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On a K-component elliptic system with the Sobolev critical exponent in high dimensions: the repulsive case

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Abstract Study the following K-component elliptic system

$$\begin{cases} -\left(a_i + b_i \sum_{j=1}^{k} b_j \int_\Omega |\nabla u_j|^2 dx\right)\Delta u_i \\ \quad = \lambda_i u_i + |u_i|^{2^*-2}u_i + \beta\left(\sum_{j=1, j\neq i}^{k} |u_j|^{\frac{2^*}{2}}\right)|u_i|^{\frac{2^*}{2}-2}u_i, \\ \quad u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k. \end{cases}$$

Here $k \geq 2$ is a integer and $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a bounded domain with smooth boundary $\partial\Omega$, $a_i, \lambda_i > 0$, $b_i \geq 0$ for all $i = 1, 2, \ldots, k$ and $\beta < 0$, $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent. By the variational method, we obtain a nontrivial solution of this system. The concentration behavior of this nontrivial solution as $\overrightarrow{\mathbf{b}} \to \overrightarrow{\mathbf{0}}$ and $\beta \to -\infty$ are both studied and the phase separation is exhibited for $N \geq 6$, where $\overrightarrow{\mathbf{b}} = (b_1, b_2, \ldots, b_k)$ is a vector. Our results extend and generalize the results in Chen and Zou (Arch Ration Mech Anal 205:515–551, 2012; Calc Var Partial Differ Equ 52:423–467, 2015). Moreover, by studying the phase separation, we also prove some existence and multiplicity results of the sign-changing solutions to the following Brezís–Nirenberg problem of the Kirchhoff type

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + |u|^{2^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $N \geq 6$, $a, \lambda > 0$ and $b \geq 0$. These results can be seen as an extension of the results in Cerami et al. (J Funct Anal 69:289–306, 1986). The concentration behaviors of the sign-changing solutions to the above equation as $b \to 0^+$ are also obtained.

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1 Introduction

In this paper, we consider the following K-component elliptic system

$$\begin{cases} -\left(a_i + b_i \sum_{j=1}^k b_j \int_\Omega |\nabla u_j|^2 dx\right)\Delta u_i \\ \quad = \lambda_i u_i + |u_i|^{2^*-2}u_i + \beta\left(\sum_{j=1,j\neq i}^k |u_j|^{\frac{2^*}{2}}\right)|u_i|^{\frac{2^*}{2}-2}u_i, \\ \quad u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k. \end{cases} \tag{$\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k}$}$$

Here $k \geq 2$ is a integer and $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a bounded domain with smooth boundary $\partial\Omega$. $a_i, \lambda_i > 0$, $b_i \geq 0$ for all $i = 1, 2, \ldots, k$ and $\beta < 0$. $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent and $\overrightarrow{\mathbf{b}} = (b_1, b_2, \ldots, b_k)$ is a vector.

Let $a_i = 1$ and $b_i = 0$ for all $i = 1, 2, \ldots, k$. Then by $2^* = 4$ for $N = 4$, we can see that System $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$ in dimension four is just the following k coupled elliptic system:

$$\begin{cases} -\Delta u_i = \lambda_i u_i + |u_i|^2 u_i + \beta\left(\sum_{j=1,j\neq i}^k |u_j|^2\right)u_i, \\ u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k. \end{cases} \tag{$\mathcal{S}^1_{\beta,k}$}$$

On the other hand, it is well known that the solution of $(\mathcal{S}^1_{\beta,k})$ in low dimensions $(1 \leq N \leq 3)$ is related to the solitary wave solutions of the following k coupled nonlinear Schrödinger equations which is also known in the literature as Gross–Pitaevskii equations (e.g. [18,37]):

$$\begin{cases} -\iota\frac{\partial}{\partial t}\Psi_i = \Delta\Psi_i - V_i(x)\Psi_i + |\Psi_i|^2\Psi_i + \beta\left(\sum_{j=1,j\neq i}^k |\Psi_j|^2\right)\Psi_i, \\ \Psi_i = \Psi_i(t,x) \in H^1(\mathbb{R}^N;\mathbb{C}), \quad N = 1, 2, 3, \quad i = 1, 2, \ldots, k. \end{cases} \tag{$\mathcal{S}^2_{\beta,k}$}$$

Here, ι is the imaginary unit. Such a system appears in many different physical problems. For example, in the Hartree–Fock theory, $(\mathcal{S}^2_{\beta,k})$ can be used to describe multispecies Bose–Einstein condensation in k different hyperfine spin states (cf. [7]) and such a condensation has been experimentally observed in the triplet states (cf. [34]). The solutions Ψ_i are the ith condensate amplitudes, the functions V_i represent the trapping magnetic potentials and β is the interaction of the states, where the interaction is attractive if $\beta > 0$ and repulsive if $\beta < 0$. For $k = 2$, the Gross–Pitaevskii equation also arises in nonlinear optics (cf. [1]). To obtain the solitary wave solutions, we set $\Psi_i(t,x) = e^{-\iota t\lambda_i}u_i(x)$ for all $i = 1, 2, \ldots, k$. Then u_i satisfy the following system

$$\begin{cases} -\Delta u_i + V_i(x)u_i = \lambda_i u_i + |u_i|^2 u_i + \beta\left(\sum_{j=1,j\neq i}^k |u_j|^2\right)u_i, \\ u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k. \end{cases} \tag{$\mathcal{S}^3_{\beta,k}$}$$

By taking $V_i(x) \equiv 0$ for all $i = 1, 2, \ldots, k$, we can see that System $(\mathcal{S}^3_{\beta,k})$ becomes System $(\mathcal{S}^1_{\beta,k})$. Due to the important applications in physics, the System $(\mathcal{S}^1_{\beta,k})$ in low dimensions $(1 \leq N \leq 3)$ has been studied extensively in the last decades. We refer the readers to [4,8,13,27,30,35–37,40] and the references therein, where various existence theorems of the solitary wave solutions were established.

Recently, the System $(\mathcal{S}^1_{\beta,k})$ in high dimensions ($N \geq 4$) has begun to attract attention (cf. [10,39]). Note that the cubic nonlinearities and the coupled terms of the system are all critical for $N = 4$ and even super critical for $N \geq 5$ with respect to the Sobolev critical exponent. Thus, the study on the System $(\mathcal{S}^1_{\beta,k})$ in high dimensions ($N \geq 4$) is much more complicated than that in low dimensions in the view point of calculus of variation. By applying the truncation argument, Tavares and Terracini [39] proved that the System $(\mathcal{S}^1_{\beta,k})$ has infinitely many sign-changing solution for all $N \geq 2$ and $k \geq 2$ with $\lambda_i < 0$ for all $i = 1, 2, \ldots, k$ being the Lagrange multipliers and $\beta < 0$. The phase separation is also studied in [39]. In [10], by establishing the threshold for the compactness of the (PS) sequence to $(\mathcal{S}^1_{\beta,2})$ and making some careful and complicated analysis, Chen and Zou proved that the System $(\mathcal{S}^1_{\beta,2})$ has a positive ground state solution for $N = 4$ and $0 < \lambda_i < \sigma_1$ for all $i = 1, 2$, where σ_1 is the first eigenvalue of $-\Delta$ in $L^2(\Omega)$. Moreover, the authors of [10] also studied the phenomenon of the phase separation of this positive ground state solution. However, only an alternative theorem is given in [10] which can not assert that the phase separation of the System $(\mathcal{S}^1_{\beta,2})$ in dimension four must happen. In order to study the phenomenon of the phase separation to the elliptic system with Sobolev critical exponent, Chen and Zou [12] studied the following more general elliptic system with Sobolev critical exponent in:

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + |u_1|^{2^*-2}u_1 + \beta|u_2|^{\frac{2^*}{2}}|u_1|^{\frac{2^*}{2}-2}u_1, & \text{in } \Omega, \\ -\Delta u_2 = \lambda_2 u_2 + |u_2|^{2^*-2}u_2 + \beta|u_1|^{\frac{2^*}{2}}|u_2|^{\frac{2^*}{2}-2}u_2, & \text{in } \Omega, \\ u_1 = u_2 = 0, & \text{on } \partial\Omega. \end{cases} \qquad (\mathcal{S}^4_{\beta,2})$$

By following the framework in [10] and making much more careful and complicated analysis, the authors proved that $(\mathcal{S}^4_{\beta,2})$ has a positive ground state solution for $N \geq 5$ and $0 < \lambda_i < \sigma_1$ for all $i = 1, 2$. Moreover, the phase separation of this positive ground state solution is expected for $N \geq 6$. By the phase separation, the relation between this positive ground state solution and the least energy sign-changing solution of the well known Brezś–Nirenberg problem (cf. [5]) is also established. It is worth to point out that the system $(\mathcal{S}^4_{\beta,2})$ and non-cubic nonlinearities (e.g. quintic) also have a physics background, see the survey articles [16,26].

There are also some studies on other elliptic systems with Sobolev critical exponent, see for example [2,11–14,29] and the references therein. Most of these studies are devoted to the two coupled case and only the very recent work [29] considered the k coupled case to the best of our knowledge. Luo and Zou [29] studied the following the elliptic system with Sobolev critical exponent:

$$\begin{cases} -\Delta u_i - \frac{\lambda_i}{|x|^2}u_i = |u_i|^{2^*-2}u_i + \beta\left(\sum_{j=1, j\neq i}^k |u_j|^{\frac{2^*}{2}}\right)|u_i|^{\frac{2^*}{2}-2}u_i, \\ u_i \in D^{1,2}(\mathbb{R}^N), \quad i = 1, 2, \ldots, k, \end{cases} \qquad (\mathcal{S}^5_{\beta,k})$$

where $\lambda_i \in (0, \frac{(N-2)^2}{4})$ for all $i = 1, 2, \ldots, k$. By the variational method, the authors proved that $(\mathcal{S}^5_{\beta,k})$ has no ground state solution for $\beta < 0$ and arbitrary $k \geq 2$. Moreover, by establishing the threshold for the compactness of the (PS) sequence to $(\mathcal{S}^5_{\beta,k})$ with $\beta > 0$, the authors also proved that $(\mathcal{S}^5_{\beta,k})$ has a positive ground state solution for $\beta > 0$ large enough.

Inspired by the above facts, we wonder what happens to the system $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$ for arbitrary $k \geq 2$? To the best of our knowledge, the system $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$ has not been studied yet in the literatures. Thus, we shall explore this problem in this paper. On the other hand, the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ can be seen as the system $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$ coupled with the nonlocal terms

$b_i \sum_{j=1}^k b_j \int_\Omega |\nabla u_j|^2 dx$. Such nonlocal term was first proposed by Kirchhoff in 1883 as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings (cf. [25]). Due to this reason, such operators are always called as the Kirchhoff operators, and the equations involving the Kirchhoff operators are always called as the Kirchhoff equations. Since the Kirchhoff equations of the elliptic type always have a variational structure, the variational method becomes a powerful tool to deal with such equations. However, it is worth to point out that, from the view point of the calculus of variation, a typical difficulty in studying the Kirchhoff equations of the elliptic type by the variational method is that, the weak limit of the (PS) sequence to the corresponding functional is not the weak solution of equations in general, which is coursed by the Kirchhoff type nonlocal term. If the embedding map of the chosen Sobolev space is compact, this difficulty can be overcome by introducing a related auxiliary functional (cf. [3,15,19,28,33,41–43] and the references therein). However, for the noncompact case, the situation is quite different and this difficulty is hard to overcome and some special ideas and technique are needed (cf. [17,20,22–24,31] and references therein). Note that the embedding map for the natural choice of the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ is also noncompact due to the Sobolev critical exponent and to the best of our knowledge, the elliptic system coupled with the Kirchhoff type nonlocal term (the Kirchhoff type system for short) has also not been studied in the literatures yet. Thus, we also wonder what happens to the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$? We remark that by our choice of the Kirchhoff type nonlocal terms, the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ is not only coupled by the local terms $\left(\sum_{j=1,j\neq i}^k |u_j|^{\frac{2^*}{2}}\right)|u_i|^{\frac{2^*}{2}-2}u_i$ but also coupled by the nonlocal terms $b_i \sum_{j=1}^k b_j \int_\Omega |\nabla u_j|^2 dx \Delta u_i$ (The reason for coupling such kind of Kirchhoff type nonlocal term will be given below). Thus, the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ is actually "double" coupled.

1.1 The existence result

We mainly consider the repulsive case $\beta < 0$ in this paper and our method is variational. Let $k \geq 2$. For every $i = 1, 2, \ldots, k$, let $\mathcal{H}_i$ be the Hilbert space of $H_0^1(\Omega)$ equipped with the inner product

$$\langle u, v\rangle_i = \int_\Omega a_i \nabla u \nabla v - \lambda_i uv dx.$$

If $\lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, then $\mathcal{H}_i$ are also the Hilbert spaces and the corresponding norms are given by $\|u\|_i = \langle u, u\rangle_i^{\frac{1}{2}}$ respectively. Set $\mathcal{H} = \prod_{i=1}^k \mathcal{H}_i$. Then $\mathcal{H}$ is a Hilbert space with the inner product

$$\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\rangle = \sum_{i=1}^k \langle u_i, v_i\rangle_i.$$

The corresponding norm is given by $\|\overrightarrow{\mathbf{u}}\| = \langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{u}}\rangle^{\frac{1}{2}}$. Here, u_i, v_i are the ith component of $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$ respectively. Define

$$\mathcal{J}(\overrightarrow{\mathbf{u}}) = \sum_{i=1}^k \left(\frac{1}{2}\|u_i\|_i^2 - \frac{1}{2^*}\mathcal{B}_{u_i,2^*}^{2^*}\right) - \frac{2\beta}{2^*} \sum_{i,j=1,i\neq j}^k \mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1} + \frac{1}{4}\left(\sum_{i=1}^k b_i \mathcal{B}_{\nabla u_i,2}^2\right)^2,$$

where $\mathcal{B}_{u,p}^{p}=\int_{\Omega}|u|^{p}dx$. Then it is easy to see that $\mathcal{J}(\overrightarrow{\mathbf{u}})$ is of C^1 in $\mathcal{H}$ and $\mathcal{J}(\overrightarrow{\mathbf{u}})$ is the corresponding functional of the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$.

Definition 1.1 $\overrightarrow{\mathbf{u}}\in\mathcal{H}$ is called as a nontrivial solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ if $\mathcal{J}'(\overrightarrow{\mathbf{u}})=0$ in $\mathcal{H}^{-1}$ with $\prod_{i=1}^{k}u_i\not\equiv 0$ and $\overrightarrow{\mathbf{u}}\in\mathcal{H}$ is called as a semi-trivial solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ if $\mathcal{J}'(\overrightarrow{\mathbf{u}})=0$ in $\mathcal{H}^{-1}$ with $\sum_{i=1}^{k}|u_i|>0$, where $\mathcal{J}'(\overrightarrow{\mathbf{u}})$ is the Fréchet derivative of $\mathcal{J}(\overrightarrow{\mathbf{u}})$ and $\mathcal{H}^{-1}$ is the dual space of $\mathcal{H}$. $\overrightarrow{\mathbf{u}}\in\mathcal{H}$ is called as a nonnegative solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ if $\overrightarrow{\mathbf{u}}$ is a nontrivial solution and $u_i\geq 0$ for all $i=1,2,\ldots,k$. $\overrightarrow{\mathbf{u}}\in\mathcal{H}$ is called as a ground state solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ if $\overrightarrow{\mathbf{u}}$ is a nontrivial solution and $\mathcal{J}(\overrightarrow{\mathbf{u}})\leq\mathcal{J}(\overrightarrow{\mathbf{v}})$ for all nontrivial solutions $\overrightarrow{\mathbf{v}}$.

Now, our first result can be stated as follows.

Theorem 1.1 *Let* $0<\lambda_i<a_i\sigma_1$ *for all* $i=1,2,\ldots,k$ *and* $\beta<0$. *Then there exists* $b_k^*>0$ *such that the system* $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ *has a nonnegative solution* $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},\beta,k}$ *for* $|\overrightarrow{\mathbf{b}}|<b_k^*$. *Moreover,* $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{0}},\beta,k}$ *is also a ground state solution of the system* $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$.

Remark 1.1 (i) It is worth to point out that even though $2^*<4$ for $N\geq 5$, the functional $\mathcal{J}(\overrightarrow{\mathbf{u}})$ is coercive in $\mathcal{H}$ only with $b_i\neq 0$ for all $i=1,2,\ldots,k$ by the Sobolev embedding theorem. Thus, $\mathcal{J}(\overrightarrow{\mathbf{u}})$ has a global minimum point in $\mathcal{H}$ in this case. However, since $\beta<0$, this global minimum point may be semi-trivial, which is different from the single Kirchhoff type equation. Since this global minimum point also may be nontrivial, we do not know that whether the nontrivial solution $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},\beta,k}$ of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ for $|\overrightarrow{\mathbf{b}}|<b_k^*$ is also a ground state solution.

(ii) We mainly follow the strategies in [10,12] and use technique of the Nehari manifold to prove Theorem 1.1. Since the functional $\mathcal{J}(\overrightarrow{\mathbf{u}})$ might be coercive in $\mathcal{H}$, some truncation arguments are needed to define a "well" Nehari manifold. Thus, we first borrow some ideas from [3] to modify the functional $\mathcal{J}(\overrightarrow{\mathbf{u}})$ and define a "well" Nehari manifold. Next, we make some observations on this "well" Nehari manifold of the modified functional and drive some basic properties. We remark that these observations are much more complicated than that in [10,12] since $k\geq 2$ is arbitrary and $\overrightarrow{\mathbf{b}}\neq\overrightarrow{\mathbf{0}}$ and we also need to borrow some ideas from [15] to carry out these observations. Thirdly, by making some careful and complicated analysis, we establish a threshold for the compactness of the (PS) sequence to the modified functional. Unlike the attractive case ($\beta>0$) (cf. [29]), we find out that the threshold for the repulsive case ($\beta<0$) has a property of iteration which is similar to that of the radial sign-changing solutions of the well known Brezís–Nirenberg problem (cf. [6]). Finally, also by making some careful and complicated analysis, we control the minimum of modified functional in the "well" Nehari manifold under this threshold and show that the minimum point is also a nonnegative solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ for $|\overrightarrow{\mathbf{b}}|$ small enough.

(iii) The existence of b_k^* seems to be necessary in Theorem 1.1. Indeed, by the standard argument, we can see that $\mathcal{B}_{\nabla u_i,2}^{2}\geq\frac{a_i\sigma_1-\lambda_i}{\sigma_1}$ for all $i=1,2,\ldots,k$ if $\overrightarrow{\mathbf{u}}$ is a nontrivial solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$. Thus, if $b_i\neq 0$ then by the Sobolev inequality and $\beta<0$, we can see that

$$b_i\left(\sum_{j=1}^{k}b_j\mathcal{B}_{\nabla u_j,2}^{2}\right)\mathcal{B}_{\nabla u_i,2}^{2}+\|u_i\|_i^2-\mathcal{B}_{u_i,2^*}^{2^*}-\beta\sum_{j=1,j\neq i}^{k}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1}>0$$

for $b_i > 0$ large enough. It contradicts to the fact that $\overrightarrow{\mathbf{u}}$ is a nontrivial solution of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$.

1.2 The concentration behaviors

Since $\overrightarrow{\mathbf{b}}$ is the parameter of the Kirchhoff type nonlocal term in $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$, it is natural to conjecture that the solutions of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ is related to that of $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$ as the single Kirchhoff type equation in the literatures. Thus, we first study the concentration behavior of $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},\beta,k}$ as $\overrightarrow{\mathbf{b}} \to \overrightarrow{\mathbf{0}}$. For the simplicity and clarity, we re-denote $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},\beta,k}$ and $\mathcal{J}(\overrightarrow{\mathbf{u}})$ by $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}}$ and $\mathcal{J}_{\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$ respectively. Let

$$m(\overrightarrow{\mathbf{b}}) = \mathcal{J}_{\overrightarrow{\mathbf{b}}}\left(\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}}\right) \quad \text{and} \quad \overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}} = \left(u_i^{\overrightarrow{\mathbf{b}}}\right)_{i=1,2,\ldots,k}.$$

Then our result in this aspect can be stated as follows.

Theorem 1.2 *Let* $0 < \lambda_i < a_i\sigma_1$ *for all* $i = 1, 2, \ldots, k$ *and* $\beta < 0$. *Then* $m(\overrightarrow{\mathbf{b}}) = m(\overrightarrow{\mathbf{0}}) + O(|\overrightarrow{\mathbf{b}}|)$ *as* $|\overrightarrow{\mathbf{b}}| \to 0$ *and* $m(\overrightarrow{\mathbf{b}})$ *is derivable for a.e.* $|\overrightarrow{\mathbf{b}}| < b_k^*$ *with*

$$\frac{\partial m(\overrightarrow{\mathbf{b}})}{\partial b_i} = \frac{1}{2}\mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}}},2}\sum_{i=1}^{k} b_i\mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}}},2} \quad \text{for all } i = 1, 2, \ldots, k.$$

Moreover, for every $\overrightarrow{\mathbf{b}}_n \to \overrightarrow{\mathbf{0}}$ *as* $n \to \infty$, *we have* $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}_n} = \overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{0}}} + o_n(1)$ *strongly in* $\mathcal{H}$ *up to a subsequence, where* $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{0}}}$ *is a nonnegative ground state solution of* $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$.

Remark 1.2 The concentration behavior of $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}}$ stated in Theorem 1.2 is as expected. Moreover, Theorem 1.2 also gives some other precise properties of $m(\overrightarrow{\mathbf{b}})$. The proof of the concentration behavior of $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}}$ is based on the concentration-compactness principle and the threshold established for Theorem 1.1 while the precise properties of $m(\overrightarrow{\mathbf{b}})$ are based on the method dealing with the parameters introduced by Chen and Zou [9] and a mini-max description of $m(\overrightarrow{\mathbf{b}})$ established in this paper.

When the interaction is repulsive ($\beta < 0$), it is expected that the phenomenon of phase separation will happen, that is, the k components of the system tend to separate in different regions as the interaction tends to infinity. Our result in this aspect is the following.

Theorem 1.3 *Let* $0 < \lambda_i < a_i\sigma_1$ *for all* $i = 1, 2, \ldots, k$ *with* $k \geq 2$. *Then for every* $\beta_n \to -\infty$, *we have the following.*

(1) *If* $k = 2$, $N \geq 6$ *and* $|\overrightarrow{\mathbf{b}}| < b_2^*$, *then we have*

  (i) $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},2,\beta_n} = \overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},2} + o_n(1)$ *strongly in* $\mathcal{H}$ *up to a subsequence for some* $\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},2} \neq \overrightarrow{\mathbf{0}}$. *Moreover,* $\widehat{u}_1^{\overrightarrow{\mathbf{b}},2}\widehat{u}_2^{\overrightarrow{\mathbf{b}},2} = 0$ *in* Ω.

  (ii) $\beta_n\mathcal{B}_{|u_i^{\overrightarrow{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}}|u_j^{\overrightarrow{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}},1} = o_n(1)$.

(iii) $\widehat{u}_i^{\overrightarrow{\mathbf{b}},2}$ *are countious and* $\Omega_i \neq \emptyset$ *for all* $i = 1, 2$, *where* $\Omega_i = \{x \in \Omega \mid \widehat{u}_i^{\overrightarrow{\mathbf{b}},2} > 0\}$. *Moreover,* $\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},2}$ *is a solution of the following system*

$$\begin{cases} -\left(a_i + b_i \sum_{j=1}^{2} b_j \mathcal{B}^2_{\nabla u_j,2}\right)\Delta u_i = \lambda_i u_i + |u_i|^{2^*-2}u_i, & \text{in } \Omega_i, \\ u_i = 0, \quad \text{on } \partial\Omega_i, \quad i = 1, 2. \end{cases} \qquad (\mathcal{S}^*_{\overrightarrow{\mathbf{b}},2})$$

(2) *If* $N \geq 9$ *and* $|\overrightarrow{\mathbf{b}}| < b_k^*$, *then the conclusion of* (1) *holds for all* $k \geq 2$ *with* $a_1 = a_2 = \cdots = a_k$ *and* $b_1 = b_2 = \cdots = b_k$.

Remark 1.3 (i) To the best of our knowledge, Theorem 1.3 seems to be the first result devoted to the phase separation of the K–component system in the critical case.

(ii) Due to the "double" coupled property, we can see from Theorem 1.3 that the limiting segregated states of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ still satisfy an elliptic system which is coupled by the Kirchhoff type nonlocal terms. On the other hand, it is well known that the limiting segregated states of elliptic systems always solve an optimal partition problem (cf. [37, 39] and the references therein). In this paper, we can also observe such phenomenon for $\overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{0}}$. Indeed, let

$$\mathcal{G} = \Big\{\mathcal{O} = \{\Omega_i^{\mathcal{O}}\}_{i=1,2,\ldots,k} \mid \cup_{i=1}^k \Omega_i^{\mathcal{O}} = \Omega \text{ and } \Omega_i^{\mathcal{O}} \cap \Omega_j^{\mathcal{O}} = \emptyset \\ \text{for all } i, j = 1, 2, \ldots, k \text{ and } i \neq j\Big\}.$$

Then it is easy to see that $\bigoplus_{i=1}^k H_0^1(\Omega_i^{\mathcal{O}}) \subset H_0^1(\Omega)$ for all $\mathcal{O} \in \mathcal{G}$. Set

$$c_{\mathcal{O}} = \inf_{v \in \bigoplus_{i=1}^k (H_0^1(\Omega_i^{\mathcal{O}})\setminus\{0\})} \max_{\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^k} \mathcal{E}\left(\sum_{j=1}^k t_j v_j\right).$$

and $c = \inf_{\mathcal{O}\in\mathcal{G}} c_{\mathcal{O}}$, then $\mathcal{E}(\sum_{i=1}^k \zeta_i \widehat{u}_i^{\overrightarrow{\mathbf{b}}_0,k}) = c$ (see Lemma 4.3 below for more details), where $\overrightarrow{\mathbf{b}}_0 = (\sqrt{b}, \sqrt{b}, \ldots, \sqrt{b})$, $\overrightarrow{\zeta} = (\zeta_i)_{i=1,2,\ldots,k} \in (\mathbb{Z}_2)^k$, v_i is the projection of v in $H_0^1(\Omega_i^{\mathcal{O}})$ and

$$\mathcal{E}(u) = \frac{a}{2}\mathcal{B}^2_{\nabla u,2} - \frac{\lambda}{2}\mathcal{B}^2_{u,2} - \frac{1}{2^*}\mathcal{B}^{2^*}_{u,2^*}.$$

To our best knowledge, this is also the first result devoted to such topic in the critical case for arbitrary $k \geq 2$.

(iii) The main idea of proving Theorem 1.3 comes from [12]. But as we will see, the nonlocal case $\overrightarrow{\mathbf{b}} \neq \overrightarrow{\mathbf{0}}$ is somewhat different and more complicated than the local case $\overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{0}}$. Thus, some new ideas and modifications are needed to deal with this case. On the other hand, since the limiting segregated states of elliptic systems is also critical, the threshold for the compactness of the (PS) sequence to the corresponding functional is also need to be established. Similar to that of Theorem 1.1, we find out that this threshold also has the property of iteration. This fact also makes Theorem 1.3 say nothing for $N = 7, 8$ and $k \geq 3$. Indeed, if we want to obtain the compactness of the (PS) sequence to the corresponding functional for $N = 7, 8$ and $k \geq 3$ in our method, then we need to establish a uniformly Lipschisz continuity of the solutions due to the the property of iteration. However, we only observe a uniformly Hölder continuity of the solutions in the appendix by following the arguments in [13,30].

(iv) Our method to obtain the uniformly Hölder continuity of the solutions in the appendix only works for $a_1 = a_2 = \cdots = a_k$ and $b_1 = b_2 = \cdots = b_k$ in the case $k \geq 3$ and this leads that the conditions $a_1 = a_2 = \cdots = a_k$ and $b_1 = b_2 = \cdots = b_k$ is requested in (2) of Theorem 1.3. However, we believe that the conditions $a_1 = a_2 = \cdots = a_k$ and $b_1 = b_2 = \cdots = b_k$ for $k \geq 3$ in (2) of Theorem 1.3 is only technique and not necessary.

1.3 Sign-changing solutions of $(\mathcal{P}_{a,b,\lambda})$

Let us consider the following Brezís–Nirenberg problem of the Kirchhoff type

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + |u|^{2^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_{a,b,\lambda}$}$$

where $a, b, \lambda > 0$. To the best of our knowledge, the existence and multiplicity of positive solutions to $(\mathcal{P}_{a,b,\lambda})$ has already been studied in the literatures, see for example [22,24,31] and the references therein. However, there are few results about the existence of sign-changing solutions to such problem in the literatures. On the other hand, it is well known that the limiting segregated states of elliptic systems can always be used to generate the sign-changing solutions of the related elliptic equations (cf. [12,37,40]). Thus, based on Theorem 1.3, we also obtain a result on the sign-changing solutions to $(\mathcal{P}_{a,b,\lambda})$ which can be stated as follows.

Theorem 1.4 *Let a and $\lambda > 0$ with $0 < \lambda < a\sigma_1$. Then we have the following.*

(1) *If $N \geq 6$ then there exists $b_2^{**} > 0$ such that $(\mathcal{P}_{a,b,\lambda})$ has a sign-changing solution $u_{b,2}$ with two nodal domains for $b < b_2^{**}$. Moreover, if $N \geq 7$ and $\Omega = \mathbb{B}_R$, then $u_{b,2}$ can be radial symmetric.*

(2) *If $N \geq 9$ and $\Omega = \mathbb{B}_R$, then for arbitrary $k \geq 2$, there exists $b_k^{**} > 0$ such that $(\mathcal{P}_{a,b,\lambda})$ with $b < b_k^{**}$ has a radial sign-changing solution $u_{b,m}^R$ which changes sign exactly $m-1$ times for all $m = 2, \ldots, k$.*

Remark 1.4 (i) To the best of our knowledge, this is the first result which is devoted to the sign-changing solutions of Brezís–Nirenberg problem of the Kirchhoff type.

