Minimal graphic functions on manifolds of non-negative Ricci curvature

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Abstract

We study minimal graphic functions on complete Riemannian manifolds Σ with non-negative Ricci curvature, Euclidean volume growth and quadratic curvature decay. We derive global bounds for the gradients for minimal graphic functions of linear growth only on one side. Then we can obtain a Liouville type theorem with such growth via splitting for tangent cones of Σ at infinity. When, in contrast, we do not impose any growth restrictions for minimal graphic functions, we also obtain a Liouville type theorem under a certain non-radial Ricci curvature decay condition on Σ. In particular, the borderline for the Ricci curvature decay is sharp by our example in the last section.

1 Introduction

The minimal surface equation on a Euclidean space,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0 \tag{1.1}$$

has been investigated extensively by many mathematicians. Let us recall some famous results that constitute a background for our present work. In 1961, J. Moser [37] derived Harnack inequalities for uniformly elliptic equations that imply weak Bernstein results for minimal graphs in any dimension. In 1969, Bombieri-De Giorgi-Miranda [4] showed interior gradient estimates for solutions to the minimal surface equation (see also the exposition in chapter 16 of [26]); the two-dimensional case had already been obtained by Finn [22] in 1954.

In this paper, we study the non-linear partial differential equation describing minimal graphs over complete Riemannian manifolds of non-negative Ricci curvature. Formally, the equation for a minimal graph over a Riemannian manifold Σ is the same as for Euclidean space,

$$\mathrm{div}_\Sigma\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0, \tag{1.2}$$

where the divergence operator and the norm now are defined in terms of the Riemannian metric of Σ.

The geometric content of (1.2) is that a solution u is the height function in the product manifold $N=\Sigma\times\mathbb{R}$ of a minimal graph M in N. We therefore call a solution to (1.2) a *a minimal graphic function.*

Of course, the Riemannian equation (1.2) is more difficult than its Euclidean version (1.1). In order to obtain strong results, one needs to restrict the class of underlying Riemannian manifolds. The linear analogue, the equation for harmonic functions on Riemannian manifolds, suggests that non-negative Ricci curvature should be a good geometric condition. In fact, harmonic functions on complete manifolds with non-negative Ricci curvature have been very successfully studied by S. T. Yau [44], Colding-Minicozzi [18], P. Li [32] and many others. Our problem can be considered as a non-linear generalization of harmonic functions on complete manifolds with non-negative Ricci curvature.

More precisely, we consider complete non-compact $n-$dimensional Riemannian manifolds Σ satisfying the three conditions:

C1) Non-negative Ricci curvature;
C2) Euclidean volume growth;
C3) Quadratic decay of the curvature tensor.

Fischer-Colbrie and Schoen [23] studied stable minimal surfaces in 3-dimensional manifolds with nonnegative scalar curvature, and showed their rigidity. In our companion paper [21], we study minimal hypersurfaces in Σ and obtain existence and non-existence results for area-minimizing hypersurfaces in such an Σ. Here, we restrict the dimension $n+1\geq 4$ for ambient product manifolds and investigate minimal graphs from the PDE point of view.

Cheeger and Colding [10, 11, 12] studied the structure of pointed Gromov-Hausdorff limits of sequences $\{M_i^n,p_i)\}$ of complete connected Riemannian manifolds with $Ric_{M_i^n}\geq-(n-1)$. They showed that the singular set $\mathscr{S}$ of such a space has dimension no bigger than $n-2$. Subsequently, Cheeger-Colding-Tian [14] showed the stronger statement that $\mathscr{S}$ has dimension no bigger than $n-4$ under some additional assumption. We should point out that conditions C1) –C3) still permit some nasty behavior of the manifold Σ. For instance, as G. Perelman pointed out, the tangent cones at infinity need not be unique. Our conditions C2) and C3) also have appeared in the investigation of the uniqueness of tangent cones at infinity for Ricci flat manifolds by Cheeger and Tian [15]. This theory has recently been further developed by Colding and Minicozzi [19].

In the last decade, minimal graphs in product manifolds received considerable attention. Concerning gradient estimates, J. Spruck [42] obtained interior gradient estimates via the maximum principle. He went on to apply them to the Dirichlet problem for constant mean curvature graphs. Recently, Rosenberg-Schulze-Spruck [39] obtained a new gradient estimate and then showed that there is no trivial positive entire minimal graph over any manifold with nonnegative Ricci curvature and curvature bounded below.

For a complete manifold Σ with conditions C1), C2) and C3) we obtain the gradient estimates for minimal graphic functions by integral methods, see Theorem 3.3. Such results in Euclidean space were given by [4], [5]. Our results and methods are different from those in [42].

Let us now describe our results in more precise PDE terms. Theorem 3.3 enables us to obtain global bounds for gradients when the growth for the minimal graphic functions is linearly constrained only on one side. So linear growth is equivalent to linear growth on one side for minimal graphic functions. Since the Sobolev inequality and the Neumann-Poincaré inequality both hold on Σ, and thus also on the minimal graph M in $\Sigma \times \mathbb{R}$ represented by a minimal graphic function with bounded gradient, this will then yield mean value inequalities for subharmonic functions on the domains of M. Hence, we have the mean value equalities both in the extrinsic balls or the intrinsic balls of M. Therefore, we obtain a Liouville type theorem for minimal graphic functions with sub-linear growth on one side, see Theorem 3.6. It is interesting to compare this Liouville type theorem with the half-space theorem of Rosenberg-Schulze-Spruck [39]. The corresponding result for harmonic functions on manifolds with non-negative Ricci curvature is due to S. Y. Cheng [7]. Furthermore, we can relax sub-linear growth to linear growth in the above Liouville type theorem if Σ is not Euclidean space, and obtain the following theorem.

Theorem 1.1. *Let u be an entire solution to* (2.1) *on a complete Riemannian manifold Σ with conditions C1), C2), C3). If u has at most linear growth on one side, then u must be a constant unless Σ is isometric to Euclidean space.*

For showing Theorem 1.1, using the harmonic coordinates of Jost-Karcher [31], we first obtain scale-invariant Schauder estimates for minimal graphic functions u. Then combining this with estimates for the Green function and a Bochner type formula, we get integral decay estimates for the Hessian of u. The Schauder estimates then imply point-wise decay estimates for the Hessian of u. Finally, by re-scaling the manifold Σ and the graphic function u we can show that Σ is asymptotic to a product of a Euclidean factor $\mathbb{R}$ and the level set of u at the value 0. This will allow us to deduce Theorem 1.1. By the example in the last section, the assumption of linear growth cannot be removed. Moreover, one should compare this theorem with the result of Cheeger-Colding-Minicozzi [13] on the splitting of the tangent cone at infinity for complete manifolds with nonnegative Ricci curvature supporting linear growth harmonic functions.

We shall also investigate minimal graphic functions without growth restrictions. Analogously to [21] we prove that for any scaling sequence of a minimal graph M in $\Sigma\times\mathbb{R}$ there exists a subsequence that converges to an area-minimizing cone T in $\Sigma_\infty\times\mathbb{R}$, where Σ_∞ is some tangent cone at infinity (not necessarily unique) of Σ satisfying conditions C1), C2) and C3). However, our proof here is more complicated than [21] as the ambient manifold $\Sigma\times\mathbb{R}$ does no longer satisfy condition C3) unless Σ is flat. In the light of stability inequalities for minimal graphs, one might expect that T is vertical, namely, $T=\mathscr{X}\times\mathbb{R}$ for some cone $\mathscr{X}\in N_\infty$. However, since we lack Sobolev and Neumann-Poincaré inequalities on minimal graphs whose gradient is not globally bounded, it seems difficult to show verticality, although such inequalities hold for ambient Euclidean space [35], [5]. Fortunately, we are able to show that T is asymptotically vertical at infinity. This also suffices to estimate the measure of a 'bad' set and employ stability arguments, as in [21], to eliminate the unbounded situation of $|Du|$, when the lower bound κ for the curvature decay (see below) is large, that is, when the curvature can only slowly decay to 0 at infinity (more precisely, the curvature decay is quadratic, but the value of the lower bound must be above some critical threshold; actually that threshold is sharp showed in the last section). Thus combining Theorem 1.1 we obtain a Liouville type theorem for minimal graphic functions.

Theorem 1.2. *Let Σ be a complete $n-$dimensional Riemannian manifold satisfying conditions C1), C2), C3) and with non-radial Ricci curvature satisfying* $\inf_{\partial B_\rho} Ric_\Sigma\left(\xi^T,\xi^T\right)\geq\kappa\rho^{-2}|\xi^T|^2$ *for some constant κ, for sufficiently large $\rho>0$, where ρ is the distance function from a fixed point in Σ and ξ^T stands for the part that is tangential to the geodesic sphere ∂B_ρ (at least away from the cut locus of the center), of a tangent vector ξ of Σ at the considered point. If $\kappa>\frac{(n-3)^2}{4}$, then any entire solution to* (1.2) *on Σ must be constant.*

In the last section, we construct a nontrivial minimal graph to show that the constant $\frac{(n-3)^2}{4}$ in Theorem 1.2 is sharp. Our approach is inspired by the method developed by Simon [41], where for each strictly minimizing isoparametric cone C in $\mathbb{R}^n$ he constructed an entire minimal graph in $\mathbb{R}^{n+1}$ converging to $C\times\mathbb{R}$ as tangent cylinder at ∞. Geometrically, our method is different from that of Simon, but analytically, it is quite similar.

2 Preliminaries

Let (Σ,σ) be an n-dimensional complete non-compact manifold with Riemannian metric $\sigma=\sum_{i,j=1}^n\sigma_{ij}dx_idx_j$ in a local coordinate. Set (σ^{ij}) be the inverse matrix of (σ_{ij}) and E_i be the dual frame of dx_i. Denote $Du=\sum_{i,j}\sigma^{ij}u_iE_j$ and $|Du|^2=\sum_{i,j}\sigma^{ij}u_iu_j$. Let div_Σ be the divergence of Σ. We shall study the following quasi-linear elliptic equation for a minimal graphic function on Σ

$$\operatorname{div}_\Sigma\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \triangleq \frac{1}{\sqrt{\det\sigma_{kl}}}\partial_j\left(\sqrt{\det\sigma_{kl}}\frac{\sigma^{ij}u_i}{\sqrt{1+|Du|^2}}\right)=0. \tag{2.1}$$

A solution $\{(x,u(x))\in\Sigma\times\mathbb{R}|\ x\in\Sigma\}$ represents a minimal hypersurface in the product manifold $N\triangleq\Sigma\times\mathbb{R}$ with the product metric

$$ds^2=dt^2+\sigma=dt^2+\sum_{i,j}\sigma_{ij}dx_idx_j.$$

Let M denote this hypersurface, i.e., $M=\{(x,u(x))\in N|\ x\in\Sigma\}$, with the induced metric g from N

$$g=\sum_{i,j}g_{ij}dx_idx_j=\sum_{i,j}(\sigma_{ij}+u_iu_j)dx_idx_j,$$

where $u_i=\frac{\partial u}{\partial x_j}$ and and $u_{ij}=\frac{\partial^2 u}{\partial x_i\partial x_j}$ in the sequel. Moreover, $\det g_{ij}=(1+|Du|^2)\det\sigma_{ij}$ and $g^{ij}=\sigma^{ij}-\frac{u^iu^j}{1+|Du|^2}$ with $u^i=\sum_j\sigma^{ij}u_j$.

Let Δ, ∇ be the Laplacian operator and Levi-Civita connection of (M,g), respectively. The equation (2.1) then becomes

$$\Delta u=\frac{1}{\sqrt{\det g_{kl}}}\partial_j\left(\sqrt{\det g_{kl}}g^{ij}u_i\right)=0. \tag{2.2}$$

Thus, u is a harmonic function on the hypersurface$M$, which in turn depends on u.

Similar to the Euclidean case ([43] or by Lemma 2.1 in [21]), any minimal graph over a bounded domain Ω is an area-minimizing hypersurface in $\Omega\times\mathbb{R}$. From the proof of Lemma 2.1 in [21], it is not hard to see that any minimal graph over Σ is an area-minimizing hypersurface in $\Sigma\times\mathbb{R}$.

Let $\overline{\nabla}$ and $\overline{R}$ be the Levi-Civita connection and curvature tensor of (N,h). Let $\langle\cdot,\cdot\rangle$ be the inner product on N with respect to its metric. Let ν denote the unit normal vector field of M in N defined by

$$\nu=\frac{1}{\sqrt{1+|Du|^2}}(-Du+E_{n+1}). \tag{2.3}$$

Choose a local orthonormal frame field $\{e_i\}_{i=1}^n$ in M. Set the coefficients of the second fundamental form $h_{ij}=\langle\overline{\nabla}_{e_i}e_j,\nu\rangle$ and the squared norm of the second fundamental form $|B|^2=\sum_{i,j}h_{ij}h_{ij}$. Then the mean curvature $H=\sum_i h_{ii}=0$ as M is minimal.

Let $\overline{Ric}$ be the Ricci curvature of $\Sigma\times\mathbb{R}$. Due to the Codazzi equation $h_{ijk}=h_{kji}-\langle\overline{R}_{e_ke_i}e_j,\nu\rangle$ (see [43], for example), we obtain a Bochner type formula

$$\Delta\langle E_{n+1},\nu\rangle=-\left(|B|^2+\overline{Ric}(\nu,\nu)\right)\langle E_{n+1},\nu\rangle. \tag{2.4}$$

Let Ric denote the Ricci curvature of Σ. With (2.3), then

$$-\Delta\log\langle E_{n+1},\nu\rangle=|B|^2+\frac{Ric(Du,Du)}{1+|Du|^2}+|\nabla\log\langle E_{n+1},\nu\rangle|^2. \tag{2.5}$$

The above formula will play a similar role as Bochner's formula for the squared length of the gradient of a harmonic function.

In the present paper we usually suppose that Σ is an $n(\geq 3)$-dimensional complete non-compact Riemannian manifold satisfying the following three conditions:

C1) Nonnegative Ricci curvature: $\mathrm{Ric} \geq 0$;

C2) Euclidean volume growth: for the geodesic balls $B_r(x)$ in Σ,

$$V_\Sigma \triangleq \lim_{r\to\infty} \frac{Vol\big(B_r(x)\big)}{r^n} > 0;$$

C3) Quadratic decay of the curvature tensor: for sufficiently large $\rho(x) = d(x,p)$, the distance from a fixed point in N,

$$|R(x)| \leq \frac{c}{\rho^2(x)}.$$

By the Bishop-Gromov volume comparison theorem $\lim_{r\to\infty} \frac{Vol(B_r(x))}{r^n}$ is monotonically nonincreasing, and hence

$$V_\Sigma r^n \leq Vol\big(B_r(x)\big) \leq \omega_n r^n \qquad \text{for all } x \in \Sigma \text{ and } r > 0, \tag{2.6}$$

where ω_n is the volume of the standard n-dimensional unit ball in Euclidean space. The above three conditions have been used in [21] to study minimal hypersurfaces in such manifolds. Now we list some properties of Σ satisfying conditions C1), C2), C3) (see [21] for completeness), which will be employed in the following text.

- By [8], there is a sufficiently small constant $\delta_0 > 0$ depending only on n, c, V_Σ so that for any $0 < \delta < \delta_0$ the injectivity radius at $q \in \partial B_{\Omega r}(p)$ satisfies $i(q) \geq r$, where $\Omega = \Big(\frac{\sqrt{c}}{\delta} + 1\Big)$.
- Let $G(p,\cdot)$ be the Green function on Σ with $\lim_{r\to 0} \sup_{\partial B_r(p)} \big|Gr^{n-2} - 1\big| = 0$ and $b = G^{\frac{1}{2-n}}$(see [36] and [17] for details). Then

$$\Delta_\Sigma b^2 = 2n|Db|^2 \tag{2.7}$$

with $|Db| \leq 1$ and $c'r \leq b(x) \leq r$ for any $n \geq 3$, $x \in \partial B_r(p)$ and some constant $c' > 0$. Moreover, we have asymptotic estimates

$$\limsup_{r\to\infty} \left(\sup_{x\in\partial B_r} \left(\left| \frac{b}{r} - \left(\frac{V_\Sigma}{\omega_n}\right)^{\frac{1}{n-2}} \right| + \left| Db - \left(\frac{V_\Sigma}{\omega_n}\right)^{\frac{1}{n-2}} \right| \right) \right) = 0, \tag{2.8}$$

and

$$\limsup_{r\to\infty} \left(\sup_{x\in\partial B_r} \left| \mathrm{Hess}_{b^2} - 2\left(\frac{V_\Sigma}{\omega_n}\right)^{\frac{2}{n-2}} \sigma \right| \right) = 0. \tag{2.9}$$

- By Gromov's compactness theorem [28] and Cheeger-Colding's theory [9], for any sequence $\bar{\varepsilon}_i \to 0$ there is a subsequence $\{\varepsilon_i\}$ converging to zero such that $\varepsilon_i\Sigma = (\Sigma, \varepsilon_i\sigma, p)$ converges to a metric cone $(\Sigma_\infty, d_\infty)$ with vertex o in the pointed Gromov-Hausdorff sense. Σ_∞ is called the tangent cone at infinity and
$\Sigma_\infty = CX = \mathbb{R}^+ \times_\rho X$ for some $(n-1)$-dimensional smooth compact manifold X of Diam $(X) \le \pi$ and the metric $s_{ij}d\theta_i d\theta_j$ with $s_{ij} \in C^{1,\alpha}(X)$ (see also [27][38]). For any compact domain $K \subset \Sigma_\infty \setminus \{o\}$, there exists a diffeomorphism $\Phi_i : K \to \Phi_i(K) \subset \varepsilon_i\Sigma$ such that $\Phi_i^*(\varepsilon_i\sigma)$ converges as $i \to \infty$ to σ_∞ in the $C^{1,\alpha}$-topology on K.

3 Gradient estimates and applications

Let u be a minimal graphic function on a Riemannian manifold Σ, and

$$v \triangleq \frac{1}{\langle E_{n+1}, \nu\rangle} = \sqrt{1+|Du|^2}.$$

Let $\tilde{v}(z) = v(x)$ for any $z = (x, u(x)) \in M$, and we usually denote $\tilde{v}$ by v, which will not cause confusion from the context in general.

Let $B_r(x)$ be the geodesic ball in Σ with radius r and centered at $x \in \Sigma$. Sometimes we write B_r instead of $B_r(p)$ for simplicity. Let $d\mu$ be the volume element of M.

