Qualitative analysis for an elliptic system in the punctured space*

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Abstract

In this paper, we investigate the qualitative properties of positive solutions for the following two-coupled elliptic system in the punctured space:

$$\begin{cases} -\Delta u = \mu_1 u^{2q+1} + \beta u^q v^{q+1} \\ -\Delta v = \mu_2 v^{2q+1} + \beta v^q u^{q+1} \end{cases} \quad \text{in } \mathbb{R}^n \backslash \{0\},$$

where μ_1, μ_2 and β are all positive constants, $n \geq 3$. We establish a monotonicity formula that completely characterizes the singularity of positive solutions. We prove a sharp global estimate for both components of positive solutions. We also prove the nonexistence of positive semi-singular solutions, which means that one component is bounded near the singularity and the other component is unbounded near the singularity.

Key words: Elliptic system, isolated singularity, semi-singular solutions, asymptotic behavior.

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1 Introduction and Main results

In this paper, we investigate the qualitative properties of positive solutions for the following two-coupled elliptic system

$$\begin{cases} -\Delta u = \mu_1 u^{2q+1} + \beta u^q v^{q+1} \\ -\Delta v = \mu_2 v^{2q+1} + \beta v^q u^{q+1} \end{cases} \quad \text{in } \mathbb{R}^n \backslash \{0\}, \tag{1.1}$$

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where μ_1, μ_2 and β are all positive constants, $n \geq 3$ and $p := 2q + 1 > 1$. We say that (u, v) is a positive solution of (1.1) if $u, v > 0$ in $\mathbb{R}^n \backslash \{0\}$, and that (u, v) is a nonnegative solution of (1.1) if $u, v \geq 0$ in $\mathbb{R}^n \backslash \{0\}$.

System (1.1) is related to the following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^{2q+1} + \beta u^q v^{q+1} & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^{2q+1} + \beta v^q u^{q+1} & \text{in } \Omega, \\ u, v > 0 \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a domain. In the case $p = 3$ $(q = 1)$, the cubic system (1.2) arises in mathematical models from various physical phenomena, such as nonlinear optics and Bose-Einstein condensation. We refer the reader for these to the survey articles [17, 21], which also contain information about the physical relevance of non-cubic nonlinearities. In the subcritical case $p < \frac{n+2}{n-2}$, the system (1.2) on a bounded smooth domain or on $\mathbb{R}^N$ has been widely investigated by variational methods and topological methods in the last decades, see [1, 14, 23, 25, 28, 33, 34] and references therein. In the critical case $p = \frac{n+2}{n-2}$, the system (1.2) on a bounded domain has been investigated in [11–13].

In a recent paper [10], Chen and Lin studied the positive singular solutions of system (1.1) with the critical case $p = \frac{n+2}{n-2}$. They proved a sharp result about the removable singularity and the nonexistence of positive semi-singular solutions. In this paper, we are concerned with these problems in the subcritical case $p < \frac{n+2}{n-2}$.

When $1 < p \leq \frac{n}{n-2}$, by a Liouville type theorem in [4] (also see [32]), we know that the system (1.1) has only the trivial nonnegative solution $u = v \equiv 0$ in $\mathbb{R}^n \backslash \{0\}$. Hence, we just need to consider the case $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.

In order to motivate our results on (1.1), we first recall the classical results for the single elliptic equation

$$-\Delta u = u^p, \qquad u > 0 \quad \text{in } \mathbb{R}^n \backslash \{0\}. \tag{1.3}$$

The asymptotic behaviors of solutions of equation (1.3) near 0 and near ∞ were studied in [6, 19] for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and in [6, 16, 22] for $p = \frac{n+2}{n-2}$. In the other case of p, see Lions [24] for $1 < p < \frac{n}{n-2}$, Aviles [2] for $p = \frac{n}{n-2}$ and Bidaut-Véron and Véron [5] for $p > \frac{n+2}{n-2}$, where the local behavior of solutions of equation (1.3) in a punctured ball was studied. When $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, it is well know that the function

$$u(x) = C_0 |x|^{-\frac{2}{p-1}} \tag{1.4}$$

with

$$C_0 = \left[\frac{2(n-2)}{(p-1)^2} \left(p - \frac{n}{n-2} \right) \right]^{1/(p-1)} \tag{1.5}$$

is an explicit singular solution for (1.3). Besides (1.4), there exists another (oneparameter family) solution of (1.3) which satisfies

$$u(x) \sim \begin{cases} C_0|x|^{-\frac{2}{p-1}} & \text{near } x = 0, \\ \lambda|x|^{-(n-2)} & \text{near } x = +\infty, \end{cases} \tag{1.6}$$

where C_0 is given by (1.5) and $\lambda > 0$. See Appendix A in [19]. On the other hand, we see easily that both components of the positive solution (u, v) of (1.1) satisfy

$$-\Delta w \geq w^p, \qquad w > 0 \quad \text{in } \mathbb{R}^n \backslash \{0\} \tag{1.7}$$

up to a multiplication. It follows from Corollary II in [32] that the inequality (1.7) has a positive solution $w_0 \in C^2(\mathbb{R}^n)$ if $p > \frac{n}{n-2}$. Furthermore, w_0 satisfies $-\Delta w_0 \geq w_0^p$ in entire space $\mathbb{R}^n$.

The purpose of the current paper is to study the positive solutions of system (1.1) with $\dfrac{n}{n-2} < p < \dfrac{n+2}{n-2}$. For any nonnegative solution (u, v) of (1.1), we can prove that $\lim\limits_{|x|\to 0} u(x)$ and $\lim\limits_{|x|\to 0} v(x)$ make sense (the limit may be $+\infty$), see Theorem 2.1 in Sect. 2. Hence, there are three possibilities in general:

(1) both $\lim\limits_{|x|\to 0} u(x) < +\infty$ and $\lim\limits_{|x|\to 0} v(x) < +\infty$, i.e., (u, v) is an entire solution of (1.1) in $\mathbb{R}^n$. By a Liouville type theorem, cf. Theorem 2 in [30], $u = v \equiv 0$ in this case.

(2) $\lim\limits_{|x|\to 0} u(x) = \lim\limits_{|x|\to 0} v(x) = +\infty$, i.e., the origin is a non-removable singularity of both u and v, and we call solutions of this type *positive both-singular solutions at 0*.

(3) either

$$\lim_{|x|\to 0} u(x) < \lim_{|x|\to 0} v(x) = +\infty$$

or

$$\lim_{|x|\to 0} v(x) < \lim_{|x|\to 0} u(x) = +\infty,$$

and we call solutions of this type *positive semi-singular solutions at 0*.

The paper [10] only studied the critical problem, where the Pohozaev identity plays a very important role in characterizing the positive singular solutions. More precisely, let (u, v) be a positive solution of (1.1) with $p = \frac{n-2}{n+2}$. Denote $B_r := \{x \in \mathbb{R}^n : |x| < r\}$. By the Pohozaev identity, one has $K(r; u, v) = K(s; u, v)$ for $0 < s < r$, where

$$\begin{aligned} K(r; u, v) := \int_{\partial B_r} \Bigg[& \frac{n-2}{2} \left(u \frac{\partial u}{\partial \nu} + v \frac{\partial v}{\partial \nu} \right) - \frac{r}{2} (|\nabla u|^2 + |\nabla v|^2) \\ & + r \left| \frac{\partial u}{\partial \nu} \right|^2 + r \left| \frac{\partial v}{\partial \nu} \right|^2 + \frac{r}{2^*} \left(\mu_1 u^{2^*} + \mu_2 v^{2^*} + 2\beta u^{\frac{2^*}{2}} v^{\frac{2^*}{2}} \right) \Bigg] d\sigma, \end{aligned}$$

and ν is the unit outer normal of ∂B_r. Hence, $K(r;u,v)$ is a constant independent of r, and we denote this constant by $K(u,v)$. Then the result of removable singularity in [10] is as follows.

Theorem A [10] *Let $\mu_1, \mu_2, \beta > 0$ and (u,v) be a nonnegative solution of* (1.1) *with $p = \frac{n-2}{n+2}$. Then $K(u,v) \le 0$. Furthermore, $K(u,v) = 0$ if and only if $u, v \in C^2(\mathbb{R}^n)$, namely both u and v are smooth at 0.*

However, there are some differences between the subcritical case and the critical case. The classical Pohozaev identity seems to be unavailable to characterize the positive singular solutions in subcritical case.

Here we will establish a monotonicity formula to character the positive singular solutions of (1.1) in the case $\dfrac{n}{n-2} < p < \dfrac{n+2}{n-2}$. For $r > 0$, define

$$\begin{aligned} E(r;u,v) :=& r^\tau \int_{\partial B_r} \left[\frac{2}{p-1}\left(u\frac{\partial u}{\partial \nu} + v\frac{\partial v}{\partial \nu}\right) - \frac{r}{2}(|\nabla u|^2 + |\nabla v|^2) \right. \\ & \left. + r\left|\frac{\partial u}{\partial \nu}\right|^2 + r\left|\frac{\partial v}{\partial \nu}\right|^2 + \frac{\tau}{r(p-1)}(u^2+v^2) \right] d\sigma \\ & + r^\tau \int_{\partial B_r} \frac{r}{p+1}\left(\mu_1 u^{p+1} + \mu_2 v^{p+1} + 2\beta u^{q+1} v^{q+1}\right) d\sigma, \end{aligned} \tag{1.8}$$

where $\tau = \frac{4}{p-1} - n + 2$. Let (u,v) be a nonnegative solution of (1.1) with $\dfrac{n}{n-2} < p < \dfrac{n+2}{n-2}$. Then we will prove that $E(r;u,v)$ is nondecreasing and bounded for $r \in (0, +\infty)$ (see Proposition 3.1 and Lemma 3.2 in Sect. 3). Hence, we may define the limits

$$E(0;u,v) := \lim_{r\to 0^+} E(r;u,v) \quad \text{and} \quad E(\infty;u,v) := \lim_{r\to +\infty} E(r;u,v).$$

Our first result is about the singularity of positive solutions. We partly classify the nonnegative solutions of (1.1) by $E(r;u,v)$.

