THE WEIL-PETERSSON CURVATURE OPERATOR ON THE UNIVERSAL TEICHMÜLLER SPACE

ZHENG HUANG AND YUNHUI WU

ABSTRACT. The universal Teichmüller space is an infinitely dimensional generalization of the classical Teichmüller space of Riemann surfaces. It carries a natural Hilbert structure, on which one can define a natural Riemannian metric, the Weil-Petersson metric. In this paper we investigate the Weil-Petersson Riemannian curvature operator $\tilde{Q}$ of the universal Teichmüller space with the Hilbert structure, and prove the following:

(i) $\tilde{Q}$ is non-positive definite.
(ii) $\tilde{Q}$ is a bounded operator.
(iii) $\tilde{Q}$ is not compact; the set of the spectra of $\tilde{Q}$ is not discrete.

As an application, we show that neither the Quaternionic hyperbolic space nor the Cayley plane can be totally geodesically immersed in the universal Teichmüller space endowed with the Weil-Petersson metric.

1. Introduction

1.1. The Weil-Petersson geometry on classical Teichmüller space.

Moduli theory of Riemann surfaces and their generalizations continue to be inspiration for ideas and questions for many different mathematical fields since the times of Gauss and Riemann. In this paper, we study the Weil-Petersson geometry of the universal Teichmüller space.

Let S_g be a closed oriented surface of genus g where $g \geq 2$, and $\mathcal{T}_g(S)$ be the Teichmüller space of S_g (space of hyperbolic metrics on S_g modulo orientation preserving diffeomorphisms isotopic to the identity). The Teichmüller space $\mathcal{T}_g(S)$ is a manifold of complex dimension $3g-3$, with its cotangent space at $(S, \sigma(z)|dz|^2) \in \mathcal{T}_g(S)$ identified as the space of holomorphic quadratic differentials $\phi(z)dz^2$ on the conformal structure of the hyperbolic metric $\sigma(z)|dz|^2$. The Weil-Petersson metric on Teichmüller space is obtained by duality from the natural L^2 pairing of holomorphic quadratic differentials. The Weil-Petersson geometry of Teichmüller space has been extensively studied: it is a Kählerian metric [Ahl61], incomplete [Chu76, Wol75] yet geodesically convex [Wol87]. Many features of the

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curvature property were also studied in detail by many authors (see a comprehensive survey [**Wol11**] and the book [**Wol10**]). Since intuitively we consider the universal Teichmüller space contains Teichmüller spaces of all genera, among those Weil-Petersson curvature features; it is known that the Weil-Petersson metric has negative sectional curvature, with an explicit formula for the Riemannian curvature tensor due to Tromba-Wolpert [**Tro86, Wol86**], strongly negative curvature in the sense of Siu [**Sch86**], dual Nakano negative curvature [**LSY08**], various curvature bounds in terms of the genus [**Hua07b, Teo09, Wu17**], good behavior of the Riemannian curvature operator on Teichmüller space [**Wu14, WW15**]. One can also refer to [**BF06, Hua05, Hua07a, LSY04, LSYY13, Wol11, Wol10, Wol12b**] for other aspects of the curvatures of the Weil-Petersson metric.

1.2. **Main results.** There are several well-known models of universal Teichmüller spaces. We will adapt the approach in [**TT06**] and use the disk model to define the universal Teichmüller space $T(1)$ as a quotient of the space of bounded Beltrami differentials on the unit disk $\mathbb{D}$. Unlike the case in the classical Teichmüller space, the Petersson pairing for the bounded Beltrami differentials on $\mathbb{D}$ is not well-defined on the whole tangent space of the universal Teichmüller space $T(1)$. To ramify this, Takhtajan-Teo [**TT06**] defined a Hilbert structure on $T(1)$ such that the Petersson pairing is now meaningful on the tangent space at any point in this Hilbert structure. We denote the universal Teichmüller space with this Hilbert structure by $T_H(1)$. The resulting metric is the Weil-Petersson metric on $T_H(1)$. All terms will be defined rigorously in §2.

The Riemannian geometry of this infinitely dimensional deformation space $T_H(1)$ is very intriguing. Takhtajan-Teo showed the Weil-Petersson metric on $T_H(1)$ has negative sectional curvature, and constant Ricci curvature [**TT06**], and Teo [**Teo09**] proved the holomorphic sectional curvature has no negative upper bound.

We are interested in the Weil-Petersson curvature operator on $T_H(1)$. In general there are some fundamental questions regarding linear operators on manifolds: whether the operator is signed, whether it is bounded, and the behavior of its eigenvalues. In this paper, we investigate the Weil-Petersson curvature operator along these question lines. In particular, we prove:

Theorem 1.1. *Let $\tilde{Q}$ be the Weil-Petersson Riemannian curvature operator on the universal Teichmüller space $T_H(1)$, then*

(i) *$\tilde{Q}$ is non-positive definite on $\wedge^2 TT_H(1)$.*
(ii) *For $C \in \wedge^2 TT_H(1)$, $\tilde{Q}(C, C) = 0$ if and only if there is an element $E \in \wedge^2 TT_H(1)$ such that $C = E - \mathbf{J} \circ E$, where $\mathbf{J}\circ$ is defined above.*

As a direct corollary, we have:

Corollary 1.2. [**TT06**] *The sectional curvature of the Weil-Petersson metric on $T_H(1)$ is negative.*

Our second result is:

Theorem 1.3. *The curvature operator $\tilde{Q}$ is bounded. More precisely, for any $V \in \wedge^2 TT_H(1)$ with $\|V\|_{eu} = 1$, we have $|\tilde{Q}(V,V)| \leq 16\sqrt{\frac{3}{\pi}}$, where $\|\cdot\|_{eu}$ is the Euclidean norm for the wedge product defined in (4.1).*

A direct consequence of Theorem 1.3 is:

Corollary 1.4. [**GBR15**] *The Riemannian Weil-Petersson curvature tensor (defined in (2.18)) is bounded.*

Being bounded and non-positively definite are properties for the Weil-Petersson curvature operator on certain part of the classical Teichmüller space as well [**Wu14, WW15**], but noncompactness of $\tilde{Q}$ is a more distinctive feature for $T_H(1)$. Our next result is:

Theorem 1.5. *The curvature operator $\tilde{Q}$ is not a compact operator, more specifically, the set of spectra of $\tilde{Q}$ is not discrete on the interval $[-16\sqrt{\frac{3}{\pi}}, 0)$.*

As an important application, in the last part of this paper we will address some rigidity questions on harmonic maps from certain symmetric spaces into $T_H(1)$. For harmonic map from a domain, which is either the Quaternionic hyperbolic space or the Cayley plane, into a non-positive curved target space, many beautiful rigidity results were established in [**DM15, GS92, JY97, MSY93**] and others. We prove the following:

Theorem 1.6. *Let Γ be a lattice in a semisimple Lie group G which is either $Sp(m,1)$ or F_4^{-20}, and let $\mathsf{Isom}(T_H(1))$ be the isometry group of $T_H(1)$ with respect to the Weil-Petersson metric. Then, any twisted harmonic map f from G/Γ into $T_H(1)$ must be a constant, with respect to each homomorphism $\rho : \Gamma \to \mathsf{Isom}(T_H(1))$. Here the twisted map f with respect to ρ means that $f(\gamma \circ Y) = \rho(\gamma) \circ f(Y)$, for all $\gamma \in \Gamma$.*

1.3. **Methods in the proofs.** An immediate difficulty we have to cope with is that $T_H(1)$ is an infinite dimensional manifold. There is however a basis for tangent vectors for the Hilbert structure that we can work with. With this basis, the Weil-Petersson Riemannian curvature tensor takes an explicit form. To prove the first two results, we need to generalize techniques developed in [**Wu14, WW15**] carefully and rigorously to the case of infinite dimensional Hilbert spaces.

Proof of the Theorem 1.5 is different. We prove a key estimate for the operator on an n-dimensional subspace (Proposition 5.4), then bound the spectra of the curvature operator by the corresponding spectra of its projection onto this subspace to derive a contradiction.

1.4. **Plan of the paper.** The organization of the paper is as follows: in §2, we set up notations and preliminaries, in particular, we restrict ourselves in the classical setting to define Teichmüller space of closed surfaces and the Weil-Petersson metric in §2.1, its curvature operator on Teichmüller space is set up in §2.2, then we define the universal Teichmüller space and its Hilbert structure in §2.3, and introduce the basis for tangent vectors for the $T_H(1)$, and describe the Weil-Petersson Riemannian curvature operator on the universal Teichmüller space in §2.4. Main theorems are proved in sections §3, §4 and §5. And in the last section §6 we prove Theorem 1.6.

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2. Preliminaries

2.1. **Teichmüller space and its Weil-Petersson metric.** Let $\mathbb{D}$ be the unit disk with the Poincaré metric, and S be a closed oriented surface of genus $g > 1$. Then by the uniformization theorem we have a hyperbolic structure $X = \mathbb{D}\backslash\Gamma$ on S, where $\Gamma \subset PSL(2,\mathbb{R})$ is a Fuchsian group, and $PSL(2,\mathbb{R})$ is the group of orientation preserving isometries of $\mathbb{D}$. Writing $\{z\}$ as the complex coordinate on $\mathbb{D}$, the Poincaré metric is explicitly given as

$$\rho(z) = \frac{4}{(1-|z|^2)^2}dzd\bar{z}.$$

It descends to a hyperbolic metric on the Riemann surface $X = \mathbb{D}\backslash\Gamma$, which we denote by $\sigma(z)|dz|^2$. Spaces of Beltrami differentials and holomorphic quadratic differentials on Riemann surfaces play a fundamental role in Teichmüller theory, and let us describe these spaces.

(i) $\mathcal{A}^{-1,1}(X)$: the space of bounded Beltrami differentials on $X = \mathbb{D}\backslash\Gamma$. A Beltrami differential on a Riemann surface is a $(-1,1)$ form in the form of $\mu(z)\frac{d\bar{z}}{dz}$, where $\mu(z)$ is a function on $\mathbb{D}$ satisfying:

$$\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z), \quad \forall\gamma \in \Gamma.$$

(ii) $\mathcal{B}^{-1,1}(X)$: the unit ball of $\mathcal{A}^{-1,1}(X)$, namely,

$$\mathcal{B}^{-1,1}(X) = \{\mu(z)\frac{d\bar{z}}{dz} \in \mathcal{A}^{-1,1}(X) : \|\mu\|_\infty = sup_{z\in\mathbb{D}}|\mu(z)| < 1\}.$$

(iii) $Q(X)$: the space of holomorphic quadratic differentials on X. A holomorphic quadratic differential is a $(2,0)$ form taking the form $q(z)dz^2$, where $q(z)$ is a holomorphic function on $\mathbb{D}$ satisfying:

$$q(\gamma(z))[\gamma'(z)]^2 = q(z), \quad \forall\gamma \in \Gamma.$$

It is a basic fact in Riemann surface theory that $Q(X)$ is a Banach space of real dimension $6g-6$.

