

Asymptotic behavior of positive singular solutions to fractional Hardy-Hénon equations*

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Abstract

In this paper, we study the asymptotic behavior of positive solutions of fractional Hardy-Hénon equation

$$(-\Delta)^\sigma u = |x|^\alpha u^p \quad \text{in } B_1 \setminus \{0\}$$

with an isolated singularity at the origin, where $\sigma \in (0, 1)$ and the punctured unit ball $B_1 \setminus \{0\} \subset \mathbb{R}^n$ with $n \geq 2$. When $-2\sigma < \alpha < 2\sigma$ and $\frac{n+\alpha}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$, we give a classification of isolated singularities of positive solutions near $x = 0$. Further, we prove the asymptotic behavior of positive singular solutions as $x \rightarrow 0$. These results parallel those known for the Laplacian counterpart proved by Caffarelli, Gidas and Spruck (Caffarelli, Gidas and Spruck in *Comm Pure Appl Math*, 1981, 1989), but the technique is very different, since the ODEs analysis is a missing ingredient in the fractional case. Our proofs are based on a monotonicity formula, combined with a blow up (down) argument, the Kelvin transformation and uniqueness of solutions of related degenerate equations on semi spherical surface \mathbb{S}_+^n . We also investigate isolated singularities located at infinity of fractional Hardy-Hénon equation.

Key words: Isolated singularities; asymptotic behavior; positive singular solutions; fractional Hardy-Hénon equations

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1 Introduction and Main Results

In the classical paper [18], Gidas and Spruck studied the asymptotic behavior of positive solutions of the following equation

$$-\Delta u = |x|^\alpha u^p \quad \text{in } B_1 \setminus \{0\} \tag{1.1}$$

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with an isolated singularity at the origin, where the punctured unit ball $B_1 \setminus \{0\} \subset \mathbb{R}^n$ with $n \geq 3$. Eq. (1.1) is called the Hardy (resp. Lane-Emden, or Hénon) equation for $\alpha < 0$ (resp. $\alpha = 0, \alpha > 0$). More specifically, assume

$$-2 < \alpha < 2 \quad \text{and} \quad \frac{n + \alpha}{n - 2} < p < \frac{n + 2}{n - 2}.$$

Let u be a positive C^2 solution of (1.1) in $B_1 \setminus \{0\}$. Then Gidas-Spruck [18] proved that either the singularity at $x = 0$ is removable, or there exist positive constants c_1, c_2 such that

$$\frac{c_1}{|x|^{(2+\alpha)/(p-1)}} \leq u(x) \leq \frac{c_2}{|x|^{(2+\alpha)/(p-1)}} \quad \text{near } x = 0. \quad (1.2)$$

Further, assume additionally that $p \neq \frac{n+2+2\alpha}{n-2}$, then they used the ODEs method and (1.2) to derive the asymptotic behavior of positive singular solutions of (1.1)

$$|x|^{(2+\alpha)/(p-1)} u(x) \rightarrow C_0 \quad \text{as } x \rightarrow 0,$$

where

$$C_0 = \left\{ \frac{(2 + \alpha)(n - 2)}{(p - 1)^2} \left(p - \frac{n + \alpha}{n - 2} \right) \right\}^{1/(p-1)}.$$

When $\alpha = 0$, Caffarelli-Gidas-Spruck [6] found that every positive C^2 solution of (1.1) with $\frac{n}{n-2} \leq p \leq \frac{n+2}{n-2}$ is asymptotically radially symmetric

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \quad \text{as } x \rightarrow 0,$$

where $\bar{u}(|x|) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u(|x|\omega) d\omega$ is the spherical average of u . From this asymptotic symmetry, they used the classical ODEs analysis to get the asymptotic behavior of positive singular solutions of (1.1) with $\frac{n}{n-2} \leq p \leq \frac{n+2}{n-2}$.

Li [24] proved the asymptotic radial symmetry of positive solutions of (1.1) with

$$-2 < \alpha \leq 0 \quad \text{and} \quad 1 < p \leq \frac{n + 2 + \alpha}{n - 2}.$$

In some other cases for α and p , the asymptotic behavior of positive singular solutions of (1.1) has also been very well understood, see Lions [27] for $\alpha = 0$ and $1 < p < \frac{n}{n-2}$, Zhang-Zhao [32] for $-2 < \alpha < 2$ and $1 < p < \frac{n+\alpha}{n-2}$, Aviles [3] for $-2 < \alpha < 2$ and $p = \frac{n+\alpha}{n-2}$, Korevaar-Mazzeo-Pacard-Schoen [23] for $\alpha = 0$ and $p = \frac{n+2}{n-2}$, and Bidaut-Véron and Véron [4] for $\alpha = 0$ and $p > \frac{n+2}{n-2}$.

In recent years, the fractional Hardy-Hénon equation

$$(-\Delta)^\sigma u = |x|^\alpha u^p \quad \text{in } B_1 \setminus \{0\} \quad (1.3)$$

has attracted a great deal of interest since problem (1.3) arises both in physics and in geometry, and is a model fractional semilinear elliptic equation, such as see [1, 2, 7,

9, 11–14, 16, 19–22, 25] and references therein. Here $\sigma \in (0, 1)$ and the fractional Laplacian operator $(-\Delta)^\sigma$ is defined as

$$(-\Delta)^\sigma u(x) = c_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(\xi)}{|x - \xi|^{n+2\sigma}} d\xi, \quad (1.4)$$

where $c_{n,\sigma}$ is a normalization constant depending only on n and σ and $P.V.$ stands for the Cauchy principal value. In particular, when $\alpha = 0$ and $p = \frac{n+2\sigma}{n-2\sigma}$, Caffarelli, Jin, Sire and Xiong [7] obtained the sharp blow up rate of positive solutions of (1.3) with a non-removable singularity and showed that every positive solution of (1.3) is asymptotically radially symmetric

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \quad \text{as } x \rightarrow 0,$$

where $\bar{u}(|x|)$ is the spherical average of u . Li-Bao [25] extended this asymptotic radial symmetry of positive solutions to (1.3) with

$$-2\sigma < \alpha \leq 0 \quad \text{and} \quad \frac{n + \alpha}{n - 2\sigma} < p \leq \frac{n + 2\sigma + 2\alpha}{n - 2\sigma}.$$

However, since the classical ODEs analysis is a missing ingredient in the fractional case to further analyze the solutions of (1.3) compared to the case when $\sigma = 1$, the asymptotic behavior of positive singular solutions of the fractional equation (1.3) is an open question.

One of the goals of this paper is to prove the asymptotic behavior of positive singular solutions to (1.3) with

$$-2\sigma < \alpha \leq 0, \quad \frac{n + \alpha}{n - 2\sigma} < p \leq \frac{n + 2\sigma + \alpha}{n - 2\sigma} \quad \text{and} \quad p \neq \frac{n + 2\sigma + 2\alpha}{n - 2\sigma}$$

We assume that $u \in C^2(B_1 \setminus \{0\})$ and

$$u \in \mathcal{L}_\sigma(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2\sigma}} dx < +\infty \right\},$$

then $(-\Delta)^\sigma u(x)$ is well-defined at every point $x \in B_1 \setminus \{0\}$. Our first main result is the following

Theorem 1.1. *Assume $n \geq 2$. Let $u \in C^2(B_1 \setminus \{0\}) \cap \mathcal{L}_\sigma(\mathbb{R}^n)$ be a positive solution of (1.3) with $-2\sigma < \alpha \leq 0$, $\frac{n+\alpha}{n-2\sigma} < p \leq \frac{n+2\sigma+\alpha}{n-2\sigma}$ and $p \neq \frac{n+2\sigma+2\alpha}{n-2\sigma}$. Then either the singularity at $x = 0$ is removable, or*

$$|x|^{(2\sigma+\alpha)/(p-1)} u(x) \rightarrow C_{p,\sigma,\alpha} \quad \text{as } x \rightarrow 0, \quad (1.5)$$

where

$$C_{p,\sigma,\alpha} = \left\{ \Lambda \left(\frac{n - 2\sigma}{2} - \frac{2\sigma + \alpha}{p - 1} \right) \right\}^{\frac{1}{p-1}} \quad (1.6)$$

and the function $\Lambda(\tau)$ is defined by

$$\Lambda(\tau) = 2^{2\sigma} \frac{\Gamma(\frac{n+2\sigma+2\tau}{4})\Gamma(\frac{n+2\sigma-2\tau}{4})}{\Gamma(\frac{n-2\sigma-2\tau}{4})\Gamma(\frac{n-2\sigma+2\tau}{4})}. \quad (1.7)$$

Under the assumption of Theorem 1.1, if u is a positive solution of (1.3) with a non-removable singularity, then Theorem 1.1 shows that u is asymptotic to a radial solution $u^*(|x|)$ to the same equation in $\mathbb{R}^n \setminus \{0\}$, where $u^*(|x|)$ is

$$u^*(|x|) \equiv C_{p,\sigma,\alpha} |x|^{-\frac{2\sigma+\alpha}{p-1}}.$$

For when $\sigma = 1$, Theorem 1.1 was proved in [18] by Gidas and Spruck. We may also see Caffarelli-Gidas-Spruck [6] for the case $\sigma = 1$ and $\alpha = 0$. Unlike the proofs of [6, 18] where the ODEs analysis is an important ingredient, the proof of Theorem 1.1 is based on a monotonicity formula, combined with a blow up (down) argument, the Kelvin transformation and uniqueness of solutions of related degenerate equations on semi spherical surface \mathbb{S}_+^n . A similar monotonicity formula for fractional Lane-Emden equation ((1.3) with $\alpha = 0$) was established and used in our recent papers [30, 31], where Theorem 1.1 was obtained for $\alpha = 0$ and $\sigma \in (0, 1)$. We introduce the Hardy-Sobolev exponent

$$p_S(\alpha) := \frac{n + 2\sigma + 2\alpha}{n - 2\sigma}.$$

This exponent plays a critical role in the equation (1.3). Remark that, when $-2\sigma < \alpha < 0$, Theorem 1.1 also holds in the Hardy-Sobolev supercritical range

$$p_S(\alpha) < p \leq \frac{n + 2\sigma + \alpha}{n - 2\sigma}.$$

We emphasize that the proof of Theorem 1.1 in the supercritical case is very different from that in subcritical case. One significant difference is that the energy integral (3.1) is non-decreasing in the subcritical case and is non-increasing in the supercritical case, the other difference is that the singular positive solutions of (1.3) in $\mathbb{R}^n \setminus \{0\}$ may not be radially symmetric in the supercritical case. For the Hardy-Sobolev critical case $p = p_S(\alpha)$ ($-2 < \alpha < 0$) and Hénon's case $0 < \alpha < 2\sigma$, we have the following classification of isolated singularities of positive solutions to (1.3) near $x = 0$.

