# THE RIEMANNIAN SECTIONAL CURVATURE OPERATOR OF THE WEIL-PETERSSON METRIC AND ITS APPLICATION

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## Abstract

Fix a number $g > 1$, let $S$ be a close surface of genus $g$, and let Teich(S) be the Teichmüller space of $S$ endowed with the Weil-Petersson metric. In this paper we show that the Riemannian sectional curvature operator of Teich(S) is non-positive definite. As an application we show that any twist harmonic map from rank-one hyperbolic spaces $H_{Q,m} = Sp(m,1)/Sp(m) \cdot Sp(1)$ or $H_{O,2} = F_4^{-20}/SO(9)$ into Teich(S) is a constant.

## 1. Introduction

Let $S$ be a closed surface of genus $g$ where $g > 1$, and $T_g$ be the Teichmüller space of $S$. $T_g$ carries various metrics that have respective properties. For example, the Teichmüller metric is a complete Finsler metric. The McMullen metric, Ricci metric, and perturbed Ricci metric have bounded geometry [**16**, **17**, **18**]. The Weil-Petersson metric is Kähler [**1**] and incomplete [**5**, **26**]. There are also some other metrics on $T_g$ like the Bergman metric, Caratheodory metric, Kähler-Einstein metric, Kobayashi metric, and so on. In [**16**, **17**], the authors showed that some metrics listed above are comparable. In this paper we focus on the Weil-Petersson case. Throughout this paper, we let Teich(S) denote $T_g$ endowed with the Weil-Petersson metric. The geometry of the Weil-Petersson metric has been well studied in the past decades. One can refer to Wolpert's recent nice book [**29**] for details.

The curvature aspect of Teich(S) is very interesting, which plays an important role in the geometry of Weil-Petersson metric. This aspect has been studied over the past several decades. Ahlfors in [**1**] showed that the holomorphic sectional curvatures are negative. Tromba [**24**] and Wolpert [**27**] independently showed the sectional curvature of Teich(S) is negative. Moreover, in [**27**] the author proved the Royden's conjecture, which says that the holomorphic curvatures are bounded above by a negative number that only depends on the topology of the surface, by establishing the curvature formula (see theorem 2.4). Wolf in

[**25**] used harmonic tools to give another proof of this curvature formula. After that, people has been applying this formula to study the curvature of Teich(S) in more detail. For example, in [**21**] Schumacher showed that Teich(S) has strongly negative curvature in the sense of Siu (see [**22**]) which is stronger than negative sectional curvature. Huang in [**10**] showed there is no negative upper bound for the sectional curvature. In [**15**] Liu-Sun-Yau also used Wolpert's curvature formula to show that Teich(S) has dual Nakano negative curvature, which says that the complex curvature operator on the dual tangent bundle is positive in some sense. For some other related problems one can refer to [**3**, **10**, **11**, **16**, **17**, **23**, **28**, **30**].

Let $X \in$ Teich(S). We can view $X$ as a hyperbolic metric on $S$. One of our purposes in this paper is to study the Riemannian sectional curvature operator of Teich(S) at $X$. The method in this paper is highly influenced by the methods in [**15**, **21**, **27**], which essentially applied the curvature formula, the Cauchy-Schwarz inequality and the positivity of the Green function for the operator $(\Delta - 2)^{-1}$, where $\Delta$ is the Beltrami-Laplace operator on $X$. What we need more in this paper is the symmetry of the Green function for $(\Delta - 2)^{-1}$.

Before giving any statements let us state some neccessary background. Let $X$ be a point in Teich(S), and $T_X$ Teich(S) be the tangent space that is identified with the harmonic Beltrami differentials at $X$. Assume that $\{\mu_i\}_{i=1}^{3g-3}$ is a basis for $T_X$ Teich(S), and $\frac{\partial}{\partial t_i}$ is the vector fields corresponding to $\mu_i$. Locally, $t_i$ is a holomorphic coordinate around $X$; let $t_i = x_i + \mathbf{i}y_i$. $(x_1, x_2, \ldots, x_{3g-3}, y_1, y_2, \ldots, y_{3g-3})$ gives a real smooth coordinate around $X$. Since Teich(S) is a Riemannian manifold, it is natural to define the curvature tensor on it, which is denoted by $R(\cdot, \cdot, \cdot, \cdot)$. Let $T$ Teich(S) be the real tangent bundle of Teich(S) and $\wedge^2 T$ Teich(S) be the wedge product of two copies of $T$ Teich(S). The *curvature operator* $Q$ is defined on $\wedge^2 T$ Teich(S) by $Q(V_1 \wedge V_2, V_3 \wedge V_4) = R(V_1, V_2, V_3, V_4)$ and extended linearly, where $V_i$ are real vectors. It is easy to see that $Q$ is a bilinear symmetric form (one can see more details in [**12**]).

Now we can state our first result.

**Theorem 1.1.** *Let $S = S_g$ be a closed surface of genus $g > 1$ and* Teich(S) *be the Teichmüller space of $S$ endowed with the Weil-Petersson metric. And let $\boldsymbol{J}$ be the almost complex structure on* Teich(S) *and $Q$ be the curvature operator of* Teich(S)*. Then, for any $X \in$* Teich(S)*, we have*

(1) *$Q$ is non-positive definite, i.e., $Q(A, A) \leq 0$ for all $A \in \wedge^2 T_X$* Teich(S)*.*
(2) *$Q(A, A) = 0$ if and only if there exists an element $B$ in $\wedge^2 T_X$* Teich(S) *such that $A = B - \boldsymbol{J} \circ B$*

*where $\boldsymbol{J} \circ B$ is defined in section 4.*

A direct corollary is that the sectional curvature of Teich(S) is negative [1, 24, 27]. Normally a metric of negative curvature may not have non-positive definite curvature operator (see [2]).

In the second part of this paper we will study harmonic maps from certain rank-one spaces into Teich(S). For harmonic maps, there are a lot of very beautiful results when the target is either a complete Riemannian manifold with non-positive curvature operator or a complete non-positive curved metric space (see [6, 7, 31]). In particular, if the domain is either the Quaternionic hyperbolic space or the Cayley plane, different rigid results for harmonic maps were established in [9, 13, 19]. For harmonic maps into Teich(S), one can refer to the nice survey [8]. In this paper we establish the following rigid result.

**Theorem 1.2.** *Let $\Gamma$ be a lattice in a semisimple Lie group $G$ which is either $Sp(m,1)$ or $F_4^{-20}$, and $\mathrm{Mod}(S)$ be the mapping class group of* Teich(S). *Then, any twist harmonic map $f$ from $G/\Gamma$ into* Teich(S) *with respect to each homomorphism $\rho : \Gamma \to \mathrm{Mod}(S)$ must be a constant.*

The twist map $f$ with respect to $\rho$ means that $f(\gamma \circ Y) = \rho(\gamma) \circ f(Y)$ for all $\gamma \in \Gamma$.