(iv) $\Omega^{-1,1}(X)$: the space of *harmonic Beltrami differentials* on X. A Beltrami differential $\nu(z)\frac{d\bar{z}}{dz} \in \mathcal{A}^{-1,1}(X)$ is harmonic if there is a holomorphic quadratic differential $q(z)dz^2 \in Q(X)$ such that

$$\nu(z)\frac{d\bar{z}}{dz} = \frac{\overline{q(z)dz^2}}{\sigma(z)|dz|^2}, \tag{2.1}$$

where $\sigma(z)|dz|^2$ is the hyperbolic metric on X. Seeing from $\mathbb{D}$, the space $\Omega^{-1,1}(X)$ consists of functions

$$\nu(z) = \frac{(1-|z|^2)^2}{4}\overline{q(z)}. \tag{2.2}$$

The Teichmüller space $\mathcal{T}_g(S)$ is the space of hyperbolic metrics on the surface S, modulo orientation preserving biholomorphisms. Real analytically $\mathcal{T}_g(S)$ is isomorphic to $\mathcal{B}^{-1,1}(X)\backslash \sim$, where two Beltrami differentials are equivalent if the unique quasiconformal maps between the extended complex plane coincide on the unit circle. At each point $X \in \mathcal{T}_g(S)$, its tangent space is identified as the space $\Omega^{-1,1}(X)$, while the cotangent space at X is identified as the space $Q(X)$.

Given two tangent vectors $\mu(z)\frac{d\bar{z}}{dz}$ and $\nu(z)\frac{d\bar{z}}{dz}$ in $\Omega^{-1,1}(X)$, the Weil-Petersson metric is defined as the following (Petersson) pairing:

$$\langle\mu,\nu\rangle_{WP} = \int_{X=\mathbb{D}\backslash\Gamma} \mu\bar{\nu}dA, \tag{2.3}$$

where $dA = \sigma|dz|^2$ is the hyperbolic area element on X. Writing as a metric tensor, we have

$$g_{i\bar{j}} = \int_X \mu_i\bar{\nu}_j dA,$$

This is a Riemannian metric with many nice properties. There is an explicit formula for its curvature tensor due to Tromba-Wolpert ([**Tro86**, **Wol86**]):

$$R_{i\bar{j}k\bar{\ell}} = \int_X D(\mu_i\bar{\mu}_j)(\mu_k\bar{\mu}_\ell)dA + \int_X D(\mu_i\bar{\mu}_\ell)(\mu_k\bar{\mu}_j)dA. \tag{2.4}$$

Here the operator D is defined as

$$D = -2(\Delta - 2)^{-1}, \tag{2.5}$$

where $\Delta = \frac{-4}{\sigma(z)}\partial_z\partial_{\bar{z}}$ is the Laplace operator on X with respect to the hyperbolic metric $\sigma(z)dA$. This operator D is fundamental in Teichmüller theory, and the following is well-known (see for instance [**Wol86**]):

Proposition 2.1. *The operator $D = -2(\Delta-2)^{-1}$ is a positive, self-adjoint operator on $C^\infty(X)$. Furthermore, let $G(w,z)$ be a Green's function for D, then $G(w,z)$ is positive, and $G(w,z) = G(z,w)$: $\forall f \in C^\infty(X)$, we have*

$$D(f)(z) = \int_{w\in X} G(z,w)f(w)dA(w). \tag{2.6}$$

To simplify our calculations, we introduce the following notation:

Definition 2.2. *For any element μ's in the tangent space $\Omega^{-1,1}(X)$, we set:*

$$(i\bar{j}, k\bar{\ell}) = \int_X D(\mu_i\bar{\mu}_j)(\mu_k\bar{\mu}_\ell)dA. \tag{2.7}$$

Using this notation, the Weil-Petersson curvature tensor formula on Teichmüller space becomes

$$R_{i\bar{j}k\bar{\ell}} = (i\bar{j}, k\bar{\ell}) + (i\bar{\ell}, k\bar{j}). \tag{2.8}$$

2.2. **The Weil-Petersson curvature operator on Teichmüller space.** We now introduce the Riemannian curvature operator for the Weil-Petersson metric on Teichmüller space $\mathcal{T}_g(S)$. Note that this is a matrix of the real order $(6g-6)^2 \times (6g-6)^2$, whose diagonal entries are the sectional curvatures.

Let U be a neighborhood of X in Teichmüller space $\mathcal{T}_g(S)$, and we have $\{t_1, t_2, ..., t_{3g-3}\}$ as a local holomorphic coordinate on U, where $t_i = x_i + \mathbf{i}y_i (1 \leq i \leq 3g-3)$. Then $\{x_1, x_2, ..., x_{3g-3}, y_1, y_2, ..., y_{3g-3}\}$ forms a real smooth coordinate in U, and

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \bar{t}_i}, \quad \frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \bar{t}_i}).$$

Let $T\mathcal{T}_g(S)$ be the real tangent bundle of $\mathcal{T}_g(S)$ and $\wedge^2 T\mathcal{T}_g(S)$ be the exterior wedge product of $T\mathcal{T}_g(S)$ and itself. For any $X \in U$, we have

$$T_X\mathcal{T}_g(S) = Span\{\frac{\partial}{\partial x_i}(X), \frac{\partial}{\partial y_i}(X)\}_{1\leq i,j\leq 3g-3},$$

and

$$\wedge^2 T_X\mathcal{T}_g(S) = Span\{\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial y_\ell}, \frac{\partial}{\partial y_m}\wedge\frac{\partial}{\partial y_n}\}. \tag{2.9}$$

Definition 2.3. *The Weil-Petersson curvature operator $\tilde{Q}$ on Teichmüller space is defined on $\wedge^2 T\mathcal{T}_g(S)$ by*

$$\tilde{Q}(V_1\wedge V_2, V_3\wedge V_4) = R(V_1,V_2,V_3,V_4),$$

where V's are tangent vectors at X, and R is the curvature tensor.

If we take a real orthonormal basis $\{e_i\}_{i=1,2,\cdots,6g-6}$ for $T_X\mathcal{T}_g(S)$, and set $R_{ijk\ell} = \langle R(e_i,e_j)e_k, e_\ell\rangle$, then

$$\wedge^2 T_X\mathcal{T}_g(S) = Span\{e_i\wedge e_j\}_{1\le i<j\le(6g-6)},$$

and the curvature operator $\tilde{Q}: \wedge^2 T_X\mathcal{T}_g(S)\to\wedge^2 T_X\mathcal{T}_g(S)$, for real coefficients a_{ij}, can be expressed as follows:

$$\tilde{Q}(\sum_{1\le i<j\le(6g-6)} a_{ij}e_i\wedge e_j) = \sum_{1\le i<j\le(6g-6)}\sum_{1\le k<\ell\le(6g-6)} a_{ij}R_{ijk\ell}e_k\wedge e_\ell. \tag{2.10}$$

In [**Wu14**] the second named author proved the curvature operator $\tilde{Q}$ is non-positively definite on Teichmüller space. Further analysis on $\tilde{Q}$ was studied in [**WW15**]. We will generalize this fundamental operator to the setting of the universal Teichmüller space and reveal some geometric features for the Weil-Petersson metric on the universal Teichmüller space.

2.3. **The universal Teichmüller space and its Hilbert structure.** Introduced by Bers ([**Ber65**]), the universal Teichmüller space $T(1)$ is a central subject for the theory of univalent functions. It contains all Teichmüller spaces $\mathcal{T}_g(S)$ of closed surfaces which are complex submanifolds.

Recall that every Riemann surface (or hyperbolic structure) X on a closed surface S is quotient of the Poincaré disk with a Fuchsian group Γ: $X=\mathbb{D}\backslash\Gamma$. Previously in §2.1, we have Teichmüller space $\mathcal{T}_g(S)$ isomorphic to a quotient space $\mathcal{B}^{-1,1}(X)\backslash\sim$, where $\mathcal{B}^{-1,1}(X)$ is the space of bounded Beltrami differentials on X with super-norm less than one, and two such Beltrami differentials are equivalent if the unique quasiconformal maps induced by them between the extended complex plane coincide on the unit circle.

Let us set up some notations before we proceed. Letting Γ be the identity group, we work in the Poincaré disk $\mathbb{D}$, we have similarly with §2.1:

(i) $\mathcal{A}^{-1,1}(\mathbb{D})$: the space of bounded functions on $\mathbb{D}$.
(ii) $\mathcal{B}^{-1,1}(\mathbb{D})$: the unit ball of $\mathcal{A}^{-1,1}(\mathbb{D})$, namely,

$$\mathcal{B}^{-1,1}(\mathbb{D}) = \{\mu(z)\in\mathcal{A}^{-1,1}(D): \|\mu\|_\infty = sup_{z\in\mathbb{D}}|\mu(z)|<1\}.$$

(iii) We will need two spaces of holomorphic functions on $\mathbb{D}$, both are analog to the space $Q(X)$, the space of holomorphic quadratic differentials on X. Let us define

$$A_\infty(\mathbb{D}) = \{q(z) : \bar{\partial} q = 0, \|q\|_\infty = sup_{z\in\mathbb{D}} \frac{|q(z)|}{\rho(z)} < \infty\}, \tag{2.11}$$

where $\rho(z) = \frac{4}{(1-|z|^2)^2} dz d\bar{z}$ is the hyperbolic metric on $\mathbb{D}$. This is the space of holomorphic functions on $\mathbb{D}$ with finite super-norm defined within (2.11).

We also define

$$A_2(\mathbb{D}) = \{q(z) : \bar{\partial} q = 0, \|q\|_2^2 = \int_{\mathbb{D}} \frac{|q(z)|^2}{\rho(z)} |dz|^2 < \infty\}. \tag{2.12}$$

This is the space of holomorphic functions on $\mathbb{D}$ with finite L^2-norm defined within (2.12).

