{j,n}}(\overline{\mathbf{u}}). \tag{3.8}$$

3.1 The case $\mathcal{A}_2 > 0$

Recall that $\lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\lambda_2^n}} = \mathcal{A}_2$. Thus, by (3.6), we have $\Omega^*_{j,n} \to \Omega^*_j$ as $n \to \infty$ and Ω^*_j are bounded smooth domains in $\mathbb{R}^4$ for both $j = 1, 2$, where $\Omega^*_{j,n}$ are given by (3.6). Let

$$\mathcal{I}_{i,\Omega^*_{j,n}}(u) = \frac{1}{2}(\|\nabla u\|^2_{\Omega^*_{j,n},2} + \frac{\lambda_i^n}{\lambda_2^n}\|u\|^2_{\Omega^*_{j,n},2}) - \frac{\alpha_i}{p}\|u\|^p_{\Omega^*_{j,n},p}, \quad i, j = 1, 2.$$

Then it is easy to see that $\mathcal{I}^*_{i,\Omega^*_{j,n}}(u)$ is of class C^2 in $H^1_0(\Omega^*_{j,n})$ and that it is the Euler Lagrange functional to the following equations

$$\begin{cases} -\Delta u + \frac{\lambda_i^n}{\lambda_2^n}u = \alpha_i|u|^{p-2}u & \text{in } \Omega^*_{j,n}, \\ u = 0 & \text{on } \partial\Omega^*_{j,n}. \end{cases} \tag{3.9}$$

It is well known that (3.9) has a positive ground state solution $\widetilde{u}_i^{j,n}$. We also set $\lim_{n\to\infty} \frac{\lambda_i^n}{\lambda_2^n} = \lambda_i^*$ and

$$\widetilde{\mathcal{I}}_{i,\Omega^*_j}(u) = \frac{1}{2}(\|\nabla u\|^2_{\Omega^*_j,2} + \lambda_i^*\|u\|^2_{\Omega^*_j,2}) - \frac{\alpha_i}{p}\|u\|^p_{\Omega^*_j,p}, \quad i, j = 1, 2.$$

Then it is easy to see that $\widetilde{\mathcal{I}}_{i,\Omega^*_j}(u)$ is of class C^2 in $H^1_0(\Omega^*_j)$ and that it is the Euler Lagrange functional to the following equations

$$\begin{cases} -\Delta u + \lambda_i^* u = \alpha_i|u|^{p-2}u & \text{in } \Omega^*_j, \\ u = 0 & \text{on } \partial\Omega^*_j. \end{cases} \tag{3.10}$$

It is also well known that (3.10) has a positive ground state solution $\widetilde{u}_i^j$.

Lemma 3.1. *There holds* $\limsup_{n\to\infty} e_{i,n}^j \leq e_i^j$*, where* $e_{i,n}^j = \mathcal{I}_{i,\Omega_{j,n}^*}(\widetilde{u}_{i,n}^j)$ *and* $e_i^j = \widetilde{\mathcal{I}}_{i,\Omega_j^*}(\widetilde{u}_i^j)$.

Proof. By the classical L^p-estimates (cf. [6]), $\widetilde{u}_i^j \in W^{2,p}(\Omega_j^*)$ for all $p \geq 2$. It follows from the Sobolev embedding theorem that $\widetilde{u}_i^j \in C^{1,\gamma}(\overline{\Omega_j^*})$ for some $\gamma \in (0,1)$. Since Ω_j^* is bounded, as [2, (5.113)], we have

$$\widetilde{u}_i^j(x) \leq C\delta \quad \text{for } x \in \Omega_j^* \backslash \Omega_{j,3\delta}^* \tag{3.11}$$

with $\delta > 0$ small enough, where

$$\Omega_{j,3\delta}^* = \{x \in \Omega_j^* \mid dist(x, \partial\Omega_j^*) \geq 3\delta\}.$$

Since $\Omega_{j,n}^* \to \Omega_j^*$ as $n \to \infty$, $\Omega_{j,\delta}^* \subset \Omega_{j,n}^*$ for n large enough. Let $\varphi(x)$ be a smooth cut-off function such that

$$\varphi(x) = \begin{cases} 1, & x \in \Omega_{j,3\delta}^*, \\ 0, & x \in \Omega_{j,n}^* \backslash \Omega_{j,2\delta}^*. \end{cases}$$

Moreover, we also require $\varphi(x) \in [0,1]$ and $|\nabla\varphi(x)| \leq \frac{C}{\delta}$. Define $\widetilde{u}_{i,\delta}^j = \widetilde{u}_i^j\varphi$. Then we have $\widetilde{u}_{i,\delta}^j \in H_0^1(\Omega_{j,n}^*)$ for n large enough. By direct calculations, we have from (3.11) that

$$\|\nabla\widetilde{u}_{i,\delta}^j\|_{\Omega_{j,n}^*,2}^2 = \|\nabla\widetilde{u}_i^j\|_{\Omega_j^*,2}^2 + O(\delta) \text{ and } \|\widetilde{u}_{i,\delta}^j\|_{\Omega_{j,n}^*,r}^r = \|\widetilde{u}_{i,}^j\|_{\Omega_j^*,r}^r + O(\delta), \tag{3.12}$$

where $O(\delta) \to 0$ as $\delta \to 0$ and $O(\delta)$ is independent of n large enough. It follows that $\mathcal{I}'_{i,\Omega_{j,n}^*}(\widetilde{u}_{i,\delta}^j)\widetilde{u}_{i,\delta}^j = O(\delta)$, which together with a standard argument, implies that there exists $t_\delta = 1 + O(\delta)$ such that $\mathcal{I}'_{i,\Omega_{j,n}^*}(t_\delta\widetilde{u}_{i,\delta}^j)t_\delta\widetilde{u}_{i,\delta}^j = 0$. By (3.12) once more, we can see that

$$\mathcal{I}_{i,\Omega_{j,n}^*}(\widetilde{u}_{i,n}^j) \leq \mathcal{I}_{i,\Omega_{j,n}^*}(t_\delta\widetilde{u}_{i,\delta}^j) = \widetilde{\mathcal{I}}_{i,\Omega_j^*}(\widetilde{u}_i^j) + O(\delta).$$