(ii) The sign-changing solutions obtained in Theorem 1.4 is constructed by the limiting segregated states of $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$, which is mainly inspiring by [12,37,40]. On the other hand, let

$$\mathcal{E}_b(u) = \frac{a}{2}\mathcal{B}_{\nabla u,2}^2 + \frac{b}{4}\mathcal{B}_{\nabla u,2}^4 - \frac{\lambda}{2}\mathcal{B}_{u,2}^2 - \frac{1}{2^*}\mathcal{B}_{u,2^*}^{2^*}.$$

Then it is easy to see that $\mathcal{E}_b(u)$ is the corresponding functional of $(\mathcal{P}_{a,b,\lambda})$. In the nonlocal case $b > 0$, we can see that $\mathcal{E}_b(u)$ has the following decomposition

$$\mathcal{E}_b\left(\sum_{i=1}^l u_i\right) = \sum_{i=1}^l \mathcal{E}_b(u_i) + \frac{b}{2}\sum_{i,j=1,j\neq j}^l \mathcal{B}_{\nabla u_i,2}^2\mathcal{B}_{\nabla u_j,2}^2,$$

where $l \geq 2$ is an integer and $u_i \in H_0^1(\Omega)$ with $u_iu_j = 0$ for all $i, j = 1, 2, \ldots, l$ and $i \neq j$. It follows that the sign-changing solutions of $(\mathcal{P}_{a,b,\lambda})$ may satisfy an elliptic system similar to $(\mathcal{S}_{\overrightarrow{\mathbf{b}},2}^*)$. Due to this reason, we choose the Kirchhoff operators $-\Big(a_i + b_i\sum_{j=1}^k b_j\int_\Omega |\nabla u_j|^2 dx\Big)\Delta u_i$ in the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$.

(iii) The existence of b_k^{**} also seems to be necessary in Theorem 1.4 and the reason is similar to that of b_k^* stated in (iv) of Remark 4.1. On the other hand, due to Theorem 1.3 and (iii) of Remark 1.3, Theorem 1.4 also says nothing for $N = 7, 8$ and $k \geq 3$.

Since b is the parameter of the Kirchhoff type nonlocal term in $(\mathcal{P}_{a,b,\lambda})$ and it is well known that the solutions of $(\mathcal{P}_{a,b,\lambda})$ is related to that of $(\mathcal{P}_{a,0,\lambda})$, as in the literatures, it is natural to study the concentration behavior of sign-changing solutions to $(\mathcal{P}_{a,b,\lambda})$ as $b \to 0^+$. Let $c_2(b) = \mathcal{E}_b(u_{b,2})$ and $\omega_m(b) = \mathcal{E}_b(u_{b,m}^R)$. Then our results in this aspect can be stated as follows.

Theorem 1.5 *Let $a, \lambda > 0$ with $0 < \lambda < a\sigma_1$. Then we have the following.*

(1) *If $N \geq 6$, then $\frac{1}{c_2(b)} - \frac{1}{c_2(0)} = O(b)$ as $b \to 0^+$ and $c_2(b)$ is derivable for a.e. $b < b_2^{**}$ with $c_2'(b) = \frac{1}{4}\mathcal{B}^4_{\nabla u_{b,2},2}$. Moreover, for every $b_n \to 0^+$, we have $u_{b_n,2} = u_{*,2} + o_n(1)$ strongly in $H_0^1(\Omega)$ up to a subsequence, where $u_{*,2}$ is a least energy sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$ with two nodal domains.*

(2) *If $N \geq 9$ and $\Omega = \mathbb{B}_R$, then for arbitrary $k \geq 2$ and all $m = 2, \ldots, k$, $\frac{1}{\omega_m(b)} - \frac{1}{\omega_m(0)} = O(b)$ as $b \to 0^+$ and $\omega_m(b)$ is derivable for a.e. $b < b_m^{**}$ with $\omega_m'(b) = \frac{1}{4}\mathcal{B}^4_{\nabla u_{b,m}^R,2}$. Moreover, for every $b_n \to 0^+$, we have $u_{b_n,m}^R = u_{*,m}^R + o_n(1)$ strongly in $H_0^1(\mathbb{B}_R)$ up to a subsequence, where $u_{*,m}^R$ is a least energy radial sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$ which changes sign exactly $m - 1$ times.*

Remark 1.5 The concentration behaviors of $u_{b,2}$ and $u_{b,m}^R$ stated in Theorem 1.5 is as expected. Moreover, Theorem 1.5 is also give some other precise properties of $c_2(b)$ and $\omega_m(b)$. Similar to Theorem 1.2, the proofs to the concentration behaviors of $u_{b,2}$ and $u_{b,m}^R$ is based on the concentration-compactness principle and some thresholds established in [6]. It is worth to point out that the precise properties of $c_2(b)$ and $\omega_m(b)$ stated in Theorem 1.5 can also be obtained by applying the method dealing with the parameters introduced by Chen and Zou in [9] directly, since $c_2(b)$ and $\omega_m(b)$ have mini-max descriptions similar to that of $m(\overrightarrow{\mathbf{b}})$. However, due to the precise properties of $m(\overrightarrow{\mathbf{b}})$ stated in Theorem 1.2, we can give more simple proofs, which reveals the relation between $m(\overrightarrow{\mathbf{b}})$ and $c_2(b)$ and $\omega_m(b)$.

This paper is organized as follows. In section 2, we will introduce and study an auxiliary functional related to $\mathcal{J}(\overrightarrow{\mathbf{u}})$. Theorem 1.1 will also be proved in this section. In section 3, we will study the concentration behaviors and prove Theorems 1.2 and 1.3. In section 4, we will study the existence and multiplicity of sign-changing solutions to $(\mathcal{P}_{a,b,\lambda})$ and prove Theorem 1.5.

Through this paper, C and C_i are indiscriminately used to denote various positive constants while $O(|\overrightarrow{\mathbf{b}}|)$ is used to denote the quantities who tend towards zero as $|\overrightarrow{\mathbf{b}}| \to 0$. We also denote $\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}} = (t_1u_1, t_2u_2, \ldots, t_ku_k)$, where $\overrightarrow{\mathbf{t}} = (t_i)_{i=1,2,\ldots,k}$ and $\overrightarrow{\mathbf{u}} = (u_i)_{i=1,2,\ldots,k}$ are two vectors.

2 An auxiliary functional

Let $\chi(s)$ be a smooth function in $[0, +\infty)$ such that

$$\chi(s) = \begin{cases} 1, & 0 \leq s \leq 1, \\ 0, & s \geq 2. \end{cases}$$

Moreover, we also request $-2 \leq \chi'(s) \leq 0$ in $[0, +\infty)$. Define

$$\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) = \sum_{i=1}^{k}\left(\frac{1}{2}\|u_i\|_i^2 - \frac{1}{2^*}\mathcal{B}_{u_i,2^*}^{2^*}\right) - \frac{2\beta}{2^*}\sum_{i,j=1,i\neq j}^{k}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1} + \frac{1}{4}\chi\left(\frac{\sum_{i=1}^{k}\mathcal{B}_{\nabla u_i,2}^2}{T^2}\right)\left(\sum_{i=1}^{k}b_i\mathcal{B}_{\nabla u_i,2}^2\right)^2, \tag{2.1}$$

where $T > 0$ is a constant specified later. Then it is easy to see that $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$ is also of C^2 in $\mathcal{H}$ and $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) = \mathcal{J}(\overrightarrow{\mathbf{u}})$ for $\sum_{i=1}^{k}\mathcal{B}_{\nabla u_i,2}^2 \leq T^2$.

Lemma 2.1 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$ and $\beta < 0$. Then for every $T > 0$, there exists $b_T > 0$ such that any critical value of $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$ must be nonnegative with $|\overrightarrow{\mathbf{b}}| < b_T$.*

Proof By a direct calculation, for every $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \in \mathcal{H}$, we have

$$\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{v}} = \left\langle \overrightarrow{\mathcal{Q}}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) \circ \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right\rangle - \int_\Omega \sum_{i=1}^{k}\left(\lambda_i u_i v_i + |u_i|^{2^*-2}u_i v_i\right)dx - \beta\sum_{i,j=1,i\neq j}^{k}\int_\Omega |u_j|^{\frac{2^*}{2}}|u_i|^{\frac{2^*}{2}-2}u_i v_i dx,$$

where $\overrightarrow{\mathcal{Q}}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) = (\mathcal{Q}^i_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}))_{i=1,2,\ldots,k}$ with

$$\mathcal{Q}^i_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) = a_i + b_i\chi\left(\frac{\sum_{j=1}^{k}\mathcal{B}_{\nabla u_i,2}^2}{T^2}\right)\sum_{i=1}^{k}b_j\mathcal{B}_{\nabla u_j,2}^2 + \frac{\left(\sum_{i=1}^{k}b_i\mathcal{B}_{\nabla u_i,2}^2\right)^2}{2T^2}\chi'\left(\frac{\sum_{i=1}^{k}\mathcal{B}_{\nabla u_i,2}^2}{T^2}\right) \tag{2.2}$$

and $\overrightarrow{\mathcal{Q}}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) \circ \overrightarrow{\mathbf{u}} = (\mathcal{Q}^i_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})u_i)_{i=1,2,\ldots,k}$. By the Cauchy inequality, for every $T > 0$ and $i = 1, 2, \ldots, k$, we can see from (2.2) that there exists $b_T > 0$ such that

$$\mathcal{Q}^i_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) \geq \frac{1}{2}\left(a_i + \frac{\lambda_i}{\sigma_1}\right) \quad \text{for all } \overrightarrow{\mathbf{u}} \in \mathcal{H} \text{ with } |\overrightarrow{\mathbf{b}}| < b_T. \tag{2.3}$$

Let $\overrightarrow{\mathbf{u}}_0 = (u_1^0, u_2^0, \ldots, u_k^0)$ be a critical point of $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$. Without loss of generality, we assume that $\overrightarrow{\mathbf{u}}_0 \neq \overrightarrow{\mathbf{0}}$. Let $\overrightarrow{\mathbf{u}}_{1,0} = (u_1^0, \ldots, 0)$ and $\overrightarrow{\mathbf{u}}_{i,0} = (0, \ldots, u_i^0, \ldots, 0)$ for $i = 2, 3, \ldots, k$. Since $0 < \lambda_i < a_i\sigma_1$ and $\beta < 0$, by multiplying $\mathcal{J}'_{T,b}(\overrightarrow{\mathbf{u}}_0) = 0$ with $\overrightarrow{\mathbf{u}}_{i,0}$ for all $i = 1, 2\ldots, k$ respectively, we can see from the Sobolev inequality and (2.3) that

$$\frac{1}{2}\left(a_i - \frac{\lambda_i}{\sigma_1}\right)\mathcal{B}_{\nabla u_i^0,2}^2 \leq \mathcal{S}^{-\frac{2^*}{2}}\mathcal{B}_{\nabla u_i^0,2}^{2^*} \quad \text{for all } i = 1, 2, \ldots, k. \tag{2.4}$$

It follows from the construction of $\chi(s)$ and $N \geq 4$ that

$$\begin{aligned}
&\mathcal{J}_{T,\vec{\mathbf{b}}}(\vec{\mathbf{u}}_0) - \frac{1}{2^*}\mathcal{J}'_{T,\vec{\mathbf{b}}}(\vec{\mathbf{u}}_0)\vec{\mathbf{u}}_0 \\
&= \frac{1}{N}\sum_{i=1}^k \|u_i\|_i^2 - \frac{N-4}{4N}\chi\left(\frac{\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^0,2}}{T^2}\right)\left(\sum_{i=1}^k b_i \mathcal{B}^2_{\nabla u_i^0,2}\right)^2 \\
&\quad -\frac{N-2}{4NT^2}\chi'\left(\frac{\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^0,2}}{T^2}\right)\left(\sum_{i=1}^k b_i \mathcal{B}^2_{\nabla u_i^0,2}\right)^2 \sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^0,2} \\
&\geq \sum_{i=1}^2 \frac{a_i\sigma_1 - \lambda_i}{N\sigma_1}\mathcal{B}^2_{\nabla u_i^0,2} - |\vec{\mathbf{b}}|CT^4.
\end{aligned} \tag{2.5}$$

Thus, by choosing b_T small enough if necessary, we can see from (2.4) that $\mathcal{J}_{T,\vec{\mathbf{b}}}(\vec{\mathbf{u}}_0) > 0$ for $|\vec{\mathbf{b}}| < b_T$. □

2.1 The Nehari manifold

Let $\mathcal{H}^* = \prod_{i=1}^k(\mathcal{H}_i\backslash\{0\})$, $\vec{\mathbf{u}}_1 = (u_1, \ldots, 0)$ and $\vec{\mathbf{u}}_i = (0, \ldots, u_i, \ldots, 0)$ for $i = 2, 3, \ldots, k$. Let $\Phi_{\vec{\mathbf{u}},T,\vec{\mathbf{b}}}(\vec{\mathbf{t}}) = \mathcal{J}_{T,\vec{\mathbf{b}}}(\vec{\mathbf{t}} \circ \vec{\mathbf{u}})$ and

$$\mathcal{W}_{T,\vec{\mathbf{b}}} = \left\{\vec{\mathbf{u}} \in \mathcal{H}^* \mid \mathcal{J}'_{T,\vec{\mathbf{b}}}(\vec{\mathbf{u}})\vec{\mathbf{u}}_i = 0, \quad i = 1, 2, \ldots, k\right\},$$

where $\vec{\mathbf{t}} \in (\mathbb{R}^+)^k$ and $\vec{\mathbf{t}} \circ \vec{\mathbf{u}} = (t_1u_1, t_2u_2, \ldots, t_ku_k)$. Since $\mathcal{J}_{T,\vec{\mathbf{b}}}(\vec{\mathbf{u}})$ is of C^2 in $\mathcal{H}$, it is easy to see that $\Phi_{\vec{\mathbf{u}},T,\vec{\mathbf{b}}}(\vec{\mathbf{t}})$ is of C^2 in $(\mathbb{R}^+)^k$ and

$$\frac{\partial\Phi_{\vec{\mathbf{u}},T,\vec{\mathbf{b}}}(\vec{\mathbf{t}})}{\partial t_i} = t_i^{-1}\mathcal{J}'_{T,\vec{\mathbf{b}}}(\vec{\mathbf{t}} \circ \vec{\mathbf{u}})(\vec{\mathbf{t}} \circ \vec{\mathbf{u}})_i$$

for all $i = 1, 2, \ldots, k$. It follows that

$$\frac{\partial\Phi_{\vec{\mathbf{u}},T,\vec{\mathbf{b}}}(\vec{\mathbf{t}})}{\partial t_i} = 0 \quad \text{for all } i = 1, 2, \ldots, k$$

if and only if $\vec{\mathbf{t}} \circ \vec{\mathbf{u}} \in \mathcal{W}_{T,\vec{\mathbf{b}}}$. In particular,

$$\frac{\partial\Phi_{\vec{\mathbf{u}},T,\vec{\mathbf{b}}}(\vec{\mathbf{1}})}{\partial t_i} = 0 \quad \text{for all } i = 1, 2, \ldots, k$$

if and only if $\vec{\mathbf{1}} \circ \vec{\mathbf{u}} = \vec{\mathbf{u}} \in \mathcal{W}_{T,\vec{\mathbf{b}}}$.

Let

$$\Theta = \left\{\vec{\mathbf{u}} \in \mathcal{H}^* \mid \beta\sum_{j=1,j\neq i}^k \mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1} + \mathcal{B}^{2^*}_{u_i,2^*} > 0, \quad i = 1, 2, \ldots, k\right\}. \tag{2.6}$$

Then it is easy to see that $\Theta \neq \emptyset$.

Lemma 2.2 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$ and $\beta < 0$. If $\overrightarrow{\mathbf{u}} \in \Theta$, then there exists a unique $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{0}}} \in (\mathbb{R}^+)^k$ such that $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{0}}} \circ \overrightarrow{\mathbf{u}} \in \mathcal{W}_{T,\overrightarrow{\mathbf{0}}}$ and*

$$\Phi_{\overrightarrow{\mathbf{u}},T,\overrightarrow{\mathbf{0}}}\left(\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{0}}}\right) = \max_{\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^k} \Phi_{\overrightarrow{\mathbf{u}},T,\overrightarrow{\mathbf{0}}}\left(\overrightarrow{\mathbf{t}}\right).$$

Proof Let $\mu \in [-1, 1]$ and $\overrightarrow{\mathbf{u}} \in \mathcal{H}^*$. We consider the following system in $(\mathbb{R}^+)^k$:

$$\overrightarrow{\mathbf{M}}\left(\mu, \overrightarrow{\mathbf{t}}\right) = \overrightarrow{\mathbf{0}},$$

where $\overrightarrow{\mathbf{M}}(\mu, \overrightarrow{\mathbf{t}}) = (M_i(\mu, \overrightarrow{\mathbf{t}}))_{i=1,2,\ldots,k}$ with

$$M_i(\mu, \overrightarrow{\mathbf{t}}) = -\mu \sum_{j=1, j\neq i}^{k} \beta\mathcal{B}_{|t_iu_i|^{\frac{2^*}{2}}|t_ju_j|^{\frac{2^*}{2}},1} + \|t_iu_i\|_i^2 - \mathcal{B}_{t_iu_i,2^*}^{2^*} = 0.$$

Clearly, there exists a unique $\overrightarrow{\mathbf{t}}_0 \in (\mathbb{R}^+)^k$ with

$$t_i^0 = \left(\frac{\|u_i\|_i^2}{\mathcal{B}_{u_i,2^*}^{2^*}}\right)^{\frac{1}{2^*-2}} \quad \text{for all } i = 1, 2, \ldots, k$$

such that $M_i(0, \overrightarrow{\mathbf{t}}_0) = 0$ for all $i = 1, 2, \ldots, k$. That is, the system $\overrightarrow{\mathbf{M}}(0, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$ is unique solvable in $(\mathbb{R}^+)^k$. Define

$$\mathcal{Z} = \left\{\mu \in [0, 1] \mid \overrightarrow{\mathbf{M}}(\mu, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}} \text{ is uniquely solvable in } (\mathbb{R}^+)^k\right\}.$$

Claim 1. $\mathcal{Z}$ is open in $[0, 1]$.

Indeed, let $\mu_0 \in \mathcal{Z}$, then there exists a unique $\overrightarrow{\mathbf{s}} \in (\mathbb{R}^+)^k$ such that $\overrightarrow{\mathbf{M}}(\mu_0, \overrightarrow{\mathbf{s}}) = \overrightarrow{\mathbf{0}}$. Set

$$M_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}) = \frac{\partial M_i(\mu_0, \overrightarrow{\mathbf{s}})}{\partial t_j}, \quad i, j = 1, 2, \ldots, k.$$

Then by a direct calculation, we have

$$\begin{aligned} M_{i,i}(\mu_0, \overrightarrow{\mathbf{s}}) &= \frac{1}{s_i}\left(2\|s_iu_i\|_i^2 - 2^*\mathcal{B}_{s_iu_i,2^*}^{2^*} - \frac{2^*\mu_0\beta}{2} \sum_{j=1,j\neq i}^{k} \mathcal{B}_{|s_iu_i|^{\frac{2^*}{2}}|s_ju_j|^{\frac{2^*}{2}},1}\right) \\ &= \frac{1}{s_i}\left((2-2^*)\mathcal{B}_{s_iu_i,2^*}^{2^*} - \beta\mu_0\left(\frac{2^*}{2} - 2\right) \sum_{j=1,j\neq i}^{k} \mathcal{B}_{|s_iu_i|^{\frac{2^*}{2}}|s_ju_j|^{\frac{2^*}{2}},1}\right) \end{aligned}$$

and

$$M_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}) = -\frac{2^*\mu_0\beta}{2} s_j^{\frac{2^*}{2}-1} \sum_{j=1,j\neq i}^{k} \mathcal{B}_{|s_iu_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1},$$

$i, j = 1, 2, \ldots, k$ and $i \neq j$. Consider the matrix

$$M_{\mu_0}(\overrightarrow{\mathbf{s}}) = (M_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}))_{i,j=1,2,\ldots,k}. \tag{2.7}$$

Then it is well known that

$$\det(M_{\mu_0}(\overrightarrow{\mathbf{s}})) = \frac{(-1)^k}{\prod_{i=1}^k s_i} \det(\widetilde{M}_{\mu_0}(\overrightarrow{\mathbf{s}})),$$

where the matrix $\widetilde{M}_{\mu_0}(\overrightarrow{\mathbf{s}}) = (\widetilde{M}_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}))_{i,j=1,2,\ldots,k}$ with

$$\widetilde{M}_{i,i}(\mu_0, \overrightarrow{\mathbf{s}}) = (2^* - 2)\mathcal{B}^{2^*}_{s_i u_i, 2^*} + \beta\mu_0\left(\frac{2^*}{2} - 2\right)\sum_{j=1, j\neq i}^{k} \mathcal{B}_{|s_i u_i|^{\frac{2^*}{2}}|s_j u_j|^{\frac{2^*}{2}}, 1}$$

and

$$\widetilde{M}_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}) = \frac{2^*\mu_0\beta}{2}\sum_{j=1, j\neq i}^{k} \mathcal{B}_{|s_i u_i|^{\frac{2^*}{2}}|s_j u_j|^{\frac{2^*}{2}}, 1},$$

$i, j = 1, 2, \ldots, k$ and $i \neq j$. By $\beta < 0$, we can see that $\widetilde{M}_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}) < 0$ for all $i, j = 1, 2, \ldots, k$ and $j \neq i$. Moreover, by $M_i(\mu_0, \overrightarrow{\mathbf{s}}) = 0$ for all $i = 1, 2, \ldots, k$, we have

$$\begin{aligned}\sum_{j=1}^{k}\widetilde{M}_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}) &= (2^* - 2)\mathcal{B}^{2^*}_{s_i u_i, 2^*} + \beta\mu_0(2^* - 2)\sum_{j=1, j\neq i}^{k} \mathcal{B}_{|s_i u_i|^{\frac{2^*}{2}}|s_j u_j|^{\frac{2^*}{2}}, 1} \\ &= (2^* - 2)\|s_i u_i\|_i^2 \\ &> 0\end{aligned} \tag{2.8}$$

for all $i = 1, 2, \ldots, k$. Thus, the matrix $\widetilde{M}_{\mu_0}(\overrightarrow{\mathbf{s}}) = (\widetilde{M}_{i,j}(\mu_0, \overrightarrow{\mathbf{s}}))_{i,j=1,2,\ldots,k}$ is strictly diagonally dominant. It follows that $\widetilde{M}_{\mu_0}(\overrightarrow{\mathbf{s}})$ is nonsingular and $\det(\widetilde{M}_{\mu_0}(\overrightarrow{\mathbf{s}})) \neq 0$, which implies the matrix $M_{\mu_0}(\overrightarrow{\mathbf{s}})$ is also nonsingular and $\det(M_{\mu_0}(\overrightarrow{\mathbf{s}})) \neq 0$. Now, by the implicit function theorem, there exist an open set $\mathcal{U}_0 \times \mathcal{T}_0 \subset [0, 1] \times (\mathbb{R}^+)^k$ with $(\mu_0, \overrightarrow{\mathbf{s}}) \in \mathcal{U}_0 \times \mathcal{T}_0$ and $\overrightarrow{\mathbf{t}}(\mu) \in C^1(\mathcal{U}_0, \mathcal{T}_0)$ such that $\overrightarrow{\mathbf{t}}(\mu_0) = \overrightarrow{\mathbf{s}}$ and $(\mu, \overrightarrow{\mathbf{t}}(\mu))$ is the unique solution of the system $\overrightarrow{\mathbf{M}}(\mu, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$ in $\mathcal{U}_0 \times \mathcal{T}_0$. Suppose that $\mathcal{U}_0 \not\subset \mathcal{Z}$, then there exists $\mu_1 \in \mathcal{U}_0$ such that the system $\overrightarrow{\mathbf{M}}(\mu_1, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$ has a second solution $\overrightarrow{\mathbf{s}}_1$. Clearly, we must have $\overrightarrow{\mathbf{s}}_1 \notin \mathcal{T}_0$. Without loss of generality, we assume $\mu_1 > \mu_0$. Note that $(\mu_1, \overrightarrow{\mathbf{s}}_1)$ also solves the system $\overrightarrow{\mathbf{M}}(\mu, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$, then by applying the implicit function theorem in a similar way for $(\mu_1, \overrightarrow{\mathbf{s}})$, we can show that there exist an open set $\mathcal{U}_1 \times \mathcal{T}_1 \subset [0, 1] \times (\mathbb{R}^+)^k$ with $(\mu_1, \overrightarrow{\mathbf{s}}_1) \in \mathcal{U}_1 \times \mathcal{T}_1$, $\mathcal{U}_1 \subset \mathcal{U}_0$, $\mathcal{T}_1 \cap \mathcal{T}_0 = \emptyset$ and $\overrightarrow{\mathbf{t}}_1(\mu) \in C^1(\mathcal{U}_1, \mathcal{T}_1)$ such that $\overrightarrow{\mathbf{t}}_1(\mu_1) = \overrightarrow{\mathbf{s}}_1$ and $(\mu_1, \overrightarrow{\mathbf{t}}_1(\mu))$ is the unique solution of the system $\overrightarrow{\mathbf{M}}(\mu, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$ in $\mathcal{U}_1 \times \mathcal{T}_1$. By the extension theorem, one of the following three cases must happen:

(1) $t_i(\mu)$ blow up at some $\mu < \mu_0$ for some $i = 1, 2, \ldots, k$;
(2) $t_i(\mu) \to 0$ as $\mu \to \mu_*^+$ at some $\mu_* \in (0, \mu_0)$ for some $i = 1, 2, \ldots, k$;
(3) $\overrightarrow{\mathbf{t}}(\mu) \to \overrightarrow{\mathbf{t}}_0$ as $\mu \to 0^+$, where $\overrightarrow{\mathbf{t}}_0$ is the unique one in $(\mathbb{R}^+)^k$ satisfying the system $\overrightarrow{\mathbf{M}}(0, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$.

Let

$$\theta_{i,j}(\overrightarrow{\mathbf{u}}) = \beta_{i,j}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}}, 1}, \quad i, j = 1, 2, \ldots, k,$$

where $\beta_{i,i} = 1$ and $\beta_{i,j} = \mu\beta$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$. Since $\overrightarrow{\mathbf{u}} \in \Theta$, we can see that $\theta_{i,j}(\overrightarrow{\mathbf{u}}) < 0$ and $\sum_{j=1}^{k}\theta_{i,j}(\overrightarrow{\mathbf{u}}) > 0$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$. Thus, the matrix $\theta(\overrightarrow{\mathbf{u}}) = (\theta_{i,j}(\overrightarrow{\mathbf{u}}))_{i,j=1,2,\ldots,k}$ is strictly diagonally dominant. It follows that

$$\sum_{i=1}^{k}\left(\sum_{j=1, j\neq i}^{k}\mu\beta\mathcal{B}_{|t_i(\mu)u_i|^{\frac{2^*}{2}}|t_j(\mu)u_j|^{\frac{2^*}{2}}, 1} + \mathcal{B}^{2^*}_{t_i(\mu)u_i, 2^*}\right) \geq C\sum_{i=1}^{k}(t_i(\mu))^{2^*}, \tag{2.9}$$

which implies $\sum_{i=1}^k M_i(\mu, \overrightarrow{\mathbf{t}}) < 0$ if $|\overrightarrow{\mathbf{t}}(\mu)|$ large enough. On the other hand, since $2^* > 2$ and $\beta < 0$, by a standard argument, $M_i(\mu, \overrightarrow{\mathbf{t}}) > 0$ for all i if $|t_i(\mu)|$ small enough. Thus, we must have $\overrightarrow{\mathbf{t}}(\mu) \to \overrightarrow{\mathbf{t}}_0$ as $\mu \to 0^+$. Similarly, we also have $\overrightarrow{\mathbf{t}}_1(\mu) \to \overrightarrow{\mathbf{t}}_0$ as $\mu \to 0^+$. Hence, $\overrightarrow{\mathbf{t}}(\mu)$ and $\overrightarrow{\mathbf{t}}_1(\mu)$ are two different branches bifurcated at 0. On the other hand, it is easy to see that $\widetilde{M}_{i,j}(0, \overrightarrow{\mathbf{t}}_0) = 0$ and $\sum_{j=1}^k \widetilde{M}_{i,j}(0, \overrightarrow{\mathbf{t}}_0) = (2^* - 2)\|t_i^0 u_i\|_i^2 > 0$ for all $i, j = 1, 2, \ldots, k$ and $j \neq i$. It follows from the implicit function theorem once more that there exists only one branch bifurcated at 0, which is a contradiction. Therefore, we must have $\mathcal{T}_0 = (\mathbb{R}^+)^k$ and $\mathcal{U}_0 \subset \mathcal{Z}$, that is, $\mathcal{Z}$ is open in [0, 1].

Claim 2. $\mathcal{Z}$ is closed in [0, 1].

Indeed, let $\{\mu_n\} \subset \mathcal{Z}$ such that $\mu_n \to \mu_0$ as $n \to \infty$. By (2.9), we can see that $\sum_{i=1}^k M_i(\mu, \overrightarrow{\mathbf{t}}) < 0$ if $|\overrightarrow{\mathbf{t}}|$ large enough, thus, $\{\overrightarrow{\mathbf{t}}_n\}$ is bounded in $(\mathbb{R}^+)^k$, where $\overrightarrow{\mathbf{t}}_n$ is the unique solution of the system $\overrightarrow{\mathbf{M}}(\mu_n, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$. Passing to a subsequence, we can see that $\overrightarrow{\mathbf{t}}_n \to \overrightarrow{\mathbf{t}}^0$ as $n \to \infty$. Clearly, $\overrightarrow{\mathbf{t}}^0$ is a solution of the system $\overrightarrow{\mathbf{M}}(\mu_0, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}}$. By a similar argument as used in Claim 1, we can show that $\overrightarrow{\mathbf{t}}^0$ is also the unique solution of this system in $(\mathbb{R}^+)^k$. Due to the uniqueness of $\overrightarrow{\mathbf{t}}^0$, we can see that every subsequence of $\overrightarrow{\mathbf{t}}_n$ must convergence to $\overrightarrow{\mathbf{t}}^0$. It follows that $\mu_0 \in \mathcal{Z}$, that is, $\mathcal{Z}$ is closed in [0, 1].