Lemma 3.1. *Suppose Σ has nonnegative Ricci curvature and $u(p) = 0$, then for any constant $\beta > 0$*

$$\int_{B_r\cap\{|u|<\beta r\}} \log v\, d\mu \le (1+10\beta)\left(2+\beta+r^{-1}\sup_{B_{3r}} u\right) Vol(B_{3r}). \tag{3.1}$$

Proof. We define a function u_s by

$$u_s = \begin{cases} \beta s & \text{if } \ u \ge \beta s \\ u & \text{if } \ |u| < \beta s \\ -\beta s & \text{if } \ u \le -\beta s. \end{cases}$$

$\rho(x)$ is a global Lipschitz function with $|D\rho| \equiv 1$ almost everywhere. We define a Lipschitz function $\zeta(r)$ on $[0,\infty)$ satisfying supp$\zeta \subset [0,2r]$, $\zeta\big|_{[0,r)} \equiv 1$ and $|D\zeta| \le \frac{1}{r}$. Then $\eta(x) = \zeta(\rho(x))$ is a Lipschitz function with supp$\eta \subset \bar{B}_{2r}$, $\eta\big|_{B_r} \equiv 1$ and $|D\eta| \le \frac{1}{r}$. Then by using (2.1) and integrating by parts we have

$$0 = -\int \mathrm{div}_\Sigma\left(\frac{Du}{v}\right)\eta u_r d\mu_\Sigma \ge \int_{B_r\cap\{|u|<\beta r\}} \frac{|Du|^2}{v} d\mu_\Sigma + \int \frac{Du\cdot D\eta}{v} u_r d\mu_\Sigma, \tag{3.2}$$

which implies

$$\begin{aligned}\int_{B_r\cap\{|u|<\beta r\}} 1 d\mu \leq & \int_{B_r\cap\{|u|<\beta r\}} \left(\frac{|Du|^2}{v}+1\right) d\mu_\Sigma \\ \leq & Vol(B_r)+\beta r\int_{B_{2r}} |D\eta| d\mu_\Sigma \leq (1+\beta)Vol(B_{2r}).\end{aligned} \tag{3.3}$$

From integrating by parts, we deduce

$$\begin{aligned}0 = & -\int \mathrm{div}_\Sigma\left(\frac{Du}{v}\right)\cdot(u_r+\beta r)\eta\log v\, d\mu_\Sigma \\ \geq & \int_{B_r\cap\{|u|<\beta r\}} \frac{|Du|^2}{v}\log v\, d\mu_\Sigma + \int \frac{Du\cdot D\eta}{v}(u_r+\beta r)\log v\, d\mu_\Sigma \\ & + \int \frac{Du\cdot D\log v}{v}(u_r+\beta r)\eta d\mu_\Sigma \\ \geq & \int_{B_r\cap\{|u|<\beta r\}} \frac{|Du|^2}{v}\log v\, d\mu_\Sigma - 2\beta r\int_{B_{2r}\cap\{u>-\beta r\}} |D\eta|\log v\, d\mu_\Sigma \\ & -2\beta r\int_{B_{2r}\cap\{u>-\beta r\}} |D\log v|\eta d\mu_\Sigma,\end{aligned} \tag{3.4}$$

then we obtain

$$\begin{aligned}&\int_{B_r\cap\{|u|<\beta r\}} \log v d\mu \leq \int_{B_r\cap\{|u|<\beta r\}} \left(\frac{|Du|^2}{v}+1\right)\log v\, d\mu_\Sigma \\ \leq & (1+2\beta)\int_{B_{2r}\cap\{u>-\beta r\}} \log v\, d\mu_\Sigma + 2\beta r\int_{B_{2r}\cap\{u>-\beta r\}} |D\log v|\eta d\mu_\Sigma.\end{aligned} \tag{3.5}$$

Obviously, (2.5) implies $\Delta\log v \geq |\nabla\log v|^2$, then for any $\xi\in C_0^1(B_{2r}\times\mathbb{R})$ we have

$$\begin{aligned}&\int |\nabla\log v|^2\xi^2 d\mu \leq \int \xi^2\Delta\log v d\mu = -2\int \xi\nabla\xi\cdot\nabla\log v d\mu \\ \leq & \frac{1}{2}\int |\nabla\log v|^2\xi^2 d\mu + 2\int |\nabla\xi|^2 d\mu.\end{aligned} \tag{3.6}$$

Set $\xi(x,t)=\eta(x)\tau(t)$ for $(x,t)\in\Sigma\times\mathbb{R}$, where $0\leq\tau\leq 1$, $\tau\equiv 1$ in $(-\beta r, \sup_{B_{2r}} u)$, $\tau\equiv 0$ outside $(-(1+\beta)r, r+\sup_{B_{2r}} u)$, $|\tau'|<\frac{1}{r}$. Then

$$\begin{aligned}\int |\nabla\log v|^2\xi^2 d\mu \leq & 4\int |\nabla\xi|^2 d\mu \leq 8\int \left(|\nabla\eta|^2\tau^2+|\nabla\tau|^2\eta^2\right) d\mu \\ \leq & \frac{16}{r^2}\int_{B_{2r}\cap\{u>-(1+\beta)r\}} 1 d\mu.\end{aligned} \tag{3.7}$$

At any considered point $z=(x,u(x))\in M$,

$$\nabla\log\tilde{v}(z) = \sum_{i,j}\left(g^{ij}\partial_i\log v(x)\right)\left(E_j+u_j(x)E_{n+1}\right),$$

then

$$|\nabla \log \tilde{v}(z)|^2 = \sum_{i,j} g^{ij} \partial_i \log v(x) \cdot \partial_j \log v(x). \tag{3.8}$$

So we obtain

$$|\nabla \log v|^2 = |D \log v|^2 - \frac{|Du \cdot D \log v|^2}{1+|Du|^2} \geq \frac{|D \log v|^2}{1+|Du|^2}. \tag{3.9}$$

Together with (3.9) and (3.7) it follows that

$$\begin{aligned}
&\int_{B_{2r}\cap\{u>-\beta r\}} |D \log v| \eta \, d\mu_\Sigma \leq \int_{B_{2r}\cap\{u>-\beta r\}} |\nabla \log v| \eta v \, d\mu_\Sigma \\
\leq &\int_{B_{2r}\cap\{u>-\beta r\}} \left(\frac{|\nabla \log v|^2 \eta^2 r}{8} + \frac{2}{r} \right) v \, d\mu_\Sigma \\
\leq &\frac{r}{8} \int |\nabla \log v|^2 \xi^2 d\mu + \frac{2}{r} \int_{B_{2r}\cap\{u>-\beta r\}} v \, d\mu_\Sigma \\
\leq &\frac{2}{r} \int_{B_{2r}\cap\{u>-(1+\beta)r\}} 1 \, d\mu + \frac{2}{r} \int_{B_{2r}\cap\{u>-\beta r\}} v \, d\mu_\Sigma \\
\leq &\frac{4}{r} \int_{B_{2r}\cap\{u>-(1+\beta)r\}} v \, d\mu_\Sigma.
\end{aligned} \tag{3.10}$$

Note that $\log v \leq v$ as $v \geq 1$, substituting (3.10) into (3.5) we obtain

$$\begin{aligned}
&\int_{B_r\cap\{|u|<\beta r\}} \log v \, d\mu \leq (1+2\beta) \int_{B_{2r}\cap\{u>-\beta r\}} v \, d\mu_\Sigma + 8\beta \int_{B_{2r}\cap\{u>-(1+\beta)r\}} v \, d\mu_\Sigma \\
&\leq (1+10\beta) \int_{B_{2r}\cap\{u>-(1+\beta)r\}} v \, d\mu_\Sigma.
\end{aligned} \tag{3.11}$$

Let $\tilde{\eta}$ be a Lipschitz function with $\mathrm{supp}\tilde{\eta} \subset \bar{B}_{3r}$ with $\tilde{\eta}\big|_{B_{2r}} \equiv 1$ and $|D\tilde{\eta}| \leq \frac{1}{r}$. Then

$$\begin{aligned}
0 = &-\int_{B_{3r}} \mathrm{div}_\Sigma \left(\frac{Du}{v} \right) \tilde{\eta} \cdot \max\{u + (1+\beta) r, 0\} d\mu_\Sigma \\
\geq &\int_{B_{2r}\cap\{u>-(1+\beta)r\}} \frac{|Du|^2}{v} d\mu_\Sigma + \int_{B_{3r}} \frac{Du \cdot D\tilde{\eta}}{v} \max\{u + (1+\beta) r, 0\} d\mu_\Sigma \\
\geq &\int_{B_{2r}\cap\{u>-(1+\beta)r\}} \frac{|Du|^2}{v} d\mu_\Sigma - \frac{1}{r} \int_{B_{3r}} \max\{u + (1+\beta) r, 0\} d\mu_\Sigma,
\end{aligned} \tag{3.12}$$

which implies

$$\begin{aligned}
\int_{B_{2r}\cap\{u>-(1+\beta)r\}} v \, d\mu_\Sigma &\leq \int_{B_{2r}\cap\{u>-(1+\beta)r\}} \left(\frac{|Du|^2}{v} + 1 \right) d\mu_\Sigma \\
&\leq \int_{B_{2r}} d\mu_\Sigma + \frac{1}{r} \int_{B_{3r}} \left(\sup_{B_{3r}} u + (1+\beta) r \right) d\mu_\Sigma \\
&\leq \left(2 + \beta + r^{-1} \sup_{B_{3r}} u \right) Vol(B_{3r}).
\end{aligned} \tag{3.13}$$

Combining (3.11) and (3.13) we complete the proof of the Lemma. □

For any $z_i = (x_i, t_i)$, $i = 1, 2$, denote the distance function $\bar{\rho}$ on $\Sigma \times \mathbb{R}$ by

$$\bar{\rho}_{z_1}(z_2) = \sqrt{\rho_{x_1}^2(x_2) + (t_2 - t_1)^2}.$$

For any $q, x \in \Sigma$ there are $\tilde{q} = (q, u(q)) \in M$ and $\tilde{x} = (x, u(x)) \in M$ such that $\bar{\rho}_{\tilde{q}}(\tilde{x}) = \sqrt{\rho_q^2(x) + (u(x) - u(q))^2}$.

Lemma 3.2. *Suppose the sectional curvature $R(x) \le K^2$ on $B_{i(q)}(q)$ with injective radius $i(q)$ at q, then we have*

$$\Delta \bar{\rho}_{\tilde{q}}^2(\tilde{x}) \ge 2 + 2(n-1)K\rho_q(x)\cot(K\rho_q(x)) \qquad \text{for } 0 < \rho_q(x) < \min\left\{i(q), \frac{\pi}{2K}\right\}.$$

Proof. By the Hessian comparison theorem, for any $\xi \perp \frac{\partial}{\partial \rho_q}$ we have

$$\mathrm{Hess}_{\rho_q}(\xi, \xi) \ge K \cot(K\rho_q)|\xi|^2.$$

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field of M. Note that M is minimal, we obtain

$$\begin{aligned}
\Delta \bar{\rho}_{\tilde{q}}^2 =& \sum_{i=1}^n \left(\nabla_{e_i}\nabla_{e_i}\bar{\rho}_{\tilde{q}}^2 - (\nabla_{e_i}e_i)\,\bar{\rho}_{\tilde{q}}^2\right)\\
=& \sum_{i=1}^n \left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\bar{\rho}_{\tilde{q}}^2 - \left(\overline{\nabla}_{e_i}e_i\right)\bar{\rho}_{\tilde{q}}^2\right) + \sum_{i=1}^n \left(\overline{\nabla}_{e_i}e_i - \nabla_{e_i}e_i\right)\bar{\rho}_{\tilde{q}}^2\\
=& \Delta_N \bar{\rho}_{\tilde{q}}^2 - \overline{\mathrm{Hess}}_{\bar{\rho}_{\tilde{q}}^2}(\nu, \nu)\\
=& \Delta_\Sigma \rho_q^2 + 2 - \frac{1}{v^2}\mathrm{Hess}_{\rho_q^2}(Du, Du) - \frac{2}{v^2}.
\end{aligned} \tag{3.14}$$

If $Du \ne 0$, we set $(Du)^T = Du - \left\langle Du, \frac{\partial}{\partial \rho_q}\right\rangle \frac{\partial}{\partial \rho_q}$. Let $\{E_\alpha\}_{\alpha=1}^{n-1} \cup \frac{\partial}{\partial \rho_q}$ be an orthonormal basis of $T\Sigma$ with $E_1 = (Du)^T \left|(Du)^T\right|^{-1}$. Combining $\mathrm{Hess}_{\rho_q^2}\left(E_\alpha, \frac{\partial}{\partial \rho_q}\right) = 0$ and $\mathrm{Hess}_{\rho_q^2}\left(\frac{\partial}{\partial \rho_q}, \frac{\partial}{\partial \rho_q}\right) = 2$ give

$$\mathrm{Hess}_{\rho_q^2}(Du, Du) = \mathrm{Hess}_{\rho_q^2}\left((Du)^T, (Du)^T\right) + 2\left\langle Du, \frac{\partial}{\partial \rho_q}\right\rangle^2. \tag{3.15}$$

Hence
(3.16)

$$
\begin{aligned}
\Delta\bar{\rho}_{\tilde{q}}^2 =& \sum_\alpha \mathrm{Hess}_{\rho_q^2}(E_\alpha,E_\alpha)+4-\frac{1}{v^2}\mathrm{Hess}_{\rho_q^2}\left((Du)^T,(Du)^T\right)-\frac{2}{v^2}\left\langle Du,\frac{\partial}{\partial\rho_q}\right\rangle^2-\frac{2}{v^2}\\
=&\sum_\alpha \mathrm{Hess}_{\rho_q^2}(E_\alpha,E_\alpha)+2-\frac{\left|(Du)^T\right|^2}{v^2}\mathrm{Hess}_{\rho_q^2}(E_1,E_1)+\frac{2}{v^2}\left|(Du)^T\right|^2\\
\geq& 2(n-2)K\rho_q\cot(K\rho_q)+2+\left(2-2\frac{\left|(Du)^T\right|^2}{v^2}\right)K\rho_q\cot(K\rho_q)+\frac{2}{v^2}\left|(Du)^T\right|^2\\
\geq& 2(n-1)K\rho_q\cot(K\rho_q)+2.
\end{aligned}
$$

If $|Du|=0$, clearly $\Delta\bar{\rho}_{\tilde{q}}^2=\Delta_\Sigma\rho_q^2\geq 2(n-1)K\rho_q\cot(K\rho_q)+2$. □

Suppose that Σ satisfies conditions C1), C2) and C3). For sufficiently small $\delta>0$ depending only on n,c,V_Σ and any fixed $q\in\partial B_{\Omega r}(p)$ with $\Omega=\left(\frac{\sqrt{c}}{\delta}+1\right)$, by [8] the injectivity radius at q satisfies $i(q)\geq r$ and

$$d(p,x)\geq\frac{\sqrt{c}}{\delta}\,r,\qquad \text{for any }\ x\in B_r(q).$$

Then by condition C3)

$$|R(x)|\leq\frac{\delta^2}{r^2},\qquad \text{for any }\ x\in B_r(q). \tag{3.17}$$

Hence $\rho_q(x)$ is smooth for $x\in B_r(q)\setminus\{q\}$. For $\tilde{q}=(q,u(q))\in M$, we denote $\mathbb{B}_s(\tilde{q})=\{z\in N|\ \bar{\rho}_{\tilde{q}}(z)<s\}$ and $D_s(\tilde{q})=\mathbb{B}_s(\tilde{q})\cap M$. If $\tilde{x}=(x,u(x))\in D_s(\tilde{q})$, then obviously $x\in B_s(q)$.

For any $t\in[0,1)$ we have $\cos t\geq 1-t$, then

$$\left(\tan t-\frac{t}{1-t}\right)'=\frac{1}{\cos^2 t}-\frac{1}{(1-t)^2}\leq 0.$$

So on $[0,1)$

$$\tan t\leq\frac{t}{1-t}.$$

Hence on $D_r(\tilde{q})$ we have

$$\Delta\bar{\rho}_{\tilde{q}}^2(\tilde{x})\geq 2+2(n-1)\left(1-\frac{\delta}{r}\rho_q(x)\right)\geq 2n-\frac{2n\delta\rho_q(x)}{r}.$$

For any smooth function f on M, combining the above inequalities we get
(3.18)
$$\begin{aligned}&2n\int_{D_s(\tilde q)} f\left(1-\frac{\delta\rho_q}{r}\right)\le\int_{D_s(\tilde q)} f\Delta\bar\rho_{\tilde q}^2=\int_{D_s(\tilde q)}\operatorname{div}\left(f\nabla\bar\rho_{\tilde q}^2\right)-\int_{D_s(\tilde q)}\nabla f\cdot\nabla\bar\rho_{\tilde q}^2\\
=&\int_{\partial D_s(\tilde q)} f\nabla\bar\rho_{\tilde q}^2\cdot\frac{\nabla\bar\rho_{\tilde q}}{|\nabla\bar\rho_{\tilde q}|}-\int_{D_s(\tilde q)}\operatorname{div}\left((\bar\rho_{\tilde q}^2-s^2)\nabla f\right)+\int_{D_s(\tilde q)}(\bar\rho_{\tilde q}^2-s^2)\Delta f\\
=&2s\int_{\partial D_s(\tilde q)} f|\nabla\bar\rho_{\tilde q}|+\int_{D_s(\tilde q)}(\bar\rho_{\tilde q}^2-s^2)\Delta f.\end{aligned}$$
Since
(3.19)
$$\begin{aligned}|\nabla\bar\rho_{\tilde q}|^2=&\frac{1}{4\bar\rho_{\tilde q}^2}g^{ij}\partial_i(\rho_q^2+(u-u(q))^2)\cdot\partial_j(\rho_q^2+(u-u(q))^2)\\
=&\frac{1}{\bar\rho_{\tilde q}^2}\left(\rho_q^2\left(1-\frac{|Du\cdot D\rho_q|^2}{v^2}\right)+2\rho_q(u-u(q))\frac{Du\cdot D\rho_q}{v^2}+(u-u(q))^2\frac{|Du|^2}{v^2}\right)\\
\le&\frac{1}{\bar\rho_{\tilde q}^2}\left(\rho_q^2+(u-u(q))^2\frac{1}{v^2}+(u-u(q))^2\frac{|Du|^2}{v^2}\right)\\
=&1,\end{aligned}$$
we have
$$\begin{aligned}&\frac{\partial}{\partial s}\left(s^{-n}\int_{D_s(\tilde q)}\log v\right)=-ns^{-n-1}\int_{D_s(\tilde q)}\log v+s^{-n}\int_{\partial D_s(\tilde q)}\frac{\log v}{|\nabla\bar\rho_{\tilde q}|}\\
\ge&-ns^{-n-1}\int_{D_s(\tilde q)}\log v+s^{-n}\int_{\partial D_s(\tilde q)}\log v|\nabla\bar\rho_{\tilde q}|\\
\ge&-ns^{-n-1}\int_{D_s(\tilde q)}\log v+s^{-n}\frac{n}{s}\int_{D_s(\tilde q)}\left(1-\frac{\delta\rho_q}{r}\right)\log v\\
\ge&-\frac{n\delta}{r}s^{-n}\int_{D_s(\tilde q)}\log v.\end{aligned}\tag{3.20}$$
Let ω_n be the volume of the standard n-dimensional unit ball in Euclidean space. For $0<s\le r$ integrating the above inequality implies
$$\log v(\tilde q)\le\frac{e^{\frac{n\delta s}{r}}}{\omega_n s^n}\int_{D_s(\tilde q)}\log v.\tag{3.21}$$
We say that a function f has *at most linear growth* on Σ if
$$\limsup_{x\to\infty}\frac{|f(x)|}{\rho(x)}<\infty.$$
and say that f has *linear growth* on Σ if
$$0<\limsup_{x\to\infty}\frac{|f(x)|}{\rho(x)}<\infty.$$
Denote $f_+=\max\{f,0\}$ and $f_-=\min\{f,0\}$.

Theorem 3.3. *Let u be a minimal graphic function on a complete Riemannian manifold Σ which satisfies conditions C1), C2) and C3). Then we have gradient estimates*

$$|Du(x)| \leq C_1 e^{C_2 r^{-1}\left(u(x)-\sup_{y\in B_{(\Omega+1)r}(p)} u(y)\right)} \tag{3.22}$$

for any $r>0$ and $x \in \partial B_{\Omega r}(p)$, where C_1, C_2 are positive constants depending only n, and Ω is a constant depending only on n, c, V_Σ. Moreover, if u_+ (or u_-) has at most linear growth, then $|Du|$ is uniformly bounded on all of Σ.

Proof. For any $p \in \Sigma$ fixed, let $\Omega = \left(\frac{\sqrt{c}}{\delta}+1\right)$ as before. For any $r>0$, $x \in \partial B_{\Omega r}(p)$, combining (3.21) and (3.1) with $\beta = 1$ and $u(x)-u$ replacing u, we have

$$\begin{aligned}
\log|Du(x)| \leq & \log v(\tilde{x}) \leq \frac{e^{n\delta}}{\omega_n r^n}\int_{D_r(\tilde{x})} \log v \\
\leq & 11 e^{n\delta}\left(3+\frac{\sup_{y\in B_r(x)}(u(x)-u(y))}{r}\right)\frac{Vol(B_{3r})}{\omega_n r^n} \\
\leq & 11 e^{n\delta}\left(3+\frac{\sup_{y\in B_{(\Omega+1)r}(p)}(u(x)-u(y))}{r}\right)\frac{Vol(B_{3r})}{\omega_n r^n},
\end{aligned} \tag{3.23}$$

where $\tilde{x} = (x, u(x))$. Take $0<\delta<1$, then invoking (2.6) we obtain

$$\log|Du(x)| \leq 11(3e)^n\left(3+\frac{u(x)-\sup_{y\in B_{(\Omega+1)r}(p)} u(y)}{r}\right). \tag{3.24}$$

Thus (3.22) holds. Obviously, we can substitute u in (3.22) by $-u$. Hence if u_+ or u_- has at most linear growth, then letting $r \to \infty$ implies that $|Du|$ is uniformly bounded on all of Σ. $\square$

Lemma 3.4. *Let Σ be a Riemannian manifold with conditions C1), C2) C3), and u be a smooth solution to* (2.1) *on Σ with at most linear growth. Then for any nonnegative subharmonic(superharmonic) function $\varphi^+(\varphi^-)$ on M, we have*

$$\sup_{D_{\lambda r}(q)} \varphi^+ \leq \frac{C_1}{r^n}\int_{D_r(q)} \varphi^+ \qquad \textit{and} \qquad \inf_{D_{\lambda r}(q)} \varphi^- \geq \frac{C_2}{r^n}\int_{D_r(q)} \varphi^-$$

for arbitrary $r>0$ and some constant $0<\lambda<1$ independent of r. Here $C_1, C_2 > 0$ are constants depending only on n, c, V_Σ, λ.