We completes the proof by letting $\delta \to 0^+$. □

With Lemma 3.1 in hands, we have the following.

Lemma 3.2. *There holds* $\limsup_{n\to\infty} \overline{c}_{j,n} \leq \sum_{i=1}^2 e_i^j$.

Proof. Thanks to Lemma 3.1, it suffices to show that

$$\limsup_{n\to\infty} \overline{c}_{j,n} \leq \limsup_{n\to\infty} \sum_{i=1}^2 e_{i,n}^j. \tag{3.13}$$

Let

$$\widetilde{\mathbf{u}}_{n,j} = (\widetilde{u}_{1,n}^j, \widetilde{u}_{2,n}^j).$$

Thanks to Lemma 3.1 and the boundedness of Ω_j^* once more, it is standard to show that $\widetilde{u}_{i,n}^j \to \widetilde{u}_{i,*}^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$, where $\widetilde{u}_{i,*}^j$ is a positive ground state solution of (3.10). For every $n \in \mathbb{N}$, we consider the system $\mathbf{\Phi}_{n,j}(\mathbf{t}, \tau) = \mathbf{0}$, where $\mathbf{\Phi}_{n,j}(\mathbf{t}, \tau) = (\Phi_1^{n,j}(\mathbf{t}, \tau), \Phi_2^{n,j}(\mathbf{t}, \tau))$ with

$$\begin{aligned}\Phi_i^{n,j}(\mathbf{t}, \tau) &= \|\nabla(t_i\widetilde{u}_{i,n}^j)\|_{\Omega_{j,n}^*,2}^2 + \frac{\lambda_i^n}{\lambda_2^n}\|t_i\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,2}^2 - \alpha_i\|t_i\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,p}^p \\ &\quad -\tau(\lambda_2^n)^{\frac{4-p}{p-2}}(\mu_i\|t_i\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,4}^4 + \beta\|(t_1\widetilde{u}_{1,n}^j)^2(t_2\widetilde{u}_{2,n}^j)^2\|_{\Omega_{j,n}^*,1}).\end{aligned}$$

Clearly, $\mathbf{\Phi}_{n,j}(\mathbf{t}, \tau)$ is of C^1. Moreover, since $\widetilde{u}_{i,n}^j$ solves (3.9), we also have that $\mathbf{1}$ is the unique solution of $\mathbf{\Phi}_{n,j}(\mathbf{t}, 0) = \mathbf{0}$. It follows that $\mathcal{Z}_{n,j} \neq \emptyset$, where

$$\mathcal{Z}_{n,j} = \{\tau \in [0, 1] \mid \mathbf{\Phi}_{n,j}(\mathbf{t}, \tau) = \mathbf{0} \text{ is uniquely solvable for } \mathbf{t} \in (\mathbb{R}^+)^2\}.$$

Let $\tau \in \mathcal{Z}_n$ and $\mathbf{t}_{n,j}(\tau)$ be the unique solution of $\mathbf{\Phi}_{n,j}(\mathbf{t}, \tau) = \mathbf{0}$. Then we have

$$\begin{aligned}0 &= \sum_{i=1}^{2}(\|\nabla(t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j)\|_{\Omega_{j,n}^*,2}^2 + \frac{\lambda_i^n}{\lambda_2^n}\|t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,2}^2 - \alpha_i\|t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,p}^p) \\ &\quad -\tau(\lambda_2^n)^{\frac{4-p}{p-2}}(\sum_{i=1}^{2}\mu_i\|t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,4}^4 + \beta\|(t_1^{n,j}(\tau)\widetilde{u}_{1,n}^j)^2(t_2^{n,j}(\tau)\widetilde{u}_{2,n}^j)^2\|_{\Omega_{j,n}^*,1}).\end{aligned}$$

By $\beta \geq -\sqrt{\mu_1\mu_2}$, we can see from the fact that $\widetilde{u}_{i,n}^j \to \widetilde{u}_{i,*}^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$ with $\widetilde{u}_{i,*}^j \neq 0$ that $\{\mathbf{t}_{n,j}(\tau)\}$ is uniformly bounded in $(\mathbb{R}^+)^2$ for n and τ. Thus, by the boundedness of $\{\widetilde{u}_{i,n}^j\}$ in $H^1(\mathbb{R}^4)$ and a similar argument for (2.13), we can show that

$$\alpha_i\|t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,p}^p \geq \left(\frac{\mathcal{S}_p}{2}\right)^{\frac{p}{p-2}} \tag{3.14}$$

for n large enough. Moreover, by similar calculations of (2.9) and (2.10), we also have