By Claim 1 and Claim 2, $\mathcal{Z}$ is both open and closed in [0, 1]. Since $0 \in \mathcal{Z}$, we must have $\mathcal{Z} = [0, 1]$. Note that we also have

$$t_i \frac{\partial \Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})}{\partial t_i} = M_i(1, \overrightarrow{\mathbf{t}}) = 0 \quad \text{for all } i = 1, 2, \ldots, k, \tag{2.10}$$

thus, $\overrightarrow{\mathbf{t}}(1)$ is the unique critical point of $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})$ in $(\mathbb{R}^+)^k$. Since $\overrightarrow{\mathbf{u}} \in \Theta$, by the construction of $\chi(s)$ and (2.9), we can see that $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}}) \to -\infty$ as $|\overrightarrow{\mathbf{t}}| \to +\infty$. Thus, $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})$ has a global maximum point on $\overline{(\mathbb{R}^+)^k}$. Note that $M_i(\mu, \overrightarrow{\mathbf{t}}) > 0$ for all i if $|t_i(\mu)|$ small enough, thus, the global maximum point of $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})$ can not belong to $\partial\overline{(\mathbb{R}^+)^k}$ due to (2.10). Therefore, this global maximum point is also a critical point of $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})$ in $(\mathbb{R}^+)^k$. Note that $\overrightarrow{\mathbf{t}}(1) \in (\mathbb{R}^+)^k$ is the unique critical point of $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})$ in $(\mathbb{R}^+)^k$. Hence, $\overrightarrow{\mathbf{t}}(1) \in (\mathbb{R}^+)^k$ is the global maximum point of $\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{t}})$ in $(\mathbb{R}^+)^k$. □

With Lemma 2.2 in hands, we can obtain the following.

Lemma 2.3 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, $\beta < 0$ and $|\overrightarrow{\mathbf{b}}| < b_T$, where b_T is given by Lemma 2.1.*

(1) *For every $\overrightarrow{\mathbf{u}} \in \Theta$, there exists a unique $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{b}}} \in (\mathbb{R}^+)^k$ such that $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{b}}} \circ \overrightarrow{\mathbf{u}} \in \mathcal{W}_{T, \overrightarrow{\mathbf{b}}}$ and*

$$\Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{b}}}) = \max_{\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^k} \Phi_{\overrightarrow{\mathbf{u}}, T, \overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{t}}), \tag{2.11}$$

where Θ is given by (2.6). Moreover, $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{b}}} = \overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{0}}} + o(1)$ as $|\overrightarrow{\mathbf{b}}| \to 0^+$.

(2) *If $\overrightarrow{\mathbf{u}} \in \mathcal{H}$ satisfies $\sum_{i=1}^k \mathcal{B}_{\nabla u_i, 2}^2 \leq T^2$ and*

$$\mathcal{Q}^i_{T, \overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\mathcal{B}_{\nabla u_i, 2}^2 \leq \lambda_i \mathcal{B}_{u_i, 2}^2 + \mathcal{B}_{u_i, 2^*}^{2^*} + \beta \sum_{j=1, j\neq i}^k \mathcal{B}_{|u_i|^{\frac{2^*}{2}} |u_j|^{\frac{2^*}{2}}, 1} \tag{2.12}$$

for all $i = 1, 2, \ldots, k$*, then we also have that* $s_i^{\vec{\mathbf{b}}} \leq 1$ *for all* $i = 1, 2, \ldots, k$*.*

Proof (1) Let $\vec{\mathbf{u}} \in \Theta$ and consider the following system in $(\mathbb{R}^+)^k$:

$$\vec{\mathbf{M}}_*(\vec{\mathbf{b}}, \vec{\mathbf{t}}) = \vec{\mathbf{0}},$$

where $\vec{\mathbf{M}}_*(\vec{\mathbf{b}}, \vec{\mathbf{t}}) = (M_{i,*}(\vec{\mathbf{b}}, \vec{\mathbf{t}}))_{i=1,2,\ldots,k}$ with

$$\begin{aligned} M_{i,*}(\vec{\mathbf{b}}, \vec{\mathbf{t}}) &= \mathcal{Q}^i_{T,\vec{\mathbf{b}}}(\vec{\mathbf{t}} \circ \vec{\mathbf{u}})\mathcal{B}^2_{\nabla t_i u_i,2} - \lambda_i \mathcal{B}^2_{t_i u_i,2} - \mathcal{B}^{2^*}_{t_i u_i,2^*} \\ &\quad -\beta \sum_{j=1, j\neq i}^{k} \mathcal{B}_{|t_i u_i|^{\frac{2^*}{2}} |t_j u_j|^{\frac{2^*}{2}},1} \\ &= 0. \end{aligned}$$

Since $\chi(s)$ is of C^∞, for all $i = 1, 2, \ldots, k$, $M_{i,*}(\vec{\mathbf{b}}, \vec{\mathbf{t}})$ is of C^∞ for all t_j and b_j. Note that $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, then by Lemma 2.2, $\vec{\mathbf{M}}_*(\vec{\mathbf{0}}, \vec{\mathbf{t}}) = \vec{\mathbf{M}}(1, \vec{\mathbf{t}})$ is unique solvable in $(\mathbb{R}^+)^k$. On the other hand, by the construction of $\chi(s)$ once more, we can see that

$$\frac{\partial M_{i,*}(\vec{\mathbf{b}}, \vec{\mathbf{t}})}{\partial t_j} = \frac{\partial M_i(1, \vec{\mathbf{t}})}{\partial t_j} + O(|\vec{\mathbf{b}}|)$$

if $\sum_{i=1}^k \mathcal{B}^2_{\nabla t_i u_i,2} \leq 2T^2$ and

$$\frac{\partial M_{i,*}((\vec{\mathbf{b}}, \vec{\mathbf{t}}))}{\partial t_j} = \frac{\partial M_i(1, \vec{\mathbf{t}})}{\partial t_j}$$

if $\sum_{i=1}^k \mathcal{B}^2_{\nabla t_i u_i,2} > 2T^2$ for all $i, j = 1, 2, \ldots, k$, where $O(|\vec{\mathbf{b}}|)$ is independent of $\vec{\mathbf{t}} \circ \vec{\mathbf{u}}$. It follows that

$$\det\left(M_*\left(\vec{\mathbf{b}}, \vec{\mathbf{t}}\right)\right) = \det\left(M_1\left(\vec{\mathbf{t}}\right)\right) + O(|\vec{\mathbf{b}}|), \tag{2.13}$$

where $O(|\vec{\mathbf{b}}|)$ is also independent of $\vec{\mathbf{t}} \circ \vec{\mathbf{u}}$ and $M_1(\vec{\mathbf{t}})$ is given by (2.7) and

$$M_*(\vec{\mathbf{b}}, \vec{\mathbf{t}}) = \left(\frac{\partial M_{i,*}(\vec{\mathbf{b}}, \vec{\mathbf{t}})}{\partial t_j}\right)_{i,j=1,2,\ldots,k}.$$

By Lemma 2.2, $M_1(\vec{\mathbf{s}}_{\vec{\mathbf{0}}})$ is nonsingular for every $\vec{\mathbf{u}} \in \Theta$, where $\vec{\mathbf{s}}_{\vec{\mathbf{0}}}$ is also given in Lemma 2.2. Thus, by $M_1(\vec{\mathbf{t}}) = \vec{\mathbf{M}}_*(\vec{\mathbf{0}}, \vec{\mathbf{t}})$ and (2.13), we can see that the system $\vec{\mathbf{M}}_*(\vec{\mathbf{b}}, \vec{\mathbf{t}}) = \vec{\mathbf{0}}$ is unique solvable near $(\vec{\mathbf{0}}, \vec{\mathbf{s}}_{\vec{\mathbf{0}}})$ for every $\vec{\mathbf{u}} \in \Theta$. On the other hand, by the construction of $\chi(s)$ and (2.9), we also have that $\sum_{i=1}^k M_{i,*}(\vec{\mathbf{b}}, \vec{\mathbf{t}}) \to -\infty$ as $|\vec{\mathbf{t}}| \to +\infty$ for $|\vec{\mathbf{b}}| < b_T$, Hence, by applying the implicit function theorem similarly as that in Claim 1 in the proof of Lemma 2.2, we can show that the system $\vec{\mathbf{M}}_*(\vec{\mathbf{b}}, \vec{\mathbf{t}}) = \vec{\mathbf{0}}$ is unique solvable in $(\mathbb{R}^+)^k$ near $\vec{\mathbf{0}}$. By choosing b_T small enough if necessary, we may assume that the system $\vec{\mathbf{M}}_*(\vec{\mathbf{b}}, \vec{\mathbf{t}}) = \vec{\mathbf{0}}$ is unique solvable in $(\mathbb{R}^+)^k$ for $|\vec{\mathbf{b}}| < b_T$. Finally, by the construction of $\chi(s)$ and (2.9) once more, $\Phi_{\vec{\mathbf{u}},T,\vec{\mathbf{b}}}(\vec{\mathbf{t}}) \to -\infty$ as $|\vec{\mathbf{t}}| \to +\infty$ for $|\vec{\mathbf{b}}| < b_T$. It follows from a similar argument as used in the proof of Lemma 2.2 that (2.11) holds.

(2) By (2.3) and (2.12), we can see from $\lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$ that $\overrightarrow{\mathbf{u}} \in \Theta$. Thus, by (1), there exists a unique $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{b}}} \in (\mathbb{R}^+)^k$ such that $\overrightarrow{\mathbf{s}}_{\overrightarrow{\mathbf{b}}} \circ \overrightarrow{\mathbf{u}} \in \mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$. Suppose the contrary. Then there exists $\overrightarrow{\mathbf{b}}_n \to \overrightarrow{\mathbf{0}}$ as $n \to \infty$ such that $s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} > 1$ for all n, where

$$s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} = \max\left\{s_1^{\overrightarrow{\mathbf{b}}_n}, s_2^{\overrightarrow{\mathbf{b}}_n}, \ldots, s_k^{\overrightarrow{\mathbf{b}}_n}\right\}.$$

Since $\sum_{i=1}^k \mathcal{B}_{\nabla u_i,2}^2 \leq T^2$ and $\beta < 0$, we have from the construction of $\chi(s)$ and (2.12) that

$$\begin{aligned}
&\left(s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n}\right)^2 \|u_{i_0(n)}\|_{i_0(n)}^2 + \left(s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n}\right)^4 b_{i_0(n)}^n \left(\sum_{j=1}^k b_j^n \mathcal{B}_{\nabla u_j,2}^2\right) \mathcal{B}_{\nabla u_{i_0(n)},2}^2 \\
&\geq \|s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} u_{i_0(n)}\|_{i_0(n)}^2 + b_{i_0(n)}^n \left(\sum_{j=1}^k b_j^n \mathcal{B}_{\nabla s_j^{\overrightarrow{\mathbf{b}}_n} u_j,2}^2\right) \mathcal{B}_{\nabla s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} u_{i_0(n)},2}^2 \\
&= \mathcal{B}_{s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} u_{i_0(n)},2^*}^{2^*} + \beta \sum_{j=1, j\neq i_0(n)}^k \mathcal{B}_{|s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} u_{i_0(n)}|^{\frac{2^*}{2}} |s_j^{\overrightarrow{\mathbf{b}}_n} u_j|^{\frac{2^*}{2}},1} \\
&\geq \left(s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n}\right)^{2^*} \left(\mathcal{B}_{u_{i_0(n)},2^*}^{2^*} + \beta \sum_{j=1, j\neq i_0(n)}^k \mathcal{B}_{|u_{i_0(n)}|^{\frac{2^*}{2}} |u_j|^{\frac{2^*}{2}},1}\right) \\
&\geq \left(s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n}\right)^{2^*} \left(\|u_{i_0(n)}\|_{i_0(n)}^2 + b_{i_0(n)}^n \left(\sum_{j=1}^k b_j^n \mathcal{B}_{\nabla u_j,2}^2\right) \mathcal{B}_{\nabla u_{i_0(n)},2}^2\right).
\end{aligned} \tag{2.14}$$

If $b_{i_0(n)}^n = 0$, then by (2.14), we must have from $2 < 2^*$ that $s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} \leq 1$, which contradicts to $s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} > 1$. Thus, $b_{i_0(n)}^n \neq 0$ for all n. It follows from $s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} > 1$, $2 < 2^* \leq 4$ and (2.14) that

$$\left(s_{i_0}^{\overrightarrow{\mathbf{b}}_n}\right)^{2^*-2} \frac{\left(s_{i_0}^{\overrightarrow{\mathbf{b}}_n}\right)^{4-2^*} - 1}{\left(s_{i_0(n)}^{\overrightarrow{\mathbf{b}}}\right)^{2^*-2} - 1} \geq \frac{\mathcal{D}_n}{\mathcal{C}_n}. \tag{2.15}$$

Here, $\mathcal{C}_n = b_{i_0(n)}^n (\sum_{j=1}^k b_j^n \mathcal{B}_{\nabla u_j,2}^2) \mathcal{B}_{\nabla u_{i_0(n)},2}^2$ and $\mathcal{D}_n = \|u_{i_0(n)}\|_{i_0(n)}^2$. Since $s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} > 1$ for all n, we may assume that $s_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} = 1 + t_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n}$ with $t_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} > 0$. Thus, by (1) and a similar argument as used in (2.14), we can see from $|\overrightarrow{\mathbf{b}}_n| = o_n(1)$ that $t_{i_0(n)}^{\overrightarrow{\mathbf{b}}_n} = o_n(1)$. This together with the Taylor's expansion, implies

$$\left(s_{i_0}^{\overrightarrow{\mathbf{b}}_n}\right)^{2^*-2} \frac{\left(s_{i_0}^{\overrightarrow{\mathbf{b}}_n}\right)^{4-2^*} - 1}{\left(s_{i_0(n)}^{\overrightarrow{\mathbf{b}}}\right)^{2^*-2} - 1} = \frac{4 - 2^*}{2^* - 2} + o_n(1). \tag{2.16}$$

On the other hand, by (2.12) and the Sobolev inequality, we can see from $\sum_{i=1}^{k}\mathcal{B}^2_{\nabla u_i,2}\leq T^2$, $\beta<0$ and the construction of $\chi(s)$ that

$$\|u_{i_0(n)}\|^2_{i_0(n)}\geq\left(\frac{a_{i_0(n)}\sigma_1-\lambda_{i_0(n)}}{\sigma_1}\mathcal{S}\right)^{\frac{N}{2}}\geq\left(\min_{i=1,2,\dots,k}\frac{a_i\sigma_1-\lambda_i}{\sigma_1}\mathcal{S}\right)^{\frac{N}{2}}.$$

Therefore, by $\sum_{i=1}^{k}\mathcal{B}^2_{\nabla u_i,2}\leq T^2$ and $|\overrightarrow{\mathbf{b}}_n|=o_n(1)$, we have $\frac{\mathcal{D}_n}{\mathcal{C}_n}\to+\infty$ as $n\to\infty$, which contradicts to (2.15) and (2.16). Thus, $s_i^{\overrightarrow{\mathbf{b}}}\leq1$ for all $i=1,2,\dots,k$ if $|\overrightarrow{\mathbf{b}}|$ small enough. By choosing b_T small enough if necessary, we obtain the conclusion. □

We close this section by

Lemma 2.4 *Let $k\geq2$, $0<\lambda_i<a_i\sigma_1$ for all $i=1,2,\dots,k$, $\beta<0$ and $|\overrightarrow{\mathbf{b}}|<b_T$, where b_T is given by Lemma 2.1. Then $\mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$ is a natural constraint in $\mathcal{H}$.*

Proof Let $\mathcal{F}_{i,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})=\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i$, $i=1,2,\dots,k$. Then it is easy to see that $\mathcal{F}_{i,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$ is of C^1 in $\mathcal{H}$ for all $i=1,2,\dots,k$. Suppose $\overrightarrow{\mathbf{u}}$ is a critical point of $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$ in $\mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$, then by the method of Lagrange multipliers, there exists $\overrightarrow{\xi}=\{\xi_i\}_{i=1,2,\dots,k}\subset\mathbb{R}^k$ such that

$$\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})-\sum_{j=1}^{k}\xi_j\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})=0\quad\text{in }\mathcal{H}^{-1}.\tag{2.17}$$

Multiplying this equation with $\overrightarrow{\mathbf{u}}_i$ respectively, we can see that $\overrightarrow{\xi}$ is the solution of the following system

$$\sum_{j=1}^{k}\xi_j\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i=0,\quad i=1,2,\dots,k.\tag{2.18}$$

Since $|\overrightarrow{\mathbf{b}}|<b_T$, by a direct calculation, we can see from the construction of $\chi(s)$ that

$$\mathcal{F}'_{i,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i=(2-2^*)\|u_i\|_i^2-\beta\left(\frac{2^*}{2}-2^*\right)\sum_{j=1,j\neq i}^{k}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1}+O(|\overrightarrow{\mathbf{b}}|)\tag{2.19}$$

and

$$\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i=-\frac{2^*}{2}\beta\sum_{j=1,j\neq i}^{k}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1}+O(|\overrightarrow{\mathbf{b}}|)\tag{2.20}$$

if $\sum_{i=1}^{k}\mathcal{B}^2_{\nabla u_i,2}\leq2T^2$ and

$$\mathcal{F}'_{i,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i=(2-2^*)\|u_i\|_i^2-\beta\left(\frac{2^*}{2}-2^*\right)\sum_{j=1,j\neq i}^{k}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1}\tag{2.21}$$

and

$$\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i=-\frac{2^*}{2}\beta\sum_{j=1,j\neq i}^{k}\mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1}\tag{2.22}$$

if $\sum_{i=1}^{k}\mathcal{B}^2_{\nabla u_i,2} > 2T^2$ for all $i, j = 1, 2, \dots, k$ and $j \neq i$, where $O(|\overrightarrow{\mathbf{b}}|)$ is independent of $\overrightarrow{\mathbf{u}}$. Since $\beta < 0$, we can see from (2.19)–(2.22) that the matrix $(\mathcal{F}'_{j,\overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i)_{i,j=1,2,\dots,k}$ is nonsingular. By (2.19)–(2.22) once more, we also have that

$$\det\left(\left(\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i\right)_{i,j=1,2,\dots,k}\right) = \det\left(\left(\mathcal{F}'_{j,\overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i\right)_{i,j=1,2,\dots,k}\right) + O(|\overrightarrow{\mathbf{b}}|), \tag{2.23}$$

$O(|\overrightarrow{\mathbf{b}}|)$ is also independent of $\overrightarrow{\mathbf{u}}$. Thus, the matrix $(\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i)_{i,j=1,2,\dots,k}$ is nonsingular uniformly for $|\overrightarrow{\mathbf{b}}|$ small enough. By choosing b_T small enough if necessary, we may assume that the matrix $(\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i)_{i,j=1,2,\dots,k}$ is nonsingular for $|\overrightarrow{\mathbf{b}}| < b_T$. Therefore, $\overrightarrow{\mathbf{0}}$ is the unique solution of (2.18) due to the Cramer's Rule and we must have $\overrightarrow{\xi} = \overrightarrow{\mathbf{0}}$ for $|\overrightarrow{\mathbf{b}}| < b_T$. Thanks to (2.17), we have $\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}) = 0$ in $\mathcal{H}^{-1}$ for $|\overrightarrow{\mathbf{b}}| < b_T$, that is, $\mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$ is a natural constraint in $\mathcal{H}$ for $|\overrightarrow{\mathbf{b}}| < b_T$. □

2.2 Ground state

Let $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \dots, k$, $\beta < 0$ and $|\overrightarrow{\mathbf{b}}| < b_T$. By a similar argument as used in (2.5), we can see that $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$ is bounded from below and coercive on $\mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$. Thus,

$$m_{T,\overrightarrow{\mathbf{b}}} = \inf_{\overrightarrow{\mathbf{u}}\in\mathcal{W}_{T,\overrightarrow{\mathbf{b}}}} \mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$$

is well defined. Thanks to the Ekeland's principle, there exists $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$ such that

(a) $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n) = m_{T,\overrightarrow{\mathbf{b}}} + o_n(1)$;
(b) $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{v}}) \geq \mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n) - \frac{1}{n}\|\overrightarrow{\mathbf{v}} - \overrightarrow{\mathbf{u}}_n\|$ for all $\overrightarrow{\mathbf{v}} \in \mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$.

Lemma 2.5 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \dots, k$, $\beta < 0$ and $|\overrightarrow{\mathbf{b}}| < b_T$, where b_T is given by Lemma 2.1. Then $\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n) = o_n(1)$ in $\mathcal{H}^{-1}$.*

Proof By a similar argument as used in (2.4), we can see that $\|u_i^n\|_i^2 \geq C > 0$ for all $i = 1, 2, \dots, k$. It follows from similar arguments as used for (2.19)–(2.22) with some trivial modifications that

$$\det\left(\left(\mathcal{F}'_{j,\overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{u}}_n)\overrightarrow{\mathbf{u}}_i^n\right)_{i,j=1,2,\dots,k}\right) \neq 0.$$

By similar arguments as used for (2.23) with some trivial modifications, we also have

$$\det\left(\left(\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n)\overrightarrow{\mathbf{u}}_i^n\right)_{i,j=1,2,\dots,k}\right) = \det\left(\left(\mathcal{F}'_{j,\overrightarrow{\mathbf{0}}}(\overrightarrow{\mathbf{u}}_n)\overrightarrow{\mathbf{u}}_i^n\right)_{i,j=1,2,\dots,k}\right) + O(|\overrightarrow{\mathbf{b}}|),$$

where $O(|\overrightarrow{\mathbf{b}}|)$ is independent of n. Thus, $\det((\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n)\overrightarrow{\mathbf{u}}_i^n)_{i,j=1,2,\dots,k}) \neq 0$ uniformly for $|\overrightarrow{\mathbf{b}}|$ small enough. By taking b_T small enough if necessary, we can see that $\det((\mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n)\overrightarrow{\mathbf{u}}_i^n)_{i,j=1,2,\dots,k}) \neq 0$ for all $|\overrightarrow{\mathbf{b}}| < b_T$. For every $\overrightarrow{\mathbf{v}} \in \mathcal{H}$, we define $\overrightarrow{\mathbf{l}} \circ \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_n$, where $\overrightarrow{\mathbf{l}}, \overrightarrow{\mathbf{t}} \in \mathbb{R}^k$. Let

$$\overrightarrow{\mathbf{Y}}(\overrightarrow{\mathbf{l}}, \overrightarrow{\mathbf{t}}) = \left(\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{l}} \circ \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_n)(\overrightarrow{\mathbf{l}} \circ \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_n)_i\right)_{i=1,2,\ldots,k}.$$

Clearly, $\overrightarrow{\mathbf{Y}}$ is of C^1. Moreover, we also have that

$$\frac{\partial Y_j(\overrightarrow{\mathbf{0}}, \overrightarrow{\mathbf{1}})}{\partial t_i} = \mathcal{F}'_{j,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n)\overrightarrow{\mathbf{u}}_i^n \quad \text{for all } i, j = 1, 2, \ldots, k \text{ and } i \neq j.$$

Then by applying the implicit function theorem to $\overrightarrow{\mathbf{Y}}(\overrightarrow{\mathbf{l}}, \overrightarrow{\mathbf{t}})$ in a standard way (cf. [10]), we can see that there exist $\varepsilon_n > 0$ and $\overrightarrow{\mathbf{t}}_n(\overrightarrow{\mathbf{l}}) \in C^1([-\varepsilon_n, \varepsilon_n]^k, [\frac{1}{2}, \frac{3}{2}]^k)$ such that $\overrightarrow{\mathbf{l}} \circ \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{t}}_n(\overrightarrow{\mathbf{l}}) \circ \overrightarrow{\mathbf{u}}_n \in \mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$ and $\overrightarrow{\mathbf{t}}_n(\overrightarrow{\mathbf{0}}) = \overrightarrow{\mathbf{1}}$. Moreover, since $\left(\frac{\partial Y_j(\overrightarrow{\mathbf{0}}, \overrightarrow{\mathbf{1}})}{\partial t_i}\right)_{i,j=1,2,\ldots,k}$ is strictly diagonally dominant, we have from (2.8) and [38, Theorem 3.2] that $|\overrightarrow{\mathbf{t}}'_n(\overrightarrow{\mathbf{0}})| \leq C\|\overrightarrow{\mathbf{v}}\|$. Since (b) holds, by using the Taylor expansion to $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{l}} \circ \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_n)$ in a standard way (cf. [10]), we can obtain that $\mathcal{J}'_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n) = o_n(1)$ in $\mathcal{H}^{-1}$. □

We re-denote $\mathcal{H}$, $\mathcal{H}^*$, $\mathcal{H}^{-1}$, $\mathcal{J}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}})$, Θ, $\mathcal{W}_{T,\overrightarrow{\mathbf{b}}}$ and $m_{T,\overrightarrow{\mathbf{b}}}$ respectively by $\mathcal{H}_k$, $\mathcal{H}_k^*$, $\mathcal{H}_k^{-1}$, $\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}})$, Θ_k, $\mathcal{W}_{T,\overrightarrow{\mathbf{b}},k}$ and $m_{T,\overrightarrow{\mathbf{b}},k}$. For every $i_0 = 1, 2, \ldots, k$, we set $\mathcal{H}_k^{i_0,*} = \prod_{i=1,i\neq i_0}^k (\mathcal{H}_i\backslash\{0\})$ and

$$\mathcal{J}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}(\overrightarrow{\mathbf{u}}) = \sum_{i=1,i\neq i_0}^k \left(\frac{1}{2}\|u_i\|_i^2 - \frac{1}{2^*}\mathcal{B}^{2^*}_{u_i,2^*}\right) - \frac{2\beta}{2^*} \sum_{i,j=1,i,j\neq i_0,j\neq i}^k \mathcal{B}_{|u_i|^{\frac{2^*}{2}}|u_j|^{\frac{2^*}{2}},1}$$
$$+ \frac{1}{4}\chi\left(\frac{\sum_{i=1,i\neq i_0}^k \mathcal{B}^2_{\nabla u_i,2}}{T^2}\right)\left(\sum_{i=1,i\neq i_0}^k b_i\mathcal{B}^2_{\nabla u_i,2}\right)^2.$$

Let

$$\mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} = \left\{\overrightarrow{\mathbf{u}} \in \mathcal{H}_k^{i_0,*} \mid (\mathcal{J}^{i_0,*}_{T,\overrightarrow{\mathbf{b}}})'(\overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}}_i = 0, \quad i = 1, 2, \ldots, k \text{ and } i \neq i_0\right\}.$$

Then by Lemma 2.3, $\mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} \neq \emptyset$ for all $i_0 = 1, 2, \ldots, k$. Define

$$m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} = \inf_{\mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}} \mathcal{J}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}(\overrightarrow{\mathbf{u}}).$$

Lemma 2.6 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, $\beta < 0$ and $|\overrightarrow{\mathbf{b}}| < b_T$, where b_T is given by Lemma 2.1. If $m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}$ can be attained by some $\overrightarrow{\mathbf{u}}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} \in \mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}$ for all $i_0 = 1, 2, \ldots, k$, then*

$$m_{T,\overrightarrow{\mathbf{b}},k} < \min_{i_0=1,2,\ldots,k}\left\{m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} + \frac{1}{N}(a_{i_0}\mathcal{S})^{\frac{N}{2}}\right\}.$$

Proof For the simplicity and clarity, we only give the proof for $i_0 = k$ since the proofs for other i_0 are similar. In this case, we have $\mathcal{J}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}(\overrightarrow{\mathbf{u}}) = \mathcal{J}_{T,\overrightarrow{\mathbf{b}},k-1}(\overrightarrow{\mathbf{u}})$, $\mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} = \mathcal{W}_{T,\overrightarrow{\mathbf{b}},k-1}$, $m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} = m_{T,\overrightarrow{\mathbf{b}},k-1}$ and $\overrightarrow{\mathbf{u}}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} = \overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k-1}$. For the sake of clarity, we also divide this proof into two steps.

Step. 1 We prove the conclusion for $\overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{0}}$.