(iv) For the notion of generalized "harmonic Beltrami differentials" on $\mathbb{D}$, we also have two spaces to introduce:

$$\Omega^{-1,1}(\mathbb{D}) = \{\nu(z) \in \mathcal{A}^{-1,1}(\mathbb{D}) : \nu = \frac{\bar{q}}{\rho(z)} \text{ for some } q \in A_\infty(\mathbb{D})\}, \tag{2.13}$$

and

$$H^{-1,1}(\mathbb{D}) = \{\nu(z) \in \mathcal{A}^{-1,1}(\mathbb{D}) : \nu = \frac{\bar{q}}{\rho(z)} \text{ for some } q \in A_2(\mathbb{D})\}. \tag{2.14}$$

Definition 2.4. *The universal Teichmüller space $T(1) = \mathcal{B}^{-1,1}(\mathbb{D})\backslash \sim$, where $\mu \sim \nu \in \mathcal{B}^{-1,1}(\mathbb{D})$ if and only if $w_\mu = w_\nu$ on the unit circle, and w_μ is the unique quasiconformal map between extended complex planes which fixes the points $-1, -i, 1$, and solves the Beltrami equation $f_{\bar{z}} = \mu f_z$.*

At any point in the universal Teichmüller space $T(1)$, the cotangent space is naturally identified with the Banach space $A_\infty(\mathbb{D})$ defined in (2.11), and the tangent space is identified with the space $\Omega^{-1,1}(\mathbb{D})$ defined in (2.13). It is then clear the Petersson pairing of functions in the space $\Omega^{-1,1}(\mathbb{D})$ is not well-defined. However, for any $\mu, \nu \in H^{-1,1}(\mathbb{D}) \subset \Omega^{-1,1}(\mathbb{D})$, we write the Petersson pairing as the following inner product:

$$\langle \mu, \nu \rangle = \int_{\mathbb{D}} \mu \bar{\nu} \rho(z) |dz|^2. \tag{2.15}$$

Then this defines a Hilbert structure on the universal Teichmüller space $T(1)$, introduced in ([**TT06**]), namely, $T(1)$ endowed with this inner product, becomes an infinite dimensional complex manifold and Hilbert space. We denote this Hilbert manifold $T_H(1)$, which consists of all the points of the universal Teichmüller space $T(1)$, with tangent space identified as $H^{-1,1}(\mathbb{D})$, a sub-Hilbert space of the Banach space $\Omega^{-1,1}(\mathbb{D})$. We call the resulting metric from (2.15) the *Weil-Petersson metric* on $T_H(1)$. The space we are dealing with is still very complicated: in the corresponding topology induced from the inner product above, the Hilbert manifold $T_H(1)$ is a disjoint union of uncountably many components ([**TT06**]).

One of the most important tools for us is the Green's function for the operator D on the disk. We abuse our notation to denote the operator $D = -2(\Delta_\rho - 2)^{-1}$ and $G(z,w)$ its Green's function, where Δ_ρ is the Laplace operator on the Poincaré disk $\mathbb{D}$. Let us organize some properties we will use later into the following proposition.

Proposition 2.5. [**Hej76**] *The Green's function $G(z,w)$ satisfies the following properties:*

(i) *Positivity: $G(z,w) > 0$ for all $z, w \in \mathbb{D}$;*
(ii) *Symmetry: $G(z,w) = G(w,z)$ for all $z, w \in \mathbb{D}$;*
(iii) *Unit hyperbolic area: $\int_{\mathbb{D}} G(z,w)dA(w) = 1$ for all $z \in \mathbb{D}$;*
(iv) *We denote $BC^\infty(\mathbb{D})$ the space of bounded smooth functions on $\mathbb{D}$, then for $\forall f(z) \in BC^\infty(\mathbb{D})$,*

$$D(f)(z) = \int_{w\in\mathbb{D}} G(z,w)f(w)dA(w). \tag{2.16}$$

Moreover, $D(f) \in BC^\infty(\mathbb{D})$.

2.4. **The curvature operator on the universal Teichmüller space.** We have defined the Hilbert manifold $T_H(1)$ and its Riemannian metric (2.15) for its tangent space $H^{-1,1}(\mathbb{D})$ which we will work with for the rest of the paper, let us now generalize the concept of the curvature operator (Definition 2.3) for Teichmüller space to $T_H(1)$. This has been done in more abstract settings, see for instance [pages 238-239, [**Lan99**]] or [**Duc13**].

We work in the Poincaré disk $\mathbb{D}$. On one hand, without Fuchsian group action, we are forced to deal with an infinite dimensional space of certain functions, on the other hand, the hyperbolic metric is explicit. This leads to some explicit calculations that one can take advantage of. First we note that the tangent space $H^{-1,1}(\mathbb{D})$ has an explicit orthonormal basis: we set, $n \geq 2$,

$$\mu_{n-1} = \frac{(1-|z|^2)^2}{4}\sqrt{\frac{2n^3-2n}{\pi}}\bar{z}^{n-2}. \tag{2.17}$$

Lemma 2.6. [**TT06, Teo09**] *The set $\{\mu_i\}_{i\geq 1}$ forms an orthonormal basis with respect to the Weil-Petersson metric on $H^{-1,1}(\mathbb{D})$.*

Moreover, Takhtajan-Teo established the curvature tensor formula for the Weil-Petersson metric on $T_H(1)$, which takes the same form as Tromba-Wolpert's formula for Teichmüller space of closed surfaces:

Theorem 2.7. [**TT06**] *For* $\{\mu\}$*'s in* $H^{-1,1}(\mathbb{D})$*, the Riemannian curvature tensor for the Weil-Petersson metric* (2.15) *is given by:*

$$R_{i\bar{j}k\bar{\ell}} = \int_{\mathbb{D}} D(\mu_i\bar{\mu}_j)(\mu_k\bar{\mu}_\ell)dA + \int_{\mathbb{D}} D(\mu_i\bar{\mu}_\ell)(\mu_k\bar{\mu}_j)dA. \tag{2.18}$$

Here we abuse our notation to use $D = -2(\Delta_\rho - 2)^{-1}$*, where* Δ_ρ *is the Laplace operator on the Poincaré disk* $\mathbb{D}$*.*

Let U be a neighborhood of $p \in T_H(1)$ and $\{t_1, t_2, \cdots\}$ be a local holomorphic coordinate system on U such that $\{t_i(p) = \mu_i\}_{i\geq 1}$ is orthonormal at p, where μ_i's are explicitly defined in (2.17), we write $t_i = x_i + \mathbf{i}y_i \;\; (i \geq 1)$, then $\{x_1, y_1, x_2, y_2, \cdots\}$ is a real smooth coordinate system in U, and we have:

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \bar{t}_i}, \quad \frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \bar{t}_i}).$$

Let $TT_H(1)$ be the real tangent bundle of $T_H(1)$ and $\wedge^2 TT_H(1)$ be the exterior wedge product of $TT_H(1)$ and itself. For any $p \in U$, we have

$$T_pT_H(1) = Span\{\frac{\partial}{\partial x_i}(p), \frac{\partial}{\partial y_j}(p)\}_{i.j\geq 1},$$

and

$$\wedge^2 TT_H(1) = Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_\ell}, \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}.$$

Following Lang [Chapter 9, [**Lan99**]], we define:

Definition 2.8. *The Weil-Petersson curvature operator* $\tilde{Q} : \wedge^2 TT_H(1) \to \wedge^2 TT_H(1)$ *is given as*

$$\tilde{Q}(V_1 \wedge V_2, V_3 \wedge V_4) = R(V_1, V_2, V_3, V_4),$$

and extended linearly, where V_i *are real tangent vectors, and* R *is the curvature tensor for the Weil-Petersson metric.*

It is easy to see that $\tilde{Q}$ is a bilinear symmetric form.

3. Non-Positive Definiteness and Zero Level Set

In this section, we prove the first part of Theorem 1.1:

Theorem 3.1. *The operator* $\tilde{Q}$ *is non-positive definite.*

The strategy of our proof is the most direct approach, namely, lengthy but careful calculations using explicit nature of both the hyperbolic metric on $\mathbb{D}$, and the orthonormal basis on $H^{-1,1}(\mathbb{D})$ given by (2.17). We will verify the theorem by calculating with various combinations of bases elements, then extend bilinearly. We follow closely the argument in the proof of Theorem 1.1 in [**Wu14**], which was inspired by calculations in [**LSY08**].

3.1. **Preparation.** Note that the version of the operator $D = -2(\Delta - 2)^{-1}$ on a closed surface is positive and self-adjoint on $L^2(X = \mathbb{D}\backslash\Gamma)$, and it plays a fundamental role in Teichmüller theory, but in the case of $\mathbb{D}$, the operator $D = -2(\Delta_\rho - 2)^{-1}$ is noncompact, therefore we have to justify several properties carefully.

Proposition 3.2. *We have the following:*

(i) *The operator $D = -2(\Delta_\rho - 2)^{-1}$ is self-adjoint on $L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$;*
(ii) *For any $f \in L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$, we have also $D(f) \in L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$;*
(iii) *The operator $D = -2(\Delta_\rho - 2)^{-1}$ is positive on $L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$.*

Proof. (i). For all $f, h \in L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$, we have

$$\begin{aligned}\int_{\mathbb{D}} D(f) h dA(z) &= \int_{\mathbb{D}} \int_{\mathbb{D}} G(z,w) f(w) dA(w) h(z) dA(z) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} G(z,w) h(z) dA(z) f(w) dA(w) \\ &= \int_{\mathbb{D}} f D(h) dA(w).\end{aligned}$$

(ii). From Proposition 2.5, we know $D(f), D(f^2) \in BC^\infty(\mathbb{D})$. Using the positivity of the Green's function $G(z,w)$, and $\int_{\mathbb{D}} G(z,w) dA(w) = 1$, we estimate with the Cauchy-Schwarz inequality:

$$\begin{aligned}\int_{\mathbb{D}} |D(f(z))|^2 dA(z) &= \int_{\mathbb{D}} \{\int_{\mathbb{D}} |G(z,w) f(w)| dA(w)\}^2 dA(z) \\ &\leq \int_{\mathbb{D}} (\int_{\mathbb{D}} |G(z,w) f^2(w)| dA(w)) \bullet (1) dA(z) \\ &= \int_D D(|f^2|) dA(z) \\ &= \int_D |f|^2 dA(z).\end{aligned}$$

The last step we used self-adjointness of D and the fact that $D(1) = 1$. This proves $D(f) \in L^2(\mathbb{D})$.

(iii) Given any real function $f \in L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$, let us denote $u = D(f)$ and by (ii) above, it also lies in $L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$, then $f = -\frac{1}{2}(\Delta_\rho - 2)u$, and

$$\begin{aligned}\int_{\mathbb{D}} D(f) f dA &= -\frac{1}{2} \int_{\mathbb{D}} u(\Delta_\rho - 2) u dA \\ &= \int_{\mathbb{D}} u^2 dA - \frac{1}{2} \int_{\mathbb{D}} u \Delta_\rho u dA \\ &\geq \int_{\mathbb{D}} u^2 dA \geq 0.\end{aligned}$$

Here we used that Δ_ρ is negative definite on $L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$ ([**Hej76**]).