$$\begin{aligned}t_i^{n,j}(\tau)\frac{\partial\Phi_i^{n,j}(\mathbf{t}_{n,j}(\tau), \tau)}{\partial t_i} &= -(p-2)\alpha_i\|t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,p}^p \\ &\quad -2\tau\mu_i(\lambda_2^n)^{\frac{4-p}{p-2}}\|t_i^{n,j}(\tau)\widetilde{u}_{i,n}^j\|_{\Omega_{j,n}^*,4}^4 \qquad (3.15)\end{aligned}$$

respectively for $i = 1, 2$ and

$$\begin{aligned}&t_1^{n,j}(\tau)\frac{\partial\Phi_2^{n,j}(\mathbf{t}_{n,j}(\tau), \tau)}{\partial t_1} \\ =\;& t_2^{n,j}(\tau)\frac{\partial\Phi_1^{n,j}(\mathbf{t}_{n,j}(\tau), \tau)}{\partial t_2} \\ =\;& -2\tau\beta(\lambda_2^n)^{\frac{4-p}{p-2}}\|(t_1^{n,j}(\tau)\widetilde{u}_{1,n}^j)^2(t_2^{n,j}(\tau)\widetilde{u}_{2,n}^j)^2\|_{\Omega_{j,n}^*,1} \qquad (3.16)\end{aligned}$$

Set

$$\widetilde{\Theta}_{n,j} = (\widetilde{\theta}_{kl}^{n,j})_{k,l=1,2}$$

with $\widetilde{\theta}_{kl}^{n,j} = \frac{\partial \Phi_k^{n,j}(\mathbf{t}_n(\tau),\tau)}{\partial t_l}$. Thus, by (3.14) and (3.15)–(3.16), we can see that

$$\det(\widetilde{\Theta}_{n,j}) \geq (p-2)^2 \alpha_1 \alpha_2 \left(\frac{\mathcal{S}_p}{2}\right)^{\frac{2p}{p-2}}$$

for n large enough. Now, applying a similar argument as used in the proof of [14, Lemma 2.2], we can show that $\mathcal{Z}_{n,j} = [0,1]$ for n large enough. It follows that

$$\mathbf{t}_{n,j}(1) \circ \widetilde{\mathbf{u}}_{n,j} = (t_1^{n,j}(1)\widetilde{u}_{1,n}^j, t_2^{n,j}(1)\widetilde{u}_{2,n}^j) \in \mathcal{N}_{\Omega_{j,n}^*}.$$

By (3.8), we have from the boundedness of $\{\mathbf{t}_{n,j}(1) \circ \widetilde{\mathbf{u}}_{n,j}\}$ in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ that

$$\overline{c}_{j,n} \leq \overline{\mathcal{E}}_{\Omega_{j,n}^*}(\mathbf{t}_{n,j}(1) \circ \widetilde{\mathbf{u}}_{n,j}) = \sum_{i=1}^{2} \mathcal{I}_{i,\Omega_{j,n}^*}(t_i^{n,j}(1)\widetilde{u}_{i,n}^j) + o_n(1) \leq \sum_{i=1}^{2} e_{i,n}^j + o_n(1)$$

that is, (3.13) holds. □

Since $\Omega_{j,n}^*$ are smooth, by setting $\widetilde{v}_i^{j,n} \equiv 0$ outside $\Omega_{j,n}^*$, we can regard $\widetilde{v}_i^{j,n} \in H^1(\mathbb{R}^4)$. By $\beta \geq -\sqrt{\mu_1\mu_2}$ and $2 < p < 4$, we have

$$\begin{aligned}
\overline{\mathcal{E}}_{\Omega_{j,n}^*}(\widetilde{\mathbf{v}}_{j,n}) &= \frac{p-2}{2p}\sum_{i=1}^{2}(\|\nabla \widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,2}^2 + \frac{\lambda_i^n}{\lambda_2^n}\|\widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,2}^2) \\
&\quad + \frac{4-p}{4p}\sum_{i=1}^{2}\|\widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,4}^4 + \frac{4-p}{2p}\beta\|(\widetilde{v}_1^{j,n})^2(\widetilde{v}_2^{j,n})^2\|_{\Omega_{j,n}^*,1} \\
&\geq \frac{p-2}{2p}\sum_{i=1}^{2}(\|\nabla \widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,2}^2 + \frac{\lambda_i^n}{\lambda_2^n}\|\widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,2}^2). \qquad (3.17)
\end{aligned}$$

It follows from Lemma 3.2 that $\{\widetilde{v}_i^{j,n}\}$ are bounded in $D^{1,2}(\mathbb{R}^4)$ for n.

Proposition 3.2. *There holds $\widetilde{v}_i^{j,n} \to \widetilde{v}_i^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$, where $\widetilde{v}_i^j \in H_0^1(\Omega_j^*)$ is the positive ground state solution of* (3.10)*. Moreover, $u_i^n(x_i^n) = O((\lambda_2^n)^{\frac{1}{p-2}})$ as $n \to \infty$.*

