THE RIGIDITY THEOREMS FOR LAGRANGIAN SELF SHRINKERS

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Abstract. By the integral method we prove that any space-like entire graphic self-shrinking solution to Lagrangian mean curvature flow in $\mathbb{R}^{2n}_n$ with the indefinite metric $\sum_i dx_i dy_i$ is flat. This result improves the previous ones in [9] and [1] by removing the additional assumption in their results. In a similar manner, we reprove its Euclidean counterpart which is established in [1].

1. Introduction

Let M be a submanifold in $\mathbb{R}^{m+n}$. Mean curvature flow is a one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \to \mathbb{R}^{m+n}$ with corresponding images $M_t = X_t(M)$ such that

$$\begin{cases} \dfrac{d}{dt} X(x,t) = H(x,t), & x \in M \\ X(x,0) = X(x) \end{cases}$$

is satisfied, where $H(x,t)$ is the mean curvature vector of M_t at $X(x,t)$ in $\mathbb{R}^{m+n}$.

An important class of solutions to the above mean curvature flow equations are self-similar shrinkers, whose profiles, self-shrinkers, satisfy a system of quasi-linear elliptic PDE of the second order

$$H = -\frac{X^N}{2}, \tag{1.1}$$

where $(\cdots)^N$ stands for the orthogonal projection into the normal bundle NM.

In the ambient pseudo-Euclidean space we can also study the mean curvature flow (see [5] [6] [7] [11] and [8], for example). And self-shrinking graphs with high codimensions in pseudo-Euclidean space has been studied in [3]. Let $\mathbb{R}^{2n}_n$ be Euclidean space with null coordinates $(x, y) = (x_1, \cdots, x_n; y_1, \cdots, y_n)$, which means that the indefinite metric is defined by $ds^2 = \sum_i dx_i dy_i$. If $M = \{(x, Du(x)) \big| \ x \in \mathbb{R}^n\}$ is a space-like submanifold in $\mathbb{R}^{2n}_n$, then u is convex (In this paper, we say that a smooth function f is *convex*, if

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$D^2 f > 0$, i.e., hessian of f is positive definite in $\mathbb{R}^n$). The underlying Euclidean space $\mathbb{R}^{2n} = \mathbb{C}^n$ of $\mathbb{R}^{2n}_n$ has the usual complex structure. It is easily seen that M is a Lagrangian submanifold in $\mathbb{R}^{2n}$ ([10], Lemma 5.2.11), as well as in $\mathbb{R}^{2n}_n$. Moreover, if M is also a self-shrinker, namely, the convex function u satisfies (1.1). It has been shown that up to an additive constant u satisfies the elliptic equation (see [1][8][9])

$$\log\det D^2 u(x) = \frac{1}{2} x \cdot Du(x) - u(x). \tag{1.2}$$

Huang-Wang [9] and Chau-Chen-Yuan [1] have investigated the entire solutions to the above equation and showed that an entire smooth convex solution to (1.2) in $\mathbb{R}^n$ is the quadratic polynomial under the decay condition on Hessian of u.

In [1], Chau-Chen-Yuan introduce a natural geometric quantity $\phi = \log\det D^2 u$ which obeys a second order elliptic equation with an "amplifying force". Based on it, we consider an important operator: the drift Laplacian operator $\mathcal{L}$, which was introduced by Colding-Minicozzi [2], and we can also write the second order equation for ϕ in [1] as $\mathcal{L}\phi = 0$. This enables us to apply integral method to prove any entire smooth proper convex solution to (1.2) in $\mathbb{R}^n$ is the quadratic polynomial, Theorem 2.3, where the case $n = 1$ is simple.

It is worth to note that when ϕ is constant the mean curvature of M vanishes (see (8.5.7) of Chap. VIII in [10]), namely, the gradient graph of a solution u to (1.2) defines a space-like minimal Lagrangian submanifold in $\mathbb{R}^{2n}_n$.

By thoroughly analysing the convexity of u, we could prove that any solution of (1.2) is proper, which is showed in Theorem 2.6. Thus, we remove the additional condition of the corresponding results in [9] and [1]. Precisely, we obtain

Theorem 1.1. *Any entire smooth convex solution $u(x)$ to* (1.2) *in $\mathbb{R}^n$ is the quadratic polynomial* $u(0) + \frac{1}{2}\langle D^2 u(0)x, x\rangle$.

We also consider the corresponding problem in ambient Euclidean space: a Lagrangian graph $\{(x, Du(x)) \big| \; x \in \mathbb{R}^n\}$ in $\mathbb{R}^{2n}$ satisfying (1.1). Now, u is an entire solution to the following equation:

$$\arctan\lambda_1(x) + \cdots + \arctan\lambda_n(x) = \frac{1}{2} x \cdot Du(x) - u(x) \tag{1.3}$$

where $\lambda_1(x), \cdots, \lambda_n(x)$ are the eigenvalues of the Hessian D^2u of u at $x \in \mathbb{R}^n$. Chau-Chen-Yuan [1] constructed a barrier function to show that the phase function

$$\Theta = \arctan \lambda_1(x) + \cdots + \arctan \lambda_n(x)$$

on this Lagrangian graph is a constant via the maximum principles. A geometric meaning of the phase function is the summation of the all Jordan angles of the Gauss map $\gamma : M \to \mathbf{G}_{n,n}$ (see [10] Chap 7, for example). They proved the following theorem.

Theorem 1.2. *If $u(x)$ is an entire smooth solution to* (1.3) *in $\mathbb{R}^n$, then $u(x)$ is the quadratic polynomial $u(0) + \frac{1}{2}\langle D^2u(0)x, x\rangle$.*

We could also derive the phase function satisfies: $\mathcal{L}\Theta = 0$. This enables us to use the integral method to reprove the above rigidity theorem.