Theorem 1.1. *Let $\mu_1, \mu_2, \beta > 0$ and (u,v) be a nonnegative solution of* (1.1) *with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. Then*

$$\Big\{E(0;u,v), E(\infty;u,v)\Big\} \subset \left\{0, -\frac{p-1}{2(p+1)}(k^2+l^2)C_0^{p+1}\right\},$$

where C_0 is given by (1.5) *and $k, l \ge 0$ (not all 0) satisfy*

$$\mu_1 k^{2q} + \beta k^{q-1} l^{q+1} = 1, \quad \mu_2 l^{2q} + \beta l^{q-1} k^{q+1} = 1. \tag{1.9}$$

Furthermore, we get

(1) $E(0;u,v) = 0$ if and only if (u,v) is trivial, i.e., $u = v \equiv 0$.

(2) $E(r;u,v) \equiv constant \neq 0$ *if and only if* (u,v) *is of the form*

$$u(x) = kC_0|x|^{-\frac{2}{p-1}}, \quad v(x) = lC_0|x|^{-\frac{2}{p-1}}, \tag{1.10}$$

where C_0 *is given by* (1.5) *and* $k, l \geq 0$ *(not all 0) satisfy* (1.9).

Remark 1.1. *We point out that if one of* k *and* l *is 0, such as* $k = 0$*, then system of equations* (1.9) *is understood as a single equation* $\mu_2 l^{2q} = 1$.

Remark 1.2. *It is easy to see that the non-negative solution of* (1.9)*, which is not all 0, is not unique. This fact determines that the behavior of positive solutions of system* (1.1) *is much more complicated than that of single equation* (1.3).

From now on, we denote

$$A_{k,l} = \frac{p-1}{2(p+1)}(k^2+l^2)C_0^{p+1}.$$

Theorem 1.1 shows that if (u,v) is a positive solution of (1.1) with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, then it is either both-singular or semi-singular at 0, and $E(0;u,v) = -A_{k,l} < 0$.

Our second result is concerned with the asymptotic behaviors of positive solutions near 0 and near ∞. Just like the critical case in [10], a basic open question is *whether positive semi-singular solutions at 0 exist or not*. Since the system (1.1) with $p = \frac{n+2}{n-2}$ is conformally invariant, the behaviors of positive singular solutions near 0 and near ∞ is equivalent, see [10]. For the subcritical case $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, the system (1.1) is not conformally invariant, the behaviors of positive singular solutions near 0 and near ∞ is very different.

Let u be a positive solution of equation (1.3) with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, by Theorem 3.6 in [19], then either the singularity at ∞ is removable, i.e., there exists a constant $c > 0$ such that

$$u(x) \leq c|x|^{-(n-2)}, \quad |x| \geq 1, \tag{1.11}$$

or there exist constants $0 < c_1 \leq c_2$ such that

$$c_1|x|^{-\frac{2}{p-1}} \leq u(x) \leq c_2|x|^{-\frac{2}{p-1}}, \quad |x| \geq 1. \tag{1.12}$$

In order to describe the behavior of positive solutions of system (1.1) near ∞, we introduce the following definition.

Definition 1.1. *Assume* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$*. we say that* (u,v) *is a positive semi-singular solution of* (1.1) *at* ∞ *if either*

$$u(x) \leq C|x|^{-(n-2)} \quad \text{and} \quad C_1|x|^{-\frac{2}{p-1}} \leq v(x) \leq C_2|x|^{-\frac{2}{p-1}} \quad \text{near } \infty,$$

or

$$v(x) \leq C|x|^{-(n-2)} \quad \text{and} \quad C_1|x|^{-\frac{2}{p-1}} \leq u(x) \leq C_2|x|^{-\frac{2}{p-1}} \quad \text{near } \infty.$$

A new question is *whether positive semi-singular solutions of* (1.1) *at* ∞ *exist or not*.

Theorem 1.2. *Let $\mu_1, \mu_2, \beta > 0$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.*

(1) System (1.1) *has no positive semi-singular solutions at 0 whenever $n \geq 4$.*

(2) Assume $n \geq 3$. In particular, if $n = 3$, we suppose additionally that the system (1.1) *has no positive semi-singular solutions at 0.*
Then, for any positive solution (u, v) of (1.1)*, there exist constants $0 < C_1 \leq C_2$ such that*

$$C_1|x|^{-\frac{2}{p-1}} \leq u(x),\ v(x) \leq C_2|x|^{-\frac{2}{p-1}}, \quad x \in B_1\backslash\{0\}. \tag{1.13}$$

(3) System (1.1) *has no positive semi-singular solutions at ∞ whenever $n \geq 4$.*

(4) Assume $n \geq 3$. In particular, if $n = 3$, we suppose additionally that the system (1.1) *has no positive semi-singular solutions at ∞.*
Then, for any positive solution (u, v) of (1.1)*, either the singularity at ∞ is removable, i.e., there exists $C_3 > 0$ such that*

$$u(x),\ v(x) \leq C_3|x|^{-(n-2)}, \quad |x| \geq 1, \tag{1.14}$$

or there exist constants $0 < C_1 \leq C_2$ such that

$$C_1|x|^{-\frac{2}{p-1}} \leq u(x),\ v(x) \leq C_2|x|^{-\frac{2}{p-1}}, \quad |x| \geq 1. \tag{1.15}$$

Theorem 1.2 gives a classification of positive solutions to system (1.1) whenever $n \geq 4$. In a subsequent work [36], we will study the asymptotic symmetry and classification of isolated singularities of positive solutions to system (1.1) in a *punctured ball*. We remark that above theorems are very important to study these problems in a punctured ball. We also remark that the single equation (1.3) ($\frac{n}{n-2} < p < \frac{n+2}{n-2}$) in a punctured ball or in punctured space was well studied in the seminal papers [6, 19].

Theorem 1.2 shows that, for $n \geq 4$, 0 (or ∞) is a non-removable singularity of u if and only if 0 (resp. ∞) is a non-removable singularity of v, but we don't know whether this conclusion holds or not for $n = 3$. We tend to believe that it also holds for $n = 3$, by which we obtain the sharp estimate of positive solutions. These are some essential differences between the system (1.1) and the scalar equation (1.3).

Remark that, in Theorem 1.2, we get the *sharp* estimates for *both components* of positive solutions. In a very recent paper [18], Ghergu, Kim and Shahgholian studied the positive singular solutions of system

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} \tag{1.16}$$

in a punctured ball, where $\mathbf{u} = (u_1, \cdots, u_m)$. When $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, they only obtain the similar sharp estimate for $|\mathbf{u}(x)|$ as a whole *rather than* each component u_i of the positive vector solutions $\mathbf{u}$. We also point out that, the sharp estimates of singular solutions for the scalar equation (1.3) is relatively simple. However, coupled system (1.1) turns out to be much more delicate and complicated than (1.3). Further, in the first equation of (1.1), the power of u in the coupling term is q, which is >1 if $n = 3$ and < 1 if $n \geq 4$. This fact makes the argument depending heavily on the dimension. This is another important difference between (1.1) and (1.3).

Assume $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and let W be a positive solution of (1.3), then (kW, lW) is a positive both-singular solution of (1.1), where $k, l > 0$ satisfy (1.9). Conversely, there is an interesting problem remaining: *whether any positive solutions are of the form* (kW, lW)*, where* W *is a solution of* (1.3)? This question seems very tough. We remark that there are some papers to study the proportionality of components ($u/v \equiv$ constant) of *entire* positive solutions for others systems, such as see [9, 15, 26, 29], but we can not obtain the conclusion $u/v \equiv constant$ to system (1.1) via the ideas of these papers, since (1.1) does not satisfy the structural conditions in these papers, and system (1.1) has an isolated singularity 0. Therefore, system (1.1) can not be reduced to a single equation. We will prove the radial symmetry of positive singular solutions of (1.1) and then prove our result by analyzing an ODE system, which turns out to be very delicate and complicated. Remark that the ODE system corresponding to the subcritical case in this paper is very different from the critical case in [10], and some new observations and ideas are needed. In particular, a very important monotonicity property of ODE system in [10] does not exist in our problem. See Sect. 3 for a detailed explanation.

The rest of this paper is structured as follows. Sect. 2 is devoted to prove that positive singular solutions of (1.1) are radially symmetric via the method of moving planes. In Sect. 3, we establish the monotonicity formula and some crucial lemmas. Theorem 1.1 and 1.2 are proved in Sect. 4. We will see that our arguments are much more delicate than the scalar equation and are also very different with the critical system. We will denote positive constants (possibly different in different lines) by $C, \bar{C}, C_1, C_2, \cdots$.

2 Radial symmetry

In this section we apply the method of moving planes to prove the radial symmetry of positive singular solutions, which extend the classical result in Caffarelli-Gidas-Spruck [6] to system (1.1). We may also see [10] for this extension to (1.1) with $p = \frac{n+2}{n-2}$. Since we will use the Kelvin transform, it does not seem trivial to get the strictly decreasing of $u(r)$ and $v(r)$ for $r = |x|$ from this proof. We will prove this strictly decreasing property via some estimates in Lemma 3.4 in Sect. 3.

Theorem 2.1. *Let* (u, v) *be a positive solution of* (1.1) *with* $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$. *Assume that* $\limsup_{|x|\to 0} u(x) = +\infty$ *or* $\limsup_{|x|\to 0} v(x) = +\infty$. *Then both* u *and* v *are radially symmetry about the origin and are strictly decreasing about* $r = |x| > 0$.

