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On a two-component Bose–Einstein condensate with steep potential wells

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Abstract In this paper, we study the following two-component systems of nonlinear Schrödinger equations

$$\begin{cases} \Delta u - (\lambda a(x) + a_0(x))u + \mu_1 u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^3, \\ \Delta v - (\lambda b(x) + b_0(x))v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^3, \\ u, v \in H^1(\mathbb{R}^3), u, v > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda, \mu_1, \mu_2 > 0$ and $\beta < 0$ are parameters; $a(x), b(x) \geq 0$ are steep potentials and $a_0(x), b_0(x)$ are sign-changing weight functions; $a(x), b(x), a_0(x)$ and $b_0(x)$ are not necessarily to be radial symmetric. By the variational method, we obtain a ground state solution and multi-bump solutions for such systems with λ sufficiently large. The concentration behaviors of solutions as both $\lambda \to +\infty$ and $\beta \to -\infty$ are also considered. In particular, the phenomenon of phase separations is observed in the whole space $\mathbb{R}^3$. In the Hartree–Fock theory, this provides a theoretical enlightenment of phase separation in $\mathbb{R}^3$ for the 2-mixtures of Bose–Einstein condensates.

Keywords Bose–Einstein condensate · Steep potential well · Ground state solution · Multi-bump solution

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1 Introduction

In this paper, we consider the following two-component systems of nonlinear Schrödinger equations

$$\begin{cases} \Delta u-(\lambda a(x)+a_0(x))u+\mu_1 u^3+\beta v^2u=0 & \text{in } \mathbb{R}^3,\\ \Delta v-(\lambda b(x)+b_0(x))v+\mu_2 v^3+\beta u^2v=0 & \text{in } \mathbb{R}^3,\\ u,v\in H^1(\mathbb{R}^3),\ u,v>0 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{P}_{\lambda,\beta}$}$$

where $\lambda,\mu_1,\mu_2>0$ and $\beta<0$ are parameters. The potentials $a(x), b(x), a_0(x)$ and $b_0(x)$ satisfy the following conditions:

(D_1) $a(x), b(x)\in C(\mathbb{R}^3)$ and $a(x), b(x)\geq 0$ on $\mathbb{R}^3$.

(D_2) There exist $a_\infty>0$ and $b_\infty>0$ such that $\mathcal{D}_a:=\{x\in\mathbb{R}^3 \mid a(x)<a_\infty\}$ and $\mathcal{D}_b:=\{x\in\mathbb{R}^3\mid b(x)<b_\infty\}$ are nonempty and have finite measures.

(D_3) $\Omega_a=\mathrm{int}a^{-1}(0)$ and $\Omega_b=\mathrm{int}b^{-1}(0)$ are nonempty bounded sets and have smooth boundaries. Moreover, $\overline{\Omega}_a=a^{-1}(0)$, $\overline{\Omega}_b=b^{-1}(0)$ and $\overline{\Omega}_a\cap\overline{\Omega}_b=\emptyset$.

(D_4) $a_0(x), b_0(x)\in C(\mathbb{R}^3)$ and there exist $R, d_a, d_b>0$ such that

$$a_0^-(x)\leq d_a(1+a(x)) \quad\text{and}\quad b_0^-(x)\leq d_b(1+b(x)) \quad \text{for } |x|\geq R,$$

where $a_0^-(x)=\max\{-a_0(x),0\}$ and $b_0^-(x)=\max\{-b_0(x),0\}$.

(D_5) $\inf\sigma_a(-\Delta+a_0(x))>0$ and $\inf\sigma_b(-\Delta+b_0(x))>0$, where $\sigma_a(-\Delta+a_0(x))$ is the spectrum of $-\Delta+a_0(x)$ on $H_0^1(\Omega_a)$ and $\sigma_b(-\Delta+b_0(x))$ is the spectrum of $-\Delta+b_0(x)$ on $H_0^1(\Omega_b)$.

Remark 1.1 If $a_0(x), b_0(x)\in C(\mathbb{R}^3)$ are bounded, then the condition (D_4) is trivial. However, under the assumptions of $(D_4)-(D_5)$, $a_0(x)$ and $b_0(x)$ may be sign-changing and unbounded.

Two-component systems of nonlinear Schrödinger equations like $(\mathcal{P}_{\lambda,\beta})$ appear in the Hartree–Fock theory for a double condensate, that is, a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (cf. [25]), where the solutions u and v are the corresponding condensate amplitudes, μ_j are the intraspecies and interspecies scattering lengths. The interaction is attractive if $\beta>0$ and repulsive if $\beta<0$. When the interaction is repulsive, it is expected that the phenomenon of phase separations will happen, that is, the two components of the system tend to separate in different regions as the interaction tends to infinity. This kind of systems also arises in nonlinear optics (cf. [2]). Due to the important application in physics, the following system

$$\begin{cases} \Delta u-\lambda_1 u+\mu_1 u^3+\beta v^2 u=0 & \text{in } \Omega,\\ \Delta v-\lambda_2 v+\mu_2 v^3+\beta u^2 v=0 & \text{in } \Omega,\\ u,v=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ or $\mathbb{R}^3$, has attracted many attentions of mathematicians in the past decade. We refer the readers to [7–9,15,17,19,20,29–31,33,34,36,37,39,40,44]. In these literatures, various existence theories of the solutions were established for the Bose–Einstein condensates in $\mathbb{R}^2$ and $\mathbb{R}^3$. Recently, some mathematicians devoted their interest to the two coupled Schrödinger equations with critical Sobolev exponent in the high dimensions, and a number of the existence results of the solutions for such systems were also established. See for example [13–16,18].

On the other hand, if the parameter λ is sufficiently large, then $\lambda a(x)$ and $\lambda b(x)$ are called the steep potential wells under the conditions $(D_1)-(D_3)$. The depth of the wells is controlled by the parameter λ. Such potentials were first introduced by Bartsch and Wang in [3] for the scalar Schrödinger equations. An interesting phenomenon for this kind of Schrödinger equations is that one can expect to find the solutions which are concentrated at the bottom of the wells as the depth goes to infinity. Due to this interesting property, such topic for the scalar Schrödinger equations was studied extensively in the past decade. We refer the readers to [4,5,10,24,35,42,43,45,51] and the references therein. In particular, in [24], by assuming that the bottom of the steep potential wells consists of finitely many disjoint bounded domains, the authors obtained multi-bump solutions for scalar Schrödinger equations with steep potential wells, which are concentrated at any given disjoint bounded domains of the bottom as the depth goes to infinity.

We wonder what happens to the two-component Bose–Einstein condensate $(\mathcal{P}_{\lambda,\beta})$ with steep potential wells? In the current paper, we shall explore this problem to find whether the solutions of such systems are concentrated at the bottom of the wells as $\lambda \to +\infty$ and when the phenomenon of phase separations of such systems can be observed in the whole space $\mathbb{R}^3$.

We remark that the phenomenon of phase separations for (1.1) was observed in [13,16, 20,21,38,49,50] for the ground state solution when Ω is a bounded domain. In particular, this phenomenon was also observed on the whole spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ by [48], where the system is radial symmetric! However, when the system is not necessarily radial symmetric, the phenomenon of phase separations for Bose–Einstein condensates on the whole space $\mathbb{R}^3$ has not been obtained yet. For other kinds of elliptic systems with strong competition, the phenomenon of phase separations has also been well studied; we refer the readers to [11,12,22] and references therein.

We recall some definitions in order to state the main results in the current paper. We say that $(u_0, v_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ is a non-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ if (u_0, v_0) is a solution of $(\mathcal{P}_{\lambda,\beta})$ with $u_0 \neq 0$ and $v_0 \neq 0$. We say $(u_0, v_0) \in E$ is a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ if (u_0, v_0) is a non-trivial solution of $(\mathcal{P}_{\lambda,\beta})$ and

$$J_{\lambda,\beta}(u_0, v_0) = \inf\{J_{\lambda,\beta}(u, v) \mid (u, v) \text{ is a non-trivial solution of } (\mathcal{P}_{\lambda,\beta})\},$$

where $J_{\lambda,\beta}(u, v)$ is the corresponding functional of $(\mathcal{P}_{\lambda,\beta})$ and given by

$$\begin{aligned} J_{\lambda,\beta}(u, v) = & \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + (\lambda b(x) + b_0(x))v^2 \mathrm{d}x \\ & - \frac{\mu_1}{4}\int_{\mathbb{R}^3} u^4 \mathrm{d}x - \frac{\mu_2}{4}\int_{\mathbb{R}^3} v^4 \mathrm{d}x - \frac{\beta}{2}\int_{\mathbb{R}^3} u^2 v^2 \mathrm{d}x. \end{aligned} \tag{1.2}$$

Remark 1.2 In Sect. 2, we will give a variational setting of $(\mathcal{P}_{\lambda,\beta})$ and show that the solutions of $(\mathcal{P}_{\lambda,\beta})$ are equivalent to the positive critical points of $J_{\lambda,\beta}(u, v)$ in a suitable Hilbert space E.

Let $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$ be two functionals, respectively, defined on $H_0^1(\Omega_a)$ and $H_0^1(\Omega_b)$, which are given by

$$I_{\Omega_a}(u) = \frac{1}{2}\int_{\Omega_a} |\nabla u|^2 + a_0(x)u^2 \mathrm{d}x - \frac{\mu_1}{4}\int_{\Omega_a} u^4 \mathrm{d}x$$

and by

$$I_{\Omega_b}(v) = \frac{1}{2}\int_{\Omega_b} |\nabla v|^2 + b_0(x)v^2 \mathrm{d}x - \frac{\mu_2}{4}\int_{\Omega_b} v^4 \mathrm{d}x.$$

Then, by the condition (D_5), it is well known that $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$ have least energy nonzero critical points. We denote the least energy of nonzero critical points for $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$ by m_a and m_b, respectively. Now, our first result can be stated as follows.

Theorem 1.1 *Assume $(D_1)-(D_5)$. Then there exists $\Lambda_* > 0$ independent of β such that $(\mathcal{P}_{\lambda,\beta})$ has a ground state solution $(u_{\lambda,\beta}, v_{\lambda,\beta})$ for all $\lambda \geq \Lambda_*$ and $\beta < 0$, which has the following properties:*

(1) $\int_{\mathbb{R}^3\backslash\Omega_a} |\nabla u_{\lambda,\beta}|^2 + u_{\lambda,\beta}^2 \mathrm{d}x \to 0$ *and* $\int_{\mathbb{R}^3\backslash\Omega_b} |\nabla v_{\lambda,\beta}|^2 + v_{\lambda,\beta}^2 \mathrm{d}x \to 0$ *as* $\lambda \to +\infty$.
(2) $\int_{\Omega_a} |\nabla u_{\lambda,\beta}|^2 + a_0(x)u_{\lambda,\beta}^2 \mathrm{d}x \to 4m_a$ *and* $\int_{\Omega_b} |\nabla v_{\lambda,\beta}|^2 + b_0(x)v_{\lambda,\beta}^2 \mathrm{d}x \to 4m_b$ *as* $\lambda \to +\infty$.

Furthermore, for each $\{\lambda_n\} \subset [\Lambda_*, +\infty)$ satisfies $\lambda_n \to +\infty$ as $n \to \infty$ and $\beta < 0$, there exists $(u_{0,\beta}, v_{0,\beta}) \in (H^1(\mathbb{R}^3)\backslash\{0\}) \times (H^1(\mathbb{R}^3)\backslash\{0\})$ such that

(3) $(u_{0,\beta}, v_{0,\beta}) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b)$ *with* $u_{0,\beta} \equiv 0$ *outside* Ω_a *and* $v_{0,\beta} \equiv 0$ *outside* Ω_b.
(4) $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ *strongly in* $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ *as* $n \to \infty$ *up to a subsequence.*
(5) $u_{0,\beta}$ *is a least energy nonzero critical point of* $I_{\Omega_a}(u)$ *and* $v_{0,\beta}$ *is a least energy nonzero critical point of* $I_{\Omega_b}(v)$.

Remark 1.3 Roughly speaking, under the conditions $(D_1)-(D_5)$, Theorem 1.1 obtain a solution of the system $(\mathcal{P}_{\lambda,\beta})$ in the following form $(u_0 + w_{\lambda,\beta}^1, v_0 + w_{\lambda,\beta}^2)$, where u_0 and v_0 are, respectively, the least energy nonzero critical points of $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$ and $w_{\lambda,\beta}^1$ and $w_{\lambda,\beta}^2$ are two perturbations with $w_{\lambda,\beta}^1 \to 0$ and $w_{\lambda,\beta}^2 \to 0$ strongly in $H^1(\mathbb{R}^3)$ as $\lambda \to +\infty$. It is worth to point out that the assumption $\overline{\Omega}_a \cap \overline{\Omega}_b = \emptyset$ is not essential in proving Theorem 1.1. For instance, in the case $\overline{\Omega}_a = \overline{\Omega}_b = \Omega$, our method to prove Theorem 1.1 (with some necessary modifications) still works to find out a solution of the system $(\mathcal{P}_{\lambda,\beta})$ in the form $(u_\beta + w_{\lambda,\beta}^3, v_\beta + w_{\lambda,\beta}^4)$, where (u_β, v_β) is the ground state solution of the following system

$$\begin{cases} \Delta u - a_0(x)u + \mu_1 u^3 + \beta v^2 u = 0 & \text{in } \Omega, \\ \Delta v - b_0(x)v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

and $w_{\lambda,\beta}^3$ and $w_{\lambda,\beta}^4$ are also two perturbations with $w_{\lambda,\beta}^3 \to 0$ and $w_{\lambda,\beta}^4 \to 0$ strongly in $H^1(\mathbb{R}^3)$ as $\lambda \to +\infty$.

Next, we assume that the bottom of the steep potential wells consists of finitely many disjoint bounded domains. It is natural to ask whether the two-component Bose–Einstein condensate $(\mathcal{P}_{\lambda,\beta})$ with such steep potential wells has multi-bump solutions which are concentrated at any given disjoint bounded domains of the bottom as the depth goes to infinity. Our second result is devoted to this study. Similar to [24], we need the following conditions on the potentials $a(x)$, $b(x)$, $a_0(x)$ and $b_0(x)$.

(D_3') $\Omega_a = \mathrm{int}a^{-1}(0)$ and $\Omega_b = \mathrm{int}b^{-1}(0)$ satisfy $\Omega_a = \bigcup_{i_a=1}^{n_a} \Omega_{a,i_a}$ and $\Omega_b = \bigcup_{j_b=1}^{n_b} \Omega_{b,j_b}$, where $\{\Omega_{a,i_a}\}$ and $\{\Omega_{b,j_b}\}$ are all nonempty bounded domains with smooth boundaries, and $\overline{\Omega}_{a,i_a} \cap \overline{\Omega}_{a,j_a} = \emptyset$ for $i_a \neq j_a$ and $\overline{\Omega}_{b,i_b} \cap \overline{\Omega}_{b,j_b} = \emptyset$ for $i_b \neq j_b$. Moreover, $\overline{\Omega}_a = a^{-1}(0)$ and $\overline{\Omega}_b = b^{-1}(0)$ with $\overline{\Omega}_a \cap \overline{\Omega}_b = \emptyset$.

(D_5') $\inf \sigma_{a,i_a}(-\Delta + a_0(x)) > 0$ for all $i_a = 1, \ldots, n_a$ and $\inf \sigma_{b,j_b}(-\Delta + b_0(x)) > 0$ for all $j_b = 1, \ldots, n_b$, where $\sigma_{a,i_a}(-\Delta + a_0(x))$ is the spectrum of $-\Delta + a_0(x)$ on $H_0^1(\Omega_{a,i_a})$ and $\sigma_{b,j_b}(-\Delta + b_0(x))$ is the spectrum of $-\Delta + b_0(x)$ on $H_0^1(\Omega_{b,j_b})$.

Remark 1.4 Under the conditions (D_3') and (D_4), it is easy to see that the condition (D_5') is equivalent to the condition (D_5). For the sake of clarity, we use the condition (D_5') in the study of multi-bump solutions.

We define $I_{\Omega_{a,i_a}}(u)$ on $H_0^1(\Omega_{a,i_a})$ for each $i_a = 1, \ldots, n_a$ by

$$I_{\Omega_{a,i_a}}(u) = \frac{1}{2}\int_{\Omega_{a,i_a}} |\nabla u|^2 + a_0(x)u^2 \mathrm{d}x - \frac{\mu_1}{4}\int_{\Omega_{a,i_a}} u^4 \mathrm{d}x$$

and $I_{\Omega_{b,j_b}}(v)$ on $H_0^1(\Omega_{b,j_b})$ for each $j_b = 1, \ldots, n_b$ by

$$I_{\Omega_{b,j_b}}(v) = \frac{1}{2}\int_{\Omega_{b,j_b}} |\nabla v|^2 + b_0(x)v^2 \mathrm{d}x - \frac{\mu_2}{4}\int_{\Omega_{b,j_b}} v^4 \mathrm{d}x.$$

Then by the conditions (D_3') and (D_5'), it is well known that $I_{\Omega_{a,i_a}}(u)$ and $I_{\Omega_{b,j_b}}(v)$ have least energy nonzero critical points for every $i_a = 1, \ldots, n_a$ and every $j_b = 1, \ldots, n_b$, respectively. We denote the least energy of nonzero critical points for $I_{\Omega_{a,i_a}}(u)$ and $I_{\Omega_{b,j_b}}(v)$ by m_{a,i_a} and m_{b,j_b}, respectively. Now, our second result can be stated as the following.

Theorem 1.2 *Assume $\beta < 0$ and the conditions (D_1)–(D_2), (D_3'), (D_4) and (D_5') hold. If the set $J_a \times J_b \subset \{1, \ldots, n_a\} \times \{1, \ldots, n_b\}$ satisfying $J_a \neq \emptyset$ and $J_b \neq \emptyset$, then there exists $\Lambda_*(\beta) > 0$ such that $(\mathcal{P}_{\lambda,\beta})$ has a non-trivial solution $(u_{\lambda,\beta}^{J_a}, v_{\lambda,\beta}^{J_b})$ for $\lambda \geq \Lambda_*(\beta)$ with the following properties:*

(1) $\int_{\mathbb{R}^3\setminus\Omega_{a,0}^{J_a}} |\nabla u_{\lambda,\beta}^{J_a}|^2 + (u_{\lambda,\beta}^{J_a})^2 \mathrm{d}x \to 0$ *and* $\int_{\mathbb{R}^3\setminus\Omega_{b,0}^{J_b}} |\nabla v_{\lambda,\beta}^{J_b}|^2 + (v_{\lambda,\beta}^{J_b})^2 \mathrm{d}x \to 0$ *as* $\lambda \to +\infty$, *where* $\Omega_{a,0}^{J_a} = \cup_{i_a\in J_a} \Omega_{a,i_a}$ *and* $\Omega_{b,0}^{J_b} = \cup_{j_b\in J_b} \Omega_{b,j_b}$.

(2) $\int_{\Omega_{a,i_a}} |\nabla u_{\lambda,\beta}^{J_a}|^2 + a_0(x)(u_{\lambda,\beta}^{J_a})^2 \mathrm{d}x \to 4m_{a,i_a}$ *and* $\int_{\Omega_{b,j_b}} |\nabla v_{\lambda,\beta}^{J_b}|^2 + b_0(x)(v_{\lambda,\beta}^{J_b})^2 \mathrm{d}x \to 4m_{b,j_b}$ *as* $\lambda \to +\infty$ *for all* $i_a \in J_a$ *and* $j_b \in J_b$.

Furthermore, for each $\beta < 0$ and $\{\lambda_n\} \subset [\Lambda_*(\beta), +\infty)$ satisfying $\lambda_n \to +\infty$ as $n \to \infty$, there exists $(u_{0,\beta}^{J_a}, v_{0,\beta}^{J_b}) \in (H^1(\mathbb{R}^3)\setminus\{0\}) \times (H^1(\mathbb{R}^3)\setminus\{0\})$ such that

(3) $(u_{0,\beta}^{J_a}, v_{0,\beta}^{J_b}) \in H_0^1(\Omega_{a,0}^{J_a}) \times H_0^1(\Omega_{b,0}^{J_b})$ *with* $u_{0,\beta}^{J_a} \equiv 0$ *outside* $\Omega_{a,0}^{J_a}$ *and* $v_{0,\beta}^{J_b} \equiv 0$ *outside* $\Omega_{b,0}^{J_b}$.

(4) $(u_{\lambda_n,\beta}^{J_a}, v_{\lambda_n,\beta}^{J_b}) \to (u_{0,\beta}^{J_a}, v_{0,\beta}^{J_b})$ *strongly in* $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ *as* $n \to \infty$ *up to a subsequence.*

(5) *the restriction of* $u_{0,\beta}^{J_a}$ *on* Ω_{a,i_a} *lies in* $H_0^1(\Omega_{a,i_a})$ *and is a least energy nonzero critical point of* $I_{\Omega_{a,i_a}}(u)$ *for all* $i_a \in J_a$, *while the restriction of* $v_{0,\beta}^{J_b}$ *on* Ω_{b,j_b} *lies in* $H_0^1(\Omega_{b,j_b})$ *and is a least energy nonzero critical point of* $I_{\Omega_{b,j_b}}(v)$ *for all* $j_b \in J_b$.

Corollary 1.1 *Suppose $\beta < 0$ and the conditions (D_1)–(D_2), (D_3'), (D_4) and (D_5') hold. Then, $(\mathcal{P}_{\lambda,\beta})$ has at least $(2^{n_a} - 1)(2^{n_b} - 1)$ non-trivial solutions for $\lambda \geq \Lambda_*(\beta)$.*

Remark 1.5 (i) To the best of our knowledge, it seems that Theorem 1.2 is the first result for the existence of multi-bump solutions to system $(\mathcal{P}_{\lambda,\beta})$.

(ii) Under the condition (D_3'), we can see that

$$I_{\Omega_a}(u) = \sum_{i_a=1}^{n_a} I_{\Omega_{a,i_a}}(u) \quad \text{and} \quad I_{\Omega_b}(v) = \sum_{j_b=1}^{n_b} I_{\Omega_{b,j_b}}(v).$$

Let $m_{a,0} = \min\{m_{a,1}, \ldots, m_{a,n_a}\}$ and $m_{b,0} = \min\{m_{b,1}, \ldots, m_{b,n_b}\}$. Then, we must have $m_{a,0} = m_a$ and $m_{b,0} = m_b$. Without loss of generality, we assume $m_{a,0} = m_{a,1}$ and $m_{b,0} = m_{b,1}$. Now, by Theorem 1.2, we can find a solution of $(\mathcal{P}_{\lambda,\beta})$ with the same concentration behavior as the ground state solution obtained in Theorem 1.1 as $\lambda \to +\infty$. However, we do not know these two solutions are the same or not.

Next we consider the phenomenon of phase separations for System $(\mathcal{P}_{\lambda,\beta})$, i.e., the concentration behavior of the solutions as $\beta \to -\infty$. In the following theorem, we may observe such a phenomenon on the whole space $\mathbb{R}^3$.

Theorem 1.3 *Assume (D_1)–(D_5). Then, there exists $\Lambda_{**} \geq \Lambda_*$ such that $\beta^2 \int_{\mathbb{R}^3} u_{\lambda,\beta}^2 v_{\lambda,\beta}^2 \to 0$ as $\beta \to -\infty$ for $\lambda \geq \Lambda_{**}$, where $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is the ground state solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Theorem 1.1. Furthermore, for every $\{\beta_n\} \subset (-\infty, 0)$ with $\beta_n \to -\infty$ and $\lambda \geq \Lambda_{**}$, there exists $(u_{\lambda,0}, v_{\lambda,0}) \in (H^1(\mathbb{R}^3)\backslash\{0\}) \times (H^1(\mathbb{R}^3)\backslash\{0\})$ satisfying the following properties:*

(1) *$(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,0}, v_{\lambda,0})$ strongly in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $n \to \infty$ up to a subsequence.*
(2) *$u_{\lambda,0} \in C(\mathbb{R}^3)$ and $v_{\lambda,0} \in C(\mathbb{R}^3)$.*
(3) *$u_{\lambda,0} \geq 0$ and $v_{\lambda,0} \geq 0$ in $\mathbb{R}^3$ with $\{x \in \mathbb{R}^3 \mid u_{\lambda,0}(x) > 0\} = \mathbb{R}^3 \backslash \overline{\{x \in \mathbb{R}^3 \mid v_{\lambda,0}(x) > 0\}}$. Furthermore, $\{x \in \mathbb{R}^3 \mid u_{\lambda,0}(x) > 0\}$ and $\{x \in \mathbb{R}^3 \mid v_{\lambda,0}(x) > 0\}$ are connected domains.*
(4) *$u_{\lambda,0} \in H_0^1(\{u_{\lambda,0} > 0\})$ and is a least energy solution of*

$$-\Delta u + (\lambda a(x) + a_0(x))u = \mu_1 u^3, \quad u \in H_0^1(\{u_{\lambda,0} > 0\}), \tag{1.3}$$

while $v_{\lambda,0} \in H_0^1(\{v_{\lambda,0} > 0\})$ and is a least energy solution of

$$-\Delta v + (\lambda a(x) + a_0(x))v = \mu_1 v^3, \quad v \in H_0^1(\{v_{\lambda,0} > 0\}). \tag{1.4}$$

Remark 1.6 (1) Theorem 1.3 is based on Theorem 1.1, Thus, due to Remark 1.3, the assumption $\overline{\Omega}_a \cap \overline{\Omega}_b = \emptyset$ is also not necessary in Theorem 1.3 and it still holds for $\overline{\Omega}_a = \overline{\Omega}_b = \Omega$.

(2) It is natural to ask that whether the multi-bump solutions founded in Theorem 1.2 have the same phenomenon of phase separations as that of the ground state solution described in Theorem 1.3. However, by checking the proof of Theorem 1.2, we can see that $\Lambda_*(\beta) \to +\infty$ as $\beta \to -\infty$. Thus, our method to prove Theorem 1.2 is invalid to assert that the system $(\mathcal{P}_{\lambda,\beta})$ has multi-bump solutions for λ large but β fixed and β diverging. Due to this reason, we can not obtain the phenomenon of phase separations to the multi-bump solutions founded in Theorem 1.2.

Before closing this section, we would like to cite other references studying the equations with steep potential wells. For example, in [46], the Kirchhoff-type elliptic equation with a steep potential well was studied. The Schrödinger–Poisson systems with a steep potential well were considered in [32,52]. Non-trivial solutions were obtained in [26–28] for quasilinear Schrödinger equations with steep potential wells, while the multi-bump solutions were also obtained in [28] for such equations.

In this paper, we will always denote the usual norms in $H^1(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ $(p \geq 1)$ by $\|\cdot\|$ and $\|\cdot\|_p$, respectively; C and C' will be indiscriminately used to denote various positive constants; $o_n(1)$ will always denote the quantities tending toward zero as $n \to \infty$.

2 The variational setting

In this section, we mainly give a variational setting for $(\mathcal{P}_{\lambda,\beta})$. Simultaneously, an important estimate is also established in this section, which is used frequently in this paper.