Since $m_{T,\overrightarrow{\mathbf{0}},k-1}$ can be attained by $\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{0}},k-1} \in \mathcal{W}_{T,\overrightarrow{\mathbf{0}},k-1}$, by Lemma 2.4, $\mathcal{J}'_{T,\overrightarrow{\mathbf{0}},k-1}(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{0}},k-1}) = 0$ in $\mathcal{H}^{-1}_{k-1}$. Since (2.3) holds and $\beta < 0$, by the Brezís–Kato theorem, we have $u_i^{T,\overrightarrow{\mathbf{0}},k-1} \in L^q(\Omega)$ for all $q \geq 2$ and $i = 1, 2, \ldots, k-1$. It follows from the Calderon–Zygmund inequality that $u_i^{T,\overrightarrow{\mathbf{0}},k-1} \in W_0^{2,q}(\Omega)$ for all $q \geq 2$ and $i = 1, 2, \ldots, k-1$. Thanks to the Sobolev embedding theorem and the classical Shaulder's estimates, we actually have $u_i^{T,\overrightarrow{\mathbf{0}},k-1} \in C^2(\Omega)$ for all $i = 1, 2, \ldots, k-1$. Let $x_R \in \Omega$ and $R > 0$ satisfy $\mathbb{B}_{3\sqrt{a_k}R}(\sqrt{a_k}x_R) \subset \Omega$ and $\Psi \in C_0^2(\mathbb{B}_2(0))$ satisfy $0 \leq \Psi(x) \leq 1$ and $\Psi(x) \equiv 1$ in $\mathbb{B}_1(0)$. Set $\varphi_R^*(x) = \Psi\left(\frac{x-\sqrt{a_k}x_R}{\sqrt{a_k}R}\right)$ for $x \in \mathbb{B}_{2\sqrt{a_k}R}(x_R)$ and $\varphi_R^*(x) = 0$ for $x \in \Omega\backslash\mathbb{B}_{2\sqrt{a_k}R}(x_R)$. Then $\varphi_R^*(x) \in C_0^2(\Omega)$ and

$$\varphi_R(x) = \begin{cases} 1, & x \in \mathbb{B}_{\sqrt{a_k}R}(\sqrt{a_k}x_R); \\ 0, & x \in \Omega\backslash\mathbb{B}_{2\sqrt{a_k}R}(x_R), \end{cases}$$

and $|\nabla\varphi_R(x)| \leq \frac{C}{R}$. Let

$$U_{\varepsilon,x_R}(x) = \frac{(N(N-2)\varepsilon^2)^{(N-2)/4}}{(\varepsilon^2 + |x - x_R|^2)^{(N-2)/2}}.$$

Then it is well known that

$$\mathcal{B}^2_{\nabla v_\varepsilon,2} = a_k^{\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^{N-2}) \quad \text{and} \quad \mathcal{B}^{2^*}_{v_\varepsilon,2^*} = (a_k\mathcal{S})^{\frac{N}{2}} + O(\varepsilon^N). \tag{2.24}$$

Moreover, we also have

$$\mathcal{B}^2_{v_\varepsilon,2} \geq \begin{cases} C\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2), & N = 4, \\ C\varepsilon^2 + O(\varepsilon^{N-2}), & N \geq 5. \end{cases} \tag{2.25}$$

Here, $v_\varepsilon(x) = \varphi_R(x)U_{\varepsilon,x_R}(\frac{x}{\sqrt{a_k}R})$. Moreover, by $u_i^{T,\overrightarrow{\mathbf{0}},k-1} \in C^2(\Omega)$ for all $i = 1, 2, \ldots, k-1$, we also have that

$$\begin{aligned}\sum_{j=1}^{k-1} &\mathcal{B}_{|u_j^{T,\overrightarrow{\mathbf{0}},k-1}|^{\frac{2^*}{2}}|v_\varepsilon|^{\frac{2^*}{2}},1} \\ &\leq C \max_{i=1,2,\ldots,k-1} \|u_i^{T,\overrightarrow{\mathbf{0}},k-1}\|_{L^\infty(\mathbb{B}_{3\sqrt{a_k}R}(\sqrt{a_k}x_R))}\mathcal{B}^{\frac{2^*}{2}}_{v_\varepsilon,\frac{2^*}{2}}.\end{aligned} \tag{2.26}$$

For $N = 4$, we have $2^* = 4$. It follows from (2.26) that

$$\begin{aligned}\sum_{j=1}^{k-1} &\mathcal{B}_{|u_j^{T,\overrightarrow{\mathbf{0}},k-1}|^{\frac{2^*}{2}}|v_\varepsilon|^{\frac{2^*}{2}},1} \\ &\leq C \max_{i=1,2,\ldots,k-1} \|u_i^{T,\overrightarrow{\mathbf{0}},k-1}\|_{L^\infty(\mathbb{B}_{3\sqrt{a_k}R}(\sqrt{a_k}x_R))}\mathcal{B}^2_{v_\varepsilon,2}.\end{aligned} \tag{2.27}$$

For $N \geq 5$, by a similar argument as used in [12, (3.6)], we can see from (2.26) that

$$\sum_{j=1}^{k-1} \mathcal{B}_{|u_j^{T,\overrightarrow{\mathbf{0}},k-1}|^{\frac{2^*}{2}}|v_\varepsilon|^{\frac{2^*}{2}},1} \leq o(\varepsilon^2). \tag{2.28}$$

On the other hand, since $\vec{\mathbf{u}}_{T,\vec{\mathbf{0}},k-1} \in \mathcal{W}_{T,\vec{\mathbf{0}},k-1}$ and $v_\varepsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0^+$, by (2.24), we can see that $\vec{\mathbf{V}}_\varepsilon \in \Theta_k$ for ε small enough, where

$$\vec{\mathbf{V}}_\varepsilon = \left(u_1^{T,\vec{\mathbf{0}},k-1}, \ldots, u_{k-1}^{T,\vec{\mathbf{0}},k-1}, v_\varepsilon\right).$$

Thus, by Lemma 2.3, there exists $\vec{\mathbf{s}}_\varepsilon \in (\mathbb{R}^+)^k$ such that $\vec{\mathbf{s}}_\varepsilon \circ \vec{\mathbf{V}}_\varepsilon \in \mathcal{W}_{T,\vec{\mathbf{0}},k}$. Since $\vec{\mathbf{V}}_\varepsilon \in \Theta_k$ for ε small enough, we can see from a similar argument as used in (2.9) that $\vec{\mathbf{s}}_\varepsilon$ is bounded from above for ε. Note that $\beta < 0$, by a standard argument, we can also see that s_i^ε is bounded below away from 0 for all $i = 1, 2, \ldots, k$. It follows from Lemma 2.3 once more and (2.24) that

$$\begin{aligned}
m_{T,\vec{\mathbf{0}},k} &\leq \mathcal{J}_{T,\vec{\mathbf{0}},k}\left(\vec{\mathbf{s}}_\varepsilon \circ \vec{\mathbf{V}}_\varepsilon\right)\\
&\leq \mathcal{J}_{T,\vec{\mathbf{0}},k-1}\left(\vec{\mathbf{u}}_{T,\vec{\mathbf{0}},k-1}\right) - \frac{4\beta C}{2^*}\sum_{j=1}^{k-1}\mathcal{B}_{|u_j^{T,\vec{\mathbf{0}},k-1}|^{\frac{2^*}{2}}|v_\varepsilon|^{\frac{2^*}{2}},1}\\
&\quad + \frac{a_k}{2}\mathcal{B}^2_{\nabla s_k^\varepsilon v_\varepsilon,2} - \frac{\lambda_k C_1}{2}\mathcal{B}^2_{v_\varepsilon,2} - \frac{1}{2^*}\mathcal{B}^{2^*}_{s_k^\varepsilon v_\varepsilon,2^*}\\
&\leq m_{T,\vec{\mathbf{0}},k-1} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}} - \frac{\lambda_k C_1}{2}\mathcal{B}^2_{v_\varepsilon,2} + O(\varepsilon^{N-2})\\
&\quad - \frac{4\beta C}{2^*}\sum_{j=1}^{k-1}\mathcal{B}_{|u_j^{T,\vec{\mathbf{0}},k-1}|^{\frac{2^*}{2}}|v_\varepsilon|^{\frac{2^*}{2}},1}.
\end{aligned} \tag{2.29}$$

For $N = 4$, note that $u_i^{T,\vec{\mathbf{0}},k-1} \in C^2(\Omega)$ for all $i = 1, 2, \ldots, k-1$. Thus, $\max_{i=1,2,\ldots,k-1}\|u_i^{T,\vec{\mathbf{0}},k-1}\|_{L^\infty(\mathbb{B}_{3\sqrt{a_k}R}(\sqrt{a_k}x_R))}$ in (2.27) can be chosen small enough by choosing x_R close $\partial\Omega$ enough. Hence, we can see from (2.27) and (2.29) that

$$m_{T,\vec{\mathbf{0}},k} \leq m_{T,\vec{\mathbf{0}},k-1} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}} - \frac{\lambda_k C_1}{4}\mathcal{B}^2_{v_\varepsilon,2} + O(\varepsilon^{N-2}),$$

which together with (2.25), implies $m_{T,\vec{\mathbf{0}},k} < m_{T,\vec{\mathbf{0}},k-1} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}}$ for $N = 4$. In the case $N \geq 5$, by (2.28) and (2.29), we can see that

$$m_{T,\vec{\mathbf{0}},k} \leq m_{T,\vec{\mathbf{0}},k-1} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}} - \frac{\lambda_k C_1}{2}\mathcal{B}^2_{v_\varepsilon,2} + o(\varepsilon^2).$$

By (2.25) once more, we also have $m_{T,\vec{\mathbf{0}},k} < m_{T,\vec{\mathbf{0}},k-1} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}}$ for $N \geq 5$.

Step. 2 We prove the conclusion for $|\vec{\mathbf{b}}| < b_T$.

By Lemma 2.3, we can see that $\mathcal{W}_{T,\vec{\mathbf{b}},k} \subset \Theta_k$. Since $b_i \geq 0$ for all $i = 1, 2, \ldots, k$, by Lemma 2.2 and a similar argument as used in [21, Lemma 5.1], we can see that

$$m_{T,\vec{\mathbf{0}},k} \leq m_{T,\vec{\mathbf{b}},k}. \tag{2.30}$$

On other hand, for every $\vec{\mathbf{u}} \in \mathcal{W}_{T,\vec{\mathbf{0}},k}$, by Lemma 2.2, we have $\vec{\mathbf{u}} \in \Theta_k$. It follows from Lemma 2.3 that there exists a unique $\vec{\mathbf{s}}_{\vec{\mathbf{b}}} \in (\mathbb{R}^+)^k$ such that $\vec{\mathbf{s}}_{\vec{\mathbf{b}}} \circ \vec{\mathbf{u}} \in \mathcal{W}_{T,\vec{\mathbf{b}},k}$ and $\vec{\mathbf{s}}_{\vec{\mathbf{b}}} = \vec{\mathbf{1}} + o(1)$. Thus, by the construction of $\chi(s)$ and Lemma 2.2 once more, we have

$$\mathcal{J}_{T,\vec{\mathbf{0}},k}(\vec{\mathbf{u}}) \geq \mathcal{J}_{T,\vec{\mathbf{0}},k}(\vec{\mathbf{s}}_{\vec{\mathbf{b}}} \circ \vec{\mathbf{u}}) \geq \mathcal{J}_{T,\vec{\mathbf{b}},k}(\vec{\mathbf{s}}_{\vec{\mathbf{b}}} \circ \vec{\mathbf{u}}) - C|\vec{\mathbf{b}}|T^4$$

for $|\overrightarrow{\mathbf{b}}| < b_T$, which implies

$$m_{T,\overrightarrow{\mathbf{0}},k} \geq m_{T,\overrightarrow{\mathbf{b}},k} - C|\overrightarrow{\mathbf{b}}|T^4. \tag{2.31}$$

Hence, by choosing b_T small enough if necessary, we have from Step. 1 that

$$m_{T,\overrightarrow{\mathbf{b}},k} < m_{T,\overrightarrow{\mathbf{b}},k-1} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}}$$

for $|\overrightarrow{\mathbf{b}}| < b_T$. It completes the proof. □

Let

$$T_k^* = \left(\frac{\sum_{i=1}^k 2\sigma_1(a_i\mathcal{S})^{\frac{N}{2}}}{\min_{i=1,2,\ldots,k}(a_i\sigma_1 - \lambda_i)}\right)^{\frac{1}{2}}. \tag{2.32}$$

Then we have the following.

Lemma 2.7 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, $\beta < 0$ and $|\overrightarrow{\mathbf{b}}| < b_T$, where b_T is given by Lemma 2.1. Then $\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^n,2} \leq (2T_k^*)^2 + o_n(1)$ up to a subsequence. Moreover, up to a subsequence, we also have*

$$\overrightarrow{\mathcal{Q}}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n) = \left(a_i + b_i\sum_{j=1}^k b_j\mathcal{B}^2_{\nabla u_i^n,2}\right)_{i=1,2,\ldots,k}$$

for $T \geq 2T_k^*$.

Proof It is well known that $m_{T,\overrightarrow{\mathbf{0}},1} < \frac{1}{N}(a_1\mathcal{S})^{\frac{N}{2}}$. Then by Lemma 2.6 and (2.31), we can see that

$$m_{T,\overrightarrow{\mathbf{b}},k} < \sum_{i=1}^k \frac{1}{N}(a_i\mathcal{S})^{\frac{N}{2}} \tag{2.33}$$

for $|\overrightarrow{\mathbf{b}}| < b_T$. It follows from (a) and the construction of $\chi(s)$ that

$$\begin{aligned} o_n(1) + \sum_{i=1}^k (a_i\mathcal{S})^{\frac{N}{2}} &> \sum_{i=1}^k \|u_i^n\|_i^2 - C|\overrightarrow{\mathbf{b}}|T^4 \\ &\geq \min_{i=1,2,\ldots,k}\frac{a_i\sigma_1 - \lambda_i}{\sigma_1}\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^n,2} - C|\overrightarrow{\mathbf{b}}|T^4. \end{aligned} \tag{2.34}$$

Suppose that $\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^n,2} > (2T_k^*)^2$. By choosing b_T small enough if necessary, we have from (2.34) that

$$(2T_k^*)^2 \leq \sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^n,2} \leq \frac{\sum_{i=1}^k 2\sigma_1(a_i\mathcal{S})^{\frac{N}{2}}}{\min_{i=1,2,\ldots,k}(a_i\sigma_1 - \lambda_i)} + o_n(1),$$

which contradicts to (2.32). Thus, we must have $\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^n,2} \leq (2T_k^*)^2 + o_n(1)$ up to a subsequence. Thanks to the construction of $\chi(s)$, we also have that

$$\overrightarrow{\mathcal{Q}}_{T,\overrightarrow{\mathbf{b}}}(\overrightarrow{\mathbf{u}}_n) = \left(a_i + b_i\sum_{j=1}^k b_j\mathcal{B}^2_{\nabla u_i^n,2}\right)_{i=1,2,\ldots,k}$$

up to a subsequence for $T \geq 2T_k^*$. □

Proposition 2.1 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, $\beta < 0$ and $|\overrightarrow{\mathbf{b}}| < b_T$, where b_T is given by Lemma 2.1. If $T \geq 2T_k^*$, then the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ has a solution $\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k} \in \mathcal{W}_{T,\overrightarrow{\mathbf{b}},k}$ with $u_i^{T,\overrightarrow{\mathbf{b}},k} \geq 0$ and $u_i^{T,\overrightarrow{\mathbf{b}},k} \neq 0$ for all $i = 1, 2, \ldots, k$ and $\sum_{i=1}^k \mathcal{B}^2_{\nabla u_i^{T,\overrightarrow{\mathbf{b}},k},2} \leq (2T_k^*)^2$. Moreover, we also have*

$$\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}) = m_{T,\overrightarrow{\mathbf{b}},k} \quad \textit{and} \quad \mathcal{J}'_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}) = 0 \textit{ in } \mathcal{H}_k^{-1}.$$

Proof We will prove this proposition by iterating. It is well known that $m^{i_0,*}_{T,\overrightarrow{\mathbf{0}},1}$ can be attained by some $\overrightarrow{\mathbf{u}}^{i_0,*}_{T,\overrightarrow{\mathbf{0}},1} \in \mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{0}},1}$ for all $i_0 = 1, 2$. On the other hand, by the results in [24], we can see from Lemma 2.7 that $m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},1}$ can be attained by some $\overrightarrow{\mathbf{u}}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},1} \in \mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},1}$ for all $i_0 = 1, 2$ if $|\overrightarrow{\mathbf{b}}| > 0$ small enough. Thus, by choosing b_T small enough if necessary, we may assume that $m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},1}$ can be attained by some $\overrightarrow{\mathbf{u}}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},1} \in \mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},1}$ for all $i_0 = 1, 2$ and $|\overrightarrow{\mathbf{b}}| < b_T$. Now, let $k \geq 2$ and suppose that $m^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}$ can be attained by some $\overrightarrow{\mathbf{u}}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1} \in \mathcal{W}^{i_0,*}_{T,\overrightarrow{\mathbf{b}},k-1}$ for all $i_0 = 1, 2, \ldots, k$. By Lemma 2.7 once more, we can see that $\{\overrightarrow{\mathbf{u}}_n\}$ is bounded in $\mathcal{H}_k$. Without loss of generality, we assume $\overrightarrow{\mathbf{u}}_n \rightharpoonup \overrightarrow{\mathbf{u}}_*$ weakly in $\mathcal{H}_k$ as $n \to \infty$ and $\mathcal{B}^2_{\nabla u_i^n,2} = B_i + o_n(1)$ for all $i = 1, 2 \ldots, k$. Thanks to the Sobolev embedding theorem and without loss of generality once more, we assume that $\overrightarrow{\mathbf{u}}_n = \overrightarrow{\mathbf{u}}_* + o_n(1)$ strongly in $\mathcal{L}^p(\Omega) = (L^p(\Omega))^k$ for all $1 \leq p < 2^*$.

Case 1. $\overrightarrow{\mathbf{u}}_* = \overrightarrow{\mathbf{0}}$.

Since $\beta < 0$, we have from $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{W}_{T,\overrightarrow{\mathbf{b}},k}$ that

$$a_i\mathcal{B}^2_{\nabla u_i^n,2} \leq \mathcal{B}^{2^*}_{u_i^n,2^*} + o_n(1) \leq \mathcal{S}^{-\frac{2^*}{2}}\mathcal{B}^{2^*}_{\nabla u_i^n,2} + o_n(1)$$

for all $i = 1, 2, \ldots, k$. It follows that $\mathcal{B}^2_{\nabla u_i^n,2} \geq a_i^{\frac{1}{2^*-2}}\mathcal{S}^{\frac{N}{2}} + o_n(1)$ for all $i = 1, 2, \ldots, k$. Thus, by $\{\overrightarrow{\mathbf{u}}_n\} \subset \mathcal{W}_{T,\overrightarrow{\mathbf{b}},k}$ once more, we can see that

$$\begin{aligned}
\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_n) &= \frac{1}{N}\sum_{i=1}^k \|u_i^n\|_i^2 + O(|\overrightarrow{\mathbf{b}}|) \\
&= \frac{1}{N}\sum_{i=1}^k a_i\mathcal{B}^2_{\nabla u_i^n,2} + O(|\overrightarrow{\mathbf{b}}|) + o_n(1) \\
&\geq \frac{1}{N}\sum_{i=1}^k (a_i\mathcal{S})^{\frac{N}{2}} + O(|\overrightarrow{\mathbf{b}}|) + o_n(1).
\end{aligned} \tag{2.35}$$

However, by choosing b_T small enough if necessary, we can see that (2.35) is impossible for $|\overrightarrow{\mathbf{b}}| < b_T$ due to Lemma 2.6.

Case 2. $\overrightarrow{\mathbf{u}}_* \neq \overrightarrow{\mathbf{0}}$.

Without loss of generality, we may assume $u_i^* \neq 0$ for $i = 1, 2, \ldots, i_0$ and $u_i^* = 0$ for $i = i_0 + 1, \ldots, k$ with some $i_0 \in \{1, 2, \ldots, k\}$. Let $v_i^n = u_i^n - u_i^*$. Then by the Brezís–Líeb lemma and [12, Lemma 3.3], we can see from Lemma 2.5 that

$$\mathcal{B}^{2^*}_{v_i^n,2^*} + \mathcal{B}^{2^*}_{u_i^*,2^*} + o_n(1) = \mathcal{B}^{2^*}_{u_i^n,2^*} \tag{2.36}$$

and

$$\mathcal{B}_{|u_i^n|^{\frac{2^*}{2}}|u_j^n|^{\frac{2^*}{2}},1}=\mathcal{B}_{|v_i^n|^{\frac{2^*}{2}}|v_j^n|^{\frac{2^*}{2}},1}+\mathcal{B}_{|u_i^*|^{\frac{2^*}{2}}|u_j^*|^{\frac{2^*}{2}},1}+o_n(1) \tag{2.37}$$

for all $i, j = 1, 2, \ldots, k$ and $i \neq j$. By Lemmas 2.5 and 2.7, we can see from (2.36) and (2.37) that

$$\left(a_i+b_i\sum_{j=1}^k b_jB_j\right)\mathcal{B}^2_{\nabla u_i^*,2}=\lambda_i\mathcal{B}^2_{u_i^*,2}+\mathcal{B}^{2^*}_{u_i^*,2^*}+\beta\sum_{j=1,j\neq i}^{i_0}\mathcal{B}_{|u_i^*|^{\frac{2^*}{2}}|u_j^*|^{\frac{2^*}{2}},1} \tag{2.38}$$

for all $i = 1, 2, \ldots, k$. Let $\overrightarrow{\mathbf{u}}_{*,i_0} = (u_i^*)_{i=1,2,\ldots,i_0}$. Then $\overrightarrow{\mathbf{u}}_{*,i_0} \in \mathcal{H}^*_{i_0}$. It follows from $\lambda_i < a_i\sigma_1$ and (2.38) that

$$\mathcal{B}^{2^*}_{u_i^*,2^*}+\beta\sum_{j=1,j\neq i}^{i_0}\mathcal{B}_{|u_i^*|^{\frac{2^*}{2}}|u_j^*|^{\frac{2^*}{2}},1}>0$$

for all $i = 1, 2, \ldots, i_0$. By Lemma 2.3, there exists $\overrightarrow{\mathbf{t}}_{*,\overrightarrow{\mathbf{b}}} \in (\mathbb{R}^+)^{i_0}$ such that $\overrightarrow{\mathbf{t}}_{*,\overrightarrow{\mathbf{b}}} \circ \overrightarrow{\mathbf{u}}_{*,i_0} \in \mathcal{W}_{T,\overrightarrow{\mathbf{b}},i_0}$. Note that by a similar argument as used in the proof of Lemma 2.2, we can see from (2.38) that

$$\mathcal{I}_{i_0}(\overrightarrow{\mathbf{u}}_{*,i_0})=\max_{\overrightarrow{\mathbf{t}}\in(\mathbb{R}^+)^{i_0}}\mathcal{I}_{i_0}(\overrightarrow{\mathbf{t}}\circ\overrightarrow{\mathbf{u}}_{*,i_0}),$$

where

$$\begin{aligned}\mathcal{I}_{i_0}(\overrightarrow{\mathbf{u}})=\sum_{i=1}^{i_0}&\frac{a_i+b_i\sum_{j=1}^k b_jB_j}{2}\mathcal{B}^2_{\nabla u_i,2}-\frac{\lambda_i}{2}\mathcal{B}^2_{u_i^*,2}-\frac{1}{2^*}\mathcal{B}^{2^*}_{u_i^*,2^*}\\&-\frac{2\beta}{2^*}\sum_{i,j=1,j\neq i}^{i_0}\mathcal{B}_{|u_i^*|^{\frac{2^*}{2}}|u_j^*|^{\frac{2^*}{2}},1}.\end{aligned}$$

By (2.38) and Lemma 2.3, we can see that $t_i^{*,\overrightarrow{\mathbf{b}}} \leq 1$ for all $i = 1, 2\ldots, i_0$. It follows from the construction of $\chi(s)$ that

$$\begin{aligned}\mathcal{I}_{i_0}(\overrightarrow{\mathbf{u}}_{*,i_0})&\geq\mathcal{I}_{i_0}(\overrightarrow{\mathbf{t}}_{*,\overrightarrow{\mathbf{b}}}\circ\overrightarrow{\mathbf{u}}_{*,i_0})\\&\geq\mathcal{J}_{T,\overrightarrow{\mathbf{b}},i_0}(\overrightarrow{\mathbf{t}}_{*,\overrightarrow{\mathbf{b}}}\circ\overrightarrow{\mathbf{u}}_{*,i_0})+O(|\overrightarrow{\mathbf{b}}|)\\&\geq m_{T,\overrightarrow{\mathbf{b}},i_0}+O(|\overrightarrow{\mathbf{b}}|),\end{aligned} \tag{2.39}$$

where $O(|\overrightarrow{\mathbf{b}}|)$ is independent of $\overrightarrow{\mathbf{u}}_{*,i_0}$. On the other hand, for $i = i_0 + 1, \ldots, k$, we have from $\beta < 0$ that

$$\left(a_i+b_i\sum_{j=1}^k b_jB_j\right)\mathcal{B}^2_{\nabla u_i^n,2}\leq\mathcal{B}^{2^*}_{u_i^n,2^*}+o_n(1),$$

which together with the Sobolev inequality, implies that

$$\mathcal{B}^2_{\nabla u_i^n,2}\geq\left((a_i+b_i\sum_{j=1}^k b_jB_j)\mathcal{S}^{\frac{2^*}{2}}\right)^{\frac{2}{2^*-2}}+o_n(1)\quad\text{for } i=i_0+1,\ldots,k. \tag{2.40}$$

Combining (2.39) and (2.40), we have that

$$\begin{aligned}\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_n) &= \frac{1}{N}\sum_{i=1}^{k}\|u_i^n\|_i^2 + O(|\overrightarrow{\mathbf{b}}|)\\ &\geq \mathcal{I}_{i_0}(\overrightarrow{\mathbf{u}}_{*,i_0}) + \frac{1}{N}\sum_{i=i_0+1}^{k}(a_i\mathcal{S})^{\frac{N}{2}} + O(|\overrightarrow{\mathbf{b}}|) + o_n(1)\\ &\geq m_{T,\overrightarrow{\mathbf{b}},i_0} + \frac{1}{N}\sum_{i=i_0+1}^{k}(a_i\mathcal{S})^{\frac{N}{2}} + O(|\overrightarrow{\mathbf{b}}|) + o_n(1).\end{aligned}$$

Here, we also use the construction of $\chi(s)$ and Lemma 2.7. By choosing b_T small enough if necessary, we can see that it is impossible for $|\overrightarrow{\mathbf{b}}| < b_T$ if $i_0 \neq k$ due to Lemma 2.6. Thus, we must have that $u_i^* \neq 0$ for all $i = 1, 2, \ldots, k$. Let $\overrightarrow{\mathbf{v}}_n = \overrightarrow{\mathbf{u}}_n - \overrightarrow{\mathbf{u}}_*$. Then $\overrightarrow{\mathbf{v}}_n \rightharpoonup 0$ weakly in $\mathcal{H}_k$ as $n \to \infty$. Moreover, by (2.36)–(2.38), we have

$$\left(a_i + b_i\sum_{j=1}^{k} b_j B_j\right)\mathcal{B}^2_{\nabla v_i^n,2} = \mathcal{B}^{2^*}_{v_i^n,2^*} + \beta\mathcal{B}_{|v_i^n|^{\frac{2^*}{2}}|v_j^n|^{\frac{2^*}{2}},1} + o_n(1)$$

for all $i = 1, 2, \ldots, k$. This together with the Sobolev inequality and $\beta < 0$, implies that for each $i = 1, 2, \ldots, k$, either $v_i^n = o_n(1)$ strongly in $H_0^1(\Omega)$ or

$$\mathcal{B}^2_{\nabla v_i^n,2} \geq \left(a_i + b_i\sum_{j=1}^{k} b_j B_j\right)^{\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}} + o_n(1).$$

We claim that $v_i^n = o_n(1)$ strongly in $H_0^1(\Omega)$ for all $i = 1, 2, \ldots, k$. Indeed, suppose the contrary, then there exists $i_0 = 1, 2, \ldots, k$ such that

$$\mathcal{B}^2_{\nabla v_{i_0}^n,2} \geq \left(a_{i_0} + b_{i_0}\sum_{j=1}^{k} b_j B_j\right)^{\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}} + o_n(1).$$

Without loss of generality, we assume that $\mathcal{B}^2_{\nabla v_1^n,2} \geq (a_1 + b_1\sum_{j=1}^{k} b_j B_j)^{\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}} + o_n(1)$. It follows from Lemma 2.7, (2.39) and the Brezís–Líeb lemma once more that

$$\begin{aligned}m_{T,\overrightarrow{\mathbf{b}},k} + o_n(1) &= \mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_n)\\ &= \mathcal{I}_k(\overrightarrow{\mathbf{u}}_n) + O(|\overrightarrow{\mathbf{b}}|) + o_n(1)\\ &= \mathcal{I}_k(\overrightarrow{\mathbf{u}}_*) + \mathcal{I}_k(\overrightarrow{\mathbf{v}}_n) + O(|\overrightarrow{\mathbf{b}}|) + o_n(1)\\ &\geq m_{T,\overrightarrow{\mathbf{b}},k} + \frac{1}{N}\left(a_1 + b_1\sum_{j=1}^{k} b_j B_j\right)^{\frac{N}{2}}\mathcal{S}^{\frac{N}{2}} + O(|\overrightarrow{\mathbf{b}}|) + o_n(1).\end{aligned}$$

Thus, by choosing b_T small enough if necessary, this inequality can not hold for n large enough if $|\overrightarrow{\mathbf{b}}| < b_T$. Hence, we must have that $\overrightarrow{\mathbf{u}}_n = \overrightarrow{\mathbf{u}}_* + o_n(1)$ strongly in $\mathcal{H}_k$, which implies $\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_*) = m_{T,\overrightarrow{\mathbf{b}},k}$. Let $\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k} = (|u_i^*|)_{i=1,2,\ldots,k}$. Then it is easy to see that $\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k} \in \mathcal{W}_{T,\overrightarrow{\mathbf{b}},k}$ and $\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}) = m_{T,\overrightarrow{\mathbf{b}},k}$. By Lemma 2.4, we have $\mathcal{J}'_{T,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}) = 0$ in $\mathcal{H}_k^{-1}$. Finally, by the fact that $\overrightarrow{\mathbf{u}}_n = \overrightarrow{\mathbf{u}}_* + o_n(1)$ strongly in $\mathcal{H}_k$ and Lemma 2.7, we can see that $\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}$ is also a solution of the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ with $u_i^{T,\overrightarrow{\mathbf{b}},k} \geq 0$ and $u_i^{T,\overrightarrow{\mathbf{b}},k} \neq 0$ for all $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} \mathcal{B}^2_{\nabla u_i^{T,\overrightarrow{\mathbf{b}},k},2} \leq (2T_k^*)^2$. □

We close this section by

Proof of Theorem 1.1 It follows immediately from Proposition 2.1 and the construction of $\chi(s)$. □

3 The concentration behaviors

We first study the concentration behavior of the solutions as $\overrightarrow{\mathbf{b}} \to \overrightarrow{\mathbf{0}}$. Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$ and $\beta < 0$, then by Proposition 2.1, the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}},\beta,k})$ has a solution $\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}$ if $T \geq 2T_k^*$ and $|\overrightarrow{\mathbf{b}}| < b_T$. Moreover,

$$\mathcal{J}_{T,\overrightarrow{\mathbf{b}},k}\left(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}\right) = m_{T,\overrightarrow{\mathbf{b}},k} \quad \text{and} \quad \mathcal{J}'_{T,\overrightarrow{\mathbf{b}},k}\left(\overrightarrow{\mathbf{u}}_{T,\overrightarrow{\mathbf{b}},k}\right) = 0 \text{ in } \mathcal{H}_k^{-1}.$$

In this section, we always take $T = 4T_k^*$ and re-denote $\overrightarrow{\mathbf{u}}_{4T_k^*,\overrightarrow{\mathbf{b}},k}$, $m_{4T_k^*,\overrightarrow{\mathbf{b}},k}$, $\mathcal{W}_{4T_k^*,\overrightarrow{\mathbf{b}},k}$ and $\mathcal{J}_{4T_k^*,\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}})$ by $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},k}$, $m_{\overrightarrow{\mathbf{b}},k}$, $\mathcal{W}_{\overrightarrow{\mathbf{b}},k}$ and $\mathcal{J}_{\overrightarrow{\mathbf{b}},k}(\overrightarrow{\mathbf{u}})$ for the simplicity.

Definition 3.1 Let

$$\overrightarrow{\mathbf{b}}_{1,i} = (b_1, b_2, \ldots, b_{i,1}, \ldots, b_k) \quad \text{and} \quad \overrightarrow{\mathbf{b}}_{2,i} = (b_1, b_2, \ldots, b_{i,2}, \ldots, b_k).$$

We say $m_{\overrightarrow{\mathbf{b}},k}$ is increasing for b_i if $m_{\overrightarrow{\mathbf{b}}_{1,i},k} < m_{\overrightarrow{\mathbf{b}}_{2,i},k}$ with $b_{i,1} < b_{i,2}$.

