MHD Boundary Layers Theory in Sobolev Spaces Without Monotonicity I: Well-Posedness Theory

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Abstract

We study the well-posedness theory for the MHD boundary layer. The boundary layer equations are governed by the Prandtl-type equations that are derived from the incompressible MHD system with non-slip boundary condition on the velocity and perfectly conducting condition on the magnetic field. Under the assumption that the initial tangential magnetic field is not zero, we establish the local-in-time existence, uniqueness of solutions for the nonlinear MHD boundary layer equations. Compared with the well-posedness theory of the classical Prandtl equations for which the monotonicity condition of the tangential velocity plays a crucial role, this monotonicity condition is not needed for the MHD boundary layer. This justifies the physical understanding that the magnetic field has a stabilizing effect on MHD boundary layer in rigorous mathematics. © 2018 Wiley Periodicals, Inc.

Contents

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1 Introduction and Main Result

One important problem about magnetohydrodynamics (MHD) is to understand the high Reynolds number limits in a domain with boundary. In this paper, we consider the following initial boundary value problem for the two-dimensional (2D) viscous MHD equations (cf. [4, 5, 7, 37]) in a periodic domain $\{(t,x,y) : t \in [0,T], x \in \mathbb{T}, y \in \mathbb{R}_+\}$:

$$\begin{cases} \partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon - (\mathbf{H}^\epsilon \cdot \nabla)\mathbf{H}^\epsilon + \nabla p^\epsilon = \mu\epsilon\triangle\mathbf{u}^\epsilon, \\ \partial_t \mathbf{H}^\epsilon - \nabla \times (\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) = \kappa\epsilon\triangle\mathbf{H}^\epsilon, \\ \nabla \cdot \mathbf{u}^\epsilon = 0, \quad \nabla \cdot \mathbf{H}^\epsilon = 0. \end{cases} \tag{1.1}$$

Here, we assume the viscosity and resistivity coefficients have the same order of a small parameter ϵ. $\mathbf{u}^\epsilon = (u_1^\epsilon, u_2^\epsilon)$ denotes the velocity vector, $\mathbf{H}^\epsilon = (h_1^\epsilon, h_2^\epsilon)$ denotes the magnetic field, and $p^\epsilon = \widetilde{p}^\epsilon + |\mathbf{H}^\epsilon|^2/2$ denotes the total pressure with $\widetilde{p}^\epsilon$ the pressure of the fluid. On the boundary, the non-slip boundary condition is imposed on velocity field

$$\mathbf{u}^\epsilon\big|_{y=0} = \mathbf{0}, \tag{1.2}$$

and the perfectly conducting boundary condition on magnetic field

$$h_2^\epsilon\big|_{y=0} = \partial_y h_1^\epsilon\big|_{y=0} = 0. \tag{1.3}$$

The formal limiting system of (1.1) yields the ideal MHD equations when ϵ tends to 0. However, there is a mismatch in the tangential velocity between the equations (1.1) and the limiting equations on the boundary $y = 0$. This is why a boundary layer forms in the vanishing viscosity and resistivity limit process. To find out the terms in (1.1) whose contribution is essential for the boundary layer, we use the same scaling as the one used in [32],

$$t = t, \quad x = x, \quad \widetilde{y} = \epsilon^{-\frac{1}{2}} y,$$

then, set

$$\begin{cases} u_1(t,x,\widetilde{y}) = u_1^\epsilon(t,x,y), \\ u_2(t,x,\widetilde{y}) = \epsilon^{-\frac{1}{2}} u_2^\epsilon(t,x,y), \end{cases} \qquad \begin{cases} h_1(t,x,\widetilde{y}) = h_1^\epsilon(t,x,y), \\ h_2(t,x,\widetilde{y}) = \epsilon^{-\frac{1}{2}} h_2^\epsilon(t,x,y), \end{cases}$$

and

$$p(t,x,\widetilde{y}) = p^\epsilon(t,x,y).$$

Then by taking the leading order, the equations (1.1) are reduced to

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 - h_1\partial_x h_1 - h_2\partial_y h_1 + \partial_x p = \mu\partial_y^2 u_1, \\ \partial_y p = 0, \\ \partial_t h_1 + \partial_y(u_2 h_1 - u_1 h_2) = \kappa\partial_y^2 h_1, \\ \partial_t h_2 - \partial_x(u_2 h_1 - u_1 h_2) = \kappa\partial_y^2 h_2, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \end{cases} \tag{1.4}$$

in $\{t > 0, x \in \mathbb{T}, y \in \mathbb{R}^+\}$, where we have replaced $\tilde{y}$ by y for simplicity of notation.

The second equation of (1.4) implies that the leading order of boundary layers for the total pressure $p^\epsilon(t,x,y)$ is invariant across the boundary layer and should be matched to the outflow pressure $P(t,x)$ on top of the boundary layer, that is, the trace of pressure of ideal MHD flow. Consequently, we have

$$p(t,x,y) \equiv P(t,x).$$

It is worth noting that the pressure $\tilde{p}^\epsilon$ of the fluid may have the leading order of boundary layers because of the appearance of the boundary layer for the magnetic field. It is different from the general fluid in the absence of the magnetic field, for which the leading boundary layer for the pressure of the fluid always vanishes.

The tangential component $u_1(t,x,y)$ of the velocity field and $h_1(t,x,y)$ of magnetic filed, respectively, should match the outflow tangential velocity $U(t,x)$ and the outflow tangential magnetic field $H(t,x)$ on the top of the boundary layer, that is,

$$u_1(t,x,y) \to U(t,x), \quad h_1(t,x,y) \to H(t,x), \quad \text{as } y \to +\infty, \tag{1.5}$$

where $U(t,x)$ and $H(t,x)$ are the trace of the tangential velocity and magnetic field, respectively. Therefore, we have the following "matching" condition:

$$U_t + UU_x - HH_x + P_x = 0, \quad H_t + UH_x - HU_x = 0, \tag{1.6}$$

which shows that (1.5) is consistent with the first and third equations of (1.4). Moreover, on the boundary $\{y = 0\}$, the boundary conditions (1.2) and (1.3) give

$$u_1\big|_{y=0} = u_2\big|_{y=0} = \partial_y h_1\big|_{y=0} = h_2\big|_{y=0} = 0. \tag{1.7}$$

On the other hand, it is noted that equation $(1.4)_4$ is a direct consequence of equations $(1.4)_3$, $\partial_x h_1 + \partial_y h_2 = 0$ in $(1.4)_5$, and the boundary condition (1.7). Hence, we only need to study the following initial boundary value problem of the MHD boundary layer equations in $\{t \in [0,T],\ x \in \mathbb{T},\ y \in \mathbb{R}^+\}$,

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 - h_1\partial_x h_1 - h_2\partial_y h_1 = \mu\partial_y^2 u_1 - P_x, \\ \partial_t h_1 + \partial_y(u_2 h_1 - u_1 h_2) = \kappa\partial_y^2 h_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \\ u_1\big|_{t=0} = u_{10}(x,y), \quad h_1\big|_{t=0} = h_{10}(x,y), \\ (u_1, u_2, \partial_y h_1, h_2)\big|_{y=0} = \mathbf{0}, \quad \lim\limits_{y\to+\infty}(u_1, h_1) = (U,H)(t,x). \end{cases} \tag{1.8}$$

The aim of this paper is to show the local well-posedness of the system (1.8) with nonzero tangential component of that magnetic field, that is, without loss of generality, by assuming

$$h_1(t,x,y) > 0.$$

Let us first introduce some weighted Sobolev spaces for later use. Denote

$$\Omega := \{(x,y) \colon x \in \mathbb{T},\ y \in \mathbb{R}_+\}.$$

For any $l \in \mathbb{R}$, denote by $L_l^2(\Omega)$ the weighted Lebesgue space with respect to the spatial variables:

$$L_l^2(\Omega) := \Big\{ f(x,y)\colon \Omega \to \mathbb{R},\ \|f\|_{L_l^2(\Omega)} := \Big(\int_\Omega \langle y\rangle^{2l} |f(x,y)|^2 dx\, dy \Big)^{\frac{1}{2}} < +\infty \Big\}$$

with $\langle y\rangle = 1 + y$. Then, for any given $m \in \mathbb{N}$, denote by $H_l^m(\Omega)$ the weighted Sobolev spaces,

$$H_l^m(\Omega) := \{ f(x,y)\colon \Omega \to \mathbb{R},\ \|f\|_{H_l^m(\Omega)} < +\infty \} \tag{1.9}$$

with the norm

$$\|f\|_{H_l^m(\Omega)} := \Big(\sum_{m_1+m_2\le m} \big\| \langle y\rangle^{l+m_2} \partial_x^{m_1} \partial_y^{m_2} f \big\|_{L^2(\Omega)}^2 \Big)^{\frac{1}{2}}.$$

Now, we can state the main result as follows.

THEOREM 1.1. *Let $m \ge 5$ be an integer and $l \ge 0$ a real number. Assume that the outer flow $(U, H, P_x)(t,x)$ satisfies that for some $T > 0$,*

$$M_0 := \sum_{i=0}^{2m+2} \Big(\sup_{0\le t\le T} \|\partial_t^i (U,H,P)(t,\cdot)\|_{H^{2m+2-i}(\mathbb{T}_x)} \Big) < +\infty. \tag{1.10}$$

Also, we assume the initial data $(u_{10}, h_{10})(x,y)$ satisfies

$$(u_{10}(x,y) - U(0,x), h_{10}(x,y) - H(0,x)) \in H_l^{3m+2}(\Omega), \tag{1.11}$$

and the compatibility conditions up to m^{th} order. Moreover, there exists a sufficiently small constant $\delta_0 > 0$ such that

$$\begin{aligned} \big|\langle y\rangle^{l+1} \partial_y^i (u_{10}, h_{10})(x,y)\big| &\le (2\delta_0)^{-1} \quad \text{for } i = 1,2,\ (x,y) \in \Omega, \\ h_{10}(x,y) &\ge 2\delta_0. \end{aligned} \tag{1.12}$$

Then, there exist a positive time $0 < T_* \le T$ and a unique solution (u_1, u_2, h_1, h_2) to the initial boundary value problem (1.8) *such that*

$$(u_1 - U, h_1 - H) \in \bigcap_{i=0}^{m} W^{i,\infty}\big(0, T_*; H_l^{m-i}(\Omega)\big),$$

and

$$(u_2 + U_x y, h_2 + H_x y) \in \bigcap_{i=0}^{m-1} W^{i,\infty}\big(0, T_*; H_{-1}^{m-1-i}(\Omega)\big),$$

$$(\partial_y u_2 + U_x, \partial_y h_2 + H_x) \in \bigcap_{i=0}^{m-1} W^{i,\infty}\big(0, T_*; H_l^{m-1-i}(\Omega)\big).$$

Moreover, if $l > \frac{1}{2}$,

$$(u_2 + U_x y, h_2 + H_x y) \in \bigcap_{i=0}^{m-1} W^{i,\infty}\big(0, T_*; L^\infty\big(\mathbb{R}_{y,+}; H^{m-1-i}(\mathbb{T}_x)\big)\big).$$

Remark 1.2. Note that the regularity assumption on the outflow (U, H, P) and the initial data (u_{10}, h_{10}) is not optimal. Here, we need the regularity to simplify the construction of approximate solution; cf. Section 4. One may relax the regularity requirement by using other approximations.

Let us briefly explain the main idea of the proof. Similar to the Prandtl equations, the main difficulty in the analysis on the system (1.8) in the Sobolev framework is the loss of x-derivatives in the vertical components $u_2 = -\partial_x \partial_y^{-1} u_1$ and $h_2 = -\partial_x \partial_y^{-1} h_1$ with $\partial_y^{-1} f := \int_0^y f \, d\tilde{y}$ appearing in the terms $u_2 \partial_y u_1 - h_2 \partial_y h_1$ and $u_2 \partial_y h_1 - h_2 \partial_y u_1$ in the first and second equations of (1.8), respectively. This loss of x-derivative is coupled with the lack of any horizontal diffusion so that some kind of cancellation mechanism has to be used in the analysis. Our analysis is based on two new observations. One observation is that $\psi := \partial_y^{-1} h_1$ satisfies

$$\partial_t \psi + u_2 h_1 - u_1 h_2 = \kappa \partial_y^2 \psi.$$

Another observation is that under the assumption on the nondegeneracy of h_1, instead of estimating $\partial_x^m (u_1, h_1)$, $m \in \mathbb{N}$, we consider the following unknown functions to capture the cancellation:

$$u_m := \partial_x^m u_1 - \frac{\partial_y u_1}{h_1} \partial_x^m \psi, \quad h_m := \partial_x^m h_1 - \frac{\partial_y h_1}{h_1} \partial_x^m \psi.$$

With the help of (u_m, h_m), the difficulty in the analysis on $\partial_x^m u_2 \partial_y u_1 - \partial_x^m h_2 \partial_y h_1$ and $\partial_x^m u_2 \partial_y h_1 - \partial_x^m h_2 \partial_y u_1$ mentioned above can be overcome. Since $\partial_x^m (u_1, h_1)$ and (u_m, h_m) are equivalent in the Sobolev framework under the setting in this paper, the loss of th x-derivative can be avoided; see Section 3.2 for the detailed discussion. We also point out that in Section 5, a nonlinear coordinate transformation in the spirit of the classical Crocco transformation to the system (1.8) is introduced, and it provides another approach to study the system with a similar well-posedness result.

We now review some related works to the problem studied in this paper. First of all, the study on fluid around a rigid body with high Reynolds numbers is an important problem in both physics and mathematics. The classical work can be traced back to Prandtl in 1904 about the derivation of the Prandtl equations for boundary layers from the incompressible Navier-Stokes equations with non-slip boundary condition; cf. [33]. About 60 years after its derivation, the first systematic work in rigorous mathematics was achieved by Oleĭnik (cf. [31]), in which she showed that under the monotonicity condition on the tangential velocity field in the normal direction to the boundary, local-in-time well-posedness of the Prandtl system can be justified in 2D by using the Crocco transformation. This result together with some extensions are presented in Oleinik-Samokhin's classical book [32]. Recently, this well-posedness result was proved by simply using an energy method in the framework of Sobolev spaces in [1,30] independently by taking care of the cancellation in the convection terms to overcome the loss of derivative in the tangential direction. Moreover, by imposing an additional favorable condition on the pressure, a global-in-time weak solution was obtained in [39]. Some three space dimensional cases were studied for both classical and weak solutions in [23,25]. Since Oleinik's classical work, the necessity of the monotonicity condition on the velocity field for well-posedness remained as a question until 1980s when Caflisch and Sammartino [35,36] obtained the well-posedness in the framework of analytic functions without this condition (cf. [18–20,28,29,41] and the references therein). Recently the analyticity condition was further relaxed to Gevrey regularity; cf. [10,11,21,22].

When the monotonicity condition is violated, separation of the boundary layer is expected and observed for classical fluid. For this, E-Engquist constructed a finite-time blowup solution to the Prandtl equations in [8]. Recently, when the background shear flow has a nondegenerate critical point, some interesting ill-posedness (or instability) phenomena of solutions to both the linear and nonlinear Prandtl equations around the shear flow have been studied; cf. [9,12,14,15,24,27] and the references therein. All these results show that the monotonicity assumption on the tangential velocity is essential for the well-posedness except in the framework of analytic functions or Gevrey functions.

On the other hand, for electrically conducting fluid such as plasmas and liquid metals, the system of magnetohydrodynamics (denoted by MHD) is a fundamental system to describe the movement of fluid under the influence of an electromagnetic field. The study on the MHD was initiated by Alfvén [2], who showed that the magnetic field can induce current in a moving conductive fluid with a new propagation mechanism along the magnetic field, called Alfvén waves.

For plasma, the boundary layer equations can be derived from the fundamental MHD system, and they are more complicated than the classical Prandtl system because of the coupling of the magnetic field with the velocity field through the Maxwell equations. On the other hand, in physics, it is believed that the magnetic field has a stabilizing effect on the boundary layer that could provide a mechanism for containment of, for example, the high-temperature gas. If the magnetic field is transversal to the boundary, there are extensive discussions on the so-called Hartmann boundary layer; cf. [5, 16, 17]. In addition, there are works on the stability of boundary layers with minimum Reynolds number for flow with different structures to reveal the difference from the classical boundary layers without an electromagnetic field; cf. [3,6,34].

In terms of mathematical derivation when the non-slip boundary condition for the velocity is present, the boundary layer systems that capture the leading order of fluid variables around the boundary depend on three physical parameters: the magnetic Reynolds number, the Reynolds number, and their ratio, called the magnetic Prandtl number. When the Reynolds number tends to infinity while the magnetic Reynolds number is fixed, the derived boundary layer system is similar to the Prandtl system for classical fluid and its well-posedness was discussed in Oleĭnik-Samokhin's book [32], for which the monotonicity condition on the velocity field is needed. When the Reynolds number is fixed while the magnetic Reynolds number tends to infinity, which corresponds to an infinite magnetic Prandtl number, the boundary layer system is similar to an inviscid Prandtl system, and the monotonicity condition on the velocity field is not needed for well-posedness. For the case with the finite magnetic Prandtl number, i.e., both the Reynolds number and the magnetic Reynolds number tend to infinity at the same rate, the boundary layers system is totally different from the classical Prandtl system, and this is the system to be discussed in this paper. Note that for this system, no mathematical well-posedness results have been obtained so far in the Sobolev spaces. Furthermore, we mention that in [38], the authors establish the vanishing viscosity limit for the MHD system in a bounded smooth domain of $\mathbb{R}^d$, $d = 2, 3$, with a slip boundary condition, while the leading order of boundary layers for both velocity and the magnetic field vanishes because of the slip boundary conditions.

To be precise, in this paper, to capture the stabilizing effect of the magnetic field, we establish the well-posedness theory for the problem (1.8) without any monotonicity assumption on the tangential velocity. The only essential condition is that the background tangential magnetic field have a lower positive bound. Hence, the result in this paper enriches the classical local well-posedness results of the classical Prandtl equations. At the same time, it is in agreement with the general physical understanding that the magnetic field stabilizes the boundary layer.

The rest of the paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, we establish the a priori energy estimates for the nonlinear problem (1.8). The local-in-time existence and uniqueness of the solution to (1.8) in Sobolev space are given in Section 4. In Section 5, we introduce another method for the study of the well-posedness theory for (1.8) by using a nonlinear coordinate transformation in the spirit of Crocco transformation for the classical Prandtl system. Finally, the technical proof of a lemma is given in the Appendix.

2 Preliminaries

First, we introduce some notation. Use the tangential derivative operator

$$\partial_\tau^\beta = \partial_t^{\beta_1}\partial_x^{\beta_2} \quad \text{for } \beta = (\beta_1, \beta_2) \in \mathbb{N}^2,\ |\beta| = \beta_1 + \beta_2,$$

and then denote the derivative operator (in both time and space) by

$$D^\alpha = \partial_\tau^\beta \partial_y^k \quad \text{for } \alpha = (\beta_1, \beta_2, k) \in \mathbb{N}^3,\ |\alpha| = |\beta| + k.$$

Set $e_i \in \mathbb{N}^2$, $i = 1, 2$, and $E_j \in \mathbb{N}^3$, $j = 1, 2, 3$, by

$$e_1 = (1, 0) \in \mathbb{N}^2, \quad e_2 = (0, 1) \in \mathbb{N}^2,$$

and

$$E_1 = (1,0,0) \in \mathbb{N}^3, \quad E_2 = (0,1,0) \in \mathbb{N}^3, \quad E_3 = (0,0,1) \in \mathbb{N}^3.$$

Denote by ∂_y^{-1} the inverse of the derivative ∂_y, i.e., $(\partial_y^{-1} f)(y) := \int_0^y f(z)dz$. Moreover, we use the notation $[\cdot,\cdot]$ to denote the commutator, and denote a non-decreasing polynomial function by $\mathcal{P}(\cdot)$, which may differ from line to line.

For $m \in \mathbb{N}$, define the function spaces $\mathcal{H}_l^m$ of measurable functions $f(t,x,y)$: $[0,T]\times\Omega \to \mathbb{R}$ such that for any $t \in [0,T]$,

$$\|f(t)\|_{\mathcal{H}_l^m} := \Big(\sum_{|\alpha|\le m} \|\langle y\rangle^{l+k} D^\alpha f(t,\cdot)\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}} < +\infty. \tag{2.1}$$

The following inequalities will be used frequently in this paper, whose proofs are given in the Appendix.

LEMMA 2.1. *For proper functions f, g, h, the following holds*:

(i) *If* $\lim_{y\to+\infty}(fg)(x,y) = 0$, *then*

$$\left|\int_{\mathbb{T}_x} (fg)\big|_{y=0}\, dx\right| \le \|\partial_y f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|\partial_y g\|_{L^2(\Omega)}. \tag{2.2}$$

In particular, if $\lim_{y\to+\infty} f(x,y) = 0$, *then*

$$\big\|f|_{y=0}\big\|_{L^2(\mathbb{T}_x)} \le \sqrt{2}\,\|f\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_y f\|_{L^2(\Omega)}^{\frac{1}{2}}. \tag{2.3}$$

(ii) *For $l \in \mathbb{R}$ and an integer $m \ge 3$, any $\alpha = (\beta,k) \in \mathbb{N}^3$, $\widetilde{\alpha} = (\widetilde{\beta},\widetilde{k}) \in \mathbb{N}^3$, with $|\alpha| + |\widetilde{\alpha}| \le m$,*

$$\begin{aligned} \|(D^\alpha f \cdot D^{\widetilde{\alpha}} g)(t,\cdot)\|_{L^2_{l+k+\widetilde{k}}(\Omega)} &\le C\|f(t)\|_{\mathcal{H}_{l_1}^m}\|g(t)\|_{\mathcal{H}_{l_2}^m}, \\ &\forall\, l_1, l_2 \in \mathbb{R},\ l_1 + l_2 = l. \end{aligned} \tag{2.4}$$

(iii) *For any $\lambda > \frac{1}{2}$, $\widetilde{\lambda} > 0$,*

$$\begin{aligned} \big\|\langle y\rangle^{-\lambda}(\partial_y^{-1} f)(y)\big\|_{L^2_y(\mathbb{R}_+)} &\le \frac{2}{2\lambda-1}\|\langle y\rangle^{1-\lambda} f(y)\|_{L^2_y(\mathbb{R}_+)}, \\ \big\|\langle y\rangle^{-\widetilde{\lambda}}(\partial_y^{-1} f)(y)\big\|_{L^\infty_y(\mathbb{R}_+)} &\le \frac{1}{\widetilde{\lambda}}\|\langle y\rangle^{1-\widetilde{\lambda}} f(y)\|_{L^\infty_y(\mathbb{R}_+)}, \end{aligned} \tag{2.5}$$

and then, for $l \in \mathbb{R}$, an integer $m \ge 3$, and any $\alpha = (\beta,k) \in \mathbb{N}^3$, $\widetilde{\beta} = (\widetilde{\beta}_1,\widetilde{\beta}_2) \in \mathbb{N}^2$ with $|\alpha| + |\widetilde{\beta}| \le m$,

$$\big\|\big(D^\alpha g \cdot \partial_\tau^{\widetilde{\beta}}\partial_y^{-1} h\big)(t,\cdot)\big\|_{L^2_{l+k}(\Omega)} \le C\|g(t)\|_{\mathcal{H}_{l+\lambda}^m}\|h(t)\|_{\mathcal{H}_{1-\lambda}^m}. \tag{2.6}$$

In particular, for $\lambda = 1$,

$$\begin{aligned} \big\|\langle y\rangle^{-1}\big(\partial_y^{-1} f\big)(y)\big\|_{L^2_y(\mathbb{R}_+)} &\le 2\|f\|_{L^2_y(\mathbb{R}_+)}, \\ \big\|\big(D^\alpha g \cdot \partial_\tau^{\widetilde{\beta}}\partial_y^{-1} h\big)(t,\cdot)\big\|_{L^2_{l+k}(\Omega)} &\le C\|g(t)\|_{\mathcal{H}_{l+1}^m}\|h(t)\|_{\mathcal{H}_0^m}. \end{aligned} \tag{2.7}$$

For any $\lambda > \frac{1}{2}$,

$$\left\| \left(\partial_y^{-1} f\right)(y) \right\|_{L_y^\infty(\mathbb{R}_+)} \le C \| f \|_{L^2_{y,\lambda}(\mathbb{R}_+)}, \tag{2.8}$$

and then, for $l \in \mathbb{R}$, *an integer* $m \ge 2$, *and any* $\alpha = (\beta, k) \in \mathbb{N}^3$, $\widetilde{\beta} = (\widetilde{\beta}_1, \widetilde{\beta}_2) \in \mathbb{N}^2$ *with* $|\alpha| + |\widetilde{\beta}| \le m$,

$$\left\| \left(D^\alpha f \cdot \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} g \right)(t, \cdot) \right\|_{L^2_{l+k}(\Omega)} \le C \| f(t) \|_{\mathcal{H}_l^m} \| g(t) \|_{\mathcal{H}_\lambda^m}. \tag{2.9}$$

To overcome the technical difficulty originating from the boundary terms at $\{y = +\infty\}$, we introduce an auxiliary function $\phi(y) \in C^\infty(\mathbb{R}_+)$ satisfying

$$\phi(y) = \begin{cases} y, & y \ge 2R_0, \\ 0, & 0 \le y \le R_0, \end{cases}$$

for some constant $R_0 > 0$. Then, set the new unknowns:

$$\begin{aligned} u(t,x,y) &:= u_1(t,x,y) - U(t,x)\phi'(y), \\ v(t,x,y) &:= u_2(t,x,y) + U_x(t,x)\phi(y), \\ h(t,x,y) &:= h_1(t,x,y) - H(t,x)\phi'(y), \\ g(t,x,y) &:= h_2(t,x,y) + H_x(t,x)\phi(y). \end{aligned} \tag{2.10}$$