The case when f is complex valued can be proved similarly after working on real and imaginary parts separately. □

Recall that $\{\mu_1, \mu_2, \cdots\}$ forms an orthonormal basis for $H^{-1,1}(\mathbb{D})$, where μ_i's are given explicitly in (2.17). Using the coordinate system described in §2.4, we have

$$\wedge^2 TT_H(1) = Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_\ell}, \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}.$$

Naturally we will work with these three combinations. Let us define a few terms to simplify our calculations:

(i) Consider $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, where a_{ij} are real. We denote

$$F(z,w) = \sum_{i,j \geq 1} a_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}. \tag{3.1}$$

(ii) The Green's function of the operator D: $G(z,w) = G(w,z)$.

(iii) Consider $\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$, where b_{ij} are real. We denote

$$H(z,w) = \sum_{i,j \geq 1} b_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}. \tag{3.2}$$

There are three types of basis elements in $\wedge^2 TT_H(1)$, however, in terms of the curvature operator, we only have to work with the first two types because of the next lemma:

Lemma 3.3. *We have the following:*

(i) $\tilde{Q}(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_\ell}) = \tilde{Q}(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_\ell})$.

(ii) $\tilde{Q}(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_\ell}) = \tilde{Q}(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_\ell})$.

Proof. The Weil-Petersson metric on the Hilbert manifold $T_H(1)$ is Kähler-Einstein ([**TT06**]), therefore its associated complex structure $\mathbf{J}$ is an isometry on the tangent space $H^{-1,1}(\mathbb{D})$ such that $\mathbf{J}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $\mathbf{J}^2 = -\text{id}$. Now it is easy to verify:

$$\begin{aligned} \tilde{Q}(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_\ell}) &= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \mathbf{J}\frac{\partial}{\partial x_k}, \mathbf{J}\frac{\partial}{\partial x_\ell}) \\ &= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}) \\ &= \tilde{Q}(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_\ell}). \end{aligned}$$

The other equality is proved similarly. □

3.2. **Proof of Theorem 3.1.** The curvature tensor in equation (2.18) has two terms. Set

$$(ij,kl) := \int_{\mathbb{D}} D(\mu_i\overline{\mu}_j)(\mu_k\overline{\mu}_l)dA.$$

Thus, the curvature tensor satisfies

$$R_{i\bar{j}k\bar{\ell}} = (ij,kl) + (il,kj). \tag{3.3}$$

Proof of Theorem 3.1. We write

$$A = \sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \quad B = \sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}. \tag{3.4}$$

By Lemma 3.3, we only have to show $\tilde{Q}(A+B,A+B)\le 0$. We pause to give an expression for $\tilde{Q}(A+B,A+B)$. Since $\tilde{Q}(A,B)=\tilde{Q}(B,A)$, we have

$$\tilde{Q}(A+B,A+B) = \tilde{Q}(A,A) + 2\tilde{Q}(A,B) + \tilde{Q}(B,B).$$

Now we work with these terms.

Lemma 3.4. *Using above notations, we have*

$$\begin{aligned}\tilde{Q}(A,A) &= -4\int_{\mathbb{D}} D(\Im F(z,z))(\Im F(z,z))dA(z)\\ &+ 2\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)F(z,w)F(w,z)dA(w)dA(z)\}\\ &- 2\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|F(z,w)|^2 dA(w)dA(z).\end{aligned}$$

Proof of Lemma 3.4. First we recall the Weil-Petersson curvature tensor formula (2.18), notation in (2.7), and $\frac{\partial}{\partial x_i}=\frac{\partial}{\partial t_i}+\frac{\partial}{\partial \bar{t}_i}$, then we take advantage of the Green's function $G(z,w)$ for D and the fact that D is self-adjoint on $L^2(\mathbb{D})\cap BC^\infty(\mathbb{D})$ to calculate as follows:

$$\begin{aligned}\tilde{Q}(A,A) &= \sum_{i,j,k,\ell} a_{ij}a_{k\ell}(R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}\bar{k}\ell} + R_{\bar{i}jk\bar{\ell}} + R_{\bar{i}j\bar{k}\ell})\\ &= \sum_{i,j,k,\ell} a_{ij}a_{k\ell}((i\bar{j},k\bar{\ell}) + (i\bar{\ell},k\bar{j}) - (i\bar{j},\ell\bar{k}) - (i\bar{k},\ell\bar{j})\\ &- (j\bar{i},k\bar{\ell}) - (j\bar{\ell},k\bar{i}) + (j\bar{i},\ell\bar{k}) + (j\bar{k},\ell\bar{i}))\\ &= \sum_{i,j,k,\ell} a_{ij}a_{k\ell}(i\bar{j}-j\bar{i},k\bar{\ell}-\ell\bar{k}) + \sum_{i,j,k,\ell} a_{ij}a_{k\ell}((i\bar{\ell},k\bar{j}) + (\ell\bar{i},j\bar{k}))\\ &- \sum_{i,j,k,\ell} a_{ij}a_{k\ell}((i\bar{k},\ell\bar{j}) + (j\bar{\ell},k\bar{i}))\\ &= \int_{\mathbb{D}} D(F(z,z)-\overline{F(z,z)})(F(z,z)-\overline{F(z,z)})dA(z)\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\Re\{\sum_{i,j,k,\ell} a_{ij}a_{k\ell}(i\bar{\ell},k\bar{j})\} - 2\Re\{\sum_{i,j,k,\ell} a_{ij}a_{k\ell}(i\bar{k},\ell\bar{j})\} \\
= &\quad -4\int_{\mathbb{D}} D(\Im F(z,z))(\Im F(z,z))dA(z) \\
&+ \quad 2\Re\{\sum_{i,j,k,\ell} a_{ij}a_{k\ell}(i\bar{\ell},k\bar{j})\} - 2\Re\{\sum_{i,j,k,\ell} a_{ij}a_{k\ell}(i\bar{k},\ell\bar{j})\}.
\end{aligned}
$$

For the second term in the equation above,

$$
\begin{aligned}
&\Re\{\sum_{i,\ell} a_{ij}a_{k\ell}((i\bar{\ell},k\bar{j})\} \\
= &\Re\{\int_{\mathbb{D}} D(\sum_i a_{ij}\mu_i\overline{\mu_\ell})(\sum_k a_{k\ell}\mu_k\overline{\mu_j})dA(z)\} \\
= &\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\mu_\ell(w)}(\sum_k a_{k\ell}\mu_k(z)\overline{\mu_j}(z))dA(z)dA(w)\} \\
= &\Re\{\int_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\}.
\end{aligned}
$$

Similarly, we have

$$
\Re\{\sum_{i,\ell} a_{ij}a_{k\ell}((i\bar{k},\ell\bar{j})\} = \iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|F(z,w)|^2 dA(w)dA(z).
$$

Then, the lemma follows by the equations above. □

Lemma 3.5. *Using above notations, we have*

$$
\begin{aligned}
\tilde{Q}(B,B) \quad = &\quad -4\int_{\mathbb{D}} D(\Re H(z,z))(\Re H(z,z))dA(z) \\
&- \quad 2\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)H(z,w)H(w,z)dA(w)dA(z)\} \\
&- \quad 2\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|H(z,w)|^2 dA(w)dA(z).
\end{aligned}
$$

Proof of Lemma 3.5. Using $\frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \bar{t}_i})$, we have

$$
\tilde{Q}(B,B) = \tilde{Q}(\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j})
$$

$$
\begin{aligned}
= &\quad -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}(R_{i\bar{j}k\bar{\ell}} - R_{i\bar{j}\bar{k}\ell} - R_{\bar{i}jk\bar{\ell}} + R_{\bar{i}j\bar{k}\ell}) \\
= &\quad -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}(R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}\ell\bar{k}} + R_{j\bar{i}k\bar{\ell}} + R_{j\bar{i}\ell\bar{k}}) \\
= &\quad -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}((i\bar{j},k\bar{\ell}) + (i\bar{\ell},k\bar{j}) + (i\bar{j},\ell\bar{k}) + (i\bar{k},\ell\bar{j}) \\
&+ \quad (j\bar{i},k\bar{\ell}) + (j\bar{\ell},k\bar{i}) + (j\bar{i},\ell\bar{k}) + (j\bar{k},\ell\bar{i}))
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}(i\overline{j}+j\overline{i},k\overline{\ell}+\ell\overline{k}) - \sum_{i,j,k,\ell} b_{ij}b_{k\ell}((i\overline{\ell},k\overline{j})+(\ell\overline{i},j\overline{k})) \\
- \quad & \sum_{i,j,k,\ell} b_{ij}b_{k\ell}((i\overline{k},\ell\overline{j})+(j\overline{\ell},k\overline{i})).
\end{aligned}
$$

Let us work with these three terms. For the first one, we have

$$
\begin{aligned}
& -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}(i\overline{j}+j\overline{i},k\overline{\ell}+\ell\overline{k}) \\
= \quad & -\int_X D(\sum_{ij} b_{ij}\mu_i\overline{\mu_j}+\sum_{ij} b_{ij}\mu_j\overline{\mu_i})(\sum_{ij} b_{ij}\mu_i\overline{\mu_j}+\sum_{ij} b_{ij}\mu_j\overline{\mu_i})dA(z) \\
= \quad & -\int_X D(H(z,z)+\overline{H(z,z)})(H(z,z)+\overline{H(z,z)})dA(z) \\
= \quad & -4\int_{\mathbb{D}} D(\Re H(z,z))(\Re H(z,z))dA(z).
\end{aligned}
$$

For the second term, using the same argument in calculating $\tilde{Q}(A,A)$ above, we have

$$
\begin{aligned}
& -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}((i\overline{\ell},k\overline{j})+(\ell\overline{i},j\overline{k})) \\
= \quad & -2\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)H(z,w)H(w,z)dA(w)dA(z)\}.
\end{aligned}
$$

While similarly the third term yields

$$
\begin{aligned}
& -\sum_{i,j,k,\ell} b_{ij}b_{k\ell}((i\overline{k},\ell\overline{j})+(j\overline{\ell},k\overline{i})) \\
= \quad & -2\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|H(z,w)|^2 dA(w)dA(z).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tilde{Q}(B,B) \quad = \quad & -4\int_{\mathbb{D}} D(\Re H(z,z))(\Re H(z,z))dA(z) \\
- \quad & 2\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)H(z,w)H(w,z)dA(w)dA(z)\} \\
- \quad & 2\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|H(z,w)|^2 dA(w)dA(z).
\end{aligned}
$$

□

We are left to deal with the final expression $\tilde{Q}(A,B)$.