Let

$$E_a = \{u \in D^{1,2}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} (a(x) + a_0^+(x))u^2 \mathrm{d}x < +\infty\}$$

and

$$E_b = \{u \in D^{1,2}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} (b(x) + b_0^+(x))u^2 \mathrm{d}x < +\infty\},$$

where $a_0^+(x) = \max\{a_0(x), 0\}$ and $b_0^+(x) = \max\{b_0(x), 0\}$. Then, by the conditions (D_1) and (D_4), E_a and E_b are Hilbert spaces equipped with the inner products

$$\langle u, v\rangle_a = \int_{\mathbb{R}^3} \nabla u \nabla v + (a(x) + a_0^+(x))uv\mathrm{d}x \quad \text{and}$$
$$\langle u, v\rangle_b = \int_{\mathbb{R}^3} \nabla u \nabla v + (b(x) + b_0^+(x))uv\mathrm{d}x,$$

respectively. The corresponding norms of E_a and E_b are, respectively, given by

$$\|u\|_a = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + (a(x) + a_0^+(x))u^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

and by

$$\|v\|_b = \left(\int_{\mathbb{R}^3} |\nabla v|^2 + (b(x) + b_0^+(x))v^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Since the conditions $(D_1)-(D_2)$ hold, by a similar argument as that in [46], we can see that

$$\|u\| \leq \left(\max\{1 + |\mathcal{D}_a|^{\frac{2}{3}} S^{-1}, \frac{1}{a_\infty}\}\right)^{\frac{1}{2}} \|u\|_a \quad \text{for all } u \in E_a \tag{2.1}$$

and

$$\|v\| \leq \left(\max\{1 + |\mathcal{D}_b|^{\frac{2}{3}} S^{-1}, \frac{1}{b_\infty}\}\right)^{\frac{1}{2}} \|v\|_b \quad \text{for all } v \in E_b, \tag{2.2}$$

where S is the best Sobolev embedding constant from $D^{1,2}(\mathbb{R}^3)$ to $L^6(\mathbb{R}^3)$ and given by

$$S = \inf\{\|\nabla u\|_2^2 \mid u \in D^{1,2}(\mathbb{R}^3), \|u\|_6^2 = 1\}.$$

It follows that both E_a and E_b are embedded continuously into $H^1(\mathbb{R}^3)$. Moreover, by applying the Hölder and Sobolev inequalities, we also have

$$\|u\|_4 \leq \left(\max\{1 + |\mathcal{D}_a|^{\frac{2}{3}} S^{-1}, \frac{1}{a_\infty}\}\right)^{\frac{1}{2}} S^{-\frac{3}{8}} \|u\|_a \quad \text{for all } u \in E_a \tag{2.3}$$

and

$$\|v\|_4 \leq \left(\max\{1 + |\mathcal{D}_b|^{\frac{2}{3}} S^{-1}, \frac{1}{b_\infty}\}\right)^{\frac{1}{2}} S^{-\frac{3}{8}} \|v\|_b \quad \text{for all } v \in E_b. \tag{2.4}$$

On the other hand, by the conditions (D_2) and (D_3), there exist two bounded open sets Ω_a' and Ω_b' with smooth boundaries such that $\Omega_a \subset \Omega_a' \subset \mathcal{D}_a$, $\Omega_b \subset \Omega_b' \subset \mathcal{D}_b$, $\overline{\Omega_a'} \cap \overline{\Omega_b'} = \emptyset$, $\mathrm{dist}(\Omega_a, \mathbb{R}^3 \backslash \Omega_a') > 0$ and that $\mathrm{dist}(\Omega_b, \mathbb{R}^3 \backslash \Omega_b') > 0$. Furthermore, by the condition (D_4), the

Hölder and the Sobolev inequalities, there exists $\Lambda_0 > 2\max\{1, d_a + \frac{d_a + C_{a,0}}{a_\infty}, d_b + \frac{d_b + C_{b,0}}{b_\infty}\}$ such that

$$\int_{\mathbb{R}^3} a_0^-(x)u^2 dx \leq \int_{B_R(0)} C_{a,0}u^2 dx + \int_{\mathbb{R}^3 \setminus B_R(0)} d_a(1 + a(x))u^2 dx \leq \frac{\lambda}{2}\|u\|_a^2 \tag{2.5}$$

and

$$\int_{\mathbb{R}^3} b_0^-(x)v^2 dx \leq \int_{B_R(0)} C_{b,0}v^2 dx + \int_{\mathbb{R}^3 \setminus B_R(0)} d_b(1 + b(x))v^2 dx \leq \frac{\lambda}{2}\|v\|_b^2 \tag{2.6}$$

for $\lambda \geq \Lambda_0$, where $B_R(0) = \{x \in \mathbb{R}^3 \mid |x| < R\}$, $C_{a,0} = \sup_{B_R(0)} a_0^-(x)$ and $C_{b,0} = \sup_{B_R(0)} b_0^-(x)$. Combining (2.3)–(2.6) and the Hölder inequality, we can see that $(\mathcal{P}_{\lambda,\beta})$ has a variational structure in the Hilbert space $E = E_a \times E_b$ for $\lambda \geq \Lambda_0$, where E is endowed with the norm $\|(u, v)\| = (\|u\|_a^2 + \|v\|_b^2)^{\frac{1}{2}}$. The corresponding functional of $(\mathcal{P}_{\lambda,\beta})$ is given by (1.2). Furthermore, by applying (2.3)–(2.6) in a standard way, we can also see that $J_{\lambda,\beta}(u, v)$ is C^2 in E and the solution of $(\mathcal{P}_{\lambda,\beta})$ is equivalent to the positive critical point of $J_{\lambda,\beta}(u, v)$ in E for $\lambda \geq \Lambda_0$. In the case of (D_3'), we can choose Ω_a' and Ω_b' as follows:

(I) $\Omega_a' = \bigcup_{i_a=1}^{n_a} \Omega_{a,i_a}' \subset \mathcal{D}_a$, where $\Omega_{a,i_a} \subset \Omega_{a,i_a}'$ and $\text{dist}(\Omega_{a,i_a}, \mathbb{R}^3 \setminus \Omega_{a,i_a}') > 0$ for all $i_a = 1, \ldots, n_a$ and $\overline{\Omega_{a,i_a}'} \cap \overline{\Omega_{a,j_a}'} = \emptyset$ for $i_a \neq j_a$.

(II) $\Omega_b' = \bigcup_{i_b=1}^{n_b} \Omega_{b,i_b}' \subset \mathcal{D}_b$, where $\Omega_{b,i_b} \subset \Omega_{b,i_b}'$ and $\text{dist}(\Omega_{b,i_b}, \mathbb{R}^3 \setminus \Omega_{b,i_b}') > 0$ for all $j_b = 1, \ldots, n_b$ and $\overline{\Omega_{b,i_b}'} \cap \overline{\Omega_{b,j_b}'} = \emptyset$ for $i_b \neq j_b$.

(III) $\overline{\Omega_a'} \cap \overline{\Omega_b'} = \emptyset$.

Thus, (2.5)–(2.6) still hold for such Ω_a' and Ω_b' with λ sufficiently large. Without loss of generality, we may assume that (2.5)–(2.6) still hold for such Ω_a' and Ω_b' with $\lambda \geq \Lambda_0$. It follows that the solution of $(\mathcal{P}_{\lambda,\beta})$ is also equivalent to the positive critical point of the C^2 functional $J_{\lambda,\beta}(u, v)$ in E for $\lambda \geq \Lambda_0$ under the conditions $(D_1)-(D_2)$, (D_3') and (D_4).

The remaining of this section will be devoted to an important estimate, which is used frequently in this paper and essentially due to Ding and Tanaka [24].

Lemma 2.1 *Assume* $(D_1)-(D_5)$. *Then, there exist* $\Lambda_1 \geq \Lambda_0$ *and* $C_{a,b} > 0$ *such that*

$$\inf_{u \in E_a \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2 dx}{\int_{\mathbb{R}^3} u^2 dx} \geq C_{a,b}$$

and

$$\inf_{v \in E_b \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla v|^2 + (\lambda b(x) + b_0(x))v^2 dx}{\int_{\mathbb{R}^3} v^2 dx} \geq C_{a,b}$$

for all $\lambda \geq \Lambda_1$.

Proof Since the conditions $(D_1)-(D_4)$ hold, by a similar argument as [24, Lemma 2.1], we have

$$\lim_{\lambda \to +\infty} \inf \sigma_{a,*}(-\Delta + \lambda a(x) + a_0(x)) = \inf \sigma_a(-\Delta + a_0(x)),$$

where $\sigma_{a,*}(-\Delta + \lambda a(x) + a_0(x))$ is the spectrum of $-\Delta + \lambda a(x) + a_0(x)$ on $H^1(\Omega_a')$. Denote $\inf \sigma_a(-\Delta + a_0(x))$ by ν_a. Then, by the condition (D_5), there exists $\Lambda_1' \geq \Lambda_0$ such that

$$\sigma_{a,*}(-\Delta + \lambda a(x) + a_0(x)) \geq \frac{\nu_a}{2} \quad \text{for} \quad \lambda \geq \Lambda_1'. \tag{2.7}$$

On the other hand, by the conditions (D_2) and (D_4), we have $a_0^-(x) \leq C_{a,0} + d_a + d_a a_\infty$ for $x \in \mathcal{D}_a$. Let $\mathcal{D}_{a,\overline{R}} = \mathcal{D}_a \cap B^c_{\overline{R}}$, where $B^c_{\overline{R}} = \{x \in \mathbb{R} \mid |x| \geq \overline{R}\}$. Then, by the condition (D_2) once more, $|\mathcal{D}_{a,\overline{R}}| \to 0$ as $\overline{R} \to +\infty$, which then implies that there exists $\overline{R}_0 > 0$ such that $|\mathcal{D}_{a,\overline{R}_0}|S^{-1}(C_{a,0} + d_a + d_a a_\infty + 1) \leq \frac{1}{2}$. Thanks to the conditions $(D_1)-(D_4)$, there exists $\Lambda_1 = \Lambda_1(\overline{R}_0) \geq \Lambda_1'$ such that

$$\lambda a(x) + a_0(x) + (C_{a,0} + d_a + d_a a_\infty + 1)\chi_{\mathcal{D}_{a,\overline{R}_0}} \geq 1 \quad \text{for all } x \in \mathbb{R}^3\backslash\Omega_a' \text{ and } \lambda \geq \Lambda_1,$$

where $\chi_{\mathcal{D}_{a,\overline{R}_0}}$ is the characteristic function of the set $\mathcal{D}_{a,\overline{R}_0}$. It follows from the Hölder and the Sobolev inequalities that

$$\int_{\mathbb{R}^3\backslash\Omega_a'} u^2 \mathrm{d}x \leq (1 + 2|\mathcal{D}_{a,\overline{R}_0}|S^{-1}) \int_{\mathbb{R}^3\backslash\Omega_a'} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2 \mathrm{d}x \tag{2.8}$$

for all $u \in E_a\backslash\{0\}$ and $\lambda \geq \Lambda_1$. Combining (2.7)–(2.8) and the choice of Ω_a', we have

$$\int_{\mathbb{R}^3} u^2 \mathrm{d}x \leq C_a \int_{\mathbb{R}^3} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2 \mathrm{d}x \quad \text{for all } u \in E_a\backslash\{0\} \text{ and } \lambda \geq \Lambda_1,$$

where $C_a = \max\{\frac{2}{\nu_a}, 1 + 2|\mathcal{D}_{a,\overline{R}_0}|S^{-1}\}$. By similar arguments as (2.7) and (2.8), we can also have

$$\int_{\mathbb{R}^3} v^2 \mathrm{d}x \leq C_b \int_{\mathbb{R}^3} |\nabla v|^2 + (\lambda b(x) + b_0(x))v^2 \mathrm{d}x$$

for all $v \in E_b\backslash\{0\}$ and $\lambda \geq \Lambda_1$, where $C_b = \max\{\frac{2}{\nu_b}, 1 + 2|\mathcal{D}_{b,\overline{R}_0}|S^{-1}\}$, $\nu_b = \inf \sigma_b(-\Delta + b_0(x))$ and $\mathcal{D}_{b,\overline{R}_0} = \mathcal{D}_b \cap B^c_{\overline{R}_0}$. We completes the proof by taking $C_{a,b} = (\min\{C_a, C_b\})^{-1}$. □

Remark 2.1 Under the conditions $(D_1)-(D_2)$, (D_3'), (D_4) and (D_5'), we can see that

$$\nu_a = \min_{i_a=1,2,\ldots,n_a} \left\{\inf \sigma_{a,i_a}(-\Delta+a_0(x))\right\} \quad \text{and} \quad \nu_b = \min_{j_b=1,2,\ldots,n_b} \left\{\inf \sigma_{b,j_b}(-\Delta+b_0(x))\right\}.$$

Now, by a similar argument as (2.7), we get that

$$\int_{\Omega'_{a,i_a}} u^2 \mathrm{d}x \leq \frac{2}{\nu_a} \int_{\Omega'_{a,i_a}} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2 \mathrm{d}x \tag{2.9}$$

and

$$\int_{\Omega'_{b,j_b}} v^2 \mathrm{d}x \leq \frac{2}{\nu_b} \int_{\Omega'_{b,j_b}} |\nabla v|^2 + (\lambda b(x) + b_0(x))v^2 \mathrm{d}x \tag{2.10}$$

for all $i_a = 1, \ldots, n_a$ and $j_b = 1, \ldots, n_b$ if λ sufficiently large. Without loss of generality, we may assume (2.9) and (2.10) hold for $\lambda \geq \Lambda_1$. It follows that Lemma 2.1 still holds under the conditions $(D_1)-(D_2)$, (D_3'), (D_4) and (D_5').

By Lemma 2.1, we observe that $\int_{\mathbb{R}^3} |\nabla u|^2+(\lambda a(x)+a_0(x))u^2 \mathrm{d}x$ and $\int_{\mathbb{R}^3} |\nabla v|^2+(\lambda b(x)+b_0(x))v^2 \mathrm{d}x$ are norms of E_a and E_b for $\lambda \geq \Lambda_1$, respectively. Therefore, we set

$$\|u\|^2_{a,\lambda} = \int_{\mathbb{R}^3} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2 \mathrm{d}x; \quad \|v\|^2_{b,\lambda} = \int_{\mathbb{R}^3} |\nabla v|^2 + (\lambda b(x) + b_0(x))v^2 \mathrm{d}x.$$

3 A ground state solution

Our interest in this section is to find a ground state solution to $(\mathcal{P}_{\lambda,\beta})$ under the conditions $(D_1)-(D_5)$. For the sake of convenience, we always assume the conditions $(D_1)-(D_5)$ hold in this section. Since $J_{\lambda,\beta}(u,v)$, the corresponding energy functional of $(\mathcal{P}_{\lambda,\beta})$, is C^2 in E, it is well known that all non-trivial solutions of $(\mathcal{P}_{\lambda,\beta})$ lie in the Nehari manifold of $J_{\lambda,\beta}(u,v)$, which is given by

$$\begin{aligned}\mathcal{N}_{\lambda,\beta} &= \{(u,v) \in E \mid u \neq 0, v \neq 0, \langle D[J_{\lambda,\beta}(u,v)], (u,0)\rangle_{E^*,E} \\ &= \langle D[J_{\lambda,\beta}(u,v)], (0,v)\rangle_{E^*,E} = 0\},\end{aligned}$$

where $D[J_{\lambda,\beta}(u,v)]$ is the Frechét derivative of the functional $J_{\lambda,\beta}$ in E at (u,v) and E^* is the dual space of E. If we can find $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E$ such that $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{\lambda,\beta}$ and $D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^*, then $(u_{\lambda,\beta}, v_{\lambda,\beta})$ must be a ground state solution of $(\mathcal{P}_{\lambda,\beta})$, where $m_{\lambda,\beta} = \inf_{\mathcal{N}_{\lambda,\beta}} J_{\lambda,\beta}(u,v)$. In what follows, we drive some properties of $\mathcal{N}_{\lambda,\beta}$.

Let $(u,v) \in (E_a\backslash\{0\}) \times (E_b\backslash\{0\})$ and define $T_{\lambda,\beta,u,v} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $T_{\lambda,\beta,u,v}(t,s) = J_{\lambda,\beta}(tu, sv)$. These functions are called the fibering maps of $J_{\lambda,\beta}(u,v)$, which are closely linked to $\mathcal{N}_{\lambda,\beta}$. Clearly, $\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(t,s) = \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(t,s) = 0$ is equivalent to $(tu, sv) \in \mathcal{N}_{\lambda,\beta}$. In particular, $\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(1,1) = \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(1,1) = 0$ if and only if $(u,v) \in \mathcal{N}_{\lambda,\beta}$. Let

$$\mathcal{A}_\beta = \{(u,v) \in E \mid \mu_1\mu_2\|u\|_4^4\|v\|_4^4 - \beta^2\|u^2v^2\|_1^2 > 0\}. \tag{3.1}$$

Then, $\mathcal{A}_\beta \neq \emptyset$ for every $\beta < 0$. Now, our first observation on $\mathcal{N}_{\lambda,\beta}$ can be stated as follows.

Lemma 3.1 *Assume $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, we have the following.*

(1) *If $(u,v) \in \mathcal{A}_\beta$, then there exists a unique $(t_{\lambda,\beta}(u,v), s_{\lambda,\beta}(u,v)) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that*

$$(t_{\lambda,\beta}(u,v)u, s_{\lambda,\beta}(u,v)v) \in \mathcal{N}_{\lambda,\beta},$$

where $t_{\lambda,\beta}(u,v)$ and $s_{\lambda,\beta}(u,v)$ are given by

$$t_{\lambda,\beta}(u,v) = \left(\frac{\mu_2\|v\|_4^4\|u\|_{a,\lambda}^2 - \beta\|u^2v^2\|_1\|v\|_{b,\lambda}^2}{\mu_1\mu_2\|u\|_4^4\|v\|_4^4 - \beta^2\|u^2v^2\|_1^2}\right)^{\frac{1}{2}} \tag{3.2}$$

and by

$$s_{\lambda,\beta}(u,v) = \left(\frac{\mu_1\|u\|_4^4\|v\|_{b,\lambda}^2 - \beta\|u^2v^2\|_1\|u\|_{a,\lambda}^2}{\mu_1\mu_2\|u\|_4^4\|v\|_4^4 - \beta^2\|u^2v^2\|_1^2}\right)^{\frac{1}{2}}. \tag{3.3}$$

Moreover, $T_{\lambda,\beta,u,v}(t_{\lambda,\beta}(u,v), s_{\lambda,\beta}(u,v)) = \max_{t\geq 0, s\geq 0} T_{\lambda,\beta,u,v}(t,s)$.

(2) *If $(u,v) \in E\backslash\mathcal{A}_\beta$, then $\mathcal{B}_{u,v} \cap \mathcal{N}_{\lambda,\beta} = \emptyset$, where $\mathcal{B}_{u,v} = \{(tu, sv) \mid (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+\}$.*

Proof (1) The proof is similar to [20, Lemma 2.2], where $\mathcal{N}_{0,\beta}$ with $a_0(x) = a_0 > 0$ and $b_0(x) = b_0 > 0$ was studied. For the reader's convenience, we give the details here. Let $T^1_{\lambda,\beta,u,v}$ and $T^2_{\lambda,\beta,u,v}$ be two functions on $\mathbb{R}^+ \times \mathbb{R}^+$ defined by

$$T^1_{\lambda,\beta,u,v}(t,s) = \|u\|_{a,\lambda}^2 - \mu_1\|u\|_4^4 t^2 - \beta\|u^2v^2\|_1 s^2$$

and by

$$T^2_{\lambda,\beta,u,v}(t,s) = \|v\|_{b,\lambda}^2 - \mu_2\|v\|_4^4 s^2 - \beta\|u^2v^2\|_1 t^2.$$

Then, it is easy to see that

$$\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(t,s) = tT^1_{\lambda,\beta,u,v}(t,s) \quad \text{and} \quad \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(t,s) = sT^2_{\lambda,\beta,u,v}(t,s). \tag{3.4}$$

Suppose $(u, v) \in \mathcal{A}_\beta$, $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, by Lemma 2.1, the two-component systems of algebraic equations, given by

$$\begin{cases} T^1_{\lambda,\beta,u,v}(t, s) = 0, \\ T^2_{\lambda,\beta,u,v}(t, s) = 0, \end{cases} \tag{3.5}$$

has a unique nonzero solution $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$ in $\mathbb{R}^+ \times \mathbb{R}^+$, where $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$ is characterized as (3.2) and (3.3). Hence, by (3.4), $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$ is the unique one in $\mathbb{R}^+ \times \mathbb{R}^+$ such that $\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(t, s) = \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(t, s) = 0$, that is, $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$ is the unique one in $\mathbb{R}^+ \times \mathbb{R}^+$ such that $(t_{\lambda,\beta}(u, v)u, s_{\lambda,\beta}(u, v)v) \in \mathcal{N}_{\lambda,\beta}$. It remains to show that $T_{\lambda,\beta,u,v}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) = \max_{t\geq 0,s\geq 0} T_{\lambda,\beta,u,v}(t, s)$. Indeed, by a direct calculation, we have

$$\frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial t^2}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) = -2\mu_1\|u\|_4^4[t(u, v)]^2 < 0$$

and

$$\begin{aligned}
&\begin{vmatrix} \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial t^2}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) & \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial t \partial s}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) \\ \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial s \partial t}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) & \frac{\partial^2 T_{\lambda,\beta,u,v}}{\partial s^2}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) \end{vmatrix} \\
&= \begin{vmatrix} -2[t_{\lambda,\beta}(u, v)]^2\mu_1\|u\|_4^4 & -2t_{\lambda,\beta}(u, v)s_{\lambda,\beta}(u, v)\beta\|u^2v^2\|_1 \\ -2t_{\lambda,\beta}(u, v)s_{\lambda,\beta}(u, v)\beta\|u^2v^2\|_1 & -2[s_{\lambda,\beta}(u, v)]^2\mu_2\|v\|_4^4 \end{vmatrix} \\
&= 4[t_{\lambda,\beta}(u, v)]^2[s_{\lambda,\beta}(u, v)]^2(\mu_1\mu_2\|u\|_4^4\|v\|_4^4 - \beta^2\|u^2v^2\|_1^2) > 0,
\end{aligned}$$

which implies that $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$ is a local maximum of $T_{\lambda,\beta,u,v}(t, s)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. It follows from the uniqueness of $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$, $T_{\lambda,\beta,u,v}(t, s) > 0$ for $|(t, s)|$ sufficiently small and $T_{\lambda,\beta,u,v}(t, s) \to -\infty$ as $|(t, s)| \to +\infty$ that $(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v))$ must be the global maximum of $T_{\lambda,\beta,u,v}(t, s)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. Thus, we have

$$T_{\lambda,\beta,u,v}(t_{\lambda,\beta}(u, v), s_{\lambda,\beta}(u, v)) = \max_{t\geq 0,s\geq 0} T_{\lambda,\beta,u,v}(t, s).$$

(2) Suppose $(u, v) \notin \mathcal{A}_\beta$, $\lambda \geq \Lambda_1$ and $\beta < 0$. If $\mathcal{B}_{u,v} \cap \mathcal{N}_{\lambda,\beta} \neq \emptyset$, then there exists $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\frac{\partial T_{\lambda,\beta,u,v}}{\partial t}(t, s) = \frac{\partial T_{\lambda,\beta,u,v}}{\partial s}(t, s) = 0$. It follows from (3.4) that (t, s) is a solution of (3.5) in $\mathbb{R}^+ \times \mathbb{R}^+$. On the other hand, since $(u, v) \notin \mathcal{A}_\beta$, $\lambda \geq \Lambda_1$ and $\beta < 0$, by Lemma 2.1, (3.5) has no solution in $\mathbb{R}^+ \times \mathbb{R}^+$, which is a contradiction. Hence, we must have $\mathcal{B}_{u,v} \cap \mathcal{N}_{\lambda,\beta} = \emptyset$ if $(u, v) \notin \mathcal{A}_\beta$, $\lambda \geq \Lambda_1$ and $\beta < 0$. □

By Lemma 3.1, we know that $\mathcal{N}_{\lambda,\beta} \subset \mathcal{A}_\beta$ for $\lambda \geq \Lambda_1$ and $\beta < 0$. Moreover, $m_{\lambda,\beta}$ is well defined and nonnegative for $\lambda \geq \Lambda_1$ and $\beta < 0$. Let

$$I_{a,\lambda}(u) = \frac{1}{2}\|u\|_{a,\lambda}^2 - \frac{\mu_1}{4}\|u\|_4^4 \quad \text{and} \quad I_{b,\lambda}(v) = \frac{1}{2}\|v\|_{b,\lambda}^2 - \frac{\mu_2}{4}\|v\|_4^4.$$

Then, by (2.3)–(2.6), $I_{a,\lambda}(u)$ is well defined on E_a and $I_{b,\lambda}(v)$ is well defined on E_b. Moreover, by a standard argument, we can see that $I_{a,\lambda}(u)$ and $I_{b,\lambda}(v)$ are of C^2 in E_a and E_b, respectively. Denote

$$\mathcal{N}_{a,\lambda} = \{u \in E_a\backslash\{0\} \mid I'_{a,\lambda}(u)u = 0\} \quad \text{and} \quad \mathcal{N}_{b,\lambda} = \{u \in E_b\backslash\{0\} \mid I'_{b,\lambda}(u)u = 0\}.$$

Clearly, $\mathcal{N}_{a,\lambda}$ and $\mathcal{N}_{b,\lambda}$ are nonempty, which together with Lemma 2.1 implies $m_{a,\lambda} = \inf_{\mathcal{N}_{a,\lambda}} I_{a,\lambda}(u)$ and $m_{b,\lambda} = \inf_{\mathcal{N}_{b,\lambda}} I_{b,\lambda}(v)$ are well defined and nonnegative. Due to this fact, we have the following.