Proof. Without loss of generality, we assume $\widetilde{v}_i^{j,n} \rightharpoonup \widetilde{v}_i^j$ weakly in $D^{1,2}(\mathbb{R}^4)$ and strongly in $L_{loc}^r(\mathbb{R}^4)$ for $2 \leq r < 4$ as $n \to \infty$. Since $\widetilde{v}_i^{j,n} \equiv 0$ outside $\Omega_{j,n}^*$ and $\Omega_{j,n}^* \to \Omega_j^*$ as $n \to \infty$, we have $\widetilde{v}_i^j = 0$ a.e. in $\mathbb{R}^4 \backslash \Omega_j^*$, which together with the fact that $\partial\Omega_j^*$ is smooth, implies $\widetilde{v}_i^j \in H_0^1(\Omega_j^*)$ and solves (3.10). On the other hand, Since $\widetilde{\mathbf{v}}_{j,n}$ satisfies (2.3), by the Hölder and Sobolev's inequalities, we can see from the boundedness of Ω_j^* that

$$(1+o_n(1))\|\nabla \widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,2}^2 \leq \alpha_i\|\widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,p}^p \leq C\|\nabla \widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,2}^p, \qquad (3.18)$$

which implies $\|\widetilde{v}_i^{j,n}\|_{\Omega_{j,n}^*,p}^p \geq C + o_n(1)$. Thus, by the Sobolev embedding theorem, $\widetilde{v}_i^j \neq 0$. It follows that $\mathcal{I}_{i,\Omega_j^*}^*(\widetilde{v}_i^j) \geq e_i^j$. This together with Lemma 3.2 and (3.17), implies $\widetilde{v}_i^{j,n} \to \widetilde{v}_i^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$ and $\widetilde{v}_i^j$ is a ground state solution of (3.10). Thanks to the maximum principle, $\widetilde{v}_i^j$ is also positive. Now, since $\widetilde{v}_i^{j,n} \to \widetilde{v}_i^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$, by applying the modified Moser's iteration (cf. [2]), we can see from the boundedness of Ω_j^* and $\Omega_{j,n}^* \to \Omega_j^*$ as $n \to \infty$ that $\{\widetilde{v}_i^{j,n}\}$ is bounded in $L^\infty(\mathbb{R}^4)$. It follows from the classical elliptic estimates (cf. [6]) that $\{\widetilde{v}_i^{j,n}\}$ is bounded in $C_{loc}^{1,\gamma}(\mathbb{R}^4)$ for some $\gamma \in (0,1)$. Thus, $\widetilde{v}_i^{j,n}(y) \to \widetilde{v}_i^j(y)$ uniformly in any compact set of $\mathbb{R}^4$ as $n \to \infty$. In particular, $\widetilde{v}_i^{j,n}(0) \to \widetilde{v}_i^j(0)$ as $n \to \infty$. By (3.2) and (3.4), we have $u_i^n(x_i^n) = O((\lambda_2^n)^{\frac{1}{p-2}})$ as $n \to \infty$ for both $i = 1, 2$. □

3.2 The case $\mathcal{A}_2 = 0$

Recall that $\lim_{n\to\infty} \frac{\varepsilon_n}{\sqrt{\lambda_2^n}} = \mathcal{A}_2$. Thus, we have $\Omega_{j,n}^* \to \Omega_j^*$ as $n \to \infty$ and Ω_j^* are unbounded smooth domains in $\mathbb{R}^4$ for both $j = 1, 2$ in this case, where $\Omega_{j,n}^*$ are given by (3.6). We denote $\lim_{n\to\infty} \frac{\lambda_i^n}{\lambda_2^n} = \lambda_i^*$. Then by $0 < \lambda_1^n \leq \lambda_1^n$, we have $0 \leq \lambda_1^* \leq \lambda_2^* = 1$. Let $\Omega_j^{1,0}$ and $\Omega_j^{2,0}$ be two bounded domains with $\Omega_j^{1,0} \cap \Omega_j^{2,0} = \emptyset$ and $\Omega_j^{k,0} \subset \Omega_j^*$ for both $k = 1, 2$. Then it is well known that the following equation

$$\begin{cases} -\Delta u + \lambda_i^* u = \alpha_i |u|^{p-2} u & \text{in } \Omega_j^{i,0}, \\ u = 0 & \text{on } \partial\Omega_j^{i,0}, \end{cases} \tag{3.19}$$

has a positive ground state solution w_i^j for all $i, j = 1, 2$. Denote the energy of w_i^j by $\widetilde{e}_i^j$.

Lemma 3.3. *There holds* $\limsup_{n\to\infty} \overline{c}_{j,n} \leq \sum_{i=1}^2 \widetilde{e}_i^j$, *where* $\overline{c}_{j,n}$ *is given by* (3.8).

Proof. Let

$$\widetilde{\mathbf{w}}_j = (w_1^j, w_2^j).$$

Since $\Omega_{j,n}^* \to \Omega_j^*$ as $n \to \infty$, we can see that $\Omega_j^{i,0} \subset \Omega_{j,n}^*$ for n large enough. Thus, by the fact that w_i^j is the positive ground state solution of (3.19) in $\Omega_j^{i,0}$, we can see from $\Omega_j^{1,0} \cap \Omega_j^{2,0} = \emptyset$ that there exist $t_i^{j,n} \to 1$ as $n \to \infty$ such that $\mathbf{t}_{j,n} \circ \widetilde{\mathbf{w}}_j = (t_1^{j,n} w_1^j, t_2^{j,n} w_2^j) \in \overline{\mathcal{N}}_{\Omega_{j,n}^*}$, where $\overline{\mathcal{N}}_{\Omega_{j,n}^*}$ is given by (3.7). Hence, the conclusion follows immediately from (3.8). □

As in the above section, since $\Omega_{j,n}^*$ are smooth, by setting $\widetilde{v}_i^{j,n} \equiv 0$ outside $\Omega_{j,n}^*$, we can regard $\widetilde{v}_i^{j,n} \in H^1(\mathbb{R}^4)$, where $\widetilde{v}_i^{j,n}$ is given by (3.4). Now, by (3.17) and Lemma 3.3, we also have that $\{\widetilde{v}_i^{j,n}\}$ are bounded in $D^{1,2}(\mathbb{R}^4)$ for n. Without loss of generality, we assume $\widetilde{v}_i^{j,n} \rightharpoonup \widetilde{v}_i^j$ weakly in $D^{1,2}(\mathbb{R}^4)$ and strongly in $L_{loc}^r(\mathbb{R}^4)$ for $2 \leq r < 4$ as $n \to \infty$.