Lemma 3.6. *Using above notations, we have*

$$
\tilde{Q}(A,B) \quad = \quad -4\int_{\mathbb{D}} D(\Im F(z,z))(\Re H(z,z))dA(z)
$$

$$
\begin{aligned}
&- \quad 2\Im\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)H(w,z)\,dA(w)A(z)\}\\
&- \quad 2\Im\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)\overline{H(z,w)}\,dA(w)A(z)\}.
\end{aligned}
$$

Proof of Lemma 3.6.

$$
\begin{aligned}
\tilde{Q}(A,B) \quad &= \quad (-\mathbf{i})\sum_{i,j,k,\ell} a_{ij}b_{k\ell}(-R_{i\bar{j}k\bar{\ell}} + R_{i\bar{j}\bar{k}\ell} - R_{\bar{i}jk\bar{\ell}} + R_{\bar{i}j\bar{k}\ell})\\
&= \quad (-\mathbf{i})\sum_{i,j,k,\ell} a_{ij}b_{k\ell}\{-(i\bar{j},k\bar{\ell}) - (i\bar{\ell},k\bar{j}) - (i\bar{j},\ell\bar{k}) - (i\bar{k},\ell\bar{j})\\
&+ \quad (j\bar{i},k\bar{\ell}) + (j\bar{\ell},k\bar{i}) + (j\bar{i},\ell\bar{k}) + (j\bar{k},\ell\bar{i})\}\\
&= \quad (-\mathbf{i})\sum_{i,j,k,\ell} a_{ij}b_{k\ell}(j\bar{i} - i\bar{j}, k\bar{\ell} + \ell\bar{k})\\
&+ \quad (-\mathbf{i})\sum_{i,j,k,\ell} a_{ij}b_{k\ell}(-(i\bar{\ell},k\bar{j}) + (\ell\bar{i},j\bar{k}))\\
&+ \quad (-\mathbf{i})\sum_{i,j,k,\ell} a_{ij}b_{k\ell}(-(i\bar{k},\ell\bar{j}) + (j\bar{\ell},k\bar{i}))\\
&= \quad -\mathbf{i}(-2\mathbf{i})\int_{\mathbb{D}} D(\Im F(z,z))(2\Re H(z,z))\,dA(z)\\
&+ \quad (-\mathbf{i})(-2\mathbf{i})\Im\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)H(w,z)\,dA(w)A(z)\}\\
&+ \quad (-\mathbf{i})(-2\mathbf{i})\Im\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)\overline{H(z,w)}\,dA(w)A(z)\}\\
&= \quad -4\int_{\mathbb{D}} D(\Im F(z,z))(\Re H(z,z))\,dA(z)\\
&- \quad 2\Im\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)H(w,z)\,dA(w)A(z)\}\\
&- \quad 2\Im\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)\overline{H(z,w)}\,dA(w)A(z)\}.
\end{aligned}
$$

□

Proposition 3.7 (Formula for curvature operator)**.** *Using above notations, we have*

$$
\begin{aligned}
&\tilde{Q}(A+B,A+B) =\\
- \quad & 4\int_{\mathbb{D}} D(\Im\{F(z,z) + \boldsymbol{i}H(z,z)\})\cdot(\Im\{F(z,z) + \boldsymbol{i}H(z,z)\})\,dA(z)\\
- \quad & 2\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w) + \boldsymbol{i}H(z,w))|^2\,dA(w)dA(z)\\
+ \quad & 2\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)(F(z,w) + \boldsymbol{i}H(z,w))(F(w,z) + \boldsymbol{i}H(w,z))\,dA(w)dA(z)\},
\end{aligned}
$$

where $F(z,w)$ and $H(z,w)$ are defined in (3.1) and (3.2) respectively, and $G(z,w)$ is the Green's function for the operator D.

Proof of Proposition 3.7. It follows from Lemma 3.4, 3.5 and 3.6 that

$$
\begin{aligned}
& \tilde{Q}(A+B,A+B) \\
= \;& (\int_{\mathbb{D}} D(F(z,z)-\overline{F(z,z)})(F(z,z)-\overline{F(z,z)})dA(z) \\
- \;& \int_{\mathbb{D}} D(H(z,z)+\overline{H(z,z)})(H(z,z)+\overline{H(z,z)})dA(z) \\
+ \;& 2\mathbf{i}\cdot\int_{\mathbb{D}} D(F(z,z)-\overline{F(z,z)})(H(z,z)+\overline{H(z,z)})dA(z)) \\
(- \;& 2\cdot\int_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w))|^2 dA(w)dA(z) \\
- \;& 2\cdot\int_{\mathbb{D}\times\mathbb{D}} G(z,w)|H(z,w))|^2 dA(w)dA(z) \\
- \;& 4\cdot\Im\{\int_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)\overline{H(z,w)})dA(w)dA(z)\}) \\
(+ \;& 2\cdot\Re\{\int_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\} \\
- \;& 2\cdot\Re\{\int_{\mathbb{D}\times\mathbb{D}} G(z,w)H(z,w)H(w,z)dA(w)dA(z)\} \\
- \;& 4\cdot\Im\{\int_{\mathbb{D}\times\mathbb{D}} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\}).
\end{aligned}
$$

The sum of the first three terms is exactly

$$-4\int_{\mathbb{D}} D(\Im\{F(z,z)+\mathbf{i}H(z,z)\})\cdot(\Im\{F(z,z)+\mathbf{i}H(z,z)\})dA(z).$$

Just as $|a+\mathbf{i}b|^2=|a|^2+|b|^2+2\cdot\Im(a\cdot\overline{b})$, where a and b are two complex numbers, the sum of the second three terms is exactly

$$-2\cdot\int_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)+\mathbf{i}H(z,w))|^2 dA(w)dA(z).$$

For the last three terms, since

$$\Im(F(z,w)\cdot H(w,z)) = -\Re(F(z,w)\cdot(\mathbf{i}H(w,z))),$$

the sum is exactly

$$2\cdot\Re\{\int_{\mathbb{D}\times\mathbb{D}} G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)\}.$$

The proof is complete. $\square$

Now we continue with the proof of Theorem 3.1. By Proposition 3.7, there are three integrals in the expression of $\tilde{Q}(A+B, A+B)$. We first work with the last two terms by the Cauchy-Schwarz inequality and the fact that $G(z,w) = G(w,z)$ to find:

$$
\begin{aligned}
& |\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)| \\
\leq \quad & \iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))|dA(w)dA(z) \\
\leq \quad & \{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)+\mathbf{i}H(z,w)|^2 dA(w)dA(z)\}^{\frac{1}{2}} \\
\times \quad & \{\iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|F(w,z)+\mathbf{i}H(w,z)|^2 dA(w)dA(z)\}^{\frac{1}{2}} \\
= \quad & \iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|F(z,w)+\mathbf{i}H(z,w)|^2 dA(z)dA(w).
\end{aligned}
$$

Thus, we have

Proposition 3.8.

$$\tilde{Q}(A+B, A+B) \leq -4\int_{\mathbb{D}} D(\Im F(z,z))(\Im F(z,z))dA(z).$$

Theorem 3.1 follows directly from Proposition 3.2 and 3.8. □

3.3. **Zero level set.** In order to apply Theorem 3.1 to more geometrical situations later, we determine the zero level set for the operator $\tilde{Q}$.

First let us define an action on $\wedge^2 TT_H(1)$. Recall our explicit orthonormal basis from (2.17):

$$\mu_{n-1} = \frac{(1-|z|^2)^2}{4}\sqrt{\frac{2n^3-2n}{\pi}}\bar{z}^{n-2}, \quad n \geq 2.$$

For any point $P \in T_H(1)$, let $\{\frac{\partial}{\partial t_j}\}_{j\geq 1}$ be the vector field on $T_H(1)$ near P such that $\frac{\partial}{\partial t_j}|_P = \mu_j$, and we write $t_j = x_j + \mathbf{i}y_j$, then the complex structure $\mathbf{J}$ associated with the Weil-Petersson metric is an isometry on the tangent space $H^{-1,1}(\mathbb{D})$ with $\mathbf{J}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $\mathbf{J}^2 = -\text{id}$. This naturally extends to an action, which we abuse our notation to denote it by $\mathbf{J}$, on $\wedge^2 TT_H(1)$.

Definition 3.9. *The action* $(\mathbf{J}, \circ)$ *is defined as follows on a basis:*

$$
\begin{cases}
\boldsymbol{J}\circ\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} := \frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}, \\
\boldsymbol{J}\circ\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} := -\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial y_i}, \\
\boldsymbol{J}\circ\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j} := \frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},
\end{cases}
$$

and we extend it linearly.

Lemma 3.10. *We have* $\mathbf{J}\circ\mathbf{J} = id$. *Moreover,*

$$\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\rangle(P)=\langle\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\rangle(P)=\delta_{ij},\qquad \langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_j}\rangle(P)=0. \tag{3.5}$$

Proof. The identity $\mathbf{J}\circ\mathbf{J} = \mathrm{id}$ is clear by definition. We only show the first equality in (3.5):

$$\begin{aligned}\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\rangle(P) &= \Re\{\langle\frac{\partial}{\partial x_i}+\mathbf{iJ}\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}+\mathbf{iJ}\frac{\partial}{\partial x_j}\rangle(P)\}\\ &= \Re\{\langle\frac{\partial}{\partial \mu_i},\frac{\partial}{\partial \mu_j}\rangle(P)\}=\delta_{ij}.\end{aligned}$$

□

Now we treat the equality case for $\tilde{Q}\leq 0$, namely,

Theorem 3.11. *(= (ii) of Theorem 1.1) For* $C\in\wedge^2 TT_H(1)$, $\tilde{Q}(C,C)=0$ *if and only if there is an element* $E\in\wedge^2 TT_H(1)$ *such that* $C=E-\mathbf{J}\circ E$, *where* $\mathbf{J}\circ$ *is defined above.*

Proof. One direction is straightforward: If $C=E-\mathbf{J}\circ E$ for some $E\in\wedge^2 TT_H(1)$, we have that $\tilde{Q}(C,C)=0$ since $\mathbf{J}$ is an isometry on tangent space.