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2. Space-like Lagrangian self-shrinkers in pseudo-Euclidean space

Let $M = \{(x, Du(x))\big|\ x \in \mathbb{R}^n\}$ be a space-like submanifold satisfying (1.2) in ambient space $\mathbb{R}^{2n}_n$ with the induced metric $g_{ij}dx_idx_j$, where $Du = (u_1, u_2, \cdots, u_n)$. Then $g_{ij} = \partial_i\partial_j u = u_{ij}$, and let (g^{ij}) denote the inverse matrix (g_{ij}). We write $g = \det g_{ij}$ for simplicity and $\xi \cdot \eta = \langle \xi, \eta\rangle$ for any vectors $\xi, \eta \in \mathbb{R}^n$. By (1.2), we have

$$\partial_j(\log g) = \frac{1}{2}u_j + \frac{1}{2}x_iu_{ij} - u_j = \frac{1}{2}x_iu_{ij} - \frac{1}{2}u_j, \tag{2.1}$$

and

$$\begin{aligned}\partial_i(\sqrt{g}g^{ij}) =& \frac{1}{2}\sqrt{g}g^{kl}\partial_i g_{kl}g^{ij} - \sqrt{g}g^{ki}\partial_i g_{kl}g^{lj} \\ =& -\frac{1}{2}\sqrt{g}g^{kl}u_{kli}g^{ij} = -\frac{1}{2}\sqrt{g}g^{ij}\partial_i(\log g).\end{aligned} \tag{2.2}$$

Let $\mathcal{L}$ be a differential operator defined by

$$\mathcal{L}\phi = \frac{1}{\sqrt{g}}e^{\frac{1}{4}x\cdot Du}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{g}e^{-\frac{1}{4}x\cdot Du}\frac{\partial}{\partial x_j}\phi\right),$$

for any function $\phi \in C^2(\mathbb{R}^n)$. Combining (2.1) and (2.2), we have

$$\begin{aligned}\mathcal{L}\phi =&\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\phi_j) + e^{\frac{1}{4}x\cdot Du}\partial_i(e^{-\frac{1}{4}x\cdot Du})g^{ij}\phi_j\\ =&g^{ij}\phi_{ij} + \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g})\phi_j - \frac{1}{4}(u_i + x_k u_{ki})g^{ij}\phi_j\\ =&g^{ij}\phi_{ij} - \frac{1}{4}(x_k u_{ki} - u_i)g^{ij}\phi_j - \frac{1}{4}(u_i + x_k u_{ki})g^{ij}\phi_j\\ =&g^{ij}\phi_{ij} - \frac{1}{2}x_k u_{ki} g^{ij}\phi_j\\ =&g^{ij}\phi_{ij} - \frac{1}{2}x_j\phi_j.\end{aligned} \tag{2.3}$$

Remark 2.1. *The submanifold M in $\mathbb{R}^{2n}_n$ is defined by $(\mathbb{R}^n, ds^2 = u_{ij}dx_i dx_j)$. The operator $\mathcal{L}$ is also defined on M. It is precisely the drift Laplacian $\mathcal{L}$ in the version of space-like self-shrinkers in pseudo-Euclidean space, which was introduced by Colding-Minicozzi* [2] *in the ambient Euclidean space.*

Lemma 2.2. *Let Ω be a convex domain in $\mathbb{R}^n (n \geq 2)$ and u be a smooth proper convex function in Ω, then for any $\alpha > 0$*

$$\int_\Omega |x \cdot Du| e^{-\alpha u} dx < +\infty.$$

Proof. Let $\Gamma_t = \{x \in \Omega |\ u(x) = t\}$ and $\Omega_t = \{x \in \Omega |\ u(x) < t\}$ for each $t \in \mathbb{R}$. By the convexity of u, we know that $\Gamma_t \cap L$ contains two point at most, where L is any line in $\mathbb{R}^n$. Since u is proper, then Γ_t is homotopic to $(n-1)$-sphere in $\mathbb{R}^n$, which implies Ω_t is a bounded domain enclosed by Γ_t. Thus, $\inf_{x\in\Omega} u(x) > -\infty$ and $\lim_{x\to\partial\Omega} u(x) = +\infty$. By translating Ω in the plane $\mathbb{R}^n$, we can assume $0 \in \Omega$ and $u(0) = \inf_{x\in\Omega} u(x)$. Moreover, by the convexity of u, there exist constants $C, \delta > 0$ such that for any $x \in \Omega$

$$u(x) + C \geq \delta |x|. \tag{2.4}$$

It suffices to show

$$\int_\Omega |Du| e^{-\beta u} dx < +\infty \tag{2.5}$$

holds for some $0 < \beta < \alpha$.

Set $x' = (x_1, \cdots, x_{n-1})$. Let

$$\Omega' = \{x' \in \mathbb{R}^{n-1} |\ \exists x_n\ s.t.\ (x', x_n) \in \Omega\}.$$

For every fixed $x' \in \Omega'$, $u_{nn}(x', x_n) = \partial_{x_n}\partial_{x_n} u(x', x_n)$ is positive, and $u_n(x', x_n)$ is monotonic increasing in x_n. Since $\lim_{x\to\partial\Omega} u(x) = +\infty$, then there is x_n^* such that $(x', x_n^*) \in \Omega$

and $u_n(x', x_n^*) = 0$. Furthermore, we have $x_n^1, x_n^2 \in [-\infty, +\infty]$ satisfying $x_n^1 < x_n^2$ and $(x', x_n^i) \in \partial\Omega$ for $i = 1, 2$.

For each fixed $x' \in \Omega'$, we have

$$\begin{aligned}\int_{(x',x_n)\in\Omega} |u_n| e^{-\beta u} dx_n &= -\int_{x_n^1}^{x_n^*} u_n(x', x_n) e^{-\beta u(x',x_n)} dx_n + \int_{x_n^*}^{x_n^2} u_n(x', x_n) e^{-\beta u(x',x_n)} dx_n \\ &= \frac{1}{\beta}\int_{x_n^1}^{x_n^*} de^{-\beta u(x',x_n)} - \frac{1}{\beta}\int_{x_n^*}^{x_n^2} de^{-\beta u(x',x_n)} \\ &= \frac{2}{\beta} e^{-\beta u(x',x_n^*)}.\end{aligned} \tag{2.6}$$

Since $u(x', x_n^*) + C \geq \delta\sqrt{|x'|^2 + (x_n^*)^2} \geq \delta|x'|$, then by (2.6),

$$\begin{aligned}\int_\Omega |u_n| e^{-\beta u} dx &= \int_{x'\in\Omega'} \int_{(x',x_n)\in\Omega} |u_n| e^{-\beta u} dx_n dx' = \int_{x'\in\Omega'} \frac{2}{\beta} e^{-\beta u(x',x_n^*)} dx' \\ &\leq \int_{x'\in\Omega'} \frac{2}{\beta} e^{\beta C - \beta\delta|x'|} dx' < \infty.\end{aligned} \tag{2.7}$$