Proposition 3.1 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$ and $\beta < 0$. Then we have the following.*

(1) $m_{\overrightarrow{\mathbf{b}},k} = m_{\overrightarrow{\mathbf{0}},k} + O(|\overrightarrow{\mathbf{b}}|)$. *Moreover,* $\frac{\partial m_{\overrightarrow{\mathbf{b}},k}}{\partial b_i} = \frac{1}{2}\mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2} \sum_{i=1}^{k} b_i \mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2}$ *for all* $i = 1, 2, \ldots, k$ *and almost* $\overrightarrow{\mathbf{b}}$ *with* $|\overrightarrow{\mathbf{b}}| < b_{4T_k^*}$.

(2) *If* $\overrightarrow{\mathbf{b}}_n \to \overrightarrow{\mathbf{0}}$ *as* $n \to \infty$, *then* $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}_n,k} = \overrightarrow{\mathbf{u}}_k + o_n(1)$ *strongly in* $\mathcal{H}_k$ *up to a subsequence. Moreover,* $\overrightarrow{\mathbf{u}}_k$ *is a solution of* $(\mathcal{S}_{\overrightarrow{\mathbf{0}},\beta,k})$ *with*

$$\mathcal{J}_{\overrightarrow{\mathbf{0}},k}(\overrightarrow{\mathbf{u}}_k) = m_{\overrightarrow{\mathbf{0}},k}.$$

Proof (1) By (2.30) and (2.31), it is easy to see that $m_{\overrightarrow{\mathbf{b}},k} = m_{\overrightarrow{\mathbf{0}},k} + O(|\overrightarrow{\mathbf{b}}|)$. Since Lemma 2.3 holds, by a similar argument as used in the proof of [21, Lemma 5.1], we can see that $m_{\overrightarrow{\mathbf{b}},k}$ is increasing for all b_i. Thanks to the Lebesgue lemma, we can see that $\frac{\partial m_{\overrightarrow{\mathbf{b}},k}}{\partial b_i}$ exists for all $i = 1, 2, \ldots, k$ and almost $\overrightarrow{\mathbf{b}}$ with $|\overrightarrow{\mathbf{b}}| < b_{4T_k^*}$. Consider the following system in $(\mathbb{R}^+)^k$

$$\overrightarrow{\mathbf{M}}_{**}(\overrightarrow{\mu}, \overrightarrow{\mathbf{t}}) = \overrightarrow{\mathbf{0}},$$

where $\overrightarrow{\mathbf{M}}_{**}(\overrightarrow{\mu}, \overrightarrow{\mathbf{t}}) = (M_{i,**}(\overrightarrow{\mu}, \overrightarrow{\mathbf{t}}))_{i=1,2,\ldots,k}$ with

$$M_{i,**}(\overrightarrow{\mu}, \overrightarrow{\mathbf{t}}) = \mathcal{Q}^i_{4T^*_k, \overrightarrow{\mu}}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},k})\mathcal{B}^2_{\nabla t_i u_i^{\overrightarrow{\mathbf{b}},k},2} - \lambda \mathcal{B}^2_{t_i u_i^{\overrightarrow{\mathbf{b}},k},2} - \mathcal{B}^{2^*}_{t_i u_i^{\overrightarrow{\mathbf{b}},k},2^*}$$
$$-\beta \sum_{j=1, j\neq i}^{k} \mathcal{B}_{|t_i u_i^{\overrightarrow{\mathbf{b}},k}|^{\frac{2^*}{2}} |t_j u_j^{\overrightarrow{\mathbf{b}},k}|^{\frac{2^*}{2}},1}.$$

Clearly, $\overrightarrow{\mathbf{M}}_{**}(\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}) = \overrightarrow{\mathbf{0}}$. It follows from the construction of $\chi(s)$ that

$$\frac{\partial M_{i,**}((\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}))}{\partial t_i} = \frac{\beta 2^*}{2} \sum_{j=1, i\neq j}^{k} \mathcal{B}_{|u_i^{\overrightarrow{\mathbf{b}},k}|^{\frac{2^*}{2}} |u_j^{\overrightarrow{\mathbf{b}},k}|^{\frac{2^*}{2}},1}$$
$$+(2-2^*)\|u_i^{\overrightarrow{\mathbf{b}},k}\|_i^2 + O(|\overrightarrow{\mathbf{b}}|)$$

and

$$\frac{\partial M_{i,**}((\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}))}{\partial t_j} = -\frac{\beta 2^*}{2} \sum_{j=1, i\neq j}^{k} \mathcal{B}_{|u_i^{\overrightarrow{\mathbf{b}},k}|^{\frac{2^*}{2}} |u_j^{\overrightarrow{\mathbf{b}},k}|^{\frac{2^*}{2}},1} + O(|\overrightarrow{\mathbf{b}}|)$$

for all $i, j = 1, 2 \ldots, k$ and $i \neq j$, where $O(|\overrightarrow{\mathbf{b}}|)$ is only dependent on T^*_k. By $\beta < 0$, we can see that $\frac{\partial M_{i,**}((\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}))}{\partial t_j} > 0$ and $\sum_{j=1}^{k} \frac{\partial M_{i,**}((\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}))}{\partial t_j} < 0$ for all $i, j = 1, 2 \ldots, k$ and $i \neq j$ if $|\overrightarrow{\mathbf{b}}|$ small enough. Thus, by choosing $b_{4T^*_k}$ small enough if necessary, we can see that the matrix

$$M_{**}(\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}) = \left(\frac{\partial M_{i,**}((\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}}))}{\partial t_j}\right)_{i,j=1,2,\ldots,k}$$

is nonsingular for $|\overrightarrow{\mathbf{b}}| < b_{4T^*_k}$. Therefore, there exists $\overrightarrow{\mathbf{t}}(\overrightarrow{\mu}) \in C^1$ such that $(\overrightarrow{\mu}, \overrightarrow{\mathbf{t}}(\overrightarrow{\mu}))$ near $(\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{1}})$ with $\overrightarrow{\mathbf{t}}(\overrightarrow{\mathbf{b}}) = 1$ and $\overrightarrow{\mathbf{M}}_{**}(\overrightarrow{\mu}, \overrightarrow{\mathbf{t}}(\overrightarrow{\mu})) = \overrightarrow{\mathbf{0}}$ for $|\overrightarrow{\mathbf{b}}| < b_{T^*_k}$. Now, for every $i = 1, 2, \ldots, k$, we set $\overrightarrow{\mathbf{b}}_{i,*} = (b_1, b_2, \ldots, b_{i,*}, \ldots, b_k)$ such that $b_{i,*}$ close to b_i. Then by the definition of $m_{\overrightarrow{\mathbf{b}},k}$ and the Taylor expansion, we have

$$m_{\overrightarrow{\mathbf{b}}_{i,*},k} \leq \mathcal{J}_{\overrightarrow{\mathbf{b}}_{i,*},k}\left(\overrightarrow{\mathbf{t}}(\overrightarrow{\mathbf{b}}_{i,*}) \circ \overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},k}\right)$$
$$= \mathcal{J}_{\overrightarrow{\mathbf{b}},k}\left(\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},k}\right) + \frac{b_{i,*} - b_i}{2}\mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2} \sum_{i=1}^{k} b_i \mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2}$$
$$+ \mathcal{J}'_{\overrightarrow{\mathbf{b}},k}\left(\overrightarrow{\mathbf{u}}\,\overrightarrow{(\mathbf{b}), k}\right)\left(\overrightarrow{\mathbf{t}}'_{\overrightarrow{\mathbf{b}}} \circ \overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},k}\right)(b_{i,*} - b_i) + o(b_{i,*} - b_i).$$

Here, we use the fact that $\sum_{i=1}^{k} \mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2} \leq (2T^*_k)^2$ due to Proposition 2.1. Thus, by the fact that $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},k} \in \mathcal{W}_{\overrightarrow{\mathbf{b}},k}$, we can see that

$$\frac{\partial m_{\overrightarrow{\mathbf{b}},k}}{\partial b_i} = \frac{1}{2}\mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2} \sum_{i=1}^{k} b_i \mathcal{B}^2_{\nabla u_i^{\overrightarrow{\mathbf{b}},k},2}$$

for all $i = 1, 2, \ldots, k$ if $\frac{\partial m_{\overrightarrow{\mathbf{b}},k}}{\partial b_i}$ exists.

(2) Without loss of generality, we always assume $|\vec{\mathbf{b}}_n| < b_{4T_k^*}$. By (2.33), we can see from Proposition 2.1 and the construction of $\chi(s)$ that

$$\begin{aligned}\sum_{i=1}^{k}\frac{1}{N}(a_i\mathcal{S})^{\frac{N}{2}} &> \mathcal{J}_{\vec{\mathbf{b}}_n,k}\left(\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k}\right) - \frac{1}{2^*}\mathcal{J}'_{\vec{\mathbf{b}}_n,k}\left(\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k}\right)\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k}\\ &\geq \frac{1}{N}\sum_{i=1}^{k}\|u_i^{\vec{\mathbf{b}}_n,k}\|_i^2 + o_n(1)\\ &\geq \frac{1}{N}\sum_{i=1}^{k}\frac{a_i\sigma_1-\lambda_i}{\sigma_1}\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}}_n,k},2} + o_n(1).\end{aligned}$$

Thus, $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k}\}$ is bounded in $\mathcal{H}_k$. By the Sobolev embedding theorem and without loss of generality, we may assume that $\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k} \rightharpoonup \vec{\mathbf{u}}_k$ weakly in $\mathcal{H}_k \cap \mathcal{L}^{2^*}(\Omega)$ as $n \to \infty$ and $\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k} = \vec{\mathbf{u}}_k + o_n(1)$ strongly in $\mathcal{L}^r(\Omega)$ with $1 \leq r < 2^*$. Since $|\vec{\mathbf{b}}_n| = o_n(1)$, by Proposition 2.1 and (1), we can see that

$$\mathcal{J}_{\vec{\mathbf{0}},k}\left(\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k}\right) = m_{\vec{\mathbf{0}},k} + o_n(1) \quad \text{and} \quad \mathcal{J}'_{\vec{\mathbf{0}},k}\left(\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k}\right) = o_n(1) \quad \text{in } \mathcal{H}_k^{-1}.$$

Since Lemma 2.6 holds for $\vec{\mathbf{b}} = \vec{\mathbf{0}}$, by a similar argument as used in the proof of Proposition 2.1, we can show that $\vec{\mathbf{u}}_{\vec{\mathbf{b}}_n,k} = \vec{\mathbf{u}}_k + o_n(1)$ strongly in $\mathcal{H}_k$ up to a subsequence. It follows that $\vec{\mathbf{u}}_k$ is a solution of $(\mathcal{S}_{\vec{\mathbf{0}},\beta,k})$ and

$$\mathcal{J}_{\vec{\mathbf{0}},k}(\vec{\mathbf{u}}_k) = m_{\vec{\mathbf{0}},k}.$$

Clearly, $u_i^k \geq 0$ for all $i = 1, 2, \ldots, k$ by Proposition 2.1. □

Proof of Theorem 1.2 The conclusions follow immediately from Proposition 3.1. □

We next consider the concentration behavior of the solutions as $\beta \to -\infty$. For this purpose, we re-denote $\vec{\mathbf{u}}_{\vec{\mathbf{b}},k}$, $m_{\vec{\mathbf{b}},k}$, $\mathcal{W}_{\vec{\mathbf{b}},k}$ and $\mathcal{J}_{\vec{\mathbf{b}},k}(\vec{\mathbf{u}})$ by $\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta}$, $m_{\vec{\mathbf{b}},k,\beta}$, $\mathcal{W}_{\vec{\mathbf{b}},k,\beta}$ and $\mathcal{J}_{\vec{\mathbf{b}},k,\beta}(\vec{\mathbf{u}})$ respectively. We also re-denote $\mathcal{J}^{i_0,*}_{4T_k^*,\vec{\mathbf{b}},k-1}(\vec{\mathbf{u}})$, $\mathcal{W}^{i_0,*}_{4T_k^*,\vec{\mathbf{b}},k-1}$ and $m^{i_0,*}_{4T_k^*,\vec{\mathbf{b}},k-1}$ by $\mathcal{J}^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta}(\vec{\mathbf{u}})$, $\mathcal{W}^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta}$ and $m^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta}$ respectively.

Define

$$\begin{aligned}\mathcal{G} = \{\mathcal{O} = \{\Omega_i^{\mathcal{O}}\}_{i=1,2,\ldots,k} \mid\ &\cup_{i=1}^k\Omega_i^{\mathcal{O}} = \Omega \text{ and } \Omega_i^{\mathcal{O}}\cap\Omega_j^{\mathcal{O}} = \emptyset\\ &\text{for all } i, j = 1, 2, \ldots, k \text{ and } i \neq j\}.\end{aligned} \tag{3.1}$$

Then $\prod_{i=1}^k(H_0^1(\Omega_i^{\mathcal{O}})\backslash\{0\}) \subset \Theta_k$ for all $\mathcal{O} = \{\Omega_i^{\mathcal{O}}\}_{i=1,2,\ldots,k} \in \mathcal{G}$, since $\Omega_i^{\mathcal{O}} \cap \Omega_j^{\mathcal{O}} = \emptyset$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$. For each $\mathcal{O} = \{\Omega_i^{\mathcal{O}}\}_{i=1,2,\ldots,k} \in \mathcal{G}$, we set $\mathcal{H}_{k,*}^{\mathcal{O}} = \prod_{i=1}^k(H_0^1(\Omega_i^{\mathcal{O}})\backslash\{0\})$.

Lemma 3.1 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, $\beta < 0$ and $|\vec{\mathbf{b}}| < b_{4T_k^*}$. Then $\mathcal{H}_{k,*}^{\mathcal{O}} \cap \mathcal{W}_{\vec{\mathbf{b}},k,\beta} \neq \emptyset$ for all $\mathcal{O} \in \mathcal{G}$.*

Proof Let $\mathcal{O} = \{\Omega_i^{\mathcal{O}}\}_{i=1,2,\ldots,k} \in \mathcal{G}$. Since $\mathcal{H}_{k,*}^{\mathcal{O}} \subset \theta_k$, the conclusion follows immediately from Lemma 2.3. □

By Lemma 3.1 and (2.5), we can see that

$$m^{\mathcal{O}}_{\vec{\mathbf{b}},k} = \inf_{\mathcal{H}^{\mathcal{O}}_{k,*}\cap\mathcal{W}_{\vec{\mathbf{b}},k,\beta}} \mathcal{J}_{\vec{\mathbf{b}},k,\beta}(\vec{\mathbf{u}})$$

is well defined for every $\mathcal{O} \in \mathcal{G}$ and $m^{\mathcal{O}}_{\vec{\mathbf{b}},k} \geq m_{\vec{\mathbf{b}},k,\beta}$ for all $\beta < 0$. Moreover, since $\Omega_i^{\mathcal{O}} \cap \Omega_j^{\mathcal{O}} = \emptyset$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$, we can also see that $m^{\mathcal{O}}_{\vec{\mathbf{b}},k}$ is independent of β.

Lemma 3.2 *Let $k \geq 2$, $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$, $\beta < 0$ and $|\vec{\mathbf{b}}| < b_{4T_k^*}$.*

(1) *If $N \geq 9$ and $\{\vec{\mathbf{u}}^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta}\}$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$ for all $i_0 = 1, 2, \ldots, k$ and β, then*

$$\sup_{\beta<0} m_{\vec{\mathbf{b}},k,\beta} < \min_{i_0=1,2,\ldots,k} \left\{ \sup_{\beta<0} m^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta} + \frac{1}{N}(a_{i_0}\mathcal{S})^{\frac{N}{2}} \right\}.$$

(2) *If $N \geq 6$ then we have*

$$\sup_{\beta<0} m_{\vec{\mathbf{b}},2,\beta} < \min_{i_0=1,2} \left\{ m^{i_0,*}_{\vec{\mathbf{b}},1} + \frac{1}{N}(a_{i_0}\mathcal{S})^{\frac{N}{2}} \right\}.$$

Proof (1) For the simplicity and clarity, we only give the proof of $i_0 = k$ since the proofs for other i_0 are similar. In this case, we have $\mathcal{J}^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta}(\vec{\mathbf{u}}) = \mathcal{J}_{\vec{\mathbf{b}},k-1,\beta}(\vec{\mathbf{u}})$, $\mathcal{W}^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta} = \mathcal{W}_{\vec{\mathbf{b}},k-1,\beta}$ and $m^{i_0,*}_{\vec{\mathbf{b}},k-1,\beta} = m_{\vec{\mathbf{b}},k-1,\beta}$. For the sake of clarity, we also divide this proof into two steps.

Step. 1 We prove the conclusion for $\vec{\mathbf{b}} = \vec{\mathbf{0}}$.

Let $x_R \in \Omega$ and $R > 0$ satisfying $\mathbb{B}_{3R}(x_R) \subset \Omega$ and $\Psi \in C_0^2(\mathbb{B}_2(0))$ satisfying $0 \leq \Psi(x) \leq 1$ and $\Psi(x) \equiv 1$ in $\mathbb{B}_1(0)$. Set $\varphi_R(x) = 1 - \Psi\left(\frac{x-x_R}{R}\right)$ for $x \in \mathbb{B}_{2R}(x_R)$ and $\varphi_R(x) \equiv 1$ for $x \in \mathbb{R}^N\backslash\mathbb{B}_{2R}(x_R)$, then

$$\varphi_R(x) = \begin{cases} 0, \ x \in \mathbb{B}_R(x_R); \\ 1, \ x \in \mathbb{R}^N\backslash\mathbb{B}_{2R}(x_R), \end{cases}$$

and $|\nabla\varphi_R(x)| \leq \frac{C}{R}$. Let

$$\vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta,R} = \left(u_i^{\vec{\mathbf{0}},k-1,\beta,R}\right)_{i=1,2,\ldots,k-1}$$

with $u_i^{\vec{\mathbf{0}},k-1,\beta,R} = u_i^{\vec{\mathbf{0}},k-1,\beta}\varphi_R$. Then it is easy to see that $\vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta} = \vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta,R} + o_R(1)$ strongly in $\mathcal{H}_{k-1}$ and $u_i^{\vec{\mathbf{0}},k-1,\beta,R}(x)V_{R,k}(x - x_R) = 0$ for all $i = 1, 2, \ldots, k-1$, where $o_R(1) \to 0$ as $R \to 0^+$ and $V_{R,k}$ is the unique positive solution of

$$\begin{cases} -a_k\Delta u = \lambda_k u + |u|^{2^*-2}u, & \text{in } \mathbb{B}_R(0), \\ u = 0, & \text{on } \partial\mathbb{B}_R(0). \end{cases} \tag{3.2}$$

Let

$$\vec{\mathbf{V}}_{R,k}(x) = (u_1^{\vec{\mathbf{0}},k-1,\beta,R}(x), \ldots, u_k - 1^{\vec{\mathbf{0}},k-1,\beta,R}(x), V_R(x - x_R)).$$

Then by $\vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta} = \vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta,R} + o_R(1)$ strongly in $\mathcal{H}_{k-1}$ and $u_i^{\vec{\mathbf{0}},k-1,\beta,R}(x)V_{R,k}(x - x_R) = 0$ for all $i = 1, 2, \ldots, k-1$, we can see from $\vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta} \in \mathcal{W}_{\vec{\mathbf{0}},k-1,\beta}$ and the fact that $V_{R,k}$ satisfies (3.2) that $\vec{\mathbf{V}}_{R,k} \in \Theta_k$ for $R > 0$ small enough. It follows from Lemma 2.3 that there exists $\vec{\mathbf{s}}_R \in (\mathbb{R}^+)^k$ with $\vec{\mathbf{s}}_R = \vec{\mathbf{1}} + o_R(1)$ such that $\vec{\mathbf{s}}_R \circ \vec{\mathbf{V}}_{R,k} \in \mathcal{W}_{\vec{\mathbf{0}},k,\beta}$. Moreover, by $u_i^{\vec{\mathbf{0}},k-1,\beta,R}(x)V_{R,k}(x - x_R) = 0$ for all $i = 1, 2, \ldots, k-1$ and the fact that $V_{R,k}$ satisfies (3.2) once more, we also have $s_k^R = 1$. Note that $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},k-1,\beta}\}$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$ for β. Thus, by a similar argument as used in [12, Lemma 5.1], we can see that

$$\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{0}},k-1,\beta,R},2} \leq \mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{0}},k-1,\beta},2} + C\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{\frac{N-2}{2}} \tag{3.3}$$

and

$$\mathcal{B}^p_{u_i^{\vec{\mathbf{0}},k-1,\beta},p} \geq \mathcal{B}^p_{u_i^{\vec{\mathbf{0}},k-1,\beta,R},p} \geq \mathcal{B}^p_{u_i^{\vec{\mathbf{0}},k-1,\beta},p} - CR^N \tag{3.4}$$

for all $2 \leq p \leq 2^*$ and $i = 1, 2, \ldots, k-1$. Let

$$s^R_{i_0(R)} = \max\left\{s_1^R, s_2^R, \ldots, s_{k-1}^R\right\}.$$

Since $\beta < 0$, by a standard argument, we can see from $\lambda_i < a_i\sigma_1$ for all $i = 1, 2, \ldots, k$ and Proposition 2.1 that $C \leq \|u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta}\|^2_{i_0(R)} \leq C_1$ for all R small enough. On the other hand, by (3.3)–(3.4) and $u_i^{\vec{\mathbf{0}},k-1,\beta,R}(x)V_{R,k}(x - x_R) = 0$ for all $i = 1, 2, \ldots, k-1$, we can see from $\beta < 0$ and $\vec{\mathbf{s}}_R \circ \vec{\mathbf{V}}_{R,k} \in \mathcal{W}_{\vec{\mathbf{0}},k,\beta}$ that

$$\begin{aligned}
&\left(s^R_{i_0(R)}\right)^2\left(\|u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta}\|^2_{i_0(R)} + C\|u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{\frac{N-2}{2}} + CR^N\right)\\
&\geq \|s^R_{i_0(R)}u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta,R}\|^2_{i_0(R)}\\
&= \mathcal{B}^{2^*}_{s^R_{i_0(R)}u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta,R},2^*} + \beta\sum_{j=1,j\neq i_0(R)}^{k-1}\mathcal{B}_{|s^R_{i_0(R)}u_{i_0}^{\vec{\mathbf{0}},k-1,\beta,R}|^{\frac{2^*}{2}}|s_j^Ru_j^{\vec{\mathbf{0}},k-1,\beta,R}|^{\frac{2^*}{2}},1}\\
&\geq \left(s^R_{i_0(R)}\right)^{2^*}\left(\mathcal{B}^{2^*}_{u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta},2^*} + \beta\sum_{j=1,j\neq i_0(R)}^{k-1}\mathcal{B}_{|u_{i_0(R)}^{\vec{\mathbf{0}},k-1,\beta}|^{\frac{2^*}{2}}|u_j^{\vec{\mathbf{0}},k-1,\beta}|^{\frac{2^*}{2}},1} - CR^N\right),
\end{aligned}$$

which together with $s^R_{i_0(R)} = 1 + o_R(1)$, implies

$$s^R_{i_0(R)} \leq \left(1 + C\max_{i=1,2,\ldots,k-1}\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{\frac{N-2}{2}} + CR^N\right)^{\frac{1}{2^*-2}}. \tag{3.5}$$

Since $\{\vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta}\}$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$ for β, by $\vec{\mathbf{u}}_{\vec{\mathbf{0}},k-1,\beta} \in \mathcal{W}_{\vec{\mathbf{0}},k-1,\beta}$, we can see that

$$\max_{i=1,2,\ldots,k-1}\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))} \leq C \quad\text{and}\quad \sup_{\beta<0} m_{\vec{\mathbf{0}},k-1,\beta} < +\infty.$$

Thus, by (3.3)–(3.5) and the sharp estimates in [12, Theorem 5.1], we can see that

$$\begin{aligned}
m_{\vec{\mathbf{0}},k,\beta} &\le \mathcal{J}_{\vec{\mathbf{0}},k,\beta}(\vec{\mathbf{s}}_R \circ \vec{\mathbf{V}}_{R,k}) \\
&= \frac{1}{N}\sum_{i=1}^{k-1}\|s_i^R u_i^{\vec{\mathbf{0}},k-1,\beta,R}\|_i^2 + \frac{1}{N}\|V_{R,k}\|_k^2 \\
&\le \left(1 + C\max_{i=1,2,\dots,k-1}\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{\frac{N-2}{2}} + CR^N\right)^{\frac{2}{2^*-2}} \\
&\quad \times \left(m_{\vec{\mathbf{0}},k-1,\beta} + C\max_{i=1,2,\dots,k-1}\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{\frac{N-2}{2}} + CR^N\right) \\
&\quad + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}} - C_1 R^{\frac{2N-4}{N-4}} \\
&\le m_{\vec{\mathbf{0}},k-1,\beta} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}} \\
&\quad + C\max_{i=1,2,\dots,k-1}\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{\frac{N-2}{2}} - C_1 R^{\frac{2N-4}{N-4}}
\end{aligned} \tag{3.6}$$

for $N \ge 5$ and $R > 0$ small enough. Note that $\frac{N-2}{2} > \frac{2N-4}{N-4}$ if $N \ge 9$, therefore, by (3.6), we have

$$\sup_{\beta<0} m_{\vec{\mathbf{b}},k,\beta} < \sup_{\beta<0} m_{\vec{\mathbf{b}},k-1,\beta} + \frac{1}{N}(a_k\mathcal{S})^{\frac{N}{2}}.$$

Step. 2 We prove the conclusion for $|\vec{\mathbf{b}}| < b_{4T_k^*}$.

Indeed, by (2.30) and (2.31), we can see that $m_{\vec{\mathbf{0}},k,\beta} \le m_{\vec{\mathbf{b}},k,\beta} \le m_{\vec{\mathbf{0}},k,\beta} + O(|\vec{\mathbf{b}}|)$ for all $k \ge 2$, where $O(|\vec{\mathbf{b}}|)$ is independent of β. Thus, the conclusion follows immediately from Step. 1.

(2) As in (1), we only give the proof for $i_0 = 2$. Since $k = 2$, it is easy to see that

$$\mathcal{J}^{2,*}_{\vec{\mathbf{b}},1,\beta}(\vec{\mathbf{u}}) = \frac{1}{2}\|u_1\|_1^2 - \frac{1}{2^*}\mathcal{B}^{2^*}_{u_1,2^*} + \frac{1}{4}\chi\left(\frac{\mathcal{B}^2_{\nabla u_1,2}}{T^2}\right)\left(b_1\mathcal{B}^2_{\nabla u_1,2}\right)^2.$$

Thus, $\mathcal{J}^{2,*}_{\vec{\mathbf{b}},1,\beta}(\vec{\mathbf{u}})$ is independent of β, which implies $\vec{\mathbf{u}}^{2,*}_{\vec{\mathbf{b}},1,\beta} = u_{b_1,1,\beta}$ and $m^{2,*}_{\vec{\mathbf{b}},1,\beta} = m_{b_1,1,\beta}$ are also independent of β. We re-denote $u_{b_1,1,\beta}$ and $m_{b_1,1,\beta}$ by $u_{b_1,1}$ and $m_{b_1,1}$ respectively. Since $u_{b_1,1}$ is of C^1 and independent of β, we can improve (3.3) to

$$\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{0}},k-1,\beta,R},2} \le \mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{0}},k-1,\beta},2} + C\|u_i^{\vec{\mathbf{0}},k-1,\beta}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{N-2}.$$

Since $u_{b_1,1} = 0$ on $\partial\Omega$, $\|u_{b_1,1}\|_{L^\infty(B_{2R}(x_R))}$ can be chosen small enough by choosing x_R close $\partial\Omega$ enough. It follows from $N - 2 \ge \frac{2N-4}{N-4}$ for $N \ge 6$ that

$$C\|u_{0,1}\|_{L^\infty(\mathbb{B}_{2R}(x_R))}R^{N-2} - C_1 R^{\frac{2N-4}{N-4}} < 0 \quad \text{for } R > 0 \text{ small enough.}$$

Now, by repeating the argument as used in (1) for $k = 2$, we can obtain the conclusion. □

Proposition 3.2 *Let $k \ge 2$ and $0 < \lambda_i < a_i\sigma_1$ for all $i = 1, 2, \dots, k$. Then for every $\beta_n \to -\infty$, we have the following.*

(1) *If $N \ge 6$ and $|\vec{\mathbf{b}}| < b_{4T_2^*}$, then we have*

(i) $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n} = \vec{\hat{\mathbf{u}}}_{\vec{\mathbf{b}},2} + o_n(1)$ *strongly in* $\mathcal{H}_2$ *up to a subsequence. Moreover,* $\widehat{u}_1^{\vec{\mathbf{b}},2}\widehat{u}_2^{\vec{\mathbf{b}},2} = 0$ *in* Ω.

(ii) $\beta_n \mathcal{B}_{|u_i^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}}|u_j^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}},1} = o_n(1)$.

(iii) $\widehat{u}_i^{\vec{\mathbf{b}},2}$ *are continuous and* $\widehat{u}_i^{\vec{\mathbf{b}},2} > 0$ *for all* $i = 1, 2$. *Moreover,* $\vec{\hat{\mathbf{u}}}_{\vec{\mathbf{b}},2}$ *is a solution of the following system*

$$\begin{cases} -\left(a_i + b_i \sum_{j=1}^{2} b_j \mathcal{B}^2_{\nabla u_j,2}\right)\Delta u_i = \lambda_i u_i + |u_i|^{2^*-2}u_i, & in\ \Omega_i, \\ u_i = 0, \quad on\ \partial\Omega_i, \quad i = 1, 2, \end{cases} \tag{$\mathcal{S}^*_{\vec{\mathbf{b}},2}$}$$

where $\Omega_i = \{x \in \Omega \mid \widehat{u}_i^{\vec{\mathbf{b}},2} > 0\}(i = 1, 2)$.

(2) *If* $N \geq 9$ *and* $|\vec{\mathbf{b}}| < b_{4T_k^*}$, *then the conclusion of* (1) *holds for all* $k \geq 2$ *with* $a_1 = a_2 = \cdots = a_k$ *and* $b_1 = b_2 = \cdots = b_k$.