Conversely, let $C\in\wedge^2 TT_H(1)$ with $\tilde{Q}(C,C)=0$. We write

$$C=\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}+b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}+c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}.$$

Applying the identities in the proof of Lemma 3.3, we have

$$\tilde{Q}(C,C)=\tilde{Q}(\sum_{ij} d_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}+b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j},\sum_{ij} d_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}+b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}), \tag{3.6}$$

where $d_{ij}=a_{ij}+c_{ij}$. This enables us to write $C=A+B$, where

$$A=\sum_{ij} d_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},\quad B=\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}.$$

From the proof of $\tilde{Q}(A+B,A+B)\leq 0$ in Proposition 3.7, we find that $Q(A+B,A+B)=0$ if and only if there exists a constant k such that both of the following hold:

$$\begin{cases}\Im\{F(z,z)+\mathbf{i}H(z,z)\}=0,\\ F(z,w)+\mathbf{i}H(z,w)=k\cdot\overline{(F(w,z)+\mathbf{i}H(w,z))}.\end{cases}$$

Setting $z=w$, we find $k=1$. Therefore the second equation above implies

$$\sum_{ij}(d_{ij}-d_{ji}+\mathbf{i}(b_{ij}+b_{ji}))\mu_i(w)\overline{\mu_j}(z)=0.$$

Since $\{\mu_i\}_{i\geq 1}$ is a basis,

$$d_{ij} = d_{ji}, \qquad b_{ij} = -b_{ji}.$$

That is,

$$a_{ij} - a_{ji} = -(c_{ij} - c_{ji}), \quad b_{ij} = -b_{ji}.$$

We now define

$$E = \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \tag{3.7}$$

and we verify that $C = E - \mathbf{J} \circ E$. Indeed, first we have

$$\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \sum_{i<j} (a_{ij} - a_{ji}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

then we apply the definition of $\mathbf{J}\circ$ in Definition 3.9 to find

$$\begin{aligned} \mathbf{J} \circ \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} &= \sum_{i<j} (a_{ij} - a_{ji}) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \\ &= -\sum_{i<j} (c_{ij} - c_{ji}) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \\ &= -\sum c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}. \end{aligned}$$

Similarly,

$$\mathbf{J} \circ \sum (\frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}) = -\sum \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}.$$

This completes the proof. □

4. Boundedness

If we denote $\langle \cdot, \cdot \rangle$ the pairing of vectors in the space $\wedge^2 TT_H(1)$. This natural inner product on $\wedge^2 TT_H(1)$ associated to the Weil-Petersson metric on $T_H(1)$ is given as (following [**Lan99**]): $\forall V_i \in H^{-1,1}(\mathbb{D})$,

$$\langle V_1 \wedge V_2, V_3 \wedge V_4 \rangle_{eu} = \langle V_1, V_3 \rangle \langle V_2, V_4 \rangle - \langle V_1, V_4 \rangle \langle V_2, V_3 \rangle. \tag{4.1}$$

The goal of this section is to prove Theorem 1.3, namely,

Theorem 4.1. *(=Theorem 1.3) The curvature operator $\tilde{Q}$ is bounded, i.e., for any $V \in \wedge^2 TT_H(1)$ with $\|V\|_{eu} = 1$, we have $|\tilde{Q}(V, V)| \leq 16\sqrt{\frac{3}{\pi}}$, where $\|\cdot\|_{eu}$ is the Euclidean norm for the wedge product defined in (4.1).*

We follow the idea for the proof of Theorem 1.3 in [**WW15**].

4.1. **Technical lemmas.** Let us begin with a useful lemma.

Lemma 4.2. *Let $D = -2(\Delta_\rho - 2)^{-1}$ as above above. Then, for any complex-valued function $f \in L^2(\mathbb{D}) \cap BC^\infty(\mathbb{D})$,*

$$\int_{\mathbb{D}} (D(f)\bar{f})dA \leq \int_{\mathbb{D}} |f|^2 dA, \tag{4.2}$$

where $dA = \rho|dz|^2$ is the hyperbolic area element for $\mathbb{D}$.

Proof. Recall the Green's function of D on $\mathbb{D}$ is $G(z, w)$, such that, $\forall f \in L^2(\mathbb{D}, \mathbb{C})$, we have

$$D(f)(z) = \int_{w \in \mathbb{D}} G(z, w) f(w) dA(w). \tag{4.3}$$

Assuming first f is real valued, we apply the Cauchy-Schwarz inequality and the symmetry of the Green's function:

$$\begin{aligned}
& \int_{\mathbb{D}} D(f(z))f(z)dA(z) \\
= & \iint_{\mathbb{D}\times\mathbb{D}} \{G(z,w)f(w)dA(w)\}f(z)dA(z) \\
\leq & \sqrt{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)f^2(w)dA(w)dA(z)} \cdot \sqrt{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)f^2(z)dA(z)dA(w)} \\
= & \sqrt{\iint_{\mathbb{D}} D(f^2(w))dA(w)} \cdot \sqrt{\int_{\mathbb{D}} D(f^2(z))dA(z)} \\
= & \int_{\mathbb{D}} D(f^2(z))dA(z) \\
= & \int_{\mathbb{D}} f^2(z)dA(z).
\end{aligned}$$

When f is complex valued, we can write $f = f_1 + \mathbf{i} f_2$, where f_1 and f_2 are real-valued. Then using D is self-adjoint (Proposition 3.2), we find,

$$\begin{aligned}
\int_{\mathbb{D}} (D(f)\bar{f})dA &= \int_{\mathbb{D}} (D(f_1)f_1 dA + \int_{\mathbb{D}} (D(f_2)f_2 dA \\
&\leq \int_{\mathbb{D}} |f_1|^2 + |f_2|^2 dA \\
&= \int_{\mathbb{D}} |f|^2 dA.
\end{aligned}$$

∎

Recalling from (3.1) and (3.2), we write $F(z,z) = \sum_{ij} a_{ij}\mu_i(z)\bar{\mu}_j(z)$ for the expression $A = \sum_{i,j\geq 1} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, and $H(z,z) = \sum_{ij} b_{jj}\mu_i(z)\bar{\mu}_j(z)$ for the expression $B = \sum_{i,j\geq 1} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$, where $\{\mu_i\}_{i\geq 1}$ is the orthonormal basis

(2.17) for $H^{-1,1}(\mathbb{D})$. By Lemma 3.3, to show Theorem 4.1, it suffices to work with $V = A + B$.

Lemma 4.3. *Under above notation, we have the following estimate:*

$$\begin{aligned} |\tilde{Q}(V,V)| \quad &\leq \quad 8\cdot(\int_{\mathbb{D}}|F(z,z)|^2 dA + \int_{\mathbb{D}}|H(z,z)|^2 dA) \\ &+ \quad 4\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)+\boldsymbol{i}H(z,w))|^2 dA(w)dA(z). \end{aligned} \tag{4.4}$$

Proof. The expression for $\tilde{Q}(V,V)$ is shown in Proposition 3.7. We have

$$\begin{aligned} & |\tilde{Q}(V,V)| \\ \leq \quad & 4\int_{\mathbb{D}} D(\Im\{F(z,z)+\mathbf{i}H(z,z)\})\cdot(\Im\{F(z,z)+\mathbf{i}H(z,z)\})dA(z) \\ + \quad & 2\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)+\mathbf{i}H(z,w))|^2 dA(w)dA(z) \\ + \quad & 2\Re\{\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)\}. \end{aligned}$$

We work with these terms. First we apply Lemma 4.2, and the triangle inequality to find:

$$\begin{aligned} & 4\int_{\mathbb{D}} D(\Im\{F(z,z)+\mathbf{i}H(z,z)\})\cdot(\Im\{F(z,z)+\mathbf{i}H(z,z)\})dA(z) \\ \leq \quad & 4\int_{\mathbb{D}}|\Im\{F(z,z)+\mathbf{i}H(z,z)\}|^2 dA(z) \\ \leq \quad & 8(\int_{\mathbb{D}}|\{F(z,z)|^2 dA + \int_{\mathbb{D}}|H(z,z)\}|^2 dA). \end{aligned}$$

Recalling from the end of the proof of Theorem 3.1, we have established the following inequality:

$$\begin{aligned} & |\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)| \\ \leq \quad & \iint_{\mathbb{D}\times\mathbb{D}} G(w,z)|F(z,w)+\mathbf{i}H(z,w)|^2 dA(z)dA(w). \end{aligned}$$

Now (4.4) follows. ☐

We will also quote a Harnack type inequality for any $\mu(z) \in H^{-1,1}(\mathbb{D})$. We note that it works in our favor that the injectivity radius of $\mathbb{D}$ is infinity.

Proposition 4.4. [**TT06, Teo09**] *Let $\mu \in H^{-1,1}(\mathbb{D})$. Then the L^∞-norm of $\mu \in H^{-1,1}(\mathbb{D})$ can be estimated from above by its Weil-Petersson norm, namely, for all $\mu \in H^{-1,1}(\mathbb{D})$, we have*

$$sup_{z\in\mathbb{D}}|\mu(z)| \leq \sqrt{\frac{3}{4\pi}}\|\mu\|_{WP}. \tag{4.5}$$

We also derive the following estimate which is quite general, and we formulate it to the following lemma:

Lemma 4.5. *Let $\{z\}, \{w\}$ two complex coordinates on $\mathbb{D}$, if a converging series is in the form of*

$$K(z, w) = \sum_{i,j\geq 1} d_{ij}\mu_i(w)\overline{\mu_j(z)},$$

for some $d_{ij} \in \mathbb{R}$, where $\{\mu_j\}$ is the orthonormal basis (2.17) on $H^{-1,1}(\mathbb{D})$, then we have

$$\int_{\mathbb{D}} |K(z, z)|^2 dA(z) \leq \sqrt{\frac{3}{4\pi}} \sum_{i,j\geq 1} d_{ij}^2, \tag{4.6}$$

and

$$sup_{w\in\mathbb{D}}|K(z, w)|^2 \leq \sqrt{\frac{3}{4\pi}} \sum_{i,j,\ell\geq 1} d_{ij}d_{i\ell}\overline{\mu_j(z)}\mu_\ell(z). \tag{4.7}$$

Proof. We use the standard technique for this type of argument, namely, since

$$|K(z, z)|^2 \leq sup_{w\in\mathbb{D}}|K(z, w)|^2,$$

we will try to use one complex coordinate against the other. Fixing z, we note that the form $K(z, w)$ is a harmonic Beltrami differential on $\mathbb{D}$ in the coordinate w. Indeed,

$$K(z, w) = \sum_{i\geq 1}\{\sum_{j\geq 1} d_{ij}\overline{\mu_j(z)}\}\mu_i(w).$$

This enables us to apply (4.5):

$$\begin{aligned} sup_{w\in\mathbb{D}}|K(z, w)|^2 &\leq \sqrt{\frac{3}{4\pi}} \int_{\mathbb{D}} K(z, w)\overline{K(z, w)}dA(w) \\ &= \sqrt{\frac{3}{4\pi}} \int_{\mathbb{D}} \{\sum_{i,j} d_{ij}\mu_i(w)\overline{\mu_j(z)}\}\{\sum_{k,\ell} d_{k\ell}\overline{\mu_k(w)}\mu_\ell(z)\}dA(w) \\ &= \sqrt{\frac{3}{4\pi}}\{\sum_{i,j,\ell} d_{ij}d_{i\ell}\overline{\mu_j(z)}\mu_\ell(z)\}. \end{aligned}$$

We also used the basis $\{\mu_j\}$ is orthonormal with respect to the Weil-Petersson metric. Therefore we have

$$\begin{aligned} \int_{\mathbb{D}} |K(z, z)|^2 dA(z) &\leq \int_{\mathbb{D}} sup_{w\in\mathbb{D}}|K(z, w)|^2 dA(z) \\ &\leq \sqrt{\frac{3}{4\pi}} \int_{\mathbb{D}} \sum_{i,j,\ell} d_{ij}d_{i\ell}\overline{\mu_j(z)}\mu_\ell(z)dA(z) \end{aligned}$$

$$= \sqrt{\frac{3}{4\pi}} \sum_{i,j} d_{ij}^2.$$

□

4.2. $\tilde{Q}$ **is bounded.** We now prove the boundedness.