Proof. In the manifold Σ we have the Sobolev and Neumann-Poincaré inequalities. So we have those inequalities also on the manifold M by the boundedness of $|Du|$. Then we can use De Giorgi-Moser-Nash's theory and the volume doubling condition to obtain the mean value inequality(see [37] for the details). $\square$

Denote $|Du|_0 = \sup_{x\in\Sigma}|Du(x)|$. Now we want to use the above Lemma to deduce the mean value equalities for the bounded gradient of u.

Lemma 3.5. *Let Σ be a Riemannian manifold with conditions C1), C2) C3), and u be a smooth solution to* (2.1) *on Σ with at most linear growth. Then we have mean value equalities on both exterior balls and interior balls:*

$$|Du|_0^2 = \lim_{r\to\infty} \frac{1}{Vol(D_r(z))} \int_{D_r(z)} |Du|^2 d\mu, \tag{3.25}$$

and

$$|Du|_0^2 = \lim_{r\to\infty} \frac{1}{Vol(B_r(x))} \int_{B_r(x)} |Du|^2 d\mu_\Sigma. \tag{3.26}$$

Proof. Denote $\phi_{\max} \triangleq \sup_{z\in M} \log v(z)$. If u has linear growth at most, for any $\tilde{q}\in M$ we have

$$\frac{1}{r^n}\int_{D_r(\tilde{q})} (\phi_{\max} - \log v) \le C\left(\phi_{\max} - \log v(\tilde{q})\right), \tag{3.27}$$

which implies for any $z\in M$

$$\phi_{\max} = \lim_{r\to\infty} \frac{1}{Vol(D_r(z))} \int_{D_r(z)} \log v < \infty. \tag{3.28}$$

Since $e^{2\log v} - 1 = |Du|^2$ is a bounded subharmonic function on M, we obtain (3.25).

For any $0 < \varepsilon < |Du|_0$ and any fixed point $z = (x, u(x))$ there is an $r_0 > 0$ such that for any $r \ge r_0$

$$\frac{1}{Vol(D_r(z))} \int_{D_r(z)} |Du|^2 d\mu > |Du|_0^2 - \varepsilon^2.$$

Denote $\Omega_r \triangleq \{(q,u(q)) \in D_r(z) |\ |Du|^2(q) < |Du|_0^2 - \varepsilon\}$, then

$$\begin{aligned}
|Du|_0^2 - \varepsilon^2 <& \frac{1}{Vol(D_r(z))}\left(\int_{D_r(z)\setminus\Omega_r} |Du|_0^2 d\mu + \int_{\Omega_r} |Du|^2 d\mu\right)\\
\le& \left(1 - \frac{Vol(\Omega_r)}{Vol(D_r(z))}\right)|Du|_0^2 + \frac{Vol(\Omega_r)}{Vol(D_r(z))}\left(|Du|_0^2 - \varepsilon\right)\\
=& |Du|_0^2 - \varepsilon \frac{Vol(\Omega_r)}{Vol(D_r(z))},
\end{aligned}$$

which implies

$$Vol(\Omega_r) \le \varepsilon Vol(D_r(z)).$$

Let Ω_r^T be the projection from Ω_r to Σ defined by

$$\{q\in\Sigma |\ (q,u(q)) \in \Omega_r\} = \{q\in\Sigma |\ (q,u(q)) \in D_r(z),\ |Du|^2(q) < |Du|_0^2 - \varepsilon\}.$$

There exists a constant $C>1$ such that the projection of $D_{Cr}(z)$ contains $B_r(x)$ for any $r\geq r_0$. By the definition of $D_{Cr}(z)$, it follows that
(3.29)

$$
\begin{aligned}
&\frac{1}{Vol(B_r(x))}\int_{B_r(x)}|Du|^2d\mu_\Sigma\geq\frac{1}{Vol(B_r(x))}\int_{B_r(x)\setminus\Omega^T_{Cr}}\left(\sup_\Sigma|Du|^2-\varepsilon\right)d\mu_\Sigma\\
\geq&\frac{1}{Vol(B_r(x))}\int_{B_r(x)}\left(\sup_\Sigma|Du|^2-\varepsilon\right)d\mu_\Sigma-\frac{1}{Vol(B_r(x))}\int_{\Omega^T_{Cr}}\left(\sup_\Sigma|Du|^2-\varepsilon\right)d\mu_\Sigma\\
\geq&\sup_\Sigma|Du|^2-\varepsilon-\frac{Vol(\Omega_{Cr})}{Vol(B_r(x))}\left(\sup_\Sigma|Du|^2-\varepsilon\right)\\
\geq&\sup_\Sigma|Du|^2-\varepsilon-\varepsilon\frac{Vol(D_{Cr}(z))}{Vol(B_r(x))}\left(\sup_\Sigma|Du|^2-\varepsilon\right).
\end{aligned}
$$

Let $r\to\infty$, then $\varepsilon\to 0$ implies that (3.26) holds. □

Theorem 3.6. *If the minimal graphic function u on a complete manifold with conditions C1), C2) and C3) has sub-linear growth for its negative part, namely,*

$$\limsup_{x\to\infty}\frac{|u_-(x)|}{\rho(x)}=0,$$

where $u_-=\min\{u,0\}$. *Then u is a constant.*

Proof. From Theorem 3.3, $|Du|$ is globally bounded. For any small $\delta>0$, there is a $C_\delta>0$ such that $u(x)\geq -C_\delta-\delta\rho(x)$. Lemma 3.4 implies a Harnack inequality for positive harmonic functions on M. Hence there is an absolute constant $C>1$ so that for any $r>0$ and $q\in M$ we have

$$\sup_{x\in D_{\lambda r}(q)}\left(u(x)-\inf_{D_r(q)}u+1\right)\leq C\inf_{x\in D_{\lambda r}(q)}\left(u(x)-\inf_{D_r(q)}u+1\right)\leq C\left(u(q)-\inf_{D_r(q)}u+1\right).$$

Thus
(3.30)

$$\sup_{x\in D_{\lambda r}(q)}u(x)\leq Cu(q)+(C-1)\left(1-\inf_{D_r(q)}u\right)\leq Cu(q)+(C-1)\left(1+C_\delta+\delta r\right)$$

Therefore, for any $\varepsilon>0$ there is $r_0>0$ such that for any x with $\rho(x)\geq r_0$ we have

$$|u(x)|<\varepsilon\rho(x).$$

For any $r\geq r_0$, let η be a Lipschitz function on M with $\mathrm{supp}\eta\subset D_{2r}$ with $\eta\big|_{D_r}\equiv 1$ and $|\nabla\eta|\leq\frac{1}{r}$ on $D_{2r}\setminus D_r$. Due to $\Delta u=0$, we see that

$$0=-\int_M u\eta^2\Delta u=\int_M\nabla u\cdot\nabla(u\eta^2)=\int_M|\nabla u|^2\eta^2+2\int_M u\eta\nabla u\cdot\nabla\eta. \tag{3.31}$$

From the Cauchy inequality we conclude that

$$\int_{D_r}|\nabla u|^2\leq\int_M|\nabla u|^2\eta^2\leq 4\int_M|\nabla\eta|^2u^2\leq\frac{4}{r^2}\int_{D_{2r}\setminus D_r}u^2\leq 16\varepsilon^2Vol(D_{2r}). \tag{3.32}$$

With $|\nabla u|^2 = \frac{|Du|^2}{v^2}$, we get

$$
\begin{aligned}
\sup_{y\in\Sigma}|Du|^2(y) &= \lim_{r\to\infty}\frac{1}{Vol(D_r)}\int_{D_r}|Du|^2 d\mu \\
&\le \limsup_{r\to\infty}\frac{1+|Du|_0^2}{Vol(D_r)}\int_{D_r}|\nabla u|^2 d\mu \\
&\le 16\varepsilon^2(1+|Du|_0^2)\limsup_{r\to\infty}\frac{Vol(D_{2r})}{Vol(D_r)}.
\end{aligned} \tag{3.33}
$$

Forcing $\varepsilon\to 0$ gives $|Du|\equiv 0$, namely, u is a constant. □

Actually, if Σ is not Euclidean space, we can give a stronger theorem by replacing sub-linear growth by linear growth in the following section.

4 A Liouville theorem via splitting for tangent cones at infinity

Let Σ be an n-dimensional manifold with conditions C1), C2), C3), and u be a smooth linear growth solution to (2.1). We shall now derive pointwise estimates for the Hessian of u. We first rewrite (2.1), in order to apply elliptic regularity theory as can found, for instance in [26, 30]. In a local coordinate, (2.1) is

$$
\partial_j\left(\sqrt{\sigma}\frac{\sigma^{ij}u_i}{\sqrt{1+|Du|^2}}\right)=0, \tag{4.1}
$$

where $\sqrt{\sigma}=\sqrt{\det\sigma_{kl}}$. Take derivatives and set $w=\partial_k u$ to turn this equation into

$$
\partial_j\left(\frac{\sqrt{\sigma}}{\sqrt{1+|Du|^2}}\left(\sigma^{ij}-\frac{u^iu^j}{1+|Du|^2}\right)w_i\right)+\partial_j f_k^j=0, \tag{4.2}
$$

where

$$
f_k^j=\frac{u_i}{\sqrt{1+|Du|^2}}\partial_k\left(\sqrt{\sigma}\sigma^{ij}\right)-\frac{1}{2}\sqrt{\sigma}\sigma^{ij}\frac{u_iu_pu_q}{(1+|Du|^2)^{3/2}}\partial_k\sigma^{pq}.
$$

Define an operator L on $C^2(\Sigma)$ by

$$
Lf=\partial_j\left(\frac{\sqrt{\sigma}}{\sqrt{1+|Du|^2}}\left(\sigma^{ij}-\frac{u^iu^j}{1+|Du|^2}\right)f_i\right),
$$

then

$$
Lw+\partial_j f_k^j=0.
$$

Since Σ satisfies conditions C1), C2) and C3), by [3, 27, 38] there exist harmonic coordinates satisfying the estimates of [31]. That is, for a fixed point $p\in\Sigma$, there are positive constants $\alpha'=\alpha'(n,c,V_\Sigma),\theta=\theta(n,c,V_\Sigma)\in(0,1)$ and $C=C(n,c,V_\Sigma,\alpha')$ such that for any $q\in\partial B_r(p)$ and $r>0$ there is a harmonic coordinate system $\{x_i:\ i=1,\cdots,n\}$ on $B_{\theta r}(q)$ satisfying

$$
\Delta_\Sigma x_i\equiv 0,\qquad (\sigma_{ij})_{n\times n}\ge\frac{1}{C},\qquad \sigma_{ij}+r|D\sigma_{ij}|+r^{1+\alpha'}[D\sigma_{ij}]_{\alpha',B_{\theta r}(q)}\le C, \tag{4.3}
$$

where $0 < \alpha' < 1$, $\sigma_{ij} = \sigma\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and

$$[\varphi]_{\alpha', B_{\theta r}(q)} = \sup_{x,y\in B_{\theta r}(q), x\neq y} \frac{|\varphi(x)-\varphi(y)|}{d(x,y)^{\alpha'}}.$$

Lemma 4.1. *For any $q \in \partial B_r(p) \subset \Sigma$, $r > 0$ and $s \le \theta r$ we have*

$$\text{osc}_{B_s(q)} Du \le C s^{\alpha} r^{-\alpha}, \tag{4.4}$$

where $C = C(n, |Du|_0, c, V_\Sigma)$ and $\alpha = \alpha(n, |Du|_0, c, V_\Sigma) \le \alpha' < 1$ are positive constants.

Proof. For any fixed $s \le \frac{1}{4}\theta r$, denote $M_4(s) = \sup_{B_{4s}(q)} w$, $m_4(s) = \inf_{B_{4s}(q)} w$, $M_1(s) = \sup_{B_s(q)} w$, $m_1(s) = \inf_{B_s(q)} w$. Then we have

$$L(M_4 - w) = \partial_i f^i_k, \qquad L(w - m_4) = -\partial_i f^i \qquad \text{on } B_{\theta r}(q).$$

Due to (4.3), it is not hard to find out that $|f^i_k| \le \frac{C}{r}$ on $B_{\theta r}(q)$. By the weak Harnack inequality (Theorem 8.18 of [26]), we have

$$s^{-n}\int_{B_{2s}(q)} (M_4(s) - w)\, d\mu_\Sigma \le C\left(M_4(s) - M_1(s) + \frac{s}{r}\right), \tag{4.5}$$

and

$$s^{-n}\int_{B_{2s}(q)} (w - m_4(s))\, d\mu_\Sigma \le C\left(m_1(s) - m_4(s) + \frac{s}{r}\right), \tag{4.6}$$

where $C = C(n, |Du|_0, c, V_\Sigma)$. Denote $\omega(s) = \text{osc}_{B_s(q)} w = M_1(s) - m_1(s)$. Combining (2.6), (4.5) and (4.6) gives

$$2^n V_\Sigma\ \omega(4s) \le \frac{Vol(B_{2s}(q))}{s^n}\omega(4s) \le C\left(\omega(4s) - \omega(s) + \frac{2s}{r}\right), \tag{4.7}$$

which implies that there is a $\gamma \in (0,1)$ such that for all $s \in [0, \frac{1}{4}\theta r]$

$$\omega(s) \le \gamma\omega(4s) + \frac{2s}{r}.$$

By an iterative trick(see Lemma 8.23 in [26]), we complete the proof. $\square$

The above Lemma implies the following Hölder continuity for the gradient of u.

Corollary 4.2. *For any $q \in \partial B_r(p) \subset \Sigma$, $r > 0$ we have*

$$[Du]_{\alpha, B_{\theta r}(q)} \le C r^{-\alpha}, \tag{4.8}$$

where $C = C(n, |Du|_0, c, V_\Sigma)$ and $\alpha = \alpha(n, |Du|_0, c, V_\Sigma) < 1$ are positive constants.

Standard elliptic regularity theory (the scale-invariant Schauder estimates, see [26, 30]) implies that there exists a constant $C=C(n,|Du|_0,c,V_\Sigma,\alpha)$ such that for $q\in\partial B_r(p)$

$$\sup_{B_{\frac{\theta r}{2}}(q)}|D^2u|\le Cr^{-2}\sup_{B_{\theta r}(q)}|u|, \tag{4.9}$$

and

$$[D^2u]_{\alpha,B_{\frac{\theta r}{2}}(q)}\le Cr^{-2-\alpha}\sup_{B_{\theta r}(q)}|u|. \tag{4.10}$$

Theorem 4.3. *If u is solution to* (2.1) *with linear growth, then*

$$\limsup_{r\to\infty}\left(r\sup_{\partial B_r(p)}|D^2u|\right)=0. \tag{4.11}$$

Proof. For the fixed point $p\in\Sigma$, there is a constant C such that $\{(x,u(x))|\ x\in B_r(p)\}\subset D_{Cr}(p)$. Let b be the function defined on Σ in section 2. Together with (2.7) and the computation in (3.14), we have

$$\begin{aligned}\Delta b=&\Delta_\Sigma b-\frac{1}{v^2}\text{Hess}_b(Du,Du)\\=&(n-1)\frac{|Db|^2}{b}-\frac{1}{2v^2b}\left(\text{Hess}_{b^2}(Du,Du)-2\langle Db,Du\rangle^2\right).\end{aligned} \tag{4.12}$$

By the properties of the function b, there is a large r_0 so that for any x with $b(x)\ge r_0$ one has

$$|\Delta b|\le\frac{n+2}{b}.$$

Denote $U_s=\{(x,u(x))\in M|\ x\in\{b<s\}\}$ for $s>0$. Let ζ be a nonnegative smooth function on $\mathbb{R}^+$ with supp$\zeta\subset[0,2r]$, $\zeta\big|_{[0,r]}\equiv1$, $|\zeta'|\le\frac{C}{r}$ and $|\zeta''|\le\frac{C}{r^2}$. Set

$$\eta(x)=\zeta(b(x))\qquad\text{for any }x\in\Sigma.$$

Then η is a smooth function with supp$\eta\subset U_{2r}$, $\eta\big|_{U_r}\equiv1$, $|\nabla\eta|\le\frac{C}{r}$ and $|\Delta\eta|\le\frac{C}{r^2}$ for sufficiently large r. Recalling (2.5) and $b(x)\le\rho(x)$ we obtain

$$\begin{aligned}\int_{B_r}|\text{Hess}_u|^2d\mu_\Sigma\le&\int_{\{b<r\}}|\text{Hess}_u|^2d\mu_\Sigma\le\int_{U_{2r}}|\text{Hess}_u|^2\eta\,d\mu\\\le&C\int_{U_{2r}}|B|^2\eta\le C\int_{U_{2r}}\eta\Delta(\log v-\phi_{\max})\\=&C\int_{U_{2r}}(\log v-\phi_{\max})\Delta\eta\le\frac{C}{r^2}\int_{U_{2r}}(\phi_{\max}-\log v),\end{aligned} \tag{4.13}$$

where $\phi_{\max}=\sup_{z\in M}\log v(z)$ as before. Due to (3.28) we see that

$$\lim_{r\to\infty}\left(\frac{1}{r^{n-2}}\int_{B_r}|\text{Hess}_u|^2d\mu_\Sigma\right)=0. \tag{4.14}$$

From (4.9) we have

$$\sup_{\partial B_r(p)} |D^2u| \le \frac{C}{r} \tag{4.15}$$

for some fixed $C = C(n, |Du|_0, c, V_\Sigma, \alpha)$. If

$$\limsup_{r\to\infty} \left(r \sup_{\partial B_r(p)} |D^2u| \right) > 0, \tag{4.16}$$

then there exist $\varepsilon > 0$, $r_i \to \infty$ and $q_i \in \partial B_{r_i}(p)$ such that

$$r_i|D^2u(q_i)| \ge \varepsilon. \tag{4.17}$$

By (4.10), we conclude that

$$[D^2u]_{\alpha, B_{\frac{\theta r_i}{2}}(q_i)} \le C r_i^{-1-\alpha}. \tag{4.18}$$

There is a sufficiently small $\delta = \delta(n, |Du|_0, c, V_\Sigma, \alpha, \varepsilon) \in (0, \frac{\theta}{2})$ such that for any $y \in B_{\delta r_i}(q_i)$

$$|D^2u(q_i)| - |D^2u(y)| \le [D^2u]_{\alpha, B_{\frac{\theta r_i}{2}}(q_i)} d(y, q_i)^\alpha \le C r_i^{-1-\alpha} (\delta r_i)^\alpha < \frac{\varepsilon}{2r_i}, \tag{4.19}$$

which together with (4.17) implies

$$|D^2u(y)| \ge \frac{\varepsilon}{2r_i}. \tag{4.20}$$

Hence

$$\begin{aligned} &\frac{1}{r_i^{n-2}} \int_{B_{(1+\delta)r_i}(p) \setminus B_{(1-\delta)r_i}(p)} |D^2u|^2 d\mu_\Sigma \ge \frac{1}{r_i^{n-2}} \int_{B_{\delta r_i}(q_i)} |D^2u|^2 d\mu_\Sigma \\ \ge &\frac{1}{r_i^{n-2}} \int_{B_{\delta r_i}(q_i)} \frac{\varepsilon^2}{4r_i^2} d\mu_\Sigma \ge \frac{\varepsilon^2}{4} \delta^n V_\Sigma. \end{aligned} \tag{4.21}$$

Letting $r_i \to \infty$ deduces a contradiction to (4.14). This completes the proof. $\square$