Choose the above construction for (u, v, h, g) to ensure the divergence-free conditions and homogenous boundary conditions, i.e.,

$$\begin{gathered} \partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0, \\ (u, v, \partial_y h, g)|_{y=0} = \mathbf{0}, \quad \lim_{y \to +\infty} (u, h) = \mathbf{0}, \end{gathered}$$

which implies that $v = -\partial_y^{-1}\partial_x u$ and $g = -\partial_y^{-1}\partial_x h$. It is easy to get that

$$\begin{aligned} (u,h)(t,x,y) = {} & \left(u_1(t,x,y) - U(t,x), h_1(t,x,y) - H(t,x)\right) \\ & + \left(U(t,x)(1 - \phi'(y)), H(t,x)(1 - \phi'(y))\right), \end{aligned}$$

which implies that by the construction of $\phi(y)$,

$$\begin{aligned} \|(u,h)(t)\|_{\mathcal{H}_l^m} - CM_0 &\le \|(u_1 - U, h_1 - H)(t)\|_{\mathcal{H}_l^m} \\ &\le \|(u,h)(t)\|_{\mathcal{H}_l^m} + CM_0. \end{aligned} \tag{2.11}$$

By using the new unknowns (u, v, h, g) given by (2.10), we can reformulate the original problem (1.8) as follows:

$$
(2.12)\quad \begin{cases} \partial_t u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \\ \quad - \mu\partial_y^2 u + U_x\phi' u + U\phi'' v - H_x\phi' h - H\phi'' g = r_1, \\ \partial_t h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u \\ \quad - \kappa\partial_y^2 h + H_x\phi' u + H\phi'' v - U_x\phi' h - U\phi'' g = r_2, \\ \partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0, \\ (u, v, \partial_y h, g)|_{y=0} = \mathbf{0}, \\ (u, h)|_{t=0} = \big(u_{10}(x, y) - U(0, x)\phi'(y), h_{10}(x, y) - H(0, x)\phi'(y)\big) \\ \qquad\quad \triangleq (u_0, h_0)(x, y), \end{cases}
$$

where

$$
(2.13)\quad \begin{cases} r_1 = U_t[(\phi')^2 - \phi\phi'' - \phi'] + P_x\big[(\phi')^2 - \phi\phi'' - 1\big] + \mu U\phi^{(3)}, \\ r_2 = H_t[(\phi')^2 + \phi\phi'' - \phi'] + \kappa H\phi^{(3)}. \end{cases}
$$

Note that we have used the divergence-free conditions in obtaining the equations of (u, h) in (2.12), and the relations (1.6) in the calculation of (2.13). It is worth noting that by substituting (2.10) into the second equation of (1.8) directly, there is another equivalent form for the equation of h, which may be convenient to use in some situations:

$$
(2.14)\quad \partial_t h + \partial_y[(v - U_x\phi)(h + H\phi') - (u + U\phi')(g - H_x\phi)] - \kappa\partial_y^2 h = - H_t\phi' + \kappa H\phi^{(3)}.
$$

By the choice of $\phi(y)$, it is easy to get that

$$
(2.15)\quad \begin{aligned} r_1(t, x, y),\ r_2(t, x, y) &\equiv 0, \quad y \geq 2R_0, \\ r_1(t, x, y) \equiv -P_x(t, x),\ r_2(t, x, y) &\equiv 0, \quad 0 \leq y \leq R_0, \end{aligned}
$$

and then for any $t \in [0, T]$, $\lambda \geq 0$, and $|\alpha| \leq m$, by virtue of (1.10),

$$
(2.16)\quad \begin{aligned} \big\|\langle y\rangle^{\lambda}\big(D^\alpha r_1, D^\alpha r_2\big)(t)\big\|_{L^2(\Omega)} &\leq C \sum_{|\beta| \leq |\alpha|+1} \big\|\partial_\tau^\beta(U, H, P_x)(t)\big\|_{L^2(\mathbb{T}_x)} \\ &\leq CM_0. \end{aligned}
$$

Furthermore, similarly to (2.11), we have that for the initial data

$$
(2.17)\quad \begin{aligned} &\|(u_0, h_0)\|_{H_l^{2m}(\Omega)} - CM_0 \\ &\leq \|(u_{10}(x, y) - U(0, x), h_{10} - H(0, x))\|_{H_l^{2m}(\Omega)} \\ &\leq \|(u_0, h_0)\|_{H_l^{2m}(\Omega)} + CM_0. \end{aligned}
$$

Finally, from the transformation (2.10) and the relations (2.11) and (2.17), it is easy to see that Theorem 1.1 is a corollary of the following result.

THEOREM 2.2. *Let $m \geq 5$ be an integer, $l \geq 0$ a real number, and $(U, H, P_x)(t, x)$ satisfy the hypotheses given in Theorem* 1.1. *In addition, assume that for the problem* (2.12)*, the initial data $(u_0(x, y), h_0(x, y)) \in H_l^{3m+2}(\Omega)$ and satisfies the compatibility conditions up to m^{th} order. Moreover, there exists a sufficiently small constant $\delta_0 > 0$ such that for $(x, y) \in \Omega$,*

$$\begin{aligned} &\left|\langle y\rangle^{l+1}\partial_y^i(u_0, h_0)(x, y)\right| \leq (2\delta_0)^{-1}, \quad i = 1, 2, \\ &h_0(x, y) + H(0, x)\phi'(y) \geq 2\delta_0, \end{aligned} \tag{2.18}$$

Then, there exist a time $0 < T_* \leq T$ and a unique solution (u, v, h, g) to the initial boundary value problem (2.12) *such that*

$$(u, h) \in \bigcap_{i=0}^{m} W^{i,\infty}(0, T_*; H_l^{m-i}(\Omega)) \tag{2.19}$$

and

$$\begin{aligned} (v, g) &\in \bigcap_{i=0}^{m-1} W^{i,\infty}\big(0, T_*; H_{-1}^{m-1-i}(\Omega)\big), \\ (\partial_y v, \partial_y g) &\in \bigcap_{i=0}^{m-1} W^{i,\infty}\big(0, T_*; H_l^{m-1-i}(\Omega)\big). \end{aligned} \tag{2.20}$$

Moreover, if $l > \frac{1}{2}$,

$$(v, g) \in \bigcap_{i=0}^{m-1} W^{i,\infty}\Big(0, T_*; L^\infty\big(\mathbb{R}_{y,+}; H^{m-1-i}(\mathbb{T}_x)\big)\Big). \tag{2.21}$$

Therefore, our main task is to prove Theorem 2.2, and its proof will be given in the following two sections.

3 A Priori Estimates

In this section, we will establish a priori estimates for the nonlinear problem (2.12).

PROPOSITION 3.1 (Weighted estimates for $D^m(u, h)$). *Let $m \geq 5$ be an integer, $l \geq 0$ be a real number, and the hypotheses for $(U, H, P_x)(t, x)$ given in Theorem* 1.1 *hold. Assume that (u, v, h, g) is a classical solution to the problem* (2.12) *in* $[0, T]$ *satisfying that*

$$(u, h) \in L^\infty(0, T; \mathcal{H}_l^m), \quad (\partial_y u, \partial_y h) \in L^2(0, T; \mathcal{H}_l^m).$$

Moreover, for $(t, x, y) \in [0, T] \times \Omega$ and sufficiently small constant $\delta_0 > 0$,

$$\begin{aligned} h(t, x, y) + H(t, x)\phi'(y) &\geq \delta_0, \\ \langle y\rangle^{l+1}\partial_y^i(u, h)(t, x, y) &\leq \delta_0^{-1}, \quad i = 1, 2. \end{aligned} \tag{3.1}$$

Then, it holds that for small time,

$$\begin{aligned}
&\sup_{0\le s\le t} \|(u,h)(s)\|_{\mathcal{H}_l^m}\\
&\le \delta_0^{-4}\big(\mathcal{P}(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)})+CM_0^6t\big)^{\frac12}\\
&\quad\cdot\big\{1-C\delta_0^{-24}\big(\mathcal{P}\big(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}\big)+CM_0^6t\big)^2t\big\}^{-\frac14}.
\end{aligned}\tag{3.2}$$

Also, we have that for $i=1,2$,

$$\begin{aligned}
&\big\|\langle y\rangle^{l+1}\partial_y^i(u,h)(t)\big\|_{L^\infty(\Omega)}\\
&\le \big\|\langle y\rangle^{l+1}\partial_y^i(u_0,h_0)\big\|_{L^\infty(\Omega)}\\
&\quad + C\delta_0^{-4}t\big(\mathcal{P}\big(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}\big)+CM_0^6t\big)^{\frac12}\\
&\qquad\cdot\big\{1-C\delta_0^{-24}\big(\mathcal{P}\big(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}\big)+CM_0^6t\big)^2t\big\}^{-\frac14}
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
h(t,x,y)\ge h_0(x,y) &- C\delta_0^{-4}t\big(\mathcal{P}\big(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}\big)+CM_0^6t\big)^{\frac12}\\
&\cdot\big\{1-C\delta_0^{-24}\big(\mathcal{P}\big(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}\big)+CM_0^6t\big)^2t\big\}^{-\frac14}.
\end{aligned}\tag{3.4}$$

The proof of Proposition 3.1 will be given in the following two subsections. Specifically, we will obtain the weighted estimates for $D^\alpha(u,h)$ for $\alpha=(\beta,k)=(\beta_1,\beta_2,k)$ satisfying $|\alpha|=|\beta|+k\le m$ and $|\beta|\le m-1$, in the first subsection, and the weighted estimates for $\partial_\tau^\beta(u,h)$ with $|\beta|=m$ in the second subsection.

3.1 Weighted H_l^m-Estimates with Normal Derivatives

The weighted estimates on $D^\alpha(u,h)$ with $|\alpha|=|\beta|+k\le m$ and $|\beta|\le m-1$ can be obtained by the standard energy method because one order tangential regularity loss is allowed. That is, we have the following estimates:

PROPOSITION 3.2 (Weighted estimates for $D^\alpha(u,h)$ with $|\alpha|\le m, |\beta|\le m-1$). *Let $m\ge 5$ be an integer, let $l\ge 0$ be a real number, and let the hypotheses for $(U,H,P_x)(t,x)$ given in Theorem* 1.1 *hold. Assume that (u,v,h,g) is a classical solution to the problem* (2.12) *in* $[0,T]$ *and satisfies*

$$(u,h)\in L^\infty\big(0,T;\mathcal{H}_l^m\big),\quad (\partial_y u,\partial_y h)\in L^2\big(0,T;\mathcal{H}_l^m\big).$$

Then, there exists a positive constant C, depending on m, l, and ϕ, such that for any small $0 < \delta_1 < 1$,

$$\begin{aligned}\sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\Big(\frac{d}{dt}\|D^\alpha(u,h)(t)\|^2_{L^2_{l+k}(\Omega)}+\mu\|D^\alpha\partial_y u(t)\|^2_{L^2_{l+k}(\Omega)}\\+\kappa\|D^\alpha\partial_y h(t)\|^2_{L^2_{l+k}(\Omega)}\Big)\\\le\delta_1 C\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}^m_0}+C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big(1+\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big)\\+\sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\|D^\alpha(r_1,r_2)(t)\|^2_{L^2_{l+k}(\Omega)}+C\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H,P)(t)\big\|^2_{L^2(\mathbb{T}_x)}.\end{aligned}\tag{3.5}$$

PROOF. Applying the operator $D^\alpha=\partial_\tau^\beta\partial_y^k$ for $\alpha=(\beta,k)=(\beta_1,\beta_2,k)$ and satisfying $|\alpha|=|\beta|+k\le m,\ |\beta|\le m-1$ in the first two equations of (2.12) yields

$$\begin{cases}\partial_t D^\alpha u=D^\alpha r_1+\mu\partial_y^2 D^\alpha u-D^\alpha\big\{[(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]u\\\qquad-[(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]h\\\qquad+U_x\phi' u+U\phi'' v-H_x\phi' h-H\phi'' g\big\},\\\partial_t D^\alpha h=D^\alpha r_2+\kappa\partial_y^2 D^\alpha h-D^\alpha\big\{[(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]h\\\qquad-[(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]u\\\qquad+H_x\phi' u+H\phi'' v-U_x\phi' h-U\phi'' g\big\}.\end{cases}\tag{3.6}$$

Multiplying $(3.6)_1$ by $\langle y\rangle^{2l+2k}D^\alpha u$ and $(3.6)_2$ by $\langle y\rangle^{2l+2k}D^\alpha h$ and integrating them over Ω with respect to the spatial variables x and y, we obtain that

$$\begin{aligned}\frac12\frac{d}{dt}\|\langle y\rangle^{l+k}D^\alpha(u,h)(t)\|^2_{L^2(\Omega)}\\=\int_\Omega\big(D^\alpha r_1\cdot\langle y\rangle^{2l+2k}D^\alpha u+D^\alpha r_2\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy\\+\mu\int_\Omega\big(\partial_y^2 D^\alpha u\cdot\langle y\rangle^{2l+2k}D^\alpha u\big)dx\,dy\\+\kappa\int_\Omega\big(\partial_y^2 D^\alpha h\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy\\-\int_\Omega\big(I_1\cdot\langle y\rangle^{2l+2k}D^\alpha u+I_2\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy,\end{aligned}\tag{3.7}$$

where

$$\begin{cases}I_1=D^\alpha\big\{[(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]u\\\qquad-[(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]h\\\qquad+U_x\phi' u+U\phi'' v-H_x\phi' h-H\phi'' g\big\},\\I_2=D^\alpha\big\{[(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]h\\\qquad-[(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]u\\\qquad+H_x\phi' u+H\phi'' v-U_x\phi' h-U\phi'' g\big\}.\end{cases}\tag{3.8}$$

First of all, it is easy to see that by virtue of (2.16),

$$\int_\Omega \big(D^\alpha r_1\cdot\langle y\rangle^{2l+2k}D^\alpha u + D^\alpha r_2\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy \leq \frac{1}{2}\|D^\alpha(u,h)(t)\|^2_{L^2_{l+k}(\Omega)} + \frac{1}{2}\|D^\alpha(r_1,r_2)(t)\|^2_{L^2_{l+k}(\Omega)}. \tag{3.9}$$

Next, we assume that the following two estimates hold, which will be proved later: for any small $0<\delta_1<1$,

$$\begin{aligned}
&\mu\int_\Omega \big(\partial_y^2 D^\alpha u\cdot\langle y\rangle^{2l+2k}D^\alpha u\big)dx\,dy\\
&\quad+\kappa\int_\Omega \big(\partial_y^2 D^\alpha h\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy\\
&\qquad\leq -\frac{\mu}{2}\|D^\alpha\partial_y u(t)\|^2_{L^2_{l+k}(\Omega)} - \frac{\kappa}{2}\|D^\alpha\partial_y h(t)\|^2_{L^2_{l+k}(\Omega)}\\
&\qquad\quad+\delta_1\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}^m_0}\\
&\qquad\quad+C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big(1+\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big)\\
&\qquad\quad+C\sum_{|\beta|\leq m-1}\big\|\partial_\tau^\beta P_x(t)\big\|^2_{L^2(\mathbb{T}_x)},
\end{aligned}\tag{3.10}$$

and

$$-\int_\Omega \big(I_1\cdot\langle y\rangle^{2l+2k}D^\alpha u + I_2\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy \leq C\Big(\sum_{|\beta|\leq m+2}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^2(\mathbb{T}_x)} + \|(u,h)(t)\|_{\mathcal{H}^m_l}\Big)\|(u,h)(t)\|^2_{\mathcal{H}^m_l}. \tag{3.11}$$

At the moment, by plugging inequalities (3.9)–(3.11) into (3.7) and then summing over α, we obtain that there exists a constant $C_m>0$, depending only on m, such that

$$\begin{aligned}
&\sum_{\substack{|\alpha|\leq m\\|\beta|\leq m-1}}\Big(\frac{d}{dt}\big\|D^\alpha(u,h)(t)\big\|^2_{L^2_{l+k}(\Omega)} + \mu\big\|D^\alpha\partial_y u(t)\big\|^2_{L^2_{l+k}(\Omega)}\\
&\qquad+\kappa\|D^\alpha\partial_y h(t)\|^2_{L^2_{l+k}(\Omega)}\Big)\\
&\quad\leq \delta_1 C_m\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}^m_0} + C\delta_1^{-1}\big\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big(1+\|(u,h)(t)\big\|^2_{\mathcal{H}^m_l}\big)\\
&\qquad+\sum_{\substack{|\alpha|\leq m\\|\beta|\leq m-1}}\|D^\alpha(r_1,r_2)(t)\|^2_{L^2_{l+k}(\Omega)}\\
&\qquad+C\sum_{|\beta|\leq m+2}\big\|\partial_\tau^\beta(U,H,P)(t)\big\|^2_{L^2(\mathbb{T}_x)},
\end{aligned}$$

which implies the estimate (3.5) immediately. □

Now, it remains to show the estimates (3.10) and (3.11).

PROOF OF (3.10). In this part, we will first handle the term $\mu\int_\Omega(\partial_y^2 D^\alpha u \cdot \langle y\rangle^{2l+2k} D^\alpha u)dx\,dy$; the term $\kappa\int_\Omega(\partial_y^2 D^\alpha h \cdot \langle y\rangle^{2l+2k} D^\alpha h)dx\,dy$ can be estimated similarly. By integration by parts, we have

$$\begin{aligned}
&\mu\int_\Omega \big(\partial_y^2 D^\alpha u \cdot \langle y\rangle^{2l+2k} D^\alpha u\big)dx\,dy\\
&\quad= -\mu\big\|\langle y\rangle^{l+k}\partial_y D^\alpha u(t)\big\|^2_{L^2(\Omega)}\\
&\qquad+2(l+k)\mu\int_\Omega\big(\langle y\rangle^{2l+2k-1}\partial_y D^\alpha u\cdot D^\alpha u\big)dx\,dy\\
&\qquad+\mu\int_{\mathbb{T}_x}(\partial_y D^\alpha u\cdot D^\alpha u)\big|_{y=0}\,dx.
\end{aligned}\tag{3.12}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&2(l+k)\mu\int_\Omega\big(\langle y\rangle^{2l+2k-1}\partial_y D^\alpha u\cdot D^\alpha u\big)dx\,dy\le\\
&\quad\frac{\mu}{14}\big\|\langle y\rangle^{l+k}\partial_y D^\alpha u(t)\big\|^2_{L^2(\Omega)}+14\mu(l+k)^2\|\langle y\rangle^{l+k}D^\alpha u(t)\|^2_{L^2(\Omega)},
\end{aligned}\tag{3.13}$$

which implies that by plugging (3.13) into (3.12),

$$\begin{aligned}
&\mu\int_\Omega \big(\partial_y^2 D^\alpha u \cdot \langle y\rangle^{2l+2k} D^\alpha u\big)dx\,dy\\
&\quad\le-\frac{13\mu}{14}\big\|\langle y\rangle^{l+k}D^\alpha\partial_y u(t)\big\|^2_{L^2(\Omega)}+C\|u(t)\|^2_{\mathcal{H}^m_l}\\
&\qquad+\mu\int_{\mathbb{T}_x}(\partial_y D^\alpha u\cdot D^\alpha u)\big|_{y=0}\,dx.
\end{aligned}\tag{3.14}$$

The last term in (3.14), that is, the boundary integral $\mu\int_{\mathbb{T}_x}(\partial_y D^\alpha u\cdot D^\alpha u)\big|_{y=0}\,dx$, is treated in the following two cases.

Case 1. $|\alpha|\le m-1$. By the inequality (2.2), we obtain that for any small $0<\delta_1<1$,

$$\begin{aligned}
&\mu\bigg|\int_{\mathbb{T}_x}(\partial_y D^\alpha u\cdot D^\alpha u)\big|_{y=0}dx\bigg|\\
&\quad\le\mu\big\|\partial_y^2 D^\alpha u(t)\big\|_{L^2(\Omega)}\big\|D^\alpha u(t)\big\|_{L^2(\Omega)}\\
&\qquad+\mu\big\|\partial_y D^\alpha u(t)\big\|^2_{L^2(\Omega)}\delta_1\big\|\partial_y^2 D^\alpha u(t)\big\|^2_{L^2(\Omega)}\\
&\qquad+\frac{\mu^2}{4\delta_1}\|D^\alpha u(t)\|^2_{L^2(\Omega)}+\mu\|\partial_y D^\alpha u(t)\|^2_{L^2(\Omega)}\\
&\quad\le\delta_1\|\partial_y u(t)\|_{\mathcal{H}^m_0}+C\delta_1^{-1}\|u(t)\|^2_{\mathcal{H}^m_0}.
\end{aligned}\tag{3.15}$$

Case 2. $|\alpha| = |\beta| + k = m$. This equation implies that $k \geq 1$ from $|\beta| \leq m-1$. Then, if we denote $\gamma \triangleq \alpha - E_3 = (\beta, k-1)$ with $|\gamma| = |\beta| + k - 1 = m-1$, the first equation in (2.12) reads that by virtue of $\partial_y D^\alpha = \partial_y^2 D^\gamma$,

$$\begin{aligned}\mu\partial_y D^\alpha u = D^\gamma\big\{&\partial_t u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u\\ &- [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h\\ &+ U_x\phi' u + U\phi'' v - H_x\phi' h - H\phi'' g - r_1\big\}.\end{aligned}$$

Then, combining (2.15) with the fact $\phi \equiv 0$ for $y \leq R_0$, we get at $y = 0$

$$\begin{aligned}\mu\partial_y D^\alpha u &= D^\gamma[\partial_t u + (u\partial_x + v\partial_y)u - (h\partial_x + g\partial_y)h + P_x]\\ &= D^\gamma P_x + D^{\gamma+E_1}u + D^\gamma(u\partial_x u - h\partial_x h)\\ &\quad + D^\gamma(v\partial_y u - g\partial_y h).\end{aligned} \tag{3.16}$$

It is easy to obtain from (2.3) that

$$\begin{aligned}&\left|\int_{\mathbb{T}_x}(D^\gamma P_x \cdot D^\alpha u)\Big|_{y=0}\,dx\right|\\ &\leq \|D^\gamma P_x(t)\|_{L^2(\mathbb{T}_x)}\|D^\alpha u(t)|_{y=0}\|_{L^2(\mathbb{T}_x)}\\ &\leq \sqrt{2}\|D^\gamma P_x(t)\|_{L^2(\mathbb{T}_x)}\|D^\alpha u(t)\|_{L^2(\Omega)}^{\frac12}\|D^\alpha\partial_y u(t)\|_{L^2(\Omega)}^{\frac12}\\ &\leq \frac{\mu}{14}\|D^\alpha\partial_y u(t)\|_{L^2(\Omega)}^2 + C\|u(t)\|_{\mathcal{H}_0^m}^2 + C\|D^\gamma P_x(t)\|_{L^2(\mathbb{T}_x)}^2,\end{aligned} \tag{3.17}$$

provided $|\alpha| = m$. Also, by (2.2) and $|\gamma + E_1| = m$,

$$\begin{aligned}&\left|\int_{\mathbb{T}_x}(D^{\gamma+E_1}u \cdot D^\alpha u)\Big|_{y=0}\,dx\right|\\ &\leq \left\|D^{\gamma+E_1}\partial_y u(t)\right\|_{L^2(\Omega)}\|D^\alpha u(t)\|_{L^2(\Omega)}\\ &\quad + \|D^{\gamma+E_1}u(t)\|_{L^2(\Omega)}\|D^\alpha\partial_y u(t)\|_{L^2(\Omega)}\\ &\leq \frac{\delta_1}{3}\|D^{\gamma+E_1}\partial_y u(t)\|_{L^2(\Omega)}^2 + \frac{\mu}{14}\|D^\alpha\partial_y u(t)\|_{L^2(\Omega)}^2\\ &\quad + C\delta_1^{-1}\|u(t)\|_{\mathcal{H}_0^m}^2.\end{aligned} \tag{3.18}$$

Next, as we know

$$D^\gamma(u\partial_x u) = \sum_{\tilde\gamma\leq\gamma}\binom{\gamma}{\tilde\gamma}(D^{\tilde\gamma}u \cdot D^{\gamma-\tilde\gamma+E_2}u),$$

it follows that

$$
\begin{aligned}
&\left|\int_{\mathbb{T}_x}(D^\gamma(u\partial_x u)\cdot D^\alpha u)\big|_{y=0}\,dx\right| \\
&\le C\sum_{\widetilde{\gamma}\le\gamma}\Big\{\big\|\partial_y(D^{\widetilde{\gamma}}u\cdot D^{\gamma-\widetilde{\gamma}+E_2}u)\big\|_{L^2(\Omega)}\|D^\alpha u\|_{L^2(\Omega)} \\
&\qquad + \|D^{\widetilde{\gamma}}u\cdot D^{\gamma-\widetilde{\gamma}+E_2}u\|_{L^2(\Omega)}\|D^\alpha\partial_y u\|_{L^2(\Omega)}\Big\}.
\end{aligned}
\tag{3.19}
$$

Then, by using (2.4) and noting that $|\gamma| = m-1\ge 3$, we have

$$
\begin{aligned}
&\|\partial_y(D^{\widetilde{\gamma}}u\cdot D^{\gamma-\widetilde{\gamma}+E_2}u)\|_{L^2(\Omega)} \\
&\quad\le \big\|D^{\widetilde{\gamma}}\partial_y u\cdot D^{\gamma-\widetilde{\gamma}+E_2}u\big\|_{L^2(\Omega)} + \big\|D^{\widetilde{\gamma}}u\cdot D^{\gamma-\widetilde{\gamma}+E_2}\partial_y u\big\|_{L^2(\Omega)} \\
&\quad\le C\|\partial_y u(t)\|_{\mathcal{H}_0^{m-1}}\|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}} + C\|u(t)\|_{\mathcal{H}_0^{m-1}}\|\partial_{xy}^2 u(t)\|_{\mathcal{H}_0^{m-1}} \\
&\quad\le C\|u(t)\|_{\mathcal{H}_0^m}\|\partial_y u(t)\|_{\mathcal{H}_0^m} + C\|u(t)\|_{\mathcal{H}_0^m}^2
\end{aligned}
$$

and

$$
\big\|D^{\widetilde{\gamma}}u\cdot D^{\gamma-\widetilde{\gamma}+E_2}u\big\|_{L^2(\Omega)} \le C\|u(t)\|_{\mathcal{H}_0^m}\|u(t)\|_{\mathcal{H}_0^m} \le C\|u(t)\|_{\mathcal{H}_0^m}^2.
$$