Proof of Theorem 4.1. We find via the definition of the Euclidean inner product (4.1) and the symmetric properties of the curvature tensor:

$$\langle \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_\ell} \rangle_{eu}(P) = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}, \tag{4.8}$$

and

$$\langle \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_\ell} \rangle_{eu}(P) = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}, \tag{4.9}$$

and

$$\langle \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j} \rangle_{eu}(P) = \delta_{ij}, \tag{4.10}$$

and

$$\langle \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_\ell} \rangle_{eu}(P) = 0. \tag{4.11}$$

We now denote $V = A + B + C \in \wedge^2 TT_H(1)$, where $A = \sum_{i<j} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, $B = \sum_{i,j \geq 1} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$, and $C = \sum_{i<j} c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$. Then by (4.8), (4.9), (4.10), and (4.11), we have

$$\langle A, A \rangle_{eu}(P) = \sum_{i<j} a_{ij}^2, \quad \langle B, B \rangle_{eu}(P) = \sum_{i,j} b_{ij}^2, \quad \langle C, C \rangle_{eu}(P) = \sum_{i<j} c_{ij}^2. \tag{4.12}$$

and

$$\langle A, B \rangle_{eu}(P) = 0, \quad \langle A, C \rangle_{eu}(P) = 0, \quad \langle B, C \rangle_{eu}(P) = 0. \tag{4.13}$$

Assume that $\|V\|_{eu} = 1$, that is

$$\sum_{i<j} a_{ij}^2 + \sum_{i,j} b_{ij}^2 + \sum_{i<j} c_{ij}^2 = 1.$$

Recalling from Lemma 3.3 and 4.3, we have:

$$\begin{aligned} |\tilde{Q}(V,V)| \quad &\leq \quad 8 \cdot (\int_{\mathbb{D}} |F(z,z)|^2 dA + \int_{\mathbb{D}} |H(z,z)|^2 dA) \\ &+ \quad 4 \iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w) + \mathbf{i}H(z,w))|^2 dA(w) dA(z). \end{aligned}$$

Where $F(z,w) = \sum_{i<j}(a_{ij}+c_{ij})\overline{\mu_j(z)}\}\mu_i(w)$ and $H(z,w) = \sum_{i,j\geq 1} b_{ij}\overline{\mu_j(z)}\}\mu_i(w)$. We now estimate these two terms. Note that both our forms $F(z,w)$ ((3.1)) and $H(z,w)$ ((3.2)) are of the type in Lemma 4.5. Since both series $\sum_{i<j} a_{ij}^2$ and $\sum_{i,j} b_{ij}^2$ converge, we have the first term in (4.4) bounded from above as follows:

$$\begin{aligned} 8\cdot(\int_{\mathbb{D}}|F(z,z)|^2dA+\int_{\mathbb{D}}|H(z,z)|^2dA) &\leq 8\sqrt{\frac{3}{4\pi}}(\sum_{i<j}(a_{ij}+c_{ij})^2+\sum_{i,j}b_{ij}^2) \\ &\leq 8\sqrt{\frac{3}{4\pi}}\times 2(\sum_{i<j}a_{ij}^2+\sum_{i,j}b_{ij}^2+\sum_{i<j}c_{ij}^2) \\ &= 8\sqrt{\frac{3}{\pi}}. \end{aligned}$$

The second term is also bounded by applying (4.7). To see this, we use the fact that $\{\mu_i\}$'s form an orthonormal basis,

$$\begin{aligned} & \iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)|^2dA(z)dA(w) \\ \leq\ & \iint_{\mathbb{D}\times\mathbb{D}} G(z,w)sup_{w\in\mathbb{D}}(|F(z,w)|^2)dA(z)dA(w) \\ \leq\ & \iint_{\mathbb{D}\times\mathbb{D}} G(z,w)\sqrt{\frac{3}{4\pi}}\sum_{i<j}\sum_{\ell\geq 1}(a_{ij}+c_{ij})(a_{i\ell}+c_{i\ell})\overline{\mu_j(z)}\mu_\ell(z)dA(z)dA(w) \\ =\ & \int_{\mathbb{D}} D(\sqrt{\frac{3}{4\pi}}\sum_{i<j}\sum_{\ell\geq 1}(a_{ij}+c_{ij})(a_{i\ell}+c_{i\ell})\overline{\mu_j(z)}\mu_\ell(z))dA(w) \\ =\ & \int_{\mathbb{D}} \sqrt{\frac{3}{4\pi}}\sum_{i<j}\sum_{\ell\geq 1}(a_{ij}+c_{ij})(a_{i\ell}+c_{i\ell})\overline{\mu_j(z)}\mu_\ell(z)dA(w) \\ =\ & \sqrt{\frac{3}{4\pi}}\sum_{i<j}(a_{ij}+c_{ij})^2. \end{aligned}$$

Similar argument yields

$$\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|H(z,w)|^2dA(z)dA(w) \leq \sqrt{\frac{3}{4\pi}}\sum_{i,j}b_{ij}^2.$$

Therefore the second term can be estimated as follows:

$$\begin{aligned} & 4\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)+\mathbf{i}H(z,w))|^2dA(w)dA(z) \\ \leq\ & 8\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|F(z,w)|^2dA(z)dA(w) \end{aligned}$$

$$+ \quad 8\iint_{\mathbb{D}\times\mathbb{D}} G(z,w)|H(z,w)|^2 dA(z)dA(w)$$

$$\leq \qquad 8\sqrt{\frac{3}{4\pi}}(\sum_{i<j}(a_{ij}+c_{ij})^2+\sum_{i,j}b_{ij}^2)$$

$$\leq \qquad 8\sqrt{\frac{3}{\pi}}.$$

Combining with earlier same upper bound for the first term, we find

$$|\tilde{Q}(V,V)| \leq 16\sqrt{\frac{3}{\pi}}.$$

Proof is now complete. □

5. Noncompactness

In this section, we treat the question about the compactness.

Theorem 5.1. *(=Theorem 1.5) The curvature operator $\tilde{Q}$ is not a compact operator.*

We will prove this theorem by contradiction. First we proceed with several technical lemmas.

Lemma 5.2. *For any harmonic Beltrami differential $\mu \in L^2(\mathbb{D})$ on $\mathbb{D}$, we have*

$$D(|\mu|^2) \geq \frac{|\mu|^2}{3}. \tag{5.1}$$

Proof. The argument here is motivated by Lemma 5.1 in [**Wol12a**] which is for the case of a closed Riemann surface.

Recall that the curvature of a metric expressed as $\sigma(z)|dz|^2$ on a Riemannian 2-manifold is given by

$$K(\sigma(z)|dz|^2) = -\frac{1}{2}\Delta_\sigma \ln(\sigma(z)),$$

where Δ_σ is the Laplace-Beltrami operator of $\sigma(z)|dz|^2$.

Suppose that $p \in \mathbb{D}$ with $|\mu| \neq 0$. By definition one may assume that

$$|\mu(z)| = \frac{|\Phi(z)|}{\rho(z)}$$

where $\Phi(z)$ is holomorphic on $\mathbb{D}$. Let Δ be the Laplace-Beltrami operator of $\rho(z)|dz|^2$. Then using the curvature information that $K(\rho(z)|dz|^2) = -1$ and $K(|\Phi(z)||dz|^2)(p) = 0$, we see that at $p \in \mathbb{D}$,

$$\Delta \ln \frac{|\Phi(p)|^2}{\rho^2(p)} = -4.$$

On the other hand, at $p \in \mathbb{D}$ we have

$$\Delta \ln \frac{|\Phi(p)|^2}{\rho^2(p)} = \frac{\Delta \frac{|\Phi(p)|^2}{\rho^2(p)}}{\frac{|\Phi(p)|^2}{\rho^2(p)}} - \frac{|\nabla \frac{|\Phi(p)|^2}{\rho^2(p)}|^2}{\frac{|\Phi(p)|^4}{\rho^4(p)}}.$$

Thus, at $p \in \mathbb{D}$ with $\mu(p) \neq 0$, we have

$$\Delta \frac{|\Phi(p)|^2}{\rho^2(p)} \geq -4 \frac{|\Phi(p)|^2}{\rho^2(p)}. \tag{5.2}$$

If $p \in \mathbb{D}$ with $|\mu(p)| = 0$, the maximum principle gives that

$$\Delta \frac{|\Phi(p)|^2}{\rho^2(p)} \geq 0 = -4 \frac{|\Phi(p)|^2}{\rho^2(p)}.$$

Therefore, we have

$$\Delta \frac{|\Phi(z)|^2}{\rho^2(z)} \geq -4 \frac{|\Phi(z)|^2}{\rho^2(z)}, \qquad \forall z \in \mathbb{D}. \tag{5.3}$$

Rewrite it as

$$(\Delta - 2) \frac{|\Phi(z)|^2}{\rho^2(z)} \geq -6 \frac{|\Phi(z)|^2}{\rho^2(z)}, \quad \forall z \in \mathbb{D}.$$

Since the operator $D = -\frac{1}{2}(\Delta - 2)^{-1}$ is positive on $\mathbb{D}$, the conclusion follows. □

We also need the following elementary estimate:

Lemma 5.3. *For all positive integer $m \in \mathbb{Z}^+$, we have*

$$\int_0^1 (1-r)^6 \cdot r^m dr \geq \frac{45}{2^{17} m^7}. \tag{5.4}$$

Proof. The integral is the well-known beta function. Since m is a positive integer,

$$\begin{aligned} \int_0^1 (1-r)^6 \cdot r^m dr &= \frac{6! \cdot m!}{(m+7)!} \\ &\geq \frac{6!}{(8m)^7} \\ &= \frac{45}{2^{17} m^7}. \end{aligned}$$

□

We denote

$$A_i = \frac{1}{2^{\frac{i}{2}}} \left(\sum_{k=2^i}^{2^{i+1}-1} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_k} \right). \tag{5.5}$$

Proposition 5.4. *For all i large enough, we have*

$$-\tilde{Q}(A_i, A_i) \geq 2^{-30}.$$

Proof. We first observe that, from the definition (4.1) of the inner product on $\wedge^2 TT_H(1)$,

$$\langle A_i, A_j \rangle = \delta_{ij}.$$

We wish to estimate $\tilde{Q}(A_i, A_i)$ according to Proposition 3.7. For A_i in above (5.5), by the definitions in (3.1) and (3.2), the corresponding expressions F and H in Proposition 3.8 satisfy that

$$F(z, z) = 0$$

and

$$H(z, z) = \frac{1}{2^{\frac{i}{2}}}\left(\sum_{k=2^i}^{2^{i+1}-1} |\mu_k(z)|^2\right)$$

where $\mu_k(z) = \frac{(1-|z|^2)^2}{4}\sqrt{\frac{2(k+1)^3-2(k+1)}{\pi}}\bar{z}^{k-1}$.