Theorem 1.2. *Assume $n \geq 2$. Let $u \in C^2(B_1 \setminus \{0\}) \cap \mathcal{L}_\sigma(\mathbb{R}^n)$ be a positive solution of (1.3). Assume*

$$-2\sigma < \alpha < 2\sigma \quad \text{and} \quad \frac{n + \alpha}{n - 2\sigma} < p < \frac{n + 2\sigma}{n - 2\sigma}.$$

Then either the singularity at $x = 0$ is removable, or there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{|x|^{(2\sigma+\alpha)/(p-1)}} \leq u(x) \leq \frac{C_2}{|x|^{(2\sigma+\alpha)/(p-1)}} \quad \text{near } x = 0. \quad (1.8)$$

Remark 1.1. *If $\alpha < -2\sigma$, by Corollary 2.1, then Eq. (1.3) has no positive solution in any domain Ω containing the origin.*

We will use a doubling lemma of Poláčik-Quittner-Souplet [29] to obtain the upper bound in (1.8). To derive the lower bound in (1.8), one main difficulty is to prove Proposition 3.2. In Theorem 3.3 of Gidas-Spruck [18] where they proved the lower

bound in (1.2) by using the following statement:

”If $\liminf_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0$, then the Harnack inequality (a Harnack inequality similar to (2.20) in this paper) implies that

$$\lim_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0.”$$

This seems not obvious and requires more explanation. Aviles also pointed out this point on p.190 in [3]. In this paper, we will make full use of a monotonicity formula (Proposition 3.1) to prove Proposition 3.2. Remark that, our proof also applies to Eq. (1.1) and then we can give a rigorous proof of above statement. We believe that the ideas used here can be applied in other situations to deal with similar questions. We also mention that Chen-Lin [10] recently proved a result similar to Proposition 3.2 of this paper to a critical elliptic system by applying Pohozaev identity, see Corollary 4.1, Lemma 4.3 and Lemma 4.4 of [10], where the Harnack inequality also holds for $w_1 + w_2$ and the proof is very delicate and complicated.

The following two theorems treat the isolated singularities located at infinity.

Theorem 1.3. Assume $n \geq 2$. Let $u \in \mathcal{L}_\sigma(\mathbb{R}^n)$ be a nonnegative C^2 solution of

$$(-\Delta)^\sigma u = |x|^\alpha u^p \quad \text{in } |x| > 1 \quad (1.9)$$

with $\alpha > -2\sigma$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$.

- (1) If $1 < p < \frac{n+\alpha}{n-2\sigma}$, then necessarily $u(x) \equiv 0$ in $|x| > 1$.
- (2) If $\frac{n+\alpha}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$, then either the singularity at ∞ is removable, i.e., there exists $C > 0$ such that

$$u(x) \leq \frac{C}{|x|^{n-2\sigma}}, \quad \text{near } x = \infty,$$

or there exist positive constants C_1, C_2 such that

$$\frac{C_1}{|x|^{(2\sigma+\alpha)/(p-1)}} \leq u(x) \leq \frac{C_2}{|x|^{(2\sigma+\alpha)/(p-1)}} \quad \text{near } x = \infty.$$

Theorem 1.4. Assume $n \geq 2$. Let $u \in \mathcal{L}_\sigma(\mathbb{R}^n)$ be a positive C^2 solution of (1.9) with $-2\sigma < \alpha \leq 0$, $\frac{n+\alpha}{n-2\sigma} < p \leq \frac{n+2\sigma+\alpha}{n-2\sigma}$ and $p \neq \frac{n+2\sigma+2\alpha}{n-2\sigma}$. Then either there exists $\beta > 0$ such that

$$\lim_{|x| \rightarrow \infty} |x|^{n-2\sigma} u(x) = \beta,$$

or

$$\lim_{|x| \rightarrow \infty} |x|^{(2\sigma+\alpha)/(p-1)} u(x) = C_{p,\sigma,\alpha},$$

where $C_{p,\sigma,\alpha}$ is given by (1.6).

In particular, we give a complete classification of isolated singularities of positive solutions to fractional Lane-Emden equation near ∞ .

Corollary 1.1. Assume $n \geq 2$. Let $u \in \mathcal{L}_\sigma(\mathbb{R}^n)$ be a positive C^2 solution of

$$(-\Delta)^\sigma u = u^p \quad \text{in } |x| > 1 \quad (1.10)$$

with $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. Then either there exists $\beta > 0$ such that

$$\lim_{|x| \rightarrow \infty} |x|^{n-2\sigma} u(x) = \beta, \quad (1.11)$$

or

$$\lim_{|x| \rightarrow \infty} |x|^{\frac{2\sigma}{p-1}} u(x) = C_{p,\sigma,0}, \quad (1.12)$$

where $C_{p,\sigma,0}$ is given by (1.6).

Remark 1.2. Our characterization of isolated singularities near ∞ of fractional Lane-Emden equation is complemented by the existence of fast-decay solutions satisfying (1.11) which has been recently obtained by Ao-Chan-DelaTorre-Fontelos-González-Wei [1, 2]. More precisely, for some exponent $p_1 = p_1(n, \sigma) \in (\frac{n}{n-2\sigma}, \frac{n+2\sigma}{n-2\sigma})$ and for every $\beta \in (0, \infty)$, there exists a positive solution of (1.10) satisfying (1.11) which was proved in [1] when $\frac{n}{n-2\sigma} < p < p_1$ and in [2] when $p_1 \leq p < \frac{n+2\sigma}{n-2\sigma}$.

The paper is organized as follows. In Section 2, we introduce the extension formula for $(-\Delta)^\sigma$ which is due to Caffarelli-Silvestre [8] and prove some important estimates. In Section 3, we establish an important monotonicity formula and prove Theorem 1.2. Theorem 1.1 on the asymptotic behavior of positive singular solutions is proved in Section 4. In Section 5, we prove Theorems 1.3 and 1.4.

2 Preliminaries

In this section, we introduce some notations and prove some important lemmas which will be used in this paper.

We use capital letters, such as $X = (x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, to denote points in \mathbb{R}_+^{n+1} . We denote B_R as the ball in \mathbb{R}^{n+1} with radius R and center 0 and B_R as the ball in \mathbb{R}^n with radius R and center 0. We also denote \mathcal{B}_R^+ as the upper half-ball $\mathcal{B}_R \cap \mathbb{R}_+^{n+1}$, $\partial^+ \mathcal{B}_R^+ = \partial \mathcal{B}_R^+ \cap \mathbb{R}_+^{n+1}$ as the positive part of $\partial \mathcal{B}_R^+$, and $\partial^0 \mathcal{B}_R^+ = \partial \mathcal{B}_R^+ \setminus \partial^+ \mathcal{B}_R^+$ as the flat part of $\partial \mathcal{B}_R^+$ which is the ball B_R in \mathbb{R}^n . For a more general domain $\Omega \subset \mathbb{R}_+^{n+1}$, we also denote $\partial^0 \Omega$ as the interior of $\overline{\Omega} \cap \partial \mathbb{R}_+^{n+1}$ in \mathbb{R}^n .

We will study the fractional Hardy-Hénon equation (1.3) via the well known extension theorem for the fractional Laplacian $(-\Delta)^\sigma$ established by Caffarelli-Silvestre [8]. Assume $u \in C^2(B_1 \setminus \{0\}) \cap \mathcal{L}_\sigma(\mathbb{R}^n)$. For $X = (x, t) \in \mathbb{R}_+^{n+1}$, let

$$U(x, t) = \int_{\mathbb{R}^n} P_\sigma(x - y, t) u(y) dy, \quad (2.1)$$

where

$$P_\sigma(x, t) = p_{n,\sigma} \frac{t^{2\sigma}}{(|x|^2 + t^2)^{\frac{n+2\sigma}{2}}}$$

and $p_{n,\sigma}$ is a positive constant chosen such that $\int_{\mathbb{R}^n} P_\sigma(x,1)dx = 1$. Then $U \in C^2(\mathbb{R}_+^{n+1}) \cap C((\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+) \setminus \{0\})$, $t^{1-2\sigma} \partial_t U \in C((\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+) \setminus \{0\})$ and

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ U(x,0) = u(x) & \text{on } \partial^0 \mathcal{B}_1^+ \setminus \{0\}. \end{cases} \quad (2.2)$$

Moreover, by the extension formulation in [8], we obtain

$$\frac{\partial U}{\partial \nu^\sigma}(x,0) = \kappa_\sigma (-\Delta)^\sigma u(x) \quad \text{on } \partial^0 \mathcal{B}_1^+ \setminus \{0\}, \quad (2.3)$$

where $\frac{\partial U}{\partial \nu^\sigma}(x,0) := -\lim_{t \rightarrow 0^+} t^{1-2\sigma} \partial_t U(x,t)$ and the constant $\kappa_\sigma = \frac{\Gamma(1-\sigma)}{2^{2\sigma-1} \Gamma(\sigma)}$.

Therefore, instead of Eq. (1.3), we may study the following degenerate elliptic equation with the Neumann boundary condition in one dimension higher

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x,0) = \kappa_\sigma |x|^\alpha U^p(x,0) & \text{on } \partial^0 \mathcal{B}_1^+ \setminus \{0\}. \end{cases} \quad (2.4)$$

By (2.2) and (2.3), if we get the asymptotic behavior of the traces $u(x) := U(x,0)$ of nonnegative solutions $U(x,t)$ of (2.4) near the origin, then, from which we can obtain the behavior of nonnegative solutions of (1.3) near the origin.

We say that U is a nonnegative weak solution of (2.4) if U is in the weighted Sobolev space $W^{1,2}(t^{1-2\sigma}, \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\epsilon^+})$ for every $\epsilon > 0$, $U \geq 0$, and it satisfies (2.4) in the sense of distribution away from 0, that is, we have

$$\int_{\mathcal{B}_1^+} t^{1-2\sigma} \nabla U \cdot \nabla \Phi = \kappa_\sigma \int_{\partial^0 \mathcal{B}_1^+} |x|^\alpha U^p \Phi \quad (2.5)$$

for all nonnegative $\Phi \in C_c^\infty((\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+) \setminus \{0\})$. It follows from the regularity result in [5, 22] that $U(x,t)$ is locally Hölder continuous in $\overline{\mathcal{B}_{3/4}^+} \setminus \{0\}$.

We say that the origin 0 is a removable singularity of solution U of (2.4) if $U(x,0)$ can be extended as a continuous function near the origin, otherwise we say that the origin 0 is a non-removable singularity.

We say $U \in W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}_+^{n+1}})$ if $U \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+)$ for all $R > 0$, and $U \in W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}_+^{n+1}} \setminus \{0\})$ if $U \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \setminus \overline{\mathcal{B}_\epsilon^+})$ for all $R > \epsilon > 0$.

We now establish the basic singularity and decay estimates. In the case $\sigma = 1$, that is for the Laplacian, the corresponding result was proved in [18, 28].

Proposition 2.1. *Let $n \geq 2$, $\alpha \in \mathbb{R}$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$.*

(1) *Suppose that U is a nonnegative weak solution of (2.4). Then there exists a constant $C = C(n, p, \alpha, \sigma)$ such that*

$$U(x,0) \leq \frac{C}{|x|^{(2\sigma+\alpha)/(p-1)}}, \quad 0 < |x| < \frac{1}{2}. \quad (2.6)$$

(2) Suppose that U is a nonnegative weak solution of

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathbb{R}_+^{n+1} \setminus \overline{\mathcal{B}_1^+}, \\ \frac{\partial U}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha U^p(x, 0) & \text{on } B_1^c, \end{cases} \quad (2.7)$$

where $B_1^c := \{x \in \mathbb{R}^n : |x| > 1\}$. Then there exists a constant $C = C(n, p, \alpha, \sigma)$ such that

$$U(x, 0) \leq \frac{C}{|x|^{(2\sigma+\alpha)/(p-1)}}, \quad |x| > 2. \quad (2.8)$$

To prove Proposition 2.1, we need the following lemma.