Re-scale the metric by $\sigma \to r^{-2}\sigma$ and denote the inner product, norm, gradient, Hessian and volume element for this re-scaled metric by $\langle \cdot, \cdot \rangle_r$, $|\cdot|_r$, D^r, Hess^r and $d\mu_r$, respectively. Set $\tilde{u}_r = r^{-1}u$ and let B_s^r be the ball with radius s and centered at p for this re-scaled metric. Namely, B_s^r is the ball with radius s and centered at p in $r^{-2}\Sigma = (\Sigma, r^{-2}\sigma, p)$. Then for any fixed $r_0 > 0$ (3.26) implies

$$\lim_{r\to\infty} \frac{1}{Vol(B_{r_0}^r)} \int_{B_{r_0}^r} |D^r \tilde{u}_r|_r^2 d\mu_r = \sup_\Sigma |Du|^2, \tag{4.22}$$

and (4.11) implies that there is a constant $C > 0$ so that

$$\lim_{r\to\infty} \sup_{\partial B_{r_0}^r} |\mathrm{Hess}_{\tilde{u}_r}^r|_r = 0 \qquad \text{and} \qquad \sup_{\partial B_{r_0}^s} |\mathrm{Hess}_{\tilde{u}_r}^s|_s < \frac{C}{r_0} \quad \text{for any } s > 0. \tag{4.23}$$

For any $x, y \in B^r_{r_0}(p)$ there exists a minimal normal geodesic γ_{xy} connecting x and y such that $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$ and $|\dot\gamma_{xy}| = l_{xy}$ is a constant. Clearly, $l_{xy} \le 2r_0$. Parallel translating the vector $D^r\tilde u_r(x)$ along $\gamma_{xy}(t)$ produces a unique vector at y denoted by $D^r_{\pi_y}\tilde u_r(x)$. Denote by C a constant depending only on n. We have

$$\begin{aligned}\left|D^r_{\pi_y}\tilde u_r(x) - D^r\tilde u_r(y)\right|_r =& \left|D^r\tilde u_r(x) - D^r_{\pi_x}\tilde u_r(y)\right|_r \\ \le& C\cdot l_{xy}\int_0^1 |\mathrm{Hess}_{\tilde u_r}|_r(\gamma_{xy}(t))dt.\end{aligned} \tag{4.24}$$

For any fixed $\varepsilon > 0$, together with (4.22)(4.23) and the above inequality we get

$$\lim_{r\to\infty}\sup_{\gamma_{xy}\in B^r_{r_0}\setminus B^r_\varepsilon}\left|D^r_{\pi_y}\tilde u_r(x) - D^r\tilde u_r(y)\right|_r = 0, \tag{4.25}$$

and

$$\lim_{r\to\infty}\inf_{x\in B^r_{r_0}\setminus B^r_\varepsilon}|D^r\tilde u_r(x)|_r = \sup_\Sigma |Du|. \tag{4.26}$$

Note that $|Du|_0 \triangleq \sup_\Sigma |Du| > 0$. Suppose $u(p) = \tilde u_r(p) = 0$. Set

$$\Gamma_r = \tilde u_r^{-1}(0) = \{x \in r^{-2}\Sigma |\ \tilde u_r(x) = 0\}.$$

For any $x \in B^r_{r_0}$, there exists $x_0 \in B^r_{2r_0}\cap\Gamma_r$ so that

$$d_r(x, x_0) = d_r(x, \Gamma_r) \triangleq \inf_{z\in\Gamma_r} d_r(x, z).$$

Here d_r is the distance function on $r^{-2}\Sigma$. Set the signed distance function $d_{\Gamma_r}(x) = d_r(x, \Gamma_r)$ for $\tilde u_r(x) \ge 0$, and $d_{\Gamma_r}(x) = -d_r(x, \Gamma_r)$ for $\tilde u_r(x) \le 0$.

Lemma 4.4. *For any fixed $r_0 > 0$ there is*

$$\lim_{r\to\infty}\sup_{y\in B^r_{r_0}}|\tilde u_r(y) - |Du|_0\cdot d_{\Gamma_r}(y)| = 0 \tag{4.27}$$

Proof. For any $r > 0$ there is a $y_r \in B^r_{r_0}$ such that

$$\left|\tilde u_r(y_r) - |Du|_0\cdot d_{\Gamma_r}(y_r)\right| = \sup_{z\in B^r_{r_0}}\left|\tilde u_r(z) - |Du|_0\cdot d_{\Gamma_r}(z)\right|.$$

Without loss of generality, we suppose $\tilde u_r(y_r), d_{\Gamma_r}(y_r) > 0$. Clearly,

$$\tilde u_r(y_r) - |Du|_0\cdot d_{\Gamma_r}(y_r) \le 0.$$

For any $\varepsilon > 0$ we set

$$\Gamma^\varepsilon_r = \tilde u_r^{-1}(\varepsilon|Du|_0) = \{x \in r^{-2}\Sigma |\ \tilde u_r(x) = \varepsilon|Du|_0\}.$$

Hence, for any $x_r \in \Gamma^\varepsilon_r$ one has $x_r \in r^{-2}\Sigma\setminus B^r_\varepsilon$. For any fixed $\varepsilon > 0$ from (4.22) and (4.23) there is an $r_* > 0$ so that for any $r \ge r_*$ we have $|D^r\tilde u_r|_r\Big|_{\Gamma^\varepsilon_r\cap B^r_{2r_0}} \ge \frac{1}{2}|Du|_0$. Clearly there exist $z_r \in \Gamma^\varepsilon_r$ satisfying $d_r(y_r, z_r) = d_{\Gamma^\varepsilon_r}(y_r)$ and a unique normal geodesic γ_r connecting y_r, z_r with $\gamma_r(0) = z_r$, $\gamma_r(l_r) = y_r$ and $|\dot{\gamma}_r|_r = 1$, where $l_r = d_{\Gamma_r^\varepsilon}(y_r)$. Smoothness of Γ_r^ε implies $\dot{\gamma}_r(0) = -D^r\tilde{u}_r(z_r)/|D^r\tilde{u}_r(z_r)|_r$, then

$$
\begin{aligned}
\tilde{u}_r(y_r) =& \tilde{u}_r(y_r) - \tilde{u}_r(z_r) = \int_0^{l_r} \langle D^r\tilde{u}_r(\gamma_r(t)), \dot{\gamma}_r(t)\rangle_r \, dt \\
=& \int_0^{l_r} \left\langle D^r\tilde{u}_r(\gamma_r(t)) - D^r_{\pi_{\gamma(t)}}\tilde{u}_r(z_r), \dot{\gamma}_r(t) \right\rangle_r dt + \int_0^{l_r} |D^r\tilde{u}_r(z_r)|_r dt \\
\geq& -\int_0^{l_r} \left| D^r_{\pi_{\gamma_r}(t)}\tilde{u}_r(z_r) - D^r\tilde{u}_r(\gamma_r(t)) \right|_r dt + |D^r\tilde{u}_r(z_r)|_r \, d_{\Gamma_r^\varepsilon}(y_r) \\
\geq& -C\int_0^{l_r}\int_0^t |\mathrm{Hess}_{\tilde{u}_r}(\gamma_r(s))|_r \, ds dt + |D^r\tilde{u}_r(z_r)|_r \, d_{\Gamma_r^\varepsilon}(y_r).
\end{aligned}
\tag{4.28}
$$

Combining (4.23) and (4.26) implies

$$
\liminf_{r\to\infty} \left(\tilde{u}_r(y_r) - |Du|_0 \cdot d_{\Gamma_r^\varepsilon}(y_r)\right) \geq 0. \tag{4.29}
$$

Letting $\varepsilon \to 0$ completes the proof. $\square$

Remark 4.5. Analogously to the proof of Lemma 4.4, we have

$$
\lim_{r\to\infty} \sup_{y\in B^r_{r_0}} \left| \tilde{u}_r(y) - |Du|_0 \left(d_{\Gamma_r^s}(y) + s\right) \right| = 0
$$

for $s \in \mathbb{R}$, where $\Gamma_r^s = \tilde{u}_r^{-1}(s|Du|_0) = \{x \in r^{-2}\Sigma | \ \tilde{u}_r(x) = s|Du|_0\}$.

For any $x, y \in \Gamma_r^s$, let $\gamma^s_{r,xy}$ be a normal geodesic joining x to y with length $l^s_{r,xy}$. Since

$$
\frac{\partial^2}{\partial t^2}\tilde{u}_r\left(\gamma^s_{r,xy}(t)\right) = \mathrm{Hess}_{\tilde{u}_r}\left(\dot{\gamma}^s_{r,xy}(t), \dot{\gamma}^s_{r,xy}(t)\right),
$$

then by the Newton-Leibniz formula we conclude that

$$
\lim_{r\to\infty} \sup_{x,y\in\Gamma_r^s\cap B^r_{r_0}} \left(\sup_{t\in[0,l^s_{r,xy}]} \left| \tilde{u}_r\left(\gamma^s_{r,xy}(t)\right) - s|Du|_0 \right| \right) = 0. \tag{4.30}
$$

For any sequence $r_i \to \infty$ there is a subsequence r_{i_j} such that $r_{i_j}^{-1}\Sigma = (\Sigma, r_{i_j}^{-1}\sigma, p)$ converges to a regular metric cone Σ_0 with vertex o in the pointed Gromov-Hausdorff sense. Clearly, the geodesic $\gamma^s_{r_{i_j},px}$ should converge to a radial line starting from o in Σ_0. Therefore, combining (4.30) and Remark 4.5 we know that $\Gamma^s_{r_{i_j}}$ must converge to an $(n-1)$-dimensional cone $C\mathscr{Y}_s$ in Σ_0. Moreover, for any $z_1, z_2 \in C\mathscr{Y}_s$ the geodesic joining z_1 and z_2 in Σ_0 must live in $C\mathscr{Y}_s$. Let Ω be a connected component of $\Sigma_0 \setminus C\mathscr{Y}_s$ and $z_3, z_4 \in \Omega$. Then the geodesic γ_{z_3,z_4} joining z_3 and z_4 in Σ_0 satisfies $\gamma_{z_3,z_4} \setminus C\mathscr{Y}_s \subset \Omega$.

Consider $y, z \in \{x \in r^{-2}\Sigma | \ \tilde{u}_r(x) \geq 0\}$. For any $x \in \Gamma_r$, $d_{\Gamma_r}(y) \leq d_r(y,x) \leq d_r(x,z) + d_r(y,z)$. Taking all the points in Γ_r it follows that

$$
d_{\Gamma_r}(y) \leq d_{\Gamma_r}(z) + d_r(y,z).
$$

The same method implies

$$d_{\Gamma_r}(z) \le d_{\Gamma_r}(y) + d_r(y,z).$$

Hence

$$|d_{\Gamma_r}(z) - d_{\Gamma_r}(y)| \le d_r(y,z). \tag{4.31}$$

For some fixed point z with $\tilde{u}_r(z) > 0$ and small $\varepsilon > 0$, choose $\delta > 0$ sufficiently small, then there is an $x \in \Gamma_r$ such that

$$d_{\Gamma_r}(z) + \varepsilon\delta \ge d_r(z,x).$$

Let $l(t)$ be a minimal geodesic connecting z and x, and denote $y \in l(t) \cap \partial B^r_\delta$. Then

$$d_{\Gamma_r}(z) + \varepsilon\, d_r(z,y) \ge d_r(y,x) + d_r(z,y) \ge d_{\Gamma_r}(y) + d_r(z,y).$$

Combining this with (4.31) we conclude that $|D^r d_{\Gamma_r}|_r \equiv 1$ almost everywhere.

Lemma 4.6. *For any fixed $r_1 > 0$*

$$\lim_{r\to\infty} \int_{B^r_{r_1}} \left|D^r\left(\tilde{u}_r - |Du|_0 \cdot d_{\Gamma_r}\right)\right|_r^2 = 0. \tag{4.32}$$

Proof. Let η be a Lipschitz function with $\eta\big|_{B^r_{r_1}} \equiv 0$, $|D^r\eta|_r \le 1$ and $\operatorname{supp}\eta \subset \overline{B}^r_{2r_1}$. Let $\langle\cdot,\cdot\rangle_r$ and div_r be the inner product and divergence in $r^{-2}\Sigma$, then

(4.33)

$$\begin{aligned}
&\int_{B^r_{r_1}} |D^r(\tilde{u}_r - |Du|_0 \cdot d_{\Gamma_r})|_r^2 \le \int_{B^r_{2r_1}} |D^r(\tilde{u}_r - |Du|_0 \cdot d_{\Gamma_r})|_r^2\,\eta^2 \\
=&\int_{B^r_{2r_1}} \left(|Du|_0^2 \cdot |D^r d_{\Gamma_r}|_r^2 - |D^r\tilde{u}_r|_r^2\right)\eta^2 - 2\int_{B^r_{2r_1}} \langle D^r\tilde{u}_r, D^r(|Du|_0 \cdot d_{\Gamma_r} - \tilde{u}_r)\rangle_r\,\eta^2 \\
=&\int_{B^r_{2r_1}} \left(|Du|_0^2 - |D^r\tilde{u}_r|_r^2\right)\eta^2 \\
&+2\int_{B^r_{2r_1}} \left(2\eta\langle D^r\eta, D^r\tilde{u}_r\rangle_r + \eta^2 \operatorname{div}_r(D^r\tilde{u}_r)\right)\left(|Du|_0 \cdot d_{\Gamma_r} - \tilde{u}_r\right).
\end{aligned}$$

Together with (4.22)(4.23) and Lemma (4.4), we complete the proof. □

If (Z,d) is a metric space and $S_1, S_2 \subset Z$, then we set

$$d(S_1,S_2) = \inf\{d(s_1,s_2)|\ s_1 \in S_1,\ s_2 \in S_2\}, \quad B(S_1,\varepsilon) = \{z \in Z|\ d(z,S_1) < \varepsilon\}.$$

And we define *Hausdorff distance d_H* on S_1, S_2 by

$$d_H(S_1,S_2) = \inf\{\varepsilon > 0|\ S_1 \subset B(S_2,\varepsilon),\ S_2 \subset B(S_1,\varepsilon)\}.$$

If Z_1, Z_2 are both metric spaces, then an *admissible metric* on the disjoint union $Z_1 \bigsqcup Z_2$ is a metric that extends the given metrics on Z_1 and Z_2. With this one can define the *Gromov-Hausdorff distance* as

$$d_{GH}(Z_1,Z_2) = \inf\{d_H(Z_1,Z_2)|\ \text{adimissible metrics on } Z_1 \bigsqcup Z_2\}.$$

Now we consider a mapping $B^r_{r_1} \to B_{(p,0)}(r_1, \Gamma_r \times \mathbb{R}) : \; y \mapsto (y_0, |Du|_0^{-1}\tilde{u}_r(y))$, where $y_0 \in \Gamma_r$ satisfies $d_r(y, y_0) = d_{\Gamma_r}(y)$ and $B_{(p,0)}(r_1, \Gamma_r \times \mathbb{R})$ denotes the ball in $\Gamma_r \times \mathbb{R}$ with radius r_1 and centered at $(p, 0)$. Together with Lemma 4.4 and Lemma 4.6, using Theorem 3.6 in [9], we obtain

$$\lim_{r\to\infty} d_{GH}\left(B^r_{r_1}, B_{(p,0)}(r_1, \Gamma_r \times \mathbb{R})\right) = 0. \tag{4.34}$$

In fact, we can also obtain (4.34) through the following Lemma.

Lemma 4.7. *For any fixed $r_1 > 0$*

$$\lim_{r\to\infty} \sup_{y,z\in B^r_{r_1}} \left|(\tilde{u}_r(y) - \tilde{u}_r(z))^2 - |Du|_0^2 \cdot \left(d_r(y,z)^2 - d_r(y_0,z_0)^2\right)\right| = 0, \tag{4.35}$$

where $y_0, z_0 \in \Gamma_r$ satisfy $d_r(y, y_0) = d_{\Gamma_r}(y)$ and $d_r(z, z_0) = d_{\Gamma_r}(z)$.

Proof. We shall use the idea of the proof of (23.16) in [24] to show our Lemma. For any small $\delta > 0$, let $\varepsilon_i(r)$ be a general positive function satisfying $\lim_{r\to\infty} \varepsilon_i(r) = 0$, which depends only on $n, r_1, c, V_\Sigma, \delta$ for $i = 1, 2, \cdots$. It is sufficient to show that

$$\left|(\tilde{u}_r(y) - \tilde{u}_r(z))^2 - |Du|_0^2 \cdot \left(d_r(y,z)^2 - d_r(y_0,z_0)^2\right)\right| \le \varepsilon_1(r) + \delta$$

for any $y, z \in B^r_{r_1}$. Suppose $d_{\Gamma_r}(y) \ge d_{\Gamma_r}(z) \ge 0$ ($d_{\Gamma_r}(y)d_{\Gamma_r}(z) \le 0$ is similar). Let $l_1 : \; [0, d_{\Gamma_r}(y)] \to r^{-2}\Sigma$ be a normal minimal geodesic joining y_0 to y, and $l_2 : \; [0, d_{\Gamma_r}(z)] \to r^{-2}\Sigma$ be a normal minimal geodesic joining z_0 to z. When $d_{\Gamma_r}(z) \le t \le d_{\Gamma_r}(y)$, we set $l_2(t) = z$ for convenience. Let $Q(t) = d_r(l_1(t), l_2(t))$ and $\gamma_t = \gamma_{t,r} : \; [0, Q(t)] \to r^{-2}\Sigma$ be a normal minimal geodesic joining $l_2(t)$ to $l_1(t)$.

Let $h_t(s) = \tilde{u}_r(\gamma_t(s))$, then

$$\left|\frac{d^2 h_t}{ds^2}\right| = |\mathrm{Hess}_{\tilde{u}_r}(\dot{\gamma}_t(s), \dot{\gamma}_t(s))| \le |\mathrm{Hess}_{\tilde{u}_r}|_r. \tag{4.36}$$

Note that $\gamma_{t,r_{i_j}}$ converges to a normal minimal geodesic $\tilde{\gamma}_t$ as $r_{i,j}^{-1}\Sigma$ converges to a cone Σ_0. Hence due to $\Gamma^s_{r_{i_j}}$ converging to a cone $C\mathscr{Y}_s \subset \Sigma_0$ and $\tilde{\gamma}_t$ living on one side of $C\mathscr{Y}_{\frac{3t}{4|Du|_0}}$ we obtain $\gamma_{t,r_{i_j}}(s) \in r^{-2}\Sigma \setminus B^r_{\frac{\varepsilon}{2|Du|_0}}$ for any $0 \le s \le Q(t)$ and $t \ge \varepsilon$ if r is sufficiently large. Since we can choose any sequence r_i and then choose a suitable subsequence, we conclude $\gamma_{t,r} \in r^{-2}\Sigma \setminus B^r_{\frac{\varepsilon}{2|Du|_0}}$ for $t \ge \varepsilon$ and sufficiently large r. Hence combining (4.23) and the Newton-Leibniz formula we have

$$\left|h_t(Q(t)) - h_t(0) - Q(t)\frac{dh_t}{ds}(Q(t))\right| \le \varepsilon_2(r) \tag{4.37}$$

for any fixed $t \ge \varepsilon$. Note $h_t(Q(t)) = \tilde{u}_r(l_1(t))$ and $h_t(0) = \tilde{u}_r(l_2(t))$, Lemma 4.4 implies

$$\begin{aligned} |h_t(Q(t)) - |Du|_0 t| &\le \varepsilon_3(r) \qquad \text{for } 0 \le t \le d_{\Gamma_r}(y), \\ |h_t(0) - |Du|_0 t| &\le \varepsilon_4(r) \qquad \text{for } 0 \le t \le d_{\Gamma_r}(z). \end{aligned} \tag{4.38}$$

Without loss of generality, we assume $d_{\Gamma_r}(y) > \varepsilon$. So we obtain

(4.39)
$$\left|Q(t)\frac{dh_t}{ds}(Q(t))\right| \le \varepsilon_5(r) \qquad \text{for } \varepsilon \le t < \max\{d_{\Gamma_r}(z), \varepsilon\},$$
$$\left|Q(t)\frac{dh_t}{ds}(Q(t)) - |Du|_0\,(t - d_{\Gamma_r}(z))\right| \le \varepsilon_6(r) \qquad \text{for } \max\{d_{\Gamma_r}(z), \varepsilon\} \le t \le d_{\Gamma_r}(y).$$

Analogously we get

(4.40)
$$\left|Q(t)\frac{dh_t}{ds}(0)\right| + \left|Q(t)\frac{dh_t}{ds}(Q(t))\right| \le 2\varepsilon_5(r) \qquad \text{for } \varepsilon \le t \le \max\{d_{\Gamma_r}(z), \varepsilon\}.$$