**Plan of the paper.** In section 2 we provide some necessary background and some basic properties for the operator $D = -2(\Delta - 2)^{-1}$. In section 3 we establish the curvature operator formulas on different subspaces of $\wedge^2 T_X$ Teich(S) and show that the curvature operator is negative definite or non-positive definite on these different subspaces. In section 4 we establish the curvature operator formula for $Q$ on $\wedge^2 T_X$ Teich(S) to prove theorem 1.1. In section 5 we finish the proof of theorem 1.2.

## 2. Notations and Preliminaries

**2.1. Surfaces.** Let $S$ be a closed surface of genus $g \geq 2$, $M_{-1}$ denote the space of Riemannian metrics with constant curvature $-1$, and $X = (S, \sigma|dz|^2)$ be a particular element of $M_{-1}$. $Diff_0$, which is the group of diffeomorphisms isotopic to the identity, acts by pullback on $M_{-1}$. The Teichmüller space $T_g$ of $S$ is defined by the quotient space

$$M_{-1}/Diff_0.$$

The Teichmüller space has a natural complex structure, and its holomorphic cotangent space $T_X^* T_g$ is identified with the *quadratic differentials* $Q(X) = \varphi(z)dz^2$ on $X$. The *Weil-Petersson metric* is the Hermitian metric on $T_g$ arising from the the *Petersson scalar product*

$$< \varphi, \psi >= \int_S \frac{\varphi \cdot \overline{\psi}}{\sigma^2} dz d\overline{z}$$

via duality. We will concern ourselves primarily with its Riemannian part $g_{WP}$. Throughout this paper, we denote the Teichmüller space endowed with the Weil-Petersson metric by Teich(S).

Setting $D = -2(\Delta - 2)^{-1}$, where $\Delta$ is the Beltrami-Laplace operator on $X$, we have $D^{-1} = -\frac{1}{2}(\Delta - 2)$. The following property has been proved in a lot of literature; for completeness, we still state the proof here.

**Proposition 2.1.** *Let $D$ be the operator above. Then*

(1) *$D$ is self-adjoint.*
(2) *$D$ is positive.*

*Proof of (1).* Let $f$ and $g$ be two real-valued smooth functions on $X$, and $u = Df$, $v = Dg$. Then

$$\int_S Df \cdot g dA = \int_S u \cdot (-\frac{1}{2}(\Delta - 2)v)dA = -\frac{1}{2} \int_S u \cdot (\Delta - 2)v dA$$
$$= -\frac{1}{2} \int_S v \cdot (\Delta - 2)u dA = \int_S Dg \cdot f dA,$$

where the equality in the second row follows from the fact that $\Delta$ is self-adjoint on closed surfaces. For the case that $f$ and $g$ are complex-valued, one can prove it through the real and imaginary parts by using the same argument.

*Proof of (2).* Let $f$ be a real-valued smooth functions on $X$, and $u = Df$. Then

$$\int_S Df \cdot f dA = \int_S u \cdot (-\frac{1}{2}(\Delta - 2)u)dA = -\frac{1}{2} \int_S (u \cdot (\Delta u) - 2u^2)dA$$
$$= \frac{1}{2}(\int_S |\nabla u|^2 + 2u^2 dA) \geq 0,$$

where the equality in the second row follows from the Stoke's Theorem. The last equality holds if and only if $u = 0$. That is, $D$ is positive. For the case that $f$ is complex-valued, one can show it by arguing the real and imaginary parts.                                      q.e.d.

For the Green function of the operator $-2(\Delta - 2)^{-1}$, we have

**Proposition 2.2.** *Let $D$ be the operator above. Then there exists a Green function $G(w, z)$ for $D$ satisfying:*

(1) $G(w, z)$ is positive.

(2) $G(w, z)$ is symmetric, i.e, $G(w, z) = G(z, w)$.

*Proof.* One can refer to [**20**] and [**27**].                    q.e.d.

**The Riemannian tensor of the Weil-Petersson metric.** The curvature tensor is given by the following. Let $\mu_\alpha, \mu_\beta$ be two elements in the tangent space at $X$, and

$$g_{\alpha\bar\beta} = \int_X \mu_\alpha \cdot \overline{\mu_\beta} dA,$$

where dA is the area element for X.

Let us study the curvature tensor in these local coordinates. First of all, for the inverse of $(g_{i\bar j})$, we use the convention

$$g^{i\bar j} g_{k\bar j} = \delta_{ik}.$$

The curvature tensor is given by

$$R_{i\bar jk\bar l} = \frac{\partial^2}{\partial t^k \partial \overline{t^l}} g_{i\bar j} - g^{s\bar t} \frac{\partial}{\partial t^k} g_{i\bar t} \frac{\partial}{\partial \overline{t^l}} g_{s\bar j}.$$

Since Ahlfors showed that the first derivatives of the metric tensor vanish at the base point $X$ in these coordinates, at $X$ we have

(1) $$R_{i\bar jk\bar l} = \frac{\partial^2}{\partial t^k \partial \overline{t^l}} g_{i\bar j}.$$

By the same argument in Kähler geometry we have

**Proposition 2.3.** *For any indices i, j, k, l, we have*

$$(1) \quad R_{ij\overline{kl}} = R_{\overline{ij}kl} = 0.$$

$$(2) \quad R_{i\bar jk\bar l} = -R_{i\bar jl\bar k}.$$

$$(3) \quad R_{i\bar jk\bar l} = R_{k\bar ji\bar l}.$$

$$(4) \quad R_{i\bar jk\bar l} = R_{i\bar lk\bar j}.$$

*Proof.* These follow from formula (1) and the first Bianchi identity (one can refer to [**12**]).                    q.e.d.

Now let us state Wolpert's curvature formula, which is crucial in the proof of theorem 1.1.