Proof. We follow the idea in [6] to use the method of moving planes. Without loss of generality, we assume that $\limsup_{|x|\to 0} u(x) = +\infty$. Fix an arbitrary point $z \neq 0$ and define the Kelvin transform

$$U(x) = \frac{1}{|x|^{n-2}} u\left(z + \frac{x}{|x|^2}\right), \quad V(x) = \frac{1}{|x|^{n-2}} v\left(z + \frac{x}{|x|^2}\right).$$

Then (U,V) satisfies

$$\begin{cases} -\Delta U = |x|^{\alpha}\left(\mu_1 U^{2q+1} + \beta U^q V^{q+1}\right) \\ -\Delta V = |x|^{\alpha}\left(\mu_2 V^{2q+1} + \beta V^q U^{q+1}\right) \end{cases} \quad \text{in } \mathbb{R}^n \backslash \{0, z_0\}, \tag{2.1}$$

where $\alpha = p(n-2) - (n+2)$ and $z_0 = -z/|z|^2$. Clearly, U and V are singular at 0 and z_0. Define an axis going through 0 and z, we shall show that both U and V are axisymmetric. To this end, we consider any reflection direction η orthogonal to this axis. We may assume, without loss of generality, that $\eta = (0, \cdots, 0, 1)$ is the positive x_n direction. For any $\lambda > 0$, let

$$\Sigma_\lambda = \{x \in \mathbb{R}^n : x_n > \lambda\}, \qquad T_\lambda = \{x \in \mathbb{R}^n : x_n = \lambda\}.$$

We denote $x^\lambda = (x', 2\lambda - x_n)$ as the reflection of the point $x = (x', x_n)$ about the plane T_λ. Since U and V have the harmonic asymptotic expansion (see (2.6) in [6]) at infinity, by Lemma 2.3 in [6], there exist large positive constants $\lambda_0 > 10$ and $R > |z_0| + 10$ such that for any $\lambda \geq \lambda_0$, we have

$$U(x) < U(x^\lambda), \qquad V(x) < V(x^\lambda) \qquad \text{for } x \in \Sigma_\lambda, \quad |x^\lambda| > R. \tag{2.2}$$

By the maximum principle for super harmonic functions with isolated singularities, cf. Lemma 2.1 in [7], there exists $C > 0$ such that

$$U(x), V(x) \geq C \qquad \text{for } x \in \overline{B_R} \backslash \{0, z_0\}. \tag{2.3}$$

Since $U(x), V(x) \to 0$ as $|x| \to +\infty$, by (2.2) and (2.3), there exists $\lambda_1 > \lambda_0$ such that for any $\lambda \geq \lambda_1$, we have

$$U(x) \leq U(x^\lambda), \qquad V(x) \leq V(x^\lambda) \qquad \text{for } x \in \Sigma_\lambda, \quad x^\lambda \notin \{0, z_0\}. \tag{2.4}$$

Define

$$\lambda^* := \inf\{\bar{\lambda} > 0 \mid (2.4) \text{ holds for all } \lambda \geq \bar{\lambda}\}.$$

Suppose $\lambda^* > 0$. Then (2.4) also holds for $\lambda = \lambda^*$. Since $\limsup_{|x| \to 0} u(x) = +\infty$, we get $\limsup_{x \to z_0} U(x) = +\infty$ and hence $U(x) \not\equiv U(x^{\lambda^*})$. Define $W_1(x) = U(x^{\lambda^*})$ and $W_2(x) = V(x^{\lambda^*})$ for $x \in \Sigma_{\lambda^*}$ and $x^{\lambda^*} \notin \{0, z_0\}$. Then we have

$$\begin{cases} -\Delta(W_1 - U) \geq 0 \\ -\Delta(W_2 - V) \geq \beta |x|^{\alpha} V^q \big(W_1^{q+1} - U^{q+1}\big) \end{cases} \quad \text{for } x \in \Sigma_{\lambda^*}, \ x^{\lambda^*} \notin \{0, z_0\}.$$

By the strong maximum principle we obtain

$$U(x) < W_1(x), \qquad V(x) < W_2(x) \qquad \text{for } x \in \Sigma_{\lambda^*}, \quad x^{\lambda^*} \notin \{0, z_0\}. \tag{2.5}$$

Note that $0, z_0 \notin T_{\lambda^*}$ because of $\lambda^* > 0$. By the Hopf boundary lemma, we have for any $x \in T_{\lambda^*}$,

$$\frac{\partial (W_1 - U)(x)}{\partial x_n} = -2\frac{\partial U(x)}{\partial x_n} > 0, \quad \frac{\partial (W_2 - V)(x)}{\partial x_n} = -2\frac{\partial V(x)}{\partial x_n} > 0. \tag{2.6}$$

By the definition λ^*, there exists $\lambda_j \to \lambda^*$ ($\lambda_j < \lambda^*$) such that (2.4) does not hold for $\lambda = \lambda_j$. Without loss of generality and up to a subsequence, we assume that there exists $x_j \in \Sigma_{\lambda_j}$ such that $U(x_j^{\lambda_j}) < U(x_j)$. It follows Lemma 2.4 in [6] (the plane $x_n = 0$ there corresponds to $x_n = \lambda^*$ here) that $|x_j|$ are uniformly bounded. Hence, up to a subsequence, $x_j \to \bar{x} \in \overline{\Sigma_{\lambda^*}}$ with $U(\bar{x}^{\lambda^*}) \le U(\bar{x})$. By (2.5), we have $\bar{x} \in T_{\lambda^*}$ and then $\frac{\partial U}{\partial x_n}(\bar{x}) \ge 0$, a contradiction with (2.6). Hence $\lambda^* = 0$ and so both U and V are axisymmetric about the axis going through 0 and z. Since z is arbitrary, u and v are both axisymmetric about any axis going through 0, and so u, v are both radially symmetry about the origin. The strictly decreasing property of u and v will be proved via some estimates in Lemma 3.4 in Sect. 3. □

3 Monotonicity formula and crucial lemmas

In this section, we establish the monotonicity formula and some crucial lemmas. Let (u, v) be a positive solution of (1.1). By Theorem 2.1 in Sect. 2 and Theorem 2 in [30], we may assume that $u(x) = u(|x|)$ and $v(x) = v(|x|)$ are radially symmetric functions. Denote $\delta = \frac{2}{p-1}$. We use the classical change of variables as in Fowler [16]. Let $t = -\ln r$ and

$$w_1(t) := r^\delta u(r) = e^{-\delta t} u(e^{-t}), \qquad w_2(t) := r^\delta v(r) = e^{-\delta t} v(e^{-t}).$$

Then, by a direct calculation (w_1, w_2) satisfies

$$\begin{cases} w_1'' + \tau w_1' - \sigma w_1 + \mu_1 w_1^{2q+1} + \beta w_1^q w_2^{q+1} = 0 \\ w_2'' + \tau w_2' - \sigma w_2 + \mu_2 w_2^{2q+1} + \beta w_2^q w_1^{q+1} = 0 \end{cases} \quad t \in \mathbb{R}, \tag{3.1}$$

where $w_i' := \frac{dw_i}{dt}$, $w_i'' := \frac{d^2 w_i}{dt^2}$ and

$$\tau = \frac{4}{p-1} - n + 2, \qquad \sigma = \frac{2}{p-1}\left(n - 2 - \frac{2}{p-1}\right).$$

We note that $\tau, \sigma > 0$ due to $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.

When $p = \frac{n+2}{n-2}$, we see that $\tau = 0$ and $\sigma = \delta^2 = \left(\frac{n-2}{2}\right)^2$. Consider two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(t) := -\frac{1}{2}|w_1'|^2 + \frac{\delta^2}{2} w_1^2 - \frac{\mu_1}{p+1} w_1^{p+1}, \ f_2(t) := -\frac{1}{2}|w_2'|^2 + \frac{\delta^2}{2} w_2^2 - \frac{\mu_1}{p+1} w_2^{p+1}.$$

Then by (3.1) we have

$$f_1'(t) = \beta w_1^q w_2^{q+1} w_1', \qquad f_2'(t) = \beta w_2^q w_1^{q+1} w_2'.$$

That is, for $i = 1, 2$, *the monotonicity of f_i is exactly the same as the monotonicity of w_i*. This monotonicity property plays a very important role and is used frequently in [10]. However, when $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, we have $\tau > 0$, there is *no similar monotonicity property*. In addition, we note that when $\tau = 0$, system (3.1) is invariant under reflection but our problem (3.1) is *not* because $\tau > 0$. Therefore, we need some new observations and different ideas to deal with our ODE system (3.1).

Multiplying the first equation of (3.1) by w_1', the second equation of (3.1) by w_2', we easily obtain the following identity

$$\Psi'(t) = -\tau \left[(w_1')^2 + (w_2')^2\right](t), \tag{3.2}$$

where

$$\begin{aligned}\Psi(t) =& \frac{1}{2}\left(|w_1'|^2 + |w_2'|^2 - \sigma\left(w_1^2 + w_2^2\right)\right)(t) \\ &+ \frac{1}{p+1}\left(\mu_1 w_1^{p+1} + 2\beta w_1^{q+1} w_2^{q+1} + \mu_2 w_2^{p+1}\right)(t).\end{aligned} \tag{3.3}$$

Therefore, we have $\Psi'(t) \le 0$ in $\mathbb{R}$, namely $\Psi(t)$ is decreasing for $t \in \mathbb{R}$. Indeed, by an easy computation, we can deduce from (1.8) that $E(r; u, v) = \sigma_{n-1}\Psi(t)$, where $t = -\ln r$ and σ_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. Hence we have

$$\frac{d}{dr}E(r; u, v) = -\frac{\sigma_{n-1}}{r}\Psi'(t). \tag{3.4}$$

From this, we easily obtain the following monotonicity formula.

Proposition 3.1. *Let (u, v) be a nonnegative solution of* (1.1) *with $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$. Then $E(r; u, v)$ is nondecreasing for $r \in (0, \infty)$. Moreover,*

$$\frac{d}{dr}E(r; u, v) = \tau r^{\tau} \int_{\partial B_r} \left[\left(\frac{\partial u}{\partial \nu} + \frac{2}{p-1}\frac{u}{r}\right)^2 + \left(\frac{\partial v}{\partial \nu} + \frac{2}{p-1}\frac{v}{r}\right)^2\right], \tag{3.5}$$

where ν is the unit outer normal of ∂B_r.

To prove that $E(r; u, v)$ is bounded in $(0, +\infty)$, we need the following estimates. There are many ways to prove them, for example, we can use the method in Poláčik-Quittner-Souplet [27], and here we use a simple idea from Lemma 4.2 in [18].

Lemma 3.1. *Let (u, v) be a nonnegative solution of* (1.1) *with $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$. Then*

$$u(x), v(x) \le C|x|^{-\frac{2}{p-1}} \quad \textit{in } \mathbb{R}^n \backslash \{0\} \tag{3.6}$$

and

$$|\nabla u(x)|, |\nabla v(x)| \le C|x|^{-\frac{2}{p-1}-1} \quad \textit{in } \mathbb{R}^n \backslash \{0\}, \tag{3.7}$$

where both two constants C depend only on n, p, μ_1, μ_2 and β.