Substituting the above two inequalities into (3.19) gives

$$
\begin{aligned}
&\left|\int_{\mathbb{T}_x}(D^\gamma(u\partial_x u)\cdot D^\alpha u)\big|_{y=0}\,dx\right| \\
&\le C\sum_{\widetilde{\gamma}\le\gamma}\Big\{\big(\|u(t)\|_{\mathcal{H}_0^m}\|\partial_y u(t)\|_{\mathcal{H}_0^m} + \|u(t)\|_{\mathcal{H}_0^m}^2\big)\big\|D^\alpha u\big\|_{L^2(\Omega)} \\
&\qquad + \|u(t)\|_{\mathcal{H}_0^m}^2\big\|\partial_y D^\alpha u\big\|_{L^2(\Omega)}\Big\} \\
&\le \frac{\delta_1}{3}\|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \frac{\mu}{14}\|D^\alpha\partial_y u(t)\|_{L^2(\Omega)}^2 \\
&\quad + C\delta_1^{-1}\|u(t)\|_{\mathcal{H}_0^m}^4 + C\|u(t)\|_{\mathcal{H}_0^m}^2.
\end{aligned}
\tag{3.20}
$$

Similarly, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{T}_x}(D^\gamma(h\partial_x h)\cdot D^\alpha u)\big|_{y=0}\,dx\right| \\
&\le C\big(\|h(t)\|_{\mathcal{H}_0^m}\|\partial_y h(t)\|_{\mathcal{H}_0^m} + C\|h(t)\|_{\mathcal{H}_0^m}^2\big)\|D^\alpha u\|_{L^2(\Omega)} \\
&\quad + C\|h(t)\|_{\mathcal{H}_0^m}^2\|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
&\le \frac{\delta_1}{3}\|(\partial_y u,\partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + \frac{\mu}{14}\|D^\alpha\partial_y u(t)\|_{L^2(\Omega)}^2 \\
&\quad + C\delta_1^{-1}\|(u,h)(t)\|_{\mathcal{H}_0^m}^4 + C\|(u,h)(t)\|_{\mathcal{H}_0^m}^2.
\end{aligned}
\tag{3.21}
$$

We now turn to control the integral

$$
\left|\int_{\mathbb{T}_x}(D^\gamma(v\partial_y u)\cdot D^\alpha u)\big|_{y=0}\,dx\right|.
$$

Recall that $D^\gamma = \partial_\tau^\beta \partial_y^{k-1}$; by the boundary condition $v\big|_{y=0} = 0$ and divergence-free condition $u_x + v_y = 0$, we obtain that on $\{y = 0\}$,

$$\begin{aligned}
D^\gamma(v\partial_y u) &= \partial_\tau^\beta\Big(v\partial_y^k u + \sum_{i=1}^{k-1}\binom{k-1}{i}\partial_y^i v\cdot\partial_y^{k-i}u\Big)\\
&= \sum_{j=0}^{k-2}\binom{k-1}{j+1}\partial_\tau^\beta\big[-\partial_y^j\partial_x u\cdot\partial_y^{k-j-1}u\big]\\
&= -\sum_{\substack{\widetilde\beta\le\beta\\ 0\le j\le k-2}}\binom{k-1}{j+1}\binom{\beta}{\widetilde\beta}\big(\partial_\tau^{\widetilde\beta+e_2}\partial_y^j u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j-1}u\big),
\end{aligned}$$

where we denote $\binom{j}{i} = 0$ for $i > j$. Note that the right-hand side of the above equality vanishes when $k = 1$, and we only need to consider the case $k \ge 2$. Thus, from the above expression for $D^\gamma(v\partial_y u)$ at $y = 0$, we obtain that by (2.2),

$$\begin{aligned}
&\Big|\int_{\mathbb{T}_x}(D^\gamma(v\partial_y u)\cdot D^\alpha u)\big|_{y=0}dx\Big|\\
&\le C\sum_{\substack{\widetilde\beta\le\beta\\ 0\le j\le k-2}}\big\{\big\|\partial_y\big(\partial_\tau^{\widetilde\beta+e_2}\partial_y^j u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j-1}u\big)\big\|_{L^2(\Omega)}\|D^\alpha u\|_{L^2(\Omega)}\\
&\qquad + \big\|\partial_\tau^{\widetilde\beta+e_2}\partial_y^j u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j-1}u\big\|_{L^2(\Omega)}\|D^\alpha\partial_y u\|_{L^2(\Omega)}\big\}.
\end{aligned}\tag{3.22}$$

As $0 \le j \le k-2$, it follows that by (2.4),

$$\begin{aligned}
&\big\|\partial_y\big(\partial_\tau^{\widetilde\beta+e_2}\partial_y^j u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j-1}u\big)\big\|_{L^2(\Omega)}\\
&\quad\le \big\|\partial_\tau^{\widetilde\beta+e_2}\partial_y^{j+1}u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j-1}u\big\|_{L^2(\Omega)} + \big\|\partial_\tau^{\widetilde\beta+e_2}\partial_y^j u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j}u\big\|_{L^2(\Omega)}\\
&\quad\le C\|\partial_y u(t)\|_{\mathcal{H}_0^{m-1}}\|\partial_y u(t)\|_{\mathcal{H}_0^{m-1}} + C\|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}}\|\partial_y u(t)\|_{\mathcal{H}_0^{m-1}}\\
&\quad\le C\|u(t)\|^2_{\mathcal{H}_0^m}
\end{aligned}$$

and

$$\big\|\partial_\tau^{\widetilde\beta+e_2}\partial_y^j u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-j-1}u\big\|_{L^2(\Omega)} \le C\|u(t)\|_{\mathcal{H}_0^m}\|u(t)\|_{\mathcal{H}_0^m} \le C\|u(t)\|^2_{\mathcal{H}_0^m},$$

provided that $|\beta| + k = |\alpha| = m$. Substituting the above two inequalities into (3.22) gives

$$\begin{aligned}
&\Big|\int_{\mathbb{T}_x}\big(D^\gamma(v\partial_y u)\cdot D^\alpha u\big)\big|_{y=0}\,dx\Big|\\
&\le C\sum_{\substack{\widetilde\beta\le\beta\\ 0\le j\le k-2}}\big\{\|u(t)\|^2_{\mathcal{H}_0^m}\|D^\alpha u\|_{L^2(\Omega)}y\\
&\qquad + \|u(t)\|^2_{\mathcal{H}_0^m}\|\partial_y D^\alpha u\|_{L^2(\Omega)}\big\}\\
&\le \frac{\mu}{14}\|D^\alpha\partial_y u(t)\|^2_{L^2(\Omega)} + C\|u(t)\|^4_{\mathcal{H}_0^m} + C\|u(t)\|^2_{\mathcal{H}_0^m}.
\end{aligned}\tag{3.23}$$

Similarly, we can obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{T}_x} \left(D^\gamma(g\partial_y h)\cdot D^\alpha u\right)\big|_{y=0}\,dx\right| \\
&\quad \le C\|h(t)\|^2_{\mathcal{H}_0^m}\|D^\alpha u\|_{L^2(\Omega)} + C\|h(t)\|^2_{\mathcal{H}_0^m}\|\partial_y D^\alpha u\|_{L^2(\Omega)} \\
&\quad \le \frac{\mu}{14}\|D^\alpha \partial_y u(t)\|^2_{L^2(\Omega)} + C\|(u,h)(t)\|^4_{\mathcal{H}_0^m} + C\|(u,h)(t)\|^2_{\mathcal{H}_0^m}.
\end{aligned}
\tag{3.24}
$$

Therefore, from (3.16) and combining the estimates (3.17), (3.18), (3.20), (3.21), (3.23), and (3.24), we have that when $|\alpha| = |\beta| + k = m$ with $|\beta| \le m-1$,

$$
\begin{aligned}
&\left|\int_{\mathbb{T}_x} \left(\mu\partial_y D^\alpha u\cdot D^\alpha u\right)\big|_{y=0} dx\right| \\
&\quad \le \delta_1\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}_0^m} + \frac{3\mu}{7}\|D^\alpha\partial_y u(t)\|^2_{L^2(\Omega)} \\
&\qquad + C\delta_1^{-1}\|(u,h)(t)\|^4_{\mathcal{H}_0^m} + C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}_0^m} \\
&\qquad + C\|D^\gamma P_x(t)\|^2_{L^2(\mathbb{T}_x)}.
\end{aligned}
\tag{3.25}
$$

Combining (3.15) with (3.25) implies that for $|\alpha| = |\beta| + k \le m$, $|\beta| \le m-1$,

$$
\begin{aligned}
&\left|\int_{\mathbb{T}_x} \left(\mu\partial_y D^\alpha u\cdot D^\alpha u\right)\big|_{y=0}\,dx\right| \\
&\quad \le \delta_1\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}_0^m} + \frac{3\mu}{7}\|D^\alpha\partial_y u(t)\|^2_{L^2(\Omega)} \\
&\qquad + C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}_0^m}\left(1 + \|(u,h)(t)\|^2_{\mathcal{H}_0^m}\right) \\
&\qquad + C\sum_{|\beta|\le m-1}\|\partial_\tau^\beta P_x(t)\|^2_{L^2(\mathbb{T}_x)}.
\end{aligned}
\tag{3.26}
$$

Then, plugging the above estimate (3.26) into (3.14) we have

$$
\begin{aligned}
&\mu\int_\Omega \left(\partial_y^2 D^\alpha u\cdot\langle y\rangle^{2l+2k} D^\alpha u\right)dx\,dy \\
&\quad \le -\frac{\mu}{2}\|D^\alpha\partial_y u(t)\|^2_{L^2_{l+k}(\Omega)} + \delta_1\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}_0^m} \\
&\qquad + C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}_0^m}\left(1 + \|(u,h)(t)\|^2_{\mathcal{H}_0^m}\right) \\
&\qquad + C\sum_{|\beta|\le m-1}\left\|\partial_\tau^\beta P_x(t)\right\|^2_{L^2(\mathbb{T}_x)}.
\end{aligned}
\tag{3.27}
$$

On the other hand, one can get a similar estimate on the term $\kappa\int_\Omega(\partial_y^2 D^\alpha h\cdot\langle y\rangle^{2l+2k}D^\alpha h)dx\,dy$:

$$\begin{aligned}&\kappa\int_\Omega\big(\partial_y^2 D^\alpha h\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy\\ &\quad\le-\frac{\kappa}{2}\|D^\alpha\partial_y h(t)\|^2_{L^2_{l+k}(\Omega)}+\delta_1\|(\partial_y u,\partial_y h)(t)\|^2_{\mathcal{H}_0^m}\\ &\qquad+C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}_0^m}\big(1+\|(u,h)(t)\|^2_{\mathcal{H}_0^m}\big).\end{aligned}\tag{3.28}$$

Thus, we prove (3.10) by combining (3.27) with (3.28). □

PROOF OF (3.11). From the definition (3.8) of I_1 and I_2, we have

$$\begin{aligned}I_1&=[(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]D^\alpha u\\ &\quad-[(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]D^\alpha h\\ &\quad+[D^\alpha,(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]u\\ &\quad-[D^\alpha,(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]h\\ &\quad+D^\alpha[U_x\phi'u+U\phi''v-H_x\phi'h-H\phi''g]\\ &\triangleq I_1^1+I_1^2+I_1^3,\end{aligned}$$

and

$$\begin{aligned}I_2&=[(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]D^\alpha h\\ &\quad-[(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]D^\alpha u\\ &\quad+[D^\alpha,(u+U\phi')\partial_x+(v-U_x\phi)\partial_y]h\\ &\quad-[D^\alpha,(h+H\phi')\partial_x+(g-H_x\phi)\partial_y]u\\ &\quad+D^\alpha[H_x\phi'u+H\phi''v-U_x\phi'h-U\phi''g]\\ &\triangleq I_2^1+I_2^2+I_2^3.\end{aligned}$$

Thus, we divide the term

$$-\int_\Omega(I_1\cdot\langle y\rangle^{2l+2k}D^\alpha u+I_2\cdot\langle y\rangle^{2l+2k}D^\alpha h)dx\,dy$$

into three parts:

$$\begin{aligned}&-\int_\Omega\big(I_1\cdot\langle y\rangle^{2l+2k}D^\alpha u+I_2\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy\\ &\quad=-\sum_{i=1}^3\int_\Omega\big(I_1^i\cdot\langle y\rangle^{2l+2k}D^\alpha u+I_2^i\cdot\langle y\rangle^{2l+2k}D^\alpha h\big)dx\,dy\\ &\quad\triangleq G_1+G_2+G_3,\end{aligned}\tag{3.29}$$

and estimate each G_i, $i=1,2,3$, in the following.

First, note that

$$\phi(y) \equiv y, \quad \phi'(y) \equiv 1, \quad \phi^{(i)}(y) \equiv 0 \quad \text{for } y \geq 2R_0,\ i \geq 2.$$

Then, there exists some positive constant C such that

$$\|\langle y\rangle^{i-1}\phi^{(i)}(y)\|_{L^\infty(\mathbb{R}_+)}, \|\langle y\rangle^{\lambda}\phi^{(j)}(y)\|_{L^\infty(\mathbb{R}_+)} \leq C \quad \text{for } i = 0, 1,\ j \geq 2,\ \lambda \in \mathbb{R}. \tag{3.30}$$

Estimate for G_1

$$\begin{aligned} G_1 = &-\frac{1}{2}\int_\Omega \{\langle y\rangle^{2l+2k}[(u + U\phi')\partial_x + (v - U_x\phi)\partial_y](|D^\alpha u|^2 + |D^\alpha h|^2)\} dx\, dy \\ &+ \int_\Omega \{\langle y\rangle^{2l+2k}[(h + H\phi')\partial_x + (g - H_x\phi)\partial_y](D^\alpha u \cdot D^\alpha h)\} dx\, dy. \end{aligned}$$

Although the coefficient $v - U_x\phi$ in G_1 has a linear growth as $y \to +\infty$, the term $\int_\Omega \langle y\rangle^{2l+2k}(v - U_x\phi)\partial_y(|D^\alpha u|^2 + |D^\alpha h|^2) dx\, dy$ is well-defined because

$$\begin{aligned} &\left|\int_\Omega \langle y\rangle^{2l+2k}(v - U_x\phi)\partial_y\big(|D^\alpha u|^2 + |D^\alpha h|^2\big) dx\, dy\right| \\ &= 2\left|\int_\Omega \frac{v - U_x\phi}{1+y}\big[\langle y\rangle^{l+k} D^\alpha u \cdot \langle y\rangle^{l+k+1}\partial_y D^\alpha u \right. \\ &\qquad\qquad \left. + \langle y\rangle^{l+k} D^\alpha h \cdot \langle y\rangle^{l+k+1}\partial_y D^\alpha h\big] dx\, dy\right| \\ &\leq 2\left\|\frac{v - U_x\phi}{1+y}\right\|_{L^\infty(\Omega)} \|\langle y\rangle^{l+k} D^\alpha(u,h)(t)\|_{L^2(\Omega)} \\ &\qquad\qquad \cdot \big\|\langle y\rangle^{l+k+1}\partial_y D^\alpha(u,h)(t)\big\|_{L^2(\Omega)}. \end{aligned}$$

A similar argument applies to the term

$$\int_\Omega \langle y\rangle^{2l+2k}(g - H_x\phi)\partial_y(D^\alpha u \cdot D^\alpha h) dx\, dy.$$

Then, from

$$\partial_x(u + U\phi') + \partial_y(v - U_x\phi) = 0, \quad \partial_x(h + H\phi') + \partial_y(g - H_x\phi) = 0,$$

and the boundary conditions $(v - U_x\phi)|_{y=0} = (g - H_x\phi)|_{y=0} = 0$, we obtain by integration by parts that

$$\begin{aligned} G_1 = &(l+k)\int_\Omega \{\langle y\rangle^{2l+2k-1}(v - U_x\phi) \cdot (|D^\alpha u|^2 + |D^\alpha h|^2)\} dx\, dy \\ &- 2(l+k)\int_\Omega \{\langle y\rangle^{2l+2k-1}(g - H_x\phi) \cdot (D^\alpha u \cdot D^\alpha h)\} dx\, dy. \end{aligned}$$

Note that the above equality still holds for $l+k=0$. Therefore, by using that $v=-\partial_y^{-1}\partial_x u$, $g=-\partial_y^{-1}\partial_x h$, and (3.30) with $i=0$, we get by virtue of (2.5) and the Sobolev embedding inequality that

$$
\begin{aligned}
G_1 &\le (l+k)\left(\left\|\frac{v-U_x\phi}{1+y}\right\|_{L^\infty(\Omega)}+\left\|\frac{g-H_x\phi}{1+y}\right\|_{L^\infty(\Omega)}\right)\\
&\quad\cdot\|\langle y\rangle^{l+k}D^\alpha(u,h)(t)\|^2_{L^2(\Omega)}\\
&\le C(\|u_x(t)\|_{L^\infty(\Omega)}+\|h_x(t)\|_{L^\infty(\Omega)}+\|(U_x,H_x)(t)\|_{L^\infty(\mathbb{T}_x)})\\
&\quad\cdot\|\langle y\rangle^{l+k}D^\alpha(u,h)(t)\|^2_{L^2(\Omega)}\\
&\le C(\|(u,h)(t)\|_{\mathcal{H}_0^3}+\|(U_x,H_x)(t)\|_{L^\infty(\mathbb{T}_x)})\|(u,h)(t)\|^2_{\mathcal{H}_l^m}.
\end{aligned}
\tag{3.31}
$$

Estimate for G_2

For G_2, note that

$$
\begin{aligned}
G_2 \le{}& \left\|I_1^2(t)\right\|_{L^2_{l+k}(\Omega)}\|D^\alpha u(t)\|_{L^2_{l+k}(\Omega)}\\
&+\left\|I_2^2(t)\right\|_{L^2_{l+k}(\Omega)}\|D^\alpha h(t)\|_{L^2_{l+k}(\Omega)}.
\end{aligned}
\tag{3.32}
$$

Thus, we need to obtain $\|I_1^2(t)\|_{L^2_{l+k}(\Omega)}$ and $\|I_2^2(t)\|_{L^2_{l+k}(\Omega)}$. To this end, we are going to estimate only the L^2_{l+k} of I_1^2, because the L^2_{l+k}-estimate on I_2^2 can be obtained similarly.

Rewrite the quantity I_1^2 as

$$
\begin{aligned}
I_1^2 &= \left[D^\alpha, u\partial_x+v\partial_y\right]u-\left[D^\alpha, h\partial_x+g\partial_y\right]h\\
&\quad+\left[D^\alpha, U\phi'\partial_x-U_x\phi\partial_y\right]u-\left[D^\alpha, H\phi'\partial_x-H_x\phi\partial_y\right]h\\
&\triangleq I_{1,1}^2+I_{1,2}^2.
\end{aligned}
\tag{3.33}
$$

In the following, we will estimate $\|I_{1,1}^2\|_{L^2_{l+k}(\Omega)}$ and $\|I_{1,2}^2\|_{L^2_{l+k}(\Omega)}$.

L^2_{l+k}-estimate of $I_{1,1}^2$: The quantity $I_{1,1}^2$ can be expressed as

$$
\begin{aligned}
I_{1,1}^2 = \sum_{0<\tilde\alpha\le\alpha}\binom{\alpha}{\tilde\alpha}\big\{&\big(D^{\tilde\alpha}u\,\partial_x+D^{\tilde\alpha}v\,\partial_y\big)(D^{\alpha-\tilde\alpha}u)\\
&-\big(D^{\tilde\alpha}h\,\partial_x+D^{\tilde\alpha}g\,\partial_y\big)(D^{\alpha-\tilde\alpha}h)\big\}.
\end{aligned}
\tag{3.34}
$$

Let $\tilde\alpha\triangleq(\tilde\beta,\tilde k)$. Then we will study the terms in (3.34) using the two cases where $\tilde k=0$ and $\tilde k\ge 1$.

Case 1. $\widetilde{k}=0$. First, $D^{\widetilde{\alpha}}=\partial_\tau^{\widetilde{\beta}}$ and $\widetilde{\beta}\geq e_i$, $i=1,2$, since $|\widetilde{\alpha}|>0$. Then we obtain by (2.4) that

$$\begin{aligned}\left\|D^{\widetilde{\alpha}}u\cdot\partial_x D^{\alpha-\widetilde{\alpha}}u\right\|_{L^2_{l+k}(\Omega)}&=\left\|\partial_\tau^{\widetilde{\beta}-e_i}(\partial_\tau^{e_i}u)\cdot D^{\alpha-\widetilde{\alpha}}(\partial_x u)\right\|_{L^2_{l+k}(\Omega)}\\&\leq C\left\|\partial_\tau^{e_i}u(t)\right\|_{\mathcal{H}_0^{m-1}}\|\partial_x u(t)\|_{\mathcal{H}_l^{m-1}}\\&\leq C\|u(t)\|^2_{\mathcal{H}_l^m},\end{aligned}$$

provided that $m-1\geq 3$. Similarly, it also holds that

$$\left\|D^{\widetilde{\alpha}}h\cdot\partial_x D^{\alpha-\widetilde{\alpha}}h\right\|_{L^2_{l+k}(\Omega)}\leq C\|h(t)\|^2_{\mathcal{H}_l^m}.$$

On the other hand, by using $v=-\partial_y^{-1}\partial_x u$, we have

$$D^{\widetilde{\alpha}}v\cdot\partial_y D^{\alpha-\widetilde{\alpha}}u=-\partial_\tau^{\widetilde{\beta}}\partial_y^{-1}(\partial_x u)\cdot\partial_\tau^{\beta-\widetilde{\beta}}\partial_y^{k+1}u.$$

Then, when $|\alpha|=|\beta|+k\leq m-1$, applying (2.7) to the right-hand side of the above equality yields

$$\begin{aligned}\left\|D^{\widetilde{\alpha}}v\cdot\partial_y D^{\alpha-\widetilde{\alpha}}u\right\|_{L^2_{l+k}(\Omega)}&=\left\|\partial_\tau^{\widetilde{\beta}}\partial_y^{-1}(\partial_x u)\cdot\partial_\tau^{\beta-\widetilde{\beta}}\partial_y^{k}(\partial_y u)\right\|_{L^2_{l+k}(\Omega)}\\&\leq C\|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}}\|\partial_y u(t)\|_{\mathcal{H}_{l+1}^{m-1}}\\&\leq C\|u(t)\|^2_{\mathcal{H}_l^m},\end{aligned}$$

provided that $m-1\geq 3$. When $|\alpha|=|\beta|+k=m$, it implies that $k\geq 1$ since $|\beta|\leq m-1$, and consequently, we get that by (2.7),

$$\begin{aligned}&\left\|D^{\widetilde{\alpha}}v\cdot\partial_y D^{\alpha-\widetilde{\alpha}}u\right\|_{L^2_{l+k}(\Omega)}\\&\quad=\left\|\partial_\tau^{\widetilde{\beta}-e_i}\partial_y^{-1}\left(\partial_\tau^{e_i+e_2}u\right)\cdot\partial_\tau^{\beta-\widetilde{\beta}}\partial_y^{k-1}\left(\partial_y^2 u\right)\right\|_{L^2_{l+1+(k-1)}(\Omega)}\\&\quad\leq C\left\|\partial_\tau^{e_i+e_2}u(t)\right\|_{\mathcal{H}_0^{m-2}}\left\|\partial_y^2 u(t)\right\|_{\mathcal{H}_{l+2}^{m-2}}\\&\quad\leq C\|u(t)\|^2_{\mathcal{H}_l^m},\end{aligned}$$

provided that $m-2\geq 3$. Therefore, it holds that for $|\alpha|=|\beta|+k\leq m$, $|\beta|\leq m-1$,

$$\left\|D^{\widetilde{\alpha}}v\cdot\partial_y D^{\alpha-\widetilde{\alpha}}u\right\|_{L^2_{l+k}(\Omega)}\leq C\|u(t)\|^2_{\mathcal{H}_l^m}.$$

Similarly, one can obtain

$$\left\|D^{\widetilde{\alpha}}g\cdot\partial_y D^{\alpha-\widetilde{\alpha}}h\right\|_{L^2_{l+k}(\Omega)}\leq C\|h(t)\|^2_{\mathcal{H}_l^m}.$$

Thus, we conclude that for $\widetilde{k} = 0$ with $\widetilde{\alpha} = (\widetilde{\beta}, \widetilde{k})$,

$$\begin{aligned}\big\|\big(D^{\widetilde{\alpha}}u\,\partial_x + D^{\widetilde{\alpha}}v\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}u)\\ -\big(D^{\widetilde{\alpha}}h\,\partial_x + D^{\widetilde{\alpha}}g\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}h)\big\|_{L^2_{l+k}(\Omega)}\\ \le C\|(u,h)(t)\|^2_{\mathcal{H}^m_l}.\end{aligned}\tag{3.35}$$

Case 2. $\widetilde{k} \ge 1$. It follows that $\widetilde{\alpha} \ge E_3$. Then the right-hand side of (3.34) becomes

$$\begin{aligned}&\big(D^{\widetilde{\alpha}}u\,\partial_x + D^{\widetilde{\alpha}}v\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}u) - \big(D^{\widetilde{\alpha}}h\,\partial_x + D^{\widetilde{\alpha}}g\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}h)\\ &\quad= \big(D^{\widetilde{\alpha}}u\,\partial_x - D^{\widetilde{\alpha}-E_3}(\partial_x u)\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}u) - \big(D^{\widetilde{\alpha}}h\,\partial_x\\ &\qquad - D^{\widetilde{\alpha}-E_3}(\partial_x h)\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}h).\end{aligned}$$