By the same way to $\{u_i\}$ for $i = 1, \cdots, n-1$, we know (2.5) holds. This shows the Lemma. □

Theorem 2.3. *If Ω is a convex domain containing the origin in $\mathbb{R}^n (n \geq 2)$ and $u(x)$ is a smooth proper convex solution to (1.2) in Ω, then Ω is $\mathbb{R}^n$ and $u(x)$ is the quadratic polynomial $u(0) + \frac{1}{2}\langle D^2u(0)x, x\rangle$.*

Proof. Let $\phi = \log g$, then by (2.1), $\phi_{ij} = \frac{1}{2} x_k u_{ijk}$ and

$$g^{ij}\phi_{ij} = \frac{1}{2} g^{ij} x_k u_{ijk} = \frac{1}{2} x_k \phi_k. \tag{2.8}$$

(2.8) was found by Chau-Chen-Yuan [1]. Combining (2.3) and (2.8), we have

$$\mathcal{L}\phi = g^{ij}\phi_{ij} - \frac{1}{2} x_j \phi_j = 0. \tag{2.9}$$

Let F be a positive monotonic increasing C^1-function on $\mathbb{R}$, and η be a nonnegative Lipschitz function in Ω with compact support, both to be defined later. Using (1.2) and (2.9) and integrating by parts, we have

$$\begin{aligned}0 &= -\int_\Omega F(\phi)\eta^2 \mathcal{L}\phi \; e^{-\frac{1}{4}x\cdot Du}\sqrt{g}dx \\ &= \int_\Omega g^{ij}\partial_i \left(F(\phi)\eta^2\right)\phi_j e^{-\frac{1}{4}x\cdot Du}\sqrt{g}dx \\ &= \int_\Omega g^{ij}\phi_i\phi_j F'\eta^2 e^{-\frac{u}{2}}dx + 2\int_\Omega F(\phi)\eta g^{ij}\eta_i\phi_j e^{-\frac{u}{2}}dx.\end{aligned} \tag{2.10}$$

Since u is proper convex, then $\lim_{x\to\partial\Omega} u(x) = +\infty$ and we define the set $\Omega_t = \{x \in \Omega|\ u(x) < t\}$ as Lemma 2.2, which is an exhaustion of the domain Ω. Let

$$\eta(x) \triangleq \begin{cases} 1 & \text{if} \quad x \in \Omega_t \\ t+1-u(x) & \text{if} \quad x \in \Omega_{t+1}\setminus\Omega_t \\ 0 & \text{if} \quad x \in \Omega\setminus\Omega_{t+1}, \end{cases}$$

and

$$F(s) \triangleq \begin{cases} e^s & \text{if} \quad s<0 \\ 1 & \text{if} \quad s=0 \\ 1+\arctan s & \text{if} \quad s>0. \end{cases}$$

By (2.1) and (2.10), we have

$$\begin{aligned} \int_{\Omega_t} g^{ij}\phi_i\phi_j F' e^{-\frac{u}{2}}dx \le \int_{\Omega} g^{ij}\phi_i\phi_j F'\eta^2 e^{-\frac{u}{2}}dx &= -2\int_{\Omega} F(\phi)\eta g^{ij}\eta_i\phi_j e^{-\frac{u}{2}}dx \\ &=2\int_{\Omega_{t+1}\setminus\Omega_t} F(\phi)\eta g^{ij}u_i(\frac{1}{2}x_k u_{jk}-\frac{1}{2}u_j)e^{-\frac{u}{2}}dx \\ &\le\int_{\Omega_{t+1}\setminus\Omega_t} F(\phi)\eta x_i u_i e^{-\frac{u}{2}}dx \\ &\le(1+\frac{\pi}{2})\int_{\Omega_{t+1}\setminus\Omega_t}|x_i u_i|e^{-\frac{u}{2}}dx. \end{aligned} \tag{2.11}$$

By Lemma 2.2, let t go to infinity in (2.11), we know $D\phi = 0$ and $\phi = \log g$ is a constant in Ω. By the equation (1.2) and $0 \in \Omega$, as shown in [1], we know $u(x)$ is the quadratic polynomial $u(0) + \frac{1}{2}\langle D^2u(0)x, x\rangle$. Since $\lim_{x\to\partial\Omega} u(x) = +\infty$, then $\Omega = \mathbb{R}^n$. □

As for $n = 1$, the equation (1.2) gives the equation

$$u'' = e^{\frac{1}{2}xu'-u}.$$

Since $(xu' - u)' = xu''$, we have $xu' - u \ge -u(0)$ and

$$u''(x) \ge e^{-\frac{u(0)+u(x)}{2}}.$$

If $\lim_{x\to+\infty} u(x) = C_0 \in [-\infty, +\infty)$, then $u(x) \le \max\{u(0), C_0\}$ on $[0, +\infty)$. Then

$$u''(x) \ge e^{-\frac{u(0)+\max\{u(0),C_0\}}{2}}.$$

This means that

$$\lim_{x\to+\infty} u'(x) = +\infty,$$

which contradicts with $\lim_{x\to+\infty} u(x) < +\infty$. A similar argument concludes that $\lim_{x\to-\infty} u(x) = +\infty$. Thus,

$$\lim_{|x|\to\infty} u(x) = +\infty.$$

Combining (2.4), we have

$$\int_{\mathbb{R}} |xu'|e^{-\frac{u}{2}}\,dx < \infty.$$

Following the argument of Theorem 2.3, we could prove Theorem 1.1 for the case $n = 1$.

For proving Theorem 1.1 completely, it suffices to remove the proper condition of $u(x)$ in Theorem 2.3 when $\Omega = \mathbb{R}^n$. Now we give two lemmas on convex functions which will be used in Theorem 2.6 in the case $n \geq 2$. One is an algebraic property for the Hessian of convex functions, the other is on the size of Lebesgue measure of a set which arises from the equation (1.2).