Proof. By the strong maximum principle, we assume that $u > 0$ in $\mathbb{R}^n \backslash \{0\}$. Since u is superharmonic, it follows from Lemma 2.1 in [7] that

$$\liminf_{x \to 0} u(x) > 0. \tag{3.8}$$

Let $\widetilde{u} = u^{1-p}$. Then $\widetilde{u}$ satisfies

$$\Delta \widetilde{u} \ge \frac{p}{p-1}\frac{|\nabla \widetilde{u}|}{\widetilde{u}} + \mu_1(p-1), \quad \text{in } \mathbb{R}^n \backslash \{0\}.$$

Hence, the auxiliary function

$$\widetilde{w}(x) = \widetilde{u}(x) - \frac{\mu_1(p-1)}{2n}|x|^2$$

is subharmonic in $\mathbb{R}^n \backslash \{0\}$. By (3.8), $\widetilde{w}$ is bounded near the origin. Thus, for any $r > 0$, it follows from Theorem 1 in [20] that

$$0 \le \limsup_{x \to 0} \widetilde{w}(x) \le \sup_{\partial B_r} \widetilde{w} = \sup_{\partial B_r} \widetilde{u} - \frac{\mu_1(p-1)}{2n} r^2.$$

In terms of u, we have

$$\inf_{\partial B_r} u \le \left(\frac{\mu_1(p-1)}{2n} \right)^{-\frac{1}{p-1}} r^{-\frac{2}{p-1}}.$$

By the radial symmetry obtained in Theorem 2.1, we get the estimate of u in (3.6). Using a similar argument, we can get the estimate of v in (3.6).

For any $x_0 \in \mathbb{R}^n \backslash \{0\}$, take $\lambda = \frac{|x_0|}{2}$ and define

$$u_1(x) = \lambda^{\frac{2}{p-1}} u(x_0 + \lambda x), \quad v_1(x) = \lambda^{\frac{2}{p-1}} v(x_0 + \lambda x).$$

Then (u_1, v_1) satisfies the system (1.1) in B_1. By (3.6), $|u_1|, |v_1| \le C$ in B_1. Standard gradient estimate gives

$$|\nabla u_1(0)|, \ |\nabla v_1(0)| \le C.$$

Rescaling back we get (3.7). □

By the above lemma, one easily obtains the boundedness of $E(r; u, v)$ for $r \in (0, +\infty)$. Let's omit the proof.

Lemma 3.2. *Let (u, v) be a nonnegative solution of* (1.1) *with $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$. Then $E(r; u, v)$ is uniformly bounded for all $r \in (0, +\infty)$.*

Another consequence of the upper bound (3.6) is the following Harnack inequality.

Lemma 3.3. *Let (u, v) be a nonnegative solution of* (1.1) *with $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$. Then for all $r > 0$, we have*

$$\sup_{B_{2r} \backslash B_r} (u + v) \le C \inf_{B_{2r} \backslash B_r} (u + v), \tag{3.9}$$

where C depends only on n, p, μ_1, μ_2 and β.

Proof. Let $z = u + v$, then w satisfies the scalar equation

$$-\Delta z = C(x) z^p \quad \text{in } \mathbb{R}^n \backslash \{0\},$$

where $C(x)$ is a bounded function for all $x \in \mathbb{R}^n \backslash \{0\}$. For any $r > 0$, let $z_r(x) =: z(rx)$, then z_r satisfies

$$-\Delta z_r = a_r(x) z_r \quad \text{in } \mathbb{R}^n \backslash \{0\},$$

where $a_r(x) := r^2 C(rx) z^{p-1}(rx)$. By Lemma 3.1, we have

$$|a_r(x)| \leq C|x|^{-2} \leq 4C \quad \text{for } \frac{1}{2} \leq |x| \leq 4,$$

where C is a constant that depends only on n, p, μ_1, μ_2 and β. Therefore, the classical Harnack inequality gives

$$\sup_{1\leq|x|\leq 2} z_r \leq C \inf_{1\leq|x|\leq 2} z_r.$$

Consequently, we get

$$\sup_{B_{2r}\setminus B_r} (u+v) \leq C \inf_{B_{2r}\setminus B_r} (u+v).$$

□

Now we prove the strictly decreasing property of positive solutions of (1.1).

Lemma 3.4. *Let (u, v) be a positive solution of (1.1) with $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$. Then both $u'(r) < 0$ and $v'(r) < 0$ for all $r > 0$.*

Proof. By the divergence theorem, for any $0 < \epsilon < r$, we have

$$-\int_{\partial B_r} \frac{\partial u}{\partial \nu} + \int_{\partial B_\epsilon} \frac{\partial u}{\partial \nu} = \int_{B_r\setminus B_\epsilon} \mu_1 u^p + \beta u^q v^{q+1}, \tag{3.10}$$

where ν is the unit outer normal of ∂B_r and ∂B_ϵ. By Lemma 3.1,

$$\int_{\partial B_\epsilon} \left|\frac{\partial u}{\partial \nu}\right| \leq C\epsilon^{n-\frac{2}{p-1}-2} \to 0 \quad \text{as } \epsilon \to 0.$$

Thus, letting $\epsilon \to 0$ in (3.10), we get

$$-\sigma_{n-1} u'(r) r^{n-1} = \int_{B_r} \mu_1 u^p + \beta u^q v^{q+1} > 0,$$

and hence $u'(r) < 0$ for all $r > 0$. Similarly, we also have $v'(r) < 0$ for all $r > 0$. □

As a result of Lemma 3.4, we have the following important corollary.

Corollary 3.1. *Let (w_1, w_2) be a positive solution of (3.1) with $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$. Then $w_i'(t) > -\delta w_i(t)$ for all $t \in \mathbb{R}$ and $i = 1, 2$.*

Proof. Since $w_1(t) = r^\delta u(r)$ with $r = e^{-t}$, Lemma 3.4 gives

$$w_1'(t) = -(\delta r^\delta u(r) + r^{\delta+1} u'(r)) > -\delta w_1(t).$$

Similarly, we can get $w_2'(t) > -\delta w_2(t)$ for all $t \in \mathbb{R}$. □

4 Positive solutions

In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. *Step 1.* We first prove that $E(r;u,v) \equiv constant$ if and only if (u,v) is of the form (1.10).

If $E(r;u,v)$ is a constant for all $r>0$. By Proposition 3.1,

$$\left(\frac{\partial u}{\partial \nu}+\frac{2}{p-1}\frac{u}{r}\right)^2+\left(\frac{\partial v}{\partial \nu}+\frac{2}{p-1}\frac{v}{r}\right)^2=0 \quad \text{in } \mathbb{R}^n\backslash\{0\}.$$

Integrating in r we get

$$u(x)=|x|^{-\frac{2}{p-1}}u\left(\frac{x}{|x|}\right) \text{ and } v(x)=|x|^{-\frac{2}{p-1}}v\left(\frac{x}{|x|}\right).$$

This shows that (u,v) is homogeneous of degree $-\frac{2}{p-1}$. On the other hand, since (u,v) is a nonnegative solution of (1.1) in $\mathbb{R}^n\backslash\{0\}$, by Theorem 2.1, then either (u,v) is trivial, i.e., $u=v\equiv 0$, or (u,v) is of the form

$$u(x)=kC_0|x|^{-\frac{2}{p-1}},\ \ v(x)=lC_0|x|^{-\frac{2}{p-1}},$$

where C_0 is given by (1.5) and $k,l\geq 0$ (not all 0) satisfy (1.9). We notice that if one of k and l is 0, such as $k=0$, then l satisfies $\mu_2 l^{2q}=1$. On the other hand, if (u,v)has the form (1.10), then, a direct calculation shows that $E(r;u,v)\equiv constant$.

Step 2. We compute the possible values of $E(0;u,v)$ and $E(\infty;u,v)$.

By Proposition 3.1 and Lemma 3.2, we know that the limit

$$E(0;u,v)=\lim_{r\to 0}E(r;u,v)$$

exists. For any $\lambda>0$, define the blowing up sequence

$$u^\lambda(x)=\lambda^{\frac{2}{p-1}}u(\lambda x),\quad v^\lambda(x)=\lambda^{\frac{2}{p-1}}v(\lambda x).$$

Then (u^λ,v^λ) is also a nonnegative solution of (1.1) in $\mathbb{R}^n\backslash\{0\}$. By Lemma 3.1, (u^λ,v^λ) is uniformly bounded in each compact subset of $\mathbb{R}^n\backslash\{0\}$. It follows from the interior regularity that (u^λ,v^λ) is uniformly bounded in $C^{2,\gamma}(K)$ on each compact set $K\subset\mathbb{R}^n\backslash\{0\}$, for some $0<\gamma<1$. Hence, there exists a nonnegative function $(u^0,v^0)\in C^2(\mathbb{R}^n\backslash\{0\})$ and a subsequence $\lambda_j\to 0$ such that $(u^{\lambda_j},v^{\lambda_j})$ converges to (u^0,v^0) in $C^2_{loc}(\mathbb{R}^n\backslash\{0\})$, and (u^0,v^0) also satisfies (1.1) in $\mathbb{R}^n\backslash\{0\}$. Moreover, by the scaling invariance of E, for any $r>0$, we have

$$E(r;u^0,v^0)=\lim_{j\to\infty}E(r;u^{\lambda_j},v^{\lambda_j})=\lim_{j\to\infty}E(r\lambda_j;u,v)=E(0;u,v). \tag{4.1}$$

That is, $E(r;u^0,v^0)$ is a constant for all $r>0$. By Step 1, either (u^0,v^0) is trivial, i.e., $u^0=v^0\equiv 0$, or (u^0,v^0) is of the form

$$u^0(x)=kC_0|x|^{-\frac{2}{p-1}},\ \ v^0(x)=lC_0|x|^{-\frac{2}{p-1}}, \tag{4.2}$$

where C_0 is given by (1.5) and $k, l \geq 0$ (not all 0) satisfy (1.9). If (u^0, v^0) is trivial, by (4.1), then $E(0, u, v) = 0$. If (u^0, v^0) has the form (4.2), then a direct calculation shows that

$$E(r, u^0, v^0) = -\frac{p-1}{2(p+1)}(k^2 + l^2)C_0^{p+1}$$

for all $r > 0$. Thus , by (4.1), we get $E(0, u, v) = -A_{k,l}$.