By applying (2.4) to the terms on the right-hand side of the above quality, we get

$$\begin{aligned}\big\|D^{\widetilde{\alpha}}u\cdot\partial_x D^{\alpha-\widetilde{\alpha}}u\big\|_{L^2_{l+k}(\Omega)} &= \big\|D^{\widetilde{\alpha}-E_3}(\partial_y u)\cdot D^{\alpha-\widetilde{\alpha}}(\partial_x u)\big\|_{L^2_{l+1+(k-1)}(\Omega)}\\ &\le C\|\partial_y u(t)\|_{\mathcal{H}^{m-1}_{l+1}}\|\partial_x u(t)\|_{\mathcal{H}^{m-1}_0}\\ &\le C\|u(t)\|^2_{\mathcal{H}^m_l},\end{aligned}$$

$$\begin{aligned}\big\|D^{\widetilde{\alpha}-E_3}(\partial_x u)\cdot\partial_y D^{\alpha-\widetilde{\alpha}}u\big\|_{L^2_{l+k}(\Omega)} &= \big\|D^{\widetilde{\alpha}-E_3}(\partial_x u)\cdot D^{\alpha-\widetilde{\alpha}}(\partial_y u)\big\|_{L^2_{l+1+(k-1)}(\Omega)}\\ &\le C\|\partial_x u(t)\|_{\mathcal{H}^{m-1}_0}\|\partial_y u(t)\|_{\mathcal{H}^{m-1}_{l+1}}\\ &\le C\|u(t)\|^2_{\mathcal{H}^m_l},\end{aligned}$$

and similarly,

$$\big\|D^{\widetilde{\alpha}}h\cdot\partial_x D^{\alpha-\widetilde{\alpha}}h\big\|_{L^2_{l+k}(\Omega)} + \big\|D^{\widetilde{\alpha}-E_3}(\partial_x h)\cdot\partial_y D^{\alpha-\widetilde{\alpha}}h\big\|_{L^2_{l+k}(\Omega)} \le C\|h(t)\|^2_{\mathcal{H}^m_l}.$$

Consequently, we actually conclude that for $\widetilde{k} \ge 1$ with $\widetilde{\alpha} = (\widetilde{\beta}, \widetilde{k})$,

$$\begin{aligned}\big\|\big(D^{\widetilde{\alpha}}u\,\partial_x + D^{\widetilde{\alpha}}v\,\partial_y\big)\big(D^{\alpha-\widetilde{\alpha}}u\big)\\ -\big(D^{\widetilde{\alpha}}h\,\partial_x + D^{\widetilde{\alpha}}g\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}h)\big\|_{L^2_{l+k}(\Omega)}\\ \le C\|(u,h)(t)\|^2_{\mathcal{H}^m_l}.\end{aligned}\tag{3.36}$$

Finally, based on the results obtained in the above two cases, it holds that by using (3.35) and (3.36) in (3.34),

$$\big\|I^2_{1,1}(t)\big\|_{L^2_{l+k}(\Omega)} \le C\|(u,h)(t)\|^2_{\mathcal{H}^m_l}.\tag{3.37}$$

For the L^2_{l+k}-estimate on $I^2_{1,2}$, write

$$\begin{aligned}I^2_{1,2} = \sum_{0<\widetilde{\alpha}\le\alpha}\binom{\alpha}{\widetilde{\alpha}}\big\{&\big(D^{\widetilde{\alpha}}(U\phi')\,\partial_x - D^{\widetilde{\alpha}}(U_x\phi)\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}u)\\ &-\big(D^{\widetilde{\alpha}}(H\phi')\,\partial_x - D^{\widetilde{\alpha}}(H_x\phi)\,\partial_y\big)(D^{\alpha-\widetilde{\alpha}}h)\big\}.\end{aligned}$$

Let $\widetilde{\alpha} \triangleq (\widetilde{\beta}, \widetilde{k})$ and note that $|\alpha - \widetilde{\alpha}| \le |\alpha| - 1 \le m - 1$. By using (3.30), we estimate each term on the right-hand side of the above equility as follows:

$$\begin{aligned}
&\big\| D^{\widetilde{\alpha}}(U\phi') \cdot \partial_x D^{\alpha-\widetilde{\alpha}} u \big\|_{L^2_{l+k}(\Omega)} \\
&\quad \le \big\| \langle y\rangle^{\widetilde{k}} D^{\widetilde{\alpha}}(U\phi')(t) \big\|_{L^\infty(\Omega)} \big\| \langle y\rangle^{l+k-\widetilde{k}} \partial_x D^{\alpha-\widetilde{\alpha}} u(t) \big\|_{L^2(\Omega)} \\
&\quad \le C \big\| \partial_\tau^{\widetilde{\beta}} U(t) \big\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}^m_l}, \\
&\big\| D^{\widetilde{\alpha}}(U_x\phi) \cdot \partial_y D^{\alpha-\widetilde{\alpha}} u \big\|_{L^2_{l+k}(\Omega)} \\
&\quad \le \big\| \langle y\rangle^{\widetilde{k}-1} D^{\widetilde{\alpha}}(U_x\phi)(t) \big\|_{L^\infty(\Omega)} \big\| \langle y\rangle^{l+k-\widetilde{k}+1} \partial_y D^{\alpha-\widetilde{\alpha}} u(t) \big\|_{L^2(\Omega)} \\
&\quad \le C \big\| \partial_\tau^{\widetilde{\beta}} U_x(t) \big\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}^m_l},
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\big\| D^{\widetilde{\alpha}}(H\phi') \cdot \partial_x D^{\alpha-\widetilde{\alpha}} h \big\|_{L^2_{l+k}(\Omega)} + \big\| D^{\widetilde{\alpha}}(H_x\phi) \cdot \partial_y D^{\alpha-\widetilde{\alpha}} h \big\|_{L^2_{l+k}(\Omega)} \\
&\quad \le C \big(\big\| \partial_\tau^{\widetilde{\beta}} H(t) \big\|_{L^\infty(\mathbb{T}_x)} + \big\| \partial_\tau^{\widetilde{\beta}} H_x(t) \big\|_{L^\infty(\mathbb{T}_x)} \big) \|h(t)\|_{\mathcal{H}^m_l}.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\big\| I^2_{1,2}(t) \big\|_{L^2_{l+k}(\Omega)} \le C \Big(\sum_{|\beta| \le m+1} \big\| \partial_\tau^\beta (U, H)(t) \big\|_{L^\infty(\mathbb{T}_x)} \Big) \\
\cdot \|(u, h)(t)\|_{\mathcal{H}^m_l}.
\end{aligned} \tag{3.38}$$

Now, we can obtain the estimate of $\|I^2_1\|_{L^2_{l+k}(\Omega)}$. Indeed, plugging (3.37) and (3.38) into (3.33) yields

$$\begin{aligned}
&\big\| I^2_1(t) \big\|_{L^2_{l+k}(\Omega)} \le \\
&\quad C \Big(\sum_{|\beta| \le m+1} \big\| \partial_\tau^\beta (U, H)(t) \big\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}^m_l} \Big) \|(u, h)(t)\|_{\mathcal{H}^m_l}.
\end{aligned} \tag{3.39}$$

Also, one can get that, by similar arguments,

$$\begin{aligned}
&\big\| I^2_2(t) \big\|_{L^2_{l+k}(\Omega)} \le \\
&\quad C \Big(\sum_{|\beta| m+1} \big\| \partial_\tau^\beta (U, H)(t) \big\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}^m_l} \Big) \|(u, h)(t)\|_{\mathcal{H}^m_l}.
\end{aligned} \tag{3.40}$$

Then, substituting (3.39) and (3.40) into (3.32) gives

$$
\begin{aligned}
G_2 &\le C\Big(\sum_{|\beta|\le m+1}\big\|\partial_\tau^\beta(U,H)(x)\big\|_{L^\infty(\mathbb{T}_x)}+\|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)\|(u,h)(t)\|_{\mathcal{H}_l^m}\\
&\quad\cdot\|D^\alpha(u,h)(t)\|_{L^2_{l+k}(\Omega)}\\
&\le C\Big(\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^2(\mathbb{T}_x)}+\|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)\|(u,h)(t)\|^2_{\mathcal{H}_l^m}.
\end{aligned}
\tag{3.41}
$$

Estimate for G_3

For G_3, the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
G_3 \le{}& \big\|I_1^3(t)\big\|_{L^2_{l+k}(\Omega)}\|D^\alpha u(t)\|_{L^2_{l+k}(\Omega)}\\
&+\big\|I_2^3(t)\big\|_{L^2_{l+k}(\Omega)}\|D^\alpha h(t)\|_{L^2_{l+k}(\Omega)}.
\end{aligned}
\tag{3.42}
$$

It remains to estimate $\|I_1^3(t)\|_{L^2_{l+k}(\Omega)}$ and $\|I_2^3(t)\|_{L^2_{l+k}(\Omega)}$. In the following, we are going to establish the weighted estimate on I_1^3, for example, and the weighted estimate on I_2^3 can be obtained in a similar way.

Recalling that $D^\alpha=\partial_\tau^\beta\partial_y^k$, we have

$$
\begin{aligned}
I_1^3=\sum_{\widetilde\alpha\le\alpha}\binom{\alpha}{\widetilde\alpha}\big[&D^{\widetilde\alpha}u\cdot D^{\alpha-\widetilde\alpha}(U_x\phi')+D^{\widetilde\alpha}v\cdot D^{\alpha-\widetilde\alpha}(U\phi'')\\
&-D^{\widetilde\alpha}h\cdot D^{\alpha-\widetilde\alpha}(H_x\phi')-D^{\widetilde\alpha}g\cdot D^{\alpha-\widetilde\alpha}(H\phi'')\big].
\end{aligned}
\tag{3.43}
$$

Then, letting $\widetilde\alpha\triangleq(\widetilde\beta,\widetilde k)$, we estimate each term in (3.43) as follows. First, by using (3.30) we have

$$
\begin{aligned}
&\big\|D^{\widetilde\alpha}u\cdot D^{\alpha-\widetilde\alpha}(U_x\phi')\big\|_{L^2_{l+k}(\Omega)}\\
&\le\big\|\langle y\rangle^{l+\widetilde k}D^{\widetilde\alpha}u(t)\big\|_{L^2(\Omega)}\big\|\langle y\rangle^{k-\widetilde k}D^{\alpha-\widetilde\alpha}(U_x\phi')(t)\big\|_{L^\infty(\Omega)}\\
&\le C\|u(t)\|_{\mathcal{H}_l^m}\big\|\partial_\tau^{\beta-\widetilde\beta}U_x(t)\big\|_{L^\infty(\mathbb{T}_x)},
\end{aligned}
$$

and similarly,

$$
\big\|D^{\widetilde\alpha}h\cdot D^{\alpha-\widetilde\alpha}(H_x\phi')\big\|_{L^2_{l+k}(\Omega)}\le C\|h(t)\|_{\mathcal{H}_l^m}\big\|\partial_\tau^{\beta-\widetilde\beta}H_x(t)\big\|_{L^\infty(\mathbb{T}_x)}.
$$

Second, as $v=-\partial_y^{-1}\partial_x u$, we get

$$
D^{\widetilde\alpha}v\cdot D^{\alpha-\widetilde\alpha}(U\phi'')=-D^{\widetilde\alpha+E_2}\partial_y^{-1}u\cdot D^{\alpha-\widetilde\alpha}(U\phi'').
$$

Therefore, if $\widetilde k\ge1$, it follows from (3.30),

$$
\begin{aligned}
&\|D^{\widetilde\alpha}v\cdot D^{\alpha-\widetilde\alpha}(U\phi'')\|_{L^2_{l+k}(\Omega)}\\
&\quad=\big\|\partial_\tau^{\widetilde\beta+e_2}\partial_y^{\widetilde k-1}u\cdot\partial_\tau^{\beta-\widetilde\beta}\partial_y^{k-\widetilde k}(U\phi'')\big\|_{L^2_{l+k}(\Omega)}\le
\end{aligned}
$$

$$\leq \big\|\langle y\rangle^{\widetilde{k}-1}\partial_\tau^{\widetilde{\beta}+e_2}\partial_y^{\widetilde{k}-1}u(t)\big\|_{L^2(\Omega)}\big\|\langle y\rangle^{l+k-\widetilde{k}+1}\partial_\tau^{\beta-\widetilde{\beta}}\partial_y^{k-\widetilde{k}}(U\phi'')(t)\big\|_{L^\infty(\Omega)}$$
$$\leq C\|u(t)\|_{\mathcal{H}_0^m}\big\|\partial_\tau^{\beta-\widetilde{\beta}}U(t)\big\|_{L^\infty(\mathbb{T}_x)};$$

if $\widetilde{k}=0$, we obtain from (2.7) and (3.30),

$$\begin{aligned}
&\|D^{\widetilde{\alpha}}v\cdot D^{\alpha-\widetilde{\alpha}}(U\phi'')\|_{L^2_{l+k}(\Omega)}\\
&\quad=\left\|\frac{\partial_\tau^{\widetilde{\beta}+e_2}\partial_y^{-1}u}{1+y}\cdot\partial_\tau^{\beta-\widetilde{\beta}}\partial_y^k(U\phi'')\right\|_{L^2_{l+k+1}(\Omega)}\\
&\quad\leq\left\|\frac{\partial_\tau^{\widetilde{\beta}+e_2}\partial_y^{-1}u(t)}{1+y}\right\|_{L^2(\Omega)}\big\|\langle y\rangle^{l+k+1}\partial_\tau^{\beta-\widetilde{\beta}}\partial_y^k(U\phi'')(t)\big\|_{L^\infty(\Omega)}\\
&\quad\leq C\big\|\partial_\tau^{\widetilde{\beta}+e_2}u(t)\big\|_{L^2(\Omega)}\big\|\partial_\tau^{\beta-\widetilde{\beta}}U(t)\big\|_{L^\infty(\mathbb{T}_x)}\\
&\quad\leq C\|u(t)\|_{\mathcal{H}_0^m}\big\|\partial_\tau^{\beta-\widetilde{\beta}}U(t)\big\|_{L^\infty(\mathbb{T}_x)},
\end{aligned}$$

provided that $|\widetilde{\beta}|\leq|\beta|\leq m-1$. Combining the above two inequalities yields that

$$\|D^{\widetilde{\alpha}}v\cdot D^{\alpha-\widetilde{\alpha}}(U\phi'')\|_{L^2_{l+k}(\Omega)}\leq C\|u(t)\|_{\mathcal{H}_0^m}\cdot\Big(\sum_{|\beta|\leq m+1}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^\infty(\mathbb{T}_x)}\Big).$$

Similarly, we have

$$\|D^{\widetilde{\alpha}}g\cdot D^{\alpha-\widetilde{\alpha}}(H\phi'')\|_{L^2_{l+k}(\Omega)}\leq C\|h(t)\|_{\mathcal{H}_0^m}\cdot\Big(\sum_{|\beta|\leq m+1}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^\infty(\mathbb{T}_x)}\Big).$$

We take into account the above arguments to conclude that

$$\big\|I_1^3(t)\big\|_{L^2_{l+k}(\Omega)}\leq C\|(u,h)(t)\|_{\mathcal{H}_l^m}\cdot\Big(\sum_{|\beta|\leq m+1}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^\infty(\mathbb{T}_x)}\Big). \tag{3.44}$$

Then one can obtain a similar estimate of I_2^3:

$$\big\|I_2^3(t)\big\|_{L^2_{l+k}(\Omega)}\leq C\|(u,h)(t)\|_{\mathcal{H}_l^m}\cdot\Big(\sum_{|\beta|\leq m+1}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^\infty(\mathbb{T}_x)}\Big), \tag{3.45}$$

which implies, by plugging (3.44) and (3.45) into (3.42),

$$\begin{aligned}
G_3&\leq C\|D^\alpha(u,h)(t)\|_{L^2_{l+k}(\Omega)}\|(u,h)(t)\|_{\mathcal{H}_l^m}\\
&\qquad\cdot\Big(\sum_{|\beta|\leq m+1}\big\|\partial_\tau^\beta(U,H)(x)\big\|_{L^\infty(\mathbb{T}_x)}\Big)\\
&\leq C\|(u,h)(t)\|^2_{\mathcal{H}_l^m}\cdot\Big(\sum_{|\beta|\leq m+2}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^2(\mathbb{T}_x)}\Big).
\end{aligned}\tag{3.46}$$

Now, as we have completed the estimates on G_i, $i = 1, 2, 3$, given by (3.31), (3.41), and (3.46), respectively, from (3.29) the conclusion of this step follows immediately:

$$-\int_\Omega \big(I_1 \cdot \langle y\rangle^{2l+2k} D^\alpha u + I_2 \cdot \langle y\rangle^{2l+2k} D^\alpha h\big) dx\, dy \\ \leq C\Big(\sum_{|\beta|\leq m+2} \big\|\partial_\tau^\beta (U,H)(t)\big\|_{L^2(\mathbb{T}_x)} + \|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)\|(u,h)(t)\|^2_{\mathcal{H}_l^m},$$

and we complete the proof of (3.11). □

3.2 Weighted H_l^m-Estimates Only in Tangential Variables

Similar to the classical Prandtl equations, an essential difficulty for solving the problem (2.12) is the loss of one derivative in the tangential variable x in the terms $v\partial_y u - g\partial_y h$ and $v\partial_y h - g\partial_y u$. In other words, $v = -\partial_y^{-1}\partial_x u$ and $g = -\partial_y^{-1}\partial_x h$, by the divergence-free conditions, create a loss of the x-derivative that prevents us from applying the standard energy estimates.

Specifically, consider the following equations of $\partial_\tau^\beta(u,h)$ with $|\beta| = m$ by taking the m^{th}-order tangential derivatives on the first two equations of (2.12),

$$\begin{aligned} &\partial_t \partial_\tau^\beta u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\partial_\tau^\beta u \\ &\quad - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]\partial_\tau^\beta h \\ &\quad - \mu\partial_y^2\partial_\tau^\beta u + (\partial_y u + U\phi'')\partial_\tau^\beta v \\ &\quad - (\partial_y h + H\phi'')\partial_\tau^\beta g = \partial_\tau^\beta r_1 + R_u^\beta, \\ &\partial_t \partial_\tau^\beta h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\partial_\tau^\beta h \\ &\quad - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]\partial_\tau^\beta u \\ &\quad - \kappa\partial_y^2\partial_\tau^\beta h + (\partial_y h + H\phi'')\partial_\tau^\beta v \\ &\quad\quad - (\partial_y u + U\phi'')\partial_\tau^\beta g = \partial_\tau^\beta r_2 + R_h^\beta, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} R_u^\beta = {}& \partial_\tau^\beta(-U_x\phi' u + H_x\phi' h) - \big[\partial_\tau^\beta, U\phi''\big]v + \big[\partial_\tau^\beta, H\phi''\big]g \\ & - \big[\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y\big]u \\ & + \big[\partial_\tau^\beta, (h + H\phi')\partial_x - H_x\phi\partial_y\big]h \\ & - \sum_{0<\tilde\beta<\beta} \binom{\beta}{\tilde\beta}\big(\partial_\tau^{\tilde\beta} v \cdot \partial_\tau^{\beta-\tilde\beta}\partial_y u - \partial_\tau^{\tilde\beta} g \cdot \partial_\tau^{\beta-\tilde\beta}\partial_y h\big), \end{aligned} \tag{3.48}$$

$$
\begin{aligned}
R_h^\beta &= \partial_\tau^\beta(-H_x\phi' u + U_x\phi' h) - \big[\partial_\tau^\beta, H\phi''\big]v + \big[\partial_\tau^\beta, U\phi''\big]g \\
&\quad - \big[\partial_\tau^\beta, (u+U\phi')\partial_x - U_x\phi\partial_y\big]h \\
&\quad + \big[\partial_\tau^\beta, (h+H\phi')\partial_x - H_x\phi\partial_y\big]u \\
&\quad - \sum_{0<\widetilde{\beta}<\beta}\binom{\beta}{\widetilde{\beta}}\big(\partial_\tau^{\widetilde{\beta}}v\cdot\partial_\tau^{\beta-\widetilde{\beta}}\partial_y h - \partial_\tau^{\widetilde{\beta}}g\cdot\partial_\tau^{\beta-\widetilde{\beta}}\partial_y u\big).
\end{aligned}
$$

From the expression (3.48) and by using the inequalities (2.4)–(2.7), we can control the $L_l^2(\Omega)$-estimates of each term given in (3.48) and then obtain the estimates of $\|R_u^\beta(t)\|_{L_l^2(\Omega)}$ and $\|R_h^\beta(t)\|_{L_l^2(\Omega)}$. For example, for $\widetilde{\beta} > 0$, which implies that $\widetilde{\beta} \geq e_i$, $i = 1,2$, we have by virtue of (2.4),

$$
\begin{aligned}
&\big\|\big[\partial_\tau^{\widetilde{\beta}}(u+U\phi')\partial_x - \partial_\tau^{\widetilde{\beta}}(U_x\phi)\partial_y\big]\big(\partial_\tau^{\beta-\widetilde{\beta}}u\big)\big\|_{L_l^2(\Omega)} \\
&\quad\leq \big\|\partial_\tau^{\widetilde{\beta}-e_i}(\partial_\tau^{e_i}u)\cdot\partial_\tau^{\beta-\widetilde{\beta}}(\partial_x u)\big\|_{L_l^2(\Omega)} \\
&\qquad + \big\|\partial_\tau^{\widetilde{\beta}}(U\phi')(t)\big\|_{L^\infty(\Omega)}\big\|\partial_x\partial_\tau^{\beta-\widetilde{\beta}}u(t)\big\|_{L_l^2(\Omega)} \\
&\qquad + \left\|\frac{\partial_\tau^{\widetilde{\beta}}(U_x\phi)(t)}{1+y}\right\|_{L^\infty(\Omega)}\big\|\partial_y\partial_\tau^{\beta-\widetilde{\beta}}u(t)\big\|_{L_{l+1}^2(\Omega)} \\
&\quad\leq C\|\partial_\tau^{e_i}u(t)\|_{\mathcal{H}_0^{m-1}}\|\partial_x u(t)\|_{\mathcal{H}_l^{m-1}} + C\big\|\partial_\tau^{\widetilde{\beta}}(U,U_x)(t)\big\|_{L^\infty(\mathbb{T}_x)}\|u(t)\|_{\mathcal{H}_l^m} \\
&\quad\leq C\big(\big\|\partial_\tau^{\widetilde{\beta}}(U,U_x)(t)\big\|_{L^\infty(\mathbb{T}_x)} + \|u(t)\|_{\mathcal{H}_l^m}\big)\|u(t)\|_{\mathcal{H}_l^m},
\end{aligned}
$$

provided $m-1 \geq 3$ and $|\beta - \widetilde{\beta}| \leq m-1$, and (2.7) gives that for $\widetilde{\beta} < \beta$,

$$
\begin{aligned}
&\big\|\partial_\tau^{\widetilde{\beta}}v\cdot\partial_\tau^{\beta-\widetilde{\beta}}(U\phi'')\big\|_{L_l^2(\Omega)} \\
&\quad\leq \left\|\frac{\partial_\tau^{\widetilde{\beta}+e_2}\partial_y^{-1}u(t)}{1+y}\right\|_{L^2(\Omega)}\big\|\langle y\rangle^{l+1}\partial_\tau^{\beta-\widetilde{\beta}}(U\phi'')(t)\big\|_{L^\infty(\Omega)} \\
&\quad\leq C\big\|\partial_\tau^{\beta-\widetilde{\beta}}U(t)\|_{L^\infty(\mathbb{T}_x)}\|u(t)\big\|_{\mathcal{H}_0^m};
\end{aligned}
$$

moreover, for $0 < \widetilde{\beta} < \beta$, which implies that $\widetilde{\beta} \geq e_i$, $\beta - \widetilde{\beta} \geq e_j$, $i, j = 1,2$, we obtain by virtue of (2.7),

$$
\begin{aligned}
\big\|\partial_\tau^{\widetilde{\beta}}v\cdot\partial_\tau^{\beta-\widetilde{\beta}}(\partial_y u)\big\|_{L_l^2(\Omega)} &= \big\|\partial_\tau^{\widetilde{\beta}-e_i}\partial_y^{-1}(\partial_\tau^{e_i+e_2}u)\cdot\partial_\tau^{\beta-\widetilde{\beta}-e_j}\big(\partial_\tau^{e_j}\partial_y u\big)\big\|_{L_l^2(\Omega)} \\
&\leq C\|\partial_\tau^{e_i+e_2}u(t)\|_{\mathcal{H}_0^{m-2}}\big\|\partial_y\partial_\tau^{e_j}u(t)\big\|_{\mathcal{H}_{l+1}^{m-2}} \\
&\leq C\|u(t)\|_{\mathcal{H}_l^m}^2
\end{aligned}
$$

provided $m-2\geq 3$. The other terms in R_u^β and R_h^β can be estimated similarly so that

$$\|\big(R_u^\beta, R_h^\beta\big)(t)\|_{L_l^2(\Omega)} \leq C\Big(\sum_{|\beta|\leq m+2}\|\partial_\tau^\beta(U,H)(t)\|_{L^2(\mathbb{T}_x)}+\|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)\|(u,h)(t)\|_{\mathcal{H}_l^m}. \tag{3.49}$$

On the other hand, consider the equations (3.47). The main difficulty comes from the terms

$$(\partial_y u+U\phi'')\partial_\tau^\beta v-\big(\partial_y h+H\phi''\big)\partial_\tau^\beta g = -(\partial_y u+U\phi'')\cdot\big(\partial_y^{-1}\partial_\tau^{\beta+e_2}u\big)+\big(\partial_y h+H\phi''\big)\cdot\big(\partial_y^{-1}\partial_\tau^{\beta+e_2}h\big)$$

and

$$(\partial_y h+H\phi'')\partial_\tau^\beta v-(\partial_y u+U\phi'')\partial_\tau^\beta g = -(\partial_y h+H\phi'')\cdot\big(\partial_y^{-1}\partial_\tau^{\beta+e_2}u\big)+(\partial_y u+U\phi'')\cdot\big(\partial_y^{-1}\partial_\tau^{\beta+e_2}h\big),$$

which contain the $(m+1)^{\text{th}}$-order tangential derivatives, and they cannot be controlled by the standard energy method. To overcome this difficulty, we rely on the following two key observations. One is that from the equation (2.14), $\partial_y^{-1}h$ satisfies one of the following equations (see also the equation (3.54) for ψ):

$$\partial_t(\partial_y^{-1}h)+(v-U_x\phi)(h+H\phi') -(g-H_x\phi)(u+U\phi')-\kappa\partial_y h=-H_t\phi+\kappa H\phi''$$

or

$$\partial_t(\partial_y^{-1}h)+(h+H\phi')v+(u+U\phi')\partial_x(\partial_y^{-1}h) -U_x\phi h+H_x\phi u-\kappa\partial_y h=H_t\phi(\phi'-1)+\kappa H\phi'',$$

by using $g=-\partial_x\partial_y^{-1}h$ and the second relation of (1.6). This inspires us in the case of $h+H\phi'>0$ to introduce the following two quantities:

$$u_\beta := \partial_\tau^\beta u-\frac{\partial_y u+U\phi''}{h+H\phi'}\partial_\tau^\beta\partial_y^{-1}h,\quad h_\beta := \partial_\tau^\beta h-\frac{\partial_y h+H\phi''}{h+H\phi'}\partial_\tau^\beta\partial_y^{-1}h, \tag{3.50}$$

to eliminate the terms involving $\partial_\tau^\beta v$, then to avoid the loss of the x-derivative on v. Note that the new quantities (u_β,h_β) are almost equivalent to $\partial_\tau^\beta(u,h)$ in the L_l^2-norm, that is,

$$\|\partial_\tau^\beta(u,h)\|_{L_l^2(\Omega)}\lesssim\|(u_\beta,h_\beta)\|_{L_l^2(\Omega)}\lesssim\|\partial_\tau^\beta(u,h)\|_{L_l^2(\Omega)},$$

which will be proved at the end of this subsection.