Lemma 2.1. Let $n \geq 2$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Let $K \in C^1(\overline{B_1})$ satisfy

$$\|K\|_{C^1(\overline{B_1})} \leq C_1 \quad \text{and} \quad K(x) \geq C_2, \quad x \in \overline{B_1}, \quad (2.9)$$

for some constants $C_1, C_2 > 0$. Suppose that U is a nonnegative weak solution of

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x, 0) = K(x)U^p(x, 0) & \text{on } \partial^0 \mathcal{B}_1^+. \end{cases} \quad (2.10)$$

Then there exists a constant C , depending only on $n, \sigma, \gamma, p, C_1, C_2$, such that

$$U(x, 0) \leq C [\operatorname{dist}(x, \partial B_1)]^{-\frac{2\sigma}{p-1}}, \quad x \in B_1.$$

Proof. Suppose by contradiction that there exists a sequence of solutions U_i of (2.10) and a sequence of points $x_i \in B_1$ such that

$$M_i(x_i) \operatorname{dist}(x_i, \partial B_1) > 2i, \quad i = 1, 2, \dots,$$

where the functions M_i are defined by

$$M_i(x) = (U_i(x, 0))^{\frac{p-1}{2\sigma}}, \quad x \in B_1.$$

By the doubling lemma of Poláčik-Quittner-Souplet [29] there exists another sequence $y_i \in B_1$ such that

$$M_i(y_i) \operatorname{dist}(y_i, \partial B_1) > 2i, \quad M_i(y_i) \geq M_i(x_i) \quad (2.11)$$

and

$$M_i(z) \leq 2M_i(y_i) \quad \text{for any } |z - y_i| \leq i\lambda_i, \quad (2.12)$$

where $\lambda_i := M_i(y_i)^{-1}$. Note that $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$. We now define

$$\bar{U}_i(x, t) = \lambda_i^{\frac{2\sigma}{p-1}} U_i(y_i + \lambda_i x, \lambda_i t), \quad (x, t) \in \Omega_i$$

with

$$\Omega_i = \{(x, t) \in \mathbb{R}_+^{n+1} : (y_i + \lambda_i x, \lambda_i t) \in \mathcal{B}_1^+ \setminus \{0\}\}.$$

Then \bar{U}_i satisfies $\bar{U}_i(0) = 1$ and

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla\bar{U}_i) = 0 & \text{in } \Omega_i, \\ \frac{\partial\bar{U}_i}{\partial\nu^\sigma}(x, 0) = \bar{K}_i(x)\bar{U}_i(x, 0)^p & \text{on } \partial^0\Omega_i, \end{cases} \quad (2.13)$$

where $\bar{K}_i(x) = K(y_i + \lambda_i x)$ for $x \in \partial^0\Omega_i$. Moreover, by (2.12), we have

$$\bar{U}_i(x, 0) \leq 2^{\frac{2\sigma}{p-1}}, \quad x \in B_i(0) \subset \mathbb{R}^n.$$

On the other hand, by (2.9), we have $C_2 \leq \bar{K}_i(x) \leq C_1$ and, for each $R > 0$ and $i \geq i_0(R)$ large enough,

$$\|\bar{K}_i\|_{C^1(\bar{B}_R)} \leq C_1 \quad (2.14)$$

and

$$|\bar{K}_i(y) - \bar{K}_i(z)| \leq C_1|\lambda_i(y - z)| \leq C_1|y - z|, \quad y, z \in B_R(0). \quad (2.15)$$

Therefore, by Arzela-Ascoli's theorem, there exists $\bar{K} \in C(\mathbb{R}^n)$ such that, after extracting a subsequence, $\bar{K}_i \rightarrow \bar{K}$ in $C_{loc}(\mathbb{R}^n)$. Moreover, from (2.15), we have for any $y, z \in \mathbb{R}^n$,

$$|\bar{K}_i(y) - \bar{K}_i(z)| \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and hence \bar{K} is actually a constant $K_0 \geq C_2 > 0$.

It follows Corollary 2.10 and Theorem 2.15 of Jin-Li-Xiong [22] that there exists $\gamma \in (0, 1)$ such that for every $R > 1$,

$$\|\bar{U}_i\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+)} + \|\bar{U}_i\|_{C^\gamma(\bar{\mathcal{B}}_R^+)} \leq C(R),$$

where $C(R)$ is independent of i . Therefore, there is a subsequence of $i \rightarrow \infty$, still denoted by itself, and a function $\bar{U} \in W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}_+^{n+1}}) \cap C_{loc}^\gamma(\overline{\mathbb{R}_+^{n+1}})$ such that, as $i \rightarrow \infty$,

$$\begin{cases} \bar{U}_i \rightharpoonup \bar{U} & \text{weakly in } W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}_+^{n+1}}), \\ \bar{U}_i \rightarrow \bar{U} & \text{in } C_{loc}^{\gamma/2}(\overline{\mathbb{R}_+^{n+1}}). \end{cases}$$

Moreover, \bar{U} is a nonnegative solution of

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla\bar{U}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial\bar{U}}{\partial\nu^\sigma}(x, 0) = K_0\bar{U}^p(x, 0) & \text{on } \mathbb{R}^n, \end{cases} \quad (2.16)$$

and $\bar{U}(0) = 1$. Since $p < \frac{n+2\sigma}{n-2\sigma}$, this contradicts the Liouville type theorem in [22] (See Remark 1.9 of [22]). \square

Proof of Proposition 2.1. Suppose either $\Omega = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$ and $0 < |x_0| < \frac{1}{2}$, or $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$ and $|x_0| > 2$. Take

$$\lambda = \frac{|x_0|}{2}.$$

Then, for any $y \in B_1$, we have $\frac{|x_0|}{2} < |x_0 + \lambda y| < \frac{3|x_0|}{2}$. Hence $x_0 + \lambda y \in \Omega$ in either case. Define

$$W(y, t) = \lambda^{\frac{2\sigma+\alpha}{p-1}} U(x_0 + \lambda y, \lambda t).$$

Then W is a nonnegative solution of

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla W) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial W}{\partial \nu^\sigma}(y, 0) = K(y)W^p(y, 0) & \text{on } \partial^0 \mathcal{B}_1^+, \end{cases}$$

where $K(y) = |y + \frac{x_0}{\lambda}|^\alpha$ for $y \in B_1$. Clearly

$$1 \leq |y + \frac{x_0}{\lambda}| \leq 3 \quad \text{for all } y \in \overline{B_1}.$$

Therefore $\|K\|_{C^1(\overline{B_1})} \leq C_1(\alpha)$ and $K(y) \geq C_2(\alpha)$, $y \in \overline{B_1}$ for some constants $C_1(\alpha), C_2(\alpha) > 0$. By Lemma 2.1, we obtain $W(0) \leq C$. This implies that

$$U(x_0, 0) \leq C\lambda^{-\frac{2\sigma+\alpha}{p-1}} \leq C|x_0|^{-\frac{2\sigma+\alpha}{p-1}}.$$

The desired conclusion follows. \square

Corollary 2.1. *Let $n \geq 2$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Suppose that U is a nonnegative weak solution of (2.4). If $\alpha < -2\sigma$, then $U(x) \equiv 0$ in $(\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+) \setminus \{0\}$.*

Proof. By (2.6),

$$U(x, 0) \rightarrow 0 \quad \text{as } x \rightarrow 0. \quad (2.17)$$

Assume by contradiction that there exists $X_0 \in (\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+) \setminus \{0\}$ such that $U(X_0) > 0$. Then the maximum principle implies that

$$U(X) > 0 \quad \text{for all } X \in (\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+) \setminus \{0\}.$$

By Proposition 3.1 in [22], we have

$$\liminf_{X \rightarrow 0} U(X) > 0,$$

a contradiction with (2.17). \square

Now we recall a Harnack inequality. For its proof, see [5, 22].

Lemma 2.2. *Let $U \in W_{loc}^{1,2}(t^{1-2\sigma}, \mathcal{B}_1^+)$ be a nonnegative weak solution of*

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x, 0) = a(x)U(x, 0) & \text{on } \partial^0 \mathcal{B}_1^+, \end{cases} \quad (2.18)$$

If $a \in L^q(B_1)$ for some $q > \frac{n}{2\sigma}$, then we have

$$\sup_{\mathcal{B}_{1/2}^+} U \leq C \inf_{\mathcal{B}_{1/2}^+} U, \quad (2.19)$$

where C depends only on n, σ and $\|a\|_{L^q(B_1)}$.

One very useful consequence of Proposition 2.1 is the following Harnack inequality.

Lemma 2.3. *Let $n \geq 2$, $\alpha \in \mathbb{R}$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$.*

(1) *Suppose that U is a nonnegative weak solution of (2.4). Then there exists a constant $C = C(n, p, \alpha, \sigma)$ such that for all $0 < r < \frac{1}{8}$, we have*

$$\sup_{\mathcal{B}_{2r}^+ \setminus \overline{\mathcal{B}_{r/2}^+}} U \leq C \inf_{\mathcal{B}_{2r}^+ \setminus \overline{\mathcal{B}_{r/2}^+}} U \quad (2.20)$$

(2) *Suppose that U is a nonnegative weak solution of (2.7). Then there exists a constant $C = C(n, p, \alpha, \sigma)$ such that for all $r > 8$, we have*

$$\sup_{\mathcal{B}_{2r}^+ \setminus \overline{\mathcal{B}_{r/2}^+}} U \leq C \inf_{\mathcal{B}_{2r}^+ \setminus \overline{\mathcal{B}_{r/2}^+}} U \quad (2.21)$$

Proof. Let

$$V_r(X) = U(rX)$$

for $X \in \mathcal{B}_4 \setminus \overline{\mathcal{B}_{1/4}}$. Then V_r satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla V_r) = 0 & \text{in } \mathcal{B}_4 \setminus \overline{\mathcal{B}_{1/4}}, \\ \frac{\partial V_r}{\partial \nu^\sigma}(x, 0) = a_r(x) v_r(x) & \text{on } B_4 \setminus \overline{B_{1/4}}, \end{cases} \quad (2.22)$$

where $v_r(x) = V_r(x, 0)$ and $a_r(x) = r^{2\sigma+\alpha} |x|^\alpha (u(rx))^{p-1}$. By Proposition 2.1,

$$|a_r(x)| \leq C \quad \text{for all } 1/4 \leq |x| \leq 4,$$

where C is a positive constant independent of r and U . By the Harnack inequality in Lemma 2.2 and the standard Harnack inequality for uniformly elliptic equations, we have

$$\sup_{\frac{1}{2} \leq |X| \leq 2} V_r(X) \leq C \inf_{\frac{1}{2} \leq |X| \leq 2} V_r(X),$$

where C is another positive constant independent of r and U . Rescaling back we get the desired conclusion. \square

3 Classification of Isolated Singularities at $x = 0$

In this section, we classify isolated singularities of positive solutions of (2.4) near $x = 0$. To this end, we need to establish a monotonicity formula for the nonnegative solutions U of (2.4) (resp. of (2.7)). Let U be a nonnegative solution of (2.4) (resp.

of (2.7)), we define

$$\begin{aligned}
E(r; U) &:= r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n} \left[r \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} \left| \frac{\partial U}{\partial \nu} \right|^2 + \frac{2\sigma + \alpha}{p-1} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} \frac{\partial U}{\partial \nu} U \right] \\
&+ \frac{2\sigma + \alpha}{p-1} \left(\frac{2\sigma + \alpha}{p-1} - \frac{n-2\sigma}{2} \right) r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n-1} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} U^2 \\
&- \frac{1}{2} r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n+1} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} |\nabla U|^2 \\
&+ \frac{\kappa\sigma}{p+1} r^{\frac{(2\sigma+\alpha)(p+1)}{p-1}-n+1} \int_{\partial B_r} u^{p+1}.
\end{aligned} \tag{3.1}$$

We recall that the Hardy-Sobolev critical exponent is defined by

$$p_S(\alpha) := \frac{n+2\sigma+2\alpha}{n-2\sigma}.$$

Then, we have the following monotonicity formula.