Lemma 3.4. *There holds* $\Omega_j^* = \mathbb{R}^4$ *for both* $j = 1, 2$.

Proof. Since $\lim_{n\to\infty}\frac{\varepsilon_n}{\sqrt{\lambda_2^n}}=0$ and $\Omega_{j,n}^*\to\Omega_j^*$, it is well known that either $\Omega_j^*=\mathbb{R}^4$ or $\Omega_j^*=(\mathbb{R}^4)^+$ by rotations and translations, where $\Omega_{j,n}^*$ are given by (3.6). In what follows, we only give the proof of Ω_1^* since that of Ω_2^* is similar. Let $\varphi_1,\varphi_2\in C_0^\infty(\Omega_1^*)$. Then by $\Omega_{1,n}^*\to\Omega_1^*$ as $n\to\infty$, $\varphi_1,\varphi_2\in C_0^\infty(\Omega_{1,n}^*)$ for n large enough. Since $\widetilde{\mathbf{v}}_{1,n}$ satisfies (3.5), we can see from the boundedness of $\{\widetilde{v}_i^{1,n}\}$ in $D^{1,2}(\mathbb{R}^4)$ for n that $\widetilde{v}_i^1$ is a solution of the following equation

$$\begin{cases}-\Delta u+\lambda_i^*u=\alpha_i|u|^{p-2}u & \text{in }\Omega_1^*,\\ u=0 & \text{on }\partial\Omega_1^*.\end{cases}\tag{3.20}$$

Suppose the contrary that $\Omega_1^*=(\mathbb{R}^4)^+$. Then it is well known that $\widetilde{v}_i^1=0$ for both $i=1,2$.

The case $\lambda_1^*>0$

Recall that $\lambda_2^*=1$. Then by (3.17), $\{\widetilde{v}_i^{1,n}\}$ is bounded in $H^1(\mathbb{R}^4)$ for n. It follows from the fact that $H^1(\mathbb{R}^4)$ continuously embeds into $L^p(\mathbb{R}^4)$ and a similar argument as used in (3.18) that $\|\widetilde{v}_i^{1,n}\|_{\mathbb{R}^4,p}^p\geq C+o_n(1)$ for both $i=1,2$. Thus, by the Lions lemma and the Sobolev embedding theorem, there exists $\{z_i^{1,n}\}$ with $|z_i^{1,n}|\to+\infty$ as $n\to\infty$ such that $\widehat{v}_i^{1,n}(y)=\widetilde{v}_i^{1,n}(y+z_i^{1,n})\rightharpoonup\widehat{v}_i^1(y)\not\equiv0$ weakly in $H^1(\mathbb{R}^4)$ as $n\to\infty$. Let $\Omega_{1,n}^{i,0}=\Omega_{1,n}^*-z_i^{1,n}$ and assume $\Omega_{1,n}^{i,0}\to\Omega_1^{i,0}$ as $n\to\infty$. Then by a similar argument as used above, we can see that $\widehat{v}_i^1$ is a solution of (3.20) in $\Omega_1^{i,0}$ for both $i=1,2$. Since $\widehat{v}_i^1(y)\not\equiv0$, we must have that $\Omega_1^{i,0}=\mathbb{R}^4$ for both $i=1,2$. Then it is well known that the following equation

$$\begin{cases}-\Delta u+\lambda_i^*u=\alpha_i|u|^{p-2}u & \text{in }\mathbb{R}^4,\\ u\in H^1(\mathbb{R}^4),\end{cases}\tag{3.21}$$

has a unique positive ground state solution U_i for both $i=1,2$, which is also radial symmetric. Let

$$\widehat{\mathcal{I}}_{i,\mathbb{R}^4}(u)=\frac{1}{2}(\|\nabla u\|_{\mathbb{R}^4,2}^2+\lambda_i^*\|u\|_{\mathbb{R}^4,2}^2)-\frac{\alpha_i}{p}\|u\|_{\mathbb{R}^4,p}^p,\quad i=1,2.$$

Then $\widehat{\mathcal{I}}_{i,\mathbb{R}^4}(u)$ is the corresponding functional of (3.21). We claim that

$$\limsup_{n\to\infty}\overline{c}_{1,n}\leq\sum_{i=1}^2e_i^1=\widehat{\mathcal{I}}_{i,\mathbb{R}^4}(U_i).\tag{3.22}$$