Another observation is that by using the above two new unknowns (u_β, h_β) in (3.50), the regularity loss generated by $g = -\partial_y^{-1}\partial_x h$ can be cancelled by using the convection terms $-(h + H\phi')\partial_x h$ and $-(h + H\phi')\partial_x u$; more precisely,

$$\begin{aligned}
&-(h + H\phi')\partial_x\partial_\tau^\beta h - (\partial_y h + H\phi'')\partial_\tau^\beta g \\
&\quad = -(h + H\phi')\partial_x\left(h_\beta + \frac{\partial_y h + H\phi''}{h + H\phi'}\partial_\tau^\beta\partial_y^{-1}h\right) \\
&\qquad + (\partial_y h + H\phi'')\cdot\left(\partial_y^{-1}\partial_\tau^{\beta+e_2}h\right) \\
&\quad = -(h + H\phi')\partial_x h_\beta - (h + H\phi')\partial_x\left(\frac{\partial_y h + H\phi''}{h + H\phi'}\right)\cdot\partial_\tau^\beta\partial_y^{-1}h
\end{aligned}$$

and

$$\begin{aligned}
&-(h + H\phi')\partial_x\partial_\tau^\beta u - (\partial_y u + U\phi'')\partial_\tau^\beta g \\
&\quad = -(h + H\phi')\partial_x\left(u_\beta + \frac{\partial_y u + U\phi''}{h + H\phi'}\partial_\tau^\beta\partial_y^{-1}h\right) \\
&\qquad + (\partial_y u + U\phi'')\cdot\left(\partial_y^{-1}\partial_\tau^{\beta+e_2}h\right) \\
&\quad = -(h + H\phi')\partial_x u_\beta - (h + H\phi')\partial_x\left(\frac{\partial_y u + U\phi''}{h + H\phi'}\right)\cdot\partial_\tau^\beta\partial_y^{-1}h.
\end{aligned}$$

This cancellation mechanism reveals the stabilizing effect of the magnetic field on the boundary layer. Note that in the above expressions, the convection terms can be handled by the symmetric structure of the system.

Based on the above discussion, we will carry out the estimation as follows. First of all, we always assume that there exists a positive constant $\delta_0 \le 1$ such that

$$h(t,x,y) + H(t,x)\phi'(y) \ge \delta_0 \quad \text{for } (t,x,y) \in [0,T]\times\Omega. \tag{3.51}$$

Then, from the divergence-free condition $\partial_x h + \partial_y g = 0$, there exists a stream function ψ such that

$$h = \partial_y\psi, \quad g = -\partial_x\psi, \quad \psi\big|_{y=0} = 0. \tag{3.52}$$

Then, the equation (2.14) for h reads

$$\begin{aligned}
&\partial_t\partial_y\psi + \partial_y[(v - U_x\phi)(\partial_y\psi + H\phi') + (\partial_x\psi + H_x\phi)(u + U\phi')] - \kappa\partial_y^3\psi \\
&\quad = -H_t\phi' + \kappa H\phi^{(3)}.
\end{aligned} \tag{3.53}$$

By virtue of the boundary conditions

$$\partial_t\psi\big|_{y=0} = \partial_x\psi\big|_{y=0} = \partial_y^2\psi\big|_{y=0} = v\big|_{y=0} = 0$$

and $\phi(y) \equiv 0$ for $y \in [0, R_0]$, we integrate the equation (3.53) with respect to the variable y over $[0, y]$ to obtain

$$\begin{aligned}
&\partial_t\psi + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\psi + H_x\phi u \\
&\qquad\qquad + H\phi' v - \kappa\partial_y^2\psi = r_3
\end{aligned} \tag{3.54}$$

with

$$r_3 = H_t\phi(\phi' - 1) + \kappa H\phi^{(3)}. \tag{3.55}$$

Next, applying the m^{th}-order tangential derivatives operator on (3.54) and by virtue of $\partial_y\psi = h$, we obtain

$$\begin{aligned} &\partial_t\partial_\tau^\beta\psi + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\partial_\tau^\beta\psi \\ &\qquad\qquad + (h + H\phi')\partial_\tau^\beta v - \kappa\partial_y^2\partial_\tau^\beta\psi = \partial_\tau^\beta r_3 + R_\psi^\beta, \end{aligned} \tag{3.56}$$

where R_ψ^β is defined as follows:

$$\begin{aligned} R_\psi^\beta &= -\partial_\tau^\beta(H_x\phi u) - \big[\partial_\tau^\beta, H\phi'\big]v - \big[\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y\big]\psi \\ &\quad - \sum_{0<\widetilde{\beta}<\beta}\binom{\beta}{\widetilde{\beta}}\big(\partial_\tau^{\widetilde{\beta}}v \cdot \partial_\tau^{\beta-\widetilde{\beta}}\partial_y\psi\big). \end{aligned} \tag{3.57}$$

From $\psi = \partial_y^{-1}h$ and $v = -\partial_x\partial_y^{-1}u$, we get

$$\begin{aligned} R_\psi^\beta &= -\partial_\tau^\beta(H_x\phi u) + \big[\partial_\tau^\beta, H\phi'\big]\partial_x\partial_y^{-1}u - \big[\partial_\tau^\beta, (u + U\phi')\big]\partial_x\partial_y^{-1}h \\ &\quad + \big[\partial_\tau^\beta, U_x\phi\big]h + \sum_{0<\widetilde{\beta}<\beta}\binom{\beta}{\widetilde{\beta}}\big(\partial_\tau^{\widetilde{\beta}+e_2}\partial_y^{-1}u \cdot \partial_\tau^{\beta-\widetilde{\beta}}h\big) \\ &= -\sum_{\widetilde{\beta}\le\beta}\binom{\beta}{\widetilde{\beta}}\big[\partial_\tau^{\widetilde{\beta}}(H_x\phi) \cdot \partial_\tau^{\beta-\widetilde{\beta}}u\big] \\ &\quad + \sum_{0<\widetilde{\beta}\le\beta}\binom{\beta}{\widetilde{\beta}}\big[\partial_\tau^{\widetilde{\beta}}(H\phi') \cdot \partial_\tau^{\beta-\widetilde{\beta}+e_2}\partial_y^{-1}u - \partial_\tau^{\widetilde{\beta}}(u + U\phi') \cdot \partial_\tau^{\beta-\widetilde{\beta}+e_2}\partial_y^{-1}h \\ &\qquad\qquad + \partial_\tau^{\widetilde{\beta}}(U_x\phi) \cdot \partial_\tau^{\beta-\widetilde{\beta}}h\big] \\ &\quad + \sum_{0<\widetilde{\beta}<\beta}\binom{\beta}{\widetilde{\beta}}\big(\partial_\tau^{\widetilde{\beta}+e_2}\partial_y^{-1}u \cdot \partial_\tau^{\beta-\widetilde{\beta}}h\big), \end{aligned}$$

and then we can estimate $\|R_\psi^\beta(t)/(1 + y)\|_{L^2(\Omega)}$ from the above expression term by term.

For example, it is easy to get that

$$\begin{aligned} \left\|\frac{\partial_\tau^{\widetilde{\beta}}(H_x\phi) \cdot \partial_\tau^{\beta-\widetilde{\beta}}u}{1 + y}\right\|_{L^2(\Omega)} &\le \left\|\frac{\partial_\tau^{\widetilde{\beta}}(H_x\phi)(t)}{1 + y}\right\|_{L^\infty(\Omega)}\big\|\partial_\tau^{\beta-\widetilde{\beta}}u(t)\big\|_{L^2(\Omega)} \\ &\le C\,\big\|\partial_\tau^{\widetilde{\beta}}H_x(t)\big\|_{L^\infty(\mathbb{T}_x)}\|u(t)\|_{\mathcal{H}_0^m}, \end{aligned}$$

and (2.7) implies that

$$\begin{aligned}
&\left\| \frac{\partial_\tau^{\widetilde{\beta}}(H\phi') \cdot \partial_\tau^{\beta-\widetilde{\beta}+e_2}\partial_y^{-1}u}{1+y} \right\|_{L^2(\Omega)} \\
&\quad \le \left\| \partial_\tau^{\widetilde{\beta}}(H\phi')(t) \right\|_{L^\infty(\Omega)} \left\| \frac{\partial_\tau^{\beta-\widetilde{\beta}+e_2}\partial_y^{-1}u(t)}{1+y} \right\|_{L^2(\Omega)} \\
&\quad \le C \left\| \partial_\tau^{\widetilde{\beta}} H(t) \right\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}_0^m},
\end{aligned}$$

provided $|\beta - \widetilde{\beta}| \le |\beta| - 1 = m - 1$. Also, (2.7) allows us to get that for $\widetilde{\beta} \ge e_i$, $i = 1, 2$,

$$\begin{aligned}
&\left\| \frac{\partial_\tau^{\widetilde{\beta}} u \cdot \partial_\tau^{\beta-\widetilde{\beta}+e_2}\partial_y^{-1}h}{1+y} \right\|_{L^2(\Omega)} \\
&\quad = \left\| \partial_\tau^{\widetilde{\beta}-e_i}(\partial_\tau^{e_i}u) \cdot \partial_\tau^{\beta-\widetilde{\beta}}\partial_y^{-1}(\partial_x h) \right\|_{L^2_{-1}(\Omega)} \\
&\quad \le C \|\partial_\tau^{e_i}u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_x h(t)\|_{\mathcal{H}_0^{m-1}} \le C \|(u,h)(t)\|_{\mathcal{H}_0^m}^2.
\end{aligned}$$

The other terms in R_ψ^β can be estimated similarly, and we have that for (3.57),

$$\left\| \frac{R_\psi^\beta(t)}{1+y} \right\|_{L^2(\Omega)} \le C\Big(\sum_{|\beta| \le m+2} \left\| \partial_\tau^\beta (U,H)(t) \right\|_{L^2(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}_0^m} \Big) \|(u,h)\|_{\mathcal{H}_0^m}. \tag{3.58}$$

Now, combining (3.50) with (3.52), we define new functions:

$$u_\beta := \partial_\tau^\beta u - \frac{\partial_y u + U\phi''}{h + H\phi'}\partial_\tau^\beta \psi, \quad h_\beta := \partial_\tau^\beta h - \frac{\partial_y h + H\phi''}{h + H\phi'}\partial_\tau^\beta \psi, \tag{3.59}$$

and denote

$$\eta_1 := \frac{\partial_y u + U\phi''}{h + H\phi'}, \quad \eta_2 := \frac{\partial_y h + H\phi''}{h + H\phi'}. \tag{3.60}$$

Then, by noting that $\partial_\tau^\beta g = -\partial_x \partial_\tau^\beta \psi$ from (3.52) and using (3.47) and (3.56), we can derive the equations of (u_β, h_β) as follows:

$$\begin{cases}
\partial_t u_\beta + \big[(u + U\phi')\partial_x + (v - U_x\phi)\partial_y\big] u_\beta \\
\quad -[(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] h_\beta - \mu \partial_y^2 u_\beta + (\kappa - \mu)\eta_1 \partial_y h_\beta = R_1^\beta, \\
\partial_t h_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y] h_\beta \\
\quad -[(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] u_\beta - \kappa \partial_y^2 h_\beta = R_2^\beta,
\end{cases}$$

where

$$
(3.61)\quad \begin{cases} R_1^\beta = \partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3 + R_u^\beta - \eta_1 R_\psi^\beta \\ \qquad + [2\mu\partial_y\eta_1 + (g - H_x\phi)\eta_2 + (\mu-\kappa)\eta_1\eta_2]\partial_\tau^\beta h - \zeta_1\partial_\tau^\beta\psi, \\ R_2^\beta = \partial_\tau^\beta r_1 - \eta_2 \partial_\tau^\beta r_2 + R_h^\beta - \eta_2 R_\psi^\beta \\ \qquad + [2\kappa\partial_y\eta_2 + (g - H_x\phi)\eta_1]\partial_\tau^\beta h - \zeta_2\partial_\tau^\beta\psi, \end{cases}
$$

with

$$
(3.62)\quad \begin{cases} \zeta_1 = \partial_t\eta_1 + [(u+U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_1 \\ \qquad - [(h+H\phi')\partial_x + (g - H_x\phi)\partial_y]\eta_2 - \mu\partial_y^2\eta_1 \\ \qquad + (\kappa-\mu)\eta_1\partial_y\eta_2, \\ \zeta_2 = \partial_t\eta_2 + [(u+U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_2 \\ \qquad - [(h+H\phi')\partial_x + (g - H_x\phi)\partial_y]\eta_1 - \kappa\partial_y^2\eta_2. \end{cases}
$$

Also, direct calculation gives the corresponding initial boundary values as follows:

$$
(3.63)\quad \begin{cases} u_\beta|_{t=0} = \partial_\tau^\beta u(0,x,y) \\ \qquad - \dfrac{\partial_y u_0(x,y) + U(0,x)\phi''(y)}{h_0(x,y) + H(0,x)\phi'(y)} \displaystyle\int_0^y \partial_\tau^\beta h(0,x,z)dz \\ \qquad \triangleq u_{\beta 0}(x,y), \\ h_\beta|_{t=0} = \partial_\tau^\beta h(0,x,y) \\ \qquad - \dfrac{\partial_y h_0(x,y) + H(0,x)\phi''(y)}{h_0(x,y) + H(0,x)\phi'(y)} \displaystyle\int_0^y \partial_\tau^\beta h(0,x,z)dz \\ \qquad \triangleq h_{\beta 0}(x,y), \\ u_\beta|_{y=0} = 0, \quad \partial_y h_\beta|_{y=0} = 0. \end{cases}
$$

Finally, we obtain the initial boundary value problem for (u_β, h_β):

$$
(3.64)\quad \begin{cases} \partial_t u_\beta + [(u+U\phi')\partial_x + (v - U_x\phi)\partial_y]u_\beta \\ \quad -[(h+H\phi')\partial_x + (g - H_x\phi)\partial_y]h_\beta - \mu\partial_y^2 u_\beta \\ \quad +(\kappa-\mu)\eta_1\partial_y h_\beta = R_1^\beta, \\ \partial_t h_\beta + [(u+U\phi')\partial_x + (v - U_x\phi)\partial_y]h_\beta \\ \quad -[(h+H\phi')\partial_x + (g - H_x\phi)\partial_y]u_\beta \\ \quad -\kappa\partial_y^2 h_\beta = R_2^\beta, \\ (u_\beta, \partial_y h_\beta)\big|_{y=0} = 0, \quad (u_\beta, h_\beta)\big|_{t=0} = (u_{\beta 0}, h_{\beta 0})(x,y), \end{cases}
$$

with the initial data $(u_{\beta 0}, h_{\beta 0})(x, y)$ given by (3.63). Moreover, by combining $\psi = \partial_y^{-1} h$ with (2.7),

$$\big\| \langle y \rangle^{-1} \partial_\tau^\beta \psi(t) \big\|_{L^2(\Omega)} \le 2 \big\| \partial_\tau^\beta h(t) \big\|_{L^2(\Omega)}. \tag{3.65}$$

From the expression(3.60) of η_1 and η_2, and by (3.51) and the Sobolev embedding inequality, we have that for $\lambda \in \mathbb{R}$ and $i = 1, 2$,

$$\begin{aligned} \|\langle y \rangle^\lambda \eta_i\|_{L^\infty(\Omega)} &\le C\delta_0^{-1}\big(\|(U,H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}^3_{\lambda-1}}\big), \\ \|\langle y \rangle^\lambda \partial_y \eta_i\|_{L^\infty(\Omega)} &\le C\delta_0^{-2}\big(\|(U,H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}^4_{\lambda-1}}\big)^2 \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} &\|\langle y \rangle^\lambda \zeta_i\|_{L^\infty(\Omega)} \le \\ &\qquad C\delta_0^{-3}\Big(\sum_{|\beta|\le 1} \big\|\partial_\tau^\beta (U,H)(t)\big\|_{L^\infty(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}^5_{\lambda-1}}\Big)^3, \quad i = 1, 2. \end{aligned} \tag{3.67}$$

Then, for the terms R_1^β and R_2^β given by (3.61), from the inequalities (3.65)–(3.67) and the estimates (3.49) and (3.58), we obtain that for $|\beta| = m \ge 5$, $l \ge 0$,

$$\begin{aligned} &\big\|R_1^\beta(t)\big\|_{L^2_l(\Omega)} \\ &\quad \le \big\|\partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3\big\|_{L^2_l(\Omega)} + \big\|R_u^\beta\big\|_{L^2_l(\Omega)} \\ &\qquad + \|\langle y \rangle^{l+1}\eta_1\|_{L^\infty(\Omega)} \big\|\langle y \rangle^{-1} R_\psi^\beta\big\|_{L^2(\Omega)} \\ &\qquad + \big(\|2\mu\partial_y \eta_1 + (\mu-\kappa)\eta_1\eta_2\|_{L^\infty(\Omega)} \\ &\qquad\quad + \|\langle y \rangle^{-1}(g - H_x\phi)\|_{L^\infty(\Omega)} \|\langle y \rangle \eta_2\|_{L^\infty(\Omega)}\big) \big\|\partial_\tau^\beta h\big\|_{L^2_l(\Omega)} \\ &\qquad + \|\langle y \rangle^{l+1}\zeta_1\|_{L^\infty(\Omega)} \big\|\langle y \rangle^{-1}\partial_\tau^\beta \psi\big\|_{L^2(\Omega)} \\ &\quad \le \big\|\partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3\big\|_{L^2_l(\Omega)} \\ &\qquad + C\delta_0^{-3}\Big(\sum_{|\beta|\le m+2} \big\|\partial_\tau^\beta (U,H)(t)\big\|_{L^2(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}^m_l}\Big)^3 \|(u,h)(t)\|_{\mathcal{H}^m_l} \end{aligned} \tag{3.68}$$

and

$$
\begin{aligned}
&\left\|R_2^\beta(t)\right\|_{L_l^2(\Omega)} \\
&\quad\le \left\|\partial_\tau^\beta r_2-\eta_2\partial_\tau^\beta r_3\right\|_{L_l^2(\Omega)}+\left\|R_h^\beta\right\|_{L_l^2(\Omega)} \\
&\qquad+\|\langle y\rangle^{l+1}\eta_2\|_{L^\infty(\Omega)}\left\|\langle y\rangle^{-1}R_\psi^\beta\right\|_{L^2(\Omega)} \\
&\qquad+\big(\|2\kappa\partial_y\eta_2\|_{L^\infty(\Omega)} \\
&\qquad\quad+\|\langle y\rangle^{-1}(g-H_x\phi)\|_{L^\infty(\Omega)}\|\langle y\rangle\eta_1\|_{L^\infty(\Omega)}\big)\left\|\partial_\tau^\beta h\right\|_{L_l^2(\Omega)} \\
&\qquad+\|\langle y\rangle^{l+1}\zeta_2\|_{L^\infty(\Omega)}\left\|\langle y\rangle^{-1}\partial_\tau^\beta\psi\right\|_{L^2(\Omega)} \\
&\quad\le \left\|\partial_\tau^\beta r_2-\eta_2\partial_\tau^\beta r_3\right\|_{L_l^2(\Omega)} \\
&\qquad+C\delta_0^{-3}\Big(\sum_{|\beta|\le m+2}\left\|\partial_\tau^\beta(U,H)(t)\right\|_{L^2(\mathbb{T}_x)} \\
&\qquad\qquad+\|(u,h)\|_{\mathcal{H}_l^m}\Big)^3\|(u,h)(t)\|_{\mathcal{H}_l^m}.
\end{aligned}
\tag{3.69}
$$

Now, we are going to derive the following L_l^2-norms of (u_β,h_β).

PROPOSITION 3.3 (L_l^2-estimate on (u_β,h_β)). *Under the hypotheses of Proposition* 3.1, *we have that for any $t\in[0,T]$ and the quantity (u_β,h_β) given in* (3.59),

$$
\begin{aligned}
&\sum_{|\beta|=m}\bigg(\frac{d}{dt}\|(u_\beta,h_\beta)(t)\|^2_{L_l^2(\mathbb{R}_+^2)}+\mu\|\partial_y u_\beta(t)\|^2_{L_l^2(\Omega)} \\
&\qquad\qquad+\kappa\|\partial_y h_\beta(t)\|^2_{L_l^2(\Omega)}\bigg) \\
&\quad\le \sum_{|\beta|=m}\left(\left\|\partial_\tau^\beta r_1-\eta_1\partial_\tau^\beta r_3\right\|^2_{L_l^2(\Omega)}+\left\|\partial_\tau^\beta r_2-\eta_2\partial_\tau^\beta r_3\right\|^2_{L_l^2(\Omega)}\right) \\
&\qquad+C\delta_0^{-2}\Big(\sum_{|\beta|\le m+2}\|\partial_\tau^\beta(U,H)(t)\|_{L^2(\mathbb{T}_x)}+\|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)^2 \\
&\qquad\cdot\sum_{|\beta|=m}\|(u_\beta,h_\beta)(t)\|^2_{L_l^2(\Omega)} \\
&\qquad+C\delta_0^{-4}\Big(\sum_{|\beta|\le m+2}\|\partial_\tau^\beta(U,H)(t)\|_{L^2(\mathbb{T}_x)}+\|(u,h)\|_{\mathcal{H}_l^m}\Big)^4 \\
&\qquad\cdot\|(u,h)(t)\|^2_{\mathcal{H}_l^m}.
\end{aligned}
\tag{3.70}
$$

PROOF. Multiplying $(3.64)_1$ and $(3.64)_2$ by $\langle y\rangle^{2l}u_\beta$ and $\langle y\rangle^{2l}h_\beta$, respectively, and integrating them over Ω with $t\in[0,T]$, we obtain by integration by parts that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\mathbb{R}^2_+)}+\mu\|\partial_y u_\beta\|^2_{L^2_l(\Omega)}+\kappa\|\partial_y h_\beta\|^2_{L^2_l(\Omega)}\\
&\quad=2l\int_\Omega\langle y\rangle^{2l-1}\Big[(v-U_x\phi)\frac{u_\beta^2+h_\beta^2}{2}-(g-H_x\phi)u_\beta h_\beta\Big]dx\,dy\\
&\qquad+(\mu-\kappa)\int_\Omega\langle y\rangle^{2l}(\eta_1\partial_y h_\beta\cdot u_\beta)dx\,dy\\
&\qquad+\int_\Omega\langle y\rangle^{2l}\big(u_\beta R_1^\beta+h_\beta R_2^\beta\big)dx\,dy\\
&\qquad-2l\int_\Omega\langle y\rangle^{2l-1}(\mu u_\beta\partial_y u_\beta+\kappa h_\beta\partial_y h_\beta)dx\,dy,
\end{aligned}
\tag{3.71}
$$

where we have used the boundary conditions in (3.64) and $(v,g)|_{y=0}=0$.