Note initial data $Q(0) = d_r(y_0, z_0)$, then for $0 \le t \le d_{\Gamma_r}(z)$ we have

$$\begin{aligned}\left|Q^2(t) - d_r(y_0,z_0)^2\right| &\le 2\int_0^t \left|Q(s)\frac{dQ}{ds}\right| ds \\ =&2\int_0^t Q(s)\left|\left\langle \dot\gamma_s(Q(s)), \dot l_1(s)\right\rangle_r - \left\langle \dot\gamma_s(0), \dot l_2(s)\right\rangle_r\right| ds.\end{aligned} \tag{4.41}$$

Since

$$\left|\frac{dh_t}{ds}\right|_{t=0} + \left|\frac{dh_t}{ds}\right|_{t=Q(s)} = \left|\left\langle \dot\gamma_s(0), D^r\tilde u_r\big|_{l_2(s)}\right\rangle_r\right| + \left|\left\langle \dot\gamma_s(Q(s)), D^r\tilde u_r\big|_{l_1(s)}\right\rangle_r\right|,$$

then combining (4.40) and (4.41) gets

(4.42)
$$\begin{aligned}|Du|_0\left|Q^2(t) - d_r(y_0,z_0)^2\right| \le& 2\int_0^t Q(s)\left|\left\langle \dot\gamma_s(Q(s)), |Du|_0\dot l_1(s) - D^r\tilde u_r\big|_{l_1(s)}\right\rangle_r\right| ds \\ &+2\int_0^t Q(s)\left|\left\langle \dot\gamma_s(0), |Du|_0\dot l_2(s) - D^r\tilde u_r\big|_{l_2(s)}\right\rangle_r\right| ds + 4\varepsilon_5(r) + C\varepsilon,\end{aligned}$$

where C stands for a general positive constant. Lemma 4.4 indicates

$$\int_0^t \left(|Du|_0 - \left\langle \dot l_i(s), D^r\tilde u_r\big|_{l_i(s)}\right\rangle_r\right) ds \le \varepsilon_6(r) \tag{4.43}$$

for $i = 1, 2$. Then

(4.44)
$$\begin{aligned}\left(\int_0^t \left||Du|_0\dot l_i(s) - D^r\tilde u_r\big|_{l_i(s)}\right|_r ds\right)^2 \le& t\int_0^t \left||Du|_0\dot l_i(s) - D^r\tilde u_r\big|_{l_i(s)}\right|_r^2 ds \\ =&t\int_0^t \left(|Du|_0^2 + |D^r\tilde u_r(l_i(s))|_r^2 - 2|Du|_0\left\langle \dot l_i(s), D^r\tilde u_r\big|_{l_i(s)}\right\rangle_r\right) ds \\ \le& t\int_0^t \left(|D^r\tilde u_r(l_i(s))|_r^2 - |Du|_0^2\right) ds + 2t^2|Du|_0\varepsilon_6(r) \le 2t^2|Du|_0\varepsilon_6(r) \le (\varepsilon_7(r))^2.\end{aligned}$$

Hence combining (4.42) and (4.44) we obtain

$$|Du|_0\left|Q^2(t) - d_r(y_0,z_0)^2\right| \le \varepsilon_8(r) + C\varepsilon. \tag{4.45}$$

For $d_{\Gamma_r}(z) \le t \le d_{\Gamma_r}(y)$ we have
(4.46)
$$\left|\frac{dh_t}{ds}(Q(t)) - |Du|_0\frac{dQ}{dt}(t)\right| = \left|\left\langle \dot{\gamma}_t(Q(t)), D^r\tilde{u}_r\big|_{l_1(s)}\right\rangle_r - |Du|_0\left\langle \dot{\gamma}_t(Q(t)), \dot{l}_1(t)\right\rangle_r\right|,$$

then similar to the argument for $t \le \max\{d_{\Gamma_r}(z), \varepsilon\}$, employing (4.43) and (4.46), and integrating the second inequality in (4.39) give

$$\left|Q^2(d_{\Gamma_r}(y)) - Q^2(d_{\Gamma_r}(z)) - (d_{\Gamma_r}(y) - d_{\Gamma_r}(z))^2\right| \le \varepsilon_9(r) + C\varepsilon, \tag{4.47}$$

where C is a constant. Note $d_r(y,z) = Q(d_{\Gamma_r}(y))$. Combining (4.45), (4.47) and Lemma 4.4 we have

$$\begin{aligned}
&\left|(\tilde{u}_r(y) - \tilde{u}_r(z))^2 - |Du|_0^2 \cdot \left(d_r(y,z)^2 - d_r(y_0,z_0)^2\right)\right| \\
\le&\left|(\tilde{u}_r(y) - \tilde{u}_r(z))^2 - |Du|_0^2 \cdot \left(Q^2(d_{\Gamma_r}(y)) - Q^2(d_{\Gamma_r}(z))\right)\right| \\
&+ |Du|_0^2\left|Q^2(d_{\Gamma_r}(z)) - d_r(y_0,z_0)^2\right| \\
\le&|Du|_0\left(\varepsilon_8(r) + C\varepsilon\right) + |Du|_0^2\left(\varepsilon_9(r) + C\varepsilon\right) \\
&+ \left|(\tilde{u}_r(y) - \tilde{u}_r(z))^2 - |Du|_0^2 \cdot (d_{\Gamma_r}(y) - d_{\Gamma_r}(z))^2\right| \\
=&\varepsilon_{10}(r) + C\varepsilon + \left|(\tilde{u}_r(y) - \tilde{u}_r(z))^2 - (|Du|_0 \cdot d_{\Gamma_r}(y) - |Du|_0 \cdot d_{\Gamma_r}(z))^2\right| \\
\le&\varepsilon_{11}(r) + C\varepsilon.
\end{aligned} \tag{4.48}$$

Hence we complete the proof. □

For any sequence $r_i \to \infty$ there is a subsequence r_{i_j} such that $r_{i_j}^{-1}\Sigma = (\Sigma, r_{i_j}^{-1}\sigma, p)$ converges to a metric cone $C\mathfrak{X}$ with vertex o over some smooth manifold $\mathfrak{X}$ in the pointed Gromov-Hausdorff sense. Let $\mathfrak{B}_r$ be the geodesic ball with radius r and centered at o in $C\mathfrak{X}$. Then with (4.34) we get

$$\lim_{j\to\infty} d_{GH}\left(\mathfrak{B}_{r_1}, B_{(p,0)}(r_1, \Gamma_{r_{i_j}} \times \mathbb{R})\right) = 0. \tag{4.49}$$

By the previous argument, there exists an $(n-1)$-dimensional cone $\mathscr{Y}$ so that $\Gamma_{r_{i_j}}$ converges to $\mathscr{Y}$. Hence $C\mathfrak{X} = \mathscr{Y} \times \mathbb{R}$, namely, any tangent cone of Σ at infinity splits off a factor $\mathbb{R}$ isometrically.

Let $B_s^r(z)$ be the ball with radius s and centered at z in $r^{-2}\Sigma = (\Sigma, r^{-2}\sigma, p)$. For any $\varepsilon > 0$ and $s > 0$, by volume comparison theorem and condition C3), there is a sufficiently large $r_0 > 0$ such that

$$Vol\left(B_s^{r_i}(z_i)\right) \ge (\omega_n - \varepsilon)s^n$$

for each $z_i \in r_i^{-2}\Sigma$ with $d_{r_i}(z_i, p) \ge r_0 + s$ and $r_i \to \infty$. Let $\mathfrak{B}_s(z)$ be the geodesic ball with radius s and centered at z in $C\mathfrak{X}$. Taking limit in the above inequality gets

$$Vol\left(\mathfrak{B}_s(z)\right) \ge (\omega_n - \varepsilon)s^n \tag{4.50}$$

for any $z \in C\mathfrak{X}$ with $d_\infty(z,o) \geq r_0 + s$, where d_∞ is the distance function on $C\mathfrak{X}$. Let $\widetilde{\mathfrak{B}}_s(y)$ be the geodesic ball with radius s and centered at y in $C\mathscr{Y}$. Let $z = (y, t_z) \in \mathscr{Y} \times \mathbb{R}$, then (4.50) implies

$$\int_{-s}^{s} Vol\big(\widetilde{\mathfrak{B}}_{\sqrt{s^2-t^2}}(y)\big)dt \geq (\omega_n - \varepsilon)s^n. \tag{4.51}$$

Let $r_0 \to \infty$, and we fix y, then $t_z \to \infty$. So we obtain (4.51) for any $y \in \mathscr{Y}$ and $\varepsilon > 0$. Hence

$$\int_{-s}^{s} Vol\big(\widetilde{\mathfrak{B}}_{\sqrt{s^2-t^2}}(y)\big)dt \geq \omega_n s^n,$$

which means

$$Vol\big(\mathfrak{B}_s(z)\big) \geq \omega_n s^n \qquad \text{for any } z \in C\mathfrak{X} \text{ and } s > 0. \tag{4.52}$$

Since $\lim_{r\to\infty} \frac{Vol(B_r(x))}{r^n}$ is monotonically nonincreasing, then (4.52) implies

$$Vol\big(B_s(z)\big) \geq \omega_n s^n \qquad \text{for any } z \in \Sigma \text{ and } s > 0. \tag{4.53}$$

By the Bishop volume estimate, Σ is isometric to $\mathbb{R}^n$ (see also the proof of Theorem 0.3 in [16]).

Altogether, we obtain the following Liouville type theorem for minimal graphic functions with linear growth. This should be compared with the harmonic function theory in [13].

Theorem 4.8. *Let u be an entire solution to* (2.1) *on a complete Riemannian manifold Σ with conditions C1), C2), C3). If u has at most linear growth on one side, then u must be a constant unless Σ is isometric to Euclidean space.*

5 A Liouville theorem for minimal graphic functions without growth conditions

Let $G(p,\cdot)$ be the Green function on $\Sigma^n (n \geq 3)$ and $b = G^{\frac{1}{2-n}}$ as before. Now we set

$$\tilde{b} = \left(\frac{\omega_n}{V_\Sigma}\right)^{\frac{1}{n-2}} b$$

and define a function $\mathscr{R}$ in $\Sigma \times \mathbb{R}$ by

$$\mathscr{R}(x,t) = \sqrt{\tilde{b}^2(x) + t^2}, \qquad \text{for } (x,t) \in \Sigma \times \mathbb{R}.$$

Then

$$\Delta_N \mathscr{R}^2 = 2n|\nabla \tilde{b}|^2 + 2, \qquad |\overline{\nabla}\mathscr{R}|^2 = \frac{\tilde{b}^2|\nabla \tilde{b}|^2 + t^2}{\tilde{b}^2 + t^2} \leq \left(\frac{\omega_n}{V_\Sigma}\right)^{\frac{2}{n-2}}. \tag{5.1}$$

Let $\mathbb{B}_r$ be the ball in $\Sigma \times \mathbb{R}$ with radius r and centered at $(p,0)$. By the properties (2.7)(2.8)(2.9) of the function b we have

$$\Delta_N \mathscr{R}^2 - 2n|\overline{\nabla}\mathscr{R}|^2 = 2 + 2n\frac{t^2}{\mathscr{R}^2}\left(|\nabla \tilde{b}|^2 - 1\right), \tag{5.2}$$

$$\limsup_{r\to\infty}\left(\sup_{\partial\mathbb{B}_r}\left|\frac{\mathscr{R}}{r}-1\right|+\sup_{\partial\mathbb{B}_r}\left|\left|\overline{\nabla}\mathscr{R}\right|-1\right|\right)=0, \tag{5.3}$$

and

$$\limsup_{r\to\infty}\left(\sup_{\partial B_r\times\mathbb{R}}\left|\overline{\mathrm{Hess}}_{\mathscr{R}^2}-2\bar{g}\right|\right)=0. \tag{5.4}$$

Let ν be the unit normal vector on M as before. A simple calculation gives

$$\begin{aligned}\Delta\mathscr{R}^2 =&\Delta_N\mathscr{R}^2-\overline{\mathrm{Hess}}_{\mathscr{R}^2}(\nu,\nu)=2n|\nabla\tilde{b}|^2+2-\overline{\mathrm{Hess}}_{\mathscr{R}^2}(\nu,\nu)\\ =&2n|\overline{\nabla}\mathscr{R}|^2+2+2n\frac{t^2}{\mathscr{R}^2}\left(|\nabla\tilde{b}|^2-1\right)-\overline{\mathrm{Hess}}_{\mathscr{R}^2}(\nu,\nu).\end{aligned} \tag{5.5}$$

Since M is an entire graph,

$$\lim_{r_i\to\infty}\left(\inf\left\{d(x)\Big|\text{ there is a } t\in\mathbb{R} \text{ such that } (x,t)\in M\setminus\mathbb{B}_{\sqrt{r_i}}\right\}\right)=\infty.$$

Combining (5.4)(5.5)(2.8) there exists a sequence $\delta_i\to 0^+$ such that on $M\setminus\mathbb{B}_{\sqrt{r_i}}$ we have

$$\left|\Delta\mathscr{R}^2-2n|\overline{\nabla}\mathscr{R}|^2\right|\le 2\delta_i|\overline{\nabla}\mathscr{R}|^2. \tag{5.6}$$

Obviously $\Sigma\times\mathbb{R}$ has nonnegative Ricci curvature. Therefore, $Vol(\partial\mathbb{B}_r)\le |\mathbb{S}^n|r^n$, where $|\mathbb{S}^n|$ is the volume of n-dimensional unit sphere in $\mathbb{R}^{n+1}$. Since M is an area-minimizing hypersurface in $\Sigma\times\mathbb{R}$ by Lemma 2.1 in [21], then

$$Vol(M\cap\mathbb{B}_r)\le\frac{1}{2}Vol(\partial\mathbb{B}_r)\le\frac{|\mathbb{S}^n|}{2}r^n.$$

With (5.3),

$$r^{-n}\int_{M\cap\{\mathscr{R}\le r\}}|\overline{\nabla}\mathscr{R}|^2d\mu$$

is uniformly bounded for any $r\in(0,\infty)$, and so, there exists a sequence $r_i\to\infty$ such that

$$\limsup_{r\to\infty}\left(r^{-n}\int_{M\cap\{\mathscr{R}\le r\}}|\overline{\nabla}\mathscr{R}|^2d\mu\right)=\lim_{r_i\to\infty}\left(r_i^{-n}\int_{M\cap\{\mathscr{R}\le r_i\}}|\overline{\nabla}\mathscr{R}|^2d\mu\right).$$

With the proof of Lemma 5.2 in [21], we obtain the following Lemma.

Lemma 5.1. *There is a sequence $\delta_i\to 0^+$ such that for any constants $K_2>K_1>0$ and $\varepsilon\in(0,1)$ and any bounded Lipschitz function f on $N\setminus\mathbb{B}_1$ we have*

$$\begin{aligned}&\limsup_{i\to\infty}\left|\left(\frac{\delta_i}{K_2r_i}\right)^n\int_{M\cap\{\mathscr{R}\le\frac{K_2r_i}{\delta_i}\}}f|\overline{\nabla}\mathscr{R}|^2-\left(\frac{\delta_i}{K_1r_i}\right)^n\int_{M\cap\{\mathscr{R}\le\frac{K_1r_i}{\delta_i}\}}f|\overline{\nabla}\mathscr{R}|^2\right|\\ \le& C\varepsilon^n\sup_{N\setminus\mathbb{B}_1}|f|+\limsup_{i\to\infty}\int_{\frac{K_1r_i}{\delta_i}}^{\frac{K_2r_i}{\delta_i}}\left(s^{-n-1}\int_{M\cap\{\frac{\varepsilon K_1r_i}{\delta_i}<\mathscr{R}\le s\}}\mathscr{R}\nabla f\cdot\nabla\mathscr{R}\right)ds.\end{aligned} \tag{5.7}$$

There is a subsequence $\{\varepsilon_i\}$ of $\{\delta_i^2 r_i^{-2}\}$ converging to zero such that $\varepsilon_i\Sigma = (\Sigma, \varepsilon_i\sigma, p)$ converges to a metric cone $(\Sigma_\infty, d_\infty)$ with vertex o in the measured Gromov-Hausdorff sense. Denote $\varepsilon_i = \delta_i^2 r_i^{-2}$ for simplicity. The cone Σ_∞ is over some $(n-1)$-dimensional smooth compact manifold X with $C^{1,\alpha}$ Riemannian metric and Diam $X \leq \pi$, namely, $\Sigma_\infty = CX \triangleq \mathbb{R}^+ \times_\rho X$.

Let $B_r^i(x)$ be the geodesic ball with radius r and centered at x in $(\Sigma, \varepsilon_i\sigma)$, and $\mathscr{B}_r(x)$ be the geodesic ball with radius r and centered at x in Σ_∞. In particular, $X = \partial\mathscr{B}_1(o)$. Note that for convenience our definitions of $B_r^i(x)$ are different from the previous ones in section 4. Let $\mathbb{B}_r^i(x)$ be the geodesic ball with radius r and centered at x in $(\Sigma\times\mathbb{R}, \varepsilon_i(\sigma + dt^2))$, and $\widetilde{\mathscr{B}}_r(x)$ be the geodesic ball with radius r and centered at x in $\Sigma_\infty\times\mathbb{R}$.

Let $\varepsilon_i M = (M, \varepsilon_i g)$ and $D_r^i(x) = \varepsilon_i M \cap \mathbb{B}_r^i(x)$. We always omit x in $D_r^i(x)$ (or $\mathbb{B}_r^i(x), \mathscr{B}_r(x), \widetilde{\mathscr{B}}_r(x)$) if $x = p$ (or $x = (p,0), o, (o,0)$) respectively, for simplicity. Clearly, $\varepsilon_i M$ is still a minimal graph in $(\Sigma\times\mathbb{R}, \varepsilon_i(\sigma + dt^2))$.

Lemma 5.2. *There exists a subsequence $\{\varepsilon_{i_j}\} \subset \{\varepsilon_i\}$ such that $\varepsilon_{i_j}M$ converges to an area-minimizing cone $T = CY \triangleq \mathbb{R}^+ \times_\rho Y$ in $\Sigma_\infty\times\mathbb{R}$, where $Y \in \partial\widetilde{\mathscr{B}}_1(o)$ is an $(n-1)$-dimensional Hausdorff set.*

Proof. For any fixed $r > 1$ let $\Upsilon_i : \left(\overline{\mathscr{B}}_{r+1}\setminus\mathscr{B}_{\frac{1}{2r}}\right)\times\mathbb{R} \to \varepsilon_i\Sigma\times\mathbb{R}$ be a mapping defined by $\Upsilon_i(x,t) = (\Phi_i(x), t) \in \varepsilon_i\Sigma\times\mathbb{R}$, where Φ_i is a diffeomorphism from $\overline{\mathscr{B}}_{r+1}\setminus\mathscr{B}_{\frac{1}{2r}}$ to $\Phi_i(\overline{\mathscr{B}}_{r+1}\setminus\mathscr{B}_{\frac{1}{2r}}) \subset \varepsilon_i\Sigma$ such that $\Phi_i^*(\varepsilon_i\sigma)$ converges as $i\to\infty$ to σ_∞ in the $C^{1,\alpha}$-topology on $\overline{\mathscr{B}}_{r+1}\setminus\mathscr{B}_{\frac{1}{2r}}$. Thus $\Upsilon_i^*(\varepsilon_i\sigma + \varepsilon_i dt^2)$ converges as $i\to\infty$ to $\sigma_\infty + dt^2$ in the $C^{1,\alpha}$-topology on $\widetilde{\mathscr{B}}_{r+1}\setminus\widetilde{\mathscr{B}}_{\frac{1}{2r}}$. By compactness of currents (see [2], [34], [40] or [21]), there is a subsequence of ε_{i_j} such that

$$\Upsilon_{i_j}^{-1}\left(\varepsilon_{i_j}M\bigcap\left(B_r^{i_j}\setminus B_{\frac{1}{r}}^{i_j}\right)\times\mathbb{R}\right) \rightharpoonup T \qquad \text{as } j\to\infty,$$

where T is an integral-rectifiable current in $\Sigma_\infty\times\mathbb{R}$. By choosing a diagonal sequence, we can assume that the above limit holds for any $r > 1$. For convenience, we still write ε_i instead of ε_{i_j}.