Lemma 3.2 *Assume $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, $m_{\lambda,\beta} \in [m_{a,\lambda} + m_{b,\lambda}, m_a + m_b]$, where m_a and m_b are the least energy of nonzero critical points for $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$, respectively.*

Proof Suppose $(u, v) \in \mathcal{N}_{\lambda,\beta}$. Then, by Lemma 2.1 and $(u, v) \in \mathcal{N}_{\lambda,\beta} \subset \mathcal{A}_\beta$, we can see that $\|u\|^2_{a,\lambda} > 0$, $\|v\|^2_{b,\lambda} > 0$, $\|u\|_4 > 0$ and $\|v\|_4 > 0$. It follows that there exists a unique $(t^*(u), s^*(v)) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $(t^*(u)u, s^*(v)v) \in \mathcal{N}_{a,\lambda} \times \mathcal{N}_{b,\lambda}$. Note that $\beta < 0$, so by Lemma 3.1, we have

$$J_{\lambda,\beta}(u, v) \geq J_{\lambda,\beta}(t^*(u)u, s^*(v)v) \geq I_{a,\lambda}(t^*(u)u) + I_{b,\lambda}(s^*(v)v) \geq m_{a,\lambda} + m_{b,\lambda}.$$

Since $(u, v) \in \mathcal{N}_{\lambda,\beta}$ is arbitrary, we must have $m_{\lambda,\beta} \geq m_{a,\lambda} + m_{b,\lambda}$ for $\lambda \geq \Lambda_1$ and $\beta < 0$. It remains to show that $m_{\lambda,\beta} \leq m_a + m_b$ for $\lambda \geq \Lambda_1$ and $\beta < 0$. In fact, let $w_a \in H^1_0(\Omega_a)$ and $w_b \in H^1_0(\Omega_b)$ be the least energy nonzero critical points of $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$, respectively. Then, by the conditions (D_3) and (D_5), it is well known that

$$I_{\Omega_a}(w_a) = \max_{t\geq 0} I_{\Omega_a}(tw_a) \quad \text{and} \quad I_{\Omega_b}(w_b) = \max_{s\geq 0} I_{\Omega_b}(sw_b).$$

On the other hand, by the condition (D_3), we can extend w_a and w_b to $\mathbb{R}^3$ by letting $w_a = 0$ outside Ω_a and $w_b = 0$ outside Ω_b such that $w_a, w_b \in H^1(\mathbb{R}^3)$. Thanks to the condition (D_3) again, we can see that $(w_a, w_b) \in \mathcal{A}_\beta$. It follows from Lemma 3.1 that there exists $(t_{\lambda,\beta}(w_a, w_b), s_{\lambda,\beta}(w_a, w_b)) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$(t_{\lambda,\beta}(w_a, w_b)w_a, s_{\lambda,\beta}(w_a, w_b)w_b) \in \mathcal{N}_{\lambda,\beta} \quad \text{for } \lambda \geq \Lambda_1,$$

which together with the condition (D_3) once more implies

$$\begin{aligned} m_a + m_b &= I_{\Omega_a}(w_a) + I_{\Omega_b}(w_b) \\ &\geq I_{\Omega_a}(t_{\lambda,\beta}(w_a, w_b)w_a) + I_{\Omega_b}(s_{\lambda,\beta}(w_a, w_b)w_b) \\ &= J_{\lambda,\beta}(t_{\lambda,\beta}(w_a, w_b)w_a, s_{\lambda,\beta}(w_a, w_b)w_b) \\ &\geq m_{\lambda,\beta} \end{aligned}$$

for $\lambda \geq \Lambda_1$ and $\beta < 0$. □

Clearly, $m_{a,\lambda}$ and $m_{b,\lambda}$ are nondecreasing for λ. On the other hand, since Lemma 2.1 hold, by the conditions $(D_1)-(D_5)$, it is easy to show that $m_{a,\lambda}$ and $m_{b,\lambda}$ are positive for $\lambda \geq \Lambda_1$ and $m_{a,\lambda} \to m_a$ and $m_{b,\lambda} \to m_b$ as $\lambda \to +\infty$. This fact will help us to observe the following property of $\mathcal{N}_{\lambda,\beta}$, which is based on Lemma 3.2.

Lemma 3.3 *Assume $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, there exists $d_{\lambda,\beta} > 0$ such that $\mathcal{N}_{\lambda,\beta} \subset \mathcal{A}^{d_{\lambda,\beta}}_\beta$, where $\mathcal{A}^{d_{\lambda,\beta}}_\beta = \{(u, v) \in E \mid u \neq 0, v \neq 0, \mu_1\mu_2\|u\|^4_4\|v\|^4_4 - \beta^2\|u^2v^2\|^2_1 > d_{\lambda,\beta}\}$.*

Proof A similar result was obtained in [16]. But as we will see, some new ideas are needed due to the fact that $a_0(x)$ and $b_0(x)$ are sign-changing. Suppose the contrary. Since $\mathcal{N}_{\lambda,\beta} \subset \mathcal{A}_\beta$, there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ such that $\mu_1\mu_2\|u_n\|^4_4\|v_n\|^4_4 = \beta^2\|u^2_n v^2_n\|^2_1 + o_n(1)$, where $\mathcal{A}_\beta$ is given in (3.1). Clearly, one of the following two cases must happen:

(*a*) $\|u_n\|^4_4\|v_n\|^4_4 \geq C + o_n(1)$.
(*b*) $\|u_n\|^4_4\|v_n\|^4_4 = o_n(1)$ up to a subsequence.

Suppose case (*a*) happens. We claim that $\mu_1\|u_n\|^4_4 + \beta\|u^2_n v^2_n\|_1 = o_n(1)$ and $\mu_2\|v_n\|^4_4 + \beta\|u^2_n v^2_n\|_1 = o_n(1)$ up to a subsequence. If not, then up to a subsequence, we have

$$\mu_1\|u_n\|^4_4 + \beta\|u^2_n v^2_n\|_1 \geq C_1 + o_n(1) \quad \text{and} \quad \mu_2\|v_n\|^4_4 + \beta\|u^2_n v^2_n\|_1 \geq C_2 + o_n(1)$$

for $\lambda \geq \Lambda_1$ and $\beta < 0$, where C_1, C_2 are nonnegative constants with $C_1 + C_2 > 0$. It follows from $\beta < 0$ that

$$\begin{aligned}\mu_1\mu_2\|u_n\|_4^4\|v_n\|_4^4 &\geq (C_1 + o_n(1) + |\beta|\|u_n^2v_n^2\|_1)(C_2 + o_n(1) + |\beta|\|u_n^2v_n^2\|_1)\\ &\geq \beta^2\|u_n^2v_n^2\|_1^2 + (C_1 + C_2 + o_n(1))|\beta|\|u_n^2v_n^2\|_1 + C_1C_2 + o_n(1)\\ &\geq \beta^2\|u_n^2v_n^2\|_1^2 + \frac{1}{2}(C_1 + C_2)\sqrt{C} + o_n(1)\end{aligned}$$

for n large enough, which contradicts to $\mu_1\mu_2\|u_n\|_4^4\|v_n\|_4^4 = \beta^2\|u_n^2v_n^2\|_1^2 + o_n(1)$. This together with $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ implies that $\|u_n\|_{a,\lambda} = o_n(1)$ and $\|v_n\|_{b,\lambda} = o_n(1)$ up to a subsequence. Note that $J_{\lambda,\beta}(u_n, v_n) = \frac{1}{4}(\|u_n\|_{a,\lambda}^2 + \|v_n\|_{b,\lambda}^2)$. So $m_{\lambda,\beta} \leq 0$ in case (a), which is impossible for $\lambda \geq \Lambda_1$ and $\beta < 0$ due to Lemma 3.2. Now, we must have case (b). It follows that $\|u_n\|_4 = o_n(1)$ or $\|v_n\|_4 = o_n(1)$ up to a subsequence. Without loss of generality, we assume $\|u_n\|_4 = o_n(1)$. Since $\mu_1\mu_2\|u_n\|_4^4\|v_n\|_4^4 = \beta^2\|u_n^2v_n^2\|_1^2 + o_n(1)$, we also have $\beta\|u_n^2v_n^2\|_1 = o_n(1)$ in this case. These together with $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ imply $\|u_n\|_{a,\lambda} = o_n(1)$. Therefore, $J_{\lambda,\beta}(u_n, v_n) = I_{b,\lambda}(v_n) + o_n(1)$. On the other hand, since $\lambda \geq \Lambda_1$ and $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$, by Lemma 2.1 and $\mathcal{N}_{\lambda,\beta} \subset \mathcal{A}_\beta$, for all $n \in \mathbb{N}$, there exists a unique $t^*(u_n) > 0$ such that $t^*(u_n)u_n \in \mathcal{N}_{a,\lambda}$. It follows from Lemma 3.1 and $\beta < 0$ that

$$\begin{aligned}J_{\lambda,\beta}(u_n, v_n) &\geq J_{\lambda,\beta}(t^*(u_n)u_n, v_n)\\ &\geq I_{a,\lambda}(t^*(u_n)u_n) + I_{b,\lambda}(v_n)\\ &\geq m_{a,\lambda} + I_{b,\lambda}(v_n)\\ &= m_{a,\lambda} + J_{\lambda,\beta}(u_n, v_n) + o_n(1)\end{aligned}$$

for $\lambda \geq \Lambda_1$ and $\beta < 0$, which is also impossible for n large enough. Thus, there exists $d_{\lambda,\beta} > 0$ such that $\mathcal{N}_{\lambda,\beta} \subset \mathcal{A}_\beta^{d_{\lambda,\beta}}$ for $\lambda \geq \Lambda_1$ and $\beta < 0$. □

We also have the following.

Lemma 3.4 *Suppose $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, $\mathcal{N}_{\lambda,\beta}$ is a natural constraint.*

Proof Let $\varphi_{\lambda,\beta}(u, v) = \langle D[J_{\lambda,\beta}(u, v)], (u, v)\rangle_{E^*,E}$. Then, by (2.3)–(2.6), $\varphi_{\lambda,\beta}(u, v)$ is C^2 in E for $\lambda \geq \Lambda_1$ and $\beta < 0$. Since $\lambda \geq \Lambda_1$ and $(u, v) \in \mathcal{N}_{\lambda,\beta}$, we have

$$\langle D[\varphi_{\lambda,\beta}(u, v)], (u, v)\rangle_{E^*,E} = -2(\mu_1\|u\|_4^4 + \mu_2\|v\|_4^4 + 2\beta\|u^2v^2\|_1) \leq -8m_{\lambda,\beta}.$$

It follows from Lemma 3.2 that $\mathcal{N}_{\lambda,\beta}$ is a natural constraint for $\lambda \geq \Lambda_1$ and $\beta < 0$. □

Now, we can obtain a ground state solution for $(\mathcal{P}_{\lambda,\beta})$.

Proposition 3.1 *There exists $\Lambda_2 \geq \Lambda_1$ such that $(\mathcal{P}_{\lambda,\beta})$ has a ground state solution $(u_{\lambda,\beta}, v_{\lambda,\beta})$ for $\lambda \geq \Lambda_2$ and $\beta < 0$. Furthermore, we have*

$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^3\setminus\Omega_a'} |\nabla u_{\lambda,\beta}|^2 + (\lambda a(x) + a_0(x))u_{\lambda,\beta}^2 \mathrm{d}x = 0 \tag{3.6}$$

and

$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^3\setminus\Omega_b'} |\nabla v_{\lambda,\beta}|^2 + (\lambda b(x) + b_0(x))v_{\lambda,\beta}^2 \mathrm{d}x = 0. \tag{3.7}$$

Proof Let $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, for every $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ satisfying $J_{\lambda,\beta}(u_n, v_n) = m_{\lambda,\beta} + o_n(1)$, we can see from Lemma 2.1 that

$$\begin{aligned} m_{\lambda,\beta} + o_n(1) &\geq J_{\lambda,\beta}(u_n, v_n) - \frac{1}{4}\langle D[J_{\lambda,\beta}(u_n, v_n)], (u_n, v_n)\rangle_{E^*,E} \\ &= \frac{1}{4}\|u_n\|_{a,\lambda}^2 + \frac{1}{4}\|v_n\|_{b,\lambda}^2 \\ &\geq \frac{1}{4C_{a,b}}(\|u_n\|_2^2 + \|v_n\|_2^2), \end{aligned} \tag{3.8}$$

which together with the condition (D_4) and $\lambda \geq \Lambda_1$ implies

$$\begin{aligned} m_{\lambda,\beta} + o_n(1) &\geq \frac{1}{4}\|u_n\|_{a,\lambda}^2 + \frac{1}{4}\|v_n\|_{b,\lambda}^2 \\ &\geq \frac{1}{8}\|(u_n, v_n)\|^2 - C(m_{\lambda,\beta} + o_n(1)). \end{aligned} \tag{3.9}$$

It follows that $\|(u_n, v_n)\| \leq 8(C+1)(m_{\lambda,\beta} + o_n(1))$. Now, by Lemma 3.3, we can apply the implicit function theorem and the Ekeland variational principle in a standard way (cf. [13,36]) to show that there exists $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ such that $D[J_{\lambda,\beta}(u_n, v_n)] = o_n(1)$ strongly in E^* and $J_{\lambda,\beta}(u_n, v_n) = m_{\lambda,\beta} + o_n(1)$. Since $m_{\lambda,\beta} \leq m_a + m_b$, by similar arguments as (3.8) and (3.9), we have $\|(u_n, v_n)\| \leq 8(C+1)(m_a + m_b + o_n(1))$ and $(u_n, v_n) \rightharpoonup (u_{\lambda,\beta}, v_{\lambda,\beta})$ weakly in E as $n \to \infty$ for some $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E$. Clearly, $D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^*. Suppose $u_{\lambda,\beta} = 0$. Then, by the fact that E_a is embedded continuously into $H^1(\mathbb{R}^3)$, we have

$$u_n = o_n(1) \quad \text{strongly in } L^p_{loc}(\mathbb{R}^3) \quad \text{for } 2 \leq p < 6.$$

Combining with the condition (D_2) and the Hölder and the Sobolev inequalities, we get

$$\begin{aligned} \int_{\mathbb{R}^3} |u_n|^4 \mathrm{d}x &= \int_{\mathcal{D}_a} |u_n|^4 \mathrm{d}x + \int_{\mathbb{R}^3 \setminus \mathcal{D}_a} |u_n|^4 \mathrm{d}x \\ &= \int_{\mathbb{R}^3 \setminus \mathcal{D}_a} |u_n|^4 \mathrm{d}x + o_n(1) \\ &\leq \left(\frac{1}{a_\infty}\right)^{\frac{1}{2}} \int_{\mathbb{R}^3 \setminus \mathcal{D}_a} [a(x)]^{\frac{1}{2}} |u_n|^4 \mathrm{d}x + o_n(1) \\ &\leq \left(\frac{1}{a_\infty S^3}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3 \setminus \mathcal{D}_a} a(x)|u_n|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^{\frac{3}{2}} + o_n(1). \end{aligned} \tag{3.10}$$

Since $u_n = o_n(1)$ strongly in $L^p_{loc}(\mathbb{R}^3)$ for $2 \leq p < 6$, by the conditions (D_2) and (D_4), $\int_{\mathcal{D}_a} (\lambda a(x) + a_0(x)) u_n^2 \mathrm{d}x = o_n(1)$. It follows from (3.8) and (3.10) that

$$\begin{aligned} \int_{\mathbb{R}^3} |u_n|^4 \mathrm{d}x &\leq \left(\frac{1}{a_\infty S^3}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3 \setminus \mathcal{D}_a} a(x)|u_n|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^{\frac{3}{2}} + o_n(1) \\ &\leq \left(\frac{2}{a_\infty S^3 \lambda}\right)^{\frac{1}{2}} \|u_n\|_{a,\lambda} (\|u_n\|_{a,\lambda}^2 + o_n(1))^{\frac{3}{2}} + o_n(1) \\ &\leq \left(\frac{2}{a_\infty S^3 \lambda}\right)^{\frac{1}{2}} \|u_n\|_{a,\lambda}^4 + o_n(1) \end{aligned} \tag{3.11}$$

for $\lambda \geq \Lambda_1$. Note that $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$ and $\beta < 0$, from (3.8) and (3.11), we have

$$\|u_n\|_{a,\lambda}^2 \leq \left(\frac{2}{a_\infty S^3 \lambda}\right)^{\frac{1}{2}} \|u_n\|_{a,\lambda}^4 + o_n(1) \leq 4(m_a + m_b)\left(\frac{2}{a_\infty S^3 \lambda}\right)^{\frac{1}{2}} \|u_n\|_{a,\lambda}^2 + o_n(1), \tag{3.12}$$

which then implies that there exists $\Lambda_2 \geq \Lambda_1$ such that $\|u_n\|_{a,\lambda} = o_n(1)$ for $\lambda \geq \Lambda_2$ and $\beta < 0$. It follows from Lemma 2.1, the Hölder and the Sobolev inequalities and the boundedness of $\{(u_n, v_n)\}$ in E that $\|u_n\|_4 = o_n(1)$, hence, $\mu_1\mu_2\|u_n\|_4^4\|v_n\|_4^4 = o_n(1)$ for $\lambda \geq \Lambda_2$ and $\beta < 0$. However, it is impossible, since $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\beta}$, $\Lambda_2 \geq \Lambda_1$ and Lemma 3.3 holds for $\lambda \geq \Lambda_1$. Therefore, there exists $\Lambda_2 \geq \Lambda_1$ such that $u_{\lambda,\beta} \neq 0$ for $\lambda \geq \Lambda_2$ and $\beta < 0$. Similarly, we can also show that $v_{\lambda,\beta} \neq 0$ for $\lambda \geq \Lambda_2$ and $\beta < 0$. Since $(u_n, v_n) \rightharpoonup (u_{\lambda,\beta}, v_{\lambda,\beta})$ weakly in E as $n \to \infty$, by the fact that E is embedded continuously into $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, we have $(u_n, v_n) \to (u_{\lambda,\beta}, v_{\lambda,\beta})$ strongly in $L^p_{loc}(\mathbb{R}^3) \times L^p_{loc}(\mathbb{R}^3)$ as $n \to \infty$ for $2 \leq p < 6$. It follows from the boundedness of $\{(u_n, v_n)\}$ in E and the conditions (D_2) and (D_4) that $\int_{\mathcal{D}_a} a_0^-(x)u_n^2\mathrm{d}x = \int_{\mathcal{D}_a} a_0^-(x)u_{\lambda,\beta}^2\mathrm{d}x + o_n(1)$ and $\int_{\mathcal{D}_b} b_0^-(x)v_n^2\mathrm{d}x = \int_{\mathcal{D}_b} b_0^-(x)v_{\lambda,\beta}^2\mathrm{d}x + o_n(1)$, which together with $D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^*, the Fatou lemma and the conditions (D_2) and (D_4), implies

$$\begin{aligned}
m_{\lambda,\beta} &\leq J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) - \frac{1}{4}\langle D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})], (u_{\lambda,\beta}, v_{\lambda,\beta})\rangle_{E^*,E} \\
&= \frac{1}{4}(\|u_{\lambda,\beta}\|_{a,\lambda}^2 + \|v_{\lambda,\beta}\|_{b,\lambda}^2) \\
&\leq \liminf_{n\to\infty} \frac{1}{4}(\|u_n\|_{a,\lambda}^2 + \|v_n\|_{b,\lambda}^2) \\
&= \liminf_{n\to\infty}(J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) - \frac{1}{4}\langle D[J_{\lambda,\beta}(u_n, v_n)], (u_n, v_n)\rangle_{E^*,E}) \\
&= m_{\lambda,\beta} + o_n(1).
\end{aligned} \tag{3.13}$$

Therefore, $J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{\lambda,\beta}$. Since $J_{\lambda,\beta}(|u_{\lambda,\beta}|, |v_{\lambda,\beta}|) = m_{\lambda,\beta}$ and $(|u_{\lambda,\beta}|, |v_{\lambda,\beta}|) \in \mathcal{N}_{\lambda,\beta}$, $(|u_{\lambda,\beta}|, |v_{\lambda,\beta}|)$ is a local minimizer of $J_{\lambda,\beta}(u, v)$ on $\mathcal{N}_{\lambda,\beta}$. Note that by Lemma 3.4, $\mathcal{N}_{\lambda,\beta}$ is a natural constraint, we can follow the argument as used in [6, Theorem 2.3] to show that $D[J_{\lambda,\beta}(|u_{\lambda,\beta}|, |v_{\lambda,\beta}|)] = 0$ in E^*. Thus, without loss of generality, we may assume $u_{\lambda,\beta}$ and $v_{\lambda,\beta}$ both are nonnegative. Now, since $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E$, by (2.1) and (2.2), we have $u_{\lambda,\beta}, v_{\lambda,\beta} \in H^1(\mathbb{R}^3)$. It follows from the conditions (D_1) and (D_4) and the Calderon-Zygmund inequality that $u_{\lambda,\beta}, v_{\lambda,\beta} \in W^{2,2}_{loc}(\mathbb{R}^3)$. By combining the Sobolev embedding theorem and the Harnack inequality, $u_{\lambda,\beta}$ and $v_{\lambda,\beta}$ are both positive. Hence, $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a ground state solution of $(\mathcal{P}_{\lambda,\beta})$ for $\beta < 0$ and $\lambda \geq \Lambda_2$. It remains to show that (3.6) and (3.7) are true. Indeed, let Ω_a'' be a bounded domain with smooth boundary in $\mathbb{R}^3$ satisfying $\Omega_a \subset \Omega_a'' \subset \Omega_a'$, $dist(\Omega_a'', \mathbb{R}^3\backslash\Omega_a') > 0$ and $dist(\mathbb{R}^3\backslash\Omega_a'', \Omega_a) > 0$. Then, by a similar argument as (2.7), we can show that

$$\int_{\Omega_a''} |\nabla u_{\lambda,\beta}|^2 + (\lambda a(x) + a_0(x))u_{\lambda,\beta}^2\mathrm{d}x \geq \frac{\nu_a}{2}\int_{\Omega_a''} u_{\lambda,\beta}^2\mathrm{d}x \tag{3.14}$$

for λ large enough. Without loss of generality, we assume (3.14) holds for $\lambda \geq \Lambda_2$. Since $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is a ground state solution for $\lambda \geq \Lambda_2$, by combining Lemma 2.1, (3.14) and similar arguments of (3.8) and (3.9), we can see that $\|(u_{\lambda,\beta}, v_{\lambda,\beta})\|^2 \leq 8(4(C_{a,0}+d_a)C_{a,b}+1)(m_a + m_b)$ and $8(4(C_{a,0} + d_a)C_{a,b} + 1)(m_a + m_b) \geq \lambda\int_{\mathbb{R}^3\backslash\Omega_a''} a(x)u_{\lambda,\beta}^2\mathrm{d}x$ for $\lambda \geq \Lambda_2$, which together with the conditions $(D_1)-(D_3)$ imply $\int_{(\mathbb{R}^3\backslash\Omega_a'')\cap\mathcal{D}_a} u_{\lambda,\beta}^2\mathrm{d}x \to 0$ as

$\lambda \to +\infty$. It follows from the condition (D_2) and $8(4(C_{a,0}+d_a)C_{a,b}+1)(m_a+m_b) \geq \lambda \int_{\mathbb{R}^3\backslash\Omega_a''} a(x)u_{\lambda,\beta}^2\mathrm{d}x$ once more that

$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^3\backslash\Omega_a''} u_{\lambda,\beta}^2\mathrm{d}x = 0. \tag{3.15}$$

Now, we choose $\Psi \in C^\infty(\mathbb{R}^3, [0,1])$ satisfying

$$\Psi = \begin{cases} 1, & x \in \mathbb{R}^3\backslash\Omega_a', \\ 0, & x \in \Omega_a''. \end{cases} \tag{3.16}$$

Then, $u_{\lambda,\beta}\Psi \in E_a$. Note $D[J_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^* for $\lambda \geq \Lambda_2$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^3}(|\nabla u_{\lambda,\beta}|^2 + (\lambda a(x)+a_0(x))u_{\lambda,\beta}^2)\Psi\mathrm{d}x + \int_{\mathbb{R}^3}(\nabla u_{\lambda,\beta}\nabla\Psi)u_{\lambda,\beta}\mathrm{d}x\\ &\quad= \mu_1\int_{\mathbb{R}^3} u_{\lambda,\beta}^4\Psi\mathrm{d}x + \beta\int_{\mathbb{R}^3} v_{\lambda,\beta}^2u_{\lambda,\beta}^2\Psi\mathrm{d}x.\end{aligned}$$

It follows from the Hölder and the Sobolev inequalities that

$$\begin{aligned}&\int_{\mathbb{R}^3\backslash\Omega_a'}(|\nabla u_{\lambda,\beta}|^2 + (\lambda a(x)+a_0(x))u_{\lambda,\beta}^2)\mathrm{d}x\\ &\leq \mu_1\int_{\mathbb{R}^3\backslash\Omega_a''} u_{\lambda,\beta}^4\mathrm{d}x + \beta\int_{\mathbb{R}^3\backslash\Omega_a'} v_{\lambda,\beta}^2u_{\lambda,\beta}^2\mathrm{d}x + \int_{\Omega_a'\backslash\Omega_a''}|\nabla u_{\lambda,\beta}||\nabla\Psi||u_{\lambda,\beta}|\mathrm{d}x\\ &\leq \mu_1 S^{-\frac{3}{2}}\|u_{\lambda,\beta}\|_a^3\Big(\int_{\mathbb{R}^3\backslash\Omega_a''} u_{\lambda,\beta}^2\mathrm{d}x\Big)^{\frac{1}{2}} + \max_{\mathbb{R}^3}|\nabla\Psi|\|u_{\lambda,\beta}\|_a\Big(\int_{\mathbb{R}^3\backslash\Omega_a''} u_{\lambda,\beta}^2\mathrm{d}x\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.17}$$

Since $\|(u_{\lambda,\beta}, v_{\lambda,\beta})\|^2 \leq 8(4(C_{a,0}+d_a)C_{a,b}+1)(m_a+m_b)$, we can conclude from (2.8), (3.15) and (3.17) that (3.6) holds. Similarly, we can also conclude that (3.7) is true. □

We close this section by

Proof of Theorem 1.1 By Proposition 3.1, we know that there exists $\Lambda_2 \geq \Lambda_1$ such that $(\mathcal{P}_{\lambda,\beta})$ has a ground state solution $(u_{\lambda,\beta}, v_{\lambda,\beta})$ for $\lambda \geq \Lambda_2$ and $\beta < 0$. In what follows, we will show that $(u_{\lambda,\beta}, v_{\lambda,\beta})$ has the concentration behaviors for $\lambda \to +\infty$ described as (1)–(5). We first verify (3)–(5). Let $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ be the ground state solution of $(\mathcal{P}_{\lambda_n,\beta})$ obtained by Proposition 3.1 with $\lambda_n \to +\infty$ as $n \to \infty$. Then, by Lemma 3.2 and Proposition 3.1, $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}$ is bounded in E with

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3\backslash\Omega_a'}|\nabla u_{\lambda_n,\beta}|^2 + (\lambda_n a(x)+a_0(x))u_{\lambda_n,\beta}^2\mathrm{d}x = 0 \tag{3.18}$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3\backslash\Omega_b'}|\nabla v_{\lambda_n,\beta}|^2 + (\lambda_n b(x)+b_0(x))v_{\lambda_n,\beta}^2\mathrm{d}x = 0. \tag{3.19}$$

Without loss of generality, we assume $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \rightharpoonup (u_{0,\beta}, v_{0,\beta})$ weakly in E as $n \to \infty$ for some $(u_{0,\beta}, v_{0,\beta}) \in E$. By (2.1) and (2.2), $(u_{0,\beta}, v_{0,\beta}) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. For the sake of clarity, the verification of (3)–(5) is further performed through the following three steps.

Step 1 We prove that $(u_{0,\beta}, v_{0,\beta}) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b)$ with $u_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega_a$ and $v_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega_b$.