By (2.5), we get that

$$
\begin{aligned}
&2l\int_\Omega\langle y\rangle^{2l-1}\Big[(v-U_x\phi)\frac{u_\beta^2+h_\beta^2}{2}-(g-H_x\phi)u_\beta h_\beta\Big]dx\,dy\\
&\quad\le 2l\bigg(\Big\|\frac{v-U_x\phi}{1+y}\Big\|_{L^\infty(\Omega)}+\Big\|\frac{g-H_x\phi}{1+y}\Big\|_{L^\infty(\Omega)}\bigg)\|(u_\beta,h_\beta)\|^2_{L^2_l(\Omega)}\\
&\quad\le 2l\big(\|(U_x,H_x)(t)\|_{L^\infty(\mathbb{T}_x)}+\|u_x(t)\|_{L^\infty(\Omega)}+\|h_x(t)\|_{L^\infty(\Omega)}\big)\\
&\qquad\cdot\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}\\
&\quad\le C(\|(U_x,H_x)(t)\|_{L^\infty(\mathbb{T}_x)}+\|(u,h)(t)\|_{\mathcal{H}^m_l})\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}.
\end{aligned}
\tag{3.72}
$$

By integration by parts and the boundary condition $u_\beta|_{y=0}=0$, we obtain that

$$
\begin{aligned}
&(\mu-\kappa)\int_\Omega\langle y\rangle^{2l}(\eta_1\partial_y h_\beta\cdot u_\beta)dx\,dy\\
&\quad=-\mu\int_\Omega h_\beta\partial_y(\langle y\rangle^{2l}\eta_1 u_\beta)dx\,dy-\kappa\int_\Omega\langle y\rangle^{2l}(\eta_1\partial_y h_\beta\cdot u_\beta)dx\,dy\\
&\quad\le\frac{\mu}{4}\|\partial_y u_\beta(t)\|^2_{L^2_l(\Omega)}+\frac{\kappa}{4}\|\partial_y h_\beta(t)\|^2_{L^2_l(\Omega)}\\
&\qquad+C\big(1+\|\eta_1(t)\|^2_{L^\infty(\Omega)}+\|\partial_y\eta_1(t)\|_{L^\infty(\Omega)}\big)\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}\\
&\quad\le\frac{\mu}{4}\|\partial_y u_\beta(t)\|^2_{L^2_l(\Omega)}+\frac{\kappa}{4}\|\partial_y h_\beta(t)\|^2_{L^2_l(\Omega)}\\
&\qquad+C\delta_0^{-2}\big(\|(U,H)(t)\|_{L^\infty(\mathbb{T}_x)}+\|(u,h)(t)\|_{\mathcal{H}^m_l}\big)^2\\
&\qquad\quad\cdot\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)},
\end{aligned}
\tag{3.73}
$$

where we have used (3.66) in the above second inequality.

Next, it is easy to get that by (3.68) and (3.69),

$$\begin{aligned}
&\int_{\Omega} \langle y\rangle^{2l}\big(u_\beta R_1^\beta + h_\beta R_2^\beta\big)dx\,dy \\
&\le \|u_\beta(t)\|_{L_l^2(\Omega)}\big\|R_1^\beta(t)\big\|_{L_l^2(\Omega)} + \|h_\beta(t)\|_{L_l^2(\Omega)}\big\|R_2^\beta(t)\big\|_{L_l^2(\Omega)} \\
&\le \big\|\partial_\tau^\beta r_1 - \eta_1\partial_\tau^\beta r_3\big\|_{L_l^2(\Omega)}^2 + \big\|\partial_\tau^\beta r_2 - \eta_2\partial_\tau^\beta r_3\big\|_{L_l^2(\Omega)}^2 \\
&\quad + C\delta_0^{-2}\Big(\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^2(\mathbb{T}_x)} + \|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)^2 \\
&\qquad \cdot\|(u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)}^2 \\
&\quad + C\delta_0^{-4}\Big(\sum_{|\beta|\le m+2}\|\partial_\tau^\beta(U,H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}_l^m}\Big)^4 \\
&\qquad \cdot\|(u,h)(t)\|_{\mathcal{H}_l^m}^2.
\end{aligned} \tag{3.74}$$

Also,

$$\begin{aligned}
&\left|2l\int_\Omega \langle y\rangle^{2l-1}(\mu u_\beta\partial_y u_\beta + \kappa h_\beta\partial_y h_\beta)dx\,dy\right| \le \\
&\qquad \frac{\mu}{4}\|\partial_y u_\beta(t)\|_{L_l^2(\Omega)}^2 + \frac{\kappa}{4}\|\partial_y h_\beta(t)\|_{L_l^2(\Omega)}^2 + C\|(u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)}^2.
\end{aligned} \tag{3.75}$$

Substituting (3.72)–(3.75) into (3.71) yields

$$\begin{aligned}
&\frac{d}{dt}\|(u_\beta,h_\beta)(t)\|_{L_l^2(\mathbb{R}_+^2)}^2 + \mu\|\partial_y u_\beta\|_{L_l^2(\Omega)}^2 + \kappa\|\partial_y h_\beta\|_{L_l^2(\Omega)}^2 \\
&\le \big\|\partial_\tau^\beta r_1 - \eta_1\partial_\tau^\beta r_3\big\|_{L_l^2(\Omega)}^2 + \big\|\partial_\tau^\beta r_2 - \eta_2\partial_\tau^\beta r_3\big\|_{L_l^2(\Omega)}^2 \\
&\quad + C\delta_0^{-2}\Big(\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^2(\mathbb{T}_x)} + \|(u,h)(t)\|_{\mathcal{H}_l^m}\Big)^2 \\
&\qquad \cdot\|(u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)}^2 \\
&\quad + C\delta_0^{-4}\Big(\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H)(t)\big\|_{L^2(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}_l^m}\Big)^4 \\
&\qquad \cdot\|(u,h)(t)\|_{\mathcal{H}_l^m}^2;
\end{aligned} \tag{3.76}$$

thus we prove (3.70) by taking the summation over all $|\beta| = m$ in (3.76). □

Finally, we give the following result, which shows the almost equivalence in the L_l^2-norm between $\partial_\tau^\beta(u,h)$ and the quantities (u_β,h_β) given by (3.59).

LEMMA 3.4 (Equivalence between $\|\partial_\tau^\beta(u,h)\|_{L_l^2}$ and $\|(u_\beta,h_\beta)\|_{L_l^2}$). *If the smooth function* (u,h) *satisfies the problem* (2.12) *in* $[0,T]$, *and* (3.51) *holds, then for any*

$t \in [0,T]$, $l \geq 0$, any integer $m \geq 3$, and the quantity (u_β, h_β) with $|\beta| = m$ defined by (3.59)*, we have*

$$M(t)^{-1}\big\|\partial_\tau^\beta(u,h)(t)\big\|_{L_l^2(\Omega)} \leq \|(u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)} \leq M(t)\big\|\partial_\tau^\beta(u,h)(t)\big\|_{L_l^2(\Omega)} \tag{3.77}$$

and

$$\big\|\partial_y\partial_\tau^\beta(u,h)(t)\big\|_{L_l^2(\Omega)} \leq \|\partial_y(u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)} + M(t)\|h_\beta(t)\|_{L_l^2(\Omega)}, \tag{3.78}$$

where

$$\begin{aligned} M(t) := 2\delta_0^{-1}\big(C\|(U,H)(t)\|_{L^\infty(\mathbb{T}_x)} &+ \big\|\langle y\rangle^{l+1}\partial_y(u,h)(t)\big\|_{L^\infty(\Omega)} \\ &+ \big\|\langle y\rangle^{l+1}\partial_y^2(u,h)(t)\big\|_{L^\infty(\Omega)}\big). \end{aligned} \tag{3.79}$$

PROOF. First, from the definitions of u_β and h_β in (3.59), we have by using (3.65)

$$\begin{aligned} \|u_\beta(t)\|_{L_l^2(\Omega)} &\leq \big\|\partial_\tau^\beta u(t)\big\|_{L_l^2(\Omega)} \\ &\quad + \big\|\langle y\rangle^{l+1}\eta_1(t)\big\|_{L^\infty(\Omega)}\big\|\langle y\rangle^{-1}\partial_\tau^\beta\psi(t)\big\|_{L^2(\Omega)} \\ &\leq \big\|\partial_\tau^\beta u(t)\big\|_{L_l^2(\Omega)} + 2\delta_0^{-1}\big(C\|U(t)\|_{L^\infty(\mathbb{T}_x)} \\ &\quad + \big\|\langle y\rangle^{l+1}\partial_y u(t)\big\|_{L^\infty(\Omega)}\big)\big\|\partial_\tau^\beta h(t)\big\|_{L^2(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \|h_\beta(t)\|_{L_l^2(\Omega)} &\leq \big\|\partial_\tau^\beta h(t)\big\|_{L_l^2(\Omega)} + \|\langle y\rangle^{l+1}\eta_2(t)\|_{L^\infty(\Omega)}\big\|\langle y\rangle^{-1}\partial_\tau^\beta\psi(t)\big\|_{L^2(\Omega)} \\ &\leq 2\delta_0^{-1}\big(C\|H(t)\|_{L^\infty(\mathbb{T}_x)} \\ &\quad + \big\|\langle y\rangle^{l+1}\partial_y h(t)\big\|_{L^\infty(\Omega)}\big)\big\|\partial_\tau^\beta h(t)\big\|_{L_l^2(\Omega)}. \end{aligned}$$

Thus, we have by (3.79)

$$\|(u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)} \leq M(t)\big\|\partial_\tau^\beta(u,h)(t)\big\|_{L_l^2(\Omega)}. \tag{3.80}$$

On the other hand, note that from $\partial_y\psi = h$ and the expression of h_β in (3.59),

$$h_\beta = \partial_\tau^\beta h - \frac{\partial_y h + H\phi''}{h + H\phi'}\partial_\tau^\beta\psi = (h + H\phi')\cdot\partial_y\bigg(\frac{\partial_\tau^\beta\psi}{h + H\phi'}\bigg),$$

which implies that by $\partial_\tau^\beta\psi\big|_{y=0} = 0$,

$$\begin{aligned} \partial_\tau^\beta\psi(t,x,y) &= \big(h(t,x,y) + H(t,x)\phi'(y)\big) \\ &\quad \cdot\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z) + H(t,x)\phi'(z)}\,dz. \end{aligned} \tag{3.81}$$

Therefore, combining the definition (3.59) for (u_β, h_β) with (3.81), we obtain

$$
\begin{cases}
\partial_\tau^\beta u(t,x,y) = u_\beta(t,x,y) + \big(\partial_y u(t,x,y) + U(t,x)\phi''(y)\big) \\
\qquad\qquad \cdot \displaystyle\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z)+H(t,x)\phi'(z)}dz, \\
\partial_\tau^\beta h(t,x,y) = h_\beta(t,x,y) + \big(\partial_y h(t,x,y) + H(t,x)\phi''(y)\big) \\
\qquad\qquad \cdot \displaystyle\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z)+H(t,x)\phi'(z)}dz.
\end{cases} \tag{3.82}
$$

Then, by using (2.7),

$$
\begin{aligned}
\big\|\partial_\tau^\beta u(t)\big\|_{L^2_l(\Omega)} &\le \|u_\beta(t)\|_{L^2_l(\Omega)} \\
&\quad + \|\langle y\rangle^{l+1}(\partial_y u + U\phi'')(t)\|_{L^\infty(\Omega)} \\
&\qquad \cdot \left\| \frac{1}{1+y}\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z)+H(t,x)\phi'(z)}dz \right\|_{L^2(\Omega)} \\
&\le \|u_\beta(t)\|_{L^2_l(\Omega)} \\
&\quad + 2\big(C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y\rangle^{l+1}\partial_y u(t)\|_{L^\infty(\Omega)}\big) \\
&\qquad \cdot \left\| \frac{h_\beta}{h+H\phi'} \right\|_{L^2(\Omega)} \\
&\le \|u_\beta(t)\|_{L^2_l(\Omega)} \\
&\quad + 2\delta_0^{-1}\big(C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y\rangle^{l+1}\partial_y u(t)\|_{L^\infty(\Omega)}\big) \\
&\qquad \cdot \|h_\beta(t)\|_{L^2(\Omega)},
\end{aligned}
$$

and similarly,

$$
\big\|\partial_\tau^\beta h(t)\big\|_{L^2_l(\Omega)} \le 2\delta_0^{-1}\big(C\|H(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y\rangle^{l+1}\partial_y h(t)\|_{L^\infty(\Omega)}\big)\|h_\beta(t)\|_{L^2_l(\Omega)},
$$

which implies

$$
\big\|\partial_\tau^\beta (u,h)(t)\big\|_{L^2_l(\Omega)} \le M(t)\|(u_\beta, h_\beta)(t)\|_{L^2_l(\Omega)}, \tag{3.83}
$$

provided that $M(t)$ is given in (3.79). Thus, combining (3.80) with (3.83) yields (3.77).

Furthermore, by taking the derivation of (3.82) in y, we get the following forms of $\partial_y\partial_\tau^\beta u$ and $\partial_y\partial_\tau^\beta h$:

$$\begin{cases} \partial_y\partial_\tau^\beta u(t,x,y) = \partial_y u_\beta(t,x,y) + \eta_1(t,x,y)h_\beta(t,x,y) \\ \qquad\qquad + \big(\partial_y^2 u(t,x,y) + U(t,x)\phi^{(3)}(y)\big) \\ \qquad\qquad \cdot \displaystyle\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z)+H(t,x)\phi'(z)}dz, \\ \partial_y\partial_\tau^\beta h(t,x,y) = \partial_y h_\beta(t,x,y) + \eta_2(t,x,y)h_\beta(t,x,y) \\ \qquad\qquad + \big(\partial_y^2 h(t,x,y) + H(t,x)\phi^{(3)}(y)\big) \\ \qquad\qquad \cdot \displaystyle\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z)+H(t,x)\phi'(z)}dz. \end{cases}$$

Then, it follows that by (2.7) and (3.66),

$$\begin{aligned} &\big\|\partial_y\partial_\tau^\beta u(t)\big\|_{L_l^2(\Omega)} \\ &\le \|\partial_y u_\beta(t)\|_{L_l^2(\Omega)} + \|\eta_1(t)\|_{L^\infty(\Omega)}\|h_\beta(t)\|_{L_l^2(\Omega)} \\ &\quad + \big\|\langle y\rangle^{l+1}\big(\partial_y^2 u + U\phi^{(3)}\big)(t)\big\|_{L^\infty(\Omega)} \\ &\quad \cdot \left\|\frac{1}{1+y}\int_0^y \frac{h_\beta(t,x,z)}{h(t,x,z)+H(t,x)\phi'(z)}dz\right\|_{L^2(\Omega)} \\ &\le \|\partial_y u_\beta(t)\|_{L_l^2(\Omega)} \\ &\quad + \delta_0^{-1}\big(C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\partial_y u(t)\|_{L^\infty(\Omega)}\big)\|h_\beta(t)\|_{L_l^2(\Omega)} \\ &\quad + 2\big(C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \big\|\langle y\rangle^{l+1}\partial_y^2 u(t)\big\|_{L^\infty(\Omega)}\big)\left\|\frac{h_\beta}{h+H\phi'}\right\|_{L^2(\Omega)} \\ &\le \|\partial_y u_\beta(t)\|_{L_l^2(\Omega)} + M(t)\|h_\beta(t)\|_{L_l^2(\Omega)}, \end{aligned}$$

and similarly,

$$\big\|\partial_y\partial_\tau^\beta h(t)\big\|_{L_l^2(\Omega)} \le \|\partial_y h_\beta(t)\|_{L_l^2(\Omega)} + M(t)\|h_\beta(t)\|_{L_l^2(\Omega)}.$$

Combining the above two inequalities yields, by (3.79),

$$\big\|\partial_y\partial_\tau^\beta (u,h)(t)\big\|_{L_l^2(\Omega)} \le \|\partial_y (u_\beta,h_\beta)(t)\|_{L_l^2(\Omega)} + M(t)\|h_\beta(t)\|_{L_l^2(\Omega)}.$$

Thus we obtain (3.78), and this completes the proof. □

3.3 Closeness of the A Priori Estimates

In this subsection, we will prove Proposition 3.1. Before that, we need some preliminaries. First of all, from (3.1) we know

$$\big\|\langle y\rangle^{l+1}\partial_y^i (u,h)(t)\big\|_{L^\infty(\Omega)} \le \delta_0^{-1} \quad \text{for } i=1,2,\ t\in[0,T].$$

Combining this with the definitions (3.60) for η_i, $i = 1, 2$, and (3.79) for $M(t)$, it implies that for $i = 1, 2$ and δ_0 sufficiently small,

$$\|\langle y\rangle^{l+1}\eta_i(t)\|_{L^\infty(\Omega)} \le 2\delta_0^{-2},$$
$$M(t) \le 2\delta_0^{-1}\big(C\|(U,H)(t)\|_{L^\infty(\mathbb{T}_x)} + 2\delta_0^{-1}\big) \le 5\delta_0^{-2}.$$

Then, recalling that $D^\alpha = \partial_\tau^\beta\partial_y^k$, we obtain that by (3.77) and (3.78) in Lemma 3.4,

$$\begin{aligned}
&\|(u,h)(t)\|^2_{\mathcal{H}_l^m}\\
&\quad = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \|D^\alpha(u,h)(t)\|^2_{L_l^2(\Omega)} + \sum_{|\beta|=m} \big\|\partial_\tau^\beta(u,h)(t)\big\|^2_{L_l^2(\Omega)}\\
&\quad \le \sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \|D^\alpha(u,h)(t)\|^2_{L_l^2(\Omega)} + 25\delta_0^{-4}\sum_{|\beta|=m}\|(u_\beta,h_\beta)(t)\|^2_{L_l^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_y(u,h)(t)\|^2_{\mathcal{H}_l^m}\\
&\quad = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \|D^\alpha\partial_y(u,h)(t)\|^2_{L_l^2(\Omega)} + \sum_{|\beta|=m} \big\|\partial_y\partial_\tau^\beta(u,h)(t)\big\|^2_{L_l^2(\Omega)}\\
&\quad \le \sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \|D^\alpha\partial_y(u,h)(t)\|^2_{L_l^2(\Omega)} + 2\sum_{|\beta|=m}\|\partial_y(u_\beta,h_\beta)(t)\|^2_{L_l^2(\Omega)}\\
&\qquad + 50\delta_0^{-4}\sum_{|\beta|=m}\|h_\beta(t)\|^2_{L_l^2(\Omega)}.
\end{aligned}$$

Consequently, we have the following:

COROLLARY 3.5. *Under the assumptions of Proposition* 3.1*, for any* $t \in [0,T]$ *and the quantity* (u_β, h_β)*,* $|\beta| = m$*, given by* (3.59)*, it holds that*

$$\begin{aligned}
\|(u,h)(t)\|^2_{\mathcal{H}_l^m} \le &\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \|D^\alpha(u,h)(t)\|^2_{L_l^2(\Omega)}\\
&+ 25\delta_0^{-4}\sum_{|\beta|=m}\|(u_\beta,h_\beta)(t)\|^2_{L_l^2(\Omega)},
\end{aligned} \tag{3.84}$$

and

$$
\begin{aligned}
&\|\partial_y(u,h)(t)\|^2_{\mathcal{H}^m_l}\\
&\quad\le \sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}} \|D^\alpha\partial_y(u,h)(t)\|^2_{L^2_l(\Omega)} + 2\sum_{|\beta|=m}\|\partial_y(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}\\
&\qquad + 50\delta_0^{-4}\sum_{|\beta|=m}\|h_\beta(t)\|^2_{L^2_l(\Omega)}.
\end{aligned}
\tag{3.85}
$$

Now, we can derive the desired a priori estimates of (u,h) for the problem (2.12). From Proposition 3.2 and 3.3, it follows that for $m\ge 5$ and any $t\in[0,T]$,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\big\|D^\alpha(u,h)(t)\big\|^2_{L^2_l(\mathbb{R}^2_+)} + 25\delta_0^{-4}\sum_{|\beta|=m}\big\|(u_\beta,h_\beta)(t)\big\|^2_{L^2_l(\Omega)}\Big)\\
&\quad + \mu\Big(\sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\big\|D^\alpha\partial_y u(t)\big\|^2_{L^2_l(\mathbb{R}^2_+)} + 25\delta_0^{-4}\sum_{|\beta|=m}\big\|\partial_y u_\beta(t)\big\|^2_{L^2_l(\Omega)}\Big)\\
&\quad + \kappa\Big(\sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\big\|D^\alpha\partial_y h(t)\big\|^2_{L^2_l(\mathbb{R}^2_+)} + 25\delta_0^{-4}\sum_{|\beta|=m}\big\|\partial_y h_\beta(t)\big\|^2_{L^2_l(\Omega)}\Big)\\
&\le C\delta_1\|\partial_y(u,h)(t)\|^2_{\mathcal{H}^m_0} + C\delta_1^{-1}\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big(1+\|(u,h)(t)\|^2_{\mathcal{H}^m_l}\big)\\
&\quad + \sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\|D^\alpha(r_1,r_2)(t)\|^2_{L^2_{l+k}(\Omega)} + C\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H,P)(t)\big\|^2_{L^2(\mathbb{T}_x)}\\
&\quad + 25\delta_0^{-4}\sum_{|\beta|=m}\Big(\big\|\partial_\tau^\beta r_1-\eta_1\partial_\tau^\beta r_3\big\|^2_{L^2_l(\Omega)} + \big\|\partial_\tau^\beta r_2-\eta_2\partial_\tau^\beta r_3\big\|^2_{L^2_l(\Omega)}\Big)\\
&\quad + C\delta_0^{-6}\Big(\sum_{|\beta|\le m+2}\|\partial_\tau^\beta(U,H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u,h)(t)\|_{\mathcal{H}^m_l}\Big)^2\\
&\qquad\cdot\sum_{|\beta|=m}\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}\\
&\quad + C\delta_0^{-8}\Big(\sum_{|\beta|\le m+2}\|\partial_\tau^\beta(U,H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u,h)\|_{\mathcal{H}^m_l}\Big)^4\|(u,h)(t)\|^2_{\mathcal{H}^m_l}.
\end{aligned}
\tag{3.86}
$$

Plugging the inequalities (3.84) and (3.85) given in Corollary 3.5 into (3.86), and choosing δ_1 small enough, we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\|D^\alpha(u,h)(t)\|^2_{L^2_l(\mathbb{R}^2_+)} + 25\delta_0^{-4}\sum_{|\beta|=m}\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}\Big)\\
&\quad + \sum_{\substack{|\alpha|\le m\\|\beta|\le m-1}}\|D^\alpha\partial_y(u,h)(t)\|^2_{L^2_l(\mathbb{R}^2_+)} + 25\delta_0^{-4}\sum_{|\beta|=m}\|\partial_y(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)} \le
\end{aligned}
\tag{3.87}
$$

$$
\begin{aligned}
&\le C \sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \|D^\alpha(r_1,r_2)(t)\|^2_{L^2_{l+k}(\Omega)}\\
&\quad + C\delta_0^{-4}\sum_{|\beta|=m}\left(\left\|\partial_\tau^\beta(r_1,r_2)(t)\right\|^2_{L^2_l(\Omega)} + 4\delta_0^{-4}\left\|\partial_\tau^\beta r_3\right\|^2_{L^2_{-1}(\Omega)}\right)\\
&\quad + C\delta_0^{-8}\Big(1+\sum_{|\beta|\le m+2}\left\|\partial_\tau^\beta(U,H,P)(t)\right\|^2_{L^2(\mathbb{T}_x)}\Big)^3\\
&\quad + C\delta_0^{-8}\Big(\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}}\|D^\alpha(u,h)(t)\|^2_{L^2_l(\Omega)} + 25\delta_0^{-4}\sum_{|\beta|=m}\|(u_\beta,h_\beta)(t)\|^2_{L^2_l(\Omega)}\Big)^3,
\end{aligned}
$$

where we have used the fact that for $i=1,2$,

$$
\begin{aligned}
\left\|\eta_i\partial_\tau^\beta r_3\right\|_{L^2_l(\Omega)} &\le \|\langle y\rangle^{i+1}\eta_i(t)\|_{L^\infty(\Omega)}\left\|\langle y\rangle^{-1}\partial_\tau^\beta r_3\right\|_{L^2(\Omega)}\\
&\le 2\delta_0^{-2}\left\|\partial_\tau^\beta r_3\right\|_{L^2_{-1}(\Omega)}.
\end{aligned}
$$

Define

$$
F_0 := \sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}}\|D^\alpha(u,h)(0)\|^2_{L^2_l(\Omega)} + 25\delta_0^{-4}\sum_{|\beta|=m}\left\|(u_{\beta 0},h_{\beta 0})\right\|^2_{L^2_l(\Omega)} \tag{3.88}
$$

and

$$
\begin{aligned}
F(t) := C &\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}}\|D^\alpha(r_1,r_2)(t)\|^2_{L^2_{l+k}(\Omega)}\\
&+ C\delta_0^{-4}\sum_{|\beta|=m}\left(\left\|\partial_\tau^\beta(r_1,r_2)(t)\right\|^2_{L^2_l(\Omega)} + 4\delta_0^{-4}\left\|\partial_\tau^\beta r_3\right\|^2_{L^2_{-1}(\Omega)}\right)\\
&+ C\delta_0^{-8}\Big(1+\sum_{|\beta|\le m+2}\left\|\partial_\tau^\beta(U,H,P)(t)\right\|^2_{L^2(\mathbb{T}_x)}\Big)^3.
\end{aligned} \tag{3.89}
$$

By the comparison principle of ordinary differential equations in (3.87), we get

$$
\begin{aligned}
&\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}}\|D^\alpha(u,h)(t)\|^2_{L^2_l(\Omega)} + 25\delta_0^{-4}\sum_{|\beta|=m}\left\|(u_\beta,h_\beta)(t)\right\|^2_{L^2_l(\Omega)}\\
&\qquad + \int_0^t\Big(\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}}\left\|D^\alpha\partial_y(u,h)(s)\right\|^2_{L^2_l(\Omega)}\\
&\qquad\qquad + 25\delta_0^{-4}\sum_{|\beta|=m}\|\partial_y(u_\beta,h_\beta)(s)\|^2_{L^2_l(\Omega)}\Big)ds\\
&\quad \le \left(F_0+\int_0^t F(s)ds\right)\left\{1-2C\delta_0^{-8}\left(F_0+\int_0^t F(s)ds\right)^2 t\right\}^{-\frac12}.
\end{aligned} \tag{3.90}
$$