Let Ω_0 be an arbitrary bounded domain in T, and W_0 be an arbitrary bounded set with induced metric in $\Sigma_\infty\times\mathbb{R}$ with $\partial\Omega_0 = \partial W_0$. There is a constant $R > 0$ such that $\Omega_0\cup W_0 \subset \widetilde{\mathscr{B}}_R$. For any small $\delta > 0$ let $\Omega_i = \Upsilon_i(\Omega_0\setminus(\mathscr{B}_\delta\times[-R,R])) \subset \varepsilon_i M$ and $W_i = \Upsilon_i(W_0\setminus(\mathscr{B}_\delta\times[-R,R]))$ with induced metrics in $\varepsilon_i N$. Then there exists $U_0 \subset \partial(\mathscr{B}_\delta\times[-R,R])$ (possibly empty) such that $\partial\Omega_i = \partial(W_i\cup U_i)$ with $U_i = \Upsilon_i(U_0) \subset \varepsilon_i N$. Since $\varepsilon_i M$ is an area-minimizing hypersurface in $\varepsilon_i N$, then

$$\begin{aligned}
H^n(\Omega_0\setminus(\mathscr{B}_\delta\times[-R,R])) =& \lim_{i\to\infty} H^n(\Omega_i) \leq \lim_{i\to\infty} H^n(W_i\cup U_i)\\
\leq& \lim_{i\to\infty} H^n(W_i) + \lim_{i\to\infty} H^n(U_i) = H^n(W_0\setminus\widetilde{\mathscr{B}}_\delta) + H^n(U_0)\\
\leq& H^n(W_0) + H^n(\partial(\mathscr{B}_\delta\times[-R,R])).
\end{aligned} \tag{5.8}$$

Let $\delta \to 0$ to obtain

$$H^n(\Omega_0) \le H^n(W_0).$$

Namely, T is an area-minimizing set in $\Sigma_\infty \times \mathbb{R}$.

For any $f \in C^1(\partial\widetilde{\mathscr{B}}_1)$, we could extend f to $\Sigma_\infty \times \mathbb{R} \setminus \{(o,0)\}$ by defining

$$f(\rho\theta) = f(\theta)$$

for any $\rho > 0$ and $\theta \in \partial\widetilde{\mathscr{B}}_1$. Let Π_i be the map of rescaling from $(N, \sigma + dt^2)$ to $\varepsilon_i N = (N, \varepsilon_i\sigma + \varepsilon_i dt^2)$. Set $U_s = B_s \times \mathbb{R}$ for $s > 0$. Note $\varepsilon_i = \delta_i^2 r_i^{-2}$, then similar to the proof of (4.12) and (4.13) in [21], for any $K_2 > K_1 > 0$ we have

$$\limsup_{i\to\infty} \sup_{\mathbb{B}_{\frac{K_2 r_i}{\delta_i}} \setminus U_{\frac{\varepsilon K_1 r_i}{\delta_i}}} \left|\left\langle \overline{\nabla}(f \circ \Upsilon_i^{-1} \circ \Pi_i), \overline{\nabla}\mathscr{R}^2 \right\rangle\right| = 0, \tag{5.9}$$

and

$$\limsup_{i\to\infty} \sup_{\mathbb{B}_{\frac{K_2 r_i}{\delta_i}} \setminus U_{\frac{\varepsilon K_1 r_i}{\delta_i}}} \left(\mathscr{R}\left|\overline{\nabla}(f \circ \Upsilon_i^{-1} \circ \Pi_i)\right|\right) < \infty, \tag{5.10}$$

Now we can extend the function $f \circ \Upsilon_i^{-1} \circ \Pi_i$ to a uniformly bounded function F_i in $\mathbb{B}_{\frac{K_2 r_i}{\delta_i}} \setminus U_{\frac{\varepsilon K_1 r_i}{\delta_i}}$ with $F_i = f \circ \Upsilon_i^{-1} \circ \Pi_i$ on $\mathbb{B}_{\frac{K_2 r_i}{\delta_i}} \setminus U_{\frac{\varepsilon K_1 r_i}{\delta_i}}$. Obviously, we can extend F_i to a C^1-function on $\mathbb{B}_{\frac{K_2 r_i}{\delta_i}} \cap U_{\frac{\varepsilon K_1 r_i}{\delta_i}}$ with $|F_i| \le 2|f_0|_{C^0(\partial\widetilde{\mathscr{B}}_1)}$.

Note $\mathscr{R}^2(x,t) = \tilde{b}^2(x) + t^2$ for any $(x,t) \in \Sigma \times \mathbb{R}$. Due to the proof of Lemma 5.3 in [21], it is sufficient to show that there is a sequence $\tau_i \in [\varepsilon, 2\varepsilon]$ so that

$$\limsup_{i\to\infty} \int_{\frac{K_1 r_i}{\delta_i}}^{\frac{K_2 r_i}{\delta_i}} \left(\frac{1}{s^{n+1}} \int_{M\cap\{\frac{\tau_i K_1 r_i}{\delta_i} < \mathscr{R} \le s\} \cap \left(\{\tilde{b} \le \frac{\tau_i K_1 r_i}{\delta_i}\} \times \mathbb{R}\right)} \mathscr{R} \nabla F_i \cdot \nabla\mathscr{R} \right) ds < C\varepsilon \tag{5.11}$$

for some absolute constant $C > 0$.

We show (5.11) by the following consideration.

Let $\Omega_{s,i,\tau} = M \cap \{\frac{\tau K_1 r_i}{\delta_i} < \mathscr{R} \le s\} \cap \left(\{\tilde{b} \le \frac{\tau K_1 r_i}{\delta_i}\} \times \mathbb{R}\right)$ for $s \in \left(\frac{K_1 r_i}{\delta_i}, \frac{K_2 r_i}{\delta_i}\right)$. Integrating by parts implies

$$\begin{aligned} &\int_{\Omega_{s,i,\tau}} \nabla F_i \cdot \nabla\mathscr{R}^2 + \int_{\Omega_{s,i,\tau}} F_i \Delta\mathscr{R}^2 = \int_{\Omega_{s,i,\tau}} \mathrm{div}_M \left(F_i \nabla\mathscr{R}^2\right) \\ &= \int_{\partial\Omega_{s,i,\tau}} F_i \langle \nabla\mathscr{R}^2, \nu_{\partial\Omega_{s,i,\tau}} \rangle \le \int_{\partial\Omega_{s,i,\tau}} F_i |\nabla\mathscr{R}^2| \le C_1 s \int_{\partial\Omega_{s,i,\tau}} 1 \end{aligned} \tag{5.12}$$

for some absolute constant C_1. Recall $|\overline{\nabla}\mathscr{R}| \le \left(\frac{\omega_n}{V_\Sigma}\right)^{\frac{1}{n-2}}$. It is easy to see that

$$\begin{aligned}
&\left(\frac{V_\Sigma}{\omega_n}\right)^{\frac{1}{n-2}} \frac{K_1 r_i}{\delta_i}\int_\varepsilon^{2\varepsilon}\left(\int_{\partial\Omega_{s,i,\tau}} 1\right)d\tau \le \frac{K_1 r_i}{\delta_i}\int_\varepsilon^{2\varepsilon}\left(\int_{\partial\Omega_{s,i,\tau}}\frac{1}{|\nabla\mathscr{R}|}\right)d\tau\\
\le& Vol\left(M\cap\{\mathscr{R}\le s\}\cap\left(\left\{\frac{\varepsilon K_1 r_i}{\delta_i}\le\tilde{b}\le\frac{2\varepsilon K_1 r_i}{\delta_i}\right\}\times\mathbb{R}\right)\right)\\
&+Vol\left(M\cap\left\{\frac{\varepsilon K_1 r_i}{\delta_i}\le\mathscr{R}\le\frac{2\varepsilon K_1 r_i}{\delta_i}\right\}\right)\\
\le& Vol\left(\partial\left(\{\mathscr{R}\le s\}\cap\left(\left\{\frac{\varepsilon K_1 r_i}{\delta_i}\le\tilde{b}\le\frac{2\varepsilon K_1 r_i}{\delta_i}\right\}\times\mathbb{R}\right)\right)\right)\\
&+Vol\left(\partial\left(\left\{\frac{\varepsilon K_1 r_i}{\delta_i}\le\mathscr{R}\le\frac{2\varepsilon K_1 r_i}{\delta_i}\right\}\right)\right)\\
\le& C_2 s\left(\frac{\varepsilon K_1 r_i}{\delta_i}\right)^{n-1}
\end{aligned}\tag{5.13}$$

for some absolute constant C_2. Hence, for every i there exists a $\tau_i\in[\varepsilon,2\varepsilon]$ such that

$$\int_{\partial\Omega_{s,i,\tau_i}} 1 \le C_3 s\left(\frac{\varepsilon K_1 r_i}{\delta_i}\right)^{n-2}.\tag{5.14}$$

Through a simple calculation we have

$$\begin{aligned}
&\left|\int_{\Omega_{s,i,\tau_i}} F_i\Delta\mathscr{R}^2\right| \le C_4 Vol\left(\Omega_{s,i,\tau_i}\right)\\
\le& C_4 Vol\left(\partial\left(\left\{\frac{\tau K_1 r_i}{\delta_i}<\mathscr{R}\le s\right\}\bigcap\left(\left\{\tilde{b}\le\frac{\tau K_1 r_i}{\delta_i}\right\}\times\mathbb{R}\right)\right)\right)\\
\le& C_5 s\left(\frac{\varepsilon K_1 r_i}{\delta_i}\right)^{n-1}.
\end{aligned}\tag{5.15}$$

Hence

$$\int_{\Omega_{s,i,\tau_i}}\nabla F_i\cdot\nabla\mathscr{R}^2 \le -\int_{\Omega_{s,i,\tau_i}} F_i\Delta\mathscr{R}^2 + C_1 s\int_{\partial\Omega_{s,i,\tau_i}} 1 \le C_6 s^2\left(\frac{\varepsilon K_1 r_i}{\delta_i}\right)^{n-2},\tag{5.16}$$

which implies that (5.11) holds. Then, as in Lemma 5.3 in [21], we can show that

$$\frac{1}{K_2^n}\int_{T\cap\widetilde{\mathscr{B}}_{K_2}} f = \frac{1}{K_1^n}\int_{T\cap\widetilde{\mathscr{B}}_{K_1}} f\tag{5.17}$$

for arbitrary f. This means T is a cone in $\Sigma_\infty\times\mathbb{R}$. Therefore, we complete the proof. □

Set $s_i=(\varepsilon_i)^{-\frac{1}{2}}$ for convenience. The definitions of $D^{s_i}, \tilde{u}_{s_i}, \langle\cdot,\cdot\rangle_{s_i}, |\cdot|_{s_i}$ are as in section 4. We define $\rho_\infty(x)=d_\infty(o,x)$ and $\bar{\rho}_\infty(x)=\bar{d}_\infty(o,x)$ as distance functions on Σ_∞ and N_∞, respectively. Let $\rho_i(x)$ be the distance function on $\varepsilon_i\Sigma$ from o to x, and $\bar\rho_i(z)$ be the distance function on $\varepsilon_i(\Sigma\times\mathbb{R})$ from $(o,0)$ to z.

For each $x\in\varepsilon_i\Sigma$ there is a minimal normal geodesic γ_x^i from p to x such that $D^{s_i}\rho_i(x)=\dot\gamma_x^i$. When $\varepsilon_i=1$, we define $D\rho(x)$ corresponding to the normal geodesic $\dot\gamma_x$. Hence $D^{s_i}\rho_i(x)$ depends on the choice of γ_x^i. Note that $\rho_i(x)$ is just a Lipschitz function on $\varepsilon_i\Sigma$, but the definition of $D^{s_i}\rho_i(x)$ is equivalent to the common one if ρ_i is C^1 at the considered point.

Let $T_i=\varepsilon_iM\bigcap\left((B_2^i\setminus B_\varepsilon^i)\times\left[\frac{1}{\delta},\frac{1}{\delta}+\frac{2}{\varepsilon}\right]\right)$ for $\delta>0$ and a 'bad' set

(5.18)

$$E_i\triangleq\left\{z=(x,t)\in T_i\middle|\ |D^{s_i}\tilde u_{s_i}(x)|_{s_i}\le\frac{1}{\varepsilon}\text{ or }\left|(D^{s_i}\tilde u_{s_i}(x))^T\right|^2_{s_i}\le\left(1-\varepsilon^4\right)|D^{s_i}\tilde u_{s_i}(x)|^2_{s_i}\right\},$$

where $\xi^T=\xi-\langle\xi,D^{s_i}\rho_i\rangle_{s_i}D^{s_i}\rho_i$ for any local vector field ξ on $\varepsilon_i\Sigma$.

Lemma 5.3. *Suppose* $\sup_\Sigma|Du|=\infty$. *For any* $\varepsilon>0$ *there is a sufficiently small* δ_0 *such that for any* $0<\delta<\delta_0$ *there is a sufficiently large* i_0 *so that for* $i\ge i_0$ *we have*

$$H^n(E_i)<\varepsilon^n.$$

Proof. Let U_j be the subgraph of $\tilde u_{s_i}$ in $\varepsilon_i\Sigma\times\mathbb{R}$ defined by

$$\{(x,t)\in\varepsilon_i\Sigma\times\mathbb{R}|\ t<\tilde u_{s_i}(x)\}.$$

By Rellich theorem, for any compact $K\subset\Sigma_\infty$ there is a subsequence of the characteristic functions χ_{U_j} converging to $\chi_{\widetilde U}$ in $L^1(K)$ up to a diffeomorphism (see Proposition 16.5 in [25] for the Euclidean case). Clearly, $\widetilde U$ can be represented as a subgraph of some generalized function $\mathfrak{u}$ (possibly equal to $\pm\infty$ somewhere) in $\Sigma_\infty\times\mathbb{R}$, namely,

$$\widetilde U=\{(x,t)\in\Sigma_\infty\times\mathbb{R}|\ t<\mathfrak{u}(x)\}.$$

Note $T=\partial\widetilde U$. $\mathfrak{u}(x)$ is a homogeneous function of degree 1 in $\Sigma_\infty\setminus\{o\}$ as T is a cone through o.

If $\sup_\Sigma|Du|=\infty$, T contains a half line $\{o\}\times(0,\infty)$ or $\{o\}\times(-\infty,0)$. Without loss generality, we assume $\{o\}\times(0,\infty)\subset T$. Now we define a set P by $\{x\in\Sigma_\infty|\ \mathfrak{u}(x)=+\infty\}$. Since T is a cone through the point o, then P is also a cone through o in Σ_∞. In fact, for any $x\in P$ we have $\mathfrak{u}(x)=+\infty$. In particular, there is $(r,\theta)\in\mathbb{R}^+\times_\rho X=CX$ so that $x=(r,\theta)$. Then $\mathfrak{u}(tx)=+\infty$ for $tx=(tr,\theta)$, which means that P is a cone.

For any $z\in T\bigcap((\mathscr{B}_2\setminus\mathscr{B}_\varepsilon)\times\{s\})$, the slope of the line connecting z and o becomes larger and larger as s increases to infinity. Hence for any $0<\varepsilon<1$ we have

(5.19)

$$\lim_{\delta\to0}d_{GH}\left(T\bigcap\left((\mathscr{B}_2\setminus\mathscr{B}_\varepsilon)\times\left[\frac{1}{\delta},\frac{1}{\delta}+\frac{2}{\varepsilon}\right]\right),\left(\partial P\cap(\mathscr{B}_2\setminus\mathscr{B}_\varepsilon)\right)\times\left[0,\frac{2}{\varepsilon}\right]\right)=0.$$

Combining (3.18)-(3.21) and that M is area-minimizing, it is clear that there is a constant $\varepsilon_0>0$ depending only on Σ so that

$$\varepsilon_0 r^n\le\int_{M\cap\mathbb{B}_r(z)}1\le\frac{1}{2}Vol(\partial\mathbb{B}_r(z))\le\frac{|\mathbb{S}^n|}{2}r^n\qquad\text{for any } r>0,\ z\in M.$$

Since $\varepsilon_i M\rightharpoonup T$, for any $r>0$ and $y\in T$ we have

$$\varepsilon_0 r^n\le\int_{T\cap\mathscr{B}_r(y)}1\le\frac{|\mathbb{S}^n|}{2}r^n.$$

Due to (5.19) and $\{o\}\times(0,\infty)\subset T$, it is not hard to see that

$$H^{n-1}\Big(\partial P\cap(\mathscr{B}_2\setminus\mathscr{B}_\varepsilon)\Big)>0\qquad\text{and}\qquad H^{n-2}\Big(\partial P\cap\partial\mathscr{B}_1\Big)>0 \tag{5.20}$$

as P is a cone through o.

Let $\Phi_i:\overline{\mathscr{B}}_{\frac{2}{\delta}}\setminus\mathscr{B}_\varepsilon\to\Phi(\overline{\mathscr{B}}_{\frac{2}{\delta}}\setminus\mathscr{B}_\varepsilon)\subset\varepsilon_i\Sigma$ be a diffeomorphism such that $\Phi_i^*(\varepsilon_i\sigma)$ converges as $i\to\infty$ to σ_∞ in the $C^{1,\alpha}$-topology on $\overline{\mathscr{B}}_{\frac{2}{\delta}}\setminus\mathscr{B}_\varepsilon$, and $\Upsilon_i(x,t)=(\Phi_i(x),t)$ for any $x\in\overline{\mathscr{B}}_{\frac{2}{\delta}}\setminus\mathscr{B}_\varepsilon$. Note that Φ_i and Υ_i depend on δ. Obviously, $\lim_{i\to\infty}\rho_i\circ\Phi_i=\rho_\infty$ in $\overline{\mathscr{B}}_{\frac{2}{\delta}}\setminus\mathscr{B}_\varepsilon$ and $\lim_{i\to\infty}\bar\rho_i\circ\Upsilon_i=\bar\rho_\infty$ in $\Big(\mathscr{B}_{\frac{2}{\delta}}\setminus\mathscr{B}_\varepsilon\Big)\times\left(-\frac{2}{\delta},\frac{2}{\delta}\right)$. Since $\varepsilon_i M\bigcap\Big(\mathbb{B}^i_{\frac{4}{\delta}}\setminus\mathbb{B}^i_\varepsilon\Big)$ converges to $T\bigcap\Big(\widetilde{\mathscr{B}}_{\frac{4}{\delta}}\setminus\widetilde{\mathscr{B}}_\varepsilon\Big)$ in the varifold sense, for any compact set $\widetilde{K}\Subset\widetilde{\mathscr{B}}_{\frac{4}{\delta}}\setminus\widetilde{\mathscr{B}}_\varepsilon$ we have

$$0=\lim_{i\to\infty}\Big(\varepsilon_i M\llcorner\Upsilon_i(\widetilde{K})\Big)(\bar\omega^*\circ\Upsilon_i^{-1})=\lim_{i\to\infty}\int_{\varepsilon_i M\cap\Upsilon_i(\widetilde{K})}\langle\bar\omega^*\circ\Upsilon_i^{-1},\nu_i\rangle d\mu_i, \tag{5.21}$$

where $\bar\omega^*$ is the dual form of $\frac{\partial}{\partial\bar\rho_\infty}$ in TN_∞. For any compact set $K\subset(\mathscr{B}_2\setminus\mathscr{B}_\varepsilon)\times\left[\frac{1}{\delta},\frac{1}{\delta}+\frac{2}{\varepsilon}\right]$, we let $K_s=\{x|\ (x,s)\in K\}$ be a slice of K for $s\in\left[\frac{1}{\delta},\frac{1}{\delta}+\frac{2}{\varepsilon}\right]$.