Proposition 3.1. *Let $n \geq 2$, $\alpha \in \mathbb{R}$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$.*

- (1) *Assume $p \leq p_S(\alpha)$ and that U is a nonnegative weak solution of (2.4) (resp. of (2.7)). Then $E(r; U)$ is non-decreasing in $r \in (0, 1)$ (resp. in $r \in (1, \infty)$). Moreover,*

$$\frac{d}{dr} E(r; U) = J_1 r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} \left(\frac{\partial U}{\partial \nu} + \frac{2\sigma + \alpha}{p-1} \frac{U}{r} \right)^2,$$

$$\text{where } J_1 = \frac{n-2\sigma}{p-1} \left(\frac{n+2\sigma+2\alpha}{n-2\sigma} - p \right) \geq 0.$$

- (2) *Assume $p > p_S(\alpha)$ and that U is a nonnegative weak solution of (2.4) (resp. of (2.7)). Then $E(r; U)$ is non-increasing in $r \in (0, 1)$ (resp. in $r \in (1, \infty)$). Moreover,*

$$\frac{d}{dr} E(r; U) = J_1 r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} \left(\frac{\partial U}{\partial \nu} + \frac{2\sigma + \alpha}{p-1} \frac{U}{r} \right)^2,$$

$$\text{where } J_1 = \frac{n-2\sigma}{p-1} \left(\frac{n+2\sigma+2\alpha}{n-2\sigma} - p \right) < 0.$$

Proof. We shall take the standard polar coordinates in \mathbb{R}_+^{n+1} : $X = (x, t) = r\theta$, where $r = |X|$ and $\theta = \frac{X}{|X|}$. Let $\theta_1 = \frac{t}{|X|}$ denote the component of θ in the t direction and

$$\mathbb{S}_+^n = \{X \in \mathbb{R}_+^{n+1} : r = 1, \theta_1 > 0\}$$

denote the upper unit half-sphere.

Let U be a nonnegative weak solution of (2.4). Use the classical change of variable in Fowler [17],

$$V(s, \theta) = r^{\frac{2\sigma+\alpha}{p-1}} U(r, \theta), \quad s = \ln r.$$

Direct calculations show that V satisfies

$$\begin{cases} V_{ss} - J_1 V_s - J_2 V + \theta_1^{2\sigma-1} \operatorname{div}_\theta(\theta_1^{1-2\sigma} \nabla_\theta V) = 0 & \text{in } (-\infty, 0) \times \mathbb{S}_+^n, \\ -\lim_{\theta_1 \rightarrow 0^+} \theta_1^{1-2\sigma} \partial_{\theta_1} V = \kappa_\sigma V^p & \text{on } (-\infty, 0) \times \partial \mathbb{S}_+^n, \end{cases} \quad (3.2)$$

where

$$J_1 = \frac{n-2\sigma}{p-1} \left(\frac{n+2\sigma+2\alpha}{n-2\sigma} - p \right), \quad J_2 = \frac{2\sigma+\alpha}{p-1} \left(n-2\sigma - \frac{2\sigma+\alpha}{p-1} \right).$$

Multiplying (3.2) by V_s and integrating, we have

$$\begin{aligned} & \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} V_{ss} V_s - J_2 \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} V V_s - \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} \nabla_\theta V \cdot \nabla_\theta V_s + \kappa_\sigma \int_{\partial \mathbb{S}_+^n} V^p V_s \\ &= J_1 \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} (V_s)^2. \end{aligned} \quad (3.3)$$

For any $s \in (-\infty, 0)$, we define

$$\begin{aligned} \tilde{E}(s) &:= \frac{1}{2} \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} (V_s)^2 - \frac{J_2}{2} \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} V^2 - \frac{1}{2} \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} |\nabla_\theta V|^2 \\ &\quad + \frac{\kappa_\sigma}{p+1} \int_{\partial \mathbb{S}_+^n} V^{p+1}. \end{aligned}$$

Then, by (3.3), we get

$$\frac{d}{ds} \tilde{E}(s) = J_1 \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} (V_s)^2. \quad (3.4)$$

Note that

$$\begin{cases} J_1 \geq 0 & \text{when } p \leq p_S(\alpha), \\ J_1 < 0 & \text{when } p > p_S(\alpha). \end{cases}$$

Hence, $\tilde{E}(s)$ is non-decreasing in $s \in (-\infty, 0)$ if $p \leq p_S(\alpha)$ and $\tilde{E}(s)$ is non-increasing in $s \in (-\infty, 0)$ if $p > p_S(\alpha)$.

Now, rescaling back, we have

$$\begin{aligned} & \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} (V_s)^2 \\ &= \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} \left(\frac{2\sigma+\alpha}{p-1} r^{\frac{2\sigma+\alpha}{p-1}-1} U + r^{\frac{2\sigma+\alpha}{p-1}} U_r \right)^2 r^2 \\ &= r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n} \int_{\partial+\mathcal{B}_r^+} t^{1-2\sigma} \left(\frac{(2\sigma+\alpha)^2}{(p-1)^2} r^{-1} U^2 + \frac{2(2\sigma+\alpha)}{p-1} U \frac{\partial U}{\partial \nu} + r \left| \frac{\partial U}{\partial \nu} \right|^2 \right), \\ & \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} |\nabla_\theta V|^2 = r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n+1} \int_{\partial+\mathcal{B}_r^+} t^{1-2\sigma} \left(|\nabla U|^2 - \left| \frac{\partial U}{\partial \nu} \right|^2 \right), \end{aligned}$$

$$\int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} V^2 = r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n-1} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} U^2,$$

$$\int_{\partial \mathbb{S}_+^n} V^{p+1} = r^{\frac{(2\sigma+\alpha)(p+1)}{p-1}-n+1} \int_{\partial B_r} u^{p+1}.$$

Substituting these into (3.4) and noting that $s = \ln r$ is increasing in r , we easily obtain that $E(r; U)$ is non-decreasing in $r \in (0, 1)$ if $p \leq p_S(\alpha)$ and $E(r; U)$ is non-increasing in $r \in (0, 1)$ if $p > p_S(\alpha)$.

If U is a nonnegative solution of (2.7), we just need to replace $s \in (-\infty, 0)$ in the proof above with $s \in (0, \infty)$. The proof is finished. \square

By using the monotonicity of $E(r; U)$, we prove the following proposition, which will play an essential role in deriving the lower bound of positive singular solutions.

Proposition 3.2. *Let U be a nonnegative solution of (2.4) with $-2\sigma < \alpha < 2\sigma$ and $\frac{n+\alpha}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. If*

$$\liminf_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0,$$

then

$$\lim_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0.$$

Proof. We consider separately the case $p \leq p_S(\alpha)$ and the case $p > p_S(\alpha)$.

Case 1: $p \leq p_S(\alpha)$. Suppose by contradiction that

$$\liminf_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0 \quad \text{and} \quad \limsup_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = C > 0.$$

Therefore, there exist two sequences of points $\{x_i\}$ and $\{y_i\}$ satisfying

$$x_i \rightarrow 0, \quad y_i \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

such that

$$|x_i|^{\frac{2\sigma+\alpha}{p-1}} u(x_i) \rightarrow 0 \quad \text{and} \quad |y_i|^{\frac{2\sigma+\alpha}{p-1}} u(y_i) \rightarrow C > 0 \quad \text{as } i \rightarrow \infty.$$

Let $g(r) = r^{\frac{2\sigma+\alpha}{p-1}} \bar{u}(r)$, where $\bar{u}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u$ denotes the spherical average of u over ∂B_r . Then, by the Harnack inequality (2.20), we have

$$\liminf_{r \rightarrow 0} g(r) = 0 \quad \text{and} \quad \limsup_{r \rightarrow 0} g(r) = C > 0.$$

Hence, there exists a sequence of local minimum points r_i of $g(r)$ with

$$\lim_{i \rightarrow \infty} r_i = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} g(r_i) = 0.$$

Define

$$V_i(X) = \frac{U(r_i X)}{U(r_i e_1)},$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. It follows from the Harnack inequality (2.20) that V_i is locally uniformly bounded away from the origin, and V_i satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla V_i) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial V_i}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma \left(r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i e_1) \right)^{p-1} |x|^\alpha V_i^p(x, 0) & \text{on } \mathbb{R}^n \setminus \{0\}. \end{cases} \quad (3.5)$$

By the Harnack inequality (2.20), $r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i e_1) \rightarrow 0$ as $i \rightarrow \infty$. Then by Corollary 2.10 and Theorem 2.15 in [22] that there exists $\gamma \in (0, 1)$ such that for every $R > 1 > r > 0$

$$\|V_i\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} + \|V_i\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} + \|v_i\|_{C^{2,\gamma}(B_R \setminus B_r)} \leq C(R, r),$$

where $v_i(x) = V_i(x, 0)$ and $C(R, r)$ is independent of i . Then after passing to a subsequence, $\{V_i\}$ converges to a nonnegative function $V \in W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}}_+^{n+1} \setminus \{0\}) \cap C_{loc}^\gamma(\overline{\mathbb{R}}_+^{n+1} \setminus \{0\})$ satisfying

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla V) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial V}{\partial \nu^\sigma}(x, 0) = 0 & \text{on } \mathbb{R}^n \setminus \{0\}. \end{cases} \quad (3.6)$$

By a Bôcher type theorem in [22], we have

$$V(X) = \frac{a}{|X|^{n-2\sigma}} + b,$$

where a, b are nonnegative constants. Recall that r_i are local minimum of $g(r)$ for every i and note that

$$r^{\frac{2\sigma+\alpha}{p-1}} \bar{v}_i(r) = r^{\frac{2\sigma+\alpha}{p-1}} \frac{1}{|\partial B_r|} \int_{\partial B_r} v_i = \frac{1}{U(r_i e_1)} r^{\frac{2\sigma+\alpha}{p-1}} \bar{u}(r_i r) = \frac{1}{U(r_i e_1) r_i^{\frac{2\sigma+\alpha}{p-1}}} g(r_i r).$$

Hence, for every i , we have

$$\frac{d}{dr} \left[r^{\frac{2\sigma+\alpha}{p-1}} \bar{v}_i(r) \right] \Big|_{r=1} = \frac{r_i}{U(r_i e_1) r_i^{\frac{2\sigma+\alpha}{p-1}}} g'(r_i) = 0. \quad (3.7)$$

Let $v(x) = V(x, 0)$. Then we know that $v_i(x) \rightarrow v(x)$ in $C_{loc}^2(\mathbb{R}^n \setminus \{0\})$. By (3.7), we obtain

$$\frac{d}{dr} \left[r^{\frac{2\sigma+\alpha}{p-1}} \bar{v}(r) \right] \Big|_{r=1} = 0,$$

which implies that

$$a \left(\frac{2\sigma+\alpha}{p-1} - (n-2\sigma) \right) + \frac{(2\sigma+\alpha)b}{p-1} = 0. \quad (3.8)$$

On the other hand, by $V(e_1) = 1$, we have

$$a + b = 1. \quad (3.9)$$

Combine (3.8) with (3.9), we get

$$a = \frac{2\sigma + \alpha}{(p-1)(n-2\sigma)} \quad \text{and} \quad b = 1 - \frac{2\sigma + \alpha}{(p-1)(n-2\sigma)}.$$

Since $-2\sigma < \alpha < 2\sigma$ and $\frac{n+\alpha}{n-2\sigma} < p$, we have $0 < a, b < 1$. Now we compute $E(r; U)$.