Indeed, since $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ is a ground state solution of $(\mathcal{P}_{\lambda_n,\beta})$, by Lemma 3.2 and a similar argument as (3.8), we get that $4C_{a,b}(m_a+m_b) \geq (\|u_{\lambda_n,\beta}\|_2^2 + \|v_{\lambda_n,\beta}\|_2^2)$. Now, by the condition (D_4) and a similar argument as (3.8) again, we have

$$\begin{aligned} m_a + m_b &\geq J_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) - \frac{1}{4}\langle D[J_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})], (u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\rangle_{E^*,E} \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3}(\lambda_n a(x)+a_0(x))u_{\lambda_n,\beta}^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}(\lambda_n b(x)+b_0(x))v_{\lambda_n,\beta}^2 \mathrm{d}x \\ &\geq \frac{\lambda_n}{8}\int_{\mathbb{R}^3} a(x)u_{\lambda_n,\beta}^2 + b(x)v_{\lambda_n,\beta}^2\mathrm{d}x - C(\|(u_{\lambda_n,\beta}\|_2^2 + \|v_{\lambda_n,\beta})\|_2^2) \\ &\geq \frac{\lambda_n}{8}\int_{\mathbb{R}^3} a(x)u_{\lambda_n,\beta}^2 + b(x)v_{\lambda_n,\beta}^2\mathrm{d}x - C'. \end{aligned}$$

It follows that $\lim_{n\to+\infty}\int_{\mathbb{R}^3} a(x)u_{\lambda_n,\beta}^2\mathrm{d}x = \lim_{n\to+\infty}\int_{\mathbb{R}^3} b(x)v_{\lambda_n,\beta}^2\mathrm{d}x = 0$. By the Fatou Lemma and the conditions (D_1) and (D_3), we can see that $u_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_a$ and $v_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_b$. Since $(u_{0,\beta}, v_{0,\beta}) \in H^1(\mathbb{R}^3)$, by the condition (D_3) again, we must have $u_{0,\beta} \in H_0^1(\Omega_a)$ and $v_{0,\beta} \in H_0^1(\Omega_b)$.

Step 2 We prove that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ strongly in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ as $n\to\infty$ up to a subsequence.

Indeed, by the choice of Ω_a' and the Sobolev embedding theorem, we can see that $u_{\lambda_n,\beta} \to u_{0,\beta}$ strongly in $L^2(\Omega_a')$ as $n\to\infty$ up to a subsequence. Without loss of generality, we may assume $u_{\lambda_n,\beta} \to u_{0,\beta}$ strongly in $L^2(\Omega_a')$. It follows from $u_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_a$, (2.8) and (3.18) that $u_{\lambda_n,\beta} \to u_{0,\beta}$ strongly in $L^2(\mathbb{R}^3)$ as $n\to\infty$. By the Hölder and the Sobolev inequalities and the boundedness of $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}$ in E, we can see that $u_{\lambda_n,\beta} \to u_{0,\beta}$ strongly in $L^4(\mathbb{R}^3)$ as $n\to\infty$. On the other hand, by a similar argument as (3.13), we have

$$\begin{aligned} \int_{\mathbb{R}^3}|\nabla u_{0,\beta}|^2 + a_0(x)u_{0,\beta}^2\mathrm{d}x &= \int_{\mathbb{R}^3}|\nabla u_{0,\beta}|^2\mathrm{d}x + (\lambda_n a(x)+a_0(x))u_{0,\beta}^2\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_{\lambda_n,\beta}|^2 + (\lambda_n a(x)+a_0(x))u_{\lambda_n,\beta}^2\mathrm{d}x, \end{aligned}$$

which together with $D[J_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})]=0$ in E^* and $\beta<0$ implies

$$\int_{\mathbb{R}^3}|\nabla u_{0,\beta}|^2 + a_0(x)u_{0,\beta}^2\mathrm{d}x \leq \liminf_{n\to\infty}\mu_1\int_{\mathbb{R}^3}u_{\lambda_n,\beta}^4\mathrm{d}x = \mu_1\int_{\Omega_a}u_{0,\beta}^4\mathrm{d}x.$$

Note that $(u_{0,\beta}, v_{0,\beta}) \in H_0^1(\Omega_a)\times H_0^1(\Omega_b)$ with $u_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_a$ and $v_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_b$ and $D[J_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})]=0$ in E^*, by the condition (D_3), we can show that $I_{\Omega_a}'(u_{0,\beta})=0$ in $H^{-1}(\Omega_a)$ and $I_{\Omega_b}'(v_{0,\beta})=0$ in $H^{-1}(\Omega_b)$. Recalling that $u_{\lambda_n,\beta}\to u_{0,\beta}$ strongly in $L^2(\mathbb{R}^3)$ as $n\to\infty$, $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}$ is bounded in E and the conditions (D_2) and (D_4) hold, we must have $\int_{\mathcal{D}_a}a_0^-(x)u_{\lambda_n,\beta}^2\mathrm{d}x = \int_{\mathcal{D}_a}a_0^-(x)u_{0,\beta}^2\mathrm{d}x + o_n(1)$. It follows from the conditions (D_2) and (D_4) again and the Fatou Lemma that

$$\begin{aligned} \int_{\mathbb{R}^3}|\nabla u_{0,\beta}|^2 + a_0(x)u_{0,\beta}^2\mathrm{d}x &= \liminf_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_{\lambda_n,\beta}|^2 + (\lambda_n a(x)+a_0(x))u_{\lambda_n,\beta}^2\mathrm{d}x \\ &\geq \liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla u_{\lambda_n,\beta}|^2 + \frac{\lambda_n}{2}a(x)u_{\lambda_n,\beta}^2\mathrm{d}x\right) \\ &\quad + \int_{\mathbb{R}^3}a_0^+(x)u_{0,\beta}^2\mathrm{d}x + \int_{\mathcal{D}_a}a_0^-(x)u_{0,\beta}^2\mathrm{d}x. \end{aligned} \tag{3.20}$$

By the conditions $(D_1)-(D_3)$, the Fatou lemma and the fact $u_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_a$, we can see from (3.20) that $\nabla u_{\lambda_n,\beta}\to\nabla u_{0,\beta}$ strongly in $L^2(\mathbb{R}^3)$ as $n\to\infty$ up to a subsequence, which then implies $\liminf_{n\to\infty}\int_{\mathbb{R}^3}\lambda_n a(x)u^2_{\lambda_n,\beta}\mathrm{d}x=0$ and $\liminf_{n\to\infty}\int_{\mathbb{R}^3}a_0(x)u^2_{\lambda_n,\beta}\mathrm{d}x=\int_{\mathbb{R}^3}a_0(x)u^2_{0,\beta}\mathrm{d}x$. These together with the conditions $(D_1)-(D_4)$, $u_{0,\beta}=0$ on $\mathbb{R}^3\backslash\Omega_a$ and $u_{\lambda_n,\beta}\to u_{0,\beta}$ strongly in $L^2(\mathbb{R}^3)$ as $n\to\infty$ imply $u_{\lambda_n,\beta}\to u_{0,\beta}$ strongly in E_a as $n\to\infty$ up to a subsequence. Without loss of generality, we assume $u_{\lambda_n,\beta}\to u_{0,\beta}$ strongly in E_a as $n\to\infty$. Similarly, we also have $v_{\lambda_n,\beta}\to v_{0,\beta}$ strongly in E_b as $n\to\infty$, that is, $(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\to(u_{0,\beta},v_{0,\beta})$ strongly in E as $n\to\infty$. Since E is embedded continuously into $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, $(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\to(u_{0,\beta},v_{0,\beta})$ strongly in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ as $n\to\infty$.

Step 3 We prove that $u_{0,\beta}$ and $v_{0,\beta}$ are least energy nonzero critical points of $I_{\Omega_a}(u)$ and $I_{\Omega_b}(v)$, respectively.

Indeed, since $(u_{\lambda_n,\beta},v_{\lambda_n,\beta})$ is the ground state solution of $(\mathcal{P}_{\lambda_n,\beta})$, by Lemma 3.2, we can see that

$$\|u_{\lambda_n,\beta}\|_{a,\lambda_n}+\|v_{\lambda_n,\beta}\|_{b,\lambda_n}=4(m_a+m_b)+o_n(1). \tag{3.21}$$

By a similar argument as used in Step 2, we can show that

$$\lambda_n\int_{\mathbb{R}^3}a(x)u^2_{\lambda_n,\beta}\mathrm{d}x=\lambda_n\int_{\mathbb{R}^3}b(x)v^2_{\lambda_n,\beta}\mathrm{d}x=o_n(1)$$

and

$$\int_{\mathbb{R}^3}a_0(x)u^2_{\lambda_n,\beta}\mathrm{d}x=\int_{\mathbb{R}^3}a_0(x)u^2_{0,\beta}\mathrm{d}x+o_n(1)\quad\text{and}$$
$$\int_{\mathbb{R}^3}b_0(x)v^2_{\lambda_n,\beta}\mathrm{d}x=\int_{\mathbb{R}^3}b_0(x)v^2_{0,\beta}\mathrm{d}x+o_n(1).$$

These together with Step 2 and (3.21) imply

$$\int_{\mathbb{R}^3}|\nabla u_{0,\beta}|^2+a_0(x)u^2_{0,\beta}\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla v_{0,\beta}|^2+b_0(x)v^2_{0,\beta}\mathrm{d}x=4(m_a+m_b). \tag{3.22}$$

We claim that

$$\int_{\mathbb{R}^3}u^4_{\lambda_n,\beta}\mathrm{d}x\ge C+o_n(1)\quad\text{and}\quad\int_{\mathbb{R}^3}v^4_{\lambda_n,\beta}\mathrm{d}x\ge C+o_n(1). \tag{3.23}$$

Indeed, suppose the contrary, we have either $\int_{\mathbb{R}^3}u^4_{\lambda_n,\beta}\mathrm{d}x=o_n(1)$ or $\int_{\mathbb{R}^3}v^4_{\lambda_n,\beta}\mathrm{d}x=o_n(1)$ up to a subsequence. Without loss of generality, we assume $\lim_{n\to\infty}\int_{\mathbb{R}^3}u^4_{\lambda_n,\beta}\mathrm{d}x=0$. By the boundedness of $\{(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\}$ in E and the Hölder and Sobolev inequalities, $\beta\int_{\mathbb{R}^3}u^2_{\lambda_n,\beta}v^2_{\lambda_n,\beta}=o_n(1)$, which implies $J_{\lambda_n,\beta}(u_{\lambda_n,\beta},v_{\lambda_n,\beta})=I_{b,\lambda_n}(v_{\lambda_n,\beta})+o_n(1)$. By Lemma 2.1 and $\mathcal{N}_{\lambda,\beta}\subset\mathcal{A}_\beta$, for every n, there exists a unique $t^*_n(\beta)>0$ such that $t^*_n(\beta)u_{\lambda_n,\beta}\in\mathcal{N}_{a,\lambda_n}$. It follows from Lemma 3.1 and $\beta<0$ that

$$\begin{aligned}J_{\lambda_n,\beta}(u_{\lambda_n,\beta},v_{\lambda_n,\beta})&\ge J_{\lambda_n,\beta}(t^*_n(\beta)u_{\lambda_n,\beta},v_{\lambda_n,\beta})\\&\ge I_{a,\lambda_n}(t^*_n(\beta)u_{\lambda_n,\beta})+I_{b,\lambda_n}(v_{\lambda_n,\beta})\\&\ge m_{a,\lambda_n}+I_{b,\lambda_n}(v_{\lambda_n,\beta})\\&=m_{a,\lambda_n}+J_{\lambda_n,\beta}(u_{\lambda_n,\beta},v_{\lambda_n,\beta})+o_n(1)\\&=m_a+J_{\lambda_n,\beta}(u_{\lambda_n,\beta},v_{\lambda_n,\beta})+o_n(1).\end{aligned} \tag{3.24}$$

Since $m_a > 0$, (3.24) is impossible for n large enough. Now, (3.23) together with Steps 1–2 implies

$$\int_{\Omega_a} u_{0,\beta}^4 \mathrm{d}x \geq C > 0 \quad \text{and} \int_{\Omega_b} v_{0,\beta}^4 \mathrm{d}x \geq C > 0. \tag{3.25}$$

Note that $I'_{\Omega_a}(u_{0,\beta}) = 0$ in $H^{-1}(\Omega_a)$ and $I'_{\Omega_b}(v_{0,\beta}) = 0$ in $H^{-1}(\Omega_b)$, by (3.25), we have

$$\int_{\mathbb{R}^3} |\nabla u_{0,\beta}|^2 + a_0(x)u_{0,\beta}^2 \mathrm{d}x \geq 4m_a \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla v_{0,\beta}|^2 + b_0(x)v_{0,\beta}^2 \mathrm{d}x \geq 4m_b.$$

It follows from (3.22) that $u_{0,\beta}$ is a least energy nonzero critical point of $I_{\Omega_a}(u)$ and $v_{0,\beta}$ is a least energy nonzero critical point of $I_{\Omega_b}(v)$.

We close the proof of Theorem 1.1 by verifying (1) and (2). Supposing the contrary, there exists $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$ such that one of the following cases must happen:

(a) $\int_{\mathbb{R}^3\setminus\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 + u_{\lambda_n,\beta}^2 \mathrm{d}x \geq C + o_n(1)$;
(b) $\int_{\mathbb{R}^3\setminus\Omega_b} |\nabla v_{\lambda_n,\beta}|^2 + v_{\lambda_n,\beta}^2 \mathrm{d}x \geq C + o_n(1)$;
(c) $|\int_{\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 + a_0(x)u_{\lambda_n,\beta}^2 \mathrm{d}x - 4m_a| \geq C + o_n(1)$;
(d) $|\int_{\Omega_b} |\nabla v_{\lambda,\beta}|^2 + b_0(x)v_{\lambda_n,\beta}^2 \mathrm{d}x - 4m_b| \geq C + o_n(1)$.

By Steps 1–3 and the condition (D_3), it is easy to see that (c) and (d) can not hold, which then implies that we must have (a) or (b). Since (3.21) holds for $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}$, by Steps 2–3 and the condition (D_3), we can see that

$$\int_{\mathbb{R}^3\setminus\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 + (\lambda_n a(x) + a_0(x))u_{\lambda_n,\beta}^2 \mathrm{d}x + \int_{\mathbb{R}^3\setminus\Omega_b} |\nabla v_{\lambda_n,\beta}|^2 + (\lambda_n b(x) + b_0(x))v_{\lambda_n,\beta}^2 \mathrm{d}x \to 0$$

as $n \to \infty$. It follows from the conditions (D_2) and (D_4) and Steps 1–2 that

$$\int_{(\mathbb{R}^3\setminus\Omega_a)\cap\mathcal{D}_a} a_0^-(x)u_{\lambda_n,\beta}^2 \mathrm{d}x = \int_{(\mathbb{R}^3\setminus\Omega_b)\cap\mathcal{D}_b} b_0^-(x)v_{\lambda_n,\beta}^2 \mathrm{d}x = o_n(1),$$

which then together with the conditions (D_2) and (D_4) and Steps 1–2 once more implies

$$\begin{aligned}
&\int_{\mathbb{R}^3\setminus\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 + (\lambda_n a(x) + a_0(x))u_{\lambda_n,\beta}^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3\setminus\Omega_b} |\nabla v_{\lambda_n,\beta}|^2 + (\lambda_n b(x) + b_0(x))v_{\lambda_n,\beta}^2 \mathrm{d}x \\
&\geq \int_{\mathbb{R}^3\setminus\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 \mathrm{d}x + \int_{\mathbb{R}^3\setminus\Omega_a} \frac{\lambda_n}{2} a(x)u_{\lambda_n,\beta}^2 \mathrm{d}x + \int_{\mathbb{R}^3\setminus\Omega_b} |\nabla v_{\lambda_n,\beta}|^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3\setminus\Omega_b} \frac{\lambda_n}{2} b(x)v_{\lambda_n,\beta}^2 \mathrm{d}x + o_n(1) \\
&= \int_{\mathbb{R}^3\setminus\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 + u_{\lambda_n,\beta}^2 \mathrm{d}x + \int_{\mathbb{R}^3\setminus\Omega_b} |\nabla v_{\lambda_n,\beta}|^2 + v_{\lambda_n,\beta}^2 \mathrm{d}x + o_n(1),
\end{aligned}$$

Thus, $\int_{\mathbb{R}^3\setminus\Omega_a} |\nabla u_{\lambda_n,\beta}|^2 + u_{\lambda_n,\beta}^2 \mathrm{d}x + \int_{\mathbb{R}^3\setminus\Omega_b} |\nabla v_{\lambda_n,\beta}|^2 + v_{\lambda_n,\beta}^2 \mathrm{d}x \to 0$ as $n \to \infty$ and it is a contradiction. We now complete the proof by taking $\Lambda_* = \Lambda_2$. □

4 Multi-bump solutions

The main task in this section is to find multi-bump solutions to $(\mathcal{P}_{\lambda,\beta})$ described as in Theorem 1.2. For the sake of convenience, in the present section, we always assume the conditions $(D_1)-(D_2)$, (D_3'), (D_4) and (D_5') hold. Due to the conditions (D_3') and (D_5'), in the present section, Ω_a' and Ω_b' will be chosen as $(I)-(III)$ given in Sect. 2.

4.1 The penalized functional and the (*PS*) condition

Since we want to find multi-bump solutions of $(\mathcal{P}_{\lambda,\beta})$ described as in Theorem 1.2, we will make some modifications on $J_{\lambda,\beta}(u,v)$. Similar technique was developed by del Pino and Felmer [23] and was also used in several other literatures, see for example [10,24,28,45] and the references therein.

Let $J_a\times J_b$ be a given subset of $\{1,\ldots,n_a\}\times\{1,\ldots,n_b\}$ with $J_a\neq\emptyset$ and $J_b\neq\emptyset$. Without loss of generality, we assume $J_a\times J_b=\{1,\ldots,k_a\}\times\{1,\ldots,k_b\}$ with $1\leq k_a\leq n_a$ and $1\leq k_b\leq n_b$. Denote $\Omega_a^{J_a}=\bigcup_{i_a=1}^{k_a}\Omega'_{a,i_a}$ and $\Omega_b^{J_b}=\bigcup_{j_b=1}^{k_b}\Omega'_{b,j_b}$. We also denote the characteristic functions of $\Omega_a^{J_a}$ and $\Omega_b^{J_b}$ by $\chi_{\Omega_a^{J_a}}$ and $\chi_{\Omega_b^{J_b}}$, respectively. Now, let

$$\delta_\beta^2=\frac{C_{a,b}}{2}\min\left\{1,\frac{1}{\mu_1+2|\beta|},\frac{1}{\mu_2+2|\beta|}\right\},\tag{4.1}$$

where $C_{a,b}$ is given by Lemma 2.1, and define $f_a(x,t)=\chi_{\Omega_a^{J_a}}(t^+)^3+(1-\chi_{\Omega_a^{J_a}})f(t)$, $f_b(x,t)=\chi_{\Omega_b^{J_b}}(t^+)^3+(1-\chi_{\Omega_b^{J_b}})f(t)$ and $h(x,t,s)=(\chi_{\Omega_b^{J_b}}+\chi_{\Omega_a^{J_a}})t^+s^++(1-\chi_{\Omega_a^{J_a}})(1-\chi_{\Omega_b^{J_b}})h(t,s)$, where $t^+=\max\{0,t\}$, $s^+=\max\{0,s\}$,

$$f(t)=\begin{cases}0, & t\leq 0,\\ t^3, & 0\leq t\leq\delta_\beta,\\ \delta_\beta^2 t, & t\geq\delta_\beta,\end{cases}\quad\text{and}\quad h(t,s)=\begin{cases}0, & \min\{t,s\}\leq 0,\\ ts, & 0\leq t,s\leq\delta_\beta,\\ \delta_\beta t, & 0\leq t\leq\delta_\beta\leq s,\\ \delta_\beta s, & 0\leq s\leq\delta_\beta\leq t,\\ \delta_\beta^2, & \delta_\beta\leq t,s.\end{cases}$$

Then, it is easy to see that $f_a(x,t)$ and $f_b(x,t)$ are the modifications of t^3 and $h(x,t,s)$ is the modification of ts. Let us consider the following functional defined on E,

$$\begin{aligned}J^*_{\lambda,\beta}(u,v)=&\frac{1}{2}\|u\|^2_{a,\lambda}+\frac{1}{2}\|v\|^2_{b,\lambda}-\mu_1\int_{\mathbb{R}^3}F_a(x,u)\mathrm{d}x\\&-\mu_2\int_{\mathbb{R}^3}F_b(x,v)\mathrm{d}x-\beta\int_{\mathbb{R}^3}H(x,u,v)\mathrm{d}x,\end{aligned}$$

where $F_a(x,u)=\int_0^u f_a(x,t)\mathrm{d}t$, $F_b(x,v)=\int_0^v f_b(x,t)\mathrm{d}t$ and $H(x,u,v)=2\int_0^u\int_0^v h(x,t,s)dsdt$. Clearly, by the construction of $f_a(x,t)$, $f_b(x,t)$ and $h(x,t,s)$, we can see that

$$0\leq\int_{\mathbb{R}^3\backslash\Omega_a^{J_a}}F_a(x,u)\mathrm{d}x\leq\frac{\delta_\beta^2}{2}\|u^+\|_2^2,\quad 0\leq\int_{\mathbb{R}^3\backslash\Omega_b^{J_b}}F_b(x,v)\mathrm{d}x\leq\frac{\delta_\beta^2}{2}\|v^+\|_2^2,$$

$$0\leq\int_{\mathbb{R}^3\backslash(\Omega_a^{J_a}\cup\Omega_b^{J_b})}H(x,u,v)\mathrm{d}x\leq 2\delta_\beta^2\|u^+\|_2\|v^+\|_2\leq\delta_\beta^2(\|u^+\|_2^2+\|v^+\|_2^2).$$

On the other hand,by Lemma 2.1, we have

$$\|u^+\|_2^2 \le C_{a,b}^{-1}\|u\|_{a,\lambda}^2 \le \lambda C_{a,b}^{-1}\|u\|_a^2 \quad \text{for } \lambda \ge \Lambda_1 \text{ and } u \in E_a$$

and

$$\|v^+\|_2^2 \le C_{a,b}^{-1}\|v\|_{b,\lambda}^2 \le \lambda C_{a,b}^{-1}\|v\|_b^2 \quad \text{for } \lambda \ge \Lambda_1 \text{ and } v \in E_b.$$

It follows that $J^*_{\lambda,\beta}(u, v)$ is well defined on E for $\lambda \ge \Lambda_1$ and $\beta < 0$. Moreover, by a standard argument, we can see that for $\lambda \ge \Lambda_1$ and $\beta < 0$, $J^*_{\lambda,\beta}(u, v)$ is C^1 on E and the critical point of $J^*_{\lambda,\beta}(u, v)$ is the solution of the following two-component systems:

$$\begin{cases} \Delta u - (\lambda a(x) + a_0(x))u + \mu_1 f_a(x, u) + 2\beta \int_0^v h(x, u, s)\mathrm{d}s = 0 & \text{in } \mathbb{R}^3, \\ \Delta v - (\lambda b(x) + b_0(x))v + \mu_2 f_b(x, v) + 2\beta \int_0^u h(x, t, v)\mathrm{d}t = 0 & \text{in } \mathbb{R}^3, \\ u, v \in H^1(\mathbb{R}^3), u, v \ge 0 & \text{in } \mathbb{R}^3. \end{cases} \qquad (\mathcal{P}^*_{\lambda,\beta})$$

In what follows, we will make some investigations on the functional $J^*_{\lambda,\beta}(u, v)$.

Lemma 4.1 *Assume* $(u, v) \in E$. *Then,*

$$\int_{\mathbb{R}^3} \frac{1}{4} f_a(x, u)u - F_a(x, u)\mathrm{d}x \ge -\frac{\delta_\beta^2}{4}\|u^+\|_2^2,$$

$$\int_{\mathbb{R}^3} \frac{1}{4} f_b(x, v)v - F_b(x, v)\mathrm{d}x \ge -\frac{\delta_\beta^2}{4}\|v^+\|_2^2,$$

$$0 \ge \int_{\mathbb{R}^3} \frac{u}{2}\int_0^v h(x, u, s)\mathrm{d}s + \frac{v}{2}\int_0^u h(x, t, v)\mathrm{d}t - H(x, u, v)\mathrm{d}x \ge -\delta_\beta^2\|u^+v^+\|_1.$$

Proof By the construction of $f_a(x, t)$, it is easy to see that $\frac{1}{4} f_a(x, t)t - F_a(x, t) = 0$ for $x \in \Omega_a^{J_a}$. If $x \notin \Omega_a^{J_a}$, then by the construction of $f_a(x, t)$, we have

$$\frac{1}{4} f_a(x, t)t - F_a(x, t) = \begin{cases} 0, & t \le \delta_\beta, \\ \frac{\delta_\beta^4}{4} - \frac{\delta_\beta^2}{4}(t^+)^2, & \delta_\beta \le t. \end{cases}$$

It follows that for every $u \in E_a$, we have

$$\int_{\mathbb{R}^3} \frac{1}{4} f_a(x, u)u - F_a(x, u)\mathrm{d}x = \int_{\{u(x)\ge\delta_\beta\}\cap(\mathbb{R}^3\setminus\Omega_a^{J_a})} \frac{\delta_\beta^4}{4} - \frac{\delta_\beta^2}{4}(u^+)^2\mathrm{d}x \ge -\frac{\delta_\beta^2}{4}\|u^+\|_2^2.$$

By a similar argument, for every $v \in E_b$, we have

$$\int_{\mathbb{R}^3} \frac{1}{4} f_b(x, v)v - F_b(x, v)\mathrm{d}x \ge -\frac{\delta_\beta^2}{4}\|v^+\|_2^2.$$

On the other hand, since $\Omega_a^{J_a} \cap \Omega_b^{J_b} = \emptyset$, by the construction of $h(x, t, s)$, we can see that $\frac{t}{2}\int_0^s h(x, t, \tau)\mathrm{d}\tau + \frac{s}{2}\int_0^t h(x, \tau, s)\mathrm{d}\tau - H(x, t, s) = 0$ for $x \in \Omega_a^{J_a} \cup \Omega_b^{J_b}$. If $x \notin \Omega_a^{J_a} \cup \Omega_b^{J_b}$, then also by the construction of $h(x, t, s)$, we have

$$\begin{aligned} &\frac{t}{2}\int_0^s h(x, t, \tau)\mathrm{d}\tau + \frac{s}{2}\int_0^t h(x, \tau, s)\mathrm{d}\tau - H(x, t, s) \\ &= \begin{cases} 0, & t, s \le \delta_\beta, \\ \frac{(t^+)^2\delta_\beta}{4}(\delta_\beta - s), & t \le \delta_\beta \le s, \\ \frac{(s^+)^2\delta_\beta}{4}(\delta_\beta - t), & s \le \delta_\beta \le t, \\ \frac{\delta_\beta^2}{4}[(t - \delta_\beta)(\delta_\beta - 2s) + (s - \delta_\beta)(\delta_\beta - 2t)], & \delta_\beta \le t, s. \end{cases} \end{aligned}$$

It follows that for every $(u, v) \in E$, we have

$$0 \geq \int_{\mathbb{R}^3} \frac{u}{2} \int_0^v h(x, u, s)\mathrm{d}s + \frac{v}{2} \int_0^u h(x, t, v)\mathrm{d}t - H(x, u, v)\mathrm{d}x \geq -\delta_\beta^2 \|u^+ v^+\|_1,$$

which completes the proof. □

With Lemma 4.1 in hands, we can verify that $J^*_{\lambda,\beta}(u, v)$ actually satisfies the (PS) condition for $\lambda \geq \Lambda_1$ and $\beta < 0$.