Lemma 2.4. *Let u be a smooth convex function in a domain of $\mathbb{R}^n$. If $\xi_1, \cdots, \xi_n$ is an arbitrary orthonormal basis of $\mathbb{R}^n$, then*

$$\det D^2 u \leq u_{\xi_1\xi_1} u_{\xi_2\xi_2} \cdots u_{\xi_n\xi_n},$$

where $u_{\xi_i\xi_j} = \mathrm{Hessian}(u)(\xi_i, \xi_j)$ in $\mathbb{R}^n$ for $1 \leq i, j \leq n$.

Proof. By an orthogonal transformation, we have

$$\det D^2 u = \det u_{\xi_i\xi_j}. \tag{2.12}$$

Let α be a $(n-1)$-dimensional vector defined by $(u_{\xi_1\xi_2}, u_{\xi_1\xi_3}, \cdots, u_{\xi_1\xi_n})$ and A be a $(n-1)\times(n-1)$ matrix $(u_{\xi_i\xi_j})_{2\leq i,j\leq n}$. Since

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{u_{\xi_1\xi_1}}\alpha^T & I_{n-1} \end{pmatrix} \begin{pmatrix} u_{\xi_1\xi_1} & \alpha \\ \alpha^T & A \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{u_{\xi_1\xi_1}}\alpha \\ 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} u_{\xi_1\xi_1} & 0 \\ 0 & A - \frac{\alpha^T\alpha}{u_{\xi_1\xi_1}} \end{pmatrix},$$

then $A - \frac{\alpha^T\alpha}{u_{\xi_1\xi_1}}$ is a positive definite matrix and

$$\det D^2 u = \det u_{\xi_i\xi_j} = u_{\xi_1\xi_1} \det\left(A - \frac{\alpha^T\alpha}{u_{\xi_1\xi_1}}\right) \leq u_{\xi_1\xi_1}\det(A). \tag{2.13}$$

By induction,

$$\det D^2 u \leq u_{\xi_1\xi_1} u_{\xi_2\xi_2} \cdots u_{\xi_n\xi_n}. \tag{2.14}$$

□

Lemma 2.5. *Let B_δ be an open ball with radius δ and centered at the origin in $\mathbb{R}^m$, v be a smooth convex function in $\overline{B_\delta}$ with $v\big|_{\partial B_\delta} \le C_1$, then there is a constant $C_2 > 0$ depending only on m, δ, C_1 such that the set*

$$E = \{x \in B_\delta |\ e^{\frac{v(x)}{2}} \det D^2 v > C_2^m\}$$

has the measure $|E| < \frac{1}{2}|B_\delta|$.

Proof. Suppose that the measure $|E| \ge \frac{1}{2}|B_\delta|$ for some sufficiently large C_2, and we will deduce the contradiction. Denote the open sets

$$E_i = \{x \in B_\delta \big|\ D_{ii}v(x) > C_2 e^{-\frac{v(x)}{2m}}\}$$

for $i = 1, \cdots, m$. By Lemma 2.4,

$$D_{11}vD_{22}v \cdots D_{mm}v \ge \det D^2 v,$$

then

$$E \subset \bigcup_{1\le i\le m} E_i.$$

Thus,

$$\frac{1}{2}|B_\delta| \le |E| \le |\bigcup_{1\le i\le m} E_i| \le \sum_{i=1}^m |E_i|,$$

which implies there is a E_i with

$$|E_i| \ge \frac{1}{2m}|B_\delta|.$$

Without loss of generalarity set $E_1 = E_i$, then there is

$$L = \{x = (x_1, \cdots, x_m) \in B_\delta \big|\ x_2 = y_2, \cdots, x_m = y_m\}$$

such that the measure of $L \cap E_1$ is no less than $C_3\delta$ for some constant $0 < C_3 < 1$ depending only on m.

Let $f(s) = v(s, y_2, y_3, \cdots, y_m)$, $I = \{s \in \mathbb{R} \big|\ (s, y_2, y_3, \cdots, y_m) \in L\}$, then $I = (-s_0, s_0)$ with $\frac{C_3\delta}{2} \le s_0 \le \delta$ and

$$E_1 = \{|s| < s_0 |\ f''(s) > C_2 e^{-\frac{f(s)}{2m}}\}.$$

Without loss of generality, we select $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N$ for some $N < \infty$ such that $L \cap E_1 \supset \bigcup_{i=1}^N (a_i, b_i) \times (y_2, \cdots, y_m)$ and $\sum_{i=1}^N (b_i - a_i) \ge \frac{C_3\delta}{3}$.

For deducing the contradiction, we need prove $f(s_0)$ is sufficiently large as C_2 is sufficiently large, which violates our assumption $v\big|_{\partial B_\delta} \leq C_1$. Since v is convex, then $\sup_{x\in\overline{B_\delta}} v(x) \leq C_1$ and $f'' = D_{11}v > 0$. By Newton-Leibnitz formula, we get

$$f(0) - C_1 \leq f(0) - f(-s_0) = \int_{-s_0}^{0} f'(s)ds \leq f'(0)s_0,$$

which implies

$$f'(0) \geq \frac{f(0) - C_1}{s_0}. \tag{2.15}$$

Let $C_4 = \frac{1}{s_0}(C_1 e^{\frac{C_1}{2m}} - f(0)e^{\frac{f(0)}{2m}})$, which depends only on m, δ and C_1. Since C_2 is sufficiently large, then there is a $c \in (a_j, b_j)$ such that

$$\sum_{i=1}^{j-1}(b_i - a_i) + c - a_j \in \left(\frac{C_4}{C_2}, \frac{2C_4}{C_2}\right). \tag{2.16}$$

If $f'(c) < 0$, then $f(0) \geq f(s)$ for $s \in [0, c]$. Combining (2.15), (2.16) and the definition of E_1 and C_4, we have

$$\begin{aligned}
f'(c) =& f'(0) + \int_0^c f''(s)ds \geq f'(0) + \sum_{i=1}^{j-1}\int_{a_i}^{b_i} f''(s)ds + \int_{a_j}^{c} f''(s)ds \\
\geq& f'(0) + \sum_{i=1}^{j-1}\int_{a_i}^{b_i} C_2 e^{-\frac{f(s)}{2m}}ds + \int_{a_j}^{c} C_2 e^{-\frac{f(s)}{2m}}ds \\
\geq& f'(0) + \sum_{i=1}^{j-1}\int_{a_i}^{b_i} C_2 e^{-\frac{f(0)}{2m}}ds + \int_{a_j}^{c} C_2 e^{-\frac{f(0)}{2m}}ds \\
\geq& \frac{f(0) - C_1}{s_0} + C_4 e^{-\frac{f(0)}{2m}} = e^{-\frac{f(0)}{2m}}\left(C_4 + \frac{1}{s_0}\big(f(0)e^{\frac{f(0)}{2m}} - C_1 e^{\frac{f(0)}{2m}}\big)\right) \\
\geq& e^{-\frac{f(0)}{2m}}\left(C_4 + \frac{1}{s_0}\big(f(0)e^{\frac{f(0)}{2m}} - C_1 e^{\frac{C_1}{2m}}\big)\right) = 0.
\end{aligned} \tag{2.17}$$