Since $\varepsilon_i M\rightharpoonup T$, (5.19) implies

$$\lim_{i\to\infty}H^n\left(\left\{(x,t)\in T_i\Big|\ |D^{s_i}\tilde u_{s_i}(x)|_{s_i}\le\frac{1}{\delta}\right\}\right)=0. \tag{5.22}$$

Thus from (5.21) we obtain

$$\limsup_{i\to\infty}\left|\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\left(\int_{\varepsilon_i M\cap\Phi_i(K_s)}\left\langle\omega^*\circ\Phi_i^{-1},\frac{D^{s_i}\tilde u_{s_i}}{\sqrt{1+|D^{s_i}\tilde u_{s_i}|^2_{s_i}}}\right\rangle_{s_i}\right)ds\right|\le C\frac{\delta}{\varepsilon} \tag{5.23}$$

for some constant C, where ω^* is the dual form of $\frac{\partial}{\partial\rho_\infty}$ in $T\Sigma_\infty$. Note that $D^{s_i}\rho_i\to\frac{\partial}{\partial\rho_\infty}$ as $i\to\infty$ on any compact set in $\Sigma_\infty\setminus\{o\}$ up to a diffeomorphism Φ_i, then it follows that

$$\limsup_{i\to\infty}\left|\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\left(\int_{\varepsilon_i M\cap\Phi_i(K_s)}\frac{\langle D^{s_i}\rho_i,D^{s_i}\tilde u_{s_i}\rangle_{s_i}}{\sqrt{1+|D^{s_i}\tilde u_{s_i}|^2_{s_i}}}\right)ds\right|\le C\frac{\delta}{\varepsilon} \tag{5.24}$$

Together with (5.22) and (5.24), we complete the proof. □

Since M is a minimal graph in $\Sigma \times \mathbb{R}$, it is stable, and $\varepsilon_i M$ is also a stable minimal hypersurface in $\varepsilon_i(\Sigma \times \mathbb{R})$. Let B^i be the second fundamental form of $\varepsilon_i M$ in $\varepsilon_i N$ and $Ric_{\varepsilon_i N}$ be Ricci curvature of $\varepsilon_i N$. For any Lipschitz function φ with compact support in $\varepsilon_i M$ we have

$$\int_{\varepsilon_i M} \left(|B^i|^2 + Ric_{\varepsilon_i N}(\nu_i, \nu_i)\right) \phi^2 \le \int_{\varepsilon_i M} |\nabla^i \phi|^2, \tag{5.25}$$

where ∇^i is the Levi-Civita connection of $\varepsilon_i M$.

Now we suppose that there exists sufficiently large $r_0 > 0$ such that the non-radial Ricci curvature of Σ satisfies

$$\inf_{\partial B_r} Ric_\Sigma \left(\xi^T, \xi^T\right) \ge \frac{\kappa}{r^2} > 0 \tag{5.26}$$

for all $r \ge r_0$ and $n \ge 3$, where ξ is a local vector field on Σ with $\xi^T = \xi - \langle \xi, D\rho \rangle D\rho$ and $|\xi^T| = 1$. The definition of $D\rho$ is as before.

Lemma 5.4. *If* $\sup_\Sigma |Du| = \infty$*, then* κ *in* (5.26) *satisfies* $\kappa \le \frac{(n-3)^2}{4}$.

Proof. By re-scaling we get

$$\inf_{\partial B^i_r} Ric_{\varepsilon_i \Sigma} \left(\eta^T, \eta^T\right) \ge \frac{\kappa}{r^2} > 0$$

for all $r \ge \sqrt{\varepsilon_i} r_0$, where $\eta \in \Gamma(T(\varepsilon_i N))$, $\eta^T = \eta - \langle \eta, D^{s_i}\rho_i \rangle_{s_i} D^{s_i}\rho_i$ with $|\eta^T|_{s_i} = 1$. Noting that conditions C1) and C3) are both invariant under scaling, we obtain

$$\begin{aligned} Ric_{\varepsilon_i N}(\nu_i, \nu_i) &= \frac{1}{1 + |D^{s_i}\tilde{u}_{s_i}|^2_{s_i}} Ric_{\varepsilon_i \Sigma}(D^{s_i}\tilde{u}_{s_i}, D^{s_i}\tilde{u}_{s_i}) \\ &\ge \frac{1}{1 + |D^{s_i}\tilde{u}_{s_i}|^2_{s_i}} \Big(Ric_{\varepsilon_i \Sigma}\left((D^{s_i}\tilde{u}_{s_i})^T, (D^{s_i}\tilde{u}_{s_i})^T\right) \\ &\qquad + 2\langle D^{s_i}\tilde{u}_{s_i}, D^{s_i}\rho_i \rangle_{s_i} Ric_{\varepsilon_i N}\left((D^{s_i}\tilde{u}_{s_i})^T, D^{s_i}\rho_i\right) \Big) \\ &\ge \frac{1}{1 + |D^{s_i}\tilde{u}_{s_i}|^2_{s_i}} \left(\kappa' \left|(D^{s_i}\tilde{u}_{s_i})^T\right|^2_{s_i} - c' \left|(D^{s_i}\tilde{u}_{s_i})^T\right|_{s_i} \langle D^{s_i}\tilde{u}_{s_i}, D^{s_i}\rho_i \rangle_{s_i} \right) \rho_i^{-2} \end{aligned} \tag{5.27}$$

for some absolute constant $c' > 0$.

Let η be the Lipschitz function on $\varepsilon_i \Sigma$ defined by

$$\eta(x) = \left(\rho_i(x)\right)^{\frac{3-n}{2}} \sin\left(\pi \frac{\log \rho_i(x)}{\log \varepsilon}\right)$$

in $B^i_1 \setminus B^i_\varepsilon$ and $\eta = 0$ in other places. Let τ be a Lipschitz function on $\mathbb{R}$ satisfying $\tau \equiv 1$ on $\left[\frac{1}{\delta} + 1, \frac{1}{\delta} + \frac{2}{\varepsilon} - 1\right]$, $\tau(t) \equiv 0$ for $t \in \left(-\infty, \frac{1}{\delta}\right] \bigcup [\frac{1}{\delta} + \frac{2}{\varepsilon}, \infty)$, and $|\tau'| \le 1$.

For $z=(x,t)\in\Sigma\times\mathbb{R}$ set $\varphi(z)=\eta(x)\tau(t)$. Let $\overline{\nabla}^i$ be the Levi-Civita connection of $\varepsilon_i N$. Then

(5.28)
$$\begin{aligned}
&\int_{\varepsilon_i M} Ric_{\varepsilon_i N}(\nu_i,\nu_i)\varphi^2 \le \int_{\varepsilon_i M}|\overline{\nabla}^i\varphi|^2=\int_{\varepsilon_i M}\left(|D^{s_i}\eta|_{s_i}^2\tau^2+\eta^2|\tau'|^2\right)\\
&\le\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\left(\int_{\varepsilon_i M\cap\varepsilon_i\Sigma\times\{t\}}|D^{s_i}\eta|_{s_i}^2\right)dt+\left(\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+1}+\int_{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\right)\left(\int_{\varepsilon_i M\cap\varepsilon_i\Sigma\times\{t\}}\eta^2\right)dt\\
&=\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\left(\int_{\varepsilon_i M\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\left(\frac{3-n}{2}\sin\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)+\frac{\pi}{\log\varepsilon}\cos\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\right)^2\rho_i^{1-n}\right)dt\\
&\quad+\left(\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+1}+\int_{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\right)\left(\int_{\varepsilon_i M\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\sin^2\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\rho_i^{3-n}\right)dt.
\end{aligned}$$

Denote E_i as (5.18). For any $z\in T_i\setminus E_i$ we have $|D^{s_i}\tilde{u}_{s_i}(z)|_{s_i}>\frac{1}{\varepsilon}$ and

$$|\langle D^{s_i}\tilde{u}_{s_i}(z),D^{s_i}\rho_i\rangle_{s_i}|\le\varepsilon^2\left|D^{s_i}\tilde{u}_{s_i}(z)\right|_{s_i}.$$

Note $Ric_{\varepsilon_i N}(\nu_i,\nu_i)=\left(1+|D^{s_i}\tilde{u}_{s_i}|_{s_i}^2\right)^{-1}Ric(D^{s_i}\tilde{u}_{s_i},D^{s_i}\tilde{u}_{s_i})$. Combining (5.27) we get

(5.29)
$$\begin{aligned}
&\int_{\varepsilon_i M}Ric_{\varepsilon_i N}(\nu_i,\nu_i)\varphi^2\ge\int_{\varepsilon_i M\cap\{\frac{1}{\delta}+1\le t\le\frac{1}{\delta}+\frac{2}{\varepsilon}-1\}}Ric_{\varepsilon_i N}(\nu_i,\nu_i)\eta^2\\
&\ge\int_{\frac{1}{\delta}+1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}\Bigg(\int_{(\varepsilon_i M\setminus E_i)\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\frac{1}{\rho_i^2}\frac{1}{1+|D^{s_i}\tilde{u}_{s_i}|_{s_i}^2}\Big(\kappa\left|(D^{s_i}\tilde{u}_{s_i})^T\right|_{s_i}^2\\
&\quad-c'\left|(D^{s_i}\tilde{u}_{s_i})^T\right|_{s_i}\langle D^{s_i}\tilde{u}_{s_i},D^{s_i}\rho_i\rangle_{s_i}\Big)\sin^2\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\rho_i^{3-n}\Bigg)dt\\
&\ge\int_{\frac{1}{\delta}+1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}\Bigg(\int_{(\varepsilon_i M\setminus E_i)\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\frac{|D^{s_i}\tilde{u}_{s_i}|_{s_i}^2}{1+|D^{s_i}\tilde{u}_{s_i}|_{s_i}^2}\Big(\kappa(1-\varepsilon^2)\\
&\quad-c'\varepsilon^2\Big)\sin^2\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\rho_i^{1-n}\Bigg)dt\\
&\ge\frac{\kappa(1-\varepsilon^2)-c'\varepsilon^2}{1+\varepsilon^2}\int_{\frac{1}{\delta}+1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}\left(\int_{(\varepsilon_i M\setminus E_i)\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\sin^2\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\rho_i^{1-n}\right)dt.
\end{aligned}$$

Due to $H^n(E_i) < \varepsilon^n$ in Lemma 5.3, it follows that

(5.30)

$$\begin{aligned}\int_{\varepsilon_i M} Ric_{\varepsilon_i N}(\nu_i,\nu_i)\varphi^2 \geq & \frac{\kappa-(\kappa+c')\varepsilon^2}{1+\varepsilon^2}\int_{\frac{1}{\delta}+1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}\Bigg(-\varepsilon^{1-n}H^n(E_i)\\ & + \int_{\varepsilon_i M\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\sin^2\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\rho_i^{1-n}\Bigg)dt\\ \geq & \frac{\kappa-(\kappa+c')\varepsilon^2}{1+\varepsilon^2}\int_{\frac{1}{\delta}+1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}\left(\int_{\varepsilon_i M\cap(B_1^i\setminus B_\varepsilon^i)\times\{t\}}\sin^2\left(\pi\frac{\log\rho_i}{\log\varepsilon}\right)\rho_i^{1-n}\right)dt\\ & -2\frac{1-\varepsilon}{1+\varepsilon^2}(\kappa-(\kappa+c')\varepsilon^2).\end{aligned}$$

Combining this with (5.28) we let $i\to\infty$ and obtain

(5.31)

$$\begin{aligned}&\frac{\kappa-(\kappa+c')\varepsilon^2}{1+\varepsilon^2}\int_{\frac{1}{\delta}+1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}\left(\int_{CY\cap(\mathscr{B}_1\setminus\mathscr{B}_\varepsilon)\times\{t\}}\sin^2\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\rho_\infty^{1-n}\right)dt\\ &\qquad -2\frac{1-\varepsilon}{1+\varepsilon^2}(\kappa-(\kappa+c')\varepsilon^2)\\ &\leq\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\left(\int_{CY\cap(\mathscr{B}_1\setminus\mathscr{B}_\varepsilon)\times\{t\}}\left(\frac{3-n}{2}\sin\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)+\frac{\pi}{\log\varepsilon}\cos\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\right)^2\rho_\infty^{1-n}\right)dt\\ &\quad+\left(\int_{\frac{1}{\delta}}^{\frac{1}{\delta}+1}+\int_{\frac{1}{\delta}+\frac{2}{\varepsilon}-1}^{\frac{1}{\delta}+\frac{2}{\varepsilon}}\right)\left(\int_{CY\cap(\mathscr{B}_1\setminus\mathscr{B}_\varepsilon)\times\{t\}}\sin^2\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\rho_\infty^{3-n}\right)dt,\end{aligned}$$

where ρ_∞ is the distance function on $\Sigma_\infty = CX$ from the fixed point o to the considered point. Letting $\delta\to 0$, and using (5.19) we get

(5.32)

$$\begin{aligned}&\frac{\kappa-(\kappa+c')\varepsilon^2}{1+\varepsilon^2}\int_{1}^{\frac{2}{\varepsilon}-1}\left(\int_{\partial P\cap(\mathscr{B}_1\setminus\mathscr{B}_\varepsilon)}\sin^2\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\rho_\infty^{1-n}\right)dt\\ &\qquad -2\frac{1-\varepsilon}{1+\varepsilon^2}(\kappa-(\kappa+c')\varepsilon^2)\\ &\leq\int_{0}^{\frac{2}{\varepsilon}}\left(\int_{\partial P\cap(\mathscr{B}_1\setminus\mathscr{B}_\varepsilon)}\left(\frac{3-n}{2}\sin\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)+\frac{\pi}{\log\varepsilon}\cos\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\right)^2\rho_\infty^{1-n}\right)dt\\ &\quad+\left(\int_{0}^{1}+\int_{\frac{2}{\varepsilon}-1}^{\frac{2}{\varepsilon}}\right)\left(\int_{\partial P\cap(\mathscr{B}_1\setminus\mathscr{B}_\varepsilon)}\sin^2\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\rho_\infty^{3-n}\right)dt.\end{aligned}$$

We calculate

(5.33)
$$\begin{aligned}\int_{\partial P\cap(\mathscr{B}_1\backslash\mathscr{B}_\varepsilon)}\sin^2\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\rho_\infty^{1-n} =& H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\int_\varepsilon^1\sin^2\left(\pi\frac{\log s}{\log\varepsilon}\right)\frac{1}{s}ds\\ =&\left(\log\frac{1}{\varepsilon}\right)H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\int_0^1\sin^2(\pi t)dt.\end{aligned}$$

Hence we have

(5.34)
$$\begin{aligned}&\frac{\kappa-(\kappa+c')\varepsilon^2}{1+\varepsilon^2}\left(\left(\frac{2}{\varepsilon}-2\right)\left(\log\frac{1}{\varepsilon}\right)H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\int_0^1\sin^2(\pi t)dt-2(1-\varepsilon)\right)\\ \le&\frac{2}{\varepsilon}\int_{\partial P\cap(\mathscr{B}_1\backslash\mathscr{B}_\varepsilon)}\left(\frac{3-n}{2}\sin\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)+\frac{\pi}{\log\varepsilon}\cos\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\right)^2\rho_\infty^{1-n}\\ &+2\int_{\partial P\cap(\mathscr{B}_1\backslash\mathscr{B}_\varepsilon)}\sin^2\left(\pi\frac{\log\rho_\infty}{\log\varepsilon}\right)\rho_\infty^{3-n}\\ \le&\frac{2}{\varepsilon}H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\int_\varepsilon^1\left(\frac{3-n}{2}\sin\left(\pi\frac{\log s}{\log\varepsilon}\right)+\frac{\pi}{\log\varepsilon}\cos\left(\pi\frac{\log s}{\log\varepsilon}\right)\right)^2\frac{1}{s}ds\\ &+2H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\int_\varepsilon^1\sin^2\left(\pi\frac{\log s}{\log\varepsilon}\right)sds\\ \le&2\left(\log\frac{1}{\varepsilon}\right)H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\int_0^1\left(\frac{1}{\varepsilon}\left(\frac{3-n}{2}\sin(\pi t)+\frac{\pi}{\log\varepsilon}\cos(\pi t)\right)^2+\sin^2(\pi t)\right)dt\\ =&2\left(\log\frac{1}{\varepsilon}\right)H^{n-2}(\partial P\cap\partial\mathscr{B}_1)\left(\frac{1}{\varepsilon}\left(\frac{(n-3)^2}{4}+\frac{\pi^2}{(\log\varepsilon)^2}\right)+1\right)\int_0^1\sin^2(\pi t)dt.\end{aligned}$$

Together with (5.20) the above inequality implies

$$\kappa\le\frac{(n-3)^2}{4}+o(\varepsilon),$$

where $\lim_{\varepsilon\to\infty}o(\varepsilon)=0$. Therefore we complete the proof. □

Finally, combining Theorem 4.8 we obtain a Liouville theorem for minimal graphic functions without growth condition.

Theorem 5.5. *Let* (Σ,σ) *be a complete* $n-$*dimensional Riemannian manifold satisfying conditions C1), C2), C3) and with its non-radial Ricci curvature satisfying* $\inf_{\partial B_\rho}Ric_\Sigma\left(\xi^T,\xi^T\right)\ge\kappa\rho^{-2}$ *for some constant* κ *for sufficiently large* $\rho>0$, *where* ξ *is a local vector field on* Σ *with* $|\xi^T|=1$ *defined in* (5.26). *If* $\kappa>\frac{(n-3)^2}{4}$, *then any entire solution to* (2.1) *on* Σ *must be a constant.*

The number $\frac{(n-3)^2}{4}$ in Theorem 5.5 is sharp, and we will construct examples to show this in the following section.

6 Nontrivial entire minimal graphs in product manifolds

Let Σ be an Euclidean space $\mathbb{R}^{n+1}$ with a conformally flat metric

$$ds_\phi^2 = e^{\phi(r)}\sum_{i=1}^{n+1} dx_i^2,$$

where $r = |x| = \sqrt{x_1^2 + \cdots + dx_{n+1}^2}$ and $\phi(|x|)$ is smooth in $\mathbb{R}^{n+1}$. Hence Σ is a smooth manifold. Set $\tilde{\phi}(r) = \int_0^r e^{\frac{\phi(r)}{2}} dr$. Let us define $\rho = \tilde{\phi}(r)$ and $\lambda(\rho) = r\tilde{\phi}'(r)$, then the Riemannian metric in Σ can be written in polar coordinates as

$$ds^2 = d\rho^2 + \lambda^2(\rho)d\theta^2,$$

where $d\theta^2$ is the standard metric on $\mathbb{S}^n(1)$. We assume $0 < \lambda' \le 1$, $\lambda'' \le 0$,

(6.1)
$$\lim_{\rho\to\infty}\frac{\lambda(\rho)}{\rho} = \kappa, \qquad \lim_{\rho\to\infty}\left(\rho^2\frac{1-(\lambda'(\rho))^2}{\lambda^2(\rho)}\right) = \frac{1-\kappa^2}{\kappa^2}, \qquad \lim_{\rho\to\infty}\left(\rho^2\frac{\lambda''(\rho)}{\lambda(\rho)}\right) = 0.$$

From [21], there are examples satisfying the above conditions for every $\kappa \in (0,1]$. Clearly, $\lim_{r\to\infty}\frac{1}{r}\Sigma = CS_\kappa$ in the Gromov-Hausdorff measure, where S_κ is an $n-$sphere in $\mathbb{R}^{n+1}$ with radius $0 < \kappa \le 1$, namely,

$$S_\kappa = \{(x_1,\cdots,x_{n+1}) \in \mathbb{R}^{n+1} |\ x_1^2 + \cdots + x_{n+1}^2 = \kappa^2\}.$$

Moreover, let $\{e_\alpha\}_{\alpha=1}^n \bigcup \{\frac{\partial}{\partial\rho}\}$ be an orthonormal basis at the considered point of Σ. we calculate the sectional curvature and Ricci curvature of Σ as follows (see Appendix A in [33] for instance).

$$\begin{aligned}
&K_\Sigma\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = -\frac{\lambda''}{\lambda}, \quad K_\Sigma(e_\alpha, e_\beta) = \frac{1-(\lambda')^2}{\lambda^2},\\
&Ric_\Sigma\left(\frac{\partial}{\partial\rho}, e_\alpha\right) = 0, \quad Ric_\Sigma\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\right) = -n\frac{\lambda''}{\lambda},\\
&Ric_\Sigma(e_\alpha, e_\beta) = \left((n-1)\frac{1-(\lambda')^2}{\lambda^2} - \frac{\lambda''}{\lambda}\right)\delta_{\alpha\beta}.
\end{aligned} \tag{6.2}$$

In particular, $Ric_\Sigma \ge 0$ and $\lim_{\rho\to\infty}\left(\rho^2 Ric_\Sigma(e_\alpha, e_\beta)\right) = \frac{(n-2)^2}{4}\delta_{\alpha\beta}$ if $\kappa = \frac{2}{n}\sqrt{n-1}$.