**Theorem 2.4.** *(see [**27**]) The curvature tensor satisfies*

$$R_{i\bar jk\bar l} = \int_X D(\mu_i \mu_{\bar j}) \cdot (\mu_k \mu_{\bar l}) dA + \int_X D(\mu_i \mu_{\bar l}) \cdot (\mu_k \mu_{\bar j}) dA.$$

**Definition 2.5.** Let $\mu_*$ be elements $\in T_X$ Teich(S). Set

$$(i\overline{j}, k\overline{l}) := \int_X D(\mu_i\mu_{\overline{j}}) \cdot (\mu_k\mu_{\overline{l}})dA.$$

We close this section by rewriting theorem 2.4 as follows,

**Theorem 2.6.**
$$R_{i\overline{j}k\overline{l}} = (i\overline{j}, k\overline{l}) + (i\overline{l}, k\overline{j}).$$

## 3. Curvature operator on subspaces of $\wedge T_X^2$ Teich(S)

Before we study the curvature operator of Teich(S), let us set some neccessary notations. Let $U$ be a neighborhood of $X$ and $(t_1, t_2, \ldots, t_{3g-3})$ be a local holomorphic coordinate on $U$, where $t_i = x_i + \mathbf{i}y_i (1 \leq i \leq 3g-3)$. Then $(x_1, x_2, \ldots, x_{3g-3}, y_1, y_2, \ldots, y_{3g-3})$ is a real smooth coordinate in $U$. Furthermore, we have

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}, \qquad \frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \overline{t_i}}).$$

Let $T$ Teich(S) be the real tangent bundle of Teich(S) and $\wedge^2 T$ Teich(S) be the exterior wedge product of $T$ Teich(S) and itself. For any $X \in U$, we have

$$T_X \text{ Teich(S)} = Span\{\frac{\partial}{\partial x_i}(X), \frac{\partial}{\partial y_j}(X)\}_{1 \leq i,j \leq 3g-3}$$

and

$$\wedge^2 T \text{ Teich(S)} = Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}.$$

Set

$$\wedge^2 T_X^1 \text{ Teich(S)} := Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\},$$

$$\wedge^2 T_X^2 \text{ Teich(S)} := Span\{\frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_l}\},$$

$$\wedge^2 T_X^3 \text{ Teich(S)} := Span\{\frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}.$$

Hence,

$\wedge^2 T_X \text{ Teich(S)} = Span\{\wedge^2 T_X^1 \text{ Teich(S)}, \wedge^2 T_X^2 \text{ Teich(S)}, \wedge^2 T_X^3 \text{ Teich(S)}\}.$

**3.1. The curvature operator on $\wedge^2 T_X^1$ Teich(S).** Let $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be an element in $\wedge^2 T_X^1$ Teich(S), where $a_{ij}$ are real. Set

$$F(z, w) = \sum_{i,j=1}^{3g-3} a_{ij}\mu_i(w) \cdot \overline{\mu_j(z)}.$$

The following proposition is influenced by theorem 4.1 in [**15**].

**Proposition 3.1.** *Let $Q$ be the curvature operator and $D = -2(\Delta - 2)^{-1}$, where $\Delta$ is the Beltrami-Laplace operator on $X$. $G$ is the Green function of $D$, and $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ is an element in $\wedge^2 T_X^1 \operatorname{Teich}(S)$, where $a_{ij}$ are real. Then we have*

$$Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j})$$

$$= \int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)}) dA(z)$$

$$- 2 \cdot \int_{X \times X} G(z,w) |F(z,w))|^2 dA(w) dA(z)$$

$$+ 2 \cdot \Re\{ \int_{X \times X} G(z,w) F(z,w) F(w,z) dA(w) dA(z) \},$$

*where $F(z,w) = \sum_{i,j=1}^{3g-3} a_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}$.*

*Proof.* Since $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \bar{t}_i}$, from proposition 2.3,

$$Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j})$$

$$= \sum_{i,j,k,l} a_{ij} a_{kl} (R_{i\bar{j}k\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\bar{k}l})$$

$$= \sum_{i,j,k,l} a_{ij} a_{kl} (R_{i\bar{j}k\bar{l}} - R_{i\bar{j}l\bar{k}} - R_{j\bar{i}k\bar{l}} + R_{j\bar{i}l\bar{k}})$$

$$= \sum_{i,j,k,l} a_{ij} a_{kl} ((i\bar{j}, k\bar{l}) + (i\bar{l}, k\bar{j}) - (i\bar{j}, l\bar{k}) - (i\bar{k}, l\bar{j})$$

$$- (j\bar{i}, k\bar{l}) - (j\bar{l}, k\bar{i}) + (j\bar{i}, l\bar{k}) + (j\bar{k}, l\bar{i})) \quad \text{(by theorem 2.6)}$$

$$= \sum_{i,j,k,l} a_{ij} a_{kl} (i\bar{j} - j\bar{i}, k\bar{l} - l\bar{k})$$

$$+ \sum_{i,j,k,l} a_{ij} a_{kl} ((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k}))$$

$$- \sum_{i,j,k,l} a_{ij} a_{kl} ((i\bar{k}, l\bar{j}) + (j\bar{l}, k\bar{i})).$$

For the first term, from definition 2.5,

$$\sum_{i,j,k,l} a_{ij} a_{kl} (i\bar{j} - j\bar{i}, k\bar{l} - l\bar{k})$$

$$= \int_X D(\sum_{ij} a_{ij} \mu_i \overline{\mu_j} - \sum_{ij} a_{ij} \mu_j \overline{\mu_i})(\sum_{ij} a_{ij} \mu_i \overline{\mu_j} - \sum_{ij} a_{ij} \mu_j \overline{\mu_i}) dA(z)$$

$$= \int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)}) dA(z).$$

For the second term, after applying the Green function $G$ we have

$$\sum_{i,l} a_{ij} a_{kl}((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) = 2 \cdot \Re\{\sum_{i,l} a_{ij} a_{kl}((i\bar{l}, k\bar{j})\}$$

$$= 2 \cdot \Re\{\int_X D(\sum_i a_{ij}\mu_i\overline{\mu_l})(\sum_k a_{kl}\mu_k\overline{\mu_j})dA(z)\}$$

$$= 2 \cdot \Re\{\int_X \int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\mu_l(w)}(\sum_k a_{kl}\mu_k(z)\overline{\mu_j}(z))dA(z)dA(w)\}.$$

From the definition of $F(z, w)$,

$$\sum_{i,j,k,l} a_{ij} a_{kl}((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k}))$$

$$= 2 \cdot \Re\{\int_{X \times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\}.$$

For the last term, we use an argument similar to that for the second term.

$$\sum_{i,k} a_{ij} a_{kl}((i\bar{k}, l\bar{j}) + (k\bar{i}, j\bar{l})) = 2 \cdot \Re\{\sum_{i,k} a_{ij} a_{kl}((i\bar{k}, l\bar{j})\}$$

$$= 2 \cdot \Re\{\int_X D(\sum_i a_{ij}\mu_i\overline{\sum_k a_{kl}\mu_k})(\mu_l\overline{\mu_j})dA(z)\}$$

$$= 2 \cdot \Re\{\int_X \int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\sum_k a_{kl}\mu_k(w)}(\mu_l(z)\overline{\mu_j}(z))dA(z)dA(w)\}$$

From the definition of $F(z, w)$,

$$\sum_{i,j,k,l} a_{ij} a_{kl}((i\bar{k}, l\bar{j}) + (k\bar{i}, j\bar{l}))$$

$$= 2 \cdot \Re\{\int_{X \times X} G(z,w)F(z,w)\overline{F(z,w)}dA(w)dA(z)\}$$

$$= 2 \cdot \int_{X \times X} G(z,w)|F(z,w)|^2 dA(w)dA(z).$$

The conclusion follows from combining the three terms above.        q.e.d.