Proof (1) By Lemma 3.1, we can see from a similar argument as used in (2.5) that $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\}$ is bounded in $\mathcal{H}_2$. Thus, Without loss of generality, we may assume $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n} \rightharpoonup \vec{\hat{\mathbf{u}}}_{\vec{\mathbf{b}},2}$ weakly in $\mathcal{H}_2 \cap \mathcal{L}^{2^*}(\Omega)$ as $n \to \infty$ and $\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2} = B_i + o_n(1)$ for all $i = 1, 2$. Thanks to the Sobolev embedding theorem and without loss of generality once more, we assume that $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n} = \vec{\hat{\mathbf{u}}}_{\vec{\mathbf{b}},2} + o_n(1)$ strongly in $\mathcal{L}^p(\Omega)$ for all $1 \leq p < 2^*$. Let $v_i^{\vec{\mathbf{b}},2,\beta_n} = u_i^{\vec{\mathbf{b}},2,\beta_n} - \widehat{u}_i^{\vec{\mathbf{b}},2}$. Then by [12, Lemma 3.3], we can see that

$$\begin{aligned} \mathcal{B}_{|u_1^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}}|u_2^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}},1} &= \mathcal{B}_{|v_1^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}}|v_2^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}},1} \\ &\quad + \mathcal{B}_{|\widehat{u}_1^{\vec{\mathbf{b}},2}|^{\frac{2^*}{2}}|\widehat{u}_2^{\vec{\mathbf{b}},2}|^{\frac{2^*}{2}},1} + o_n(1). \end{aligned} \tag{3.7}$$

On the other hand, since $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\}$ is bounded in $\mathcal{H}_2$, by $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n} \in \mathcal{W}_{\vec{\mathbf{b}},2,\beta_n}$, we can see from the Sobolev embedding theorem and $\beta_n \to -\infty$ that

$$\mathcal{B}_{|u_1^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}}|u_2^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}},1} = o_n(1).$$

Thus, by (3.7), we have $\mathcal{B}_{|\widehat{u}_1^{\vec{\mathbf{b}},2}|^{\frac{2^*}{2}}|\widehat{u}_2^{\vec{\mathbf{b}},2}|^{\frac{2^*}{2}},1} = 0$ and $\widehat{u}_1^{\vec{\mathbf{b}},2}\widehat{u}_2^{\vec{\mathbf{b}},2} = 0$ in Ω.

Case 1. $\vec{\hat{\mathbf{u}}}_{\vec{\mathbf{b}},2} = \vec{\mathbf{0}}$.

Since $\beta_n < 0$, by $\vec{\hat{\mathbf{u}}}_{\vec{\mathbf{b}},2} = \vec{\mathbf{0}}$, we have that

$$a_i \mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2} \leq \mathcal{B}^{2^*}_{u_i^{\vec{\mathbf{b}},2,\beta_n},2^*} + o_n(1) \leq \mathcal{S}^{-\frac{2^*}{2}} \mathcal{B}^{2^*}_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2} + o_n(1)$$

for all $i = 1, 2$. It follows that $\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2} \geq a_i^{\frac{2}{2^*-2}} \mathcal{S}^{\frac{N}{2}} + o_n(1)$ for all $i = 1, 2$. Thus, by $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n} \in \mathcal{W}_{\vec{\mathbf{b}},2,\beta_n}$, we can see from the construction of $\chi(s)$ that

$$\begin{aligned}\mathcal{J}_{\vec{\mathbf{b}},2,\beta_n}\left(\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\right) &= \frac{1}{N}\sum_{i=1}^{2}\|u_i^{\vec{\mathbf{b}},2,\beta_n}\|_i^2 + O(|\vec{\mathbf{b}}|)\\ &= \sum_{i=1}^{2} a_i \mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2} + o_n(1) + O(|\vec{\mathbf{b}}|)\\ &\geq \frac{1}{N}\sum_{i=1}^{2}(a_i\mathcal{S})^{\frac{N}{2}} + o_n(1) + O(|\vec{\mathbf{b}}|) \qquad (3.8)\end{aligned}$$

where $O(|\vec{\mathbf{b}}|)$ is independent of n. It is well known that $m^{1,*}_{\vec{\mathbf{0}},1} < \frac{1}{N}(a_2\mathcal{S})^{\frac{N}{2}}$ and $m^{2,*}_{\vec{\mathbf{0}},1} < \frac{1}{N}(a_1\mathcal{S})^{\frac{N}{2}}$. Note that by similar arguments as used for (2.30) and (2.31), we can see that $m^{i,*}_{\vec{\mathbf{b}},1} = m^{i,*}_{\vec{\mathbf{0}},1} + O(|\vec{\mathbf{b}}|)$. Thus, we also have $m^{1,*}_{\vec{\mathbf{b}},1} < \frac{1}{N}(a_2\mathcal{S})^{\frac{N}{2}}$ and $m^{2,*}_{\vec{\mathbf{b}},1} < \frac{1}{N}(a_1\mathcal{S})^{\frac{N}{2}}$ for $|\vec{\mathbf{b}}|$ small enough. Thus, by choosing $b_{4T_2^*}$ small enough if necessary, we can see that (3.8) is impossible for $|\vec{\mathbf{b}}| < b_{4T_2^*}$ due to Lemma 3.2.

Case 2. $\widehat{\vec{\mathbf{u}}}_{\vec{\mathbf{b}},2} = (\widehat{u}_1^{\vec{\mathbf{b}},2}, 0)$ or $\widehat{\vec{\mathbf{u}}}_{\vec{\mathbf{b}},2} = (0, \widehat{u}_2^{\vec{\mathbf{b}},2})$ with $\widehat{u}_i^{\vec{\mathbf{b}},2} \neq 0$.

Without loss of generality, we assume $\widehat{\vec{\mathbf{u}}}_{\vec{\mathbf{b}},2} = (\widehat{u}_1^{\vec{\mathbf{b}},2}, 0)$ with $\widehat{u}_1^{\vec{\mathbf{b}},2} \neq 0$. Thus, $\widehat{u}_2^{\vec{\mathbf{b}},2,\beta_n} = o_n(1)$ in $L^2(\Omega)$. By a similar argument as used in Case 1, we can see that

$$\mathcal{B}^2_{\nabla\widehat{u}_2^{\vec{\mathbf{b}},2,\beta_n},2} \geq a_2^{\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}} + o_n(1). \qquad (3.9)$$

On the other hand, by Proposition 2.1, we can see that

$$\mathcal{J}'_{\vec{\mathbf{b}},2,\beta_n}\left(\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\right)\widehat{\vec{\mathbf{u}}}_{\vec{\mathbf{b}},2} = 0.$$

It follows from $\beta_n < 0$ that

$$\left(a_1 + b_1\mathcal{B}^2_{\nabla\widehat{u}_1^{\vec{\mathbf{b}},2},2}\right)\mathcal{B}^2_{\nabla\widehat{u}_1^{\vec{\mathbf{b}},2},2} \leq \lambda_1\mathcal{B}^2_{\widehat{u}_1^{\vec{\mathbf{b}},2},2} + \mathcal{B}^{2^*}_{\widehat{u}_1^{\vec{\mathbf{b}},2},2^*}.$$

By [24, Lemma 3.1], there exists $0 < \widehat{t}_1 \leq 1$ such that $\widehat{t}_1\widehat{u}_1^{\vec{\mathbf{b}},2} \in \mathcal{W}^{2,*}_{\vec{\mathbf{b}},1}$. Hence, by $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n} \in \mathcal{W}_{\vec{\mathbf{b}},2,\beta_n}$ and the construction of $\chi(s)$ once more, we have from (3.9) that

$$\begin{aligned}\mathcal{J}_{\vec{\mathbf{b}},2,\beta_n}(\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}) &= \frac{1}{N}\sum_{i=1}^{2}\|\widehat{u}_i^{\vec{\mathbf{b}},2,\beta_n}\|_i^2 + O(|\vec{\mathbf{b}}|)\\ &\geq \frac{1}{N}\|\widehat{t}_1\widehat{u}_1^{\vec{\mathbf{b}},2}\|_1^2 + \frac{1}{N}(a_2\mathcal{S})^{\frac{N}{2}} + o_n(1) + O(|\vec{\mathbf{b}}|)\\ &\geq m^{2,*}_{\vec{\mathbf{b}},1} + \frac{1}{N}(a_2\mathcal{S})^{\frac{N}{2}} + o_n(1) + O(|\vec{\mathbf{b}}|), \qquad (3.10)\end{aligned}$$

where $O(|\overrightarrow{\mathbf{b}}|)$ is independent of n. By choosing $b_{4T_2^*}$ small enough if necessary, we can see that (3.10) is impossible for $|\overrightarrow{\mathbf{b}}| < b_{4T_2^*}$ also due to Lemma 3.2.

Case 3. $\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},2} = (\widehat{u}_1^{\overrightarrow{\mathbf{b}},2}, \widehat{u}_2^{\overrightarrow{\mathbf{b}},2})$ with $\widehat{u}_i^{\overrightarrow{\mathbf{b}},2} \neq 0$ for all $i = 1, 2$.

By Proposition 2.1, we also have that

$$\mathcal{J}'_{\overrightarrow{\mathbf{b}},2,\beta_n}\left(\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},2,\beta_n}\right)\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},2} = 0.$$

Then by $u_i^{\overrightarrow{\mathbf{b}},2,\beta_n} > 0$ for all $i = 1, 2$ and $\beta_n < 0$, we can see that

$$\left(a_i + b_i \sum_{j=1}^{2} b_j \mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2}\right)\mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2} \leq \lambda_i \mathcal{B}^2_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2} + \mathcal{B}^{2^*}_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2^*} \tag{3.11}$$

for all $i = 1, 2$. Since $\widehat{u}_1^{\overrightarrow{\mathbf{b}},2}\widehat{u}_2^{\overrightarrow{\mathbf{b}},2} = 0$ in Ω, by Lemma 2.3, there exists $\overrightarrow{\widehat{\mathbf{s}}}_{\overrightarrow{\mathbf{b}}} \in (\mathbb{R}^+)^2$ such that $\overrightarrow{\widehat{\mathbf{s}}}_{\overrightarrow{\mathbf{b}}} \circ \overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},2} \in \mathcal{W}_{\overrightarrow{\mathbf{b}},2,\beta_n}$ for all n. By (3.11) and the fact that $\widehat{u}_1^{\overrightarrow{\mathbf{b}},2}\widehat{u}_2^{\overrightarrow{\mathbf{b}},2} = 0$ in Ω, we can see from Lemma 2.3 that $\widehat{s}_i^{\overrightarrow{\mathbf{b}}} \leq 1$ for all $i = 1, 2$. Moreover, by a standard argument, we also have

$$\left(\frac{a_i \mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2} - \lambda_i \mathcal{B}^2_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2}}{\mathcal{B}^{2^*}_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2^*}}\right)^{\frac{1}{2^*-2}} \leq \widehat{s}_i^{\overrightarrow{\mathbf{b}}}$$

for all $i = 1, 2$. Set

$$g_{\overrightarrow{\mathbf{b}}}(t_1, t_2) = \sum_{i=1}^{2} \frac{1}{N}\left(a_i \mathcal{B}^2_{\nabla t_i u_i^{\overrightarrow{\mathbf{b}},2},2} - \lambda_i \mathcal{B}^2_{t_i u_i^{\overrightarrow{\mathbf{b}},2},2}\right) - \frac{N-4}{4N}\left(\sum_{i=1}^{2} b_i \mathcal{B}^2_{\nabla t_i u_i^{\overrightarrow{\mathbf{b}},2},2}\right)^2$$

and denote

$$t_i^* = \left(\frac{a_i \mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2} - \lambda_i \mathcal{B}^2_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2}}{\mathcal{B}^{2^*}_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2^*}}\right)^{\frac{1}{2^*-2}}, \quad i = 1, 2.$$

Then by $\sum_{i=1}^{2} \mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},2},2} \leq (2T_2^*)^2$, it is easy to see that $g_{\overrightarrow{\mathbf{b}}}(t_1, t_2) \leq g_{\overrightarrow{\mathbf{b}}}(1, 1)$ on $[t_1^*, 1] \times [t_2^*, 1]$ for $|\overrightarrow{\mathbf{b}}|$ small enough. In particular, $g_{\overrightarrow{\mathbf{b}}}(\widehat{s}_1^{\overrightarrow{\mathbf{b}}}, \widehat{s}_2^{\overrightarrow{\mathbf{b}}}) \leq g_{\overrightarrow{\mathbf{b}}}(1, 1)$. Similarly, for the function

$$h_{\overrightarrow{\mathbf{b}}}(t_1, t_2) = \sum_{i=1}^{2} \frac{1}{N} a_i t_i - \frac{N-4}{4N}\left(\sum_{i=1}^{2} b_i t_i\right)^2,$$

we also have $h_{\overrightarrow{\mathbf{b}}}(t_1, t_2) \leq h_{\overrightarrow{\mathbf{b}}}(t_1', t_2')$ with $0 < t_i \leq t_i'$ for $|\overrightarrow{\mathbf{b}}|$ small enough. By choosing $b_{4T_2^*}$ small enough if necessary, we may assume that $g_{\overrightarrow{\mathbf{b}}}(t_1, t_2) \leq g_{\overrightarrow{\mathbf{b}}}(1, 1)$ on $[t_1^*, 1] \times [t_2^*, 1]$ and $h_{\overrightarrow{\mathbf{b}}}(t_1, t_2) \leq h_{\overrightarrow{\mathbf{b}}}(t_1', t_2')$ with $0 < t_i \leq t_i'$ for $|\overrightarrow{\mathbf{b}}| < b_{4T_2^*}$. It follows from $\sum_{i=1}^{2} \mathcal{B}^2_{\nabla u_i^{T,b,\beta_n},2} \leq (2T_2^*)^2$ and the construction of $\chi(s)$ that

$$
\begin{aligned}
m_{\vec{\mathbf{b}},2,\beta_n} &\le \sum_{i=1}^{2}\frac{1}{N}\left(a_i\mathcal{B}^2_{\nabla\widehat{u}_i^{\vec{\mathbf{b}},2},2}-\lambda_i\mathcal{B}^2_{\widehat{u}_i^{\vec{\mathbf{b}},2},2}\right)-\frac{N-4}{4N}\left(\sum_{i=1}^{2}b_i\mathcal{B}^2_{\nabla\widehat{u}_i^{\vec{\mathbf{b}},2},2}\right)^2\\
&=\frac{1}{N}\sum_{i=1}^{2}a_i\mathcal{B}^2_{\nabla\widehat{u}_i^{\vec{\mathbf{b}},2},2}-\frac{N-4}{4N}\left(\sum_{i=1}^{2}b_i\mathcal{B}^2_{\nabla\widehat{u}_i^{\vec{\mathbf{b}},2},2}\right)^2\\
&\quad-\frac{1}{N}\sum_{i=1}^{2}\lambda_i\mathcal{B}^2_{u_i^{\vec{\mathbf{b}},2,\beta_n},2}+o_n(1)\\
&\le\frac{1}{N}\sum_{i=1}^{2}\left(a_i\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2}-\lambda_i\mathcal{B}^2_{u_i^{\vec{\mathbf{b}},2,\beta_n},2}\right)-\frac{N-4}{4N}\left(\sum_{i=1}^{2}b_i\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},2,\beta_n},2}\right)^2\\
&\quad+o_n(1)\\
&=m_{\vec{\mathbf{b}},2,\beta_n}+o_n(1).
\end{aligned}\tag{3.12}
$$

Thus, we must have $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}=\vec{\mathbf{u}}_{\vec{\mathbf{b}},2}+o_n(1)$ strongly in $\mathcal{H}_2$ and $\vec{\vec{\mathbf{s}}}_{\vec{\mathbf{b}}}=\vec{\mathbf{1}}$, which together with $\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\in\mathcal{W}_{\vec{\mathbf{b}},2,\beta_n}$ implies

$$
\beta_n\mathcal{B}_{|u_i^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}}|u_j^{\vec{\mathbf{b}},2,\beta_n}|^{\frac{2^*}{2}},1}=o_n(1).
$$

It remains to show that $\vec{\widehat{\mathbf{u}}}_{\vec{\mathbf{b}},2}$ is a solution of $(\mathcal{S}^*_{\vec{\mathbf{b}},2})$ and $\widehat{u}_i^{\vec{\mathbf{b}},2}>0$ for all $i=1,2$. Indeed, by a similar argument as used in [10, Lemma 6.1] with some trivial modifications, we can see from the boundedness of $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\}$ in $\mathcal{H}_2$ that $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},2,\beta_n}\}$ is bounded in $\mathcal{L}^\infty(\Omega)$. By Proposition 5.1 and the Ascoli–Arzelá theorem, we can see that $\widehat{u}_i^{\vec{\mathbf{b}},2}$ are continuous for all $i=1,2$. Thus, by [32, Lemma 1], $\widehat{u}_i^{\vec{\mathbf{b}},2}\in H_0^1(\Omega_i)$ for all $i=1,2$. It follows that $\vec{\widehat{\mathbf{u}}}_{\vec{\mathbf{b}},2}$ is a solution of $(\mathcal{S}^*_{\vec{\mathbf{b}},2})$. By the maximum principle and $u_i^{\vec{\mathbf{b}},2,\beta_n}>0$ for all $i=1,2$, we also have $\widehat{u}_i^{\vec{\mathbf{b}},2}>0$ for all $i=1,2$.

(2) We prove this conclusion by iterating. By (1), we may assume $k\ge 3$ and the conclusion holds for $i=2,\dots,k-1$. Since Lemma 3.1 holds, $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta_n}\}$ is bounded in $\mathcal{H}_k$. Thus, Without loss of generality, we may assume $\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta_n}\rightharpoonup\vec{\widehat{\mathbf{u}}}_{\vec{\mathbf{b}},k}$ weakly in $\mathcal{H}_k\cap\mathcal{L}^{2^*}(\Omega)$ as $n\to\infty$ and $\mathcal{B}^2_{\nabla u_i^{\vec{\mathbf{b}},k,\beta_n},2}=B_i+o_n(1)$ for all $i=1,2,\dots,k$. Thanks to the Sobolev embedding theorem and without loss of generality once more, we assume that $\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta_n}=\vec{\widehat{\mathbf{u}}}_{\vec{\mathbf{b}},k}+o_n(1)$ strongly in $\mathcal{L}^p(\Omega)=(L^p(\Omega))^k$ for all $1\le p<2^*$. Let $v_i^{\vec{\mathbf{b}},k,\beta_n}=u_i^{\vec{\mathbf{b}},k,\beta_n}-\widehat{u}_i^{\vec{\mathbf{b}},k}$. Then by a similar argument as used in (1), we can see that $\widehat{u}_i^{\vec{\mathbf{b}},k}\widehat{u}_j^{\vec{\mathbf{b}},k}=0$ for all $i,j=1,2,\dots,k$ and $i\ne j$.

Case 1. $\vec{\widehat{\mathbf{u}}}_{\vec{\mathbf{b}},k}=\vec{\mathbf{0}}$.

By a similar argument as used for (3.8), we can show that

$$
m_{\vec{\mathbf{b}},k,\beta_n}\ge\frac{1}{N}\sum_{i=1}^{k}(a_i\mathcal{S})^{\frac{N}{2}}+o_n(1)+O(|\vec{\mathbf{b}}|),\tag{3.13}
$$

where $O(|\overrightarrow{\mathbf{b}}|)$ is independent of n. Now, choosing $b_{4T_k^*}$ small enough if necessary, we can see from similar arguments as used in Case 1 of (1) that (3.13) is impossible for $|\overrightarrow{\mathbf{b}}| < b_{4T_k^*}$.

Case 2. $\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},k} \neq \overrightarrow{\mathbf{0}}$.

Without loss of generality, we may assume $\widehat{u}_i^{\overrightarrow{\mathbf{b}},k} \neq 0$ for $i = 1, 2, \dots, i_0$ and $\widehat{u}_i^{\overrightarrow{\mathbf{b}},k} = 0$ for $i = i_0 + 1, \dots, k$ with some $i_0 \in \{1, 2, \dots, k\}$. We claim that $i_0 = k$. Indeed, suppose the contrary. By a similar argument as used in Case 2 of (1), we can see that

$$\mathcal{B}^2_{\nabla \widehat{u}_2^{\overrightarrow{\mathbf{b}},k,\beta_n},i} \geq a_i^{\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}} + o_n(1) \tag{3.14}$$

for all $i = i_0 + 1, \dots, k$ and

$$\left(a_i + b_i \sum_{j=1}^k b_j \mathcal{B}^2_{\nabla \widehat{u}_j^{\overrightarrow{\mathbf{b}},k},2}\right) \mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},k},2} \leq \lambda_i \mathcal{B}^2_{\widehat{u}_1^{\overrightarrow{\mathbf{b}},k},2} + \mathcal{B}^{2^*}_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},k},2^*} \tag{3.15}$$

for all $i = 1, 2, \dots, i_0$. Let

$$\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},i_0,*} = (\widehat{u}_1^{\overrightarrow{\mathbf{b}},k}, \dots, \widehat{u}_{i_0}^{\overrightarrow{\mathbf{b}},k}).$$

Since $\widehat{u}_i^{\overrightarrow{\mathbf{b}},k} \widehat{u}_j^{\overrightarrow{\mathbf{b}},k} = 0$ in Ω for all $i, j = 1, 2, \dots, i_0$ and $i \neq j$, by Lemma 2.3, there exists $\overrightarrow{\widehat{\mathbf{t}}}_{\overrightarrow{\mathbf{b}}} \in (\mathbb{R}^+)^{i_0}$ such that $\overrightarrow{\widehat{\mathbf{t}}}_{\overrightarrow{\mathbf{b}}} \circ \overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}},i_0,*} \in \mathcal{W}_{\overrightarrow{\mathbf{b}},i_0,\beta_n}$ for all n. Moreover, by (3.15), we can see that $\widehat{t}_i^{\overrightarrow{\mathbf{b}}} \leq 1$ for all $i = 1, 2, \dots, i_0$. Thanks to Proposition 2.1, by (3.14) and a similar argument as used for (3.10), we can see that

$$m_{\overrightarrow{\mathbf{b}},k,\beta_n} \geq m_{\overrightarrow{\mathbf{b}},i_0,\beta_n} + \frac{1}{N} \sum_{i=i_0+1}^k (a_i \mathcal{S})^{\frac{N}{2}} + o_n(1) + O(|\overrightarrow{\mathbf{b}}|) \tag{3.16}$$

where $O(|\overrightarrow{\mathbf{b}}|)$ is independent of n. On the other hand, since the conclusion holds for $i = 2, \dots, k-1$, by $i_0 < k$, the conclusion also holds for $i = i_0$. By a similar argument as used in [10, Lemma 6.1] with some trivial modifications, we can see from the boundedness of $\{\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},i_0,\beta_n}\}$ in $\mathcal{H}_{i_0}$ that $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}},i_0,\beta_n}$ is also bounded in $\mathcal{L}^\infty(\Omega)$. Thus, by choosing $b_{4T_k^*}$ small enough if necessary, we can see from Lemma 3.2 that (3.16) is impossible for $|\overrightarrow{\mathbf{b}}| < b_{4T_k^*}$. Hence, we must have that $i_0 = k$ and $\widehat{u}_i^{\overrightarrow{\mathbf{b}},k} \neq 0$ for all $i = 1, 2, \dots, k$. Since (3.15) holds, by a similar argument as used in (1), we can also show that

$$\left(\frac{a_i \mathcal{B}^2_{\nabla \widehat{u}_i^{\overrightarrow{\mathbf{b}},k},2} - \lambda_i \mathcal{B}^2_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},k},2}}{\mathcal{B}^{2^*}_{\widehat{u}_i^{\overrightarrow{\mathbf{b}},k},2^*}}\right)^{\frac{1}{2^*-2}} \leq \widehat{s}_i^{\overrightarrow{\mathbf{b}}} \leq 1$$

for all $i = 1, 2, \dots, k$. Consider the following functions

$$\begin{aligned} \widetilde{g}_{\overrightarrow{\mathbf{b}}}(t_1, t_2, \dots, t_k) = &\sum_{i=1}^k \frac{1}{N} \left(a_i \mathcal{B}^2_{\nabla t_i u_i^{\overrightarrow{\mathbf{b}},k},2} - \lambda_i \mathcal{B}^2_{t_i u_i^{\overrightarrow{\mathbf{b}},k},2}\right) \\ &- \frac{N-4}{4N} \left(\sum_{i=1}^k b_i \mathcal{B}^2_{\nabla t_i u_i^{\overrightarrow{\mathbf{b}},k},2}\right)^2 \end{aligned}$$

and

$$\widetilde{h}_{\vec{\mathbf{b}}}(t_1, t_2, \ldots, t_k) = \sum_{i=1}^{k} \frac{1}{N} a_i t_i - \frac{N-4}{4N} \left(\sum_{i=1}^{k} b_i t_i \right)^2.$$

Then by a similar argument as used in (1), we also have $\widetilde{g}_{\vec{\mathbf{b}}}(t_1, t_2, \ldots, t_k) \leq \widetilde{g}_{\vec{\mathbf{b}}}(1, 1, \ldots, 1)$ and $\widetilde{h}_{\vec{\mathbf{b}}}(t_1, t_2, \ldots, t_k) \leq \widetilde{h}_{\vec{\mathbf{b}}}(1, 1, \ldots, 1)$. It follows from a similar argument as used in (3.12) that $\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta_n} = \vec{\mathbf{u}}_{\vec{\mathbf{b}},k} + o_n(1)$ strongly in $\mathcal{H}_k$ and $\beta_n \sum_{i,j=1, j\neq i}^{k} \mathcal{B}_{|u_i^{\vec{\mathbf{b}},k,\beta_n}|^{\frac{2^*}{2}} |u_j^{\vec{\mathbf{b}},k,\beta_n}|^{\frac{2^*}{2}},1}$ $= o_n(1)$. By a similar argument as used in [10, Lemma 6.1] with some trivial modifications, we can see from the boundedness of $\{\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta_n}\}$ in $\mathcal{H}_k$ that $\vec{\mathbf{u}}_{\vec{\mathbf{b}},k,\beta_n}$ is bounded in $\mathcal{L}^\infty(\Omega)$. Thanks to Proposition 5.1 and the Ascoli–Arzelá theorem, we can see that $\widehat{u}_i^{\vec{\mathbf{b}},k}$ are continuous for all $i = 1, 2, \ldots, k$. Thus, by [32, Lemma 1], $\widehat{u}_i^{\vec{\mathbf{b}},k} \in H_0^1(\Omega_i)$ for all $i = 1, 2, \ldots, k$. It follows that $\widehat{\vec{\mathbf{u}}}_{\vec{\mathbf{b}},k}$ is a solution of $(\mathcal{S}^*_{\vec{\mathbf{b}},k})$. By the maximum principle and $u_i^{\vec{\mathbf{b}},k,\beta_n} > 0$ for all $i = 1, 2, \ldots, k$, we also have $\widehat{u}_i^{\vec{\mathbf{b}},k} > 0$ for all $i = 1, 2, \ldots, k$. It completes the proof. □

We close this section by

Proof of Theorem 1.3 The conclusions follow immediately from Proposition 3.2. □

4 Sign-changing solutions to $(\mathcal{P}_{a,b,\lambda})$

Let $k \geq 2$, $a, \lambda > 0$ with $\lambda < a\sigma_1$ and $b \geq 0$. Then for $\vec{\mathbf{a}}_0 = (a, a, \ldots, a)$, $\vec{\mathbf{b}}_0 = (\sqrt{b}, \sqrt{b}, \ldots, \sqrt{b})$ and $\vec{\lambda}_0 = (\lambda, \lambda, \ldots, \lambda)$, we have

$$T_k^* = \left(\frac{k\sigma_1 (a\mathcal{S})^{\frac{N}{2}}}{a\sigma_1 - \lambda} \right)^{\frac{1}{2}},$$

where T_k^* is given by (2.32). Moreover, it is also easy to see that Propositions 2.1–3.2 holds for $\vec{\mathbf{a}}_0$, $\vec{\mathbf{b}}_0$ and $\vec{\lambda}_0$ with $b < k(b_{4T_k^*})^2$. For the simplicity, we re-denote $k(b_{4T_k^*})^2$ by b_k. On the other hand, since $\vec{\mathbf{a}}_0$, $\vec{\mathbf{b}}_0$ and $\vec{\lambda}_0$ only dependent on a, b and λ respectively, we re-denote $\vec{\mathbf{a}}_0$, $\vec{\mathbf{b}}_0$ and $\vec{\lambda}_0$ by a, b, and λ in Propositions 2.1–3.2 respectively also for the simplicity.

Let

$$\mathcal{E}_{4T_k^*,b}(u) = \frac{a}{2}\mathcal{B}^2_{\nabla u,2} + \frac{b}{4}\chi\left(\frac{\mathcal{B}^2_{\nabla u,2}}{(4T_k^*)^2} \right) \mathcal{B}^4_{\nabla u,2} - \frac{\lambda}{2}\mathcal{B}^2_{u,2} - \frac{1}{2^*}\mathcal{B}^{2^*}_{u,2^*}.$$

Then by the construction of $\chi(s)$, we can see that $\mathcal{E}_{4T_k^*,b}(u)$ is of C^2 in $H_0^1(\Omega)$.

Lemma 4.1 *Let $0 < \lambda < a\sigma_1$ and $b < b_k$. Then any critical value of $\mathcal{E}_{4T_k^*,b}(u)$ must be nonnegative.*

Proof The proof is similar to that of Lemma 2.1. □

Recall $\mathcal{G}$ given by (3.1). Then it is easy to see that $\bigoplus_{i=1}^k H_0^1(\Omega_i^{\mathcal{O}}) \subset H_0^1(\Omega)$ for all $\mathcal{O} = \{\Omega_i^{\mathcal{O}}\}_{i=1,2,\ldots,k} \in \mathcal{G}$. Let

$$\mathcal{M}_{4T_k^*,b}^{\mathcal{O}} = \left\{ v \in \bigoplus_{i=1}^{k} (H_0^1(\Omega_i^{\mathcal{O}}) \backslash \{0\}) \mid \mathcal{E}'_{4T_k^*,b}\left(\sum_{j=1}^k v_j\right) v_i = 0, i = 1, 2, \ldots, k \right\}, \quad (4.1)$$

where v_i is the projection of v in $H_0^1(\Omega_i^{\mathcal{O}})$.