It follows from Proposition 2.19 in [22] that $|\nabla_x V_i|$ and $|t^{1-2\sigma} \partial_t V_i|$ are locally uniformly bounded in $C_{loc}^\beta(\overline{\mathbb{R}_+^{n+1}} \setminus \{0\})$ for some $\beta > 0$. Hence, there exists a constant $C > 0$ such that

$$|\nabla_x U(X)| \leq Cr_i^{-1} U(r_i e_1) = o(1) r_i^{-\frac{2\sigma+\alpha}{p-1}-1} \quad \text{for all } |X| = r_i$$

and

$$|t^{1-2\sigma} \partial_t U(X)| \leq Cr_i^{-2\sigma} U(r_i e_1) = o(1) r_i^{-\frac{2\sigma+\alpha}{p-1}-2\sigma} \quad \text{for all } |X| = r_i.$$

By the Harnack inequality (2.20), we also have

$$U(X) \leq CU(r_i e_1) = o(1) r_i^{-\frac{2\sigma+\alpha}{p-1}} \quad \text{for all } |X| = r_i.$$

Thus, we estimate

$$\begin{aligned} r_i^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n+1} \int_{\partial^+ \mathcal{B}_{r_i}^+} t^{1-2\sigma} |\nabla U|^2 &\leq r_i^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n+1} \left(o(1) r_i^{-\frac{4\sigma+2\alpha}{p-1}-2} \int_{\partial^+ \mathcal{B}_{r_i}^+} t^{1-2\sigma} \right. \\ &\quad \left. + o(1) r_i^{-\frac{4\sigma+2\alpha}{p-1}-4\sigma} \int_{\partial^+ \mathcal{B}_{r_i}^+} t^{2\sigma-1} \right) \\ &\leq Co(1), \end{aligned}$$

$$r_i^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n-1} \int_{\partial^+ \mathcal{B}_{r_i}^+} t^{1-2\sigma} U^2 \leq o(1) r_i^{2\sigma-n-1} \int_{\partial^+ \mathcal{B}_{r_i}^+} t^{1-2\sigma} \leq Co(1)$$

and

$$r_i^{\frac{(2\sigma+\alpha)(p+1)}{p-1}-n+1} \int_{\partial \mathcal{B}_{r_i}} u^{p+1} \leq Co(1),$$

where the constant C is independent of i . Hence, by the definition of $E(r; U)$, we have

$$\lim_{i \rightarrow \infty} E(r_i; U) = 0.$$

Since $E(r; U)$ is non-decreasing in $r \in (0, 1)$ for this case, we obtain

$$E(r; U) \geq 0 \quad \text{for all } r \in (0, 1). \quad (3.10)$$

On the other hand, by the scaling invariance of $E(r; U)$, for every i , we have

$$0 \leq E(r_i; U) = E\left(1; r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i X)\right) = E\left(1; r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i e_1) V_i\right).$$

Hence, for every i , we have

$$\begin{aligned} 0 &\leq \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} \left| \frac{\partial V_i}{\partial \nu} \right|^2 + \frac{2\sigma + \alpha}{p-1} \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} \frac{\partial V_i}{\partial \nu} V_i \\ &\quad + \frac{2\sigma + \alpha}{p-1} \left(\frac{2\sigma + \alpha}{p-1} - \frac{n-2\sigma}{2} \right) \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} V_i^2 \\ &\quad - \frac{1}{2} \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} |\nabla V_i|^2 + \frac{\kappa_\sigma}{p+1} \int_{\partial \mathcal{B}_1} \left(r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i e_1) \right)^{p-1} V_i^{p+1}. \end{aligned}$$

Letting $i \rightarrow \infty$, we obtain

$$\begin{aligned} 0 &\leq \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} \left| \frac{\partial V}{\partial \nu} \right|^2 + \frac{2\sigma + \alpha}{p-1} \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} \frac{\partial V}{\partial \nu} V \\ &\quad + \frac{2\sigma + \alpha}{p-1} \left(\frac{2\sigma + \alpha}{p-1} - \frac{n-2\sigma}{2} \right) \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} V^2 - \frac{1}{2} \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} |\nabla V|^2 \\ &= a^2(n-2\sigma)^2 \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} - a(n-2\sigma) \frac{2\sigma + \alpha}{p-1} \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} \\ &\quad + \frac{2\sigma + \alpha}{p-1} \left(\frac{2\sigma + \alpha}{p-1} - \frac{n-2\sigma}{2} \right) \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} - \frac{1}{2} a^2(n-2\sigma)^2 \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} \\ &= \frac{1}{2} \frac{2\sigma + \alpha}{p-1} \left(\frac{2\sigma + \alpha}{p-1} - (n-2\sigma) \right) \int_{\partial^+ \mathcal{B}_1^+} t^{1-2\sigma} < 0. \end{aligned}$$

Note that in the last inequality we have used the fact $2\sigma + \alpha > 0$ and $\frac{2\sigma+\alpha}{p-1} - (n-2\sigma) < 0$ because $\alpha > -2\sigma$ and $\frac{n+\alpha}{n-2\sigma} < p$. We get a contradiction. This finishes the proof of Case 1.

Case 2: $p > p_S(\alpha)$. In this case, By Proposition 3.1 (2), $E(r; U)$ is non-increasing in $r \in (0, 1)$. If we proceed as in the proof of Case 1, then we obtain $E(r; U) \leq 0$ for $r \in (0, 1)$ in (3.10), and so we cannot get a contradiction in the final proof. Therefore, we need a new method to deal with this supercritical case. In fact, the following method can be used in the case $p \neq p_S(\alpha)$.

Step 1. If $\liminf_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0$, then

$$\lim_{r \rightarrow 0^+} E(r; U) = 0. \quad (3.11)$$

Since $\liminf_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0$, we know that there exists a sequence of points $\{x_i\}$ such that

$$x_i \rightarrow 0 \quad \text{and} \quad |x_i|^{\frac{2\sigma+\alpha}{p-1}} u(x_i) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Let $r_i := |x_i|$. By the Harnack inequality (2.20),

$$r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i e_1) \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. Define

$$W_i(X) = r_i^{\frac{2\sigma+\alpha}{p-1}} U(r_i X), \quad X \in \mathcal{B}_{1/r_i}^+ \setminus \{0\}.$$

It follows from Proposition 2.1 and the Harnack inequality (2.20) that W_i is locally uniformly bounded away from the origin. Moreover, W_i satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla W_i) = 0 & \text{in } \mathcal{B}_{1/r_i}^+, \\ \frac{\partial W_i}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha W_i^p(x, 0) & \text{on } \partial^0 \mathcal{B}_{1/r_i}^+ \setminus \{0\}, \end{cases} \quad (3.12)$$

and

$$W_i(e_1) \rightarrow 0 \quad \text{as } i \rightarrow \infty. \quad (3.13)$$

By Corollary 2.10, Theorem 2.15 and Proposition 2.19 in [22] that there exists $\gamma \in (0, 1)$ such that for every $R > 1 > r > 0$

$$\|W_i\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} + \|W_i\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} + \|t^{1-2\sigma} \partial_t W_i\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} \leq C(R, r),$$

where $C(R, r)$ is independent of i . Then after passing to a subsequence, $\{W_i\}$ converges to a nonnegative function $W \in W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}}_+^{n+1} \setminus \{0\}) \cap C_{loc}^\gamma(\overline{\mathbb{R}}_+^{n+1} \setminus \{0\})$ satisfying

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla W) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial W}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha W^p(x, 0) & \text{on } \mathbb{R}^n \setminus \{0\}. \end{cases} \quad (3.14)$$

By (3.13), we have $W(e_1) = 0$. This together with Lemma 2.2 imply that $W \equiv 0$ in $\overline{\mathbb{R}}_+^{n+1} \setminus \{0\}$. Since $E(r; U)$ is invariant under the scaling,

$$\lim_{i \rightarrow \infty} E(r_i; U) = \lim_{i \rightarrow \infty} E(1; W_i) = E(1; W) = 0.$$

By the monotonicity of $E(r; U)$ (Proposition 3.1), we obtain

$$\lim_{r \rightarrow 0^+} E(r; U) = 0.$$

Step 2. Let W be a nonnegative solution of (3.14) in $\overline{\mathbb{R}}_+^{n+1}$. If $E(r; W) \equiv 0$ for $r \in (0, \infty)$, then

$$W \equiv 0 \quad \text{in } \overline{\mathbb{R}}_+^{n+1} \setminus \{0\}.$$

Since $p \neq p_S(\alpha)$, we have $J_1 = \frac{n-2\sigma}{p-1} \left(\frac{n+2\sigma+2\alpha}{n-2\sigma} - p \right) \neq 0$. Hence, by Proposition 3.1, we get

$$\frac{\partial W}{\partial r} + \frac{2\sigma + \alpha}{p-1} \frac{W}{r} = 0 \quad \text{in } \mathbb{R}_+^{n+1}.$$

This implies that W is homogeneous of degree $-\frac{2\sigma+\alpha}{p-1}$. That is, there exists $\varphi \in C^2(\mathbb{S}_+^n)$ such that

$$W(X) = r^{-\frac{2\sigma+\alpha}{p-1}} \varphi(\theta),$$

where $X = (x, t) = r\theta$ with $r = |X|$ and $\theta = \frac{X}{|X|}$. Let $\theta_1 = \frac{t}{|X|}$ denote the component of θ in the t direction. A calculation similar to the proof of Proposition 3.1 shows that φ satisfies

$$\begin{cases} -\theta_1^{2\sigma-1} \operatorname{div}_\theta(\theta_1^{1-2\sigma} \nabla_\theta \varphi) + J_2 \varphi = 0 & \text{on } \mathbb{S}_+^n, \\ -\lim_{\theta_1 \rightarrow 0^+} \theta_1^{1-2\sigma} \partial_{\theta_1} \varphi = \kappa_\sigma \varphi^p & \text{on } \partial \mathbb{S}_+^n, \end{cases} \quad (3.15)$$

where

$$J_2 = \frac{2\sigma + \alpha}{p-1} \left(n - 2\sigma - \frac{2\sigma + \alpha}{p-1} \right).$$

Multiplying (3.15) by φ and integrating on \mathbb{S}_+^n , we obtain

$$\int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} |\nabla_\theta \varphi|^2 + J_2 \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} \varphi^2 = \kappa_\sigma \int_{\partial \mathbb{S}_+^n} \varphi^{p+1}. \quad (3.16)$$

On the other hand, from the proof of Proposition 3.1, $E(r; W) \equiv 0$ gives

$$-\frac{J_2}{2} \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} \varphi^2 - \frac{1}{2} \int_{\mathbb{S}_+^n} \theta_1^{1-2\sigma} |\nabla_\theta \varphi|^2 + \frac{\kappa_\sigma}{p+1} \int_{\partial \mathbb{S}_+^n} \varphi^{p+1} = 0. \quad (3.17)$$

Combine (3.16) and (3.17), we easily get

$$\left(1 - \frac{2}{p+1} \right) \int_{\partial \mathbb{S}_+^n} \varphi^{p+1} = 0,$$

and so $\varphi \equiv 0$ on $\partial \mathbb{S}_+^n$. By (3.16) and $J_2 > 0$, we obtain $\varphi = 0$ on \mathbb{S}_+^n and hence $W \equiv 0$ in \mathbb{R}_+^{n+1} .