Using the Green function's positivity and symmetry,

**Theorem 3.2.** *Under the same conditions in proposition* 3.1, $Q$ *is negative definite on* $\wedge^2 T_X^1$ Teich(S).

*Proof.* By proposition 3.1 we have

$$Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j})$$

$$= \int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)})dA(z)$$

$$- 2 \cdot (\int_{X \times X} G(z,w)|F(z,w))|^2 dA(w)dA(z)$$

$$- \Re\{\int_{X \times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\}).$$

For the first term, since $F(z,z) - \overline{F(z,z)} = 2\mathbf{i}\Im\{F(z,z)\}$, by the positivity of the operator $D$,

$$\int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)})dA(z)$$

$$= -4 \cdot \int_X D(\Im\{F(z,z)\})(\Im\{F(z,z)\})dA(z) \le 0.$$

For the second term, using the Cauchy-Schwarz inequality,

$$|\int_{X \times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)|$$

$$\le \int_{X \times X} |G(z,w)F(z,w)F(w,z)|dA(w)dA(z)$$

$$\le \sqrt{\int_{X \times X} |G(z,w)||F(z,w)|^2 dA(w)dA(z)}$$

$$\times \sqrt{\int_{X \times X} |G(z,w)||F(w,z)|^2 dA(w)dA(z)}$$

$$= \int_{X \times X} G(z,w)|F(z,w)|^2 dA(w)dA(z),$$

since $G$ is positive and symmetric (see proposition 2.2).

Combining these three terms, we get that $Q$ is non-positive definite on $\wedge^2 T_X^1 \text{Teich}(S)$.

Furthermore, equality holds precisely when

$$Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}) = 0;$$

that is, there exists a constant complex number $k$ such that both of the following hold:

$$\begin{cases} F(z,z) = \overline{F(z,z)}, \\ F(z,w) = k \cdot \overline{F(w,z)}. \end{cases}$$

If we let $z = w$, we get $k = 1$. Hence, the last equation is equivalent to

$$\sum_{ij}(a_{ij} - a_{ji})\mu_i(w)\overline{\mu_j}(z) = 0.$$

Since $\{\mu_i\}_{i \geq 1}$ is a basis,

$$a_{ij} = a_{ji}.$$

This means $\sum_{ij} a_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = 0$. That is, $Q$ is negative definite on $\wedge^2 T_X^1$ Teich(S).                                              q.e.d.

**3.2. The curvature operator on $\wedge^2 T_X^2$ Teich(S).** Let $b_{ij}$ be real and $\sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} \in \wedge^2 T_X^2$ Teich(S). Set

$$H(z,w) = \sum_{i,j=1}^{3g-3} b_{ij}\mu_i(w) \cdot \overline{\mu_j(z)}.$$

Using a similar computation in proposition 3.1, the formula for the curvature operator on $\wedge^2 T_X^2$ Teich(S) is given as follows.

**Proposition 3.3.** *Let $Q$ be the curvature operator and $D$ be the same operator as shown in proposition 3.1. Let $\sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^2$ Teich(S), where $b_{ij}$ are real. Then we have*

$$Q(\sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j})$$
$$= -\int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)})dA(z)$$
$$- 2 \cdot \int_{X \times X} G(z,w)|H(z,w))|^2 dA(w)dA(z)$$
$$- 2 \cdot \Re\{\int_{X \times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z), \}$$

*where $H(z,w) = \sum_{i,j=1}^{3g-3} b_{ij}\mu_i(w) \cdot \overline{\mu_j(z)}$.*

*Proof.* Since $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}$ and $\frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \overline{t_i}})$, from proposition 2.3,

$$Q(\sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j})$$

$$= -\sum_{i,j,k,l} b_{ij}b_{kl}(R_{i\bar{j}k\bar{l}} - R_{i\overline{jk}l} - R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\overline{k}l})$$

$$= -\sum_{i,j,k,l} b_{ij}b_{kl}(R_{i\bar{j}k\bar{l}} + R_{i\bar{j}l\bar{k}} + R_{j\bar{i}k\bar{l}} + R_{j\bar{i}l\bar{k}}).$$

$$= -\sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{j},k\bar{l}) + (i\bar{l},k\bar{j}) + (i\bar{j},l\bar{k}) + (i\bar{k},l\bar{j}) \quad \text{(by theorem 2.6)}$$

$$+ \quad (j\bar{i},k\bar{l}) + (j\bar{l},k\bar{i}) + (j\bar{i},l\bar{k}) + (j\bar{k},l\bar{i}))$$

$$= -\sum_{i,j,k,l} b_{ij}b_{kl}(i\bar{j} + j\bar{i}, k\bar{l} + l\bar{k})$$

$$- \quad \sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{l},k\bar{j}) + (l\bar{i},j\bar{k}))$$

$$- \quad \sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{k},l\bar{j}) + (j\bar{l},k\bar{i})).$$

For the first term, from definition 2.5,

$$- \sum_{i,j,k,l} b_{ij}b_{kl}(i\bar{j} + j\bar{i}, k\bar{l} + l\bar{k})$$

$$= -\int_X D(\sum_{ij} b_{ij}\mu_i\overline{\mu_j} + \sum_{ij} b_{ij}\mu_j\overline{\mu_i})(\sum_{ij} b_{ij}\mu_i\overline{\mu_j} + \sum_{ij} b_{ij}\mu_j\overline{\mu_i})dA(z)$$

$$= -\int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)})dA(z).$$

For the second term and the third term, using the same argument in the proof of proposition 3.1, we have

$$\sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{l},k\bar{j}) + (l\bar{i},j\bar{k}))$$

$$= \quad 2 \cdot \Re\{\int_{X \times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z)\}$$

and

$$\sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{k},l\bar{j}) + (k\bar{i},j\bar{l}))$$

$$= \quad 2 \cdot \int_{X \times X} G(z,w)|H(z,w)|^2 dA(w)dA(z).$$

Combining these three terms we get the proposition. q.e.d.