Lemma 4.2 *Let* $0 < \lambda < a\sigma_1$ *and* $b < b_k$. *Then for every* $\mathcal{O} \in \mathcal{G}$ *and* $v \in \bigoplus_{i=1}^k (H_0^1(\Omega_i^{\mathcal{O}}) \backslash \{0\})$, *there exists a unique* $\overrightarrow{\mathbf{t}}_0 \in (\mathbb{R}^+)^k$ *such that* $\sum_{i=1}^k t_i^0 v_i \in \mathcal{M}_{4T_k^*,b}^{\mathcal{O}}$ *and*

$$\mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^k t_i^0 v_i\right) = \max_{\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^k} \mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^k t_i v_i\right).$$

Proof Let $\mathcal{O} \in \mathcal{G}$. Since $\Omega_i \cap \Omega_j = \emptyset$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$, we have $\bigoplus_{i=1}^k (H_0^1(\Omega_i^{\mathcal{O}}) \backslash \{0\}) \subset \Theta_k$. Moreover, for every $v \in \bigoplus_{i=1}^k (H_0^1(\Omega_i^{\mathcal{O}}) \backslash \{0\})$, we set $\overrightarrow{\mathbf{v}} = (v_i)_{i=1,2,\ldots,k}$, where v_i is the projection of v in $H_0^1(\Omega_i^{\mathcal{O}})$. Then it is easy to see from the fact that $\Omega_i \cap \Omega_j = \emptyset$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$ that

$$\mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^k t_i v_i\right) = \mathcal{J}_{\overrightarrow{\mathbf{b}}_0,k,\beta}(\overrightarrow{\mathbf{t}} \circ \overrightarrow{\mathbf{v}}),$$

where $\overrightarrow{\mathbf{t}} = (t_i)_{i=1,2,\ldots,k} \in (\mathbb{R}^+)^k$. Thus, the conclusion follows immediately from Lemma 2.3. □

By Lemma 4.2, we can see that

$$c_{\mathcal{O}}^{4T_k^*,b} = \inf_{\mathcal{M}_{4T_k^*,b}^{\mathcal{O}}} \mathcal{E}_{4T_k^*,b}(v)$$

is well defined for every $\mathcal{O} \in \mathcal{G}$ and

$$\mathcal{E}_{4T_k^*,b}(v) = \max_{\overrightarrow{\mathbf{t}} \in (\mathbb{R}^+)^k} \mathcal{E}_{4T_k^*,b}\left(\sum_{j=1}^k t_j v_j\right) \quad \text{for every } v \in \mathcal{M}_{4T_k^*,b}^{\mathcal{O}}$$

in the case $b < b_k$.

Let

$$\widehat{\Omega}_{b,i}^k = \left\{x \in \Omega \mid \widehat{u}_i^{b,k} > 0\right\}, \quad i = 1, 2, \ldots, k,$$

where $\widehat{u}_i^{b,k}$ are given by Proposition 3.2. Then by Proposition 3.2, $\widehat{\Omega}_{b,i}^k \cap \widehat{\Omega}_{b,j}^k = \emptyset$ and $\widehat{\Omega}_{b,i} \neq \emptyset$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$. Set

$$c^{4T_k^*,b} = \inf_{\mathcal{O} \in \mathcal{G}} c_{\mathcal{O}}^{4T_k^*,b}. \quad (4.2)$$

Lemma 4.3 *Let* $0 < \lambda < a\sigma_1$ *and* $0 < b < b_k$. *If* $N \geq 6$ *then* $\widehat{\Omega}_{b,i}^k$ *are connected domains for all* $i = 1, 2, \ldots, k$ *and* $\cup_{i=1}^k \widehat{\Omega}_{b,i}^k = \Omega$ *in the following two cases*

(1) $k \geq 2$ *and* $N \geq 9$,

(2) $k = 2$ *and* $N \geq 6$.

Moreover, we also have

$$\mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^{k} \zeta_i \widehat{u}_i^{b,k}\right) = c^{4T_k^*,b} = \lim_{\beta \to -\infty} m_{b,k,\beta},$$

where $\overrightarrow{\zeta} = (\zeta_i)_{i=1,2,\ldots,k} \in (\mathbb{Z}_2)^k$.

Proof Clearly, $\mathcal{M}^{\mathcal{O}}_{4T_k^*,b} \subset \mathcal{W}_{b,k,\beta}$ for all $\beta < 0$ and $\mathcal{O} \in \mathcal{G}$. It follows that $c^{4T_k^*,b} \geq m_{b,k,\beta}$ for all $\beta < 0$. Let $\Omega^* = \Omega \backslash (\cup_{i=1}^{k-1} \widehat{\Omega}^k_{b,i})$. Then by Proposition 3.2, $\widehat{\Omega}^k_{b,k} \subset \Omega^*$ and $\mathcal{O}^* = \{\widehat{\Omega}^k_{b,1}, \ldots, \widehat{\Omega}^k_{b,k-1}, \Omega^*\} \in \mathcal{G}$. Thus, we must have $c^{4T_k^*,b}_{\mathcal{O}^*} \geq m_{b,k,\beta}$ for all $\beta < 0$. On the other hand, by Proposition 3.2 once more, we can see that $\sum_{i=1}^k \zeta_i \widehat{u}_i^{b,k} \in \mathcal{M}^{\mathcal{O}^*}_{4T_k^*,b}$ for all $\overrightarrow{\zeta} \in (\mathbb{Z}_2)^k$. It follows from Proposition 3.2 once more that

$$c^{4T_k^*,b}_{\mathcal{O}^*} \leq \mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^{k} \zeta_i \widehat{u}_i^{b,k}\right) = \mathcal{J}_{b,k,\beta}(\overrightarrow{\mathbf{u}}_{b,k}) = \lim_{\beta \to -\infty} m_{b,k,\beta},$$

which implies

$$c^{4T_k^*,b}_{\mathcal{O}^*} = \lim_{\beta \to -\infty} m_{b,k,\beta} = \mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^{k} \zeta_i \widehat{u}_i^{b,k}\right). \tag{4.3}$$

Hence, $c^{4T_k^*,b}_{\mathcal{O}^*}$ is attained by $\widehat{u}^{b,k} = \sum_{i=1}^k \zeta_i \widehat{u}_i^{b,k} \in \mathcal{M}^{\mathcal{O}^*}_{4T_k^*,b}$. Since $\mathcal{E}'_{4T_k^*,b}(v)v_i$ is of C^1 in $\bigoplus_{i=1}^k H_0^1(\Omega_i^{\mathcal{O}^*})$, by a similar argument as used in the proof of Lemma 2.4, we can show that $\mathcal{M}^{\mathcal{O}^*}_{4T_k^*,b}$ is a natural constraint in $\bigoplus_{i=1}^k H_0^1(\Omega_i^{\mathcal{O}^*})$. Thus, $\widehat{u}^{b,k}$ is a least energy critical point of $\mathcal{E}_{4T_k^*,b}(u)$ in $\bigoplus_{i=1}^k H_0^1(\Omega_i^{\mathcal{O}^*})$, which together with Lemma 4.1, implies $\widehat{\Omega}^k_{b,i}$ and Ω^* are connected domains for all $i = 1, 2, \ldots, k-1$ and $u_k^{b,k}$ is a ground state solution of the following equation

$$\begin{cases} -\left(a + b\sum_{j=1}^{k} \mathcal{B}^2_{\nabla \widehat{u}_j,2}\right)\Delta \widehat{u}_k = \lambda \widehat{u}_k + |\widehat{u}_k|^{2^*-2}\widehat{u}_k & \text{in } \Omega^*, \\ \widehat{u}_k = 0 & \text{on } \partial\Omega^*, \\ \sum_{j=1}^{k} \mathcal{B}^2_{\nabla \widehat{u}_j,2} & \leq (4T_k^*)^2. \end{cases}$$

By Proposition 3.2 and the maximum principle, we have $u_k^{b,k} > 0$ in Ω^*, which implies $\Omega^* = \widehat{\Omega}^k_{b,k}$. Thus, $\widehat{\Omega}^k_{b,k}$ is also a connected domain and $\cup_{i=1}^k \widehat{\Omega}^k_{b,i} = \Omega$. Let $\widehat{\mathcal{O}}^k = \{\widehat{\Omega}^k_{b,i}\}_{i=1,2,\ldots,k}$. Then by Proposition 3.2 once more and the fact that $\cup_{i=1}^k \widehat{\Omega}^k_{b,i} = \Omega$, we can see that $\widehat{\mathcal{O}}^k \in \mathcal{G}$. It follows from a similar argument as used in (4.3) that $\mathcal{E}_{4T_k^*,b}(\sum_{i=1}^k \zeta_i \widehat{u}_i^{b,k}) = c^{4T_k^*,b} = \lim_{\beta \to -\infty} m_{b,k,\beta}$. □

4.1 Existence results for general Ω

Let $\widetilde{u}_0^{b,2} = \widehat{u}_1^{b,k} - \widehat{u}_2^{b,k}$.

Proposition 4.1 *Let* $0 < \lambda < a\sigma_1$ *and* $0 < b < b_2$. *If* $N \geq 6$ *then* $\widetilde{u}_0^{b,2}$ *is a sign-changing solution of* $(\mathcal{P}_{a,b,\lambda})$ *with two nodal domains. Moreover,* $\widetilde{u}_0^{0,2}$ *is a least energy sign-changing solution of* $(\mathcal{P}_{a,0,\lambda})$.

Proof Let $\widehat{v}_{\overrightarrow{\mathbf{s}}} = s_1 u_1^{b,2} - s_2 u_2^{b,2}$, where $\overrightarrow{\mathbf{s}} \in (\mathbb{R}^+)^2$. Then by Lemma 4.2, we have

$$\mathcal{E}_{4T_2^*,b}(\widetilde{u}_0^{b,2}) = \mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{1}}}) = \max_{\overrightarrow{\mathbf{s}} \in (\mathbb{R}^+)^2} \mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}}) > \mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}}) \tag{4.4}$$

for all $\overrightarrow{\mathbf{s}} \in (\mathbb{R}^+)^2$ and $\overrightarrow{\mathbf{s}} \neq \overrightarrow{\mathbf{1}}$. Suppose that $\widehat{u}_0^{b,2}$ is not a solution of $(\mathcal{P}_{a,b,\lambda})$. Then by Propositions 2.1 and 3.2, there exists $\varphi \in C_0^\infty(\Omega)$ such that $\mathcal{E}'_{4T_2^*,b}(\widehat{u}_0^{b,2})\varphi \leq -1$. It follows from the fact that $\mathcal{E}_{4T_2^*,b}(u)$ is of C^2 that there exists $0 < \varepsilon_0 < \frac{1}{10}$ such that

$$\mathcal{E}'_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon\varphi)\varphi \leq -\frac{1}{2} \tag{4.5}$$

for all $\varepsilon < \varepsilon_0$ and $\overrightarrow{\mathbf{s}} \in \mathcal{F}_{\varepsilon_0} = [1 - \frac{\varepsilon_0}{2}, 1 + \frac{\varepsilon_0}{2}]^2$. Consider the continuous function $\phi(\overrightarrow{\mathbf{s}})$ defined on $\mathcal{F}_{\frac{1}{2}}$ which satisfies $0 \leq \phi(\overrightarrow{\mathbf{s}}) \leq 1$, $\phi(\overrightarrow{\mathbf{s}}) = 1$ on $\mathcal{F}_{\varepsilon_0}$ and $\phi(\overrightarrow{\mathbf{s}}) = 0$ on $\mathcal{F}_{\frac{1}{2}} \backslash \mathcal{F}_{2\varepsilon_0}$. Then by (4.5), we have

$$\mathcal{E}'_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi)\varphi \leq -\frac{1}{2} \quad \text{on } \mathcal{F}_{\varepsilon_0}. \tag{4.6}$$

Note that by the mean value theorem, we have

$$\mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi) = \mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}}) + \int_0^{\varepsilon_0} \mathcal{E}'_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \rho\phi(\overrightarrow{\mathbf{s}})\varphi)\phi(\overrightarrow{\mathbf{s}})\varphi d\rho. \tag{4.7}$$

Thus, for $\overrightarrow{\mathbf{s}} \in \mathcal{F}_{\varepsilon_0}$, we can see from (4.4)–(4.6) and Lemma 4.3 that

$$\mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi) \leq c^{4T_2^*,b} - \frac{\varepsilon_0}{2}.$$

For $\overrightarrow{\mathbf{s}} \in \mathcal{F}_{\frac{1}{2}} \backslash \mathcal{F}_{\varepsilon_0}$, we can also see from (4.4) and Lemma 4.3 that there exists $\delta > 0$ dependent on ε_0 such that $\mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}}) \leq c^{4T_2^*,b} - \delta$. Thus, by (4.5) and (4.7), we have

$$\mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi) \leq c^{4T_2^*,b} - \delta.$$

In a word, we always have that

$$\sup_{\overrightarrow{\mathbf{s}} \in \mathcal{F}_{\frac{1}{2}}} \mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi) < c^{4T_2^*,b}. \tag{4.8}$$

Let

$$\Omega^+_{\varepsilon_0, \overrightarrow{\mathbf{s}}} = \{x \in \Omega \mid \widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi \geq 0\}$$

and $\Omega^-_{\varepsilon_0, \overrightarrow{\mathbf{s}}} = \Omega \backslash \Omega^+_{\varepsilon_0, \overrightarrow{\mathbf{s}}}$. Then $\mathcal{O}^{\varepsilon_0, \overrightarrow{\mathbf{s}}} = \{\Omega^+_{\varepsilon_0, \overrightarrow{\mathbf{s}}}, \Omega^-_{\varepsilon_0, \overrightarrow{\mathbf{s}}}\} \in \mathcal{G}$ for $k = 2$. Consider the maps $h_{\varepsilon_0} : \mathcal{F}_{\frac{1}{2}} \to H_0^1(\Omega^+_{\varepsilon_0, \overrightarrow{\mathbf{s}}}) \bigoplus H_0^1(\Omega^-_{\varepsilon_0, \overrightarrow{\mathbf{s}}})$ and $H_{\varepsilon_0} : \mathcal{F}_{\frac{1}{2}} \to (\mathbb{R})^2$ respectively given by

$$h_{\varepsilon_0}(\overrightarrow{\mathbf{s}}) = \widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi$$

and

$$H_{\varepsilon_0}(\overrightarrow{\mathbf{s}}) = (\mathcal{E}'_{4T_2^*,b}(h_{\varepsilon_0}(\overrightarrow{\mathbf{s}}))h_{\varepsilon_0}(\overrightarrow{\mathbf{s}})_+, \mathcal{E}'_{4T_2^*,b}(h_{\varepsilon_0}(\overrightarrow{\mathbf{s}}))h_{\varepsilon_0}(\overrightarrow{\mathbf{s}})_-)$$

with $h_{\varepsilon_0}(\overrightarrow{\mathbf{s}})_\pm = h_{\varepsilon_0}(\overrightarrow{\mathbf{s}})\chi_{\Omega^\pm_{\varepsilon_0,\overrightarrow{\mathbf{s}}}}$. Here, $\chi_{\Omega^\pm_{\varepsilon_0,\overrightarrow{\mathbf{s}}}}$ are the characteristic functions of $\Omega^\pm_{\varepsilon_0,\overrightarrow{\mathbf{s}}}$. Since $\phi(\overrightarrow{\mathbf{s}}) = 0$ on $\partial\mathcal{F}_{\frac{1}{2}}$, we have $H_{\varepsilon_0}(\overrightarrow{\mathbf{s}}) = H_0(\overrightarrow{\mathbf{s}})$ on $\partial\mathcal{F}_{\frac{1}{2}}$. It follows from Lemma 4.2 that

$$1 = \deg(H_0, \mathcal{F}_{\frac{1}{2}}, \overrightarrow{\mathbf{0}}) = \deg(H_{\varepsilon_0}, \mathcal{F}_{\frac{1}{2}}, \overrightarrow{\mathbf{0}}).$$

Thus by (4.2), we must have

$$\sup_{\overrightarrow{\mathbf{s}}\in\mathcal{F}_{\frac{1}{2}}} \mathcal{E}_{4T_2^*,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}} + \varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi) \geq c^{4T_2,b},$$

which contradicts to (4.8). Therefore, $\widetilde{u}_0^{b,2}$ is a sign-changing solution of $(\mathcal{P}_{a,b,\lambda})$ with two nodal domains. Thanks to Lemma 4.3, it is also easy to see that $\widetilde{u}_0^{0,2}$ is a least energy sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$. □

4.2 Existence results for $\boldsymbol{\Omega = \mathbb{B}_R}$

Let

$$H^1_{0,R}(\mathbb{B}_R) = \left\{u \in H^1_0(\mathbb{B}_R) \mid u \text{ is radially symmetric}\right\}$$

and set $\mathcal{H}_{i,R}$ and $\widetilde{\mathcal{H}}_{i,R}$ be the Hilbert spaces of $H^1_0(\mathbb{B}_R)$ and $H^1_{0,R}(\mathbb{B}_R)$ with the inner product $\langle\cdot,\cdot\rangle_i$ for $i = 1, 2, \ldots, k$ respectively. Since $\Omega = \mathbb{B}_R$ is radially symmetric, by the symmetric criticality principle of Palais, the critical points of $\mathcal{J}_{\overrightarrow{\mathbf{b}},k,\beta}(\overrightarrow{\mathbf{u}})$ and $\mathcal{E}_{4T_k^*,b}(u)$ respectively in $\widetilde{\mathcal{H}}_{k,R} = \prod_1^k \widetilde{\mathcal{H}}_{i,R}$ and $H^1_{0,R}(\mathbb{B}_R)$ are also critical points of $\mathcal{J}_{\overrightarrow{\mathbf{b}},k,\beta}(\overrightarrow{\mathbf{u}})$ and $\mathcal{E}_{4T_k^*,b}(u)$ respectively in $\mathcal{H}_{k,R} = \prod_1^k \mathcal{H}_{i,R}$ and $H^1_0(\mathbb{B}_R)$. Thus, we consider the functionals $\mathcal{J}_{\overrightarrow{\mathbf{b}},k,\beta}(\overrightarrow{\mathbf{u}})$ and $\mathcal{E}_{4T_k^*,b}(u)$ respectively in $\widetilde{\mathcal{H}}_{k,R} = \prod_1^k \widetilde{\mathcal{H}}_{i,R}$ and $H^1_{0,R}(\mathbb{B}_R)$ in this section.

Remark 4.1 In the radial symmetric case, the ball $\mathbb{B}_R(x_R)$ in Lemma 3.2 must be centered at 0. Thus, $\max_{i=1,2,\ldots,k-1}\|u_i^{T,\overrightarrow{\mathbf{0}},k-1}\|_{L^\infty(\mathbb{B}_{3\sqrt{a_k}R}(\sqrt{a_k}x_R))}$ and $\|u_{b_1,1}\|_{L^\infty(\mathbb{B}_{3R}(x_R))}$ can not choose to be small enough. It follows that 2 of Lemma 3.2 and 1 of Proposition 2.1 only holds for $N \geq 7$ in the radial symmetric case. Thus, in the radial symmetric case Proposition 4.1 only holds for $N \geq 7$.

Since we consider the functional $\mathcal{J}_{\overrightarrow{\mathbf{b}}_0,k,\beta}(\overrightarrow{\mathbf{u}})$ in $\widetilde{\mathcal{H}}_{k,R} = \prod_1^k \widetilde{\mathcal{H}}_{i,R}$. The solution $\overrightarrow{\mathbf{u}}_{\overrightarrow{\mathbf{b}}_0,k,\beta}$ of the system $(\mathcal{S}_{\overrightarrow{\mathbf{b}}_0,\beta,k})$ founded in Proposition 2.1 is radial symmetric, that is, $u_i^{b,k,\beta} \in \widetilde{\mathcal{H}}_{i,R}$ for all $i = 1, 2, \ldots, k$. This together with Proposition 3.2 and Remark 4.1, implies that $\overrightarrow{\widehat{\mathbf{u}}}_{\overrightarrow{\mathbf{b}}_0,k}$ is also radial symmetric for $N \geq 9$, that is, $\widehat{u}_i^{b,k} \in \widetilde{\mathcal{H}}_{i,R}$ for all $i = 1, 2, \ldots, k$.
Define

$$\mathcal{R}_k = \left\{\overrightarrow{\mathbf{r}} = (R_i)_{i=1,2,\ldots,k} \in (\mathbb{R}^+)^k \mid 0 < R_1 < R_2 < \ldots < R_k = R\right\}.$$

Set $R_0 = 0$. Then for every $\overrightarrow{\mathbf{r}} \in \mathcal{R}_k$, we can see that

$$\mathcal{O}_{\overrightarrow{\mathbf{r}}} = \{\mathbb{B}_{R_i}\backslash\mathbb{B}_{R_{i-1}}\}_{i=1,2,\ldots,k} \in \mathcal{G}.$$

Define $\mathcal{H}_{\overrightarrow{\mathbf{r}},k} = \bigoplus_{i=1}^{k} \mathcal{H}^{i}_{\overrightarrow{\mathbf{r}},k}$, where $\mathcal{H}^{i}_{\overrightarrow{\mathbf{r}},k} = H^{1}_{0,R}(\mathbb{B}_{R_i}\backslash\mathbb{B}_{R_{i-1}})$. Then by Lemma 4.2, $\mathcal{M}^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b} \neq \emptyset$ for $b < b_k$, where $\mathcal{M}^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b} = \mathcal{M}^{\mathcal{O}_{\overrightarrow{\mathbf{r}}}}_{4T_k^*,b}$ is given by (4.1). Set

$$\gamma^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b} = \inf_{u\in\mathcal{M}^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b}} \mathcal{E}_{4T_k^*,b}(u).$$

Then by Lemma 4.2 once more, we can see that $\gamma^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b} \geq 0$ for $0 < b < b_k$. Define

$$\gamma_{4T_k^*,b} = \inf_{\overrightarrow{\mathbf{r}}\in\mathcal{R}_k} \gamma^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b}.$$

Lemma 4.4 *Let $0 < \lambda < a\sigma_1$, $k \geq 2$ and $0 < b < b_k$. If $N \geq 7$ and $\Omega = \mathbb{B}_R$ then $\widehat{\Omega}^k_{b,i}$ are annuli for all $i = 1, 2, \ldots, k$ and $\cup_{i=1}^{k}\widehat{\Omega}^k_{b,i} = \mathbb{B}_R$. Moreover, we also have*

$$\mathcal{E}_{4T_k^*,b}\left(\sum_{i=1}^{k}\zeta_i\widehat{u}_i^{b,k}\right) = \gamma_{4T_k^*,b} = \lim_{\beta\to-\infty} m_{b,k,\beta},$$

where $\overrightarrow{\zeta} = (\zeta_i)_{i=1,2,\ldots,k} \in (\mathbb{Z}_2)^k$.

Proof Since we consider the functional $\mathcal{J}_{\overrightarrow{\mathbf{b}}_0,k,\beta}(\overrightarrow{\mathbf{u}})$ in $\widetilde{\mathcal{H}}_{k,R} = \prod_1^k \widetilde{\mathcal{H}}_{i,R}$, the proof is similar to that of Lemma 4.3. □

By Lemma 4.4, without loss of generality, we may assume that there exists $\overrightarrow{\mathbf{r}}_0 \in \mathcal{R}_k$ such that $\widehat{\Omega}^k_{b,i} = \mathbb{B}_{R_i^0}\backslash\mathbb{B}_{R_{i-1}^0}$ for all $i = 1, 2, \ldots, k$. Then by Proposition 3.2, we can see that $\widetilde{u}^{b,k}_{\overrightarrow{\zeta}} = \sum_{i=1}^{k}\zeta_i\widehat{u}_i^{b,k} \in \mathcal{M}^{\overrightarrow{\mathbf{r}}_0}_{4T_k^*,b}$ for all $\overrightarrow{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_k) \in (\mathbb{Z}_2)^k$. Let

$$\omega_{k,b} = \inf_{u\in\mathcal{M}^{\overrightarrow{\mathbf{r}}_0}_{4T_k^*,b}} \mathcal{E}_{4T_k^*,b}(u).$$

Then by Lemma 4.4 once more, it is easy to see that $\omega_{k,b} = \gamma_{4T_k^*,b}$.

Lemma 4.5 *Let $0 < \lambda < a\sigma_1$, $k \geq 2$ and $0 < b < b_k$. If $N \geq 7$ and $\Omega = \mathbb{B}_R$ then $\omega_{k,b}$ is nondecreasing for k.*

Proof Take $\overrightarrow{\mathbf{r}} \in \mathcal{R}_k$ and set $R'_i = R_{i+1}$ for $i = 1, 2, \ldots, k-1$. Then $\overrightarrow{\mathbf{r}}' = (R'_1, \ldots, R'_{k-1}) \in \mathcal{R}_{k-1}$. Moreover, we also have $\bigoplus_{i=1}^{2}(H^1_{0,R}(\mathbb{B}_{R_i}\backslash\mathbb{B}_{R_{i-1}})\backslash\{0\}) \subset (H^1_0(\mathbb{B}_{R'_1}\backslash\mathbb{B}_{R_0})\backslash\{0\})$. It follows that $\mathcal{M}^{\overrightarrow{\mathbf{r}}}_{4T_k^*,b} \subset \mathcal{M}^{\overrightarrow{\mathbf{r}}'}_{4T_{k-1}^*,b}$, which together with $\omega_{k,b} = \gamma_{4T_k^*,b}$, implies $\omega_{k,b} \geq \omega_{k-1,b}$. □

Proposition 4.2 *Let $0 < \lambda < a\sigma_1$, $k \geq 2$ and $0 < b < b_k$. If $N \geq 9$ and $\Omega = \mathbb{B}_R$ then $\widetilde{u}^{b,k}_{\overrightarrow{\zeta}_k}$ is a radial sign-changing solution of $(\mathcal{P}_{a,b,\lambda})$ which changes sign exactly k times, where $\overrightarrow{\zeta}_k = ((-1)^{i-1})_{i=1,2,\ldots,k}$. Moreover, $\widetilde{u}^{0,k}_{\overrightarrow{\zeta}_k}$ is a least energy radial sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$ which changes sign exactly k times.*

Proof Suppose that $\widetilde{u}^{b,k}_{\overrightarrow{\zeta}_k}$ is not a solution of $(\mathcal{P}_{a,b,\lambda})$. Then by Propositions 2.1 and 3.2 and the symmetric criticality principle of Palais, there exists $\varphi \in C^\infty_{0,R}(\mathbb{B}_R)$ such that $\mathcal{E}'_{T,b}(\widetilde{u}^{b,k}_{\overrightarrow{\zeta}_k})\varphi \leq -1$, where

$$C^\infty_{0,R}(\mathbb{B}_R) = \{u \in C^\infty_0(\mathbb{B})_R \mid u \text{ is radial symmetric}\}.$$

Let $\widehat{v}_{\overrightarrow{\mathbf{s}}}=\sum_{i=1}^{k}s_i(-1)^{i-1}\widehat{u}_i^{b,k}$, where $\overrightarrow{\mathbf{s}}\in(\mathbb{R}^+)^k$. Then by Lemma 4.4 and a similar argument as used in the proof of Proposition 4.1, we can see that $\mathcal{E}_{T,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi)<\omega_{k,b}$ on $\mathcal{F}^k_{\frac{1}{2}}$, where $\varepsilon_0<\frac{1}{10}$, $\mathcal{F}^k_{\varepsilon_0}=[1-\frac{\varepsilon_0}{2},1+\frac{\varepsilon_0}{2}]^k$ and $\phi(\overrightarrow{\mathbf{s}})$ is a continuous function defined on $\mathcal{F}_{\frac{1}{2}}$ with $0\leq\phi(\overrightarrow{\mathbf{s}})\leq 1$, $\phi(\overrightarrow{\mathbf{s}})=1$ on $\mathcal{F}^k_{\varepsilon_0}$ and $\phi(\overrightarrow{\mathbf{s}})=0$ on $\mathcal{F}^k_{\frac{1}{2}}\backslash\mathcal{F}^k_{2\varepsilon_0}$. On the other hand, by the definition of $\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi$, we can see that $\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi\to\widehat{v}_{\overrightarrow{\mathbf{s}}}$ uniformly in $\mathbb{B}_R$ as $\varepsilon_0\to 0$. Since $\widehat{u}_i^{b,k}$ are all radial symmetric, we can see that $\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi$ has at least k nodal domains for ε_0 small enough. Now, let $\{\Omega_i^\varepsilon\}_{i=1,2,\ldots,l}$ be the nodal domains of $\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon\phi(\overrightarrow{\mathbf{s}})\varphi$, then $l\geq k$ for $\varepsilon<\varepsilon_0$ small enough. It follows that there exists $\overrightarrow{\mathbf{r}}_\varepsilon\in\mathcal{R}_l$ such that $\Omega_i^\varepsilon=\mathbb{B}_{R_i^\varepsilon}\backslash\mathbb{B}_{R_{i-1}^\varepsilon}$ for all $i=1,2,\ldots,k$. Consider the maps $h_\varepsilon:\mathcal{F}_{\frac{1}{2}}\to\bigoplus_{i=1}^{l}H^1_{0,R}(\mathbb{B}_{R_i^\varepsilon}\backslash\mathbb{B}_{R_{i-1}^\varepsilon})$ and $H_\varepsilon:\mathcal{F}_{\frac{1}{2}}\to(\mathbb{R})^l$ respectively given by

$$h_\varepsilon(\overrightarrow{\mathbf{s}})=\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon\phi(\overrightarrow{\mathbf{s}})\varphi\quad\text{and}\quad H_\varepsilon(\overrightarrow{\mathbf{s}})=(\mathcal{E}'_{4T_k^*,b}(h_\varepsilon(\overrightarrow{\mathbf{s}}))h_\varepsilon(\overrightarrow{\mathbf{s}})_i)_{i=1,2,\ldots,l}.$$

Since $\phi(\overrightarrow{\mathbf{s}})=0$ on $\partial\mathcal{F}_{\frac{1}{2}}$, we have $H_\varepsilon(\overrightarrow{\mathbf{s}})=H_0(\overrightarrow{\mathbf{s}})$ on $\partial\mathcal{F}_{\frac{1}{2}}$. It follows from Lemma 4.4 that

$$1=\deg\left(H_0,\mathcal{F}_{\frac{1}{2}},\overrightarrow{\mathbf{0}}\right)=\deg\left(H_\varepsilon,\mathcal{F}_{\frac{1}{2}},\overrightarrow{\mathbf{0}}\right).$$

Thus by Lemma 4.5, we must have $\mathcal{E}_{T,b}(\widehat{v}_{\overrightarrow{\mathbf{s}}}+\varepsilon_0\phi(\overrightarrow{\mathbf{s}})\varphi)\geq\omega_{k,b}$ for some $\overrightarrow{\mathbf{s}}\in\mathcal{F}_{\frac{1}{2}}$, which is a contradiction. Hence, $\widetilde{u}^{b,k}_{\overrightarrow{\zeta}_k}$ is a radial sign-changing solution of $(\mathcal{P}_{a,b,\lambda})$ which changes sign exactly k times for $N\geq 9$ and $\Omega=\mathbb{B}_R$. Moreover, thanks to the fact that $\omega_{k,b}=\gamma_{4T_k^*,b}$, it is easy to see that $\widetilde{u}^{0,k}_{\overrightarrow{\zeta}_k}$ is a least energy radial sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$ which changes sign exactly k times. □

We close this section by

Proof of Theorem 1.4 The conclusions follow immediately from Propositions 4.1 and 4.2 and Remark 4.1. □

4.3 The concentration behaviors of $b\to 0^+$

We re-denote $c^{4T_k^*,b}$ and $\omega_{k,b}$ by $c_k(b)$ and $\omega_k(b)$ respectively and consider them as functions of b. For the sake of clarity, we denote $\int_{\mathbb{B}_R}|u|^p dx=\mathcal{B}^p_{u,p,R}$.