Indeed, since $\Omega_{1,n}^{i,0}\to\mathbb{R}^4$ as $n\to\infty$ for both $i=1,2$, there exist $U_{i,n}\in H_0^1(\Omega_{1,n}^{i,0})$ such that $U_{i,n}\to U_i$ strongly in $H^1(\mathbb{R}^4)$ as $n\to\infty$. Let $\mathbf{U}_n=(U_{1,n},\widetilde{U}_{2,n})$, where $\widetilde{U}_{2,n}(y)=U_{2,n}(y+z_2^{1,n}-z_1^{1,n})$. Then $\mathbf{U}_n\in H_0^1(\Omega_{1,n}^{1,0})\times H_0^1(\Omega_{1,n}^{1,0})$. For every $n\in\mathbb{N}$, we consider the system $\widehat{\mathbf{\Phi}}_n(\mathbf{t},\tau)=\mathbf{0}$, where $\widehat{\mathbf{\Phi}}_n(\mathbf{t},\tau)=(\widehat{\Phi}_1^n(\mathbf{t},\tau),\widehat{\Phi}_2^n(\mathbf{t},\tau))$ with

$$\begin{aligned}\widehat{\Phi}_1^n(\mathbf{t},\tau)\;&=\;\|\nabla(t_1U_{1,n})\|_{\Omega_{1,n}^{1,0},2}^2+\frac{\lambda_1^n}{\lambda_2^n}\|t_1U_{1,n}\|_{\Omega_{1,n}^{1,0},2}^2-\alpha_1\|t_1U_{1,n}\|_{\Omega_{1,n}^{1,0},p}^p\\ &\quad-\tau(\lambda_2^n)^{\frac{4-p}{p-2}}(\mu_1\|t_1U_{1,n}\|_{\Omega_{1,n}^{1,0},4}^4+\beta\|(t_1U_{1,n})^2(t_2\widetilde{U}_{2,n})^2\|_{\Omega_{1,n}^{1,0},1}).\end{aligned}$$

and

$$\begin{aligned}\widehat{\Phi}_2^n(\mathbf{t},\tau) &= \|\nabla(t_2\widetilde{U}_{2,n})\|_{\Omega_{1,n}^{1,0},2}^2+\|t_2\widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},2}^2-\alpha_2\|t_2\widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},p}^p\\ &\quad -\tau(\lambda_2^n)^{\frac{4-p}{p-2}}(\mu_2\|t_2\widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},4}^4+\beta\|(t_1U_{1,n})^2(t_2\widetilde{U}_{2,n})^2\|_{\Omega_{1,n}^{1,0},1}).\end{aligned}$$

Let $(\mathbf{t}_n(\tau),\tau)\in(\mathbb{R}^+)^2\times[0,1]$ be the unique solution of $\widehat{\mathbf{\Phi}}_n(\mathbf{t},\tau)=\mathbf{0}$. Then by the fact that $U_{i,n}\to U_i$ strongly in $H^1(\mathbb{R}^4)$ as $n\to\infty$, we can apply a similar argument as used for (3.15) and (3.16) that

$$t_i^n(\tau)\frac{\partial\widehat{\Phi}_i^n(\mathbf{t}_n(\tau),\tau)}{\partial t_i}\geq C,\quad i=1,2,$$

for n large enough and

$$t_1^n(\tau)\frac{\partial\widehat{\Phi}_2^n(\mathbf{t}_n(\tau),\tau)}{\partial t_1}=t_2^n(\tau)\frac{\partial\widehat{\Phi}_1^n(\mathbf{t}_n(\tau),\tau)}{\partial t_2}=o_n(1).$$

Thus, by a similar argument as used in the proof of Lemma 3.2, we can show that $\widehat{\mathbf{\Phi}}_n(\mathbf{t},1)=\mathbf{0}$ has a unique solution $\mathbf{t}_n(1)\in(\mathbb{R}^+)^2$ for n large enough. Moreover, by a direct calculation, we can also see from $\lambda_n^2\to 0^+$ as $n\to\infty$ that

$$\begin{aligned}\overline{\mathcal{E}}'_{\Omega_{1,n}^{1,0}}(\mathbf{U}_n)\mathbf{U}_n^1 &= \|\nabla U_{1,n}\|_{\Omega_{1,n}^{1,0},2}^2+\frac{\lambda_1^n}{\lambda_2^n}\|U_{1,n}\|_{\Omega_{1,n}^{1,0},2}^2-\alpha_i\|U_{1,n}\|_{\Omega_{1,n}^{1,0},p}^p\\ &\quad -(\lambda_2^n)^{\frac{4-p}{p-2}}(\mu_i\|U_{1,n}\|_{\Omega_{1,n}^{1,0},4}^4+\beta\|(U_{1,n})^2(\widetilde{U}_{2,n})^2\|_{\Omega_{1,n}^{1,0},1})\\ &= o_n(1)\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{E}}'_{\Omega_{1,n}^{1,0}}(\mathbf{U}_n)\mathbf{U}_n^2 &= \|\nabla \widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},2}^2+\|\widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},2}^2-\alpha_i\|\widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},p}^p\\ &\quad -(\lambda_2^n)^{\frac{4-p}{p-2}}(\mu_i\|\widetilde{U}_{2,n}\|_{\Omega_{1,n}^{1,0},4}^4+\beta\|(U_{1,n})^2(\widetilde{U}_{2,n})^2\|_{\Omega_{1,n}^{1,0},1})\\ &= o_n(1).\end{aligned}$$

Thus, we also have $\mathbf{t}_n(1)\to\mathbf{1}$ as $n\to\infty$. Now, we have from (3.8) that

$$\overline{c}_{1,n}=\overline{\mathcal{E}}_{\Omega_{1,n}^{1,0}}(\widehat{\mathbf{v}}_{1,n})\leq\overline{\mathcal{E}}_{\Omega_{1,n}^{1,0}}(\mathbf{t}_n(1)\circ\mathbf{U}_n)\leq\sum_{i=1}^2 e_i^1+o_n(1).$$