Using the same method in theorem 3.2, one can prove the following non-positivity result.

**Theorem 3.4.** *Under the same conditions of proposition* 3.3, *then* $Q$ *is non-positive definite on* $\wedge^2 T_X^2$ Teich(S), *and the zero level subsets of* $Q(\cdot, \cdot)$ *are* $\{\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}; \ b_{ij} = -b_{ji}\}.$

*Proof.* Let $\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^2$ Teich(S). From proposition 3.3 we have

$$
Q(\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j})
$$

$$
= \ -\int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)}) dA(z)
$$

$$
- \ 2(\cdot \int_{X \times X} G(z,w)|H(z,w))|^2 dA(w) dA(z)
$$

$$
+ \ \cdot \Re\{\int_{X \times X} G(z,w)H(z,w)H(w,z) dA(w) dA(z)\}).
$$

For the first term, since $H(z,z) + \overline{H(z,z)} = 2 \cdot \Re\{H(z,z)\}$, by the positivity of the operator $D$,

$$
-\int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)}) dA(z)
$$

$$
= \ -4 \int_X D(\Re\{H(z,z)\})(\Re\{H(z,z)\}) dA(z) \le 0.
$$

For the second term, using the Cauchy-Schwarz inequality,

$$
|\int_{X \times X} G(z,w)H(z,w)H(w,z) dA(w) dA(z)|
$$

$$
\le \ \int_{X \times X} |G(z,w)H(z,w)H(w,z)| dA(w) dA(z)
$$

$$
\le \ \sqrt{\int_{X \times X} |G(z,w)||H(z,w)|^2 dA(w) dA(z)}
$$

$$
\times \ \sqrt{\int_{X \times X} |G(z,w)||H(w,z)|^2 dA(w) dA(z)}
$$

$$
= \ \int_{X \times X} G(z,w)|H(z,w)|^2 dA(w) dA(z),
$$

since $G$ is positive and symmetric.

Combining these two terms, we get $Q$ is non-positive on $\wedge^2 T_X^2$ Teich(S).

Using the same argument as in the proof of theorem 3.2,

$$
Q(\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}) = 0
$$

if and only if there exists a constant complex number $k$ such that both of the following hold:

$$\begin{cases} H(z,z) = -\overline{H(z,z)}, \\ H(z,w) = k \cdot \overline{H(w,z)}. \end{cases}$$

If we let $z = w$, we get $k = -1$. Hence, the last equation is equivalent to

$$\sum_{ij}(b_{ij} + b_{ji})\mu_i(w)\overline{\mu_j}(z) = 0.$$

Since $\{\mu_i\}_{i \geq 1}$ is a basis,

$$b_{ij} = -b_{ji}.$$

$$\text{q.e.d.}$$

**3.3.  The curvature operator on $\wedge^2 T_X^3$ Teich(S).** Let $\mathbf{J}$ be the almost complex structure on Teich(S). Since $\{t_i\}$ is a holomorphic coordinate, we have

$$\mathbf{J}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} \quad \mathbf{J}\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

Since the Weil-Petersson metric is a Kähler metric, $\mathbf{J}$ is an isometry on the tangent space. In particular we have

$$\begin{aligned} R(V_1, V_2, V_3, V_4) &= R(\mathbf{J}V_1, \mathbf{J}V_2, \mathbf{J}V_3, \mathbf{J}V_4) \\ &= R(\mathbf{J}V_1, \mathbf{J}V_2, V_3, V_4) = R(V_1, V_2, \mathbf{J}V_3, \mathbf{J}V_4), \end{aligned}$$

where $R$ is the curvature tensor and $V_i$ are real tangent vectors in $T_X$ Teich(S). Once can refer to [**12**] for more details.

Let $C = \sum_{ij} c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^3$ Teich(S), where $c_{ij}$ are real. Set

$$K(z,w) = \sum_{i,j=1}^{3g-3} c_{ij}\mu_i(w) \cdot \overline{\mu_j(z)}.$$

**Proposition 3.5.** *Let $Q$ be the curvature operator, and $\sum_{ij} c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^3$ Teich(S). Then we have*

$$\begin{aligned} &Q(\sum_{ij} c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}) \\ =\ &\int_X D(K(z,z) - \overline{K(z,z)})(K(z,z) - \overline{K(z,z)})dA(z) \\ &-\ 2 \cdot \int_{X \times X} G(z,w)|K(z,w))|^2 dA(w)dA(z) \\ &+\ 2 \cdot \Re\{\int_{X \times X} G(z,w)K(z,w)K(w,z)dA(w)dA(z)\}. \end{aligned}$$

*Proof.* Since $\frac{\partial}{\partial y_i} = \mathbf{J}\frac{\partial}{\partial x_i} = \mathbf{J}\frac{\partial}{\partial t_i} + \mathbf{J}\frac{\partial}{\partial \bar{t}_i}$ and $\mathbf{J}$ is an isometry, by proposition 2.3,

$$Q(\sum_{ij} c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j})$$

$$= \sum_{i,j,k,l} c_{ij}c_{kl}(R_{i\bar{j}k\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\bar{k}l})$$

$$= Q(\sum_{ij} c_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}).$$

By proposition 3.1,

$$Q(\sum_{ij} c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j})$$

$$= \int_X D(K(z,z) - \overline{K(z,z)})(K(z,z) - \overline{K(z,z)})dA(z)$$

$$- 2 \cdot \int_{X \times X} G(z,w)|K(z,w))|^2 dA(w)dA(z)$$

$$+ 2 \cdot \Re\{\int_{X \times X} G(z,w)K(z,w)K(w,z)dA(w)dA(z)\}.$$

q.e.d.

Using the same argument as in the proof of theorem 3.2 one can show that

**Theorem 3.6.** *Let $Q$ be the curvature operator as above, then $Q$ is a negative definite operator on $\wedge^2 T_X^3$ Teich(S).*

## 4. Curvature operator on $\wedge^2 T_X$ Teich(S)

Every element in $\wedge^2 T_X$ Teich(S) can be represented by $\sum_{ij}(a_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j})$.

**Proposition 4.1.** *Let $Q$ be the curvature operator. Then*

$$Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j},$$

$$\sum_{ij} a_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + c_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j})$$

$$= Q(\sum_{ij} d_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} d_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}),$$

*where $d_{ij} = a_{ij} + c_{ij}$.*

*Proof.* Since the almost complex structure $\mathbf{J}$ is an isometry and $\mathbf{J}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$, we have

$$
\begin{aligned}
Q(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}) &= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \mathbf{J}\frac{\partial}{\partial x_i}, \mathbf{J}\frac{\partial}{\partial x_j}) \\
&= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})
\end{aligned}
$$

and

$$
\begin{aligned}
Q(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}) &= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}, \mathbf{J}\frac{\partial}{\partial x_i}, \mathbf{J}\frac{\partial}{\partial x_j}) \\
&= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}).
\end{aligned}
$$

The conclusion follows by expanding $Q$ and applying the two equations above.                                        q.e.d.