Lemma 4.2 *Assume $\lambda \geq \Lambda_1$ and $\beta < 0$. Then, $J^*_{\lambda,\beta}(u, v)$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$, that is, every $\{(u_n, v_n)\} \subset E$ satisfying $J^*_{\lambda,\beta}(u_n, v_n) = c + o_n(1)$ and $D[J^*_{\lambda,\beta}(u_n, v_n)] = o_n(1)$ strongly in E^* has a strongly convergent subsequence in E.*

Proof Suppose $\{(u_n, v_n)\} \subset E$ satisfying $J^*_{\lambda,\beta}(u_n, v_n) = c + o_n(1)$ and $D[J^*_{\lambda,\beta}(u_n, v_n)] = o_n(1)$ strongly in E^*. Then, by $\beta < 0$, Lemmas 2.1 and 4.1 and a similar argument of (3.8), we have

$$c + o_n(1) + o_n(1)\|(u_n, v_n)\| \geq \frac{1}{4}(1 - \mu_1\delta_\beta^2 C_{a,b}^{-1})\|u_n\|_{a,\lambda}^2 + \frac{1}{4}(1 - \mu_2\delta_\beta^2 C_{a,b}^{-1})\|v_n\|_{b,\lambda}^2. \tag{4.2}$$

It follows from Lemma 2.1 and (4.1) that $c+o_n(1)+o_n(1)\|(u_n, v_n)\| \geq \frac{C_{a,b}}{8}(\|u_n\|_2^2+\|v_n\|_2^2)$. This together with Lemma 2.1 and the condition (D_4) implies

$$\begin{aligned} c + o_n(1) + o_n(1)\|(u_n, v_n)\| &\geq \frac{1}{4}(1 - \mu_1\delta_\beta^2 C_{a,b}^{-1})\|u_n\|_{a,\lambda}^2 + \frac{1}{4}(1 - \mu_2\delta_\beta^2 C_{a,b}^{-1})\|v_n\|_{b,\lambda}^2 \\ &\geq \frac{1}{4}(1 - \mu_1\delta_\beta^2 C_{a,b}^{-1})\|u_n\|_{a}^2 + \frac{1}{4}(1 - \mu_2\delta_\beta^2 C_{a,b}^{-1})\|v_n\|_{b}^2 \\ &\quad - C(\|u_n\|_2^2 + \|v_n\|_2^2) \\ &\geq \frac{1}{8}\|(u_n, v_n)\|^2 - C'(c + o_n(1) + o_n(1)\|(u_n, v_n)\|), \end{aligned}$$

since $\lambda \geq \Lambda_1$. Thus, $\{(u_n, v_n)\}$ is bounded in E and $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E as $n \to \infty$ for some $(u_0, v_0) \in E$ up to a subsequence. Without loss of generality, we assume $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E as $n \to \infty$. Since $D[J^*_{\lambda,\beta}(u_n, v_n)] = o_n(1)$ strongly in E^*, it is easy to see that $D[J^*_{\lambda,\beta}(u_0, v_0)] = 0$ in E^*, which implies

$$\begin{aligned} o_n(1) &= \langle D[J^*_{\lambda,\beta}(u_n, v_n)] - D[J^*_{\lambda,\beta}(u_0, v_0)], (u_n, v_n) - (u_0, v_0)\rangle_{E^*,E} \\ &= \|u_n - u_0\|_{a,\lambda}^2 + \|v_n - v_0\|_{b,\lambda}^2 - \mu_1 \int_{\mathbb{R}^3} (f_a(x, u_n) - f_a(x, u_0))(u_n - u_0)\mathrm{d}x \\ &\quad - \mu_2 \int_{\mathbb{R}^3} (f_b(x, v_n) - f_b(x, v_0))(v_n - v_0)\mathrm{d}x \\ &\quad - 2\beta \int_{\mathbb{R}^3} (\int_0^{v_n} h(x, u_n, s)\mathrm{d}s - \int_0^{v_0} h(x, u_0, s)\mathrm{d}s)(u_n - u_0)\mathrm{d}x \\ &\quad - 2\beta \int_{\mathbb{R}^3} (\int_0^{u_n} h(x, t, v_n)\mathrm{d}t - \int_0^{u_0} h(x, t, v_0)\mathrm{d}t)(v_n - v_0)\mathrm{d}x. \end{aligned} \tag{4.3}$$

By the construction of $f_a(x,t)$, we can see that

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}(f_a(x,u_n)-f_a(x,u_0))(u_n-u_0)\mathrm{d}x|\\
&\le \int_{\Omega_a^{J_a}}|(f_a(x,u_n)-f_a(x,u_0))(u_n-u_0)|\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3\setminus\Omega_a^{J_a}}|(f_a(x,u_n)-f_a(x,u_0))(u_n-u_0)|\mathrm{d}x\\
&\le \int_{\Omega_a^{J_a}}(|u_n|^3+|u_0|^3)|u_n-u_0|\mathrm{d}x+2\delta_\beta^2\int_{\mathbb{R}^3\setminus\Omega_a^{J_a}}|u_0||u_n-u_0|\mathrm{d}x+\delta_\beta^2\|u_n-u_0\|_2^2.
\end{aligned}
$$

Since $(u_n,v_n)\rightharpoonup(u_0,v_0)$ weakly in E as $n\to\infty$, by (2.1) and the Sobolev embedding theorem, we have

$$u_n\to u_0 \text{ strongly in } L^p_{loc}(\mathbb{R}^3) \text{ as } n\to\infty \quad \text{for } p\in[1,6). \tag{4.4}$$

Thus, $\int_{\Omega_a^{J_a}}(|u_n|^3+|u_0|^3)|u_n-u_0|\mathrm{d}x=o_n(1)$ due to the choice of $\Omega_a^{J_a}$ and the Hölder inequality. On the other hand, we also see from (4.4) and the Hölder inequality that $2\delta_\beta^2\int_{\mathbb{R}^3\setminus\Omega_a^{J_a}}|u_0||u_n-u_0|\mathrm{d}x=o_n(1)$. Hence, we have

$$|\int_{\mathbb{R}^3}(f_a(x,u_n)-f_a(x,u_0))(u_n-u_0)\mathrm{d}x|\le\delta_\beta^2\|u_n-u_0\|_2^2+o_n(1). \tag{4.5}$$

Since (2.2) holds, we can also obtain the following estimates in a similar way:

$$|\int_{\mathbb{R}^3}(f_b(x,v_n)-f_b(x,v_0))(v_n-v_0)\mathrm{d}x|\le\delta_\beta^2\|v_n-v_0\|_2^2+o_n(1). \tag{4.6}$$

On the other hand, by the construction of $h(x,t,s)$, we can see that

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\left(\int_0^{v_n}h(x,u_n,s)\mathrm{d}s-\int_0^{v_0}h(x,u_0,s)\mathrm{d}s\right)(u_n-u_0)\mathrm{d}x|\\
&\le\delta_\beta^2\int_{\mathbb{R}^3}|u_n-u_0||v_n-v_0|\mathrm{d}x+2\delta_\beta^2\int_{\mathbb{R}^3}|v_0||u_n-u_0|\mathrm{d}x+o_n(1)
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\left(\int_0^{u_n}h(x,t,v_n)\mathrm{d}t-\int_0^{u_0}h(x,t,v_0)\mathrm{d}t\right)(v_n-v_0)\mathrm{d}x|\\
&\le\delta_\beta^2\int_{\mathbb{R}^3}|u_n-u_0||v_n-v_0|\mathrm{d}x+2\delta_\beta^2\int_{\mathbb{R}^3}|u_0||v_n-v_0|\mathrm{d}x+o_n(1).
\end{aligned}
\tag{4.8}
$$

By using similar arguments of (4.5) and (4.6), we can see from (4.7) and (4.8) that

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\left(\int_0^{v_n}h(x,u_n,s)\mathrm{d}s-\int_0^{v_0}h(x,u_0,s)\mathrm{d}s\right)(u_n-u_0)\mathrm{d}x|\\
&\le\delta_\beta^2\int_{\mathbb{R}^3}|u_n-u_0||v_n-v_0|\mathrm{d}x+o_n(1),
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\left(\int_0^{u_n}h(x,t,v_n)\mathrm{d}t-\int_0^{u_0}h(x,t,v_0)\mathrm{d}t\right)(v_n-v_0)\mathrm{d}x|\\
&\le\delta_\beta^2\int_{\mathbb{R}^3}|u_n-u_0||v_n-v_0|\mathrm{d}x+o_n(1).
\end{aligned}
\tag{4.10}
$$

Combining (4.3), (4.5)–(4.6) and (4.9)–(4.10), we can conclude that

$$o_n(1) \geq \|u_n - u_0\|_{a,\lambda}^2 + \|v_n - v_0\|_{b,\lambda}^2 - \delta_\beta^2(\mu_1 + 2|\beta|)\|u_n - u_0\|_2^2 - \delta_\beta^2(\mu_2 + 2|\beta|)\|v_n - v_0\|_2^2. \tag{4.11}$$

It follows from Lemma 2.1 and (4.1) that $o_n(1) \geq \frac{C_{a,b}}{2}(\|u_n - u_0\|_2^2 + \|v_n - v_0\|_2^2)$, which implies $u_n \to u_0$ and $v_n \to v_0$ strongly in $L^2(\mathbb{R}^3)$ as $n \to \infty$. This together the condition (D_4), implies

$$o_n(1) \geq \|u_n - u_0\|_{a,\lambda}^2 + \|v_n - v_0\|_{b,\lambda}^2 \geq \|u_n - u_0\|_a^2 + \|v_n - v_0\|_b^2 + o_n(1)$$

for $\lambda \geq \Lambda_1$ and $\beta < 0$. Thus, $(u_n, v_n) \to (u_0, v_0)$ strongly in E as $n \to \infty$ for $\lambda \geq \Lambda_1$ and $\beta < 0$, which completes the proof. □

In the final of this section, we will show that $J_{\lambda,\beta}^*(u, v)$ is actually a penalized functional of $J_{\lambda,\beta}(u, v)$ in the sense that, some special critical points of $J_{\lambda,\beta}^*(u, v)$ are also critical points of $J_{\lambda,\beta}(u, v)$.

Lemma 4.3 *Assume $\lambda \geq \Lambda_1$ and $\beta < 0$. Let $M > 0$ be a constant and $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E$ satisfy $J_{\lambda,\beta}^*(u_{\lambda,\beta}, v_{\lambda,\beta}) \leq M$ and $D[J_{\lambda,\beta}^*(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^*. Then,*

(1) *There exists $M_1 > 0$ such that $\|(u_{\lambda,\beta}, v_{\lambda,\beta})\| \leq M_1$.*
(2) *$\int_{\mathbb{R}^3\backslash\Omega_a^{J_a}} |\nabla u_{\lambda,\beta}|^2 + (\lambda a(x) + a_0(x))u_{\lambda,\beta}^2 \mathrm{d}x \to 0$ and $\int_{\mathbb{R}^3\backslash\Omega_b^{J_b}} |\nabla v_{\lambda,\beta}|^2 + (\lambda b(x) + b_0(x))v_{\lambda,\beta}^2 \mathrm{d}x \to 0$ as $\lambda \to +\infty$.*
(3) *There exists $\Lambda_1^*(\beta, M) \geq \Lambda_1$ such that $|u_{\lambda,\beta}| \leq \delta_\beta$ on $\mathbb{R}^3\backslash\Omega_a^{J_a}$ and $|v_{\lambda,\beta}| \leq \delta_\beta$ on $\mathbb{R}^3\backslash\Omega_b^{J_b}$ for $\lambda \geq \Lambda_1^*(\beta, M)$.*

Proof (1) Since $\lambda \geq \Lambda_1$ and $\beta < 0$, by a similar argument as (4.2), we can conclude that

$$M \geq \frac{1}{4}(1 - \mu_1\delta_\beta^2 C_{a,b}^{-1})\|u_{\lambda,\beta}\|_{a,\lambda}^2 + \frac{1}{4}(1 - \mu_2\delta_\beta^2 C_{a,b}^{-1})\|v_{\lambda,\beta}\|_{b,\lambda}^2. \tag{4.12}$$

It follows from Lemma 2.1 and (4.1) that $8MC_{a,b}^{-1} \geq \|u_{\lambda,\beta}\|_2^2 + \|v_{\lambda,\beta}\|_2^2$. Now, applying the condition (D_4), we can see that

$$8M + 8MC_{a,b}^{-1}(C_{a,0} + d_a + C_{b,0} + d_b) \geq \|(u_{\lambda,\beta}, v_{\lambda,\beta})\|^2. \tag{4.13}$$

We complete this proof by taking $M_1 = 8M + 8MC_{a,b}^{-1}(C_{a,0} + d_a + C_{b,0} + d_b)$.

(2) Since $\mathbb{R}^3\backslash\Omega_a^{J_a} = (\mathbb{R}^3\backslash\Omega_a') \cup (\bigcup_{i_a=k_a+1}^{n_a} \Omega_{a,i_a}')$ and $\mathbb{R}^3\backslash\Omega_b^{J_b} = (\mathbb{R}^3\backslash\Omega_b') \cup (\bigcup_{j_b=k_b+1}^{n_b} \Omega_{b,j_b}')$, for the sake of clarity, we divide this proof into the following two steps.

Step 1 We prove that

$$\lim_{\lambda\to+\infty} \int_{\mathbb{R}^3\backslash\Omega_a'} |\nabla u_{\lambda,\beta}|^2 + (\lambda a(x) + a_0(x))u_{\lambda,\beta}^2 \mathrm{d}x = 0 \tag{4.14}$$

and

$$\lim_{\lambda\to+\infty} \int_{\mathbb{R}^3\backslash\Omega_b'} |\nabla v_{\lambda,\beta}|^2 + (\lambda b(x) + b_0(x))v_{\lambda,\beta}^2 \mathrm{d}x = 0. \tag{4.15}$$

Indeed, let $\{\Omega_{a,i_a}''\}$ be a sequence of bounded domains with smooth boundaries in $\mathbb{R}^3$ and satisfy

(a) $\Omega_{a,i_a} \subset \Omega_{a,i_a}'' \subset \Omega_{a,i_a}'$ for all $i_a = 1, \ldots, n_a$.

(b) $dist(\Omega''_{a,i_a},\mathbb{R}^3\backslash\Omega'_{a,i_a})>0$ and $dist(\mathbb{R}^3\backslash\Omega''_{a,i_a},\Omega_{a,i_a})>0$ for all $i_a=1,\ldots,n_a$.

Denote $\Omega''_a=\bigcup_{i_a=1}^{n_a}\Omega''_{a,i_a}$. Then, by a similar argument as (2.7), we can show that

$$\int_{\Omega''_a}|\nabla u_{\lambda,\beta}|^2+(\lambda a(x)+a_0(x))u^2_{\lambda,\beta}\mathrm{d}x\geq\frac{\nu_a}{2}\int_{\Omega''_a}u^2_{\lambda,\beta}\mathrm{d}x \tag{4.16}$$

for λ large enough. Without loss of generality, we assume (4.16) holds for $\lambda\geq\Lambda_1$. Since Lemma 2.1 and (4.12) hold, we can obtain $8((C_{a,0}+d_a)C^{-1}_{a,b}+1)M\geq\lambda\int_{\mathbb{R}^3\backslash\Omega''_a}a(x)u^2_{\lambda,\beta}\mathrm{d}x$ for $\lambda\geq\Lambda_1$ by the condition (D_4) and (4.16). Since the condition (D'_3) is contained in the condition (D_3), by a similar argument as (3.15), we can see that

$$\int_{\mathbb{R}^3\backslash\Omega''_a}u^2_{\lambda,\beta}\mathrm{d}x\to 0\quad\text{as }\lambda\to+\infty. \tag{4.17}$$

Let $\Psi\in C^\infty(\mathbb{R}^3)$ be given by (3.16). Then, $u_{\lambda,\beta}\Psi\in E_a$. Note $D[J^*_{\lambda,\beta}(u_{\lambda,\beta},v_{\lambda,\beta})]=0$ in E^*, we get that

$$\begin{aligned}&\int_{\mathbb{R}^3}(|\nabla u_{\lambda,\beta}|^2+(\lambda a(x)+a_0(x))u^2_{\lambda,\beta})\Psi\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla u_{\lambda,\beta}\nabla\Psi)u_{\lambda,\beta}\mathrm{d}x\\&=\mu_1\int_{\mathbb{R}^3}f_a(x,u_{\lambda,\beta})u_{\lambda,\beta}\Psi\mathrm{d}x+2\beta\int_{\mathbb{R}^3}(\int_0^{v_{\lambda,\beta}}h(x,u_{\lambda,\beta},s)\mathrm{d}s)u_{\lambda,\beta}\Psi\mathrm{d}x.\end{aligned}$$

Since $\beta<0$, by (2.8) and the construction of $f_a(x,t)$ and $h(x,t,s)$, we have

$$\begin{aligned}0&\leq\int_{\mathbb{R}^3\backslash\Omega'_a}|\nabla u_{\lambda,\beta}|^2+(\lambda a(x)+a_0(x))u^2_{\lambda,\beta}\mathrm{d}x\\&\leq\mu_1\int_{\Omega_a^{J_a}\cap(\mathbb{R}^3\backslash\Omega''_a)}u^4_{\lambda,\beta}\mathrm{d}x+\mu_1\delta^2_\beta\int_{(\mathbb{R}^3\backslash\Omega_a^{J_a})\cap(\mathbb{R}^3\backslash\Omega''_a)}u^2_{\lambda,\beta}\mathrm{d}x\\&\quad+\int_{\Omega'_a\backslash\Omega''_a}|\nabla u_{\lambda,\beta}||\nabla\Psi||u_{\lambda,\beta}|\mathrm{d}x\\&\leq\mu_1\delta^2_\beta\int_{\mathbb{R}^3\backslash\Omega''_a}u^2_{\lambda,\beta}\mathrm{d}x+(\mu_1S^{-\frac{3}{2}}\|u_{\lambda,\beta}\|^3_a+\max_{\mathbb{R}^3}|\nabla\Psi|\|u_{\lambda,\beta}\|_a)\Big(\int_{\mathbb{R}^3\backslash\Omega''_a}u^2_{\lambda,\beta}\mathrm{d}x\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.18}$$

Thanks to (4.13) and (4.17), we know from (4.18) that (4.14) holds. By a similar argument, we can also conclude that (4.15) is true.

Step 2 We prove that

$$\lim_{\lambda\to+\infty}\int_{\Omega'_{a,i_a}}|\nabla u_{\lambda,\beta}|^2+(\lambda a(x)+a_0(x))u^2_{\lambda,\beta}\mathrm{d}x=0\quad\text{for all }i_a\in\{1,\ldots,n_a\}\backslash J_a \tag{4.19}$$

and

$$\lim_{\lambda\to+\infty}\int_{\Omega'_{b,j_b}}|\nabla v_{\lambda,\beta}|^2+(\lambda b(x)+b_0(x))v^2_{\lambda,\beta}\mathrm{d}x=0\quad\text{for all }j_b\in\{1,\ldots,n_b\}\backslash J_b. \tag{4.20}$$

In fact, let $\{\Omega'''_{a,i_a}\}$ be a sequence of bounded domains with smooth boundaries in $\mathbb{R}^3$ and satisfy

(i) $\Omega'_{a,i_a} \subset \Omega'''_{a,i_a}$ and $\text{dist}(\Omega'_{a,i_a}, \mathbb{R}^3 \backslash \Omega'''_{a,i_a}) > 0$ for all $i_a \in \{1, \dots, n_a\}$.

(ii) $\overline{\Omega'''_{a,i_a}} \cap \overline{\Omega'''_{a,j_a}} = \emptyset$ for all $i_a \neq j_a$.

(iii) $\overline{(\bigcup_{i_a=1}^{n_a} \Omega'''_{a,i_a})} \cap \overline{\Omega'_b} = \emptyset$.

For every $i_a \in \{1, \dots, n_a\} \backslash J_a$, we choose $\Psi_{i_a} \in C^\infty(\mathbb{R}^3, [0, 1])$ satisfying

$$\Psi_{i_a} = \begin{cases} 1, & x \in \Omega'_{a,i_a}, \\ 0, & x \in \mathbb{R}^3 \backslash \Omega'''_{a,i_a}. \end{cases}$$

Then, by a similar argument as (4.18), the choice of Ω'''_{a,i_a} and the construction of $f_a(x, t)$ and $h(x, t, s)$, we can obtain that

$$\begin{aligned} &\int_{\Omega'_{a,i_a}} |\nabla u_{\lambda,\beta}|^2 + (\lambda a(x) + a_0(x)) u^2_{\lambda,\beta} \mathrm{d}x \\ &\quad \leq \mu_1 \delta^2_\beta \int_{\Omega'''_{a,i_a}} u^2_{\lambda,\beta} \mathrm{d}x + \int_{\Omega'''_{a,i_a} \backslash \Omega'_{a,i_a}} |\nabla u_{\lambda,\beta}| |\nabla \Psi_{i_a}| |u_{\lambda,\beta}| \mathrm{d}x. \end{aligned}$$

Thanks to the choice of Ω'''_{a,i_a} and (4.13), for $i_a \in \{1, \dots, n_a\} \backslash J_a$, we have

$$\begin{aligned} &\int_{\Omega'_{a,i_a}} |\nabla u_{\lambda,\beta}|^2 + (\lambda a(x) + a_0(x)) u^2_{\lambda,\beta} \mathrm{d}x \\ &\quad \leq \mu_1 \delta^2_\beta \int_{\Omega'_{a,i_a}} u^2_{\lambda,\beta} \mathrm{d}x + C \left(\int_{\mathbb{R}^3 \backslash \Omega''_a} u^2_{\lambda,\beta} \mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

It follows from (2.9), (4.1) and (4.17) that (4.19) holds. A similar argument implies that (4.20) holds too. Now, the conclusion follows immediately from (4.14)–(4.15) and (4.19)–(4.20).

(3) By (2.9) and (4.19), we have

$$\int_{\Omega'_{a,i_a}} u^2_{\lambda,\beta} \mathrm{d}x \to 0 \quad \text{as } \lambda \to +\infty \quad \text{for } i_a \in \{1, \dots, n_a\} \backslash J_a,$$

which together with (4.17) implies

$$\int_{\mathbb{R}^3 \backslash \bigcup_{i_a \in J_a} \Omega''_{a,i_a}} u^2_{\lambda,\beta} \mathrm{d}x \to 0 \quad \text{as } \lambda \to +\infty. \tag{4.21}$$

Let $r = \frac{1}{3} dist(\Omega'_a, \Omega''_a)$. Then, for every $x \in \mathbb{R}^3 \backslash \Omega^{J_a}_a$, $B_{2r}(x) \subset \mathbb{R}^3 \backslash \bigcup_{i_a \in J_a} \Omega''_{a,i_a}$. We define $\phi_L = \min\{|u_{\lambda,\beta}|^{\alpha-1}, L\} u_{\lambda,\beta} \overline{\rho}^2$, where $\overline{\rho} \in C^\infty_0(B_{2r}(x), [0, 1])$ with $\overline{\rho} = 1$ on $B_{\frac{5r}{3}}(x)$ and $|\nabla \overline{\rho}| < \frac{C}{2r - \frac{5}{3} r}$, $\alpha > 0$ and $L > 0$. Then, $\phi_L \in E_a$. Since $D[J^*_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^* and the conditions (D_1) and (D_4) hold, by multiplying $(\mathcal{P}^*_{\lambda,\beta})$ with $(\phi_L, 0)$ and letting $L \to +\infty$, we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} \overline{\rho}^2 |u_{\lambda,\beta}|^{\alpha-1} |\nabla u_{\lambda,\beta}|^2 \mathrm{d}x \leq C_\beta \left(\int_{\mathbb{R}^3} \overline{\rho}^2 u^{\alpha+3}_{\lambda,\beta} \mathrm{d}x + \int_{\mathbb{R}^3} \overline{\rho}^2 u^{\alpha+1}_{\lambda,\beta} \mathrm{d}x \right) \\ + 4 \int_{\mathbb{R}^3} |\nabla \overline{\rho}|^2 u^{\alpha+1}_{\lambda,\beta} \mathrm{d}x, \end{aligned}$$

where $C_\beta = \mu_1 + \mu_1\delta_\beta^2 + C_{a,0} + d_a$. By the Sobolev embedding theorem, we can see that

$$\left(\int_{B_{\frac{5r}{3}}(x)} u_{\lambda,\beta}^{3(\alpha+1)}\mathrm{d}x\right)^{\frac{1}{3}} \le C_\beta(\alpha+1)^2\left(\int_{B_{2r}(x)} u_{\lambda,\beta}^{\alpha+3}\mathrm{d}x + (1+\frac{24}{r^2})\int_{B_{2r}(x)} u_{\lambda,\beta}^{\alpha+1}\mathrm{d}x\right). \tag{4.22}$$

Let $\alpha_n = 3\alpha_{n-1}$ with $\alpha_0 = 2$ and $r_n = (1+(\frac{2}{3})^{n-1})r$, $n\in\mathbb{N}$. Then, (4.22) can be rewritten as

$$\begin{aligned}&\left(\int_{B_{r_1}(x)} u_{\lambda,\beta}^{3(\alpha_0+1)}\mathrm{d}x\right)^{\frac{1}{3}}\\ &\le C_\beta(\alpha_0+1)^2\left(\int_{B_{r_0}(x)} u_{\lambda,\beta}^{\alpha_0+3}\mathrm{d}x + (1+\frac{4}{|r_0-r_1|^2})\int_{B_{r_0}(x)} u_{\lambda,\beta}^{\alpha_0+1}\mathrm{d}x\right).\end{aligned} \tag{4.23}$$

We replace α_0, r_0 and r_1 in (4.23) by α_n, r_n and r_{n+1}. Then, we can obtain

$$\begin{aligned}&\left(\int_{B_{r_{n+1}}(x)} u_{\lambda,\beta}^{3(\alpha_n+1)}\mathrm{d}x\right)^{\frac{1}{3(\alpha_n+1)}}\\ &\le [C_\beta(\alpha_n+1)^2]^{\frac{1}{(\alpha_n+1)}}\left(\int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+3}\mathrm{d}x\right)^{\frac{1}{(\alpha_n+1)}}\\ &\quad+[C_\beta(\alpha_n+1)^2]^{\frac{1}{(\alpha_n+1)}}(1+\frac{4}{|r_n-r_{n+1}|^2})^{\frac{1}{(\alpha_n+1)}}\left(\int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+1}\mathrm{d}x\right)^{\frac{1}{(\alpha_n+1)}}.\end{aligned} \tag{4.24}$$

Clearly, one of the following two cases must happen:

(1) $\int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+1}\mathrm{d}x \le \int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+3}\mathrm{d}x$ up to a subsequence.
(2) $\int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+3}\mathrm{d}x \le \int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+1}\mathrm{d}x$ up to a subsequence.