To get the possible value of $E(\infty; u, v)$, the method is similar to the above argument. We define the blowing down sequence

$$u^\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x), \quad v^\lambda(x) = \lambda^{\frac{2}{p-1}} v(\lambda x),$$

but in this case we will let $\lambda \to +\infty$. As in the above argument, there exists a non-negative function $(u^\infty, v^\infty) \in C^2(\mathbb{R}^n \backslash \{0\})$ and a subsequence $\lambda_j \to +\infty$ such that $(u^{\lambda_j}, v^{\lambda_j})$ converges to (u^∞, v^∞) in $C^2_{loc}(\mathbb{R}^n \backslash \{0\})$, and (u^∞, v^∞) also satisfies (1.1) in $\mathbb{R}^n \backslash \{0\}$. By the scaling invariance of E, for any $r > 0$, we have

$$E(r; u^\infty, v^\infty) = \lim_{j \to \infty} E(r; u^{\lambda_j}, v^{\lambda_j}) = \lim_{j \to \infty} E(r\lambda_j; u, v) = E(\infty; u, v). \qquad (4.3)$$

The rest of the argument is the same as the proof for $E(0; u, v)$. We also get

$$E(\infty; u, v) \in \left\{0, -\frac{p-1}{2(p+1)}(k^2 + l^2)C_0^{p+1}\right\}.$$

Step 3. We prove that $E(0; u, v) = 0$ if and only if (u, v) is trivial.

If $E(0; u, v) = 0$, then since $E(r; u, v)$ is nondecreasing for $r > 0$ and $E(\infty; u, v) \in \{0, -A_{k,l}\}$, we must have $E(r; u, v) = 0$ for all $r > 0$. By Step 1, this implies that either (u, v) is trivial, or is of the form (1.10) with $k, l \geq 0$ but not all 0. However, the latter gives $E(0; u, v) = -A_{k,l} < 0$, a contradiction. Hence, (u, v) must be trivial. The converse is obvious.

We are now ready to prove Theorem 1.2. Let (u, v) be a positive solution of (1.1). Obviously, Theorem 1.1 yields $E(0; u, v) = -A_{k,l} < 0$. We begin with some lemmas.

Lemma 4.1. *Let (u, v) be a positive solution of* (1.1) *with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. Then*

$$\liminf_{|x| \to 0} |x|^{\frac{2}{p-1}} (u(x) + v(x)) > 0.$$

Proof. We suppose by contradiction that

$$\liminf_{|x| \to 0} |x|^{\frac{2}{p-1}} (u(x) + v(x)) = 0.$$

Then there exists a sequence of positive numbers r_i converging to 0 such that

$$r_i^{\frac{2}{p-1}} (u(r_i) + v(r_i)) \to 0 \quad \text{as } i \to \infty. \qquad (4.4)$$

Define

$$u_i(x) = \frac{u(r_i x)}{u(r_i) + v(r_i)}, \quad v_i(x) = \frac{v(r_i x)}{u(r_i) + v(r_i)}.$$

By Harnack inequality (3.9), (u_i, v_i) is locally uniformly bounded away from the origin. Moreover, (u_i, v_i) satisfies

$$\begin{cases} -\Delta u_i = \left(r_i^{\frac{2}{p-1}} \left(u(r_i) + v(r_i) \right) \right)^{p-1} \left(\mu_1 u_i^{2q+1} + \beta u_i^q v_i^{q+1} \right) \\ -\Delta v_i = \left(r_i^{\frac{2}{p-1}} \left(u(r_i) + v(r_i) \right) \right)^{p-1} \left(\mu_2 v_i^{2q+1} + \beta v_i^q u_i^{q+1} \right) \end{cases} \quad \text{in } \mathbb{R}^n \backslash \{0\}.$$

It follows from the classical gradient estimates that ∇u_i and ∇v_i are locally uniformly bounded in $C_{loc}(\mathbb{R}^n \backslash \{0\})$. Hence, there exists some $C > 0$ independent of i such that

$$|\nabla u(x)|, \ |\nabla v(x)| \le C r_i^{-1} (u(r_i) + v(r_i))) = o(1) r_i^{-\frac{2}{p-1} - 1} \quad \text{for all } |x| = r_i.$$

This together with (4.4) easily yield $\lim_{i \to \infty} E(r_i; u, v) = 0$. By the monotonicity of E, we get $\lim_{r \to 0} E(r; u, v) = 0$. This contradicts Theorem 1.1. □

Lemma 4.2. *Let $n = 3$ and (u, v) be a nonnegative solution of (1.1) with $3 < p < 5$. If*

$$\lim_{|x| \to 0} |x|^{\frac{2}{p-1}} u(x) = 0,$$

then the singularity of u at $x = 0$ is removable.

Remark 4.1. *We remark that there is no additional assumption for v in Lemma 4.2.*

Proof. We consider an auxiliary function $w(x)$ as in [2], given by

$$w(x) = |x|^s u(x), \quad s > 0.$$

Since

$$w_{x_i} = s|x|^{s-2} x_i u(x) + |x|^s u_{x_i}$$

and

$$w_{x_i x_i} = s(s-2)|x|^{s-4} x_i^2 u + s|x|^{s-2} u + 2s|x|^{s-2} x_i u_{x_i} + |x|^s u_{x_i x_i},$$

we get

$$\Delta w - \frac{2s}{|x|^2} x \cdot \nabla w = \left[s(n-2-s) - |x|^2 (\mu_1 u^{2q} + \beta u^{q-1} v^{q+1}) \right] \frac{w}{|x|^2}.$$

By our assumption and (3.6) in Lemma 3.1, we have

$$\lim_{|x| \to 0} |x|^2 (\mu_1 u^{2q} + \beta u^{q-1} v^{q+1}) = 0.$$

Hence, for any $0 < s < 1$ there exists $R_s > 0$ such that

$$\Delta w - \frac{2s}{|x|^2} x \cdot \nabla w \ge 0 \quad \text{in } B_{R_s} \backslash \{0\}.$$

On the other hand, since $u(x) \leq C|x|^{-\frac{2}{p-1}}$ and $\frac{2}{p-1} < n-2$, we deduce that there exists $\epsilon > 0$ independent of s such that $w(x) = O(|x|^{2-n+\epsilon})$. By Theorem 1 and Remark 1 in [20], we obtain

$$u(x) \leq \left(\frac{R_s}{|x|}\right)^s \max_{\partial B_{R_s}} u(x), \qquad \text{for } x \in B_{R_s}\backslash\{0\}.$$

Therefore, we have $u \in L^\gamma(B_1)$ for all $\gamma > 1$.

Now we write the first equation of (1.1) in the form $-\Delta u = f(x)u$, where $f = \mu_1 u^{2q} + \beta u^{q-1}v^{q+1}$. Since $v^{q+1} \leq C|x|^{-\frac{p+1}{p-1}}$ and $p > 3$, we get

$$v^{q+1} \in L^{\frac{3}{2-\theta}}(B_1)$$

for some $\theta > 0$ small. Hence $f \in L^{\frac{3}{2-\theta_0}}(B_1)$ for some $\theta_0 > 0$ smaller. By Theorem 11 of Serrin [31] , we obtain that 0 is a removable singularity of u, i.e., $u(x)$ can be extended to a continuous H^1 weak solution in the entire ball B_1. □

Lemma 4.3. *Let $n \geq 4$ and (w_1, w_2) be a nonnegative solution of* (3.1) *with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. If*

$$\liminf_{t\to+\infty} w_1(t) = \liminf_{t\to+\infty} w_2(t) = 0,$$

then

$$\liminf_{t\to+\infty}(w_1 + w_2)(t) = 0.$$

Proof. If there exists T such that $w_1'(t) \leq 0$ for all $t > T$, then we have

$$\lim_{t\to+\infty} w_1(t) = 0,$$

and hence $\liminf_{t\to+\infty}(w_1 + w_2)(t) = 0$. Otherwise, there exits a sequence of local minimum point t_i of w_1 such that $t_i \to +\infty$ and $w_1(t_i) \to 0$. By $w_1''(t_i) \geq 0$ and the first equation of (3.1), we have

$$\beta w_2^{q+1}(t_i) \leq \sigma w_1^{1-q}(t_i).$$

By $n \geq 4$, we have $0 < q < 1$, and so $w_2(t_i) \to 0$. Hence $(w_1 + w_2)(t_i) \to 0$. The desired conclusion follows. □

Proof of (1) in Theorem 1.2. We first prove that (1.1) has no positive semi-singular solutions at 0 for $n \geq 4$.

Let $n \geq 4$. Suppose by contradiction that (u, v) is a positive semi-singular solution at 0 of (1.1). Without loss of generality, we assume

$$\lim_{r\to 0} u(r) = +\infty \quad \text{and} \quad \lim_{r\to 0} v(r) < +\infty. \tag{4.5}$$

Then there exists $C_0 > 0$ such that

$$u(r) \geq C_0 r^{-\frac{2}{p-1}}, \qquad \text{for } r \in (0, 1]. \tag{4.6}$$

Indeed, if (4.6) does not hold, then there exists $r_i \to 0$ such that $r_n^{\frac{2}{p-1}} u(r_i) \to 0$. Since v is bounded near the origin, hence

$$\liminf_{|x|\to 0} |x|^{\frac{2}{p-1}} (u(x) + v(x)) = 0.$$

This contradicts Lemma 4.1.

On the other hand, by the decreasing property of v, there exists $C_1 > 0$ such that $v(r) > C_1$ for all $(0, 1]$. By Lemma 3.1, we have $r^{n-1} v'(r) \to 0$ as $r \to 0$. Since $v(x)$ is radially symmetry, we write the second equation of (1.1) in the form

$$-(r^{n-1} v'(r))' = r^{n-1} (\mu_2 v^{p+1} + \beta v^q u^{q+1}).$$

Integrating from 0 to r, we get

$$\begin{aligned} -r^{n-1} v'(r) &= \int_0^r \rho^{n-1} (\mu_2 v^{p+1} + \beta v^q u^{q+1}) d\rho \geq \beta \int_0^r \rho^{n-1} v^q u^{q+1} d\rho \\ &\geq C \int_0^r \rho^{n-1} \rho^{-\frac{p+1}{p-1}} d\rho = C r^{n-\frac{p+1}{p-1}}, \end{aligned}$$

that is, we have $-v'(r) \geq C r^{-\frac{2}{p-1}}$ for all $r \in (0, 1]$. For any $r \in (0, 1)$, integrating from r to 1, we obtain

$$v(r) - v(1) \geq C \int_r^1 \rho^{-\frac{2}{p-1}} d\rho = C \left(\frac{2}{p-1} - 1 \right)^{-1} (r^{1-\frac{2}{p-1}} - 1).$$

We note that $1 - \frac{2}{p-1} < 0$ if $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and $n \geq 4$. Hence we get $v(r) \to +\infty$ as $r \to 0$, a contradiction with (4.5). This finishes the proof of (1) in Theorem 1.2.