If one wants to determine whether the curvature operator $Q$ is non-positive definite on $\wedge^2 T_X$ Teich(S), by proposition 4.1 it is sufficient to see if $Q$ is non-positive definite on $Span\{\wedge^2 T_X^1$ Teich(S), $\wedge^2 T_X^2$ Teich(S)$\}$.

**Proposition 4.2.** *Let $Q$ be the curvature operator, $\sum_{ij} a_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be an element in $\wedge^2 T_X^1$ Teich(S), and $\sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^2$ Teich(S). Then we have*

$$
\begin{aligned}
&Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}) \\
=\ & \boldsymbol{i} \cdot \int_X D(F(z,z) - \overline{F(z,z)}) \cdot (H(z,z) + \overline{H(z,z)})dA(z) \\
&- 2 \cdot \Im\{\int_{X \times X} G(z,w)F(z,w)\overline{H(z,w)})dA(w)dA(z)\} \\
&- 2 \cdot \Im\{\int_{X \times X} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\}.
\end{aligned}
$$

*Proof.* Since $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}$ and $\frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \overline{t_i}})$, by proposition 2.3,

$$Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j})$$

$$= (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-R_{i\bar{j}k\bar{l}}+R_{i\overline{jk}l}-R_{\bar{i}jk\bar{l}}+R_{\overline{i}j\overline{k}l})$$

$$= (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-R_{i\bar{j}k\bar{l}}-R_{i\bar{j}l\bar{k}}+R_{j\bar{i}k\bar{l}}+R_{j\bar{i}l\bar{k}})$$

$$= (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-(i\bar{j},k\bar{l})-(i\bar{l},k\bar{j})-(i\bar{j},l\bar{k})-(i\bar{k},l\bar{j})$$

$$+ (j\bar{i},k\bar{l})+(j\bar{l},k\bar{i})+(j\bar{i},l\bar{k})+(j\bar{k},l\bar{i}))\quad\text{(by theorem 2.6)}$$

$$= (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(j\bar{i}-i\bar{j},k\bar{l}+l\bar{k})$$

$$+ (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-(i\bar{l},k\bar{j})+(l\bar{i},j\bar{k}))$$

$$+ (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-(i\bar{k},l\bar{j})+(j\bar{l},k\bar{i})).$$

For the first term, by definition 2.5,

$$\sum_{i,j,k,l} a_{ij}b_{kl}(j\bar{i}-i\bar{j},k\bar{l}+l\bar{k})$$

$$= \int_X D(\sum_{ij} a_{ij}\overline{\mu_i}\mu_j-\sum_{ij} a_{ij}\mu_i\overline{\mu_j})(\sum_{kl} b_{kl}\mu_k\overline{\mu_l}+\sum_{kl} b_{kl}\mu_l\overline{\mu_k})dA(z)$$

$$= \int_X D(\overline{F(z,z)}-F(z,z))(H(z,z)+\overline{H(z,z)})dA(z).$$

For the second term, since $D$ is self adjoint, using the Green function $G$,

$$\sum_{i,l} a_{ij}b_{kl}(-(i\bar{l},k\bar{j})+(l\bar{i},j\bar{k}))$$

$$= 2\mathbf{i}\cdot\Im\{\sum_{i,l} a_{ij}b_{kl}(-(i\bar{l},k\bar{j})\}$$

$$= -2\mathbf{i}\cdot\Im\{\int_X D(\sum_i a_{ij}\mu_i\overline{\mu_l})(\sum_k b_{kl}\mu_k\overline{\mu_j})dA(z)\}$$

$$= -2\mathbf{i}\cdot\Im\{\int_X\int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\mu_l(w)}(\sum_k b_{kl}\mu_k(z)\overline{\mu_j}(z))dA(z)dA(w)\}$$

$$= -2\mathbf{i}\cdot\Im\{\int_{X\times X} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\}.$$

For the last term,

$$\sum_{i,k} a_{ij}b_{kl}(-(i\overline{k},l\overline{j})+(k\overline{i},j\overline{l}))=-2\mathbf{i}\cdot\Im\{\sum_{i,k}a_{ij}b_{kl}(i\overline{k},l\overline{j})\}$$

$$=\quad -2\mathbf{i}\cdot\Im\{\int_X D(\sum_i a_{ij}\mu_i\overline{\sum_k b_{kl}\mu_k})(\mu_l\overline{\mu_j})dA(z)\}$$

$$=\quad -2\mathbf{i}\cdot\Im\{\int_X\int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\sum_k b_{kl}\mu_k(w)}(\mu_l(z)\overline{\mu_j}(z))dA(z)dA(w)\}$$

$$=\quad -2\mathbf{i}\cdot\Im\{\int_{X\times X}G(z,w)F(z,w)\overline{H(z,w)}dA(w)dA(z)\}.$$

Combining these three terms above, we get the lemma.          q.e.d.

The following proposition will give the formula for curvature operator $Q$ on $Span\{\wedge^2 T_X^1\,\mathrm{Teich(S)},\wedge^2 T_X^2\,\mathrm{Teich(S)}\}$. Setting

$$A=\sum_{ij}a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},\qquad B=\sum_{ij}b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j},$$

on $Span\{\wedge^2 T_X^1\,\mathrm{Teich(S)},\wedge^2 T_X^2\,\mathrm{Teich(S)}\}$, we have

**Proposition 4.3.** *Let $Q$ be the curvature operator on* Teich(S). *Let* $A=\sum_{ij}a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ *and* $B=\sum_{ij}b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}$. *Then we have*

$$Q(A+B,A+B)=$$
$$-4\quad\int_X D(\Im\{F(z,z)+\mathbf{i}H(z,z)\})\cdot(\Im\{F(z,z)+\mathbf{i}H(z,z)\})dA(z)$$
$$-2\quad\int_{X\times X}G(z,w)|F(z,w)+\mathbf{i}H(z,w))|^2 dA(w)dA(z)$$
$$+2\quad\Re\{\int_{X\times X}G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)\},$$

*where* $F(z,w)=\sum_{i,j=1}^{3g-3}a_{ij}\mu_i(w)\cdot\overline{\mu_j(z)}$ *and* $H(z,w)=\sum_{i,j=1}^{3g-3}b_{ij}\mu_i(w)\cdot\overline{\mu_j(z)}$.