In theorem 3.4 of [21], we have showed that if $n \ge 3$ and

$$\frac{2}{n}\sqrt{n-1} \le \kappa < 1,$$

then any hyperplane through the origin in Σ is area-minimizing. Now we denote $T = \{(x_1,\cdots,x_{n+1}) \in \mathbb{R}^{n+1} |\ x_{n+1} = 0\}$ in CS_κ or $\frac{1}{r}\Sigma$ for $r > 0$, and their induced metrics are determined by the ambient spaces. We will construct an entire minimal graph with non-constant graphic function in $\Sigma\times\mathbb{R}$ for every $\kappa \in [\frac{2}{n}\sqrt{n-1}, 1)$, which obviously implies that the number $\frac{(n-3)^2}{4}$ in Theorem 5.5 is sharp.

Let D be the Levi-Civita connection of Σ. Let $\{E_i\}_{i=1}^{n+1}$ be the dual vectors of $\{dx_i\}_{i=1}^{n+1}$. Let Γ_{ij}^k be the Christoffel symbols of Σ with respect to the frame E_i, i.e., $D_{E_i}E_j = \sum_k \Gamma_{ij}^k E_k$. Set $u^i = \sigma^{ij}u_j$, $|Du|^2 = \sigma^{ij}u_iu_j$, $D_iD_ju = u_{ij} - \Gamma_{ij}^k u_k$ and $v = \sqrt{1+|Du|^2}$. We introduce an operator $\mathfrak{L}$ on a domain $\Omega \subset \Sigma$ by

$$\mathfrak{L}F = \left(1+|DF|^2\right)^{\frac{3}{2}} \mathrm{div}_\Sigma \left(\frac{DF}{\sqrt{1+|DF|^2}}\right) = \left(1+|DF|^2\right)\Delta_\Sigma F - F_{i,j}F^iF^j, \tag{6.3}$$

where $F^i = \sigma^{ik}F_k$, and $F_{i,j} = F_{ij} - \Gamma_{ij}^k F_k$ is the covariant derivative.

Let $p = \frac{n}{2}\kappa - \sqrt{\frac{n^2\kappa^2}{4} - (n-1)} \ge \frac{1}{\kappa}$, then by Theorem 1.5 in [42] there is a solution $u_j \in C^\infty(B_j)$ to the Dirichlet problem

$$\begin{cases} \mathfrak{L}u_j = 0 & \text{in } B_j \\ u_j = c_jx_{n+1}r^{p-1} & \text{on } \partial B_j \end{cases}, \tag{6.4}$$

where c_j is a positive constant and $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. By symmetry, $u_j(x', x_{n+1}) + u_j(x', -x_{n+1}) = 0$ on B_j with $x' = (x_1, \cdots, x_n)$. By [21] and maximum principle, we have

$$|u_j| \ge c_j|x_{n+1}|r^{p-1} \ \text{ in } B_j. \tag{6.5}$$

If w_j is a solution of (6.4) with boundary $d_jx_{n+1}r^{p-1}$ and $0 < d_j < c_j$, we have $|u_j| > |w_j|$ on $B_j \cap \{x_{n+1} \ne 0\}$. By the uniqueness of the solution of (6.4) there is a $c_j > 0$ such that $\sup_{B_1}|Du_j| = 1$.

Let $\Gamma_j(s) = \{x \in B_j | \, u_j(x) = s\}$. We claim $u_j(x',t) \ge u_j(x',s)$ for all $(x',t),(x',s) \in B_j$ and $t > s$. If not, without loss of generality there are $t_1 < t_2 < t_3$ so that $u_j(x',t_1) = u_j(x',t_2) = \tau_j < u_j(x',t_3)$. It is not hard to see that there is a closed curve $\vartheta_{\tau_j} \subset \Gamma_j(\tau_j)$ in the half plane $\{(s,x_{n+1}) \in \mathbb{R}^2 | \; s = |x'| \ge 0\}$ such that $u_j(z) > \tau_j$ for every $z \in U_j$ and U_j is a domain in $\{(s,x_{n+1}) \in \mathbb{R}^2 | \; s = |x'| \ge 0\}$, which is enclosed by ϑ_{τ_j}. By the symmetry of u_j, rotating ϑ_{τ_j} on $x' = (x_1, \cdots, x_n)$ generates an n-dimensional set $\widetilde{\Gamma}_j(\tau_j) \subset \Gamma_j(\tau_j)$, which encloses a domain $\widetilde{U}_j \subset B_j$. Then $u_j(x) > \tau_j$ for every $x \in \widetilde{U}_j$. But this is impossible as graph_{u_j} is area-minimizing. Hence $\frac{\partial u_j}{\partial x_{n+1}} \ge 0$.

There is a subsequence $\{j'\}$ of $\{j\}$ such that $u_{j'}$ converges to a function u defined on $\Omega \subset \Sigma$ by varifold convergence, where graph_u is also area-minimizing. By the symmetry of u, we deduce that Ω is symmetric with respect to $x_1, x_2, \cdots, x_n$, and Ω is symmetric with respect to x_{n+1}. Clearly, u is smooth, $\mathfrak{L}u = 0$ and $\lim_{x \to \partial\Omega^\pm \setminus T} u(x) = \pm\infty$, where $\Omega^\pm = \Omega \cap \{\pm x_{n=1} > 0\}$. In particular, $\Omega^- = \{(x', -x_{n+1}) | \; (x', x_{n+1}) \in \Omega^+\}$, $u(x', x_{n+1}) + u(x', -x_{n+1}) = 0$, $Du(0) = 0$ and $\sup_{B_1}|Du| = 1$. Moreover, $u(x',t) \ge u(x',s)$ for all $(x',t),(x',s) \in \Omega$ and $t > s$.

We want to show $\Omega = \Sigma$. If not, $\partial\Omega \ne \emptyset$ and both of $\partial\Omega^\pm$ are area-minimizing hypersurfaces in Σ. Since also $\partial\Omega^+$ is symmetric with respect to $x_1, x_2, \cdots, x_n$, then $\partial\Omega^+$ must be a graph on a domain of the half sphere, or else we cannot have an area-minimizing hypersurface $\partial\Omega^+$. If w is the graphic function of $\partial\Omega^+$ on a closed half of unit sphere $\mathbb{S}^n$, and $\varphi = \int_1^w \frac{1}{\lambda(s)} ds$, then φ satisfies the following elliptic equation (see formula (2.9) in [20] for instance)

$$\Delta_{\mathbb{S}}\varphi - \frac{1}{1+|\nabla_{\mathbb{S}}\varphi|^2}\mathrm{Hess}_{\mathbb{S}}\varphi(\nabla_{\mathbb{S}}\varphi,\nabla_{\mathbb{S}}\varphi) - n\lambda'(w) = 0, \tag{6.6}$$

where $\Delta_{\mathbb{S}}$, $\mathrm{Hess}_{\mathbb{S}}$ and $\nabla_{\mathbb{S}}$ are Laplacian, Hessian and Levi-Civita connection of $\mathbb{S}^n$, respectively. By the definition of λ and regularity of elliptic equations, $\partial\Omega^+$ is a smooth hypersurface in Σ. $\frac{1}{r}\partial\Omega^+$ is also an area-minimizing hypersurface in $\frac{1}{r}\Sigma$, then $\lim_{r\to\infty}\frac{1}{r}\partial\Omega^+$ will converge to an area-minimizing cone CX over X in CS_κ, where X is contained in a closed half sphere. Hence X must be an equator and CX is just a hyperplane T.

Let $S = \partial\Omega^+$. The second variation formula implies that there is a Jacobi field operator L_S given by

$$L_S h = \Delta_S h + \left(|B|^2 + Ric_\Sigma(\nu,\nu)\right) h, \tag{6.7}$$

where Δ_S, B are the Laplacian and second fundamental form for S relative to the metric σ, and Ric_Σ is the Ricci curvature of Σ relative to σ. Let $\mathbb{S}^{n-1}$ be an $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$. Since $\frac{1}{t}S$ converges to $T = \mathbb{R}_+ \times_{\kappa\rho} \mathbb{S}^{n-1}$ as $t\to\infty$, in terms of the coordinates $(\rho,\alpha)\in(0,\infty)\times\mathbb{S}^{n-1}$ we consider

$$\begin{aligned} L_T h =& \Delta_T h + \frac{n-1}{\rho^2}\left(\frac{1}{\kappa^2}-1\right)h \\ =& \frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial h}{\partial\rho}\right) + \frac{1}{\kappa^2\rho^2}\Delta_{\mathbb{S}^{n-1}}h + \frac{n-1}{\rho^2}\left(\frac{1}{\kappa^2}-1\right)h \\ =& \frac{1}{\rho^2}\left(\rho^2\frac{\partial^2}{\partial\rho^2} + (n-1)\rho\frac{\partial}{\partial\rho} + \frac{1}{\kappa^2}\Delta_{\mathbb{S}^{n-1}} + (n-1)\left(\frac{1}{\kappa^2}-1\right)\right)h. \end{aligned} \tag{6.8}$$

The only positive solutions of $L_T h = 0$ on T are

$$h = \bar{c}_1\rho^{-\lambda_-} + \bar{c}_2\rho^{-\lambda_+}, \tag{6.9}$$

where $\bar{c}_1,\bar{c}_2$ are constants and $\lambda_\pm = \frac{n-2}{2} \pm \sqrt{\frac{n^2}{4} - \frac{n-1}{\kappa^2}} > 0$. By [6, 29], the equation $L_T w = f_0$ with $|f_0| \le c\rho^{-2-\lambda_- -\delta}$, $\delta > 0$ has a nonnegative solution

$$w = (\bar{c}_3 + \bar{c}_4\log\rho)\rho^{-\lambda_-} + O(\rho^{-\lambda_- -\delta'})(\delta'>0) \qquad \text{as } \rho\to\infty, \tag{6.10}$$

where $\bar{c}_3,\bar{c}_4$ are constants, and $\bar{c}_4 = 0$ in case $\lambda_+ > \lambda_-$, namely, $\kappa > \frac{2}{n}\sqrt{n-1}$.

Now we see S as a graph $\{(x', f(x'))|\, x' \in T\setminus K\}$ outside some bounded domain in $(\mathbb{R}^{n+1}, ds_\phi^2)$, where $x' = (x_1,\cdots,x_n)$, and

$$f(x') = \int_0^{h(x')} e^{-\frac{1}{2}\phi(\sqrt{|x'|^2+s^2})} ds$$

with $|x'|^2 = x_1^2 + \cdots + x_n^2$. Let $\gamma_{\rho\alpha}$ be a unit normal smooth line in the upper plane of Σ which corresponds to $\{(x', x_{n+1}) \in \mathbb{R}^{n+1} |\, x_{n+1} \geq 0,\ x' = x_\alpha\} \subset (\mathbb{R}^{n+1}, ds_\phi^2)$ with x_α being $\rho\alpha$ in the polar coordinate and $\gamma_{\rho\alpha}(0) \in T$. Now $S = \{\gamma_{\rho\alpha}(w(\rho\alpha)) |\, \rho\alpha \in T\setminus K\}$, where $w(\rho\alpha) = h(x')$ if x' is $\rho\alpha$ in the polar coordinate. We can embed isometrically Σ into $(n+2)$-dimensional Euclidean space, then S is an n-dimensional submanifold in $\mathbb{R}^{n+2}$ with mean curvature decaying as $O(\frac{1}{r})$. Hence by the Allard regularity theorem (see [1] or [40]), for any $\varepsilon > 0$ there is a $\rho_0 > 0$ such that

$$\rho^{-1}|w(\rho\alpha)| + |\nabla_T w(\rho\alpha)| \leq \varepsilon \qquad \text{for every } \rho \geq \rho_0,\ \alpha \in \mathbb{S}^{n-1},$$

where ∇_T is the Levi-Civita connection of T with induced metric in CS_κ. Since S is a graph with graphic function f outside a compact set in $(\mathbb{R}^{n+1}, ds_\phi^2)$, then the mean curvature is $H = \frac{n}{2}\frac{\phi'(r_f)}{r_f} X^N$ with $r_f = \sqrt{|x'|^2 + f^2(x')}$, and $(\cdots)^N$ is the projection onto the normal bundle NS, namely, f satisfies (after a simple computation)

$$\sum_{i,j=1}^{n} g^{ij} f_{ij} = \frac{n}{2}\frac{\phi'}{r_f}\left(-\sum_{i=1}^{n} x_i f_i + f\right), \tag{6.11}$$

where $g_{ij} = \delta_{ij} + f_i f_j$ and (g^{ij}) is the inverse matrix of (g_{ij}). Then by the estimates (6.10) and the Schauder estimates we obtain

$$\tilde{r}^{-1}|f(x')| + |\nabla_{\mathbb{R}} f(x')| + \tilde{r}|\nabla^2_{\mathbb{R}} f(x')| \leq C\tilde{r}^{-\varepsilon} \tag{6.12}$$

with $\tilde{r} = \sqrt{x_1^2 + \cdots + x_n^2}$ and some $\varepsilon, C > 0$, where $\nabla_{\mathbb{R}}$ is the standard Levi-Civita connection of Euclidean space. By the definition of w and applying a coordinate transformation, we deduce

$$\rho^{-1}|w(\rho\alpha)| + |\nabla_T w(\rho\alpha)| + \rho|\nabla^2_T w(\rho\alpha)| \leq C'\rho^{-\varepsilon'} \tag{6.13}$$

for some $\varepsilon', C' > 0$. In particular, w satisfies (6.10).

The graph$_u$ has a unique tangent cone at infinity: a cylinder $T \times \mathbb{R}$. Let $\Gamma_y = \Gamma_y(u) = \{x \in \Sigma |\ u(x) = y\}$ for any $y \in \mathbb{R}$. Since $\lim_{x \to \partial\Omega^\pm \setminus T} u(x) = \pm\infty$ and $u(x', t) \geq u(x', s)$ for each $(x', t), (x', s) \in \Omega$ and $t > s$, we deduce $\text{dist}(\Gamma_{y_2}, 0) \geq \text{dist}(\Gamma_{y_1}, 0) > 0$ for $|y_2| \geq |y_1|$ and sufficiently large $|y_1|$. Moreover, for any $\varepsilon > 0$ there is $y_0 = y_0(\varepsilon)$ such that Γ_y is within ε of S for any $y \geq y_0$. Now we use $\rho\omega$ to represent the element of Σ with metric $\sigma = d\rho^2 + \lambda^2(\rho)d\theta^2$, and claim that

$$|Du(\rho\omega)| \to \infty \qquad \text{as} \qquad |u(\rho\omega)| + \rho \to \infty,\ (\rho, \omega) \in ((0, \infty) \times_\lambda \mathbb{S}^n) \cap \Omega. \tag{6.14}$$

If we embed Σ into $\mathbb{R}^{n+2}$ isometrically, then graph$_u$ is an $(n+1)$-dimensional submanifold in $\mathbb{R}^{n+3}$ with codimension 2. We check that the Allard regularity theorem still works in our case. Invoking elliptic regularity theory, if the minimal hypersurfaces M_k converge to a cone C in varifold sense, then the convergence is C^2 near regular points of C. For any $\mu_k, \lambda_k \to \infty$ it is clear that graph$_{u-\mu_k} = \{(x, u(x) - \mu_k) |\ x \in \Sigma\}$ converges to $S \times \mathbb{R} \subset \Sigma \times \mathbb{R}$ in the varifold sense and

$\frac{1}{\lambda_k}\left(\mathrm{graph}_{u-\mu_k}\right)$ converges to $T\times\mathbb{R}\subset CS_\kappa\times\mathbb{R}$ in the varifold sense. So we can show the above claim by C^2 convergence (see also the proof of Theorem 4 in [41]).

Denote $\widetilde{T}=T\times\mathbb{R}$. In terms of the coordinates $(\rho,\alpha,y)\in(0,\infty)\times\mathbb{S}^{n-1}\times\mathbb{R}$ we can write the operator $L_{\widetilde{T}}$ as

$$
\begin{aligned}
L_{\widetilde{T}}h=&\Delta_T h+\frac{\partial^2 h}{\partial y^2}+\frac{n-1}{\rho^2}\left(\frac{1}{\kappa^2}-1\right)h\\
=&\frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial h}{\partial\rho}\right)+\frac{\partial^2 h}{\partial y^2}+\frac{1}{\kappa^2\rho^2}\Delta_{\mathbb{S}^{n-1}}h+\frac{n-1}{\rho^2}\left(\frac{1}{\kappa^2}-1\right)h.
\end{aligned}
\tag{6.15}
$$

Let

$$
v(\rho,y)=\rho^{\lambda_-}\int_{\mathbb{S}^{n-1}}h(\rho\alpha,y)d\alpha\qquad\text{and}\qquad f(\rho,y)=\rho^{\lambda_-}\int_{\mathbb{S}^{n-1}}\tilde f(\rho\alpha,y)d\alpha
$$

with $\lambda_-=\frac{n-2}{2}-\sqrt{\frac{n^2}{4}-\frac{n-1}{\kappa^2}}>0$, then $L_{\widetilde{T}}=\tilde f$ implies that

$$
\rho^{-1-\beta}\frac{\partial}{\partial\rho}\left(\rho^{1+\beta}\frac{\partial v}{\partial\rho}\right)+\frac{\partial^2 v}{\partial y^2}=f \tag{6.16}
$$

with $\beta=2\sqrt{\frac{n^2}{4}-\frac{n-1}{\kappa^2}}$. The left of the above equation is a uniform elliptic operator for $\rho\ge c$ with any positive constant c.

By the Allard regularity theorem, there are a constant $\rho_1>0$ and a domain

$$
G=\{(\rho\alpha,y)|\ \rho\ge\rho_1,\ y\in\mathbb{R},\ \rho\alpha\in T\}\subset T\times\mathbb{R}=\widetilde{T}
$$

such that graph_u can be written as a graph in G with graphical function W outside some compact set $\widetilde{K}$ in $\Sigma\times\mathbb{R}$. Namely, $\mathrm{graph}_u\setminus\widetilde{K}=\mathrm{graph}_W=\{(\gamma_{\rho\alpha}(W(\rho\alpha,y)),y)|\ \rho\alpha\in T,\ \rho\ge\rho_1,\ y\in\mathbb{R}\}$ and $\gamma_{\rho\alpha}$ is defined as before. Similar to (6.13) we have

$$
\rho^{-1}|W(\rho\alpha)|+|\nabla_{\widetilde{T}}W(\rho\alpha)|+\rho|\nabla^2_{\widetilde{T}}W(\rho\alpha)|\le C\rho^{-\delta} \tag{6.17}
$$

for some $\delta,C>0$, where $\nabla_{\widetilde{T}}$ is the Levi-Civita connection of $\widetilde{T}$ with induced metric in $CS_\kappa\times\mathbb{R}$. Then by Theorem 1 in [41], we obtain for any $\varepsilon\in(0,1)$

$$
|y|^{\varepsilon}\rho^{\lambda_-}\frac{\partial W}{\partial y}(\rho\alpha,y)\ge C_2\quad\text{for all } y\in\mathbb{R},\ \alpha\in\mathbb{S}^{n-1},\ y\ge\rho\ge C_1, \tag{6.18}
$$

where C_1,C_2 are constants independent of y and ρ. It is clear that

$$
\frac{\partial W}{\partial y}(\rho\alpha,y)=\frac{1}{|Du(\xi)|} \tag{6.19}
$$

where $(\rho\alpha,y)\in G$, $y=u(\xi)$ and $\xi=\gamma_{\rho\alpha}(W(\rho\alpha,y))$. Fix ρ, we have $|Du(\xi)|\le C_3|u(\xi)|^{\varepsilon}$ with constant C_3 depending on ρ. Hence $|D(u(\xi))^{1-\varepsilon})|$ is bounded when ξ approaches S in $\gamma_{\rho\alpha}$, which contradicts $\lim_{x\to\partial\Omega^\pm\setminus T}u(x)=\pm\infty$. Therefore, we deduce $\Omega=\Sigma$, namely, we get a smooth entire minimal graph $\{(x,u(x))|\ x\in\Sigma\}$ on Σ.

Theorem 6.1. *Let Σ be an $(n+1)$-dimensional Riemannian manifold described in the front of this section. If $n \geq 3$ and $\frac{2}{n}\sqrt{n-1} \leq \kappa < 1$, then there exists a smooth entire minimal graph $\{(x, u(x)) | \, x \in \Sigma\}$ in $\Sigma \times \mathbb{R}$, where u is not a constant.*

Appendix

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