Step 3. End of Proof. For $\lambda > 0$ small, define

$$U^\lambda(X) = \lambda^{\frac{2\sigma+\alpha}{p-1}} U(\lambda X).$$

Then U^λ is also a nonnegative solution of (2.4) in $\mathcal{B}_{1/\lambda}^+$. It follows from Proposition 2.1 and the Harnack inequality (2.20) that U^λ is locally uniformly bounded away from the origin. By Corollary 2.10, Theorem 2.15 and Proposition 2.19 in [22] that there exists $\gamma \in (0, 1)$ such that for every $R > 1 > r > 0$

$$\|U^\lambda\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} + \|U^\lambda\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} + \|t^{1-2\sigma} \partial_t U^\lambda\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}}_r^+)} \leq C(R, r),$$

where $C(R, r)$ is independent of λ . Hence, there is a subsequence λ_i of $\lambda \rightarrow 0$, such that $\{U^{\lambda_i}\}$ converges to a nonnegative function $U^0 \in W_{loc}^{1,2}(t^{1-2\sigma}, \overline{\mathbb{R}_+^{n+1}} \setminus \{0\}) \cap C_{loc}^\gamma(\overline{\mathbb{R}_+^{n+1}} \setminus \{0\})$ satisfying

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla U^0) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial U^0}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha (U^0(x, 0))^p & \text{on } \mathbb{R}^n \setminus \{0\}. \end{cases}$$

Moreover, by the scaling invariance of E and Step 1, we have for any $r > 0$ that

$$E(r; U^0) = \lim_{i \rightarrow \infty} E(r; U^{\lambda_i}) = \lim_{i \rightarrow \infty} E(\lambda_i r; U) = \lim_{r \rightarrow 0^+} E(r; U) = 0.$$

By Step 2 we have $U^0 \equiv 0$ in $\overline{\mathbb{R}_+^{n+1}} \setminus \{0\}$. Since this holds for the limit of any sequence $\lambda \rightarrow 0$, we obtain

$$\lim_{\lambda \rightarrow 0} U^\lambda = 0 \quad \text{in } C_{loc}^{\gamma/2}(\overline{\mathbb{R}_+^{n+1}} \setminus \{0\}).$$

In particular,

$$\lim_{\lambda \rightarrow 0} \lambda^{\frac{2\sigma+\alpha}{p-1}} u(\lambda x) = 0 \quad \text{uniformly in } x \in \partial B_1,$$

which immediately gives $\lim_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0$. \square

Proposition 3.3. *Let U be a nonnegative solution of (2.4) with $-2\sigma < \alpha < 2\sigma$ and $\frac{n+\alpha}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. If*

$$\lim_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = 0,$$

then the singularity at $x = 0$ is removable, i.e., $u(x)$ can be extended to a continuous function near the origin 0.

Proof. First, from the Harnack inequality (2.20), we have

$$\lim_{|X| \rightarrow 0} |X|^{\frac{2\sigma+\alpha}{p-1}} U(X) = 0. \quad (3.18)$$

For any $0 < \mu < n - 2\sigma$ and $0 < \delta < \frac{1}{2}$, define

$$\Psi_\mu(X) := |X|^{-\mu} \left(1 - \delta \left(\frac{t}{|X|} \right)^{2\sigma} \right),$$

where $X = (x, t) \neq 0$. Then Ψ_μ satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla \Psi_\mu(X)) = t^{1-2\sigma} |X|^{-(\mu+2)} \left(\mu(n-2\sigma-\mu) - \frac{\delta(\mu+2\sigma)(n-\mu)t^{2\sigma}}{|X|^{2\sigma}} \right), \\ -\lim_{t \rightarrow 0} t^{1-2\sigma} \partial_t \Psi_\mu(x, t) = 2\sigma\delta |x|^{-2\sigma} \Psi_\mu(x, 0), \quad x \neq 0. \end{cases}$$

Let $\mu_0 = \frac{2\sigma+\alpha}{p-1}$ and $\tau \in (0, \frac{2\sigma+\alpha}{p-1})$ be fixed. Note that $0 < \frac{2\sigma+\alpha}{p-1} < n - 2\sigma$ due to $-2\sigma < \alpha$ and $\frac{n+\alpha}{n-2\sigma} < p$. Let

$$\Psi = \epsilon \Psi_{\mu_0} + C \Psi_\tau,$$

where ϵ, C are positive constants. Then we can choose $\delta = \delta(\tau, \alpha, \sigma, p, n) \in (0, \frac{1}{2})$ such that

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla \Psi) \geq 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial \Psi}{\partial \nu^\sigma}(x, 0) = 2\sigma\delta |x|^{-2\sigma} \Psi(x, 0) & \text{on } \partial^0 \mathcal{B}_1^+ \setminus \{0\}. \end{cases}$$

Let $a(x) := \kappa_\sigma |x|^\alpha u^{p-1}(x)$. Then by the assumption we have $\lim_{|x| \rightarrow 0} a(x)|x|^{2\sigma} = 0$. Hence, there exists $r_0 \in (0, 1)$ such that

$$a(x) \leq 2\sigma\delta|x|^{-2\sigma} \quad \text{for } 0 < |x| \leq r_0.$$

Therefore

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla(\Psi - U)) \geq 0 & \text{in } \mathcal{B}_{r_0}^+, \\ \frac{\partial(\Psi - U)}{\partial\nu^\sigma}(x, 0) \geq 2\sigma\delta|x|^{-2\sigma}(\Psi(x, 0) - U(x, 0)) & \text{on } \partial^0\mathcal{B}_{r_0}^+ \setminus \{0\}. \end{cases}$$

Furthermore, we note that

$$\Psi(X) \geq \frac{\epsilon}{2}|X|^{-\frac{2\sigma+\alpha}{p-1}} \quad \text{for } X \in \mathcal{B}_1^+ \setminus \{0\}.$$

Hence, for any $\epsilon > 0$, by (3.18) there exists $r_\epsilon > 0$ small such that

$$\Psi \geq U \quad \text{in } \overline{\mathcal{B}_{r_\epsilon}^+} \setminus \{0\}.$$

On the other hand, we can choose constant $C = C(\tau, r_0, U)$ sufficiently large such that

$$\Psi \geq U \quad \text{on } \partial^+\mathcal{B}_{r_0}^+.$$

The maximum principle gives that

$$\Psi \geq U \quad \text{in } \overline{\mathcal{B}_{r_0}^+} \setminus \{0\}.$$

Now letting $\epsilon \rightarrow 0$, we get

$$U(X) \leq C(\tau, r_0, U)\Psi_\tau(X) \leq C(\tau, r_0, U)|X|^{-\tau} \quad \text{in } \overline{\mathcal{B}_{r_0}^+} \setminus \{0\}. \quad (3.19)$$

By the standard rescaling argument (such as see the proof of Lemma 4.1) and Proposition 2.19 in [22], we have

$$|\nabla_x U(X)| \leq C(\tau, r_0, U)|X|^{-\tau-1} \quad \text{in } \mathcal{B}_{r_0/2}^+ \setminus \{0\} \quad (3.20)$$

and

$$|t^{1-2\sigma}\partial_t U(X)| \leq C(\tau, r_0, U)|X|^{-\tau-2\sigma} \quad \text{in } \mathcal{B}_{r_0/2}^+ \setminus \{0\}. \quad (3.21)$$

Since $\tau \in (0, \frac{2\sigma+\alpha}{p-1})$ is arbitrary, it is not difficult to verify that $U \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_1^+)$. Next we will prove that U is a nonnegative weak solution of

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial\nu^\sigma}(x, 0) = \kappa_\sigma|x|^\alpha U^p(x, 0) & \text{on } \partial^0\mathcal{B}_1^+. \end{cases} \quad (3.22)$$

In fact, for $\epsilon > 0$ small, let $\eta_\epsilon \in C^\infty(\mathbb{R}^{n+1})$ be a cut-off function satisfying

$$\eta_\epsilon(X) = \begin{cases} 0 & \text{for } |X| \leq \epsilon, \\ 1 & \text{for } |X| \geq 2\epsilon, \end{cases}$$

and

$$|\nabla \eta_\epsilon(X)| \leq C\epsilon^{-1}.$$

For any $\psi \in C_c^\infty((\mathcal{B}_1^+ \cup \partial^0 \mathcal{B}_1^+))$, using $\psi \eta_\epsilon$ as a test function in (2.5) gives

$$\int_{\mathcal{B}_1^+} t^{1-2\sigma} \nabla U \cdot \nabla(\psi \eta_\epsilon) = \kappa_\sigma \int_{B_1} |x|^\alpha u^p(x) \psi \eta_\epsilon. \quad (3.23)$$

But

$$\begin{aligned} \left| \int_{\mathcal{B}_1^+} t^{1-2\sigma} \psi \nabla U \cdot \nabla \eta_\epsilon \right| &\leq C\epsilon^{-1} \left(\int_{\mathcal{B}_{2\epsilon}^+ \setminus \mathcal{B}_\epsilon^+} t^{1-2\sigma} |\nabla U| \right)^{1/2} \left(\int_{\mathcal{B}_{2\epsilon}^+ \setminus \mathcal{B}_\epsilon^+} t^{1-2\sigma} \right)^{1/2} \\ &\leq C\epsilon^{\frac{n-2\sigma}{2}} \left(\int_{\mathcal{B}_{2\epsilon}^+ \setminus \mathcal{B}_\epsilon^+} t^{1-2\sigma} |\nabla U| \right)^{1/2} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

By (3.19) and $\alpha > -2\sigma$, we have $\int_{B_1} |x|^\alpha u^p(x) \in L_{loc}^1(B_1)$. Letting $\epsilon \rightarrow 0$ in (3.23), we get

$$\int_{\mathcal{B}_1^+} t^{1-2\sigma} \nabla U \cdot \nabla \psi = \kappa_\sigma \int_{B_1} |x|^\alpha u^p(x) \psi.$$

Hence U is a nonnegative weak solution of (3.22). Again, (3.19) and $\alpha > -2\sigma$ imply that

$$|x|^\alpha u^{p-1}(x) \in L^q(B_{1/2}).$$

for some $q > \frac{n}{2\sigma}$. It follows from Proposition 2.6 in [22] that U is Hölder continuous in $\overline{\mathcal{B}_{1/2}^+}$. \square

Proof of Theorem 1.2. The proof of Theorem 1.2 is now just a combination of Propositions 2.1, 3.2 and 3.3. \square

4 Asymptotic Behavior

In this section we prove Theorem 1.1. We first prove the boundedness of $E(r; U)$.