*Proof.* Since $Q(A,B)=Q(B,A)$,

$$Q(A+B,A+B)=Q(A,A)+2Q(A,B)+Q(B,B).$$

By proposition 3.1, proposition 3.3, and proposition 4.2 we have

$$Q(A+B, A+B)$$

$$= \ (\int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)})dA(z)$$

$$- \ \int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)})dA(z)$$

$$+ \ 2\mathbf{i} \cdot \int_X D(F(z,z) - \overline{F(z,z)})(H(z,z) + \overline{H(z,z)})dA(z))$$

$$( \ - \ 2 \cdot \int_{X \times X} G(z,w)|F(z,w))|^2 dA(w)dA(z)$$

$$- \ 2 \cdot \int_{X \times X} G(z,w)|H(z,w))|^2 dA(w)dA(z)$$

$$- \ 4 \cdot \Im\{\int_{X \times X} G(z,w)F(z,w)\overline{H(z,w)})dA(w)dA(z)\})$$

$$( \ + \ 2 \cdot \Re\{\int_{X \times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\}$$

$$- \ 2 \cdot \Re\{\int_{X \times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z)\}$$

$$- \ 4 \cdot \Im\{\int_{X \times X} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\}).$$

The sum of the first three terms is exactly

$$-4\int_X D(\Im\{F(z,z) + \mathbf{i}H(z,z)\}) \cdot (\Im\{F(z,z) + \mathbf{i}H(z,z)\})dA(z).$$

Just as $|a + \mathbf{i}b|^2 = |a|^2 + |b|^2 + 2 \cdot \Im(a \cdot \overline{b})$, where $a$ and $b$ are two complex numbers, the sum of the second three terms is exactly

$$-2 \cdot \int_{X \times X} G(z,w)|F(z,w) + \mathbf{i}H(z,w))|^2 dA(w)dA(z).$$

For the last three terms, since

$$\Im(F(z,w) \cdot H(w,z)) = -\Re(F(z,w) \cdot (\mathbf{i}H(w,z))),$$

the sum is exactly

$$2 \cdot \Re\{\int_{X \times X} G(z,w)(F(z,w) + \mathbf{i}H(z,w))(F(w,z) + \mathbf{i}H(w,z))dA(w)dA(z)\}.$$

q.e.d.

Furthermore, we have

**Theorem 4.4.** *Under the same conditions as in proposition 4.3, Q is non-positive definite on* $Span\{\wedge^2 T_X^1 \ Teich(S), \wedge^2 T_X^2 \ Teich(S)\}$, *and*

*the zero level subsets of* $Q(\cdot, \cdot)$ *are* $\{\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}; \ b_{ij} = -b_{ji}\}$ *in* $Span\{\wedge^2 T_X^1 \text{ Teich}(S), \wedge^2 T_X^2 \text{ Teich}(S)\}$.

*Proof.* Let us estimate the terms in proposition 4.3 separately. For the first term, since $D$ is a positive operator,

$$-\int_X D(\Im\{F(z,z) + \mathbf{i}H(z,z)\}) \cdot (\Im\{F(z,z) + \mathbf{i}H(z,z)\}) dA(z) \leq 0.$$

For the third term, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\left| \int_{X \times X} G(z,w)(F(z,w) + \mathbf{i}H(z,w))(F(w,z) + \mathbf{i}H(w,z)) dA(w) dA(z) \right| \\
\leq \ & \int_{X \times X} G(z,w)|(F(z,w) + \mathbf{i}H(z,w))(F(w,z) + \mathbf{i}H(w,z))| dA(w) dA(z) \\
\leq \ & \sqrt{\int_{X \times X} G(z,w)|(F(z,w) + \mathbf{i}H(z,w))|^2 dA(w) dA(z)} \\
\times \ & \sqrt{\int_{X \times X} G(z,w)|(F(w,z) + \mathbf{i}H(w,z))|^2 dA(w) dA(z)} \\
= \ & \int_{X \times X} G(z,w)|(F(z,w) + \mathbf{i}H(z,w))|^2 dA(w) dA(z).
\end{aligned}
$$

The last equality follows from $G(z,w) = G(w,z)$.

Combining the two inequalities above and the second term in proposition 4.3, we see that on $Span\{\wedge^2 T_X^1 \text{ Teich}(S), \wedge^2 T_X^2 \text{ Teich}(S)\}$ $Q$ is non-positive definite. Furthermore, $Q(A+B, A+B) = 0$ if and only if there exists a constant $k$ such that both of the following hold:

$$\begin{cases} Im\{F(z,z) + \mathbf{i}H(z,z)\} = 0, \\ F(z,w) + \mathbf{i}H(z,w) = k \cdot \overline{(F(w,z) + \mathbf{i}H(w,z))}. \end{cases}$$

If we let $z = w$, we get $k = 1$. Hence, the second equation is equivalent to

$$\sum_{ij} (a_{ij} - a_{ji} + \mathbf{i}(b_{ij} + b_{ji})) \mu_i(w) \overline{\mu_j}(z) = 0.$$

Since $\{\mu_i\}_{i \geq 1}$ is a basis,

$$a_{ij} = a_{ji}, b_{ij} = -b_{ji}.$$

That is, $A = 0$ and $B = \sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$, where $b_{ij} = -b_{ji}$.

Conversely, if $A = 0$ and $B \in \{\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}; \ b_{ij} = -b_{ji}\}$, it is not hard to apply proposition 4.3 to show that $Q(A+B, A+B) = 0$. q.e.d.

Before we prove the main theorem, let us define a natural action of $\mathbf{J}$ on $\wedge^2 T_X$ Teich(S) by

$$
\begin{cases}
\mathbf{J} \circ \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} := \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \\
\mathbf{J} \circ \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} := -\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_i}, \\
\mathbf{J} \circ \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} := \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},
\end{cases}
$$

and extend it linearly. It is easy to see that $\mathbf{J} \circ \mathbf{J} = id$.

Now we are ready to prove theorem 1.1.

*Proof of Theorem* 1.1. It follows from proposition 4.1 and theorem 4.4 that $Q$ is non-positive definite.

If $A = C - \mathbf{J} \circ C$ for some a $C \in \wedge^2 T_X$ Teich(S), then it is easy to see that $Q(A, A) = 0$, since $\mathbf{J}$ is an isometry.