That is, (3.22) holds. Since $\widehat{v}_i^1(y)$ is a nontrivial solution of (3.21) for both $i=1,2$, we must have $\widehat{\mathcal{I}}_{i,\mathbb{R}^4}^*(\widehat{v}_i^1)\geq e_i^1$. Therefore, by (3.17) and (3.22), we must have $\widehat{v}_i^{1,n}\to\widehat{v}_i^1$ strongly in $H^1(\mathbb{R}^4)$ as $n\to\infty$ for both $i=1,2$. Let $\widehat{w}_i^{1,n}=\widehat{v}_i^{1,n}-\widehat{v}_i^1$. Then by the Taylor's expansion, we have from $\lambda_2^n\to 0^+$ that

$$-\Delta\widehat{w}_i^{1,n}+(\lambda_i^*+o_n(1))\widehat{w}_i^{1,n}\leq\alpha_i((\widehat{v}_i^{1,n})^{p-2}+(\widehat{v}_i^1)^{p-2})\widehat{w}_i^{1,n}\quad\text{in }\mathbb{R}^4$$

in the weak sense. Since $\widehat{w}_i^{1,n} \to 0$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$ for both $i = 1, 2$, by applying the modified Moser's iteration (cf. [2]), we can show that $\widehat{w}_i^{1,n} \to 0$ strongly in $L^q(\mathbb{R}^4)$ with any $q \geq 2$ as $n \to \infty$ for both $i = 1, 2$. It follows from the classical elliptic estimates (cf. [6]) that $\widehat{w}_i^{1,n}(y) \to 0$ as $|y| \to +\infty$ uniformly for n. Since it is well known that $\widehat{v}_i^1(y) \to 0$ as $|y| \to +\infty$, we have $\widehat{v}_i^{1,n}(y) \to 0$ as $|y| \to +\infty$ uniformly for n. In particular, recalling that $|z_i^{1,n}| \to +\infty$, we have $\widetilde{v}_i^{1,n}(0) = \widehat{v}_i^{1,n}(-z_i^{1,n}) \to 0$ as $n \to \infty$. On the other hand, recalling that $\widetilde{v}_i^{1,n}(0) \geq \widetilde{v}_i^{1,n}(y)$ for all $y \in \mathbb{R}^4$ by (3.2) and (3.4), thus, we have from classical elliptic regularity theories (cf. [6]) and (3.5) that

$$(\lambda_i^* + o_n(1))\widetilde{v}_i^{1,n}(0) \leq \alpha_i(\widetilde{v}_i^{1,n}(0))^{p-1}, \tag{3.23}$$

which is a contradiction. Hence, we must have $\Omega_1^* = \mathbb{R}^4$.

The case $\lambda_1^* = 0$

Since the argument is similar to that of the case $\lambda_1^* > 0$, we only sketch it and point out the differences. By $\lambda_2^* = 1$, we still have $\|\widetilde{v}_2^{1,n}\|_{\mathbb{R}^4,p}^p \geq C + o_n(1)$. Thus, we also have from the Lions lemma and the Sobolev embedding theorem that $\widehat{v}_i^{1,n}(y) = \widetilde{v}_i^{1,n}(y + z_2^{1,n}) \rightharpoonup \widehat{v}_i^1(y)$ weakly in $H^1(\mathbb{R}^4)$ and $\Omega_{1,n}^{2,0} \to \mathbb{R}^4$ as $n \to \infty$, where $|z_2^{1,n}| \to +\infty$ and $\widehat{v}_2^1 \not\equiv 0$ is a solution of (3.21) for $i = 2$. We claim that

$$\limsup_{n\to\infty} \overline{c}_{1,n} \leq e_2^1, \tag{3.24}$$

where e_2^1 is given by (3.22). Indeed, let $R > 0$ and u_R be the positive ground state solution of the following equation

$$\begin{cases} -\Delta u = \alpha_1 |u|^{p-2}u & \text{in } \mathbb{B}_R, \\ u = 0 & \text{on } \partial\mathbb{B}_R, \end{cases}$$

Let $\widehat{\mathbf{U}}_n = (u_R, U_{2,n})$, where $U_{2,n}$ is given in the case $\lambda_1^* > 0$ satisfying $U_{2,n} \to U_2$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$. Then by applying a similar argument for $\mathbf{U}_n = (U_{1,n}, \widetilde{U}_{2,n})$ in the case $\lambda_1^* > 0$ to $\widehat{\mathbf{U}}_n$ with some trivial modifications, we can see from $\lambda_1^* = 0$ that there exists $\mathbf{t}_n(1) \to \mathbf{1}$ as $n \to \infty$ such that

$$\overline{c}_{1,n} = \overline{\mathcal{E}}_{\Omega_{1,n}^{2,0}}(\widehat{\mathbf{v}}_{1,n}) \leq \overline{\mathcal{E}}_{\Omega_{1,n}^{2,0}}(\mathbf{t}_n(1) \circ \widehat{\mathbf{U}}_n) \leq e_2^1 + (1 + o_n(1))C_R + o_n(1), \tag{3.25}$$

where C_R is the energy of u_R. By the uniqueness of u_R, we can see from the canonical rescaling that