Lemma 4.1. *Let $n \geq 2$, $-2\sigma < \alpha < 2\sigma$ and $\frac{n+\alpha}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. Assume that U is a nonnegative weak solution of (2.4) (resp. of (2.7)). Then $E(r; U)$ is uniformly bounded in $r \in (0, \frac{1}{8})$ (resp. in $r \in (8, \infty)$). Further, the limit*

$$\lim_{r \rightarrow 0^+} E(r; U) \quad (\text{resp.} \quad \lim_{r \rightarrow +\infty} E(r; U))$$

makes sense.

Proof. Let U be a nonnegative weak solution of (2.4) and define

$$V(X) = r^{\frac{2\sigma+\alpha}{p-1}} U(rX)$$

for any $r \in (0, \frac{1}{8})$ and $\frac{1}{2} \leq |X| \leq 2$. Then V satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla V) = 0 & \text{in } \mathcal{B}_2 \setminus \overline{\mathcal{B}}_{1/2}, \\ \frac{\partial V}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha v^p(x) & \text{on } B_2 \setminus \overline{B}_{1/2}, \end{cases}$$

where $v(x) = V(x, 0)$. It follows from Proposition 2.1 and Lemma 2.3 that

$$|V(X)| \leq C \quad \text{for all } \frac{1}{2} \leq |X| \leq 2,$$

where C is a positive constant depending only on n, p, σ and α . By Proposition 2.19 in [22], we have

$$\sup_{\frac{3}{4} \leq |X| \leq \frac{3}{2}} |\nabla_x V| + \sup_{\frac{3}{4} \leq |X| \leq \frac{3}{2}} |t^{1-2\sigma} \partial_t V| \leq C,$$

Hence, there exists $C > 0$, depending only on n, p, σ and α , such that

$$|\nabla_x U(X)| \leq C |X|^{-\frac{2\sigma}{p-1}-1} \quad \text{in } \mathcal{B}_{1/8}^+ \setminus \{0\}$$

and

$$|t^{1-2\sigma} \partial_t U(X)| \leq C |X|^{-\frac{2\sigma}{p-1}-2\sigma} \quad \text{in } \mathcal{B}_{1/8}^+ \setminus \{0\}.$$

Thus, a direct computation gives

$$\begin{aligned} r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n+1} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} |\nabla U| &\leq C, \\ r^{\frac{2(p+1)\sigma+2\alpha}{p-1}-n-1} \int_{\partial^+ \mathcal{B}_r^+} t^{1-2\sigma} U^2 &\leq C, \\ r^{\frac{(p+1)(2\sigma+\alpha)}{p-1}-n+1} \int_{\partial B_r} u^{p+1} &\leq C, \end{aligned}$$

where C is a positive constant depending only on n, p, σ and α . Now we easily conclude that $E(r; U)$ is uniformly bounded in $r \in (0, \frac{1}{8})$. By the monotonicity of $E(r; U)$, we obtain that the limit

$$\lim_{r \rightarrow 0^+} E(r; U)$$

makes sense.

Similarly, let U be a nonnegative weak solution of (2.7), we can prove that $E(r; U)$ is uniformly bounded in $r \in (8, \infty)$ and then the limit $\lim_{r \rightarrow +\infty} E(r; U)$ makes sense. \square

Next we give an elementary lemma. For its proof, such as see Lemma 3.1 of Fall [15].

Lemma 4.2. Assume $\sigma \in (0, 1)$, $-2\sigma < \alpha < 2\sigma$ and $p > \frac{n+\alpha}{n-2\sigma}$. Let $u(x) = |x|^{-\frac{2\sigma+\alpha}{p-1}}$ for $x \in \mathbb{R}^n \setminus \{0\}$. Then

$$(-\Delta)^\sigma u(x) = C_{p,\sigma,\alpha} |x|^\alpha u^p(x) \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

where $C_{p,\sigma,\alpha}$ is given by (1.6).

Proof of Theorem 1.1. Suppose that $u \in C^2(B_1 \setminus \{0\}) \cap \mathcal{L}_\sigma(\mathbb{R}^n)$ is a nonnegative solution of (1.3) and the origin 0 is a non-removable singularity, we only need to prove (1.5). Let U be the extension of u solving (2.4). We consider separately the subcritical case $p < p_S(\alpha)$ and the supercritical case $p > p_S(\alpha)$.

Case 1. $p < p_S(\alpha)$. We define the scaling

$$U^\lambda(X) = \lambda^{\frac{2\sigma+\alpha}{p-1}} U(\lambda X).$$

Then U^λ satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla U^\lambda) = 0 & \text{in } \mathcal{B}_{1/\lambda}^+, \\ \frac{\partial U^\lambda}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha (U^\lambda(x, 0))^p & \text{on } \partial^0 \mathcal{B}_{1/\lambda}^+ \setminus \{0\}. \end{cases}$$

By Theorem 1.2 and Lemma 2.3, there exist $C_1, C_2 > 0$ such that

$$C_1 |X|^{-\frac{2\sigma+\alpha}{p-1}} \leq U^\lambda(X) \leq C_2 |X|^{-\frac{2\sigma+\alpha}{p-1}} \quad \text{in } \overline{\mathcal{B}_{1/(2\lambda)}^+} \setminus \{0\}. \quad (4.1)$$

Thus U^λ is locally uniformly bounded away from the origin. By Corollary 2.10 and Theorem 2.15 in [22] that there exists $\gamma > 0$ such that for every $R > 1 > r > 0$

$$\|U^\lambda\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \setminus \overline{\mathcal{B}_r^+})} + \|U^\lambda\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}_r^+})} + \|u^\lambda\|_{C^{2,\gamma}(B_R \setminus B_r)} \leq C(R, r),$$

where $u^\lambda(x) = U^\lambda(x, 0)$ and $C(R, r)$ is independent of λ . Then there is a subsequence $\lambda_k \rightarrow 0$, such that $\{U^{\lambda_k}\}$ converges to a nonnegative function $U^0 \in W_{loc}^{1,2}(t^{1-2\sigma}, \mathbb{R}_+^{n+1} \setminus \{0\}) \cap C_{loc}^\gamma(\mathbb{R}_+^{n+1} \setminus \{0\})$ satisfying

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla U^0) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial U^0}{\partial \nu^\sigma}(x, 0) = \kappa_\sigma |x|^\alpha (U^0(x, 0))^p & \text{on } \mathbb{R}^n \setminus \{0\}, \end{cases}$$

and $u^0(x) := U^0(x, 0) \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \mathcal{L}_\sigma(\mathbb{R}^n)$ satisfying

$$(-\Delta)^\sigma u^0 = |x|^\alpha (u^0)^p \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (4.2)$$

By (4.1) we have

$$C_1 |X|^{-\frac{2\sigma+\alpha}{p-1}} \leq U^0(X) \leq C_2 |X|^{-\frac{2\sigma+\alpha}{p-1}} \quad \text{in } \overline{\mathbb{R}_+^{n+1}} \setminus \{0\}. \quad (4.3)$$

Moreover, by the scaling invariance of $E(r; U)$ and Lemma 4.1, we have for any $r > 0$ that

$$E(r; U^0) = \lim_{k \rightarrow \infty} E(r; U^{\lambda_k}) = \lim_{k \rightarrow \infty} E(r \lambda_k; U) = E(0^+; U).$$

That is, $E(r; U^0)$ is a constant. It follows from Proposition 3.1 that U^0 is homogeneous of degree $-\frac{2\sigma+\alpha}{p-1}$. Hence, there exists $\varphi^0 \in C^2(\mathbb{S}_+^n) \cap C(\overline{\mathbb{S}_+^n})$ such that

$$U^0(X) = r^{-\frac{2\sigma+\alpha}{p-1}} \varphi^0(\theta),$$

where $X = (x, t) = r\theta$ with $r = |X|$ and $\theta = \frac{X}{|X|}$. A calculation similar to the proof of Proposition 3.1 shows that φ^0 satisfies

$$\begin{cases} -\theta_1^{2\sigma-1} \operatorname{div}_\theta(\theta_1^{1-2\sigma} \nabla_\theta \varphi^0) + J_2 \varphi^0 = 0 & \text{on } \mathbb{S}_+^n, \\ -\lim_{\theta_1 \rightarrow 0^+} \theta_1^{1-2\sigma} \partial_{\theta_1} \varphi^0 = \kappa_\sigma(\varphi^0)^p & \text{on } \partial\mathbb{S}_+^n, \end{cases} \quad (4.4)$$

where $\theta_1 = \frac{t}{|X|}$ denotes the component of θ in the t direction. Recall that

$$J_2 = \frac{2\sigma + \alpha}{p-1} \left(n - 2\sigma - \frac{2\sigma + \alpha}{p-1} \right).$$

Moreover, by (4.3), φ^0 also satisfies

$$0 < C_1 \leq \varphi^0(\theta) \leq C_2 \quad \text{on } \overline{\mathbb{S}_+^n}.$$

On the other hand, since $p < p_S(\alpha)$, $U^0(x, t)$ is cylindrically symmetric about the origin 0, for this, such as see Theorem 1.1 in [26]. So φ^0 is a positive constant on $\partial\mathbb{S}_+^n$. By Lemma 4.2, we know that

$$\varphi^0 \equiv C_{p,\sigma,\alpha} \quad \text{on } \partial\mathbb{S}_+^n,$$

where $C_{p,\sigma,\alpha}$ is given by (1.6). Therefore

$$u^0(x) = C_{p,\sigma,\alpha} |x|^{-\frac{2\sigma+\alpha}{p-1}} \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Since this function $u^0(x)$ is unique, we conclude that $u^\lambda(x) \rightarrow u^0(x)$ for any sequence $\lambda \rightarrow 0$ in $C_{loc}^\gamma(\mathbb{R}^n \setminus \{0\})$. Hence

$$|\lambda x|^{\frac{2\sigma+\alpha}{p-1}} u(\lambda x) = |x|^{\frac{2\sigma+\alpha}{p-1}} u^\lambda(x) \rightarrow C_{p,\sigma,\alpha} \quad \text{as } \lambda \rightarrow 0$$

in $B_2 \setminus B_{1/2}$. We immediately conclude that

$$\lim_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = C_{p,\sigma,\alpha}.$$

Case 2. $p_S(\alpha) < p \leq \frac{n+2\sigma+\alpha}{n-2\sigma}$. We consider the Kelvin transform

$$\begin{aligned} Y &= \frac{X}{|X|^2}, \\ \tilde{U}(Y) &= |X|^{n-2\sigma} U(X). \end{aligned}$$

Then \tilde{U} satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla\tilde{U}) = 0 & \text{in } \mathbb{R}_+^{n+1}\setminus\overline{\mathcal{B}_1^+}, \\ \frac{\partial\tilde{U}}{\partial\nu^\sigma}(y,0) = \kappa_\sigma|y|^\vartheta\tilde{U}^p(y,0) & \text{on } B_1^c, \end{cases} \quad (4.5)$$

where $\vartheta := p(n-2\sigma) - (n+2\sigma+\alpha)$ and $B_1^c = \{x \in \mathbb{R}^n : |x| > 1\}$. Moreover, by Theorem 1.2 and Lemma 2.3, we have

$$\frac{C_1}{|Y|^{(2\sigma+\vartheta)/(p-1)}} \leq \tilde{U}(Y) \leq \frac{C_2}{|Y|^{(2\sigma+\vartheta)/(p-1)}} \quad \text{for } |Y| \text{ large.} \quad (4.6)$$