Thus, $f'(c) \geq 0$. Together with $f'' > 0$, we have

$$0 \leq f'(s_1) \leq f'(s_2) \text{ and } f(s_1) \leq f(s_2) \quad \text{for } c \leq s_1 \leq s_2 \leq s_0. \tag{2.18}$$

Denote $\delta_j = b_j - c$ and $\delta_k = b_k - a_k$ for $k = j+1, \cdots, N$. By (2.18) and the definition of E_1, for $t \in (c, b_j)$ we obtain

$$f'(t) = f'(c) + \int_c^t f''(s)ds \geq C_2\int_c^t e^{-\frac{f(s)}{2m}}ds \geq C_2(t - c)e^{-\frac{f(t)}{2m}},$$

then

$$e^{\frac{f(t)}{2m}} = e^{\frac{f(c)}{2m}} + \int_c^t \frac{f'(s)}{2m}e^{\frac{f(s)}{2m}}ds \geq \int_c^t \frac{C_2}{2m}(s - c)ds = \frac{C_2}{4m}(t - c)^2.$$

So we claim

$$f'(b_k) \geq C_2 \sum_{i=j}^{k} \delta_i e^{-\frac{f(b_k)}{2m}}, \quad \text{and} \quad e^{\frac{f(b_k)}{2m}} \geq \frac{C_2}{4m}\Big(\sum_{i=j}^{k} \delta_i\Big)^2 \quad \text{for } k = j, \cdots, N. \tag{2.19}$$

If (2.19) holds for some $k < N$, then $f'(a_{k+1}) \geq f'(b_k)$ and $f(a_{k+1}) \geq f(b_k)$ by (2.18). For any $t \in (a_{k+1}, b_{k+1})$, we get

$$\begin{aligned} f'(t) =& f'(a_{k+1}) + \int_{a_{k+1}}^{t} f''(s)ds \geq C_2 \sum_{i=j}^{k} \delta_i e^{-\frac{f(b_k)}{2m}} + C_2 \int_{a_{k+1}}^{t} e^{-\frac{f(s)}{2m}} ds \\ \geq& C_2\Big(t - a_{k+1} + \sum_{i=j}^{k} \delta_i\Big) e^{-\frac{f(t)}{2m}}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} e^{\frac{f(t)}{2m}} =& e^{\frac{f(a_{k+1})}{2m}} + \int_{a_{k+1}}^{t} \frac{f'(s)}{2m} e^{\frac{f(s)}{2m}} ds \\ \geq& \frac{C_2}{4m}\Big(\sum_{i=j}^{k} \delta_i\Big)^2 + \int_{a_{k+1}}^{t} \frac{C_2}{2m}\Big(s - a_{k+1} + \sum_{i=j}^{k} \delta_i\Big) ds \\ =& \frac{C_2}{4m}\Big(t - a_{k+1} + \sum_{i=j}^{k} \delta_i\Big)^2. \end{aligned} \tag{2.21}$$

By induction, we complete this claim. Combining the selection of a_i, b_i and (2.16)(2.18)(2.19), we conclude

$$C_1 \geq f(s_0) \geq f(b_N) \geq 2m \log \frac{C_2}{4m} + 4m \log \sum_{i=j}^{N} \delta_i \geq 2m \log \frac{C_2}{4m} + 4m \log(\frac{C_3 \delta}{3} - \frac{2C_4}{C_2}),$$

which is impossible for sufficiently large C_2. □

Theorem 2.6. *Any entire smooth convex solution u to (1.2) in $\mathbb{R}^n$ is proper.*

Proof. To prove the result, it suffices to show $\lim_{|x|\to\infty} u(x) = +\infty$ for $n \geq 2$. Let B_r^n be an open ball in $\mathbb{R}^n$ with radius r and centered at the origin. Suppose that

$$\liminf_{|x|\to\infty} u(x) < +\infty. \tag{2.22}$$

Since $\frac{\partial}{\partial r}\Big(r\langle \beta, Du(r\beta)\rangle - u(r\beta)\Big) = r u_{ij}\beta_i\beta_j > 0$ for every $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{S}^{n-1}(1)$, then

$$r\partial_r u(r\beta) - u(r\beta) \geq -u(0),$$

and

$$\Big(\frac{u(r\beta) - u(0)}{r}\Big)' = \frac{r\partial_r u(r\beta) - u(r\beta) + u(0)}{r^2} \geq 0.$$

So $\lim_{r\to\infty}\frac{u(r\beta)}{r}$ always exists (may be infinity) and is denoted by κ_β. Let $\Lambda=\{\beta\in\mathbb{S}^{n-1}(1)|\ \kappa_\beta\le 0\}$. If $\Lambda=\emptyset$, then for any $\beta\in\mathbb{S}^{n-1}(1)$, there is a $r_\beta>0$ such that $u(r_\beta\beta)-u(0)\ge\frac{1}{2}\tilde\kappa_\beta r_\beta$, where $\tilde\kappa_\beta=\min\{\kappa_\beta,1\}>0$. By the continuity of u, there is an open domain $S_\beta\subset\mathbb{S}^{n-1}(1)$ containing β such that $u(r_\beta\gamma)-u(0)\ge\frac{1}{4}\tilde\kappa_\beta r_\beta$ for each $\gamma\in S_\beta$. Since u is convex, then $\partial_r u(r\gamma)\ge\frac{1}{4}\tilde\kappa_\beta$ for $r\ge r_\beta$, which implies

$$u(r\gamma)-u(0)=\int_{r_\beta}^{r}\partial_s u(s\gamma)ds+u(r_\beta\gamma)-u(0)\ge\frac{1}{4}\tilde\kappa_\beta(r-r_\beta)+\frac{1}{4}\tilde\kappa_\beta r_\beta=\frac{1}{4}\tilde\kappa_\beta r$$

for each $\gamma\in S_\beta$ and $r\ge r_\beta$. By finite cover lemma, there is a sequence $\{\tilde\beta_i\}_{i=1}^N$ such that $\mathbb{S}^{n-1}(1)\subset\bigcup_{1\le i\le N}S_{\tilde\beta_i}$. Let $r^*=\max_{1\le i\le N}r_{\tilde\beta_i}$ and $\kappa^*=\min_{1\le i\le N}\tilde\kappa_{\tilde\beta_i}>0$, then for any $\beta\in\mathbb{S}^{n-1}(1)$ and $r\ge r^*$, we have $u(r\beta)-u(0)\ge\frac{1}{4}\kappa^* r$. This contradicts with (2.22). Therefore, Λ is nonempty.