If case (1) happens, then by (4.24), we can see that

$$\begin{aligned}&\left(\int_{B_{r_{n+1}}(x)} u_{\lambda,\beta}^{3(\alpha_n+1)}\right)^{\frac{1}{3(\alpha_n+1)}}\\ &\le \left(4C_\beta(\alpha_n+1)^2(1+\frac{1}{|r_n-r_{n+1}|^2})\right)^{\frac{1}{(\alpha_n+1)}}\left(\int_{B_{r_n}(x)} u_{\lambda,\beta}^{\alpha_n+3}\right)^{\frac{1}{\alpha_n+1}}.\end{aligned}$$

By iterating (4.24) and using the choice of r_n and α_n, we have

$$\begin{aligned}&\lim_{n\to+\infty}\left(\int_{B_{r_{n+1}}(x)} u_{\lambda,\beta}^{3(\alpha_n+1)}\right)^{\frac{1}{3(\alpha_n+1)}}\\ &\le \left(\prod_{n=1}^{\infty}\left(4C_\beta(\alpha_n+1)^2(1+\frac{1}{|r_n-r_{n+1}|^2})\right)^{\frac{1}{(\alpha_n+1)}}\left(\int_{\mathbb{R}^3\backslash\underset{i_a\in J_a}{\cup}\Omega''_{a,i_a}} u_{\lambda,\beta}^5\mathrm{d}x\right)^{\frac{1}{5}}\right)^{\prod_{n=1}^{+\infty}\frac{\alpha_n+3}{\alpha_n+1}}\\ &\le C'_\beta\left(\int_{\mathbb{R}^3\backslash\underset{i_a\in J_a}{\cup}\Omega''_{a,i_a}} u_{\lambda,\beta}^5\mathrm{d}x\right)^{\frac{C}{5}},\end{aligned} \tag{4.25}$$

where C'_β is a constant independent of λ and x. If case (2) happens, then by iterating (4.24) and using the choice of r_n and α_n once more, we have

$$\begin{aligned}&\lim_{n\to+\infty}\left(\int_{B_{r_{n+1}}(x)} u_{\lambda,\beta}^{3(\alpha_n+1)}\right)^{\frac{1}{3(\alpha_n+1)}}\\&\leq\prod_{n=1}^{\infty}\left(4C_\beta(\alpha_n+1)^2(1+\frac{1}{|r_n-r_{n+1}|^2})|B_{r_n}(x)|^{\frac{2}{\alpha_n+3}}\right)^{\frac{1}{(\alpha_n+1)}}\left(\int_{\mathbb{R}^3\backslash\underset{i_a\in J_a}{\cup}\Omega''_{a,i_a}} u_{\lambda,\beta}^5 dx\right)^{\frac{1}{5}}\\&\leq C'_\beta\left(\int_{\mathbb{R}^3\backslash\underset{i_a\in J_a}{\cup}\Omega''_{a,i_a}} u_{\lambda,\beta}^5 dx\right)^{\frac{1}{5}}.\end{aligned}\tag{4.26}$$

By the Hölder and the Sobolev inequalities and (4.13) and (4.21), we can conclude that

$$\left(\int_{\mathbb{R}^3\backslash\underset{i_a\in J_a}{\cup}\Omega''_{a,i_a}} u_{\lambda,\beta}^5 dx\right)^{\frac{1}{5}}\to 0\quad\text{as } \lambda\to+\infty.$$

It follows from (4.25) and (4.26) that $\|u_{\lambda,\beta}\|_{L^\infty(B_r(x))}\to 0$ as $\lambda\to+\infty$, which then implies $\|u_{\lambda,\beta}\|_{L^\infty(\mathbb{R}^3\backslash\Omega_a^{J_a})}\to 0$ as $\lambda\to+\infty$. By similar arguments, we also have $\|v_{\lambda,\beta}\|_{L^\infty(\mathbb{R}^3\backslash\Omega_b^{J_b})}\to 0$ as $\lambda\to+\infty$. Now, we can choose $\Lambda_1^*(\beta,M)\geq\Lambda_1$ such that $|u_{\lambda,\beta}|\leq\delta_\beta$ a.e. on $\mathbb{R}^3\backslash\Omega_a^{J_a}$ and $|v_{\lambda,\beta}|\leq\delta_\beta$ a.e. on $\mathbb{R}^3\backslash\Omega_b^{J_b}$ for $\lambda\geq\Lambda_1^*(\beta)$. Note that by a similar argument as used in Theorem 1.1, we can see that $(u_{\lambda,\beta},v_{\lambda,\beta})\in C(\mathbb{R}^3)\times C(\mathbb{R}^3)$. Hence, we must have $|u_{\lambda,\beta}|\leq\delta_\beta$ on $\mathbb{R}^3\backslash\Omega_a^{J_a}$ and $|v_{\lambda,\beta}|\leq\delta_\beta$ on $\mathbb{R}^3\backslash\Omega_b^{J_b}$ for $\lambda\geq\Lambda_1^*(\beta,M)$. □

4.2 Construction of critical points

In this section, we will construct critical values of $J^*_{\lambda,\beta}(u,v)$ by a minimax argument. The idea of such a construction traces back to Séré [41] and also was applied in [10,24,28,45].

We first recall some well-known results, which are useful in this construction. For all $i_a=1,\ldots,n_a$ and $j_b=1,\ldots,n_b$, we define $\mathcal{E}_{\Omega'_{a,i_a}}$ on $H^1(\Omega'_{a,i_a})$ and $\mathcal{E}_{\Omega'_{b,j_b}}$ on $H^1(\Omega'_{b,j_b})$ as follows:

$$\begin{aligned}\mathcal{E}_{\Omega'_{a,i_a}}(u)&=\frac{1}{2}\int_{\Omega'_{a,i_a}}|\nabla u|^2+(\lambda a(x)+a_0(x))u^2 dx-\frac{\mu_1}{4}\int_{\Omega'_{a,i_a}}u^4 dx,\\\mathcal{E}_{\Omega'_{b,j_b}}(v)&=\frac{1}{2}\int_{\Omega'_{b,j_b}}|\nabla v|^2+(\lambda b(x)+b_0(x))v^2 dx-\frac{\mu_2}{4}\int_{\Omega'_{b,j_b}}v^4 dx.\end{aligned}$$

By (2.9) and (2.10), $\mathcal{E}_{\Omega'_{a,i_a}}(u)$ and $\mathcal{E}_{\Omega'_{b,j_b}}(v)$ have the least energy nonzero critical point for all $i_a=1,\ldots,n_a$ and $j_b=1,\ldots,n_b$ if $\lambda\geq\Lambda_1$. We denote the ground state level of $\mathcal{E}_{\Omega'_{a,i_a}}(u)$ and $\mathcal{E}_{\Omega'_{b,j_b}}(v)$ by $m_{a,i_a,\lambda}$ and $m_{b,j_b,\lambda}$, respectively. Since $\{\Omega'_{a,i_a}\}$ and $\{\Omega'_{b,j_b}\}$ are two sequences of bounded domains, by the conditions $(D_1)-(D_2)$, (D'_3), (D_4) and (D'_5) and (2.9)–(2.10), it is easy to show that $m_{a,i_a,\lambda}$ and $m_{b,j_b,\lambda}$ are positive for $\lambda\geq\Lambda_1$. It follows that

$$m_{a,i_a,\lambda}=\inf\left\{\mathcal{E}_{\Omega'_{a,i_a}}(u)\mid\int_{\Omega'_{a,i_a}}u^4 dx=\frac{4m_{a,i_a,\lambda}}{\mu_1}\right\}\quad\text{for all } i_a=1,\ldots,n_a\tag{4.27}$$

and

$$m_{b,j_b,\lambda} = \inf \left\{ \mathcal{E}_{\Omega'_{b,j_b}}(v) \mid \int_{\Omega'_{b,j_b}} v^4 \mathrm{d}x = \frac{4m_{b,j_b,\lambda}}{\mu_2} \right\} \quad \text{for all } j_b = 1, \ldots, n_b. \tag{4.28}$$

On the other hand, let $W_{a,i_a} \in H_0^1(\Omega_{a,i_a})$ and $W_{b,j_b} \in H_0^1(\Omega_{b,j_b})$ be the least energy nonzero critical points of $I_{\Omega_{a,i_a}}(u)$ and $I_{\Omega_{b,j_b}}(v)$, respectively. Then, by the conditions (D_3') and (D_5'), it is well known that

$$I_{\Omega_{a,i_a}}(W_{a,i_a}) = \max_{t \geq 0} I_{\Omega_{a,i_a}}(tW_{a,i_a}) \quad \text{and} \quad I_{\Omega_{b,j_b}}(W_{b,j_b}) = \max_{t \geq 0} I_{\Omega_{b,j_b}}(tW_{b,j_b}). \tag{4.29}$$

Let $\gamma_{0,a} : [0,1]^{k_a} \to E_a$ and $\gamma_{0,b} : [0,1]^{k_b} \to E_b$ be

$$\gamma_{0,a}(t_1, \ldots, t_{k_a}) = \sum_{i_a=1}^{k_a} t_{i_a} R W_{a,i_a} \tag{4.30}$$

and

$$\gamma_{0,b}(s_1, \ldots, s_{k_b}) = \sum_{j_b=1}^{k_b} s_{j_b} R W_{b,j_b}, \tag{4.31}$$

where $R > 2$ is a large constant satisfying

$$I_{\Omega_{a,i_a}}(RW_{a,i_a}) \leq 0, \quad R^4 \int_{\Omega_{a,i_a}} W_{a,i_a}^4 \mathrm{d}x \geq 2\frac{4m_{a,i_a}}{\mu_1}, \tag{4.32}$$

$$I_{\Omega_{b,j_b}}(RW_{b,j_b}) \leq 0, \quad R^4 \int_{\Omega_{b,j_b}} W_{b,j_b}^4 \mathrm{d}x \geq 2\frac{4m_{b,j_b}}{\mu_2}. \tag{4.33}$$

for all $i_a = 1, \ldots, k_a$ and $j_b = 1, \ldots, k_b$. By the condition (D_3'), we can extend W_{a,i_a} and W_{b,j_b} to the whole space $\mathbb{R}^3$ by letting $W_{a,i_a} = 0$ on $\mathbb{R}^3 \backslash \Omega_{a,i_a}$ and $W_{b,j_b} = 0$ on $\mathbb{R}^3 \backslash \Omega_{b,j_b}$ such that $W_{a,i_a} \in H^1(\mathbb{R}^3)$ and $W_{b,j_b} \in H^1(\mathbb{R}^3)$ for all $i_a = 1, \ldots, n_a$ and $j_b = 1, \ldots, n_b$. Now, we can define a minimax value of $J^*_{\lambda,\beta}(u,v)$ for $\lambda \geq \Lambda_1$ and $\beta < 0$ as follows:

$$m_{J_a,J_b,\lambda,\beta} = \inf_{(\gamma_a,\gamma_b) \in \Gamma} \sup_{[0,1]^{k_a} \times [0,1]^{k_b}} J^*_{\lambda,\beta}(\gamma_a, \gamma_b),$$

where

$$\Gamma = \Big\{ (\gamma_a, \gamma_b) \mid (\gamma_a, \gamma_b) \in C([0,1]^{k_a} \times [0,1]^{k_b}, E_a \times E_b),$$
$$(\gamma_a, \gamma_b) = (\gamma_{0,a}, \gamma_{0,b}) \text{ on } \partial([0,1]^{k_a} \times [0,1]^{k_b}) \Big\}.$$

$m_{J_a,J_b,\lambda,\beta}$ may be a critical value of $J^*_{\lambda,\beta}(u,v)$. In order to show it, we need the following.

Lemma 4.4 *Assume* $(\gamma_a, \gamma_b) \in \Gamma$ *and*

$$(\xi_1, \ldots, \xi_{k_a}, \eta_1, \ldots, \eta_{k_b}) \in \left[0, R^4 \int_{\Omega_{a,1}} W_{a,1}^4 \mathrm{d}x\right] \times \ldots \times \left[0, R^4 \int_{\Omega_{a,k_a}} W_{a,k_a}^4 \mathrm{d}x\right]$$
$$\times \left[0, R^4 \int_{\Omega_{b,1}} W_{b,1}^4 \mathrm{d}x\right] \times \ldots \times \left[0, R^4 \int_{\Omega_{b,k_b}} W_{b,k_b}^4 \mathrm{d}x\right].$$

Then, there exist $(t'_1, \ldots, t'_{k_a}) \in [0, 1]^{k_a}$ *and* $(s'_1, \ldots, s'_{k_b}) \in [0, 1]^{k_b}$ *such that*

$$\int_{\Omega'_{a,i_a}} [\gamma_a(t'_1, \ldots, t'_{k_a})]^4(x)\mathrm{d}x = \xi_{i_a} \quad \textit{for all } i_a = 1, \ldots, k_a$$

and

$$\int_{\Omega'_{b,j_b}} [\gamma_b(s'_1, \ldots, s'_{k_b})]^4(x)\mathrm{d}x = \eta_{j_b} \quad \textit{for all } j_b = 1, \ldots, k_b.$$

Proof For every $(\gamma_a, \gamma_b) \in \Gamma$, we define a map $\widetilde{\gamma} : [0, 1]^{k_a} \times [0, 1]^{k_b} \to \mathbb{R}^{k_a} \times \mathbb{R}^{k_b}$ as follows:

$$\begin{aligned}
&\widetilde{\gamma}(t_1, \ldots, t_{k_a}, s_1, \ldots, s_{k_b}) \\
&= \Bigg(\int_{\Omega'_{a,1}} [\gamma_a(t_1, \ldots, t_{k_a})]^4(x)\mathrm{d}x, \ldots, \int_{\Omega'_{a,k_a}} [\gamma_a(t_1, \ldots, t_{k_a})]^4(x)\mathrm{d}x, \\
&\quad \int_{\Omega'_{b,1}} [\gamma_b(s_1, \ldots, s_{k_b})]^4(x)\mathrm{d}x, \ldots, \int_{\Omega'_{b,k_b}} [\gamma_b(s_1, \ldots, s_{k_b})]^4(x)\mathrm{d}x \Bigg).
\end{aligned}$$

Note that for every $(\gamma_a, \gamma_b) \in \Gamma$, we have

$$(\gamma_a(t_1, \ldots, t_{k_a}), \gamma_b(s_1, \ldots, s_{k_b})) = (\gamma_{0,a}(t_1, \ldots, t_{k_a}), \gamma_{0,b}(s_1, \ldots, s_{k_b}))$$

if $(t_1, \ldots, t_{k_a}, s_1, \ldots, s_{k_b}) \in \partial([0, 1]^{k_a} \times [0, 1]^{k_b})$. Then, by the construction of $\{\Omega'_{a,i_a}\}$ and $\{\Omega'_{b,j_b}\}$, we can see that

$$\int_{\Omega'_{a,i_a}} [\gamma_a(t_1, \ldots, t_{k_a})]^4(x)\mathrm{d}x = t_{i_a}^4 R^4 \int_{\Omega_{a,i_a}} W_{a,i_a}^4 \mathrm{d}x$$

and

$$\int_{\Omega'_{b,j_b}} [\gamma_b(s_1, \ldots, s_{k_b})]^4(x)\mathrm{d}x = s_{j_b}^4 R^4 \int_{\Omega_{b,j_b}} W_{b,j_b}^4 \mathrm{d}x$$

for all $i_a = 1, \ldots, k_a$, $j_b = 1, \ldots, k_b$ and $(t_1, \ldots, t_{k_a}, s_1, \ldots, s_{k_b}) \in \partial([0, 1]^{k_a} \times [0, 1]^{k_b})$. It follows that

$$deg(\widetilde{\gamma}, [0, 1]^{k_a} \times [0, 1]^{k_b}, (\xi_1, \ldots, \xi_{k_a}, \eta_1, \ldots, \eta_{k_b})) = 1,$$

which completes the proof. □

With Lemma 4.4 in hands, we can obtain the following energy estimate, which can be viewed as a linking structure of $J^*_{\lambda,\beta}(u, v)$.

Lemma 4.5 *Assume* $\lambda \geq \Lambda_1$ *and* $\beta < 0$. *Then, we have the following results.*

(1) *If* $(t_1, \ldots, t_{k_a}, s_1, \ldots, s_{k_b}) \in \partial([0, 1]^{k_a} \times [0, 1]^{k_b})$, *then*

$$\begin{aligned}
&J^*_{\lambda,\beta}(\gamma_{0,a}(t_1, \ldots, t_{k_a}), \gamma_{0,b}(s_1, \ldots, s_{k_b})) \\
&\quad \leq \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b} - \min\{m_{a,1}, \ldots, m_{a,k_a}, m_{b,1}, \ldots, m_{b,k_b}\}.
\end{aligned}$$

(2) $\sum_{i_a=1}^{k_a} m_{a,i_a,\lambda} + \sum_{j_b=1}^{k_b} m_{b,j_b,\lambda} \leq m_{J_a,J_b,\lambda,\beta} \leq \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b}.$

Proof (1) Since $(t_1, \dots, t_{k_a}, s_1, \dots, s_{k_b}) \in \partial([0,1]^{k_a} \times [0,1]^{k_b})$, there exists $i'_a \in \{1, \dots, k_a\}$ or $j'_b \in \{1, \dots, k_b\}$ such that $t_{i'_a} \in \{0, 1\}$ or $s_{j'_b} \in \{0, 1\}$. Without loss of generality, we assume $t_1 = 1$. It follows from (4.29)–(4.33) and the condition (D'_3) that

$$
\begin{aligned}
& J^*_{\lambda,\beta}(\gamma_{0,a}(t_1, \dots, t_{k_a}), \gamma_{0,b}(s_1, \dots, s_{k_b})) \\
& = I_{a,1}(RW_{a,1}) + \sum_{i_a=2}^{k_a} I_{a,i_a}(t_{i_a} RW_{a,i_a}) + \sum_{j_b=1}^{k_b} I_{b,j_b}(s_{j_b} RW_{b,j_b}) \\
& \le \sum_{i_a=2}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b} \\
& \le \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b} - \min\{m_{a,1}, \dots, m_{a,k_a}, m_{b,1}, \dots, m_{b,k_b}\}.
\end{aligned}
$$

(2) Since $(\gamma_{0,a}, \gamma_{0,b}) \in \Gamma$ and $R > 2$, by the condition (D'_3), we must have

$$
\begin{aligned}
m_{J_a,J_b,\lambda,\beta} & \le \sup_{[0,1]^{k_a} \times [0,1]^{k_b}} J^*_{\lambda,\beta}\left(\sum_{i_a=1}^{k_a} t_{i_a} RW_{a,i_a}, \sum_{j_b=1}^{k_b} s_{j_b} RW_{b,j_b}\right) \\
& \le \sum_{i_a=1}^{k_a} I_{a,i_a}(W_{a,i_a}) + \sum_{j_b=1}^{k_b} I_{b,j_b}(W_{b,j_b}) \\
& = \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b}.
\end{aligned}
$$

On the other hand, since the condition (D'_3) holds, by the construction of Ω'_{a,i_a} and Ω'_{b,j_b}, it is easy to show that $m_{a,i_a,\lambda} \le m_{a,i_a}$ and $m_{b,j_b,\lambda} \le m_{b,j_b}$ for all $i_a = 1, \dots, n_a$, $j_b = 1, \dots, n_b$ and $\lambda > 0$. This together with (4.32)–(4.33) and Lemma 4.4 implies that for every $(\gamma_a, \gamma_b) \in \Gamma$, there exist $(t'_1, \dots, t'_{k_a}) \in [0,1]^{k_a}$ and $(s'_1, \dots, s'_{k_b}) \in [0,1]^{k_b}$ such that

$$
\int_{\Omega'_{a,i_a}} [\gamma_a(t'_1, \dots, t'_{k_a})]^4(x) dx = \frac{4m_{a,i_a,\lambda}}{\mu_1} \quad \text{for all } i_a = 1, \dots, k_a \tag{4.34}
$$

and

$$
\int_{\Omega'_{b,j_b}} [\gamma_b(s'_1, \dots, s'_{k_b})]^4(x) dx = \frac{4m_{b,j_b,\lambda}}{\mu_2} \quad \text{for all } j_b = 1, \dots, k_b. \tag{4.35}
$$

Denote $\gamma_a(t'_1, \dots, t'_{k_a})$ and $\gamma_b(s'_1, \dots, s'_{k_b})$ by u_* and v_*. Then, by (2.8)–(2.10), we have

$$
\int_{\mathbb{R}^3 \setminus \Omega_a^{J_a}} u_*^2 dx \le C_{a,b}^{-1} \int_{\mathbb{R}^3 \setminus \Omega_a^{J_a}} |\nabla u_*|^2 + (\lambda a(x) + a_0(x)) u_*^2 dx
$$

and

$$
\int_{\mathbb{R}^3 \setminus \Omega_b^{J_b}} v_*^2 dx \le C_{a,b}^{-1} \int_{\mathbb{R}^3 \setminus \Omega_b^{J_b}} |\nabla v_*|^2 + (\lambda b(x) + b_0(x)) v_*^2 dx
$$

for $\lambda \ge \Lambda_1$. Note that $\{\Omega'_{a,i_a}\}$ and $\{\Omega'_{b,j_b}\}$ are two sequences of bounded domains with smooth boundaries, so the restriction of u_* on Ω'_{a,i_a} lies in $H^1(\Omega'_{a,i_a})$ for every $i_a = 1, \dots, n_a$, while the restriction of v_* on Ω'_{b,j_b} lies in $H^1(\Omega'_{b,j_b})$ for every $j_b = 1, \ldots, n_b$. Now, by $\beta < 0$, (4.1) and the construction of $f_a(x,t)$, $f_b(x,t)$ and $h(x,t,s)$, we have

$$\begin{aligned}
J^*_{\lambda,\beta}(u_*, v_*) &\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_*|^2 + (\lambda a(x) + a_0(x))u_*^2 dx - \mu_1 \int_{\mathbb{R}^3} F_a(x, u_*)dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_*|^2 + (\lambda b(x) + b_0(x))v_*^2 dx - \mu_2 \int_{\mathbb{R}^3} F_b(x, v_*)dx \\
&\geq \frac{1}{2}\left(\int_{\mathbb{R}^3\setminus\Omega_a^{J_a}} |\nabla u_*|^2 + (\lambda a(x) + a_0(x))u_*^2 dx\right. \\
&\quad \left. -\delta_\beta^2 \int_{\mathbb{R}^3\setminus\Omega_a^{J_a}} u_*^2 dx\right) + \sum_{i_a=1}^{k_a} \mathcal{E}_{\Omega'_{a,i_a},\lambda}(u_*) \\
&\quad + \frac{1}{2}\left(\int_{\mathbb{R}^3\setminus\Omega_b^{J_b}} |\nabla v_*|^2 + (\lambda b(x) + b_0(x))v_*^2 dx\right. \\
&\quad \left. -\delta_\beta^2 \int_{\mathbb{R}^3\setminus\Omega_b^{J_b}} v_*^2 dx\right) + \sum_{j_b=1}^{k_b} \mathcal{E}_{\Omega'_{b,j_b},\lambda}(v_*) \\
&\geq \sum_{i_a=1}^{k_a} \mathcal{E}_{\Omega'_{a,i_a},\lambda}(u_*) + \sum_{j_b=1}^{k_b} \mathcal{E}_{\Omega'_{b,j_b},\lambda}(v_*). \qquad (4.36)
\end{aligned}$$

Thanks to (4.27)–(4.28) and (4.34)–(4.35), (4.36) implies $J^*_{\lambda,\beta}(u_*, v_*) \geq \sum_{i_a=1}^{k_a} m_{a,i_a,\lambda} + \sum_{j_b=1}^{k_b} m_{b,j_b,\lambda}$ for $\lambda \geq \Lambda_1$ and $\beta < 0$. Since $(\gamma_a, \gamma_b) \in \Gamma$ is arbitrary, we must have $\sum_{i_a=1}^{k_a} m_{a,i_a,\lambda} + \sum_{j_b=1}^{k_b} m_{b,j_b,\lambda} \leq m_{J_a,J_b,\lambda,\beta}$ for $\lambda \geq \Lambda_1$ and $\beta < 0$, which completes the proof. □

Let $m_{a,b} := \sum_{i_a=1}^{n_a} m_{a,i_a} + \sum_{j_b=1}^{n_b} m_{b,j_b}$. Then, we can obtain the following proposition.

Proposition 4.1 *Suppose $\beta < 0$. Then, there exists $\Lambda_2^*(\beta) \geq \Lambda_1^*(\beta, m_{a,b})$ such that $m_{J_a,J_b,\lambda,\beta}$ is a critical value of $J^*_{\lambda,\beta}(u,v)$ for $\lambda \geq \Lambda_2^*(\beta)$, that is, for all $\lambda \geq \Lambda_2^*(\beta)$, there exists $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E$ satisfying $D[J^*_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^* and $J^*_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{J_a,J_b,\lambda,\beta}$, where $\Lambda_1^*(\beta, m_{a,b})$ is given by Lemma 4.3. Furthermore, for every $\{\lambda_n\} \subset [\Lambda_2^*(\beta), +\infty)$ satisfying $\lambda_n \to +\infty$ as $n \to \infty$, there exists $(u_{0,\beta}^{J_a}, v_{0,\beta}^{J_b}) \in E$ such that*

1. *$(u_{0,\beta}^{J_a}, v_{0,\beta}^{J_b}) \in H_0^1(\Omega_{a,0}^{J_a}) \times H_0^1(\Omega_{b,0}^{J_b})$ with $u_{0,\beta}^{J_a} = 0$ on $\mathbb{R}^3\setminus\Omega_{a,0}^{J_a}$ and $v_{0,\beta}^{J_b} = 0$ on $\mathbb{R}^3\setminus\Omega_{b,0}^{J_b}$.*
2. *$(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}^{J_a}, v_{0,\beta}^{J_b})$ strongly in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $n \to \infty$ up to a subsequence.*
3. *The restriction of $u_{0,\beta}^{J_a}$ on Ω_{a,i_a} lies in $H_0^1(\Omega_{a,i_a})$ and is a critical point of $I_{\Omega_{a,i_a}}(u)$ for every $i_a \in J_a$, while the restriction of $v_{0,\beta}^{J_b}$ on Ω_{b,j_b} lies in $H_0^1(\Omega_{b,j_b})$ and is a critical point of $I_{\Omega_{b,j_b}}(v)$ for every $j_b \in J_b$.*

Proof Since the conditions $(D_1)-(D_2)$, (D_3'), (D_4) and (D_5') hold, by a similar argument as [24, Lemma 3.1], we can see that $\lim_{\lambda\to+\infty} m_{a,i_a,\lambda} = m_{a,i_a}$ and $\lim_{\lambda\to+\infty} m_{b,j_b,\lambda} = m_{b,j_b}$ for all $i_a = 1, \ldots, n_a$ and $j_b = 1, \ldots, n_b$. Note that $\beta < 0$, so by Lemma 4.5, there exists $\Lambda_2^*(\beta) \geq \Lambda_1^*(\beta, m_{a,b})$ such that $m_{J_a,J_b,\lambda,\beta} > J^*_{\lambda,\beta}(\gamma_{0,a}(t_1, \ldots, t_{k_a}), \gamma_{0,b}(s_1, \ldots, s_{k_b}))$

for $\lambda \geq \Lambda_2^*(\beta)$. Thanks to the construction of $m_{J_a,J_b,\lambda,\beta}$ and Lemma 4.2, we can use the linking theorem (cf. [1]) to show that $m_{J_a,J_b,\lambda,\beta}$ is a critical value of $J^*_{\lambda,\beta}(u,v)$ for $\lambda \geq \Lambda_2^*(\beta)$, that is, there exists $(u_{\lambda,\beta}, v_{\lambda,\beta}) \in E$ satisfying $D[J^*_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta})] = 0$ in E^* and $J^*_{\lambda,\beta}(u_{\lambda,\beta}, v_{\lambda,\beta}) = m_{J_a,J_b,\lambda,\beta}$ for all $\lambda \geq \Lambda_2^*(\beta)$. In what follows, we will show that (1)–(3) hold. Suppose $\{\lambda_n\} \subset [\Lambda_2^*(\beta), +\infty)$ satisfying $\lambda_n \to +\infty$ as $n \to \infty$. Then, by Lemmas 4.3 and 4.5, $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}$ is bounded in E with

$$\int_{\mathbb{R}^3\backslash\Omega_a^{J_a}} |\nabla u_{\lambda_n,\beta}|^2 + (\lambda_n a(x) + a_0(x))u^2_{\lambda_n,\beta}\mathrm{d}x \to 0 \quad \text{as } n \to \infty \tag{4.37}$$

and

$$\int_{\mathbb{R}^3\backslash\Omega_b^{J_b}} |\nabla v_{\lambda_n,\beta}|^2 + (\lambda_n b(x) + b_0(x))v^2_{\lambda_n,\beta}\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{4.38}$$

Without loss of generality, we assume $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \rightharpoonup (u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta})$ weakly in E as $n \to \infty$ for some $(u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta}) \in E$. For the sake of clarity, we divide the following proof into two steps.