Note that

$$-2\sigma < \vartheta \leq 0$$

due to $\frac{n+\alpha}{n-2\sigma} < p$ and $p \leq \frac{n+2\sigma+\alpha}{n-2\sigma}$. Moreover,

$$\begin{aligned} p > \frac{n+\vartheta}{n-2\sigma} & \Leftrightarrow \alpha > -2\sigma, \\ p < \frac{n+2\sigma+2\vartheta}{n-2\sigma} & \Leftrightarrow p > \frac{n+2\sigma+2\alpha}{n-2\sigma}. \end{aligned}$$

Therefore, after the Kelvin transform, ϑ satisfies

$$-2\sigma < \vartheta \leq 0 \quad \text{and} \quad \frac{n+\vartheta}{n-2\sigma} < p < \frac{n+2\sigma+2\vartheta}{n-2\sigma}.$$

For any $\lambda > 0$, define

$$\tilde{U}^\lambda(Y) = \lambda^{\frac{2\sigma+\vartheta}{p-1}} \tilde{U}(\lambda Y).$$

Then \tilde{U}^λ satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla\tilde{U}^\lambda) = 0 & \text{in } \mathbb{R}_+^{n+1}\setminus\overline{\mathcal{B}_{1/\lambda}^+}, \\ \frac{\partial\tilde{U}^\lambda}{\partial\nu^\sigma}(y,0) = \kappa_\sigma|y|^\vartheta(\tilde{U}^\lambda(y,0))^p & \text{on } \mathbb{R}^n\setminus B_{1/\lambda}. \end{cases}$$

By (4.6),

$$C_1|Y|^{-\frac{2\sigma+\vartheta}{p-1}} \leq \tilde{U}^\lambda(X) \leq C_2|Y|^{-\frac{2\sigma+\vartheta}{p-1}} \quad \text{in } \mathbb{R}_+^{n+1}\setminus\overline{\mathcal{B}_{1/(2\lambda)}^+}. \quad (4.7)$$

It follows from Corollary 2.10 and Theorem 2.15 in [22] that there exists $\gamma > 0$ such that for every $R > 1 > r > 0$

$$\|\tilde{U}^\lambda\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \setminus \overline{\mathcal{B}_r^+})} + \|\tilde{U}^\lambda\|_{C^\gamma(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}_r^+})} + \|\tilde{u}^\lambda\|_{C^{2,\gamma}(B_R \setminus B_r)} \leq C(R, r),$$

where $\tilde{u}^\lambda(y) = \tilde{U}^\lambda(y,0)$ and $C(R, r)$ is independent of λ . Then there is a subsequence λ_k of $\lambda \rightarrow +\infty$, such that $\{\tilde{U}^{\lambda_k}\}$ converges to a nonnegative function $\tilde{U}^\infty \in W_{loc}^{1,2}(t^{1-2\sigma}, \mathbb{R}_+^{n+1} \setminus \{0\}) \cap C_{loc}^\gamma(\mathbb{R}_+^{n+1} \setminus \{0\})$ satisfying

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma}\nabla\tilde{U}^\infty) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial\tilde{U}^\infty}{\partial\nu^\sigma}(y,0) = \kappa_\sigma|y|^\vartheta(\tilde{U}^\infty(y,0))^p & \text{on } \mathbb{R}^n \setminus \{0\}, \end{cases}$$

and $\tilde{u}^\infty(y) := \tilde{U}^\infty(y, 0) \in C^2(\mathbb{R}^n \setminus \{0\}) \cap \mathcal{L}_\sigma(\mathbb{R}^n)$ satisfying

$$(-\Delta)^\sigma \tilde{u}^\infty = |y|^\vartheta (\tilde{u}^\infty)^p \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (4.8)$$

By (4.7) we have

$$C_1 |Y|^{-\frac{2\sigma+\vartheta}{p-1}} \leq \tilde{U}^\infty(Y) \leq C_2 |Y|^{-\frac{2\sigma+\vartheta}{p-1}} \quad \text{in } \overline{\mathbb{R}^{n+1}} \setminus \{0\}. \quad (4.9)$$

Moreover, by the scaling invariance of $E(r; \tilde{U})$ and Lemma 4.1, we have for any $r > 0$ that

$$E(r; \tilde{U}^\infty) = \lim_{k \rightarrow \infty} E(r; \tilde{U}^{\lambda_k}) = \lim_{k \rightarrow \infty} E(r \lambda_k; \tilde{U}) = \lim_{r \rightarrow +\infty} E(r; \tilde{U}).$$

That is, $E(r; \tilde{U}^\infty)$ is a constant. It follows from Proposition 3.1 that \tilde{U}^∞ is homogeneous of degree $-\frac{2\sigma+\vartheta}{p-1}$. Notice that we have $p < p_S(\vartheta)$, by a similar argument as that of Case 1, we obtain that \tilde{u}^∞ has the form

$$\tilde{u}^\infty(y) = C_{p,\sigma,\vartheta} |y|^{-\frac{2\sigma+\vartheta}{p-1}} \quad \text{for } y \in \mathbb{R}^n \setminus \{0\},$$

where $C_{p,\sigma,\vartheta}$ is given by (1.6). Since this function $\tilde{u}^\infty(y)$ is unique, we conclude that $\tilde{u}^\lambda(y) \rightarrow \tilde{u}^\infty(y)$ for any sequence $\lambda \rightarrow +\infty$ in $C_{loc}^\gamma(\mathbb{R}^n \setminus \{0\})$. In particular,

$$|\lambda y|^{\frac{2\sigma+\vartheta}{p-1}} \tilde{u}(\lambda y) = |y|^{\frac{2\sigma+\vartheta}{p-1}} \tilde{u}^\lambda(y) \rightarrow C_{p,\sigma,\vartheta} \quad \text{as } \lambda \rightarrow +\infty$$

in $B_2 \setminus B_{1/2}$. Hence we have

$$\lim_{|y| \rightarrow +\infty} |y|^{\frac{2\sigma+\vartheta}{p-1}} \tilde{u}(y) = C_{p,\sigma,\vartheta}.$$

From (1.7) we note that $\Lambda(\tau) = \Lambda(-\tau)$. Therefore

$$C_{p,\sigma,\vartheta} = \left\{ \Lambda \left(\frac{n-2\sigma}{2} - \frac{2\sigma+\vartheta}{p-1} \right) \right\}^{\frac{1}{p-1}} = \left\{ \Lambda \left(\frac{2\sigma+\alpha}{p-1} - \frac{n-2\sigma}{2} \right) \right\}^{\frac{1}{p-1}} = C_{p,\sigma,\alpha}.$$

By the Kelvin transform, we now easily get

$$\lim_{|x| \rightarrow 0} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = C_{p,\sigma,\alpha}.$$

This completes the proof of Theorem 1.1. \square

5 Isolated Singularities at ∞

In this section we prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let $u(x)$ be a nonnegative solution of (1.9) with $\alpha > -2\sigma$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Define the Kelvin transform

$$\tilde{u}(y) = \frac{1}{|y|^{n-2\sigma}} u\left(\frac{y}{|y|^2}\right) \quad \text{for } 0 < |y| < 1. \quad (5.1)$$

Then $\tilde{u}(y)$ satisfies

$$(-\Delta)^\sigma \tilde{u} = |y|^\rho \tilde{u}^p \quad \text{in } B_1 \setminus \{0\},$$

where $\rho := p(n - 2\sigma) - (n + 2\sigma + \alpha)$.

(1) If $1 < p < \frac{n+\alpha}{n-2\sigma}$, then $\rho < -2\sigma$. Hence, by Corollary 2.1, $\tilde{u}(y) \equiv 0$ in $B_1 \setminus \{0\}$ which implies that $u(x) \equiv 0$ for $|x| > 1$.

(2) If $\frac{n+\alpha}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$, then $-2\sigma < \rho < -\alpha < 2\sigma$ and

$$\frac{n + \rho}{n - 2\sigma} = p - \frac{2\sigma + \alpha}{n - 2\sigma} < p.$$

By Theorem 1.2, either the singularity at $y = 0$ is removable or there exist $c_1, c_2 > 0$ such that

$$\frac{c_1}{|y|^{(2\sigma+\rho)/(p-1)}} \leq \tilde{u}(y) \leq \frac{c_2}{|y|^{(2\sigma+\rho)/(p-1)}} \quad \text{near } y = 0. \quad (5.2)$$

If the singularity at $y = 0$ is removable, then

$$u(x) = O\left(\frac{1}{|x|^{n-2\sigma}}\right).$$

If (5.2) holds, then

$$\frac{c_1}{|x|^{(2\sigma+\alpha)/(p-1)}} \leq u(x) \leq \frac{c_2}{|x|^{(2\sigma+\alpha)/(p-1)}} \quad \text{near } x = \infty.$$

This completes the proof. \square

Proof of Theorem 1.4. Let $u(x)$ be a positive solution of (1.9) with $-2\sigma < \alpha \leq 0$, $\frac{n+\alpha}{n-2\sigma} < p \leq \frac{n+2\sigma+\alpha}{n-2\sigma}$ and $p \neq \frac{n+2\sigma+2\alpha}{n-2\sigma}$. We define the Kelvin transform $\tilde{u}(y)$ of $u(x)$ as in (5.1). Then $\tilde{u}(y)$ satisfies

$$(-\Delta)^\sigma \tilde{u} = |y|^\rho \tilde{u}^p \quad \text{in } B_1 \setminus \{0\}, \quad (5.3)$$

where $\rho := p(n - 2\sigma) - (n + 2\sigma + \alpha)$. Note that

$$\begin{aligned} -2\sigma < \rho &\Leftrightarrow \frac{n + \alpha}{n - 2\sigma} < p, \\ \rho \leq 0 &\Leftrightarrow p \leq \frac{n + 2\sigma + \alpha}{n - 2\sigma}, \\ \frac{n + \rho}{n - 2\sigma} < p &\Leftrightarrow -2\sigma < \alpha, \\ p \leq \frac{n + 2\sigma + \rho}{n - 2\sigma} &\Leftrightarrow \alpha \leq 0, \\ p \neq \frac{n + 2\sigma + 2\rho}{n - 2\sigma} &\Leftrightarrow p \neq \frac{n + 2\sigma + 2\alpha}{n - 2\sigma}. \end{aligned}$$

Hence, under the assumption of Theorem 1.4, we have

$$-2\sigma < \rho \leq 0, \quad \frac{n + \rho}{n - 2\sigma} < p \leq \frac{n + 2\sigma + \rho}{n - 2\sigma} \quad \text{and} \quad p \neq \frac{n + 2\sigma + 2\rho}{n - 2\sigma}.$$

Thus Theorem 1.1 applies to the equation (5.3). This implies that either the singularity near $y = 0$ is removable, or

$$\lim_{|y| \rightarrow 0} |y|^{\frac{2\sigma+\varrho}{p-1}} \tilde{u}(y) = C_{p,\sigma,\varrho}. \quad (5.4)$$

If the singularity near $y = 0$ is removable, then $\tilde{u}(y)$ can be extended to a continuous function near the origin 0. Hence, there exists $\beta > 0$ such that

$$\lim_{|x| \rightarrow \infty} |x|^{n-2\sigma} u(x) = \beta.$$

If (5.4) holds, then

$$\lim_{|x| \rightarrow \infty} |x|^{\frac{2\sigma+\alpha}{p-1}} u(x) = C_{p,\sigma,\alpha}.$$

This completes the proof. \square

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