There is a sequence $\{\bar\beta_i\}\subset\Lambda$ such that

$$\lim_{i\to\infty}\kappa_{\bar\beta_i}=\inf_{\beta\in\Lambda}\kappa_\beta.$$

And we can assume $\lim_{i\to\infty}\bar\beta_i=\theta$ for some $\theta=(\theta_1,\cdots,\theta_n)\in\mathbb{S}^{n-1}(1)$. For every fixed $r>0$, there is a $i_0>0$ such that for all $i\ge i_0$, $u(r\bar\beta_i)\ge u(r\theta)-1$. Then

$$0\ge\kappa_{\bar\beta_i}\ge\frac{u(r\bar\beta_i)-u(0)}{r}\ge\frac{u(r\theta)-u(0)-1}{r}.$$

Hence $u(r\theta)\le u(0)+1$, and

$$\lim_{i\to\infty}\kappa_{\bar\beta_i}\ge\lim_{r\to\infty}\frac{u(r\theta)-u(0)-1}{r}=\kappa_\theta.$$

Therefore $\kappa_\theta=\inf_{\beta\in\Lambda}\kappa_\beta\le 0$. Let $\kappa=\kappa_\theta$ for simplicity. For each $\beta\in\mathbb{S}^{n-1}(1)$, we obtain

$$\kappa=\lim_{r\to\infty}\frac{u(r\theta)}{r}=\lim_{r\to\infty}\langle\theta,Du(r\theta)\rangle\le\lim_{r\to\infty}\frac{u(r\beta)}{r}. \tag{2.23}$$

Let

$$U=\{x\in\mathbb{R}^n\big|\ u(x)<\kappa\langle\theta,x\rangle+u(0)\}.$$

Since u is an entire convex function in $\mathbb{R}^n$, then U is a convex domain in $\mathbb{R}^n$. The definition of κ implies $r\theta\in U$ for any $r>0$. We then can find a slim column region around the ray $r\theta$ inside the convex domain U. Precisely, there exist $r_0,\delta>0$ such that

$$\mathscr{C}_\theta\triangleq\{r\theta+\alpha\in\mathbb{R}^n\big|\ r\ge r_0,\ \alpha\perp\theta\text{ and }|\alpha|<\delta\}\subset U.$$

Let

$$\mathscr{S}_r=\{r\theta+\alpha\big|\ \alpha\perp\theta\text{ and }|\alpha|<\delta\}$$

be a slice of $\mathscr{C}_\theta$. Let $u_\theta(r\theta+\alpha)=\frac{\partial}{\partial r}u(r\theta+\alpha)=\langle\theta,Du(r\theta+\alpha)\rangle$ denote the $\theta-$directional derivative of u and

$$u_{\theta\theta}(r\theta+\alpha)=\frac{\partial^2}{\partial r^2}u(r\theta+\alpha)=\sum_{i,j}u_{ij}(r\theta+\alpha)\theta_i\theta_j.$$

By $\mathscr{C}_\theta\subset U$ and (2.23), we conclude $\lim_{r\to\infty}u_\theta(r\theta+\alpha)=\kappa$ for any $\alpha\perp\theta,|\alpha|<\delta$. We don't have the pointwise estimate for $u_{\theta\theta}$ in $\mathscr{C}_\theta$, but have the following integral estimate

$$\begin{aligned}\int_r^\infty\int_{\mathscr{S}_s}u_{\theta\theta}dV_{\mathscr{S}_s}ds&=\int_r^\infty\int_{\alpha\perp\theta,|\alpha|<\delta}u_{\theta\theta}(s\theta+\alpha)dV_\alpha ds\\&=\int_{\alpha\perp\theta,|\alpha|<\delta}\int_r^\infty u_{\theta\theta}(s\theta+\alpha)dsdV_\alpha\\&=\int_{\alpha\perp\theta,|\alpha|<\delta}\big(\kappa-u_\theta(r\theta+\alpha)\big)dV_\alpha<\infty.\end{aligned}\tag{2.24}$$

Let ω_{n-1} be the standard volume of $(n-1)$-dimensional unit balls. From (2.24), we can find a sequence $\{r_i\}_{i=1}^\infty\subset\mathbb{R}$ with $\lim_{i\to\infty}r_i=+\infty$ such that the open set

$$\widetilde{\mathscr{S}_{r_i}}\triangleq\{x\in\mathscr{S}_{r_i}\big|\ u_{\theta\theta}(x)<\frac{1}{r_i}\}$$

has measure

$$|\widetilde{\mathscr{S}_{r_i}}|\geq\frac{1}{2}|\mathscr{S}_{r_i}|=\frac{1}{2}\omega_{n-1}\delta^{n-1}.$$

Here, the factor $\frac{1}{2}$ is not essential and could be replaced by any positive constant which is less than 1.