Step 1 We prove that $(u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta}) \in H_0^1(\Omega^{J_a}_{a,0}) \times H_0^1(\Omega^{J_b}_{b,0})$ with $u^{J_a}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega^{J_a}_{a,0}$ and $v^{J_b}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega^{J_b}_{b,0}$.

Indeed, since $\beta < 0$, by Lemmas 4.1 and 4.5 and a similar argument as used in Step 1 of the proof for Theorem 1.1, we can conclude that $u^{J_a}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega_a$ and $v^{J_b}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega_b$. On the other hand, combining (2.9) and (4.37), we can see that $\int_{\Omega'_{a,i_a}} u^2_{\lambda_n,\beta}\mathrm{d}x \to 0$ for $i_a \in \{1, \ldots, n_a\}\backslash J_a$ as $n \to \infty$, which together with the Fatou lemma implies $u^{J_a}_{0,\beta} = 0$ on Ω'_{a,i_a} for $i_a \in \{1, \ldots, n_a\}\backslash J_a$. Since (4.38) holds, by a similar argument, we also have $v^{J_b}_{0,\beta} = 0$ on Ω'_{b,j_b} for $j_b \in \{1, \ldots, n_b\}\backslash J_b$. Note that $\{\Omega_{a,i_a}\}$ and $\{\Omega_{b,j_b}\}$ are two sequences of disjoint bounded domains with smooth boundaries. So $(u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta}) \in H_0^1(\Omega^{J_a}_{a,0}) \times H_0^1(\Omega^{J_b}_{b,0})$ with $u^{J_a}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega^{J_a}_{a,0}$ and $v^{J_b}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega^{J_b}_{b,0}$.

Step 2 We prove that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta})$ strongly in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $n \to \infty$ up to a subsequence, and the restriction of $u^{J_a}_{0,\beta}$ on Ω_{a,i_a} lies in $H_0^1(\Omega_{a,i_a})$ and is a critical point of $I_{\Omega_{a,i_a}}(u)$ for every $i_a \in J_a$, while the restriction of $v^{J_b}_{0,\beta}$ on Ω_{b,j_b} lies in $H_0^1(\Omega_{b,j_b})$ and is a critical point of $I_{\Omega_{b,j_b}}(v)$ for every $j_b \in J_b$.

Indeed, since $(u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta}) \in H_0^1(\Omega^{J_a}_{a,0}) \times H_0^1(\Omega^{J_b}_{b,0})$ with $u^{J_a}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega^{J_a}_{a,0}$ and $v^{J_b}_{0,\beta} = 0$ on $\mathbb{R}^3\backslash\Omega^{J_b}_{b,0}$, by $D[J^*_{\lambda_n,\beta}(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})] = 0$ in E^* and the condition (D_3'), we can see that the restriction of $u^{J_a}_{0,\beta}$ on Ω_{a,i_a}, denoted by $u^{J_a}_{i_a,\beta}$, lies in $H_0^1(\Omega_{a,i_a})$ and $I'_{\Omega_{a,i_a}}(u^{J_a}_{i_a,\beta}) = 0$ in $H^{-1}(\Omega_{a,i_a})$ for all $i_a \in J_a$, while the restriction of $v^{J_b}_{0,\beta}$ on Ω_{b,j_b}, denoted by $v^{J_b}_{j_b,\beta}$, lies in $H_0^1(\Omega_{b,j_b})$ and $I'_{\Omega_{b,j_b}}(v^{J_b}_{j_b,\beta}) = 0$ in $H^{-1}(\Omega_{b,j_b})$ for all $j_b \in J_b$. Now, since $\beta < 0$ and the condition (D_3) contains the condition (D_3'), by the construction of $f_a(x,t)$, $f_b(x,t)$ and $h(x,t,s)$ and a similar argument as used in Step 2 of the proof for Theorem 1.1, we can conclude that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u^{J_a}_{0,\beta}, b^{J_b}_{0,\beta})$ in E as $n \to \infty$. Since E is embedded continuously into $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u^{J_a}_{0,\beta}, v^{J_b}_{0,\beta})$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $n \to \infty$. □

In the following part, we will use a deformation argument to obtain the solution described by Theorem 1.2. Let $\varepsilon_0 = \frac{1}{4}\min\{2\sqrt{m_{a,1}}, \ldots, 2\sqrt{m_{a,k_a}}, 2\sqrt{m_{b,1}}, \ldots, 2\sqrt{m_{b,k_b}}\}$. For $0 < \varepsilon < \varepsilon_0$, we define

$$\mathcal{D}_{a,\varepsilon}=\Bigg\{u\in E_a \mid \left(\int_{\mathbb{R}^3\backslash\Omega_{a,0}^{J_a}}|\nabla u|^2+u^2\mathrm{d}x\right)^{\frac{1}{2}}<\varepsilon,\\ \left|\left(\int_{\Omega_{a,i_a}}|\nabla u|^2+a_0(x)u^2\mathrm{d}x\right)^{\frac{1}{2}}-2\sqrt{m_{a,i_a}}\right|<\varepsilon,\forall i_a\in J_a\Bigg\}$$

and

$$\mathcal{D}_{b,\varepsilon}=\Bigg\{v\in E_b \mid \left(\int_{\mathbb{R}^3\backslash\Omega_{b,0}^{J_b}}|\nabla v|^2+v^2\mathrm{d}x\right)^{\frac{1}{2}}<\varepsilon,\\ \left|\left(\int_{\Omega_{b,j_b}}|\nabla v|^2+b_0(x)v^2\mathrm{d}x\right)^{\frac{1}{2}}-2\sqrt{m_{b,j_b}}\right|<\varepsilon,\forall j_b\in J_b\Bigg\}.$$

Let $\mathcal{D}_\varepsilon=\mathcal{D}_{a,\varepsilon}\cap\mathcal{D}_{b,\varepsilon}$ and $J_{\lambda,\beta}^{m_{J_a,J_b}}=\Big\{(u,v)\in E \mid J_{\lambda,\beta}^*(u,v)\le m_{J_a,J_b}\Big\}$, where $m_{J_a,J_b}=\sum_{i_a=1}^{k_a}m_{a,i_a}+\sum_{j_b=1}^{k_b}m_{b,j_b}$. Then, we have the following.

Proposition 4.2 *Assume $\beta<0$ and $0<\varepsilon<\varepsilon_0$. Then, there exists $\Lambda_3^*(\beta,\varepsilon)\ge\Lambda_2^*(\beta)$ such that $J_{\lambda,\beta}^*(u,v)$ has a critical point in $\mathcal{D}_\varepsilon\cap J_{\lambda,\beta}^{m_{J_a,J_b}}$ for $\lambda\ge\Lambda_3^*(\beta,\varepsilon)$.*

Proof Suppose the contrary, since Lemma 4.2 holds, there exist $\{\lambda_n\}$ and $\{c_{n,\beta}\}$ with $\lambda_n\to+\infty$ as $n\to\infty$ and $c_{n,\beta}>0$ for all n such that

$$\|D[J_{\lambda_n,\beta}^*(u,v)]\|_{E^*}\ge c_{n,\beta}\quad\text{for all }(u,v)\in\mathcal{D}_\varepsilon\cap J_{\lambda_n,\beta}^{m_{J_a,J_b}}. \tag{4.39}$$

For the sake of clarity, we divide the following proof into several steps.

Step 1 We prove that there exists $N\in\mathbb{N}$ and a constant $\sigma_0>0$ such that $\|J_{\lambda_n,\beta}^*(u,v)\|_{E^*}\ge\sigma_0$ for every $(u,v)\in(\mathcal{D}_{3\varepsilon}\backslash\mathcal{D}_\varepsilon)\cap J_{\lambda_n,\beta}^{m_{J_a,J_b}}$ and $n\ge N$.

Suppose the contrary, there exists a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, such that $\|J_{\lambda_n,\beta}^*(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\|_{E^*}\to0$ as $n\to\infty$ for some $(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\in(\mathcal{D}_{3\varepsilon}\backslash\mathcal{D}_\varepsilon)\cap J_{\lambda_n,\beta}^{m_{J_a,J_b}}$. By a similar argument as used in Proposition 4.1, we can see that $(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\to(u_{0,\beta}^{J_a},v_{0,\beta}^{J_b})$ strongly in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ and E as $n\to\infty$ for some $(u_{0,\beta}^{J_a},v_{0,\beta}^{J_b})\in H_0^1(\Omega_{a,0}^{J_a})\times H_0^1(\Omega_{b,0}^{J_b})$ with $u_{0,\beta}^{J_a}=0$ on $\mathbb{R}^3\backslash\Omega_{a,0}^{J_a}$ and $v_{0,\beta}^{J_b}=0$ on $\mathbb{R}^3\backslash\Omega_{b,0}^{J_b}$. The restriction of $u_{0,\beta}^{J_a}$ on Ω_{a,i_a}, denoted by $u_{i_a,\beta}^{J_a}$, lies in $H_0^1(\Omega_{a,i_a})$ and $I'_{\Omega_{a,i_a}}(u_{i_a,\beta}^{J_a})=0$ in $H^{-1}(\Omega_{a,i_a})$ for all $i_a\in J_a$, while the restriction of $v_{0,\beta}^{J_b}$ on Ω_{b,j_b}, denoted by $v_{j_b,\beta}^{J_b}$, lies in $H_0^1(\Omega_{b,j_b})$ and $I'_{\Omega_{b,j_b}}(v_{j_b,\beta}^{J_b})=0$ in $H^{-1}(\Omega_{b,j_b})$ for all $j_b\in J_b$. Clearly, one of the following two cases must occur:

(1*) $\int_{\Omega_{a,i_a}}(u_{i_a,\beta}^{J_a})^4\mathrm{d}x\ge C$ for all $i_a\in J_a$ and $\int_{\Omega_{b,j_b}}(v_{j_b,\beta}^{J_b})^4\mathrm{d}x\ge C$ for all $j_b\in J_b$.

(2*) There exists $i'_a\in J_a$ or $j'_b\in J_b$ such that $\int_{\Omega_{a,i'_a}}(u_{i'_a,\beta}^{J_a})^4\mathrm{d}x=0$ or $\int_{\Omega_{b,j'_b}}(v_{j'_b,\beta}^{J_b})^4\mathrm{d}x=0$.

If case (1*) happens, then we must have $I_{\Omega_{a,i_a}}(u_{i_a,\beta}^{J_a})\ge m_{a,i_a}$ and $I_{\Omega_{b,j_b}}(v_{j_b,\beta}^{J_b})\ge m_{b,j_b}$ for all $i_a\in J_a$ and $j_b\in J_b$. Since $(u_{\lambda_n,\beta},v_{\lambda_n,\beta})\in J_{\lambda_n,\beta}^{m_{J_a,J_b}}$, by a similar argument as used in Step 3 of the proof for Theorem 1.1, we can show that $I_{\Omega_{a,i_a}}(u_{i_a,\beta}^{J_a})=m_{a,i_a}$ and $I_{\Omega_{b,j_b}}(v_{j_b,\beta}^{J_b})=m_{b,j_b}$ for all $i_a\in J_a$ and $j_b\in J_b$. It follows from the condition (D_4) that

$$\int_{\Omega_{a,i_a}}|\nabla u_{\lambda_n,\beta}|^2+a_0(x)u_{\lambda_n,\beta}^2\mathrm{d}x\to4m_{a,i_a}\quad\text{for all }i_a\in J_a$$

and

$$\int_{\Omega_{b,j_b}} |\nabla v_{\lambda_n,\beta}|^2 + b_0(x) v_{\lambda_n,\beta}^2 \mathrm{d}x \to 4m_{b,j_b} \quad \text{for all } j_b \in J_b$$

as $n \to \infty$, which then implies $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \in \mathcal{D}_\varepsilon$ for n large enough. It is impossible since $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \in (\mathcal{D}_{3\varepsilon} \backslash \mathcal{D}_\varepsilon) \cap J_{\lambda_n,\beta}^{m_{J_a,J_b}}$ for all n. Thus, we must have the case (2*). Without loss of generality, we assume $\int_{\Omega_{a,1}} (u_{1,\beta}^{J_a})^4 \mathrm{d}x = 0$. It follows from the condition (D_4) that

$$\left| \left(\int_{\Omega_{a,1}} |\nabla u_{\lambda_n,\beta}|^2 + a_0(x) u_{\lambda_n,\beta}^2 \mathrm{d}x \right)^{\frac{1}{2}} - 2\sqrt{m_{a,1}} \right| \to 2\sqrt{m_{a,1}} = 4\varepsilon_0 \quad \text{as } n \to \infty,$$

which implies $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \in E \backslash \mathcal{D}_{3\varepsilon}$ for n large enough. It also contradicts to the fact that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \in (\mathcal{D}_{3\varepsilon} \backslash \mathcal{D}_\varepsilon) \cap J_{\lambda_n,\beta}^{m_{J_a,J_b}}$ for all n.

Step 2 We construct a descending flow on $J_{\lambda_n,\beta}^{m_{J_a,J_b}}$ for every $n \geq N$.

Let $\eta : E \to [0, 1]$ be a local Lipschitz continuous function and satisfy

$$\eta(u, v) = \begin{cases} 1, & (u, v) \in \mathcal{D}_{\frac{3}{2}\varepsilon}, \\ 0, & (u, v) \in E \backslash \mathcal{D}_{2\varepsilon}. \end{cases}$$

Since $J_{\lambda_n,\beta}^*(u, v)$ is C^1 for every $n \geq N$, there exists a pseudo-gradient vector field of $J_{\lambda_n,\beta}^*(u, v)$, denoted by $D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}] = (D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}]_1, D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}]_2)$, which satisfies

(a_*) $\langle D[J_{\lambda_n,\beta}^*(u, v)], D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}] \rangle_{E^*,E^*} \geq \frac{1}{2} \| D[J_{\lambda_n,\beta}^*(u, v)] \|_{E^*}^2$ for all $(u, v) \in E$;

(b_*) $\| D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}] \|_{E^*} \leq 2 \| D[J_{\lambda_n,\beta}^*(u, v)] \|_{E^*}$ for all $(u, v) \in E$.

Let $\overrightarrow{\mathcal{V}}_n : J_{\lambda_n,\beta}^{m_{J_a,J_b}} \to E^*$ be a continuous map and given by

$$\begin{aligned} \overrightarrow{\mathcal{V}}_n(u, v) &= (\mathcal{V}_{1,n}(u, v), \mathcal{V}_{2,n}(u, v)) \\ &= -\frac{\eta(u, v)}{\| D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}] \|_{E^*}} (D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}]_1, D[\widetilde{J_{\lambda_n,\beta}^*(u, v)}]_2) \end{aligned}$$

for $(u, v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}} \backslash \mathcal{K} = \{(u, v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}} \mid D[J_{\lambda_n,\beta}^*(u, v)] \neq 0\}$ and

$$\overrightarrow{\mathcal{V}}_n(u, v) = (\mathcal{V}_{1,n}(u, v), \mathcal{V}_{2,n}(u, v)) = (0, 0)$$

for $(u, v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}} \cap \mathcal{K} = \{(u, v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}} \mid D[J_{\lambda_n,\beta}^*(u, v)] = 0\}$. Clearly, $\| \overrightarrow{\mathcal{V}}_n(u, v) \|_{E^*} \leq 1$ for all $(u, v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}}$. Furthermore, by (4.39), Step 1 and the definition of η, we can see that $\overrightarrow{\mathcal{V}}_n(u, v)$ is locally Lipschitz. Now, let us consider the flow $\overrightarrow{\rho}_n(\tau) = (\rho_{1,n}(\tau), \rho_{2,n}(\tau))$ given by the following two-component system of ODE

$$\begin{cases} \frac{d\rho_{1,n}(\tau)}{\mathrm{d}\tau} = \mathcal{V}_{1,n}(\rho_{1,n}(\tau), \rho_{2,n}(\tau)), \\ \frac{d\rho_{2,n}(\tau)}{\mathrm{d}\tau} = \mathcal{V}_{2,n}(\rho_{1,n}(\tau), \rho_{2,n}(\tau)), \\ (\rho_{1,n}(0), \rho_{2,n}(0)) = (u, v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}}. \end{cases}$$

By (a_*), (b_*) and a direct calculation, we can see that

$$\begin{aligned}\frac{dJ^*_{\lambda_n,\beta}(\rho_{1,n}(\tau),\rho_{2,n}(\tau))}{\mathrm{d}\tau} &= \left\langle \left(\frac{\partial J^*_{\lambda_n,\beta}}{\partial u}(\rho_{1,n},\rho_{2,n}), \frac{\partial J^*_{\lambda_n,\beta}}{\partial v}(\rho_{1,n},\rho_{2,n})\right), (\mathcal{V}_{1,n}(\rho_{1,n},\rho_{2,n}), \mathcal{V}_{2,n}(\rho_{1,n},\rho_{2,n}))\right\rangle_{E^*,E^*} \\ &\le -\frac{1}{4}\eta(\rho_{1,n},\rho_{2,n})\|D[J^*_{\lambda_n,\beta}(\rho_{1,n},\rho_{2,n})]\|_{E^*} \\ &\le 0.\end{aligned}$$

It follows that $\overrightarrow{\rho}_n(\tau) = (\rho_{1,n}(\tau), \rho_{2,n}(\tau))$ is a descending flow on $J^{m_{J_a,J_b}}_{\lambda_n,\beta}$. Furthermore, for every $\tau > 0$, we have $\overrightarrow{\rho}_n(\tau) = \overrightarrow{\rho}_n(0)$ if $\overrightarrow{\rho}_n(0) = (u, v) \in E\backslash\mathcal{D}_{2\varepsilon}$.

Step 3 For every $n \ge N$, we construct a map $\overrightarrow{\rho}^{\,0}_n(\tau) = (\rho^0_{1,n}(\tau), \rho^0_{2,n}(\tau)) \in \Gamma$ for all $\tau > 0$ such that

$$\sup_{[0,1]^{k_a}\times[0,1]^{k_b}} J^*_{\lambda_n,\beta}(\rho^0_{1,n}(\tau_n), \rho^0_{2,n}(\tau_n)) < m_{J_a,J_b} - \sigma^*_0 \quad \text{for some } \tau_n > 0, \tag{4.40}$$

where $\sigma^*_0 > 0$ is a constant.

Indeed, let $n \ge N$ and $\gamma_{0,a}$ and $\gamma_{0,b}$ be given by (4.30) and (4.31). We consider $\overrightarrow{\rho}^{\,0}_n(\tau) = (\rho^0_{1,n}(\tau), \rho^0_{2,n}(\tau))$, where $(\rho^0_{1,n}(0), \rho^0_{2,n}(0)) = (\gamma_{0,a}, \gamma_{0,b})$. Since W_{a,i_a} and W_{b,j_b} are the least energy nonzero critical points of $I_{\Omega_{a,i_a}}(u)$ and $I_{\Omega_{b,j_b}}(v)$ for all $i_a \in J_a$ and $j_b \in J_b$, by the choice of R and ε, we can see that

$$(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b})) \in E\backslash\mathcal{D}_{2\varepsilon}$$

for every $(t_1,\ldots,t_{k_a}, s_1,\ldots,s_{k_b}) \in \partial([0,1]^{k_a}\times[0,1]^{k_b})$. It follows from the construction of $(\rho^0_{1,n}(\tau), \rho^0_{2,n}(\tau))$ that

$$(\rho^0_{1,n}(\tau)(t_1,\ldots,t_{k_a}), \rho^0_{2,n}(\tau)(s_1,\ldots,s_{k_b})) = (\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b}))$$

for every $(t_1,\ldots,t_{k_a}, s_1,\ldots,s_{k_b}) \in \partial([0,1]^{k_a}\times[0,1]^{k_b})$ and $\tau > 0$. Thus, $\overrightarrow{\rho}^{\,0}_n(\tau) \in \Gamma$ for all $\tau > 0$. It remains to show (4.40) holds. For every $(t_1,\ldots,t_{k_a}, s_1,\ldots,s_{k_b}) \in [0,1]^{k_a}\times[0,1]^{k_b}$, one of the following two cases must occur:

(a^*) $(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b})) \in E\backslash\mathcal{D}_\varepsilon$.
(b^*) $(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b})) \in \mathcal{D}_\varepsilon$.

If case (a^*) happens, then by Step 2, we must have

$$\begin{aligned}&J^*_{\lambda_n,\beta}(\rho^0_{1,n}(\tau)(t_1,\ldots,t_{k_a}), \rho^0_{2,n}(\tau)(s_1,\ldots,s_{k_b})) \\ &\quad\le J^*_{\lambda_n,\beta}(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b}))\end{aligned}$$

for all $\tau > 0$. Moreover, by (4.29) and the choice of $\{W_{a,i_a}\}$ and $\{W_{b,j_b}\}$, we can see that

$$J^*_{\lambda_n,\beta}(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b})) = \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b}$$

if and only if $t_{i_a} = s_{j_b} = \frac{1}{R}$ for all $i_a \in J_a$ and $j_b \in J_b$. Since $(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b})) \in E\backslash\mathcal{D}_\varepsilon$ in this case, there exists $i'_a \in J_a$ or $j'_b \in J_b$ such that $t_{i'_a} \ne \frac{1}{R}$ or $s_{j'_b} \ne \frac{1}{R}$. It follows the construction of $\gamma_{0,a}$ and $\gamma_{0,b}$ and the condition (D'_3) that

$$m^*_{a,b} = \sup_{(u,v)\in\mathbb{P}} J^*_{\lambda_n,\beta}(u,v) = \sup_{(u,v)\in\mathbb{P}} (I_{\Omega_a}(u) + I_{\Omega_b}(v)) < \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b}, \tag{4.41}$$

where $\mathbb{P} = (\gamma_{0,a}([0,1]^{k_a}) \times \gamma_{0,b}([0,1]^{k_b}))\backslash\mathcal{D}_\varepsilon$. If case (b^*) happens, then two subcases may occur:

- (b_1^*) $(\rho_{1,n}^0(\tau)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau)(s_1,\ldots,s_{k_b})) \in \mathcal{D}_{\frac{3}{2}\varepsilon}$ for all $\tau > 0$.
- (b_2^*) There exists $\tau_n^* > 0$ such that $(\rho_{1,n}^0(\tau_n^*)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau_n^*)(s_1,\ldots,s_{k_b})) \in E\backslash\mathcal{D}_{\frac{3}{2}\varepsilon}$.

In the subcase (b_1^*), by Step 2 and the Taylor expansion, we can calculate that

$$\begin{aligned} &J_{\lambda_n,\beta}^*(\rho_{1,n}^0(\tau)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau)(s_1,\ldots,s_{k_b})) \\ &\quad\le J_{\lambda_n,\beta}^*(\gamma_{0,a}(t_1,\ldots,t_{k_a}), \gamma_{0,b}(s_1,\ldots,s_{k_b})) \\ &\qquad - \int_0^\tau \frac{1}{4}\eta(\rho_{1,n},\rho_{2,n})\|D[J_{\lambda_n,\beta}^*(\rho_{1,n},\rho_{2,n})]\|_{E^*} d\nu \\ &\quad\le \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b} - \frac{1}{4}\tau\min\{c_{n,\beta},\sigma_0\}, \end{aligned} \tag{4.42}$$

where $c_{n,\beta}$ is given by (4.39) and σ_0 is given by Step 1. It follows that

$$J_{\lambda_n,\beta}^*(\rho_{1,n}^0(\tau)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau)(s_1,\ldots,s_{k_b})) < \sum_{i_a=1}^{k_a} m_{a,i_a} + \sum_{j_b=1}^{k_b} m_{b,j_b} - \sigma_0$$

for $\tau \ge \tau_n^0 = \frac{8\sigma_0}{\min\{c_{n,\beta},\sigma_0\}}$. In the subcase (b_2^*), there must exist $0 \le \tau_{n,a}^* < \tau_{n,b}^* \le \tau_n^*$ such that

$$(\rho_{1,n}^0(\tau_{n,a}^*)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau_{n,a}^*)(s_1,\ldots,s_{k_b})) \in \partial\mathcal{D}_\varepsilon$$

and

$$(\rho_{1,n}^0(\tau_{n,b}^*)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau_{n,b}^*)(s_1,\ldots,s_{k_b})) \in \partial\mathcal{D}_{\frac{3}{2}\varepsilon}.$$

For the sake of convenience, we, respectively, denote $(\rho_{1,n}^0(\tau_{n,a}^*)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau_{n,a}^*)(s_1,\ldots,s_{k_b}))$ and $(\rho_{1,n}^0(\tau_{n,b}^*)(t_1,\ldots,t_{k_a}), \rho_{2,n}^0(\tau_{n,b}^*)(s_1,\ldots,s_{k_b}))$ by $(u_{n,1}, v_{n,1})$ and $(u_{n,2}, v_{n,2})$. Since Lemma 2.1 holds for $\lambda \ge \Lambda_1$, by a similar argument as used for (4.14) in [24], we can see that one of the following four cases must happen:

- (a^{**}) $\int_{\Omega_a} |\nabla(u_{n,1}-u_{n,2})|^2 + a_0(x)(u_{n,1}-u_{n,2})^2 dx \ge \frac{\varepsilon^2}{4}$.
- (b^{**}) $\int_{\Omega_b} |\nabla(v_{n,1}-v_{n,2})|^2 + b_0(x)(v_{n,1}-v_{n,2})^2 dx \ge \frac{\varepsilon^2}{4}$.
- (c^{**}) $\int_{\mathbb{R}^3\backslash\Omega_{a,0}^{J_a}} |\nabla(u_{n,1}-u_{n,2})|^2 + (u_{n,1}-u_{n,2})^2 dx \ge \frac{\varepsilon^2}{4}$.
- (d^{**}) $\int_{\mathbb{R}^3\backslash\Omega_{b,0}^{J_b}} |\nabla(v_{n,1}-v_{n,2})|^2 + (v_{n,1}-v_{n,2})^2 dx \ge \frac{\varepsilon^2}{4}$.