Proof of (2) in Theorem 1.2. Let (u, v) be a positive solution of (1.1). We will prove that (1.13) holds under the assumption of (2) in Theorem 1.2. By Lemma 3.1, we only need to prove

$$\liminf_{|x|\to 0} |x|^{\frac{2}{p-1}} u(x) > 0 \quad \text{and} \quad \liminf_{|x|\to 0} |x|^{\frac{2}{p-1}} v(x) > 0. \tag{4.7}$$

This is equivalent to prove

$$\liminf_{t\to +\infty} w_1(t) > 0 \quad \text{and} \quad \liminf_{t\to +\infty} w_2(t) > 0. \tag{4.8}$$

We prove (4.8) by considering two cases separately.

Case 1. $n \geq 4$.

Suppose that (4.8) does not hold. Then by Lemma 4.1 and 4.3, we may assume, without loss of generality, that

$$\liminf_{t\to +\infty} w_1(t) = 0 \quad \text{and} \quad \liminf_{t\to +\infty} w_2(t) = C_1 > 0. \tag{4.9}$$

If $\limsup_{t\to+\infty} w_1(t) > 0$, then there exits a sequence of local minimum point t_i of w_1 such that $t_i \to +\infty$ and $w_1(t_i) \to 0$. By $w_1''(t_i) \ge 0$ and the first equation of (3.1), we have

$$\beta w_2^{q+1}(t_i) \le \sigma w_1^{1-q}(t_i).$$

By $n \ge 4$, we have $0 < q < 1$, and so $w_2(t_i) \to 0$, a contradiction with (4.9). Hence, we get

$$\lim_{t\to+\infty} w_1(t) = 0. \tag{4.10}$$

It follows from the first equation of (3.1), Corollary 3.1 and (4.9) that

$$\begin{aligned} w_1''(t) &= -\tau w_1'(t) + \sigma w_1(t) - (\mu_1 w_1^{2q+1} + \beta w_1^q w_2^{q+1})(t) \\ &\le \tau\delta w_1(t) + \sigma w_1(t) - (\mu_1 w_1^{2q+1} + \beta w_1^q w_2^{q+1})(t) \\ &= \delta^2 w_1(t) - (\mu_1 w_1^{2q+1} + \beta w_1^q w_2^{q+1})(t) \\ &= w_1^q(t)(\delta^2 w_1^{1-q} - \mu_1 w_1^{q+1} - \beta w_2^{q+1})(t) < 0 \quad \text{ for all } t \ge T_0 \end{aligned} \tag{4.11}$$

with some $T_0 > 0$ large. If $w_1'(t) > 0$ for all $t > T_0$, then $w_1(t) \ge w_1(T_0) > 0$ for all $t > T_0$, a contradiction with (4.10). So there exists $T_1 > T_0$ such that $w_1'(T_1) \le 0$. By (4.11), we have $w_1'(t) \le 0$ for all $t \ge T_1$. By Corollary 3.1,

$$|w_1'(t)| = -w_1'(t) \le \delta w_1(t) \quad \text{for all } t \ge T_1.$$

So we get $|w_1'(t)| \to 0$ as $t \to +\infty$ by (4.10). We easily deduce from (4.11) that $w_1'(t) > 0$ for all $t \ge T_0$, a contradiction with (4.10). Therefore (4.8) holds.

Case 2. $n = 3$ and assume that (1.1) has no positive semi-singular solutions at 0.

Under this assumption, we have

$$\lim_{r\to 0} u(r) = \lim_{r\to 0} v(r) = +\infty. \tag{4.12}$$

Suppose by contradiction that (4.8) does not hold. Without loss of generality, we assume that $\liminf_{t\to+\infty} w_1(t) = 0$. Note that $1 < q < 2$ by $n = 3$. By Lemma 3.1, we know that w_1 and w_2 are uniformly bounded for $t \in \mathbb{R}$. Then there exists $c > 0$ such that

$$w_1''(t) + \tau w_1'(t) \ge \sigma w_1(t) - c w_1^q(t) \qquad \text{in } \mathbb{R}. \tag{4.13}$$

If $\limsup_{t\to+\infty} w_1(t) > 0$, then there exits a sequence of local minimum point t_i of w_1 such that $t_i \to +\infty$ and $w_1(t_i) \to 0$. By (4.13), there exists $\epsilon > 0$ small such that

$$\frac{d}{dt}\left(e^{\tau t} w_1'(t)\right) = e^{\tau t}(w_1''(t) + \tau w_1'(t)) > 0 \tag{4.14}$$

whenever $w_1(t) \le 2\epsilon$. Hence, there exist $t_i^* < t_i$ such that $w_1(t_i^*) = \epsilon$ and $w_1'(t) < 0$ for $t \in [t_i^*, t_i)$. So we have (making ϵ smaller if necessary)

$$w_1''(t) \ge \sigma w_1(t) - c w_1^q(t) \ge \frac{\sigma}{2} w_1(t) \qquad \text{in } [t_i^*, t_i). \tag{4.15}$$

Hence $(w_1')^2 - \frac{\sigma}{2}w_1^2$ is nonincreasing in $[t_i^*, t_i)$. In particular, we have

$$(w_1')^2(t) - \frac{\sigma}{2}w_1^2(t) \geq -\frac{\sigma}{2}w_1^2(t_i)$$

for $t \in [t_i^*, t_i)$. Integrating the inequality above, we get

$$t_i - t_i^* \leq \sqrt{\frac{2}{\sigma}} \int_{w_1(t_i)}^{w_1(t_i^*)} \frac{dw}{\sqrt{w^2 - w_1^2(t_i)}} \leq \sqrt{\frac{2}{\sigma}} \log \frac{2w_1(t_i^*)}{w_1(t_i)}.$$

Therefore,

$$w_1(t_i) \leq 2\epsilon e^{-\sqrt{\frac{\sigma}{2}}(t_i - t_i^*)}.$$

Let

$$W_1(t) := 2\epsilon e^{-\sqrt{\frac{\sigma}{2}}(t - t_i^*)}.$$

Then we have

$$\begin{cases} -w_1''(t) + \frac{\sigma}{2}w_1(t) \leq 0 = -W_1''(t) + \frac{\sigma}{2}W_1(t) & \text{in } [t_i^*, t_i), \\ w_1(t_i^*) \leq W_1(t_i^*), \\ w_1(t_i) \leq W_1(t_i). \end{cases} \tag{4.16}$$

The maximum principle gives

$$w_1(t) \leq W_1(t) = 2\epsilon e^{-\sqrt{\frac{\sigma}{2}}(t - t_i^*)} \qquad \text{in } [t_i^*, t_i). \tag{4.17}$$

By Lemma 3.1, there exists $C > 0$ such that $w_1, w_2, |w_1'|, |w_2'| \leq C$ for all $t \in \mathbb{R}$. Hence, up to a subsequence, $w_1(\cdot + t_i^*) \to \widetilde{w}_1 \geq 0$ and $w_2(\cdot + t_i^*) \to \widetilde{w}_2 \geq 0$ uniformly in $C^2_{loc}(\mathbb{R})$, where $\widetilde{w}_1(0) = 1$ and $(\widetilde{w}_1, \widetilde{w}_2)$ satisfies

$$\begin{cases} \widetilde{w}_1'' + \tau\widetilde{w}_1' - \sigma\widetilde{w}_1 + \mu_1\widetilde{w}_1^{2q+1} + \beta\widetilde{w}_1^q\widetilde{w}_2^{q+1} = 0 \\ \widetilde{w}_2'' + \tau\widetilde{w}_2' - \sigma\widetilde{w}_2 + \mu_2\widetilde{w}_2^{2q+1} + \beta\widetilde{w}_2^q\widetilde{w}_1^{q+1} = 0 \end{cases} \quad t \in \mathbb{R}.$$

Furthermore,

$$\begin{aligned} &\frac{1}{2}\left(|\widetilde{w}_1'|^2 + |\widetilde{w}_2'|^2 - \sigma\left(\widetilde{w}_1^2 + \widetilde{w}_2^2\right)\right)(t) \\ &+ \frac{1}{p+1}\left(\mu_1\widetilde{w}_1^{p+1} + 2\beta\widetilde{w}_1^{q+1}\widetilde{w}_2^{q+1} + \mu_2\widetilde{w}_2^{p+1}\right)(t) \equiv \Psi(+\infty). \end{aligned}$$

Here Ψ is defined in (3.3). Clearly, the limit $\Psi(+\infty) := \lim_{t\to+\infty}\Psi(t)$ exists. Therefore, we have

$$-\tau\left[(\widetilde{w}_1')^2 + (\widetilde{w}_2')^2\right](t) \equiv 0 \quad t \in \mathbb{R}.$$

So $\widetilde{w}_1(t) \equiv 1$ for $t \in \mathbb{R}$. On the other hand, since $w_1'(t) \leq 0$ in $[t_i^*, t_i]$, by Corollary 3.1, we obtain $|w_1'(t)| = -w_1'(t) \leq \delta w_1(t)$ for all $t \in [t_i^*, t_i]$. By the mean value theorem, we have

$$t_i - t_i^* \geq \frac{1}{\delta}\ln\frac{\epsilon}{w_1(t_i)} \to +\infty.$$

This together with (4.17) yield

$$\widetilde{w}_1(t) \le 2\epsilon e^{-\sqrt{\frac{\sigma}{2}}t}, \quad \text{for } t > 0.$$

This is a contradiction with $\widetilde{w}_1(t) \equiv 1$ for $t \in \mathbb{R}$. Therefore, $\lim_{t\to+\infty} w_1(t) = 0$, namely, $\lim_{r\to 0} r^\delta u(r) = 0$. By Lemma 4.2, the singularity of u at 0 is removable, and so $\lim_{r\to 0} u(r) < +\infty$, a contradiction with (4.12). This finishes the proof of (2) in Theorem 1.2.