Then this implies, by combining (3.84) with (3.90), that

$$\sup_{0\le s\le t}\|(u,h)(s)\|_{\mathcal{H}_l^m} \le \left(F_0+\int_0^t F(s)ds\right)^{\frac12}\left\{1-2C\delta_0^{-8}\left(F_0+\int_0^t F(s)ds\right)^2 t\right\}^{-\frac14}. \tag{3.91}$$

We know that for $i=1,2$,

$$\begin{aligned}\langle y\rangle^{l+1}\partial_y^i(u,h)(t,x,y) &= \langle y\rangle^{l+1}\partial_y^i(u_0,h_0)(x,y)\\ &\quad+\int_0^t \langle y\rangle^{l+1}\partial_t\partial_y^i(u,h)(s,x,y)ds,\end{aligned}$$

and

$$h(t,x,y)=h_0(x,y)+\int_0^t \partial_t h(s,x,y)ds.$$

Then, by the Sobolev embedding inequality and (3.91), we have, for $i=1,2$,

$$\begin{aligned}&\left\|\langle y\rangle^{l+1}\partial_y^i(u,h)(t)\right\|_{L^\infty(\Omega)}\\ &\le \left\|\langle y\rangle^{l+1}\partial_y^i(u_0,h_0)\right\|_{L^\infty(\Omega)}+\int_0^t\left\|\langle y\rangle^{l+1}\partial_t\partial_y^i(u,h)(s)\right\|_{L^\infty(\Omega)}ds\\ &\le \left\|\langle y\rangle^{l+1}\partial_y^i(u_0,h_0)\right\|_{L^\infty(\Omega)}+C\big(\sup_{0\le s\le t}\|(u,h)(s)\|_{\mathcal{H}_l^5}\big)\,t\\ &\le \left\|\langle y\rangle^{l+1}\partial_y^i(u_0,h_0)\right\|_{L^\infty(\Omega)}\\ &\quad+Ct\left(F_0+\int_0^t F(s)ds\right)^{\frac12}\left\{1-2C\delta_0^{-8}\left(F_0+\int_0^t F(s)ds\right)^2 t\right\}^{-\frac14}.\end{aligned} \tag{3.92}$$

Similarly, one can obtain that

$$\begin{aligned}h(t,x,y)&\ge h_0(x,y)-\int_0^t\|\partial_t h(s)\|_{L^\infty(\Omega)}ds\\ &\ge h_0(x,y)-C\big(\sup_{0\le s\le t}\|h(s)\|_{\mathcal{H}_0^3}\big)\cdot t\\ &\ge h_0(x,y)\\ &\quad-Ct\left(F_0+\int_0^t F(s)ds\right)^{\frac12}\left\{1-2C\delta_0^{-8}\left(F_0+\int_0^t F(s)ds\right)^2 t\right\}^{-\frac14}.\end{aligned} \tag{3.93}$$

Therefore, we obtain the following:

PROPOSITION 3.6. *Under the assumptions of Proposition* 3.1*, there exists a constant* $C > 0$*, depending only on* m, M_0*, and* ϕ*, such that*

$$\begin{aligned}&\sup_{0\le s\le t}\|(u,h)(s)\|_{\mathcal{H}_l^m}\\ &\quad\le\Big(F_0+\int_0^t F(s)ds\Big)^{\frac12}\cdot\Big\{1-2C\delta_0^{-8}\Big(F_0+\int_0^t F(s)ds\Big)^2 t\Big\}^{-\frac14}\end{aligned}\tag{3.94}$$

for small time, where the quantities F_0 *and* $F(t)$ *are defined by* (3.88) *and* (3.89)*, respectively. Also, we have that for* $i = 1, 2$*,*

$$\begin{aligned}&\big\|\langle y\rangle^{l+1}\partial_y^i(u,h)(t)\big\|_{L^\infty(\Omega)}\\ &\quad\le\big\|\langle y\rangle^{l+1}\partial_y^i(u_0,h_0)\big\|_{L^\infty(\Omega)}+Ct\cdot\big(\sup_{0\le s\le t}\|(u,h)(s)\|_{\mathcal{H}_l^5}\big)\\ &\quad\le\big\|\langle y\rangle^{l+1}\partial_y^i(u_0,h_0)\big\|_{L^\infty(\Omega)}\\ &\qquad+Ct\Big(F_0+\int_0^t F(s)ds\Big)^{\frac12}\Big\{1-2C\delta_0^{-8}\Big(F_0+\int_0^t F(s)ds\Big)^2 t\Big\}^{-\frac14}\end{aligned}\tag{3.95}$$

and

$$\begin{aligned}h(t,x,y)&\ge h_0(x,y)-C\big(\sup_{0\le s\le t}\|h(s)\|_{\mathcal{H}_0^3}\big)\cdot t\\ &\ge h_0(x,y)\\ &\quad-Ct\cdot\Big(F_0+\int_0^t F(s)ds\Big)^{\frac12}\Big\{1-2C\delta_0^{-8}\Big(F_0+\int_0^t F(s)ds\Big)^2 t\Big\}^{-\frac14}.\end{aligned}\tag{3.96}$$

From Proposition 3.6, we are ready to prove Proposition 3.1. Indeed, by using (1.10), (2.16), and the fact $\|\partial_\tau^\beta r_3\|_{L^2_{-1}(\Omega)} \le CM_0$ from the expression (3.55), it follows that from the definition (3.89) for $F(t)$,

$$F(t)\le C\delta_0^{-8}M_0^6.\tag{3.97}$$

Next, by direct calculation we know that $D^\alpha(u,h)(0,x,y)$, $|\alpha|\le m$, can be expressed by the spatial derivatives of initial data (u_0,h_0) up to order $2m$. Then, combining this with (3.63) we get that F_0, given by (3.88), is a polynomial of $\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}$, and consequently,

$$F_0\le\delta_0^{-8}\,\mathcal{P}\big(M_0+\|(u_0,h_0)\|_{H_l^{2m}(\Omega)}\big).\tag{3.98}$$

Plugging (3.97) and (3.98) into (3.94)–(3.96), we derive the estimates (3.2)–(3.4), and then obtain the proof of Proposition 3.1.

4 Local-in-Time Existence and Uniqueness

In this section, we will establish the local-in-time existence and uniqueness of solutions to the nonlinear problem (2.12).

4.1 Existence

For this, we consider a parabolic regularized system for problem (2.12), from which we can obtain the local (in time) existence of a solution by using classical energy estimates. Specifically, for a small parameter $0 < \epsilon < 1$, we investigate the following problem:

$$
\begin{cases}
\partial_t u^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]u^\epsilon \\
\quad - [(h^\epsilon + H\phi')\partial_x + (g^\epsilon - H_x\phi)\partial_y]h^\epsilon + U_x\phi' u^\epsilon + U\phi'' v^\epsilon \\
\quad - H_x\phi' h^\epsilon - H\phi'' g^\epsilon = \epsilon\partial_x^2 u^\epsilon + \mu\partial_y^2 u^\epsilon + r_1^\epsilon, \\
\partial_t h^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]h^\epsilon \\
\quad - [(h^\epsilon + H\phi')\partial_x + (g^\epsilon - H_x\phi)\partial_y]u^\epsilon + H_x\phi' u^\epsilon + H\phi'' v^\epsilon \\
\quad - U_x\phi' h^\epsilon - U\phi'' g^\epsilon = \epsilon\partial_x^2 h^\epsilon + \kappa\partial_y^2 h^\epsilon + r_2^\epsilon, \\
\partial_x u^\epsilon + \partial_y v^\epsilon = 0, \quad \partial_x h^\epsilon + \partial_y g^\epsilon = 0, \\
(u^\epsilon, h^\epsilon)\big|_{t=0} = (u_0, h_0)(x, y), \quad (u^\epsilon, v^\epsilon, \partial_y h^\epsilon, g^\epsilon)\big|_{y=0} = 0,
\end{cases}
\tag{4.1}
$$

where the source term

$$
(r_1^\epsilon, r_2^\epsilon)(t, x, y) = (r_1, r_2) + \epsilon(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t, x, y). \tag{4.2}
$$

Here, (r_1, r_2) is the source term of the original problem (2.12), and $(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)$ is constructed to ensure that the initial data (u_0, h_0) also satisfies the compatibility conditions of (4.1) up to the order of m. Actually, we can use the given functions $\partial_t^i(u, h)(0, x, y)$, $0 \le i \le m$, which can be derived from the equations and initial data of (2.12) by induction with respect to i, and it follows that $\partial_t^i(u, h)(0, x, y)$ can be expressed as polynomials of the spatial derivatives, up to order $2i$, of the initial data (u_0, h_0). Then, we may choose the corrector $(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)$ in the following form:

$$
(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t, x, y) := -\sum_{i=0}^{m}\left(\frac{t^i}{i!}\partial_x^2\partial_t^i(u, h)(0, x, y)\right), \tag{4.3}
$$

which yields that by direct calculation,

$$
\partial_t^i(u^\epsilon, h^\epsilon)(0, x, y) = \partial_t^i(u, h)(0, x, y), \quad 0 \le i \le m.
$$

Likewise, we can derive that $\psi^\epsilon := \partial_y^{-1}h^\epsilon$ satisfies

$$
\begin{aligned}
\partial_t\psi^\epsilon + \big[(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y\big]\psi^\epsilon& \\
+ H_x\phi u^\epsilon + H\phi' v^\epsilon - \kappa\partial_y^2\psi^\epsilon &= r_3^\epsilon,
\end{aligned}
$$

where

$$
r_3^\epsilon = r_3 - \epsilon\sum_{i=0}^{m}\left(\frac{t^i}{i!}\int_0^y \partial_x^2\partial_t^i h(0, x, z)dz\right) := r_3 + \epsilon\tilde{r}_3 \tag{4.4}
$$

with r_3 given by (3.55). Moreover, we have for $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ with $|\alpha| \le m$,

$$\|D^\alpha(\widetilde{r}_1^\epsilon, \widetilde{r}_2^\epsilon)(t)\|_{L^2_{l+k}(\Omega)} + \big\|\partial_\tau^\beta \widetilde{r}_3^\epsilon(t)\big\|_{L^2_{-1}(\Omega)} \le \sum_{\beta_1 \le i \le m} t^{i-\beta_1}\mathcal{P}\big(M_0 + \|(u_0, h_0)\|_{H_l^{2i+2+\beta_2+k}}\big). \tag{4.5}$$

Based on the a priori energy estimates established in Proposition 3.6, we can obtain the following:

PROPOSITION 4.1. *Under the hypotheses of Theorem* 2.2*, there exist a time* $0 < T_* \le T$*, independent of* ϵ*, and a solution* $(u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)$ *to the initial boundary value problem* (4.1) *with* $(u^\epsilon, h^\epsilon) \in L^\infty(0, T_*; \mathcal{H}_l^m)$*, that satisfies the following uniform estimates in* ϵ*:*

$$\sup_{0 \le t \le T_*} \|(u^\epsilon, h^\epsilon)(t)\|_{\mathcal{H}_l^m} \le 2F_0^{\frac{1}{2}}, \tag{4.6}$$

where F_0 *is given by* (3.88)*). Moreover, for* $t \in [0, T_*]$*,* $(x, y) \in \Omega$*,*

$$\begin{aligned} &\big\|\langle y\rangle^{l+1}\partial_y^i(u^\epsilon, h^\epsilon)(t)\big\|_{L^\infty(\Omega)} \le \delta_0^{-1}, \quad i = 1, 2, \\ &h^\epsilon(t, x, y) + H(t, x)\phi'(y) \ge \delta_0. \end{aligned} \tag{4.7}$$

PROOF. Since problem (4.1) is a parabolic system, it is standard to show that (4.1) admits a solution in a time interval $[0, T_\epsilon]$ (T_ϵ may depend on ϵ) satisfying the estimates (4.7). Indeed, one can establish a priori estimates for (4.1) and then obtain the local existence of a solution by the standard iteration and weak convergence methods.

On the other hand, we can derive similar a priori estimates as in Proposition 3.6 for (4.1), so by the standard continuity argument we can obtain the existence of a solution in a time interval $[0, T_*]$, $T_* > 0$, independent of ϵ. Therefore, we only determine the uniform lifespan T_* and verify the estimates (4.6) and (4.7).

According to Proposition 3.6, we can obtain the estimates for (u^ϵ, h^ϵ) similarly to (3.94),

$$\sup_{0 \le s \le t} \|(u^\epsilon, h^\epsilon)(s)\|_{\mathcal{H}_l^m} \le \left(F_0 + \int_0^t F^\epsilon(s)ds\right)^{\frac{1}{2}} \cdot \left\{1 - 2C\delta_0^{-8}\left(F_0 + \int_0^t F^\epsilon(s)ds\right)^2 t\right\}^{-\frac{1}{4}}, \tag{4.8}$$

as long as the quantity in $\{\cdot\}$ on the right-hand side of (3.2) is positive, where the quantity F_0 is given by (3.88), and $F^\epsilon(t)$ is defined as follows (similar to (3.89)):

$$
\begin{aligned}
F^\epsilon(t) := C &\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \big\|D^\alpha\big(r_1^\epsilon, r_2^\epsilon\big)(t)\big\|^2_{L^2_{l+k}(\Omega)}\\
&+ C\delta_0^{-4}\sum_{|\beta|=m}\big(\big\|\partial_\tau^\beta\big(r_1^\epsilon, r_2^\epsilon\big)(t)\big\|^2_{L^2_l(\Omega)} + 4\delta_0^{-4}\big\|\partial_\tau^\beta r_3^\epsilon\big\|^2_{L^2_{-1}(\Omega)}\big)\\
&+ C\delta_0^{-8}\bigg(1+\sum_{|\beta|\le m+2}\big\|\partial_\tau^\beta(U,H,P)(t)\big\|^2_{L^2(\mathbb{T}_x)}\bigg)^3.
\end{aligned}
\tag{4.9}
$$

Substituting (4.2)–(4.4) into (4.9) and recalling $F(t)$ defined by (3.89), we get

$$
\begin{aligned}
F^\epsilon(t) = F(t) + C\epsilon^2 &\sum_{\substack{|\alpha|\le m\\ |\beta|\le m-1}} \big\|D^\alpha\big(\tilde r_1^\epsilon, \tilde r_2^\epsilon\big)(t)\big\|^2_{L^2_{l+k}(\Omega)}\\
&+ C\epsilon^2\delta_0^{-4}\sum_{|\beta|=m}\big(\big\|\partial_\tau^\beta\big(\tilde r_1^\epsilon, \tilde r_2^\epsilon\big)(t)\big\|^2_{L^2_l(\Omega)} + 4\delta_0^{-4}\big\|\partial_\tau^\beta \tilde r_3^\epsilon\big\|^2_{L^2_{-1}(\Omega)}\big),
\end{aligned}
$$

which implies that from (3.97) and (4.5),

$$
\begin{aligned}
F^\epsilon(t) &\le C\delta_0^{-8}M_0^6 + \epsilon^2\delta_0^{-8}\mathcal{P}\big(M_0 + \|(u_0,h_0)\|_{H_l^{3m+2}}\big)\\
&\le \delta_0^{-8}\mathcal{P}\big(M_0 + \|(u_0,h_0)\|_{H_l^{3m+2}}\big).
\end{aligned}
$$

Therefore, by choosing

$$
T_1 := \min\left\{\frac{\delta_0^8 F_0}{\mathcal{P}\big(M_0 + \|(u_0,h_0)\|_{H_l^{3m+2}}\big)},\ \frac{3\delta_0^8}{32CF_0^2}\right\}
$$

in (4.8), we obtain (4.6) for $T_* \le T_1$.

On the other hand, similarly to the estimates (3.95) and (3.96) given in Proposition 3.6, we have the following bounds for $\langle y\rangle^{l+1}\partial_y^i(u^\epsilon, h^\epsilon)$, $i = 1, 2$, and h^ϵ:

$$
\begin{aligned}
&\big\|\langle y\rangle^{l+1}\partial_y^i(u^\epsilon, h^\epsilon)(t)\big\|_{L^\infty(\Omega)} \le\\
&\qquad \big\|\langle y\rangle^{l+1}\partial_y^i(u_0, h_0)\big\|_{L^\infty(\Omega)} + Ct\cdot\big(\sup_{0\le s\le t}\|(u^\epsilon,h^\epsilon)(s)\|_{\mathcal{H}_l^5}\big),
\end{aligned}
$$

and

$$
h^\epsilon(t,x,y) \ge h_0(x,y) - Ct\cdot\big(\sup_{0\le s\le t}\|(u^\epsilon,h^\epsilon)(s)\|_{\mathcal{H}_0^3}\big).
$$

Then, from the assumptions (2.18) for the initial data (u_0, h_0) and the choice of T_1 above, we obtain by (4.8)

$$
\big\|\langle y\rangle^{l+1}\partial_y^i(u^\epsilon, h^\epsilon)(t)\big\|_{L^\infty(\Omega)} \le (2\delta_0)^{-1} + 2CF_0^{\frac12}t, \quad i = 1, 2,
$$

and

$$h^\epsilon(t,x,y) + H(t,x)\phi'(y) \geq 2\delta_0 + (H(t,x) - H(0,x))\phi'(y) - 2CF_0^{\frac{1}{2}} t$$
$$\geq 2\delta_0 - C\big(M_0 + 2F_0^{\frac{1}{2}}\big)t.$$

So, let us choose

$$T_2 := \min\left\{T_1, \frac{1}{4C\delta_0 F_0^{1/2}}, \frac{\delta_0}{C\big(M_0 + 2F_0^{1/2}\big)}\right\};$$

then, (4.7) holds for $T_* = T_2$. Therefore, we find the lifespan $T_* = T_2$ and establish the estimates (4.6) and (4.7), and consequently complete the proof of this proposition. □

From Proposition 4.1 we obtain the local existence of solutions $(u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)$ to the problem (4.1) and their uniform estimates in ϵ. Now, by letting $\epsilon \to 0$ we will obtain the solution to the original problem (2.12) through some compactness arguments. Indeed, from the uniform estimate (4.6), by the Lions-Aubin lemma, and the compact embedding of $H_l^m(\Omega)$ in $H_{\rm loc}^{m'}$ for $m' < m$ (see [30, lemma 6.2]), we know that there exists

$$(u,h) \in L^\infty\big(0, T_*; \mathcal{H}_l^m\big) \cap \Big(\bigcap_{m'<m-1} C^1\big([0,T_*]; H_{\rm loc}^{m'}(\Omega)\big)\Big),$$

such that, up to a subsequence,

$$\begin{aligned} &\partial_t^i(u^\epsilon, h^\epsilon) \overset{*}{\rightharpoonup} \partial_t^i(u,h) \quad \text{in } L^\infty\big(0,T_*; H_l^{m-i}(\Omega)\big),\ 0 \leq i \leq m,\\ &(u^\epsilon, h^\epsilon) \to (u,h) \qquad \text{in } C^1\big([0,T_*]; H_{\rm loc}^{m'}(\Omega)\big). \end{aligned}$$

Then, by using the uniform convergence of $(\partial_x u^\epsilon, \partial_x h^\epsilon)$ because of $(\partial_x u^\epsilon, \partial_x h^\epsilon) \in \mathrm{Lip}(\Omega_{T_*})$, we get the pointwise convergence for (v^ϵ, g^ϵ), i.e.,

$$\begin{aligned} (v^\epsilon, g^\epsilon) &= \left(-\int_0^y \partial_x u^\epsilon\, dz, -\int_0^y \partial_x h^\epsilon\, dz\right)\\ &\to \left(-\int_0^y \partial_x u\, dz, -\int_0^y \partial_x h\, dz\right) := (v,g). \end{aligned} \tag{4.10}$$

Now, we can pass the limit $\epsilon \to 0$ in the problem (4.1) and obtain that (u,v,h,g), v and g given by (4.10) solve the original problem (2.12). As $(u,h) \in L^\infty(0,T_*; \mathcal{H}_l^m)$, it is easy to get that $(u,h) \in \bigcap_{i=0}^m W^{i,\infty}(0,T; H_l^{m-i}(\Omega))$, and consequently (2.19) is proven. Moreover, the relations (2.20) and (2.21) follow immediately by combining the divergence-free conditions $v = -\partial_y^{-1}\partial_x u$, $g = -\partial_y^{-1}\partial_x h$, with (2.7) and (2.8), respectively. Thus, we prove the local existence result of Theorem 2.2.

4.2 Uniqueness

We will show the uniqueness of the obtained solution to (2.12). Let (u^1, v^1, h^1, g^1) and (u^2, v^2, h^2, g^2) be two solutions in $[0, T_*]$, constructed in the previous subsection, with respect to the initial data (u_0^1, h_0^1) and (u_0^2, h_0^2), respectively. Set

$$(\widetilde{u}, \widetilde{v}, \widetilde{h}, \widetilde{g}) = (u^1 - u^2, v^1 - v^2, h^1 - h^2, g^1 - g^2),$$

then we have

$$\begin{cases} \partial_t \widetilde{u} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\widetilde{u} \\ \quad - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\widetilde{h} \\ \quad - \mu\partial_y^2\widetilde{u} + (\partial_x u^2 + U_x\phi')\widetilde{u} + (\partial_y u^2 + U\phi'')\widetilde{v} \\ \quad - (\partial_x h^2 + H_x\phi')\widetilde{h} - (\partial_y h^2 + H\phi'')\widetilde{g} = 0, \\ \partial_t \widetilde{h} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\widetilde{h} \\ \quad - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\widetilde{u} \\ \quad - \kappa\partial_y^2\widetilde{h} + (\partial_x h^2 + H_x\phi')\widetilde{u} + (\partial_y h^2 + H\phi'')\widetilde{v} \\ \quad - (\partial_x u^2 + U_x\phi')\widetilde{h} - (\partial_y u^2 + U\phi'')\widetilde{g} = 0, \\ \partial_x\widetilde{u} + \partial_y\widetilde{v} = 0, \quad \partial_x\widetilde{h} + \partial_y\widetilde{g} = 0, \\ (\widetilde{u}, \widetilde{h})|_{t=0} = (u_0^1 - u_0^2, h_0^1 - h_0^2), \quad (\widetilde{u}, \widetilde{v}, \partial_y\widetilde{h}, \widetilde{g})|_{y=0} = \mathbf{0}. \end{cases} \tag{4.11}$$

Denote $\widetilde{\psi} := \partial_y^{-1}\widetilde{h} = \partial_y^{-1}(h^1 - h^2)$; then from the second equation $(4.11)_2$ of $\widetilde{h}$ and the divergence-free conditions, we know that $\widetilde{\psi}$ satisfies the following equation:

$$\begin{aligned} &\partial_t\widetilde{\psi} + \big[(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y\big]\widetilde{\psi} \\ &\quad - \big(g^2 - H_x\phi)\widetilde{u} + (h^2 + H\phi')\widetilde{v} - \kappa\partial_y^2\widetilde{\psi} = 0. \end{aligned}$$

Similarly as for(3.50), we introduce the new quantities

$$\bar{u} := \widetilde{u} - \frac{\partial_y u^2 + U\phi''}{h^2 + H\phi'}\widetilde{\psi}, \quad \bar{h} := \widetilde{h} - \frac{\partial_y h^2 + H\phi''}{h^2 + H\phi'}\widetilde{\psi}, \tag{4.12}$$

and then we have

$$\bar{u} := u^1 - u^2 - \eta_1^2\,\partial_y^{-1}(h^1 - h^2), \quad \bar{h} := h^1 - h^2 - \eta_2^2\,\partial_y^{-1}(h^1 - h^2), \tag{4.13}$$

where we define

$$\eta_1^2 := \frac{\partial_y u^2 + U\phi''}{h^2 + H\phi'}, \quad \eta_2^2 := \frac{\partial_y h^2 + H\phi''}{h^2 + H\phi'}.$$

Next, we can obtain that through direct calculation, $(\overline{u}, \overline{h})$ admits the following initial boundary value problem:

$$
\begin{cases}
\partial_t \overline{u} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\overline{u} \\
\quad - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\overline{h} - \mu\partial_y^2\overline{u} \\
\quad + (\kappa - \mu)\eta_1^2\partial_y\overline{h} + a_1\overline{u} + b_1\overline{h} + c_1\widetilde{\psi} = 0, \\
\partial_t \overline{h} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\overline{h} \\
\quad - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\overline{u} - \kappa\partial_y^2\overline{h} \\
\quad + a_2\overline{u} + b_2\overline{h} + c_2\widetilde{\psi} = 0, \\
(\overline{u}, \partial_y\overline{h})|_{y=0} = 0, \\
(\overline{u}, \overline{h})|_{t=0} = \big(u_0^1 - u_0^2 - \eta_{10}^2\, \partial_y^{-1}(h_0^1 - h_0^2), \\
\qquad\qquad h_0^1 - h_0^2 - \eta_{20}^2\, \partial_y^{-1}(h_0^1 - h_0^2)\big),
\end{cases}
\tag{4.14}
$$

where

$$
\begin{cases}
a_1 = \partial_x u^2 + U_x\phi' + (g^2 - H_x\phi)\eta_1^2, \\
b_1 = (\kappa - \mu)\eta_1^2\eta_2^2 - 2\mu\partial_y\eta_1^2 - (\partial_x h^2 + H_x\phi') - (g^2 - H_x\phi)\eta_2^2, \\
c_1 = \big[\partial_t + (u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y - \mu\partial_y^2\big]\eta_1^2 \\
\qquad - \big[(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y\big]\eta_2^2 - 2\mu\eta_2^2\partial_y\eta_1^2 \\
\qquad + (\kappa - \mu)\eta_1^2\big[(\eta_2^2)^2 + \partial_y\eta_2^2\big] + (g^2 - H_x\phi)\big[(\eta_1^2)^2 - (\eta_2^2)^2\big] \\
\qquad + (\partial_x u^2 + U_x\phi')\eta_1^2 - (\partial_x h^2 + H_x\phi')\eta_2^2, \\
a_2 = \partial_x h^2 + H_x\phi' + (g^2 - H_x\phi)\eta_2^2, \\
b_2 = -2\kappa\partial_y\eta_2^2 - (\partial_x u^2 + U_x\phi') - (g^2 - H_x\phi)\eta_1^2, \\
c_2 = \big[\partial_t + (u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y - \kappa\partial_y^2\big]\eta_2^2 \\
\qquad - \big[(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y\big]\eta_1^2 - 2\kappa\eta_2^2\partial_y\eta_2^2 \\
\qquad + (\partial_x h^2 + H_x\phi')\eta_1^2 - (\partial_x u^2 + U_x\phi')\eta_2^2,
\end{cases}
$$

and

$$
\eta_{10}^2(x, y) := \frac{\partial_y u_0^2 + U(0, x)\phi''(y)}{h_0^2 + H(0, x)\phi'(y)}, \quad \eta_{20}^2(x, y) := \frac{\partial_y h_0^2 + H(0, x)\phi''(y)}{h_0^2 + H(0, x)\phi'(y)}.
$$