Lemma 4.6 *Let* $0<\lambda<a\sigma_1$.

(1) *If* $N\geq 6$, *then* $c_2'(b)=\frac{1}{4}\mathcal{B}^4_{\nabla\widetilde{u}_0^{b,2},2}$ *for a.e.* $b\in(0,b_2)$. *Moreover,* $\frac{1}{c_2(b)}-\frac{1}{c_2(0)}=O(b)$ *as* $b\to 0^+$.

(2) *If* $N\geq 9$ *and* $\Omega=\mathbb{B}_R$, *then* $\omega_k'(b)=\frac{1}{4}\mathcal{B}^4_{\nabla\widetilde{u}^{b,k}_{\overrightarrow{\zeta}_k},2,R}$ *for all* $k\geq 2$ *and a.e.* $b\in(0,b_k)$. *Moreover,* $\frac{1}{\omega_k(b)}-\frac{1}{\omega_k(0)}=O(b)$ *as* $b\to 0^+$ *for all* $k\geq 2$.

Proof (1) Let $\beta_n\to-\infty$ as $n\to\infty$. Then by Lemma 4.3, we have that $c_2(b)=\lim_{n\to\infty}m_{b,2,\beta_n}$. It follows from $\overrightarrow{\mathbf{b}}_0=(\sqrt{b},\sqrt{b},\ldots,\sqrt{b})$ and Proposition 3.1 that

$$c_2(b)=\lim_{n\to\infty}\int_0^b\frac{dm_{\tau,2,\beta_n}}{d\tau}d\tau=\lim_{n\to\infty}\int_0^b\frac{1}{4}\left(\sum_{i=1}^{2}\mathcal{B}^2_{\nabla u_i^{\tau,2,\beta_n},2}\right)^2d\tau\tag{4.9}$$

for $b \in [0, b_2)$. By Proposition 2.1, we have that $\sum_{i=1}^{2} \mathcal{B}^2_{\nabla u_i^{\tau,2,\beta_n},2} \leq 2T_2^*$. Thus, thanks to the Lebesgue dominated convergence theorem, we can see from (4.9) and Proposition 3.2 that

$$c_2(b) = \frac{1}{4}\int_0^b \lim_{n\to\infty}\left(\sum_{i=1}^{2} \mathcal{B}^2_{\nabla u_i^{\tau,2,\beta_n},2}\right)^2 d\tau = \frac{1}{4}\int_0^b \left(\sum_{i=1}^{2} \mathcal{B}^2_{\nabla \widetilde{u}_j^{\tau,2},2}\right)^2 d\tau$$

for $N \geq 6$, which together with Proposition 4.1, implies that $c_2'(b) = \frac{1}{4}\mathcal{B}^4_{\nabla \widetilde{u}_0^{b,2},2}$ for $N \geq 6$ and a.e. $b \in (0, b_2)$. It remains to show that $\frac{1}{c_2(b)} - \frac{1}{c_2(0)} = O(b)$ as $b \to 0^+$. Since $\widetilde{u}_0^{b,2}$ is a solution of $(\mathcal{P}_{a,b,\lambda})$, we can easy to see from Propositions 2.1 and 3.2 that

$$c_2(b) = \frac{1}{N}\left(a\mathcal{B}^2_{\nabla \widetilde{u}_0^{b,2},2} - \lambda \mathcal{B}^2_{\widetilde{u}_0^{b,2},2}\right) - \frac{(N-4)b}{4N}\mathcal{B}^4_{\nabla \widetilde{u}_0^{b,2},2}. \tag{4.10}$$

By Proposition 2.1 once more, we can see that $\frac{(N-4)b}{4N}\mathcal{B}^4_{\nabla \widetilde{u}_0^{b,2},2} = O(b)$ as $b \to 0^+$. It follows from $N \geq 6$ and (4.10) that

$$\frac{a\sigma_1 - \lambda}{N\sigma_1}(c_2'(b))^{\frac{1}{2}} = \frac{a\sigma_1 - \lambda}{2N\sigma_1}\mathcal{B}^2_{\nabla \widetilde{u}_0^{b,2},2} \leq c_2(b) \leq \frac{a}{N}\mathcal{B}^2_{\nabla \widetilde{u}_0^{b,2},2} = \frac{2a}{N}(c_2'(b))^{\frac{1}{2}}$$

for b small enough. Thus, $\frac{1}{c_2(b)} - \frac{1}{c_2(0)} = O(b)$ as $b \to 0^+$.

(2) Since we consider the functional $\mathcal{J}_{\vec{\mathbf{b}}_{0,k,\beta}}(\vec{\mathbf{u}})$ in $\widetilde{\mathcal{H}}_{k,R} = \prod_1^k \widetilde{\mathcal{H}}_{i,R}$ for the case $\Omega = \mathbb{B}_R$, by Remark 4.1, the proof is similar to that of (1). □

Proposition 4.3 *Let* $0 < \lambda < a\sigma_1$.

(1) *If* $N \geq 6$*, then for every* $b_n \to 0^+$*, we have* $\widetilde{u}_0^{b_n,2} = \widetilde{u}_0^{*,2} + o_n(1)$ *strongly in* $H_0^1(\Omega)$ *up to a subsequence. Moreover,* $\widetilde{u}_0^{*,2}$ *is a least energy sign-changing solution of* $(\mathcal{P}_{a,0,\lambda})$.

(2) *If* $N \geq 9$ *and* $\Omega = \mathbb{B}_R$*, then for every* $b_n \to 0^+$*, we have* $\widetilde{u}_{\vec{\zeta}_k}^{b_n,k} = \widetilde{u}_{\vec{\zeta}_k}^{*,k} + o_n(1)$ *strongly in* $H_{0,R}^1(\mathbb{B}_R)$ *up to a subsequence. Moreover,* $\widetilde{u}_{\vec{\zeta}_k}^{*,k}$ *is a least energy radial sign-changing solution of* $(\mathcal{P}_{a,0,\lambda})$*, which changes sign exactly* k *times.*

Proof (1) By Propositions 2.1, 3.2 and 4.1, we can see that $\mathcal{B}^2_{\nabla \widetilde{u}_0^{b_n,2},2} \leq 2T_2^*$ and $\mathcal{E}'_{4T_2^*,b_n}(\widetilde{u}_0^{b_n,2}) = 0$ in $H^{-1}(\Omega)$. It follows from $b_n = 0_n(1)$ that $\mathcal{B}^2_{\nabla \widetilde{u}_0^{b_n,2},2} \leq 2T_2^*$ and $\mathcal{E}'_{4T_2^*,0}(\widetilde{u}_0^{b_n,2}) = o_n(1)$ in $H^{-1}(\Omega)$. Thanks to [6, Lemma 3.1], we can see that $\widetilde{u}_0^{b_n,2} = \widetilde{u}_0^{*,2} + o_n(1)$ strongly in $H_0^1(\Omega)$ for some $\widetilde{u}_0^{*,2} \in H_0^1(\Omega)$ up to a subsequence, which implies $\widetilde{u}_0^{*,2}$ is a solution of $(\mathcal{P}_{a,0,\lambda})$. Since $\lambda < a\sigma_1$, by a standard argument, we also have that $\widetilde{u}_0^{*,2}$ is sign-changing. This together with Lemma 4.6, implies that $\widetilde{u}_0^{*,2}$ is actually a least energy sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$.

(2) By Propositions 2.1, 3.2 and 4.2, we can see that $\mathcal{B}^2_{\nabla \widetilde{u}_{\vec{\zeta}_k}^{b_n,k},2} \leq 2T_k^*$ and $\mathcal{E}'_{4T_k^*,b_n}(\widetilde{u}_{\vec{\zeta}_k}^{b_n,k}) = 0$ in $H^{-1}(\Omega)$ for all $k \geq 2$. It follows from $b_n = o_n(1)$ that $\mathcal{B}^2_{\nabla \widetilde{u}_{\vec{\zeta}_k}^{b_n,k},2} \leq 2T_k^*$ and $\mathcal{E}'_{4T_k^*,0}(\widetilde{u}_{\vec{\zeta}_k}^{b_n,k}) = o_n(1)$ in $H^{-1}(\Omega)$. In particular, we have $\mathcal{B}^2_{\nabla \widetilde{u}_{\vec{\zeta}_k}^{b_n,k},2} \leq 2T_k^*$ and $\mathcal{E}'_{4T_k^*,0}(\widetilde{u}_{\vec{\zeta}_k}^{b_n,k}) = o_n(1)$ in $\mathcal{H}^{-1}_{\vec{\mathbf{r}}_{0,k}}$, where $\mathcal{H}^{-1}_{\vec{\mathbf{r}}_{0,k}}$ is the dual space of $\mathcal{H}_{\vec{\mathbf{r}}_{0,k}}$. On the other

hand, by Proposition 4.2 once more, we can see from a similar argument as used for [6, Lemma 4.1] that

$$\omega_k(0) < \omega_{k-1}(0) + \frac{1}{N}(a\mathcal{S})^{\frac{N}{2}} \quad \text{for all } k \geq 2. \tag{4.11}$$

Since $\mathcal{E}_{4T_k^*,0}(u) = \mathcal{J}_{\vec{\mathbf{0}},k,\beta}(\vec{\mathbf{u}})$ and $\mathcal{E}'_{4T_k^*,0}(u) = \mathcal{J}'_{\vec{\mathbf{0}},k,\beta}(\vec{\mathbf{u}})$ respectively in $\mathcal{H}_{\vec{\mathbf{r}}_{0,k}}$ and $\mathcal{H}^{-1}_{\vec{\mathbf{r}}_{0,k}}$ for all $\beta < 0$, by (4.11) and a similar argument as used for Proposition 2.1, we can show that $\widetilde{u}^{b_n,k}_{\vec{\zeta}_k} = \widetilde{u}^{*,k}_{\vec{\zeta}_k} + o_n(1)$ strongly in $H^1_{0,R}(\mathbb{B}_R)$ for some $\widetilde{u}^{*,k}_{\vec{\zeta}_k} \in H^1_{0,R}(\mathbb{B}_R)$ for all $k \geq 2$. Similar to (1), by $\lambda < a\sigma_1$ and Lemma 4.6, we can also show that $\widetilde{u}^{*,k}_{\vec{\zeta}_k}$ is a least energy radial sign-changing solution of $(\mathcal{P}_{a,0,\lambda})$, which changes sign exactly k times. □

We close this section by

Proof of Theorem 1.5 The conclusions follow immediately from Lemma 4.6 and Proposition 4.3. □

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5 Appendix: A useful result from [30]

Noris et al. [30] studied the following cubic system

$$\begin{cases} -\Delta u_1 = \lambda_{1,\beta} u_1 + \mu_1 |u_1|^2 u_1 + \beta |u_2|^2 u_1, & \text{in } \Omega, \\ -\Delta u_2 = \lambda_{2,\beta} u_2 + \mu_2 |u_2|^2 u_2 + \beta |u_1|^2 u_2, & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}_1$}$$

The author established the following uniform Hölder bounds of L^∞ bounded positive solutions to this system.

Theorem 5.1 ([30], Theorem 1.1) *Let $N = 2, 3$. If $(\lambda_{1,\beta}, \lambda_{2,\beta})$ is uniformly bounded in $\mathbb{R}$ and (u_β, v_β) is a positive solution of $(\mathcal{P}_1)$ uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$, then for every $\alpha \in (0, 1)$, there exists a constant C, independent of β, such that*

$$\|(u_\beta, v_\beta)\|_{C^{0,\alpha}(\Omega)} \leq C \quad \textit{for all } \beta < 0.$$

Moreover the author pointed out that Theorem 5.1 also holds for positive solutions of system $(\mathcal{P}_1)$ with $N \geq 3$ and subcritical cases; see the paragraph above Theorem 1.1 in the Introduction of [30]. They also pointed out that Theorem 5.1 also holds for positive solutions of the K–component system; see for example ([30], Remark 3.11). Later, Chen et al. pointed out in [9,13] that Theorem 5.1 also holds for positive solutions of $(\mathcal{P}_1)$ with $N \geq 3$ and the critical case, since all the integrals that appear in the proof of [30] are well defined, and the compactness of Sobolev embedding for the subcritical cases is not used; see ([9], Theorem 6.2) and ([13], Theorem A.2). Therefore, together with all these comments, we should have the following result.

Theorem 5.2 *Let $k \geq 2$, $2 < p \leq 2^*$, $N \geq 3$ and $\vec{\mathbf{u}}_{\beta,k}$ be a positive solution of the following system*

$$\begin{cases} -\Delta u_i = \lambda_{i,\beta} u_i + \mu_{i,\beta} |u_i|^{p-2} u_i + \beta \Big(\sum_{j=1, j\neq i}^{k} |u_j|^{\frac{p}{2}} \Big) |u_i|^{\frac{p}{2}-2} u_i, \\ u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k. \end{cases} \qquad (\mathcal{S}_{\beta,k})$$

If $\{\lambda_{i,\beta}\}$, $\{\mu_{i,\beta}\}$ are uniformly bounded in $\mathbb{R}$ and $\vec{\mathbf{u}}_{\beta,k}$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$, then then for every $\alpha \in (0,1)$, there exists a constant C, independent of β, such that

$$\|\vec{\mathbf{u}}_{\beta,k}\|_{C^{0,\alpha}(\Omega)} \leq C \quad \text{for all } \beta < 0.$$

In the rest of this appendix, for the sake of completeness and for the reader's convenience, we will sketch the proof of Theorem 5.2. First, we need a Liouville-type result.

Lemma 5.1 *Let $k \geq 2$, $2 < p \leq 2^*$, $N \geq 3$ and $\vec{\mathbf{u}}_k$ be a positive solution of the following system*

$$-\Delta u_i = -\Big(\sum_{j=1, j\neq i}^{k} |u_j|^{\frac{p}{2}} \Big) |u_i|^{\frac{p}{2}-2} u_i, \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

$i = 1, 2, \ldots, k$. If there holds

$$\max_{1\leq i\leq k} \left\{ \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|u_i^k(x) - u_i^k(y)|}{|x-y|^\alpha} \right\} < +\infty$$

for some $\alpha \in (0,1)$, then there exists i_0 such that $u_{i_0}^k$ is a constant and $u_i^k = 0$ for $i \neq i_0$.

Proof This result has been proved in [30, Proposition 2.7] for $p = 4$, $N = 2, 3$ and arbitrary $k \geq 2$ while in [13, Proposition A.1], it has been proved for $N \geq 3$, $p = 2^*$ and $k = 2$. Now, since $2 < p \leq 2^*$, we can see that the argument of [13, Proposition A.1] still works for this lemma with $k = 2$. For $k \geq 3$, we set $\vec{\mathbf{U}}_{i,j} = (u_i^k, u_j^k)$. Then $\vec{\mathbf{U}}_{i,j}$ is a positive subsolution of (5.1) with $k = 2$. Note that all the argument of [13, Proposition A.1] actually comes from [30, Proposition 2.6] and these arguments still work for positive subsolutions, which is actually pointed out in the proof of [30, Proposition 2.7]. Thus, the conclusion also true for $k \geq 3$. □

Let $\vec{\mathbf{u}}_{\beta,k}$ be a positive solution of $(\mathcal{S}_{k,\beta})$ and suppose that $\{\lambda_{i,\beta}\}$, $\{\mu_{i,\beta}\}$ are uniformly bounded in $\mathbb{R}$ and $\vec{\mathbf{u}}_{\beta,k}$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$. Then by classical elliptic regularity theory it holds that $u_i^{k\beta} \in C^{2,\alpha}(\Omega)$ for all $\alpha \in (0,1)$. As in [13,30], we set

$$L_\beta = \max_{1\leq i\leq k} \left\{ \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|u_i^{k,\beta}(x) - u_i^{k,\beta}(y)|}{|x-y|^\alpha} \right\}$$

and assume $L_\beta \to +\infty$ as $\beta \to -\infty$. Without loss of generality, we assume that

$$L_\beta = \frac{|u_1^{k,\beta}(x_\beta) - u_1^{k,\beta}(y_\beta)|}{|x_\beta - y_\beta|^\alpha}$$

for some $x_\beta, y_\beta \in \mathbb{R}^N$. As in [13,30], we have $|x_\beta - y_\beta| \to 0$ as $\beta \to -\infty$. Define

$$\overline{u}_i^{k,\beta}(x) = \frac{1}{L_\beta \gamma_\beta^\alpha} u_i^{k,\beta}(x_\beta + \gamma_\beta x) \quad \text{for } x \in \Omega_\beta = \frac{\Omega - x_\beta}{\gamma_\beta},$$

where $\gamma_\beta \to 0$ as $\beta \to -\infty$ and will be chosen later. Clearly, $\Omega_\beta \to \Omega_\infty$ as $\beta \to -\infty$, where Ω_∞ is either $\mathbb{R}^N$ or the half space. Moreover,

$$\max_{1\le i\le k}\left\{\sup_{x,y\in\mathbb{R}^N,x\ne y}\frac{|\overline{u}_i^{k,\beta}(x)-\overline{u}_i^{k,\beta}(y)|}{|x-y|^\alpha}\right\}=\frac{|\overline{u}_1^{k,\beta}(0)-\overline{u}_1^{k,\beta}(\frac{y_\beta-x_\beta}{\gamma_\beta})|}{|\frac{y_\beta-x_\beta}{\gamma_\beta}|^\alpha}=1$$

and $\overrightarrow{\mathbf{u}}_{\beta,k}$ satisfies the following system in Ω_β:

$$\begin{cases} -\Delta\overline{u}_i^{k,\beta}=\lambda_{i,\beta}\gamma_\beta^2\overline{u}_i^{k,\beta}+\mu_{i,\beta}M_\beta|\overline{u}_i^{k,\beta}|^{p-2}\overline{u}_i^{k,\beta}\\ +\beta M_\beta\left(\sum_{j=1,j\ne i}^k|\overline{u}_j^{k,\beta}|^{\frac{p}{2}}\right)|\overline{u}_i^{k,\beta}|^{\frac{p}{2}-2}\overline{u}_i^{k,\beta},\\ \overline{u}_i^{k,\beta}\in H_0^1(\Omega_\beta),\quad i=1,2,\ldots,k,\end{cases}\qquad(\overline{\mathcal{S}}_{\beta,k})$$

where $M_\beta=L_\beta^{p-2}\gamma_\beta^{\alpha(p-2)+2}$. Since $\{\lambda_{i,\beta}\}$, $\{\mu_{i,\beta}\}$ are uniformly bounded in $\mathbb{R}$, as in [13], we have

$$\max_{1\le i\le k}\{\lambda_{i,\beta}\gamma_\beta^2(\overline{u}_i^{k,\beta})^q\}\to 0\quad\text{in } L^\infty(\Omega_\beta),\quad\forall q\in(0,2]$$

and

$$\max_{1\le i\le k}\{\mu_{i,\beta}M_\beta(\overline{u}_i^{k,\beta})^q\}\to 0\quad\text{in } L^\infty(\Omega_\beta),\quad\forall q\in[p-2,p].$$

Lemma 5.2 *Let $\gamma_\beta\to 0$ as $\beta\to-\infty$ such that $|x_\beta-y_\beta|/\gamma_\beta\le R'$ for some $R'>0$ and $\beta M_\beta\not\to 0$. Then $\max_{1\le i\le k}\{\overline{u}_i^{k,\beta}(0)\}$ is uniformly bounded for all $\beta<0$.*

Proof This result has been proved in [30, Lemma 3.4] for $p=4$,$N=2,3$ and $k=2$ while in [13, Lemma A.2], it has been proved for $N\ge 3$, $p=2^*$ and $k=2$. Since now $k\ge 2$ is arbitrary, we need to give some minor modifications. Suppose the contrary and without loss of generality, we assume

$$\max_{1\le i\le k}\{\overline{u}_i^{k,\beta}(0)\}=\overline{u}_{i_0}^{k,\beta}(0)\to+\infty\quad\text{for some } i_0\in\{1,2,\ldots,k\}.$$

Since $\overline{u}_{i_0}^{k,\beta}$ is Hölder continuous, we can choose $R>R'$ such that $B_R(0)\subset\Omega_\beta$ for $|\beta|$ large enough and

$$\inf_{B_R(0)}\overline{u}_{i_0}^{k,\beta}\to+\infty.$$

Case 1 $i_0=1$.

Let $I_\beta=|\beta|M_\beta(\overline{u}_{i_0}^{k,\beta})^{\frac{p}{2}-1}$. Since $p>2$ and $|\beta|M_\beta\not\to 0$, we have $I_\beta\to+\infty$. Now, since $p\in(2,2^*]\cap(2,4)$, as in [13, Lemma A.2], we have

$$\begin{aligned}|\beta|M_\beta\left(\sum_{j=1,j\ne i}^k|\overline{u}_j^{k,\beta}|^{\frac{p}{2}}\right)|\overline{u}_i^{k,\beta}|^{\frac{p}{2}-2}\overline{u}_i^{k,\beta}&\ge|\beta|M_\beta|\overline{u}_1^{k,\beta}|^{\frac{p}{2}}|\overline{u}_i^{k,\beta}|^{\frac{p}{2}-1}\\&\ge I_\beta|\overline{u}_i^{k,\beta}|\end{aligned}$$

for all $i=2,3,\ldots,k$. By [8, Lemma 4.4] (see also [13, Lemma A.1]), we have $\|\overline{u}_i^{k,\beta}\|_{L^\infty(\mathbb{B}_R(0))}\le C\exp(-C'\sqrt{I_\beta})$ for all $i=2,3,\ldots,k$. By similar arguments as used in [13, Lemma A.2], we can see that $\|\Delta\overline{u}_1^{k,\beta}\|_{L^\infty(\mathbb{B}_R(0))}\to 0$ as $\beta\to-\infty$. Now, the same argument as used in [30, Lemma 3.4] yields a contradiction.

Case 2 $i_0 \neq 1$.

Let $\widetilde{I}_\beta = |\beta| M_\beta \left(\sum_{j=2}^k |\overline{u}_j^{k,\beta}|^{\frac{p}{2}} \right)^{\frac{p-2}{p}}$. Clearly, we also have $\widetilde{I}_\beta \to +\infty$. Moreover, since $\frac{2}{p} > \frac{4-p}{2}$, similar to Case 1, we have

$$|\beta| M_\beta \left(\sum_{j=2}^k |\overline{u}_j^{k,\beta}|^{\frac{p}{2}} \right) |\overline{u}_1^{k,\beta}|^{\frac{p}{2}-2} \overline{u}_1^{k,\beta} \geq \widetilde{I}_\beta \overline{u}_1^{k,\beta}.$$

It follows from [8, Lemma 4.4] (see also [13, Lemma A.1]) that $\|\overline{u}_1^{k,\beta}\|_{L^\infty(\mathbb{B}_R(0))} \leq C \exp(-C' \sqrt{\widetilde{I}_\beta})$. Since $|\beta| M_\beta \not\to 0$, by similar arguments as used in [13, Lemma A.2], we also have $\|\Delta \overline{u}_1^{k,\beta}\|_{L^\infty(\mathbb{B}_R(0))} \to 0$ as $\beta \to -\infty$. Now, the same argument as used in [30, Lemma 3.4] also yields a contradiction. □

Now, since Lemmas 5.1 and 5.2 hold, by following the same arguments of [30, Lemmas 3.6 and 3.7] with some minor modifications, we can obtain the following.

Lemma 5.3 *Let $\gamma_\beta = |x_\beta - y_\beta|$. Then $|\beta| M_\beta \to +\infty$. Moreover, there exist $\overline{u}_i^{k,\infty} \in C^{0,\alpha}(\mathbb{R}^N)$, $i = 1, 2, \ldots, k$ such that, as $\beta \to -\infty$, there holds*

(1) $\overline{u}_i^{k,\beta} \to \overline{u}_i^{k,\infty}$ *uniformly in any compact set of $\Omega_\infty = \mathbb{R}^N$ for all $i = 1, 2, \ldots, k$,*
(2) *for any fixed $R > 0$ and $x_0 \in \mathbb{R}^N$, $|\beta| M_\beta \int_{\mathbb{B}_R(x_0)} |\overline{u}_j^{k,\beta}|^{\frac{p}{2}} |\overline{u}_i^{k,\beta}|^{\frac{p}{2}} \to 0$ and $\|\overline{u}_i^{k,\beta} - \overline{u}_i^{k,\infty}\|_{H^1(\mathbb{B}_R(x_0))} \to 0$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$, as a consequence, $\overline{u}_j^{k,\infty} \overline{u}_i^{k,\infty} \equiv 0$ for all $i, j = 1, 2, \ldots, k$ and $i \neq j$.*
(3) $\max_{x,y \in \partial \mathbb{B}_1(0)} |\overline{u}_1^{k,\infty}(x) - \overline{u}_1^{k,\infty}(y)| = 1$ *(in particular, $\overline{u}_1^{k,\infty}$ is not a constant),*
(4) $\overline{u}_i^{k,\infty}$ *is harmonic in $\{\overline{u}_i^{k,\infty} > 0\}$ for all $i = 1, 2, \ldots, k$.*

Completed of the proof of Theorem 5.2 As pointed out in [30, Remark 3.10], by considering $E(r) = \frac{1}{r^{N-2}} \int_{\mathbb{B}_r(0)} \sum_{i=1}^k |\nabla \overline{u}_1^{k,\infty}|^2$ and $H(r) = \frac{1}{r^{N-1}} \int_{\partial \mathbb{B}_r(0)} \sum_{i=1}^k |\overline{u}_1^{k,\infty}|^2$, we can follow the same arguments as used in the proof of [30, Theorem 1.3] to show that $\overline{u}_i^{k,\infty}$ are all harmonic in $\mathbb{R}^N$, which is contradicts to Lemma 5.3. □

We close this section by

Proposition 5.1 *Let $k \geq 2$, $2 < p \leq 2^*$, $N \geq 3$ and $\overrightarrow{\mathbf{u}}_{\beta,k}$ be a nonnegative solution of the following system*

$$\begin{cases} -\left(a_i + b_i \sum_{j=1}^k b_j \int_\Omega |\nabla u_j|^2 dx \right) \Delta u_i \\ = \lambda_i u_i + |u_i|^{p-2} u_i + \beta \left(\sum_{j=1, j \neq i}^k |u_j|^{\frac{p}{2}} \right) |u_i|^{\frac{p}{2}-2} u_i, \\ u_i \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k, \end{cases}$$

where $a_i > 0$ and $b_i \geq 0$ for all $i = 1, 2, \ldots, k$. If $\overrightarrow{\mathbf{u}}_{\beta,k}$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$, then for every $\alpha \in (0, 1)$, there exists a constant C, independent of β, such that

$$\|\overrightarrow{\mathbf{u}}_{\beta,k}\|_{C^{0,\alpha}(\Omega)} \leq C \quad \text{for all } \beta < 0$$

in one of the following two cases

(1) $k = 2$;
(2) $a_1 = a_2 = \cdots = a_k = a_0$, $b_1 = b_2 = \cdots = b_k = b_0$ *and* $k \geq 3$.

Proof By a direct calculation, we can see that $\overrightarrow{\mathbf{v}}_{\beta,k} = \overrightarrow{\mathbf{t}_\beta} \circ \overrightarrow{\mathbf{u}}_{\beta,k}$ satisfies the following system

$$\begin{cases} -\Delta v_i^{k,\beta} = \lambda_{i,\beta} v_i^{k,\beta} + \mu_{i,\beta} |v_i^{k,\beta}|^{p-2} v_i^{k,\beta} + \beta\left(\sum_{j=1,j\neq i}^k |v_j^{k,\beta}|^{\frac{p}{2}} \right) |v_i^{k,\beta}|^{\frac{p}{2}-2} v_i^{k,\beta}, \\ v_i^{k,\beta} \in H_0^1(\Omega), \quad i = 1, 2, \ldots, k, \end{cases}$$

where $\lambda_{i,\beta} = \frac{\lambda_i}{a_i + b_i \sum_{j=1}^k b_j \int_\Omega |\nabla u_j^{k,\beta}|^2 dx}$, $\mu_{i,\beta} =$ and $\mu_{i,\beta} = (t_i^\beta)^{p-2}$ with

$$t_1^\beta = \left(a_1 + b_1 \sum_{j=1}^2 b_j \int_\Omega |\nabla u_j^{k,\beta}|^2 dx \right)^{\frac{2-p}{2(p-1)}} \left(a_2 + b_2 \sum_{j=1}^2 b_j \int_\Omega |\nabla u_j^{k,\beta}|^2 dx \right)^{\frac{p}{2(p-1)}}$$

and

$$t_2^\beta = \left(a_2 + b_2 \sum_{j=1}^2 b_j \int_\Omega |\nabla u_j^{k,\beta}|^2 dx \right)^{\frac{2-p}{2(p-1)}} \left(a_1 + b_1 \sum_{j=1}^2 b_j \int_\Omega |\nabla u_j^{k,\beta}|^2 dx \right)^{\frac{p}{2(p-1)}}$$

for $k = 2$ and

$$t_i^\beta = \left((k-1)^{-1} (a_0 + b_0^2 \sum_{j=1}^k \int_\Omega |\nabla u_j^{k,\beta}|^2 dx) \right)^{\frac{1}{p-1}}$$

for $a_1 = a_2 = \cdots = a_k = a_0$, $b_1 = b_2 = \cdots = b_k = b_0$ and $k \geq 3$. Thanks to Lemma 3.2, we can see that $\{\lambda_{i,\beta}\}$ and $\{\mu_{i,\beta}\}$ are uniformly bounded in $\mathbb{R}$. Thus, by Theorem 5.2, for all $\alpha \in (0, 1)$, we have $\|\overrightarrow{\mathbf{v}}_{\beta,k}\|_{C^{0,\alpha}(\Omega)} \leq C$. Note that t_i^β are also uniformly bounded away from 0 and bounded from above, thus, we must have that $\|\overrightarrow{\mathbf{u}}_{\beta,k}\|_{C^{0,\alpha}(\Omega)} \leq C$, which completes the proof. □

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