$$u_R(x) = (\frac{1}{R})^{\frac{2}{p-2}} u_1(\frac{x}{R}).$$

Thus, $u_R \to 0$ in $D^{1,2}(\mathbb{R}^4) \cap L^p(\mathbb{R}^4)$ as $R \to +\infty$. It follows that $C_R \to 0$ as $R \to +\infty$. Hence, by letting $n \to \infty$ first and $R \to \infty$ next in (3.25), we can obtain (3.24). Now, by similar arguments as used in the case $\lambda_1^* > 0$, we can show that $\widehat{v}_i^{1,n} \to \widehat{v}_i^1$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$. Thanks to (3.24) and $\widehat{v}_2^1 \not\equiv 0$, we must have $\widehat{v}_1^1 = 0$. In what follows, by repeating the argument as used in the case $\lambda_1^* > 0$ with some trivial modifications, we can obtain a contradiction. Thus, we must have $\Omega_1^* = \mathbb{R}^4$. □

Now, we have the following.

Proposition 3.3. *There holds $\widetilde{v}_i^{j,n} \to \widetilde{v}_i^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$, where $\widetilde{v}_2^j \in H^1(\mathbb{R}^4)$ is the positive ground state solution of* (3.21) *for $i = 2$. Moreover, we also have*

(1) $\widetilde{v}_1^j \in H^1(\mathbb{R}^4)$ *is the positive ground state solution of* (3.21) *for $i = 1$ in the case $\lambda_1^* > 0$ while $\widetilde{v}_1^j = 0$ in the case $\lambda_1^* = 0$.*

(2) $u_2^n(x_2^n) = O((\lambda_2^n)^{\frac{1}{p-2}})$ *while* $u_1^n(x_1^n) = O((\lambda_2^n)^{\frac{1}{p-2}})$ *for* $\lambda_1^* > 0$ *and* $u_1^n(x_1^n) = o((\lambda_2^n)^{\frac{1}{p-2}})$ *for* $\lambda_1^* = 0$.

(3) $\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n) \to 0$ *as* $n \to \infty$.

Proof. (1) By Lemma 3.4, we have $\Omega_{j,n}^* \to \mathbb{R}^4$ as $n \to \infty$. Now, repeating the argument as used in the proof of Lemma 3.4 with some trivial modifications, we can show that $\widetilde{v}_i^{j,n} \to \widetilde{v}_i^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$ for both $i, j = 1, 2$. Moreover, $\widetilde{v}_i^j \neq 0$ are the positive ground state solution of (3.21) for both $i, j = 1, 2$ in the case $\lambda_1^* > 0$ while $\widetilde{v}_2^j \neq 0$ is the positive ground state solution of (3.21) and $\widetilde{v}_1^j = 0$ for both $j = 1, 2$ in the case $\lambda_1^* = 0$.

(2) By (1), we always have that $\widetilde{v}_2^{j,n} \to \widetilde{v}_2^j$ strongly in $H^1(\mathbb{R}^4)$ as $n \to \infty$ for both $\lambda_1^* > 0$ and $\lambda_1^* = 0$, where $\widetilde{v}_2^j \in H^1(\mathbb{R}^4)$ is the positive ground state solution of (3.21) for $i = 2$. Thus, by applying a similar argument as used in the proof of Proposition 3.2, we have $u_2^n(x_2^n) = O((\lambda_2^n)^{\frac{1}{p-2}})$. Since $\widetilde{v}_1^j \neq 0$ in the case $\lambda_1^* > 0$ and $\widetilde{v}_1^j = 0$ in the case $\lambda_1^* = 0$ for both $j = 1, 2$, we have from similar argument as used for $u_2^n(x_2^n)$ that $u_1^n(x_1^n) = O((\lambda_2^n)^{\frac{1}{p-2}})$ for $\lambda_1^* > 0$ and $u_1^n(x_1^n) = o((\lambda_2^n)^{\frac{1}{p-2}})$ for $\lambda_1^* = 0$.

(3) By repeating the argument as used in the proof of Lemma 3.4 with some trivial modifications, we have $\widetilde{v}_2^{j,n}(y) \to 0$ as $|y| \to +\infty$ uniformly for n. It follows from $\widetilde{v}_2^{2,n}(0) = \widetilde{v}_2^{1,n}(-\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n))$ and a similar argument as used for (3.23) that $\{\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n)\}$ is bounded. Without loss of generality, we assume that $\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n) \to p_0$ as $n \to \infty$. If $p_0 \neq 0$, then by the uniform convergence of $\widetilde{v}_2^{j,n}$ in compact sets of $\mathbb{R}^4$, we have from $\widetilde{v}_2^{1,n}(0) = \widetilde{v}_2^{2,n}(\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n))$, $\widetilde{v}_2^{2,n}(0) = \widetilde{v}_2^{1,n}(-\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n))$ and the well known Gidas-Ni-Nirenberg theorem [7] (see also [8]) that

$$\widetilde{v}_2^2(0) = \widetilde{v}_2^1(-p_0) < \widetilde{v}_2^1(0) = \widetilde{v}_2^2(p_0) < \widetilde{v}_2^2(0).$$

This is a contradiction. Thus, we must have $\frac{\sqrt{\lambda_2^n}}{\varepsilon_n}(x_1^n - x_2^n) \to 0$ as $n \to \infty$. □

We close this section by

Proof of Theorem 1.1: By employing the proof of Proposition 3.1 in Proposition 2.1, we can obtain the conclusions from Proposition 2.1 and Propositions 3.2–3.3. □

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