Combining (4.12) with the fact $\widetilde{\psi} = \partial_y^{-1}\widetilde{h}$, we get that

$$
\overline{h} = (h^2 + H\phi') \cdot \partial_y\left(\frac{\widetilde{\psi}}{h^2 + H\phi'}\right),
$$

and then, from $\widetilde{\psi}|_{y=0} = 0$, we have

$$
\begin{aligned}
\widetilde{\psi}(t, x, y) = {} & (h^2(t, x, y) + H(t, x)\phi'(y)) \\
& \cdot \int_0^y \frac{\overline{h}(t, x, z)}{h^2(t, x, z) + H(t, x)\phi'(z)} dz.
\end{aligned}
\tag{4.15}
$$

Since $h^2 + H\phi' \geq \delta_0$, applying (2.7) in (4.15) gives

$$\left\| \frac{\widetilde{\psi}(t)}{1+y} \right\|_{L^2(\Omega)} \leq 2\delta_0^{-1} \|h^2 + H\phi'\|_{L^\infty([0,T_*]\times\Omega)} \|\overline{h}(t)\|_{L^2(\Omega)}. \tag{4.16}$$

Moreover, through a similar process of getting the estimates (3.67), we can obtain that there exists a constant

$$C = C\big(T_*, \delta_0, \phi, U, H, \|(u^1, h^1)\|_{\mathcal{H}_l^5}, \|(u^2, h^2)\|_{\mathcal{H}_l^5}\big) > 0$$

such that

$$\begin{gathered} \|a_i\|_{L^\infty([0,T_*]\times\Omega)}, \quad \|b_i\|_{L^\infty([0,T_*]\times\Omega)}, \quad \|(1+y)c_i\|_{L^\infty([0,T_*]\times\Omega)} \leq C, \\ i = 1, 2. \end{gathered} \tag{4.17}$$

Thus, we have from (4.16) and (4.17),

$$\|(c_i \widetilde{\psi})(t)\|_{L^2(\Omega)} \leq C \|\overline{h}(t)\|_{L^2(\Omega)}, \quad i = 1, 2. \tag{4.18}$$

PROPOSITION 4.2. *Let (u^1, v^1, h^1, g^1) and (u^2, v^2, h^2, g^2) be two solutions of problem* (2.12) *with respect to the initial data (u_0^1, h_0^1) and (u_0^2, h_0^2), respectively, satisfying that $(u^j, h^j) \in \bigcap_{i=0}^m W^{i,\infty}(0, T; H_l^{m-i}(\Omega))$ for $m \geq 5$, $j = 1, 2$. Then, there exists a positive constant*

$$C = C\big(T_*, \delta_0, \phi, U, H, \|(u^1, h^1)\|_{\mathcal{H}_l^5}, \|(u^2, h^2)\|_{\mathcal{H}_l^5}\big) > 0$$

such that for the quantities $(\overline{u}, \overline{h})$ given by (4.13),

$$\frac{d}{dt} \|(\overline{u}, \overline{h})(t)\|_{L^2(\Omega)}^2 + \|(\partial_y \overline{u}, \partial_y \overline{h})(t)\|_{L^2(\Omega)}^2 \leq C \|(\overline{u}, \overline{h})\|_{L^2(\Omega)}^2. \tag{4.19}$$

Proposition 4.2 can be proved by the standard energy method and the estimates (4.17) and (4.18); here we omit the proof for brevity of presentation. Then, by virtue of Proposition 4.2 we can prove the uniqueness of solutions to (2.12) as follows.

First, if the initial data satisfies $(u^1, h^1)\big|_{t=0} = (u^2, h^2)\big|_{t=0}$, then we know that from (4.14), $(\overline{u}, \overline{h})$ admits the zero initial data, which implies that $(\overline{u}, \overline{h}) \equiv 0$ by applying Gronwall's lemma to (4.19). Second, plugging $\overline{h} \equiv 0$ into (4.15) yields $\widetilde{\psi} \equiv 0$. Then, from (4.13) we have $(u^1, h^1) \equiv (u^2, h^2)$ immediately through the following calculation:

$$(u^1, h^1) - (u^2, h^2) = (\widetilde{u}, \widetilde{h}) = (\overline{u}, \overline{h}) + \big(\eta_1^2, \eta_2^2\big)\widetilde{\psi} \equiv 0.$$

Finally, we obtain $(v^1, g^1) \equiv (v^2, g^2)$, since $v^i = -\partial_y^{-1}\partial_x u^i$ and $g^i = -\partial_y^{-1}\partial_x h^i$ for $i = 1, 2$, and show the uniqueness of solutions.

Remark 4.3. We mention that in the independent recent work [13], the authors give a systematic derivation of MHD boundary layer models and consider the linearization for a similar system to (1.8) around some shear flow. By using the analogous transformation to (3.59), they obtain the linear stability for the system in the Sobolev framework.

5 A Coordinate Transformation

In this section, we will introduce another method to study the initial boundary value problem considered in this paper:

$$
\begin{cases}
\partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 = h_1\partial_x h_1 + h_2\partial_y h_1 + \mu\partial_y^2 u_1,\\
\partial_t h_1 + \partial_y(u_2 h_1 - u_1 h_2) = \kappa\partial_y^2 h_1,\\
\partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0,\\
(u_1, u_2, \partial_y h_1, h_2)\big|_{y=0} = 0, \quad \lim_{y\to+\infty}(u_1, h_1) = (U, H).
\end{cases} \tag{5.1}
$$

As we mentioned in Section 3.2, by the divergence-free condition

$$\partial_x h_1 + \partial_y h_2 = 0,$$

there exists a stream function ψ such that

$$h_1 = \partial_y\psi, \quad h_2 = -\partial_x\psi, \quad \psi\big|_{y=0} = 0. \tag{5.2}$$

Moreover, ψ satisfies

$$\partial_t\psi + u_1\partial_x\psi + u_2\partial_y\psi = \kappa\partial_y^2\psi. \tag{5.3}$$

Under the assumptions that

$$h_1(t,x,y) > 0 \quad \text{or} \quad \partial_y\psi(t,x,y) > 0, \tag{5.4}$$

we can introduce the following transformation:

$$\tau = t, \quad \xi = x, \quad \eta = \psi(t,x,y), \tag{5.5}$$

and then (5.1) can be written in the new coordinates as follows:

$$
\begin{cases}
\partial_\tau u_1 + u_1\partial_\xi u_1 - h_1\partial_\xi h_1 + (\kappa-\mu)h_1\partial_\eta h_1\partial_\eta u_1 = \mu h_1^2\partial_\eta^2 u_1,\\
\partial_\tau h_1 - h_1\partial_\xi u_1 + u_1\partial_\xi h_1 = \kappa h_1^2\partial_\eta^2 h_1,\\
(u_1, h_1\partial_\eta h_1)|_{y=0} = 0, \quad \lim_{\eta\to+\infty}(u_1, h_1) = (U, H).
\end{cases} \tag{5.6}
$$

Remark 5.1. The equations (5.6) are quasi-linear equations, and there is no loss-of-regularity term in (5.6), so we can use the classical Picard iteration scheme to establish the local existence. However, in order to guarantee that the coordinates transformation (5.5) is valid, one needs to assume that $h_1(t,x,y) > 0$. Moreover, one can obtain the stability of solutions to (5.6) in the new coordinates (τ, ξ, η). It is necessary to transfer the well-posedness of solutions to the original equations (5.1), and then there will be some loss of regularity.

Remark 5.2. Based on the well-posedness result for the MHD boundary layer in the Sobolev framework given in this paper, we will show the validity of the vanishing limit of the viscous MHD equations (1.1) as $\epsilon \to 0$ in a future work [26], that is, to show the solution to (1.1) converges to a solution of ideal MHD equations, corresponding to $\epsilon = 0$ in (1.1), outside the boundary layer, and to a boundary layer profile studied in this paper inside the boundary layer.

Appendix: Some Inequalities

In this appendix, we will prove the inequalities given in Lemma 2.1. Such inequalities can be found in [30, 40]; here we give a proof for the reader's convenience.

PROOF OF LEMMA 2.1.

(i) From $\lim_{y\to+\infty}(fg)(x,y)=0$, we get

$$\begin{aligned}\left|\int_{\mathbb{T}_x}(fg)\big|_{y=0}\,dx\right| &= \left|\int_\Omega \partial_y(fg)dx\,dy\right| \\ &\le \int_\Omega |\partial_y f\cdot g|dx\,dy + \int_\Omega |f\cdot\partial_y g|dx\,dy \\ &\le \|\partial_y f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|\partial_y g\|_{L^2(\Omega)},\end{aligned}$$

and we get (2.2). (2.3) follows immediately by letting $g=f$ in (2.2).

(ii) From $m\ge 3$ and $|\alpha|+|\widetilde{\alpha}|\le m$, we know that there must be $|\alpha|\le m-2$ or $|\widetilde{\alpha}|\le m-2$. Without loss of generality, we assume that $|\alpha|\le m-2$; then for any $l_1,l_2\ge 0$ with $l_1+l_2=l$, we have by using the Sobolev embedding inequality that

$$\begin{aligned}&\big\|\big(D^\alpha f\cdot D^{\widetilde{\alpha}}g\big)(t,\cdot)\big\|_{L^2_{l+k+\widetilde{k}}(\Omega)} \\ &\quad\le \big\|\langle y\rangle^{l_1+k}D^\alpha f(t,\cdot)\big\|_{L^\infty(\Omega)}\cdot\big\|\langle y\rangle^{l_2+\widetilde{k}}D^{\widetilde{\alpha}}g(t,\cdot)\big\|_{L^2(\Omega)} \\ &\quad\le C\big\|\langle y\rangle^{l_1+k}D^\alpha f(t,\cdot)\big\|_{H^2(\Omega)}\|g(t)\|_{\mathcal{H}^{|\widetilde{\alpha}|}_{l_2}} \\ &\quad\le C\big\|f(t)\big\|_{\mathcal{H}^{|\alpha|+2}_{l_1}(\Omega)}\|g(t)\|_{\mathcal{H}^m_{l_2}},\end{aligned}$$

which implies (2.4) because of $|\alpha|+2\le m$.

(iii) For $\lambda>\frac{1}{2}$, it follows that by integration by parts,

$$\begin{aligned}&\big\|\langle y\rangle^{-\lambda}\big(\partial_y^{-1}f\big)(y)\big\|^2_{L^2_y(\mathbb{R}_+)} \\ &\quad= \int_0^{+\infty}\frac{\big[\big(\partial_y^{-1}f\big)(y)\big]^2}{1-2\lambda}d(1+y)^{1-2\lambda} \\ &\quad= \frac{2}{2\lambda-1}\int_0^{+\infty}(1+y)^{1-2\lambda}f(y)\cdot\big(\partial_y^{-1}f\big)(y)dy \\ &\quad\le \frac{2}{2\lambda-1}\big\|\langle y\rangle^{-\lambda}\big(\partial_y^{-1}f\big)(y)\big\|_{L^2_y(\mathbb{R}_+)}\cdot\|\langle y\rangle^{1-\lambda}f(y)\|_{L^2_y(\mathbb{R}_+)},\end{aligned}$$

which implies the first inequality of (2.5).

On the other hand, note that for $\widetilde{\lambda} > 0$,

$$\left|\left(\partial_y^{-1} f\right)(y)\right| \leq \int_0^y |f(z)|dz \leq \|(1+z)^{1-\widetilde{\lambda}} f(z)\|_{L^\infty(0,y)} \cdot \int_0^y (1+z)^{\widetilde{\lambda}-1} dz$$
$$\leq \frac{(1+y)^{\widetilde{\lambda}} - 1}{\widetilde{\lambda}} \|(1+y)^{1-\widetilde{\lambda}} f(y)\|_{L_y^\infty(\mathbb{R}_+)},$$

which implies the second inequality of (2.5) immediately.

Next, as $m \geq 3$ and $|\alpha| + |\widetilde{\beta}| \leq m$, we also get $|\alpha| \leq m-2$ or $|\widetilde{\beta}| \leq m-2$. If $|\alpha| \leq m-2$, by using Sobolev embedding inequality and the first inequality of (2.5), we have for any $\lambda > \frac{1}{2}$,

$$\begin{aligned}
&\left\|\left(D^\alpha g \cdot \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} h\right)(t,\cdot)\right\|_{L^2_{l+k}(\Omega)} \\
&\leq \|\langle y\rangle^{l+\lambda+k} D^\alpha g(t,\cdot)\|_{L^\infty(\Omega)} \cdot \left\|\langle y\rangle^{-\lambda} \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} h(t,\cdot)\right\|_{L^2(\Omega)} \\
&\leq C\|\langle y\rangle^{l+\lambda+k} D^\alpha g(t,\cdot)\|_{H^2(\Omega)} \cdot \left\|\langle y\rangle^{1-\lambda} \partial_\tau^{\widetilde{\beta}} h(t,\cdot)\right\|_{L^2(\Omega)} \\
&\leq C\|g(t)\|_{\mathcal{H}_{l+\lambda}^{|\alpha|+2}} \|h(t)\|_{\mathcal{H}_{1-\lambda}^{|\widetilde{\beta}|}}.
\end{aligned}$$

If $|\widetilde{\beta}| \leq m-2$, by the Sobolev embedding inequality and the second inequality of (2.5),

$$\begin{aligned}
&\left\|\left(D^\alpha g \cdot \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} h\right)(t,\cdot)\right\|_{L^2_{l+k}(\Omega)} \\
&\leq \|\langle y\rangle^{l+\lambda+k} D^\alpha g(t,\cdot)\|_{L^2(\Omega)} \cdot \left\|\langle y\rangle^{-\lambda} \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} h(t,\cdot)\right\|_{L^\infty(\Omega)} \\
&\leq C\|g(t)\|_{\mathcal{H}_{l+\lambda}^{|\alpha|}} \cdot \left\|\langle y\rangle^{1-\lambda} \partial_\tau^{\widetilde{\beta}} h(t,\cdot)\right\|_{H^2(\Omega)} \\
&\leq C\|g(t)\|_{\mathcal{H}_{l+\lambda}^{|\alpha|}} \|h(t)\|_{\mathcal{H}_{1-\lambda}^{|\widetilde{\beta}|+2}}.
\end{aligned}$$

Therefore, we get the proof of (2.6) and then (2.7) follows by letting $\lambda = 1$ in (2.6).

(iv) For any $\lambda > \frac{1}{2}$,

$$\left|(\partial_y^{-1} f)(y)\right| \leq \|f(y)\|_{L_y^1(\mathbb{R}_+^2)} \leq \|\langle y\rangle^{-\lambda}\|_{L_y^2(\mathbb{R}_+)} \|\langle y\rangle^\lambda f\|_{L_y^2(\mathbb{R}_+)}$$
$$\leq C\|\langle y\rangle^\lambda f\|_{L_y^2(\mathbb{R}_+)},$$

and we get (2.8).

For $m \geq 2$ and $|\alpha| + |\widetilde{\beta}| \leq m$, we get that $|\alpha| \leq m-1$ or $|\widetilde{\beta}| \leq m-1$. If $|\alpha| \leq m-1$, by using the Sobolev embedding inequality and (2.8), we have that for any $\lambda > \frac{1}{2}$,

$$\begin{aligned}
&\left\| \left(D^\alpha f \cdot \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} g\right)(t,\cdot) \right\|_{L^2_{l+k}(\Omega)} \\
&\quad \le \|\langle y\rangle^{l+k} D^\alpha f(t,\cdot)\|_{L^\infty_x L^2_y(\Omega)} \cdot \left\| \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} g(t,\cdot) \right\|_{L^2_x L^\infty_y(\Omega)} \\
&\quad \le C \|\langle y\rangle^{l+k} D^\alpha f(t,\cdot)\|_{H^1(\Omega)} \cdot \left\| \langle y\rangle^\lambda \partial_\tau^{\widetilde{\beta}} g(t,\cdot) \right\|_{L^2(\Omega)} \\
&\quad \le C \|f(t)\|_{\mathcal{H}_l^{|\alpha|+1}} \|g(t)\|_{\mathcal{H}_\lambda^{|\widetilde{\beta}|}}.
\end{aligned}$$

If $|\widetilde{\beta}| \le m-1$, by Sobolev embedding inequality and (2.8),

$$\begin{aligned}
&\left\| \left(D^\alpha f \cdot \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} g\right)(t,\cdot) \right\|_{L^2_{l+k}(\Omega)} \\
&\quad \le \|\langle y\rangle^{l+k} D^\alpha f(t,\cdot)\|_{L^2(\Omega)} \cdot \left\| \partial_\tau^{\widetilde{\beta}} \partial_y^{-1} g(t,\cdot) \right\|_{L^\infty(\Omega)} \\
&\quad \le C \|f(t)\|_{\mathcal{H}_l^{|\alpha|}} \cdot \left\| \langle y\rangle^\lambda \partial_\tau^{\widetilde{\beta}} g(t,\cdot) \right\|_{H_1^x L^2_y(\Omega)} \\
&\quad \le C \|f(t)\|_{\mathcal{H}_l^{|\alpha|}} \|g(t)\|_{\mathcal{H}_\lambda^{|\widetilde{\beta}|+1}}.
\end{aligned}$$

Thus, we get (2.9), and so complete the proof of this lemma. □

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Bibliography

[1] Alexandre, R.; Wang, Y.-G.; Xu, C.-J.; Yang, T. Well-posedness of the Prandtl equation in Sobolev spaces. *J. Amer. Math. Soc.* **28** (2015), no. 3, 745–784. doi:10.1090/S0894-0347-2014-00813-4

[2] Alfvén, H. Existence of electromagnetic-hydrodynamic waves. *Nature* **150** (1942), 405–406. doi:10.1038/150405d0

[3] Arhipov, V. N. Influence of a magnetic field on boundary layer stability. *Dokl. Akad. Nauk SSSR* **129**, 751–753 (Russian); translated as *Soviet Physics. Dokl.* **4** (1959), 1199–1201.

[4] Cowling, T. G. *Magnetohydrodynamics.* Interscience Tracts on Physics and Astronomy, 4. Interscience, New York; Interscience, London, 1957.

[5] Davidson, P. A. *An introduction to magnetohydrodynamics.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001. doi:10.1017/CBO9780511626333

[6] Drazin, P. G. Stability of parallel flow in a parallel magnetic field at small magnetic Reynolds numbers. *J. Fluid Mech.* **8** (1960), 130–142. doi:10.1017/S0022112060000475

[7] Duvaut, G.; Lions, J.-L. Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.* **46** (1972), 241–279. doi:10.1007/BF00250512

[8] E, W.; Engquist, B. Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.* **50** (1997), no. 12, 1287–1293. doi:10.1002/(SICI)1097-0312(199712)50:12<1287::AID-CPA4>3.0.CO;2-4

[9] Gérard-Varet, D.; Dormy, E. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.* **23** (2010), no. 2, 591–609. doi:10.1090/S0894-0347-09-00652-3

[10] Gérard-Varet, D.; Maekawa, Y.; Masmoudi, N. Gevrey stability of Prandtl expansions for 2D Navier-Stokes flows. Preprint, 2016. arXiv:1607.06434 [math.AP]

[11] Gérard-Varet, D.; Masmoudi, N. Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Éc. Norm. Supér. (4)* **48** (2015), no. 6, 1273–1325. doi:10.24033/asens.2270

[12] Gérard-Varet, D.; Nguyen, T. Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* **77** (2012), no. 1-2, 71–88.

[13] Gérard-Varet, D.; Prestipino, M. Formal derivation and stability analysis of boundary layer models in MHD. *Z. Angew. Math. Phys.* **68** (2017), no. 3, Art. 76, 16 pp. doi:10.1007/s00033-017-0820-x

[14] Grenier, E. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53** (2000), no. 9, 1067–1091. doi:10.1002/1097-0312(200009)53:9<1067::AID-CPA1>3.3.CO;2-H

[15] Guo, Y.; Nguyen, T. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.* **64** (2011), no. 10, 1416–1438. doi:10.1002/cpa.20377

[16] Hartmann, J. Theory of the laminar flow of an electrically conductive liquid in a homogeneous magnetic field. *K. Dan. Vidensk. Selsk. Mat. Fys. Medd.* **15** (1937), no. 6, 1–28.

[17] Hartmann, J.; Lazarus, F. Experimental investigations on the flow of mercury in a homogeneous magnetic field. *K. Dan. Vidensk. Selsk. Mat. Fys. Medd.* **15** (1937), no. 7, 1–45.

[18] Ignatova, M.; Vicol, V. Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.* **220** (2016), no. 2, 809–848. doi:10.1007/s00205-015-0942-2

[19] Kukavica, I.; Masmoudi, N.; Vicol, V.; Wong, T. K. On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.* **46** (2014), no. 6, 3865–3890. doi:10.1137/140956440

[20] Kukavica, I.; Vicol, V. On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.* **11** (2013), no. 1, 269–292. doi:10.4310/CMS.2013.v11.n1.a8

[21] Li, W.-X.; Wu, D.; Xu, C.-J. Gevrey class smoothing effect for the Prandtl equation. *SIAM J. Math. Anal.* **48** (2016), no. 3, 1672–1726. doi:10.1137/15M1020368

[22] Li, W.-X.; Yang, T. Well-posedness in Gevrey space for the Prandtl equations with non-degenerate points. *J. Eur. Math. Soc.*, in press.

[23] Liu, C.-J.; Wang, Y.-G.; Yang, T. Global existence of weak solutions to the three-dimensional Prandtl equations with a special structure. *Discrete Contin. Dyn. Syst. Ser. - S* **9** (2016), no. 6, 2011–2029. doi:10.3934/dcdss.2016082

[24] Liu, C.-J.; Wang, Y.-G.; Yang, T. On the ill-posedness of the Prandtl equations in three-dimensional space. *Arch. Ration. Mech. Anal.* **220** (2016), no. 1, 83–108. doi:10.1007/s00205-015-0927-1

[25] Liu, C.-J.; Wang, Y.-G.; Yang, T. A well-posedness theory for the Prandtl equations in three space variables. *Adv. Math.* **308** (2017), 1074–1126. doi:10.1016/j.aim.2016.12.025

[26] Liu, C.-J.; Xie, F.; Yang, T. MHD boundary layers in Sobolev spaces without monotonicity. II. Convergence theory. Preprint, 2017. arXiv:1704.00523 [math.AP]

[27] Liu, C.-J.; Yang, T. Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. *J. Math. Pures Appl. (9)* **108** (2017), no. 2, 150–162. doi:10.1016/j.matpur.2016.10.014

[28] Lombardo, M. C.; Cannone, M.; Sammartino, M. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.* **35** (2003), no. 4, 987–1004. doi:10.1137/S0036141002412057

[29] Maekawa, Y. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* **67** (2014), no. 7, 1045–1128. doi:10.1002/cpa.21516
[30] Masmoudi, N.; Wong, T. K. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.* **68** (2015), no. 10, 1683–1741. doi:10.1002/cpa.21595
[31] Oleĭnik, O. A. On the system of Prandtl equations in boundary-layer theory. *Dokl. Akad. Nauk SSSR* **150** (1963), 28–31.
[32] Oleinik, O. A.; Samokhin, V. N. *Mathematical models in boundary layer theory.* Applied Mathematics and Mathematical Computation, 15. Chapman & Hall/CRC, Boca Raton, Fla., 1999.
[33] Prandtl, L. Über Flüssigkeits-bewegung bei sehr kleiner Reibung. *Actes du 3ème Congrés international des Mathématiciens, Heidelberg*, 484–491. Teubner, Leipzig, 1904.
[34] Rossow, V. J. *Boundary layer stability diagrams for electrically conducting fluids in the presence of a magnetic field*. NACA Technical Note, 4282. NACA, Washington, 1958.
[35] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192** (1998), no. 2, 433–461. doi:10.1007/s002200050304
[36] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* **192** (1998), no. 2, 463–491. doi:10.1007/s002200050305
[37] Sermange, M.; Temam, R. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* **36** (1983), no. 5, 635–664. doi:10.1002/cpa.3160360506
[38] Xiao, Y.; Xin, Z.; Wu, J. Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition. *J. Funct. Anal.* **257** (2009), no. 11, 3375–3394. doi:10.1016/j.jfa.2009.09.010
[39] Xin, Z.; Zhang, L. On the global existence of solutions to the Prandtl's system. *Adv. Math.* **181** (2004), no. 1, 88–133. doi:10.1016/S0001-8708(03)00046-X
[40] Xu, C.-J.; Zhang, X. Long time well-posedness of the Prandtl equations in Sobolev space. *J. Differential Equations* **263** (2017), no. 12, 8749–8803. doi:10.1016/j.jde.2017.08.046
[41] Zhang, P.; Zhang, Z. Long time well-posedness of Prandtl system with small and analytic initial data. *J. Funct. Anal.* **270** (2016), no. 7, 2591–2615. doi:10.1016/j.jfa.2016.01.004

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