By similar arguments as (2.1)–(2.2), we can see that in any case, there exists a constant $C(\varepsilon) > 0$ such that $\|(u_{n,1}, v_{n,1}) - (u_{n,2}, v_{n,2})\| \ge C(\varepsilon)$. On the other hand, by Step 2, we can see that $(u_{n,1}, v_{n,1}) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}}$ and $(u_{n,2}, v_{n,2}) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}}$. It follows from $\|\overrightarrow{\mathcal{V}}_n(u,v)\|_{E^*} \le 1$ for all $(u,v) \in J_{\lambda_n,\beta}^{m_{J_a,J_b}}$ and the Taylor expansion that $\tau_{n,b}^* - \tau_{n,a}^* \ge C(\varepsilon)$. Now, by Step 1, we have

$$\begin{aligned}
&J_{\lambda_n,\beta}(\rho^0_{1,n}(\tau^*_n)(t_1,\ldots,t_{k_a}),\rho^0_{2,n}(\tau^*_n)(s_1,\ldots,s_{k_b}))\\
&\quad\le J_{\lambda_n,\beta}(\gamma_{0,a}(t_1,\ldots,t_{k_a}),\gamma_{0,b}(s_1,\ldots,s_{k_b}))\\
&\qquad-\int_{\tau^*_{n,a}}^{\tau^*_{n,b}}\frac{1}{4}\eta(\rho_{1,n},\rho_{2,n})\|D[J_{\lambda_n,\beta}(\rho_{1,n},\rho_{2,n})]\|_{E^*}d\nu\\
&\quad\le\sum_{i_a=1}^{k_a}m_{a,i_a}+\sum_{j_b=1}^{k_b}m_{b,j_b}-\frac{1}{4}C(\varepsilon)\sigma_0.
\end{aligned}\tag{4.43}$$

Let $\tau_n=\max\{\tau^0_n,\tau^*_n\}$ and $\sigma^*_0=\min\{\frac{1}{4}C(\varepsilon)\sigma_0,\sigma_0,\sum_{i_a=1}^{k_a}m_{a,i_a}+\sum_{j_b=1}^{k_b}m_{b,j_b}-m^*_{a,b}\}$. Then, (4.40) follows from (4.41)–(4.43) and Step 2.

Since $\overrightarrow{\rho}^{\,0}_n(\tau)=(\rho^0_{1,n}(\tau),\rho^0_{2,n}(\tau))\in\Gamma$ for all $\tau>0$, by the definition of $m_{J_a,J_b,\lambda_n,\beta}$ and Step 3, we can see that $m_{J_a,J_b,\lambda_n,\beta}\le m_{J_a,J_b}-\sigma^*_0$, which is impossible since Lemma 4.5 holds and $m_{a,i_a,\lambda_n}\to m_{a,i_a}$ and $m_{b,j_b,\lambda_n}\to m_{b,j_b}$ as $n\to\infty$ for all $i_a\in J_a$ and $j_b\in J_b$. □

We close this section by

Proof of Theorem 1.2 Suppose $\beta<0$. Then, by Proposition 4.2, for every $\varepsilon\in(0,\varepsilon_0)$, there exists $\Lambda^*_3(\beta,\varepsilon)>\Lambda^*_2(\beta)$ such that $J^*_{\lambda,\beta}(u,v)$ has a critical point $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})\in\mathcal{D}_\varepsilon\cap J^{m_{J_a,J_b}}_{\lambda,\beta}$ for all $\lambda\ge\Lambda^*_3(\beta,\varepsilon)$. Thanks to Lemma 4.3 and the choice of $\Lambda^*_3(\beta,\varepsilon)$, $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})$ is also a critical point of $J_{\lambda,\beta}(u,v)$. Since $u^{J_a}_{\lambda,\beta}$ and $v^{J_b}_{\lambda,\beta}$ are both nonnegative by the construction of $J^*_{\lambda,\beta}(u,v)$, we can use a similar argument as used in the proof of Theorem 1.1 to show that $u^{J_a}_{\lambda,\beta}$ and $v^{J_b}_{\lambda,\beta}$ are both positive, which implies $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})$ is a solution of $(\mathcal{P}_{\lambda,\beta})$. Clearly, $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})$ satisfies the concentration behaviors of (1) and (2), since $\Lambda^*_3(\beta,\varepsilon)\to+\infty$ as $\varepsilon\to0$ and $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})\in\mathcal{D}_\varepsilon$. Furthermore, since $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})\in J^{m_{J_a,J_b}}_{\lambda,\beta}$, by a similar argument as used in Proposition 4.1, we can see that the properties (3) and (4) are also hold. It remains to show that the concentration behavior (5) is also true. Indeed, by a similar argument as used in Proposition 4.1, the restriction of $u^{J_a}_{0,\beta}$ on Ω_{a,i_a}, denoted by $u^{J_a}_{i_a,\beta}$, lies in $H^1_0(\Omega_{a,i_a})$ and is a critical point of $I_{\Omega_{a,i_a}}(u)$ for every $i_a\in J_a$, while the restriction of $v^{J_b}_{0,\beta}$ on Ω_{b,j_b}, denoted by $v^{J_b}_{j_b,\beta}$, lies in $H^1_0(\Omega_{b,j_b})$ and is a critical point of $I_{\Omega_{b,j_b}}(v)$ for every $j_b\in J_b$. Since $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})\in\mathcal{D}_\varepsilon$ for all $\lambda\ge\Lambda^*_3(\beta,\varepsilon)$, it is easy to see that $u^{J_a}_{i_a,\beta}$ and $v^{J_b}_{j_b,\beta}$ are nonzero for all $i_a\in J_a$ and $j_b\in J_b$. It follows from $(u^{J_a}_{\lambda,\beta},v^{J_b}_{\lambda,\beta})\in J^{m_{J_a,J_b}}_{\lambda,\beta}$ that $u^{J_a}_{i_a,\beta}$ and $v^{J_b}_{j_b,\beta}$ are the least energy nonzero critical points of $I_{\Omega_{a,i_a}}(u)$ and $I_{\Omega_{b,j_b}}(v)$ for every $i_a\in J_a$ and $j_b\in J_b$, respectively. The proof of this theorem can be finished by taking $\Lambda_*(\beta)=\Lambda^*_3(\beta,\frac{\varepsilon_0}{2})$. □

5 The phenomenon of phase separations

In this section, we study the phenomenon of phase separations to $(\mathcal{P}_{\lambda,\beta})$, that is, we study the concentration behavior of the solutions for $(\mathcal{P}_{\lambda,\beta})$ as $\beta\to-\infty$.

Proof of Theorem 1.3 Suppose $\lambda\ge\Lambda_*$ and $\{\beta_n\}\subset(-\infty,0)$ satisfying $\beta_n\to-\infty$ as $n\to\infty$. Let $(u_{\lambda,\beta_n},v_{\lambda,\beta_n})$ be the ground state solution of $(\mathcal{P}_{\lambda,\beta_n})$ obtained by Theorem 1.1. Then, by Lemma 3.2 and a similar argument of (3.9), we can see that $\{(u_{\lambda,\beta_n},v_{\lambda,\beta_n})\}$ is bounded in E. Without loss of generality, we assume $(u_{\lambda,\beta_n},v_{\lambda,\beta_n})\rightharpoonup(u_{\lambda,0},v_{\lambda,0})$ weakly in E as $n\to\infty$ for some $(u_{\lambda,0},v_{\lambda,0})\in E$. Since E is embedded continuously into $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, we have $(u_{\lambda,0},v_{\lambda,0})\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$. Clearly, $u_{\lambda,0}\ge0$ and $v_{\lambda,0}\ge0$ in $\mathbb{R}^3$.

In what follows, we verify that $(u_{\lambda,0}, v_{\lambda,0})$ satisfies (1)–(4). For the sake of clarity, we divide the following proof into several steps.

Step 1 We prove that there exists $\Lambda_{**} \geq \Lambda_*$ such that $u_{\lambda,0} \neq 0$ and $v_{\lambda,0} \neq 0$ in $\mathbb{R}^3$ for $\lambda \geq \Lambda_{**}$ in the sense of almost everywhere.

Indeed, suppose $u_{\lambda,0} = 0$ a.e. in $\mathbb{R}^3$. Since $u_{\lambda,\beta_n} \rightharpoonup u_{\lambda,0}$ weakly in E_a as $n \to \infty$ and $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ is bounded in E, by a similar argument as (3.12), we can see that

$$\|u_{\lambda,\beta_n}\|_{a,\lambda}^2 \leq C\lambda^{-\frac{1}{2}}\|u_{\lambda,\beta_n}\|_{a,\lambda}^2 + o_n(1).$$

It follows that there exists $\Lambda_{**} \geq \Lambda_*$ such that $\|u_{\lambda,\beta_n}\|_{a,\lambda}^2 = o_n(1)$ for $\lambda \geq \Lambda_{**}$, which together Lemma 2.1 and the boundedness of $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ in E implies $\|u_{\lambda,\beta_n}\|_4 = o_n(1)$. Thanks to the fact that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})$ is the ground state solution of $(\mathcal{P}_{\lambda,\beta_n})$ with $\beta_n < 0$ for every n, we also have $\beta_n \int_{\mathbb{R}^3} u_{\lambda,\beta_n}^2 v_{\lambda,\beta_n}^2 \mathrm{d}x \to 0$ as $n \to \infty$. Hence, $J_{\lambda,\beta_n}(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) = I_{b,\lambda}(v_{\lambda,\beta_n}) + o_n(1)$. On the other hand, by Lemma 2.1, for every $n \in \mathbb{N}$, there exists $t_n > 0$ such that $t_n u_{\lambda,\beta_n} \in \mathcal{N}_{a,\lambda}$ with $\lambda \geq \lambda_{**}$. It follows from Lemma 3.1 and $\beta_n < 0$ that

$$\begin{aligned} J_{\lambda,\beta_n}(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) &\geq J_{\lambda,\beta_n}(t_n u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \\ &\geq I_{a,\lambda}(t_n u_{\lambda,\beta_n}) + I_{b,\lambda}(v_{\lambda,\beta_n}) \\ &\geq m_{a,\lambda} + J_{\lambda,\beta_n}(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) + o_n(1). \end{aligned} \tag{5.1}$$

Since $m_{a,\lambda} > 0$ for $\lambda \geq \lambda_{**}$, (5.1) is impossible for n large enough. By a similar argument, we can also show that $v_{\lambda,0} \neq 0$ in $\mathbb{R}^3$ for $\lambda \geq \Lambda_{**}$ in the sense of almost everywhere.

Step 2 We prove that $\{u_{\lambda,\beta_n}\}, \{v_{\lambda,\beta_n}\} \subset C(\mathbb{R}^3)$ and $\|u_{\lambda,\beta_n}\|_{C(\mathbb{R}^3)} \leq C_0$ and $\|v_{\lambda,\beta_n}\|_{C(\mathbb{R}^3)} \leq C_0$ for some $C_0 > 0$.

Indeed, by a similar argument as used in the proof of Theorem 1.1, we have $\{u_{\lambda,\beta_n}\}, \{v_{\lambda,\beta_n}\} \subset C(\mathbb{R}^3)$. On the other hand, since $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})$ is the ground state solution of $(\mathcal{P}_{\lambda,\beta_n})$ obtained by Theorem 1.1 and $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ is bounded in E, we can use a similar argument as used in (3) of Lemma 4.3 to show that $\{u_{\lambda,\beta_n}\}$ and $\{v_{\lambda,\beta_n}\}$ are bounded in $L^\infty(\mathbb{R}^3)$, that is, $\|u_{\lambda,\beta_n}\|_{L^\infty(\mathbb{R}^3)} \leq C_0$ and $\|v_{\lambda,\beta_n}\|_{L^\infty(\mathbb{R}^3)} \leq C_0$ for some $C_0 > 0$, which implies $\|u_{\lambda,\beta_n}\|_{C(\mathbb{R}^3)} \leq C_0$ and $\|v_{\lambda,\beta_n}\|_{C(\mathbb{R}^3)} \leq C_0$.

Step 3 We prove that $u_{\lambda,0}, v_{\lambda,0} \in C(\mathbb{R}^3)$ and are all local Lipschitz in $\mathbb{R}^3$.

Indeed, since the conditions $(D_1)-(D_5)$ hold, by [47, Theorem 1.7] and Step 2, we can see that $\{\nabla u_{\lambda,\beta_n}\}$ and $\{\nabla v_{\lambda,\beta_n}\}$ are bounded in $L^\infty(\mathbb{R}^3)$. On the other hand, for every n, by a similar argument as used in (3) of Lemma 4.3, we can show that $u_{\lambda,\beta_n}, v_{\lambda,\beta_n} \in L^\gamma(\mathbb{R}^3)$ for all $\gamma \geq 2$. Thanks to the Calderon-Zygmund inequality and conditions $(D_1)-(D_5)$, we have $u_{\lambda,\beta_n}, v_{\lambda,\beta_n} \in W_{loc}^{2,\gamma}(\mathbb{R}^3)$ for all $\gamma \geq 2$. Together with the Sobolev embedding theorem, it implies $u_{\lambda,\beta_n}, v_{\lambda,\beta_n} \in C^1(\mathbb{R}^3)$. It follows that $\{u_{\lambda,\beta_n}\}$ and $\{v_{\lambda,\beta_n}\}$ are bounded in $C^1(\mathbb{R}^3)$. Now, by applying the Ascoli-Arzelá theorem, we can conclude that $u_{\lambda,\beta_n} \to u_{\lambda,0}$ and $v_{\lambda,\beta_n} \to v_{\lambda,0}$ strongly in $C_{loc}(\mathbb{R}^3)$ as $n \to \infty$ with $u_{\lambda,0}, v_{\lambda,0} \in C(\mathbb{R}^3)$. This together with the boundness of $\{u_{\lambda,\beta_n}\}$ and $\{v_{\lambda,\beta_n}\}$ in $C^1(\mathbb{R}^3)$ again implies $u_{\lambda,0}$ and $v_{\lambda,0}$ are all local Lipschitz in $\mathbb{R}^3$.

Step 4 We prove that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,0}, v_{\lambda,0})$ strongly in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $n \to \infty$. Furthermore, $u_{\lambda,0} \in H_0^1(\{u_{\lambda,0} > 0\})$ and is a least energy solution of (1.3), while $v_{\lambda,0} \in H_0^1(\{v_{\lambda,0} > 0\})$ and is a least energy solution of (1.4).

Indeed, since $u_{\lambda,0} \in C(\mathbb{R}^3)$ and is local Lipschitz in $\mathbb{R}^3$, we can conclude that $\partial\{u_{\lambda,0} > 0\}$, the boundary of the set $\{u_{\lambda,0} > 0\}$, is local Lipschitz. It follows from $u_{\lambda,0} \in H^1(\mathbb{R}^3)$ and $u_{\lambda,0} = 0$ in $\mathbb{R}^3 \backslash \{u_{\lambda,0} > 0\}$ that $u_{\lambda,0} \in H_0^1(\{u_{\lambda,0} > 0\})$. Similarly, we have $v_{\lambda,0} \in H_0^1(\{v_{\lambda,0} > 0\})$. Let $I_{a,\lambda}^*(u)$ and $I_{b,\lambda}^*(v)$, respectively, be the corresponding functional of

(1.3) and (1.4). By a similar argument as used in Lemma 2.1, we can show that

$$C\int_{\{u_{\lambda,0}>0\}} u^2\mathrm{d}x \le \int_{\{u_{\lambda,0}>0\}} |\nabla u|^2 + (\lambda a(x) + a_0(x))u^2\mathrm{d}x \quad \text{for all } u \in H_0^1(\{u_{\lambda,0} > 0\}) \tag{5.2}$$

and

$$C\int_{\{v_{\lambda,0}>0\}} v^2\mathrm{d}x \le \int_{\{v_{\lambda,0}>0\}} |\nabla v|^2 + (\lambda b(x) + b_0(x))v^2\mathrm{d}x \quad \text{for all } v \in H_0^1(\{v_{\lambda,0} > 0\}) \tag{5.3}$$

if $\lambda \ge \Lambda_1$. It follows that the Nehari manifolds of $I^*_{a,\lambda}(u)$ and $I^*_{b,\lambda}(v)$ are both well defined if $\lambda \ge \Lambda_1$. Let

$$m^*_{a,\lambda} = \inf_{\mathcal{N}^*_{a,\lambda}} I^*_{a,\lambda}(u) \quad \text{and} \quad m^*_{b,\lambda} = \inf_{\mathcal{N}^*_{b,\lambda}} I^*_{b,\lambda}(v),$$

where $\mathcal{N}^*_{a,\lambda}$ and $\mathcal{N}^*_{b,\lambda}$ are, respectively, the Nehari manifolds of $I^*_{a,\lambda}(u)$ and $I^*_{b,\lambda}(v)$. Then, $m^*_{a,\lambda} > 0$ and $m^*_{b,\lambda} > 0$ if $\lambda \ge \Lambda_1$. For every $\varepsilon > 0$, there exist $u_\varepsilon \in \mathcal{N}^*_{a,\lambda}$ and $v_\varepsilon \in \mathcal{N}^*_{b,\lambda}$ such that

$$I^*_{a,\lambda}(u_\varepsilon) < m^*_{a,\lambda} + \varepsilon \quad \text{and} \quad I^*_{b,\lambda}(v_\varepsilon) < m^*_{b,\lambda} + \varepsilon. \tag{5.4}$$

Since $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}$ is bounded in E, by the fact that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})$ is the ground state solution $(\mathcal{P}_{\lambda,\beta_n})$ and $\beta_n \to -\infty$, we have $\int_{\mathbb{R}^3} u^2_{\lambda,\beta_n} v^2_{\lambda,\beta_n}\mathrm{d}x \to 0$ as $n \to \infty$. It follows from the Fatou lemma that $\int_{\mathbb{R}^3} u^2_{\lambda,0} v^2_{\lambda,0}\mathrm{d}x = 0$, which implies $\{u_{\lambda,0} > 0\} \cap \{v_{\lambda,0} > 0\} = \emptyset$. Hence, by $u_\varepsilon \in \mathcal{N}^*_{a,\lambda}$ and $v_\varepsilon \in \mathcal{N}^*_{b,\lambda}$, we can see that $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_{\lambda,\beta_n}$ for all n. Now, by (5.4), we have

$$2\varepsilon + m^*_{a,\lambda} + m^*_{b,\lambda} \ge I^*_{a,\lambda}(u_\varepsilon) + I^*_{b,\lambda}(v_\varepsilon) = J_{\lambda,\beta_n}(u_\varepsilon, v_\varepsilon) \ge m_{\lambda,\beta_n} \quad \text{for all } n.$$

Since $\varepsilon > 0$ and $n \in \mathbb{N}$ is arbitrary, we can conclude that

$$m^*_{a,\lambda} + m^*_{b,\lambda} \ge \limsup_{n\to\infty} m_{\lambda,\beta_n}. \tag{5.5}$$

On the other hand, note that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})$ is the ground state solution $(\mathcal{P}_{\lambda,\beta_n})$ and $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \rightharpoonup (u_{\lambda,0}, v_{\lambda,0})$ weakly in E as $n \to \infty$, by $\beta_n < 0$, we can see that

$$\int_{\{u_{\lambda,0}>0\}} |\nabla u_{\lambda,0}|^2 + (\lambda a(x) + a_0(x))u^2_{\lambda,0}\mathrm{d}x \le \mu_1 \int_{\{u_{\lambda,0}>0\}} u^4_{\lambda,0}\mathrm{d}x$$

and

$$\int_{\{v_{\lambda,0}>0\}} |\nabla v_{\lambda,0}|^2 + (\lambda b(x) + b_0(x))v^2_{\lambda,0}\mathrm{d}x \le \mu_2 \int_{\{v_{\lambda,0}>0\}} v^4_{\lambda,0}\mathrm{d}x.$$

It follows from (5.2) and (5.3) that there exist $0 < t_0 \le 1$ and $0 < s_0 \le 1$ such that $t_0 u_{\lambda,0} \in \mathcal{N}^*_{a,\lambda}$ and $s_0 v_{\lambda,0} \in \mathcal{N}^*_{b,\lambda}$. Now, since $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \rightharpoonup (u_{\lambda,0}, v_{\lambda,0})$ weakly in E as $n \to \infty$, by a similar argument as (3.13), we can see that

$$\begin{aligned}\liminf_{n\to+\infty} m_{\lambda,\beta_n} &= \liminf_{n\to+\infty} J_{\lambda,\beta_n}(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\\ &= \frac{1}{4}\liminf_{n\to+\infty}(\|u_{\lambda,\beta_n}\|_{a,\lambda}^2 + \|v_{\lambda,\beta_n}\|_{b,\lambda}^2)\\ &\geq \frac{1}{4}(\|u_{\lambda,0}\|_{a,\lambda}^2 + \|v_{\lambda,0}\|_{b,\lambda}^2)\\ &\geq \frac{1}{4}(\|t_0 u_{\lambda,0}\|_{a,\lambda}^2 + \|s_0 v_{\lambda,0}\|_{b,\lambda}^2)\\ &= I_{a,\lambda}^*(t_0 u_{\lambda,0}) + I_{b,\lambda}^*(s_0 v_{\lambda,0})\\ &\geq m_{a,\lambda}^* + m_{b,\lambda}^*. \end{aligned} \tag{5.6}$$

Hence, by combining (5.5) and (5.6), we must have the following results:

(a^{***}) $\lim_{n\to\infty}\|u_{\lambda,\beta_n}\|_{a,\lambda}^2 = \|u_{\lambda,0}\|_{a,\lambda}^2$ and $\lim_{n\to\infty}\|v_{\lambda,\beta_n}\|_{b,\lambda}^2 = \|v_{\lambda,0}\|_{b,\lambda}^2$.
(b^{***}) $u_{\lambda,0}\in\mathcal{N}_{a,\lambda}^*$ with $I_{a,\lambda}^*(u_{\lambda,0}) = m_{a,\lambda}^*$ and $v_{\lambda,0}\in\mathcal{N}_{b,\lambda}^*$ with $I_{b,\lambda}^*(v_{\lambda,0}) = m_{b,\lambda}^*$.

By (a^{***}) and Lemma 2.1, we know that $\|u_{\lambda,\beta_n} - u_{\lambda,0}\|_{a,\lambda} = \|v_{\lambda,\beta_n} - v_{\lambda,0}\|_{b,\lambda} = o_n(1)$. Thanks to Lemma 2.1 once more and the condition (D_4), we observe that $\|u_{\lambda,\beta_n} - u_{\lambda,0}\|_a = \|v_{\lambda,\beta_n} - v_{\lambda,0}\|_b = o_n(1)$ for $\lambda\geq\Lambda_1$. Since by (5.2) and (5.3), $\mathcal{N}_{a,\lambda}^*$ and $\mathcal{N}_{b,\lambda}^*$ are a natural constraint in $H_0^1(\{u_{\lambda,0}>0\})$ and $H_0^1(\{v_{\lambda,0}>0\})$, respectively, which together with (b^{***}) implies $u_{\lambda,0}$ and $v_{\lambda,0}$ are a least energy solution of (1.3) and (1.4), respectively.

Step 5 We prove that $\{x\in\mathbb{R}^3 \mid u_{\lambda,0}(x)>0\}$ and $\{x\in\mathbb{R}^3 \mid v_{\lambda,0}(x)>0\}$ are connected domains and $\{x\in\mathbb{R}^3 \mid u_{\lambda,0}(x)>0\} = \mathbb{R}^3\backslash\overline{\{x\in\mathbb{R}^3 \mid v_{\lambda,0}(x)>0\}}$.

Indeed, since $u_{\lambda,0}$ and $v_{\lambda,0}$ are, respectively, a least energy solution of (1.3) and (1.4), we must have $\{x\in\mathbb{R}^3 \mid u_{\lambda,0}(x)>0\}$ and $\{x\in\mathbb{R}^3 \mid v_{\lambda,0}(x)>0\}$ are connected domains. It remains to show that $\{x\in\mathbb{R}^3 \mid u_{\lambda,0}(x)>0\} = \mathbb{R}^3\backslash\overline{\{x\in\mathbb{R}^3 \mid v_{\lambda,0}(x)>0\}}$. To the contrary, we suppose that there exists an open set Ω satisfying $\{x\in\mathbb{R}^3 \mid u_{\lambda,0}(x)>0\}\subsetneqq\Omega\subsetneqq\mathbb{R}^3\backslash\overline{\{x\in\mathbb{R}^3 \mid v_{\lambda,0}(x)>0\}}$. Furthermore, it has a locally lipschitz boundary. Since $\Omega\subsetneqq\mathbb{R}^3\backslash\overline{\{x\in\mathbb{R}^3 \mid v_{\lambda,0}(x)>0\}}$, by a similar argument as in Step 3, we can show that $u_{\lambda,0}$ is a least energy solution of the following equation:

$$-\Delta u + (\lambda a(x) + a_0(x))u = \mu_1 u^3, \quad u\in H_0^1(\Omega).$$

Since $u_{\lambda,0}\geq 0$ in Ω, by a similar argument as in the proof of Theorem 1.1, we can conclude that $u_{\lambda,0}>0$ on Ω, which contradicts $\{x\in\mathbb{R}^3 \mid u_{\lambda,0}(x)>0\}\subsetneqq\Omega$.

We complete the proof by showing that $\beta^2\int_{\mathbb{R}^3} u_{\lambda,\beta}^2 v_{\lambda,\beta}^2 \to 0$ as $\beta\to-\infty$, where $(u_{\lambda,\beta}, v_{\lambda,\beta})$ is the ground state solution of $(\mathcal{P}_{\lambda,\beta})$ obtained by Theorem 1.1. Indeed, if not, then there exists $\{\beta_n\}\subset(-\infty,0)$ such that $\beta_n^2\int_{\mathbb{R}^3} u_{\lambda,\beta_n}^2 v_{\lambda,\beta_n}^2 \leq -C$. By (5.5) and (5.6), we must have $\beta_n^2\int_{\mathbb{R}^3} u_{\lambda,\beta_n}^2 v_{\lambda,\beta_n}^2 \to 0$ as $\beta_n\to-\infty$ up to a subsequence, which is a contradiction. □

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