Since $\frac{\partial}{\partial r}\bigg(r\langle\beta,Du(r\beta)\rangle-u(r\beta)\bigg)=ru_{ij}\beta_i\beta_j>0$ for every $\beta\in\mathbb{S}^{n-1}(1)$, then

$$x\cdot Du(x)-u(x)\geq-u(0),$$

and

$$\det D^2u(x)=e^{\frac{1}{2}x\cdot Du(x)-u(x)}\geq e^{-\frac{u(0)+u(x)}{2}}.\tag{2.25}$$

Let $D^2_{\mathscr{S}}u$ be the $n-1$-Hessian matrix of u in $\mathscr{S}_r$ for each $r\geq r_0$, then $D^2_{\mathscr{S}}u>0$. By (2.13) and (2.25), we get

$$\det D^2_{\mathscr{S}}u\geq u_{\theta\theta}^{-1}\det D^2u\geq u_{\theta\theta}^{-1}e^{-\frac{u(0)+u(x)}{2}}.$$

The definition of U implies that $u(x)\leq u(0)$ for any $x\in\mathscr{S}_{r_i}\subset U$. Combining the measure $|\widetilde{\mathscr{S}_{r_i}}|=|\{x\in\mathscr{S}_{r_i}\big|\ u_{\theta\theta}^{-1}(x)>r_i\}|\geq\frac{1}{2}|\mathscr{S}_{r_i}|$ and Lemma 2.5, we arrive at a contradiction if i goes to infinity. Therefore, $\lim_{|x|\to\infty}u(x)=+\infty$ when $n\geq2$. We complete the proof. □

Proof of Theorem 1.1. Noting the case $n = 1$ and Combining Theorem 2.3 and Theorem 2.6, we finish the proof. □

Let $a > 0$, c be constant numbers and $b \in \mathbb{R}^n$ be a constant vector. The entire solution to the following general type equation

$$\log\det D^2u(x) = a\big(\frac{1}{2}x \cdot Du(x) - u(x)\big) + b \cdot x + c$$

is a quadratic polynomial. In fact, let $w(x) = au(x) - 2b \cdot x - c - n\log a$, then w satisfies the equation (1.2).

3. Application to other equations

Let's prove Theorem 1.2 by the integral method, which is similar to the previous section.

Proof of Theorem 1.2. Let M be a Lagrangian submanifold satisfying (1.3) in $\mathbb{R}^{2n}$ with induced metric $g_{ij}dx_idx_j$. Then $g_{ij} = \delta_{ij} + \sum_k u_{ik}u_{jk}$, and denote $g = \det g_{ij}$ for short. Then

$$\begin{aligned}\partial_i(g^{ij}\sqrt{g}) =&\frac{1}{2}\sqrt{g}g^{kl}\partial_i g_{kl}g^{ij} - \sqrt{g}g^{ki}\partial_i g_{kl}g^{lj}\\ =&\frac{1}{2}\sqrt{g}g^{kl}g^{ij}(u_{ks}u_{lsi} + u_{ksi}u_{ls}) - \sqrt{g}g^{ki}g^{lj}(u_{ks}u_{lsi} + u_{ksi}u_{ls})\\ =&-\sqrt{g}g^{kl}g^{ij}u_{kls}u_{is}.\end{aligned} \tag{3.1}$$

Define the differential operator $\mathcal{L}$ on $C^2(\mathbb{R}^n)$ by

$$\mathcal{L}\phi = \frac{1}{\sqrt{g}}e^{\frac{|x|^2+|Du|^2}{4}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{g}e^{-\frac{|x|^2+|Du|^2}{4}}\frac{\partial}{\partial x_j}\phi\right),$$

which is the same as the drift Laplacian in [2].

Let $\Theta = \arctan\lambda_1(x) + \cdots + \arctan\lambda_n(x)$, which is the phase function on Lagrangian submanifold $M \in \mathbb{R}^{2n}$. By [1],

$$\Theta_k = g^{ij}u_{ijk} = -\frac{1}{2}u_k + \frac{1}{2}x \cdot Du_k \tag{3.2}$$

and we have

$$\begin{aligned}\mathcal{L}\Theta =&g^{ij}\Theta_{ij} + \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g})\Theta_j - \frac{1}{2}g^{ij}(x_i + u_ku_{ki})\Theta_j\\ =&g^{ij}\Theta_{ij} - g^{kl}g^{ij}u_{kls}u_{is}\Theta_j - \frac{1}{2}g^{ij}(x_k\delta_{ki} + u_ku_{ki})\Theta_j\\ =&g^{ij}\Theta_{ij} + g^{ij}\big(\frac{1}{2}u_s - \frac{1}{2}x_ku_{ks}\big)u_{is}\Theta_j - \frac{1}{2}g^{ij}\big(x_k(g_{ki} - u_{ks}u_{is}) + u_ku_{ki}\big)\Theta_j\\ =&g^{ij}\Theta_{ij} - \frac{1}{2}g^{ij}x_kg_{ki}\Theta_j = g^{ij}\Theta_{ij} - \frac{1}{2}x_j\Theta_j.\end{aligned} \tag{3.3}$$

By (3.2), $\Theta_{kl} = \frac{1}{2}x_s u_{skl}$. Then $g^{kl}\Theta_{kl} = g^{kl}\frac{1}{2}x_s u_{skl} = \frac{1}{2}x_j\Theta_j$ (see also [1]), which implies

$$\mathcal{L}\Theta = 0. \tag{3.4}$$

Let ∇ and $d\mu$ be Levi-Civita connection and volume element of M with respect to the metric $g_{ij}dx_i dx_j$, and $\rho = e^{-\frac{|x|^2+|Du|^2}{4}}$. If η is a smooth function in M with compact support, then by integral by parts we have

$$\begin{aligned} 0 = & -\int_M \eta^2\Theta\mathcal{L}\Theta\rho d\mu = 2\int_M \eta\Theta\nabla\eta\cdot\nabla\Theta\rho d\mu + \int_M |\nabla\Theta|^2\eta^2\rho d\mu \\ \geq & -2\int_M |\nabla\eta|^2\Theta^2\rho d\mu - \frac{1}{2}\int_M |\nabla\Theta|^2\eta^2\rho d\mu + \int_M |\nabla\Theta|^2\eta^2\rho d\mu, \end{aligned} \tag{3.5}$$

which implies

$$\int_M |\nabla\Theta|^2\eta^2\rho d\mu \leq 4\int_M |\nabla\eta|^2\Theta^2\rho d\mu. \tag{3.6}$$

Since Θ is a bounded function and M has Euclidean volume growth [4], then we obtain Θ is a constant. Then, as shown in [1], we obtain Theorem 1.2. □