Next we are going to prove (3) and (4) in Theorem 1.2. To this end, we define the Kelvin transform

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right), \qquad \bar{v}(x) = \frac{1}{|x|^{n-2}} v\left(\frac{x}{|x|^2}\right).$$

Then $(\bar{u}, \bar{v})$ satisfies

$$\begin{cases} -\Delta \bar{u} = |x|^\alpha (\mu_1 \bar{u}^{2q+1} + \beta \bar{u}^q \bar{v}^{q+1}) \\ -\Delta \bar{v} = |x|^\alpha (\mu_2 \bar{v}^{2q+1} + \beta \bar{v}^q \bar{u}^{q+1}) \end{cases} \quad \text{in } \mathbb{R}^n \backslash \{0\}, \tag{4.18}$$

where $\alpha := p(n-2) - (n+2) \in (-2, 0)$. By Lemma 3.1, we also have

$$\bar{u}(x),\ \bar{v}(x) \le C|x|^{-\left(n-2-\frac{2}{p-1}\right)} = C|x|^{-\frac{2+\alpha}{p-1}} \quad \text{in } \mathbb{R}^n \backslash \{0\} \tag{4.19}$$

and

$$|\nabla \bar{u}(x)|,\ |\nabla \bar{v}(x)| \le C|x|^{-\left(n-1-\frac{2}{p-1}\right)} = C|x|^{-\frac{2+\alpha}{p-1}-1} \quad \text{in } \mathbb{R}^n \backslash \{0\}. \tag{4.20}$$

Denote $\delta_0 = \frac{2+\alpha}{p-1}$. Let $t = -\ln r$ and

$$\bar{w}_1(t) := r^{\delta_0} \bar{u}(r) = e^{-\delta_0 t} \bar{u}(e^{-t}), \qquad \bar{w}_2(t) := r^{\delta_0} \bar{v}(r) = e^{-\delta_0 t} \bar{v}(e^{-t}).$$

Then by a direct calculation $(\bar{w}_1, \bar{w}_2)$ satisfies

$$\begin{cases} \bar{w}_1'' + \tau_0 \bar{w}_1' - \sigma_0 \bar{w}_1 + \mu_1 \bar{w}_1^{2q+1} + \beta \bar{w}_1^q \bar{w}_2^{q+1} = 0 \\ \bar{w}_2'' + \tau_0 \bar{w}_2' - \sigma_0 \bar{w}_2 + \mu_2 \bar{w}_2^{2q+1} + \beta \bar{w}_2^q \bar{w}_1^{q+1} = 0 \end{cases} \quad t \in \mathbb{R}, \tag{4.21}$$

where

$$\tau_0 = \frac{n-2}{p-1}\left(\frac{n+2+2\alpha}{n-2} - p\right) < 0, \quad \sigma_0 = \frac{(2+\alpha)(n-2)}{(p-1)^2}\left(p - \frac{n+\alpha}{n-2}\right) > 0.$$

We note that there are some differences between system (4.21) and system (3.1). In particular, the coefficient τ_0 in (4.21) is *less than* 0, and the coefficient τ in (3.1) is *greater than* 0. By Lemma 2.1, both $\bar{u}$ and $\bar{v}$ are also radially symmetric. Similar to the proof of Lemma 3.4 and Corollary 3.1, we have

Lemma 4.4. *Assume that* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ *and* $\alpha = p(n-2) - (n+2)$.

(1) Let $(\bar u, \bar v)$ be a positive solution of (4.18). *Then both $\bar u'(r) < 0$ and $\bar v'(r) < 0$ for all $r > 0$.*

(2) Let $(\bar w_1, \bar w_2)$ be a positive solution of (4.21). *Then $\bar w_i'(t) > -\delta_0 \bar w_i(t)$ for all $t \in \mathbb{R}$ and $i = 1, 2$.*

We define

$$\begin{aligned}\bar\Psi(t; \bar w_1, \bar w_2) :=& \frac{1}{2}\left(|\bar w_1'|^2 + |\bar w_2'|^2 - \sigma_0\left(\bar w_1^2 + \bar w_2^2\right)\right)(t)\\ &+ \frac{1}{p+1}\left(\mu_1 \bar w_1^{p+1} + 2\beta \bar w_1^{q+1}\bar w_2^{q+1} + \mu_2 \bar w_2^{p+1}\right)(t).\end{aligned} \tag{4.22}$$

Multiplying the first equation of (4.21) by $\bar w_1'$, the second equation of (4.21) by $\bar w_2'$, we easily obtain

Lemma 4.5. *Let $(\bar w_1, \bar w_2)$ be a nonnegative solution of* (4.21) *with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. Then $\bar\Psi$ is nondecreasing and uniformly bounded for $t \in \mathbb{R}$. Moreover,*

$$\frac{d}{dt}\bar\Psi(t; \bar w_1, \bar w_2) = -\tau_0\left[(\bar w_1')^2 + (\bar w_2')^2\right](t). \tag{4.23}$$

Hence the limit $\bar\Psi(+\infty; \bar w_1, \bar w_2) = \lim_{t\to+\infty}\bar\Psi(t; \bar w_1, \bar w_2)$ exists. Further, we also have

Lemma 4.6. $\bar\Psi(+\infty; \bar w_1, \bar w_2) \in \left\{0, -\frac{p-1}{2(p+1)}(k^2 + l^2)\sigma_0^{\frac{p+1}{p-1}}\right\}$, *where $k, l \geq 0$ (not all 0) satisfy* (1.9).

Proof. Given any sequence $t_i \to +\infty$, up to a subsequence, $\bar w_1(\cdot + t_i) \to z_1 \geq 0$ and $\bar w_2(\cdot + t_i) \to z_2 \geq 0$ in $C^2_{loc}(\mathbb{R})$. Then (z_1, z_2) satisfies (4.21) and

$$\bar\Psi(t; z_1, z_2) = \lim_{i\to\infty}\bar\Psi(t; \bar w_1(\cdot + t_i), \bar w_2(\cdot + t_i)) \tag{4.24}$$

$$= \lim_{i\to\infty}\bar\Psi(t + t_i; \bar w_1, \bar w_2) = \bar\Psi(+\infty; \bar w_1, \bar w_2). \tag{4.25}$$

That is, $\bar\Psi(t; z_1, z_2)$ is a constant for all $t \in \mathbb{R}$. By (4.22), both z_1 and z_2 are also constant. Then, either $z_1 = z_2 \equiv 0$, or $z_1 = k\sigma_0^{\frac{1}{p-1}}$ and $z_2 = l\sigma_0^{\frac{1}{p-1}}$, where $k, l \geq 0$ (not all 0) satisfy (1.9). The conclusion follows easily by (4.24). □

Proof of (3) in Theorem 1.2. We prove the nonexistence of positive semi-singular solutions at ∞ for $n \geq 4$.

Let $n \geq 4$. Suppose by contradiction that (u, v) is a positive semi-singular solution of (1.1) at ∞. Without loss of generality, we assume

$$u(r) \leq Cr^{-(n-2)} \quad \text{and} \quad v(r) \geq Cr^{-\frac{2}{p-1}} \qquad \text{for } r \text{ large}. \tag{4.26}$$

Then

$$\bar u(r) \leq C \quad \text{and} \quad \bar v(r) \geq Cr^{-(n-2-\frac{2}{p-1})} \qquad \text{for } 0 < r < r_0 \tag{4.27}$$

with some $0 < r_0 < 1$. Clearly there exists $C_0 > 0$ such that $\bar{u}(r) \geq C_0$ for all $r \in (0, r_0]$. By (4.20), we have $r^{n-1}\bar{u}'(r) \to 0$ as $r \to 0$. Since

$$-(r^{n-1}\bar{u}'(r))' = r^{n-1}r^{\alpha}(\mu_1\bar{u}^{p+1} + \beta\bar{u}^q\bar{v}^{q+1}).$$

Integrating from 0 to r, we get

$$\begin{aligned} -r^{n-1}\bar{u}'(r) &= \int_0^r \rho^{n-1}\rho^{\alpha}(\mu_1\bar{u}^{p+1} + \beta\bar{u}^q\bar{v}^{q+1})d\rho \geq \beta\int_0^r \rho^{n-1+\alpha}\bar{u}^q\bar{v}^{q+1}d\rho \\ &\geq C\int_0^r \rho^{n-1+\alpha}\rho^{-(n-2-\frac{2}{p-1})\frac{p+1}{2}}d\rho = Cr^{\frac{p-1}{2}(n-2)+\frac{2}{p-1}-1}, \end{aligned}$$

namely $-\bar{u}'(r) \geq Cr^{\frac{p-1}{2}(n-2)+\frac{2}{p-1}-n}$ for all $r \in (0, r_0]$. If $n \geq 4$, then a simple analysis gives $\frac{p-1}{2}(n-2) + \frac{2}{p-1} - n < -1$ for all $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. This implies for any $r \in (0, r_0)$ that

$$\bar{u}(r) - \bar{u}(r_0) \geq C\int_r^{r_0} \rho^{\frac{p-1}{2}(n-2)+\frac{2}{p-1}-n}d\rho \geq C\int_r^{r_0} \rho^{-1}d\rho = C(-\ln r + \ln r_0).$$

Hence, $\bar{u}(r) \to +\infty$ as $r \to 0$, a contradiction with (4.27). This completes the proof of (3) in Theorem 1.2.

Proof of (4) in Theorem 1.2. We now prove (4) in Theorem 1.2 by discussing two steps separately.

Step 1. $\bar{\Psi}(+\infty; \bar{w}_1, \bar{w}_2) = 0$.