Assume that $A \in \wedge^2 T_X$ Teich(S) such that $Q(A, A) = 0$. Since $\wedge^2 T$ Teich(S) $= Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}$, there exists $a_{ij}$, $b_{ij}$, and $c_{ij}$ such that

$$
A = \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.
$$

Since $Q(A, A) = 0$, by proposition 4.1 and theorem 4.4 we must have

$$
a_{ij} + c_{ij} = a_{ji} + c_{ji}, b_{ij} = -b_{ji}.
$$

That is,

$$
a_{ij} - a_{ji} = -(c_{ij} - c_{ji}), b_{ij} = -b_{ji}.
$$

Set

$$
C = \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}.
$$

**Claim.** $A = C - \mathbf{J} \circ C$.

Since $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \sum_{i<j} (a_{ij} - a_{ji}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we have

$$
\mathbf{J} \circ \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \sum_{i<j} (a_{ij} - a_{ji}) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}
$$

$$
= -\sum_{i<j} (c_{ij} - c_{ji}) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} = -\sum c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.
$$

Similarly,

$$
\mathbf{J} \circ \sum \left( \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} \right) = -\sum \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}.
$$

The claim follows from the two equations above.                          q.e.d.

## 5. Harmonic maps into Teich(S)

In this section we study the twist-harmonic maps from some domains into the Teichmüller space. Before we go to the rank-one hyperbolic space case, let us state the following lemma, which is influenced by lemma 5 in [**32**].

**Lemma 5.1.** *The rank-one Hyperbolic spaces* $H_{Q,m} = Sp(m,1)/Sp(m) \cdot Sp(1)$ *and* $H_{O,2} = F_4^{-20}/SO(9)$ *cannot be totally geodesically immersed into* Teich(S).

*Proof.* On quaternionic hyperbolic manifolds $H_{Q,m} = Sp(m,1)/Sp(m)$, assume that there is a totally geodesic immersion of $H_{Q,m}$ into Teich(S). We may select $p \in H_{Q,m}$. Choose a quaternionic line $l_Q$ on $T_p H_{Q,m}$, and we may assume that $l_Q$ is spanned over $R$ by $v$, $Iv$, $Jv$, and $Kv$. Without loss of generality, we may assume that $J$ on $l_Q \subset T_p H_{Q,m}$ is the same as the complex structure on Teich(S). Choose an element

$$v \wedge Jv + Kv \wedge Iv \in \wedge^2 T_p H_{Q,m}.$$

Let $Q^{H_{Q,m}}$ be the curvature operator on $H_{Q,m}$.

$$Q^{H_{Q,m}}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) =$$
$$R^{H_{Q,m}}(v, Jv, v, Jv) + R^{H_{Q,m}}(Kv, Iv, Kv, Iv) + 2 \cdot R^{H_{Q,m}}(v, Jv, Kv, Iv).$$

Since $I$ is an isometry, we have

$$\begin{aligned} R^{H_{Q,m}}(Kv, Iv, Kv, Iv) &= R^{H_{Q,m}}(IKv, IIv, IKv, IIv) \\ &= R^{H_{Q,m}}(-Jv, -v, -Jv, -v) \\ &= R^{H_{Q,m}}(v, Jv, v, Jv). \end{aligned}$$

Similarly,

$$\begin{aligned} R^{H_{Q,m}}(v, Jv, Kv, Iv) &= R^{H_{Q,m}}(v, Jv, IKv, IIv) \\ &= R^{H_{Q,m}}(v, Jv, -Jv, -v) \\ &= -R^{H_{Q,m}}(v, Jv, v, Jv). \end{aligned}$$

Combining the terms above, we have

$$Q^{H_{Q,m}}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) = 0.$$

Since $f$ is a geodesical immersion,

$$Q^{\text{Teich(S)}}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) = 0.$$

On the other hand, by theorem 1.1, there exists $C$ such that

$$v \wedge Jv + Kv \wedge Iv = C - \mathbf{J} \circ C.$$

Hence,

$$
\begin{aligned}
(2) \qquad & \mathbf{J} \circ (v \wedge Jv + Kv \wedge Iv) \\
= \;\; & \mathbf{J} \circ (C - \mathbf{J} \circ C) = \mathbf{J} \circ C - \mathbf{J} \circ \mathbf{J} \circ C = \mathbf{J} \circ C - C \\
= \;\; & -(v \wedge Jv + Kv \wedge Iv).
\end{aligned}
$$

On the other hand, since $\mathbf{J}$ is the same as $J$ in $H_{Q,m}$, we also have

$$
\begin{aligned}
(3) \qquad & \mathbf{J} \circ (v \wedge Jv + Kv \wedge Iv) = (Jv \wedge JJv + JKv \wedge JIv) \\
= \;\; & Jv \wedge (-v) + Iv \wedge (-Kv) = v \wedge Jv + Kv \wedge Iv.
\end{aligned}
$$

From equations (2) and (3) we get

$$
v \wedge Jv + Kv \wedge Iv = 0,
$$

which is a contradiction since $l_Q$ is spanned over $R$ by $v$, $Iv$, $Jv$, and $Kv$.

In the case of the Cayley hyperbolic plane $H_{O,2} = F_4^{20}/SO(9)$, the argument is similar by replacing a quaternionic line by a Cayley line ([**4**]).                                        q.e.d.

Now we are ready to prove theorem 1.2.

*Proof of theorem* 1.2. Since the sectional curvature operator on Teich(S) is non-positive definite, Teich(S) also has non-positive Riemannian sectional curvature in the complexified sense as stated in [**19**]. Suppose that $f$ is not constant. From theorem 2 in [**19**] (also see [**6**]), we know that $f$ should be a totally geodesic immersion, which contradicts lemma 5.1. Hence, $f$ must be a constant.                      q.e.d.

REMARK 5.1. In [**32**] it is shown that the image of any homomorphism $\rho$ from $\Gamma$ to Mod($S$) is finite. Hence, $\rho(\Gamma)$ must have a fixed point in Teich(S) from the Nielsen realization theorem (one can see [**14**, **30**]). If we assume that there exists a twist harmonic map $f$ with respect to this homomorphism, then by theorem 1.2 we know $\rho(\Gamma) \subset \text{Mod}(S)$ will fix the point $f(G/\Gamma) \in \text{Teich(S)}$.

REMARK 5.2. Conversely, if one can prove that for any homomorphism $\rho$ from $\Gamma$ to $\overline{\text{Mod}(S)}$ there exists a twist harmonic map $f$ from $G$ into the completion $\overline{\text{Teich(S)}}$ of Teich(S) such that the image $f(G) \subset \text{Teich(S)}$, theorem 1.2 tells us that the image $\rho(\Gamma)$ fixes a point in Teich(S); hence, the image $\rho(\Gamma)$ is finite because Mod($S$) acts properly on Teich(S).

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