Now, let's consider another equation. If v is a smooth subharmonic function on $\mathbb{R}^n$ satisfying

$$\log\Delta v = \frac{1}{2}x\cdot Dv - v. \tag{3.7}$$

Let $\phi = \log\Delta v$, then $\phi_i = -\frac{1}{2}v_i + \frac{1}{2}x_j v_{ij}$ and $\phi_{ii} = \frac{1}{2}x_j v_{iij}$. We have

$$\Delta\phi = \frac{1}{2}x_j\partial_j(\Delta v) = \frac{1}{2}e^{\phi}x\cdot D\phi. \tag{3.8}$$

Theorem 3.1. *Let $\phi(x)$ be an entire smooth solution to (3.8) in $\mathbb{R}^n$ and η be a Lipschitz function in $\mathbb{R}^n$ with compact support and $\eta\big|_{B_r} \equiv 1$. If*

$$\lim_{r\to+\infty}\int_{\mathbb{R}^n\setminus B_r} \frac{|D\eta|^2}{|x|^2}e^{-\phi}e^{-\frac{|x|^2}{4}e^{\phi}} = 0,$$

then ϕ is a constant.

Proof. Let η be a Lipschitz function on $\mathbb{R}^n$ with compact support and $\eta\mid_{B_r}\equiv 1$, then we multiply $\eta^2 e^{-\frac{|x|^2}{4}e^{\phi}}$ on both sides of (3.8) and integral by parts,

$$\begin{aligned} \int_{\mathbb{R}^n}\frac{1}{2}x\cdot D\phi e^{\phi}\eta^2 e^{-\frac{|x|^2}{4}e^{\phi}} &= -\int_{\mathbb{R}^n} D\phi\cdot D(\eta^2 e^{-\frac{|x|^2}{4}e^{\phi}}) \\ &= \int_{\mathbb{R}^n} D\phi\cdot(\frac{1}{2}x + \frac{|x|^2}{4}D\phi)e^{\phi}\eta^2 e^{-\frac{|x|^2}{4}e^{\phi}} - 2\int_{\mathbb{R}^n}\eta D\eta\cdot D\phi e^{-\frac{|x|^2}{4}e^{\phi}}. \end{aligned} \tag{3.9}$$

Hence we have

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}^n}|x|^2|D\phi|^2e^{\phi}\eta^2e^{-\frac{|x|^2}{4}e^{\phi}} &= 2\int_{\mathbb{R}^n}\eta D\eta\cdot D\phi e^{-\frac{|x|^2}{4}e^{\phi}}\\ &\leq\frac{1}{4}\int_{\mathbb{R}^n\setminus B_r}|x|^2|D\phi|^2e^{\phi}\eta^2e^{-\frac{|x|^2}{4}e^{\phi}}+4\int_{\mathbb{R}^n\setminus B_r}\frac{|D\eta|^2}{|x|^2}e^{-\phi}e^{-\frac{|x|^2}{4}e^{\phi}},\end{aligned} \tag{3.10}$$

then

$$\int_{B_r}|x|^2|D\phi|^2e^{\phi}e^{-\frac{|x|^2}{4}e^{\phi}}\leq 16\int_{\mathbb{R}^n\setminus B_r}\frac{|D\eta|^2}{|x|^2}e^{-\phi}e^{-\frac{|x|^2}{4}e^{\phi}}. \tag{3.11}$$

Let $r\to\infty$, then (3.11) implies ϕ is a constant. □

For the case $n\geq 3$, let

$$\eta(x)\triangleq\begin{cases}1 & \text{if}\quad x\in B_r\\ 2-\frac{|x|}{r} & \text{if}\quad x\in B_{2r}\setminus B_r\\ 0 & \text{if}\quad x\in\mathbb{R}^n\setminus B_{2r}.\end{cases}$$

If $e^{\phi(x)}\geq 4(n-2)\frac{\log|x|}{|x|^2}$ for $|x|\geq r$, then

$$\int_{\mathbb{R}^n\setminus B_r}\frac{|D\eta|^2}{|x|^2}e^{-\phi}e^{-\frac{|x|^2}{4}e^{\phi}}\leq\frac{1}{4(n-2)}\int_{B_{2r}\setminus B_r}\frac{1}{r^2|x|^{n-2}\log|x|}\leq\frac{C_n}{\log r}. \tag{3.12}$$

Here, C_n is a positive constant depending only on n.

For the case $n=2$, let

$$\eta(x)\triangleq\begin{cases}1 & \text{if}\quad x\in B_r\\ 2-\frac{\log\log|x|}{\log\log r} & \text{if}\quad x\in B_{r^{\log r}}\setminus B_r\\ 0 & \text{if}\quad x\in\mathbb{R}^n\setminus B_{r^{\log r}}.\end{cases}$$

If $|x|^2\log|x|e^{\phi}\geq C>0$ for $|x|\geq r\geq e$, then

$$\int_{\mathbb{R}^2\setminus B_r}\frac{|D\eta|^2}{|x|^2}e^{-\phi}e^{-\frac{|x|^2}{4}e^{\phi}}\leq\frac{1}{C}\int_{B_{r^{\log r}}\setminus B_r}\frac{\log|x|}{|x|^2(\log|x|)^2(\log\log r)^2}=\frac{2\pi}{C\log\log r}. \tag{3.13}$$

Hence, ϕ is a constant. By (3.7), as shown in [1], $v(x)$ is the quadratic polynomial $v(0)+\frac{1}{2}\langle D^2v(0)x,x\rangle$. For $n=2$, up to an additive constant (3.7) is equivalent to

$$\log\det\partial\bar{\partial}v(x)=\frac{1}{2}x\cdot Dv(x)-v(x).$$

Thus, our condition $\Delta v\geq\frac{C}{|x|^2\log|x|}$ for any $C>0$ as $|x|\to\infty$ is weaker than $\partial\bar{\partial}v(x)\geq\frac{1+\delta}{2|x|^2}I$ for any $\delta>0$ as $|x|\to\infty$ in [1].

Remark 3.2. *We don't know whether every entire smooth subharmonic solution to* (3.7) *is the quadratic polynomial* $v(0) + \frac{1}{2}\langle D^2 v(0)x, x\rangle$. *Here, we provide a function* $\phi(x) = \log(2n-4) - 2\log|x|$ *for* $n \geq 3$ *and* $x \in \mathbb{R}^n \setminus \{0\}$, *which satisfies* (3.8) *in* $\mathbb{R}^n \setminus \{0\}$.

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