In this case, by the proof of Lemma 4.6, we know that

$$\lim_{t\to+\infty} \bar{w}_1(t) = \lim_{t\to+\infty} \bar{w}_2(t) = 0. \tag{4.28}$$

Let $\bar{w} = \bar{w}_1 + \bar{w}_2$. It follows from (4.21) and Lemma 4.4 (2) that

$$\begin{aligned} \bar{w}'' &\geq -\tau_0\bar{w}' + \sigma_0\bar{w} - \bar{C}\bar{w}^p \\ &\geq (\tau_0\delta_0 + \sigma_0)\bar{w} - \bar{C}\bar{w}^p = \delta_0^2\bar{w} - \bar{C}\bar{w}^p, \end{aligned}$$

where $\bar{C} = \bar{C}(\mu_1, \mu_2, \beta, p)$. By (4.28), $\bar{w}''(t) > 0$ for $t > T_1$ with some $T_1 > 0$ large, and hence

$$\bar{w}'(t) < 0 \quad \text{for } t > T_1. \tag{4.29}$$

Otherwise, there exists $t_1 > T_1$ such that $\bar{w}'(t_1) > 0$, and then $\bar{w}'(t) \geq \bar{w}'(t_1) > 0$ for all $t > t_1$. We get $\bar{w}(t) \to +\infty$ as $t \to +\infty$, a contradiction with (4.28). Let $\bar{z}(t) = \bar{w}'(t) + \delta_0\bar{w}(t)$. Then

$$\bar{z}' - \delta_0\bar{z} = \bar{w}'' - \delta_0^2\bar{w} \geq -\bar{C}\bar{w}^p. \tag{4.30}$$

By (4.19) and (4.20), we easily deduce that $\bar{z}$ is bounded. It follows from (4.29) and (4.30) that

$$\bar{z}(t) \leq \bar{C}e^{\delta_0 t}\int_t^{+\infty} e^{-\delta_0\rho}\bar{w}^p(\rho)d\rho \leq \frac{\bar{C}}{\delta_0}w^p(t), \quad \forall\, t > T_1.$$

That is,

$$\bar{w}'(t) + \delta_0 \bar{w}(t) \le \frac{\bar{C}}{\delta_0} w^p(t), \quad \forall\, t > T_1.$$

From this we easily obtain

$$\frac{d}{dt}\left[(e^{\delta_0 t}\bar{w}(t))^{1-p} - \frac{\bar{C}}{\delta_0^2} e^{(1-p)\delta_0 t}\right] \ge 0, \quad \forall\, t > T_1. \tag{4.31}$$

If $\limsup_{t\to+\infty} e^{\delta_0 t}\bar{w}(t) = +\infty$, then there exists a sequence $t_i \to +\infty$ such that $(e^{\delta_0 t_i}\bar{w}(t_i))^{1-p} - \frac{\bar{C}}{\delta_0^2} e^{(1-p)\delta_0 t_i} \to 0$. By (4.31), we have

$$(e^{\delta_0 t}\bar{w}(t))^{1-p} - \frac{\bar{C}}{\delta_0^2} e^{(1-p)\delta_0 t} \le 0, \quad \forall\, t > T_1.$$

This implies that $\bar{w}(t) \ge \left(\frac{\delta_0}{\bar{C}}\right)^{p-1}$ for all $t > T_1$, a contradiction with (4.28). Therefore, we have $\limsup_{t\to+\infty} e^{\delta_0 t}\bar{w}(t) < +\infty$ and so $e^{\delta_0 t}\bar{w}(t) \le C$ uniformly for $t > 0$ large enough. We obtain that $\bar{u} + \bar{v} \le C$ uniformly for $r > 0$ small. That is,

$$u(x) + v(x) \le C|x|^{-(n-2)}$$

uniformly for $|x|$ large.

Step 2. $\bar{\Psi}(+\infty; \bar{w}_1, \bar{w}_2) < 0$.

Remark that, in this case, the limits $\lim_{t\to+\infty} \bar{w}_1(t)$ and $\lim_{t\to+\infty} \bar{w}_2(t)$ *cannot be guaranteed to exist* by the proof of Lemma 4.6. See Remark 1.2.

We claim

$$\liminf_{t\to+\infty} \bar{w}_1(t) > 0 \quad \text{and} \quad \liminf_{t\to+\infty} \bar{w}_2(t) > 0. \tag{4.32}$$

We prove this claim by considering two cases separately.

Case 1. $n \ge 4$.

We just need to modify the proof of Case 1 in Theorem 1.2 (2), the main difference is $\tau_0 < 0$ here and $\tau > 0$ there. Suppose that (4.32) does not hold. If

$$\liminf_{t\to+\infty} \bar{w}_1(t) = 0 \quad \text{and} \quad \liminf_{t\to+\infty} \bar{w}_2(t) = 0. \tag{4.33}$$

then the same argument as that of Lemma 4.3 gives $\liminf_{t\to+\infty}(\bar{w}_1+\bar{w}_2)(t) = 0$. We take $t_i \to +\infty$ such that $(\bar{w}_1+\bar{w}_2)(t_i) \to 0$. Then up to a subsequence, $\bar{w}_1(\cdot+t_i) \to \bar{z}_1$ and $\bar{w}_2(\cdot + t_i) \to \bar{z}_2$ uniformly in $C^2_{loc}(\mathbb{R})$, where $\bar{z}_1(0) = \bar{z}_2(0) = 0$ and $(\bar{z}_1, \bar{z}_2)$ satisfies (4.21). Moreover, $\bar{\Psi}(t; \bar{z}_1, \bar{z}_2) \equiv \bar{\Psi}(+\infty; \bar{w}_1, \bar{w}_2)$ for all $t \in \mathbb{R}$. By (4.22), $\bar{z}_1 = \bar{z}_2 \equiv 0$, a contradiction with $\bar{\Psi}(+\infty; \bar{w}_1, \bar{w}_2) < 0$. Hence (4.33) is impossible. Without loss of generality, we may assume that

$$\liminf_{t\to+\infty} \bar{w}_1(t) = 0 \quad \text{and} \quad \liminf_{t\to+\infty} \bar{w}_2(t) = C_2 > 0. \tag{4.34}$$

The same argument as that of Case 1 in Theorem 1.2 (2) yields $\lim_{t\to+\infty}\bar{w}_1(t)=0$. It follows from the first equation of (4.21) that

$$\bar{w}_1''+\tau_0\bar{w}_1'=\bar{w}_1^q(\sigma_0\bar{w}_1^{1-q}-\mu_1\bar{w}_1^{q+1}-\beta\bar{w}_2^{q+1})<0 \quad \text{for all } t\geq T_3 \tag{4.35}$$

with some $T_3>0$ large, and then $e^{\tau_0 t}\bar{w}_1'(t)$ is strictly decreasing for $t>T_3$. If there exists a sequence $t_i\to+\infty$ such that $\bar{w}_1'(t_i)\geq 0$, then $\bar{w}_1'(t)\geq 0$ for all $t>T_3$, a contradiction with $\lim_{t\to+\infty}\bar{w}_1(t)=0$. Hence there exists $T_4>T_3$ such that $\bar{w}_1'(t)<0$ for all $t>T_4$. By (4.35) and $\tau_0<0$, we have $\bar{w}_1''(t)<0$ for $t>T_4$. On the other hand, by Lemma 4.4 (2), $|\bar{w}_1'(t)|=-\bar{w}_1'(t)\leq\delta_0\bar{w}_1(t)\to 0$ as $t\to+\infty$. We get $\bar{w}_1'(t)>0$ for $t>T_4$, a contradiction with $\lim_{t\to+\infty}\bar{w}_1(t)=0$. Therefore, (4.32) holds.

Case 2. $n=3$ and assume that the system (1.1) has no positive semi-singular solutions at ∞.

Suppose by contradiction that (4.32) does not hold, without loss of generality, we assume $\liminf_{t\to+\infty}\bar{w}_1(t)=0$. Since $\bar{w}_1(t)$ and $\bar{w}_2(t)$ are uniformly bounded for $t\in\mathbb{R}$, by Lemma 4.4, there exists $\bar{c}>0$ such that

$$\bar{w}_1''\geq-\tau_0\bar{w}_1'+\sigma_0\bar{w}_1-\bar{c}\bar{w}_1^q\geq\delta_0^2\bar{w}_1-\bar{c}\bar{w}_1^q \quad \text{in } \mathbb{R}. \tag{4.36}$$

If $\limsup_{t\to+\infty}\bar{w}_1(t)>0$, then there exists a sequence of local minimum points t_i of $\bar{w}_1$ such that $t_i\to+\infty$ and $\bar{w}_1(t_i)\to 0$. By (4.36), there exists $\bar{\epsilon}>0$ small such that $\bar{w}_1''(t)>0$ whenever $\bar{w}_1(t)<2\bar{\epsilon}$. Therefore, there exist $\bar{t}_i<t_i$ such that $\bar{w}_1(\bar{t}_i)=\bar{\epsilon}$ and $\bar{w}_1'(t)<0$ for $t\in[\bar{t}_i,t_i)$. By the same argument as that of Case 2 in Theorem 1.2 (2), we can get a contradiction. Hence $\lim_{t\to+\infty}\bar{w}_1(t)=0$. Then there exists $\bar{T}>0$ large such that $\bar{w}_1'(t)<0$ for $t\geq\bar{T}$. For any $\epsilon>0$ small, by (4.36), we can choose $\bar{T}$ large enough such that

$$\bar{w}_1''(t)-(\delta_0-\epsilon)^2\bar{w}_1(t)\geq 0 \quad \text{for } t\geq\bar{T}.$$

By a simple comparison principle argument, we get

$$\bar{w}_1(t)\leq\bar{w}_1(\bar{T})\exp\{-(\delta_0-\epsilon)(t-\bar{T})\} \quad \text{for } t\geq\bar{T}.$$

From this we have that for any $\epsilon>0$, there exists $r_\epsilon>0$ and $c(\epsilon)>0$ such that $\bar{u}(x)\leq c(\epsilon)|x|^{-\epsilon}$ for $|x|<r_\epsilon$. Hence $\bar{u}\in L^\gamma(B_1)$ for all $\gamma>1$. Recall that $n=3$, we have

$$|x|^\alpha\bar{v}^{q+1}\leq C|x|^{\frac{p-1}{2}+\frac{2}{p-1}-4} \quad \text{for } 0<|x|<1.$$

Since $\frac{p-1}{2}+\frac{2}{p-1}-4>-2$ by $3<p<5$, we obtain $|x|^\alpha\bar{v}^{q+1}\in L^{\frac{3}{2-\theta_1}}(B_1)$ with some $\theta_1>0$ small. We write the first equation of (4.18) in the form $-\Delta\bar{u}=g(x)\bar{u}$, where $g(x)=|x|^\alpha(\mu_1\bar{u}^{2q}+\beta\bar{u}^{q-1}\bar{v}^{q+1})(x)$, then $g\in L^{\frac{3}{2-\theta_2}}(B_1)$ for some $\theta_2>0$ small. By Theorem 11 of Serrin [31] , we obtain that 0 is a removable singularity of $\bar{u}$. In particular, $\bar{u}(x)\leq C$ for $|x|$ small, where C is a positive constant. Hence we have $u(x)\leq C|x|^{-(n-2)}$ for $|x|$ large.

If $\liminf_{t\to+\infty}\bar{w}_2(t)=0$, then $\lim_{t\to+\infty}(\bar{w}_1+\bar{w}_2)(t)=0$, we easily obtain $\bar{\Psi}(+\infty;\bar{w}_1,\bar{w}_2)=0$, a contradiction. If $\liminf_{t\to+\infty}\bar{w}_2(t)>0$, then $v(x)\geq$

$C|x|^{-\frac{2}{p-1}}$ for $|x|$ large, a contradiction with the assumption. We complete the proof (4) in Theorem 1.2.

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