It follows from Proposition 3.7 that

$$\begin{aligned} -\tilde{Q}(A_i, A_i) &= \frac{4}{2^i}\int_{\mathbb{D}} D\left(\sum_{k=2^i}^{2^{i+1}-1} |\mu_k(z)|^2\right) \cdot \left(\sum_{k=2^i}^{2^{i+1}-1} |\mu_k(z)|^2\right)\rho(z)|dz|^2 \\ &\geq \frac{4}{3\cdot 2^i}\int_{\mathbb{D}}\left(\sum_{k=2^i}^{2^{i+1}-1} |\mu_k(z)|^2\right)^2\rho(z)|dz|^2, \end{aligned}$$

where we apply Lemma 5.2 for the last inequality.

Now we apply the explicit expression of

$$\mu_k(z) = \frac{(1-|z|^2)^2}{4}\sqrt{\frac{2(k+1)^3-2(k+1)}{\pi}}\bar{z}^{k-1},$$

and the fact that $2(k+1)^3 - 2(k+1) \geq (k+1)^3$,

$$\begin{aligned} -\tilde{Q}(A_i, A_i) &\geq \frac{4}{3\cdot 2^i}\sum_{2^i \leq k,j \leq 2^{i+1}-1}\int_{\mathbb{D}} \frac{(k+1)^3(j+1)^3}{4^3\cdot\pi^2}(1-|z|^2)^6|z|^{2k+2j-4}|dz|^2 \\ &= \frac{2\pi}{48\pi^2\cdot 2^i}\sum_{2^i \leq k,j \leq 2^{i+1}-1}(k+1)^3(j+1)^3\int_0^1(1-r^2)^6 r^{2k+2j-4}rdr \\ &= \frac{1}{48\pi\cdot 2^i}\sum_{2^i \leq k,j \leq 2^{i+1}-1}(k+1)^3(j+1)^3\int_0^1(1-r)^6 r^{k+j-2}dr \\ &\geq \frac{1}{48\pi\cdot 2^i}\cdot\frac{45}{2^{17}}\sum_{2^i \leq k,j \leq 2^{i+1}-1}\frac{(k+1)^3(j+1)^3}{(k+j-2)^7} \quad \textit{(by Lemma 5.3)} \\ &\geq \frac{1}{48\pi\cdot 2^i}\cdot\frac{45}{2^{17}}\sum_{2^i \leq k,j \leq 2^{i+1}-1}\frac{(k+1)^3(j+1)^3}{4^7\cdot 2^{7i}}, \end{aligned}$$

where the last inequality follows by $k+j-2 \le 2\cdot(2^{i+1}-1)-2 \le 4\cdot 2^i$.

We simplify it as

$$-\tilde{Q}(A_i, A_i) \quad \geq \quad \frac{15}{\pi\cdot 2^{35}}\cdot\frac{(\sum_{2^i\le k\le 2^{i+1}-1}(k+1)^3)^2}{2^{8i}}$$

It follows by an elementary formula

$$1^3+2^3+3^3+\cdots+m^3 = (m(m+1))^2/4$$

that we have

$$\sum_{2^i\le k\le 2^{i+1}-1}(k+1)^3 = \frac{15}{4}\cdot 2^{4i} + O(2^{3i}).$$

In particular, for large enough i we may assume that

$$\sum_{2^i\le k\le 2^{i+1}-1}(k+1)^3 > 3\cdot 2^{4i}.$$

Therefore, for large enough i we have

$$-\tilde{Q}(A_i, A_i) \geq \frac{135}{\pi\cdot 2^{35}} > 2^{-30}.$$

This completes the proof. □

Let us define $L := Span\{A_n, A_{n+1}, \cdots, A_{2n-1}\}$ which is an n-dimensional linear subspace in $\wedge^2 TT_H(1)$ and $P_L : \wedge^2 TT_H(1) \to L$ be the projection map. It is clear that $\langle A_k, A_\ell\rangle = \delta_{k\ell}$.

Lemma 5.5. *For the map*

$$P_L \circ \tilde{Q} : L \to L,$$

we have the following:

(i) $P_L \circ \tilde{Q}$ *is self-adjoint.*
(ii) $P_L \circ \tilde{Q}$ *is non-positive definite.*
(iii) $\sup_{A\in L, ||A||_{eu}=1} -\langle P_L\circ\tilde{Q}(A), A\rangle \le 16\sqrt{\frac{3}{\pi}}$.

Proof. (i). Let $A, B \in L$. Then $P_L(A) = A$ and $P_L(B) = B$. Since P_L is self-adjoint, we find

$$\begin{aligned}\langle P_L\circ\tilde{Q}(A), B\rangle &= \langle\tilde{Q}(A), P_L(B)\rangle \\ &= \langle\tilde{Q}(A), B\rangle = \langle A, \tilde{Q}(B)\rangle \\ &= \langle P_L(A), \tilde{Q}(B)\rangle \\ &= \langle A, P_L\circ\tilde{Q}(B)\rangle.\end{aligned}$$

Thus, $P_L \circ \tilde{Q}$ is self-adjoint.

(ii). For all $A \in L$,

$$\langle P_L\circ\tilde{Q}(A), A\rangle \quad = \quad \langle\tilde{Q}(A), P_L(A)\rangle$$

$$= \langle \tilde{Q}(A), A \rangle \leq 0,$$

where we apply the non-positivity of $\tilde{Q}$ in the last inequality. Thus, $P_L \circ \tilde{Q}$ is non-positive definite.

(iii). For all $A \in L$ with $||A||_{eu} = 1$,

$$\begin{aligned} -\langle P_L \circ \tilde{Q}(A), A \rangle &= -\langle \tilde{Q}(A), P_L(A) \rangle \\ &= -\langle \tilde{Q}(A), A \rangle \leq 16\sqrt{\frac{3}{\pi}}, \end{aligned}$$

where we apply Theorem 1.3 for the last inequality. □

We now prove the main theorem in this section:

Proof of Theorem 5.1. Assume that $\tilde{Q}$ is a compact operator. Then the well-known Spectral Theorem for compact operators [**Sch12**, Theorem A.3] guarantees that there exists a sequence of eigenvalues $\{\sigma_j\}_{j\geq 1}$ of $\tilde{Q}$ with $\sigma_j \leq \sigma_{j+1}$ and $\sigma_j \to 0$ as $j \to \infty$.

By the Cauchy Interlacing Theorem ([**Sch12**, Proposition 12.5]), we have

$$\sigma_i \leq \lambda_i, \ \forall 1 \leq i \leq n,$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 0$ are the eigenvalues of $P_L \circ \tilde{Q}$.

Since $\{A_i\}_{n \leq i \leq (2n-1)}$ is orthonormal, the trace of $P_L \circ \tilde{Q}$ is

$$Trace(P_L \circ \tilde{Q}) = \sum_{n \leq i \leq (2n-1)} \tilde{Q}(A_i, A_i) = \sum_{1 \leq i \leq n} \lambda_i.$$

By the part (iii) of Lemma 5.5, we have:

$$\lambda_1 \geq -16\sqrt{\frac{3}{\pi}}.$$

By Proposition 5.4,

$$\begin{aligned} -2^{-30} \cdot n &\geq \sum_{n \leq i \leq (2n-1)} \tilde{Q}(A_i, A_i) = \sum_{1 \leq i \leq n} \lambda_i \\ &= \sum_{i=1}^{[\sqrt{n}]-1} \lambda_i + \sum_{[\sqrt{n}]}^{n} \lambda_i \\ &\geq ([\sqrt{n}] - 1)\lambda_1 + (n - [\sqrt{n}] + 1)\lambda_{[\sqrt{n}]} \\ &\geq ([\sqrt{n}] - 1)(-16\sqrt{\frac{3}{\pi}}) + (n - [\sqrt{n}] + 1)\lambda_{[\sqrt{n}]}. \end{aligned}$$

Divided by n for the inequality above and for large $n >> 1$, we have

$$\lambda_{[\sqrt{n}]} \leq -2^{-31}.$$

Since λ_i is increasing,

$$\lambda_j \leq -2^{-31},\ \forall\ 1 \leq j \leq [\sqrt{n}].$$

Therefore,

$$\sigma_j \leq -2^{-31},\ \forall\ 1 \leq j \leq [\sqrt{n}].$$

Let $n \to \infty$, this contradicts with the fact that $\sigma_j \to 0$ as $j \to \infty$. ☐

6. Twisted Harmonic map into $T_H(1)$

Harmonic maps theory is an important topic in geometry and analysis. In this section, we consider harmonic maps into the universal Teichmüller space with the Weil-Petersson metric. With newly obtained curvature information about $T_H(1)$, we study harmonic maps into $T_H(1)$ and prove some rigidity results. We follow a similar argument as the proof of Theorem 1.2 in [**Wu14**].

Proof of Theorem 1.6. Since the curvature operator on $T_H(1)$ is non-positive definite, $T_H(1)$ also has non-positive Riemannian sectional curvature in the complexified sense as stated in [**MSY93**]. Suppose that f is not constant. From [**MSY93**, Theorem 2] (one may also see [**Cor92**]), we know that f is a totally geodesic immersion. We remark here that the target space in [**MSY93**, Theorem 2] is stated to be a finite dimensional complex manifold. Actually, the proof goes through in the case that the target space has infinite dimension, without modification. Similar arguments are applied in [**Duc15**, Section 5].

On quaternionic hyperbolic manifolds $H_{Q,m} = Sp(m,1)/Sp(m)$, since f is totally geodesic, we identify the image $f(H_{Q,m} = Sp(m,1)/Sp(m))$ with $H_{Q,m} = Sp(m,1)/Sp(m)$. We may select $p \in H_{Q,m}$. Choose a quaternionic line l_Q on $T_pH_{Q,m}$, and we may assume that l_Q is spanned over $\mathbb{R}$ by v, Iv, Jv and Kv. Without loss of generality, we may assume that J on $l_Q \subset T_pH_{Q,m}$ is the same as the complex structure on $T_H(1)$.

Let $Q^{H_{Q,m}}$ be the curvature operator on $H_{Q,m}$, and we choose an element

$$v \wedge Jv + Kv \wedge Iv \in \wedge^2 T_pH_{Q,m}.$$

Then we have

$$Q^{H_{Q,m}}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) =$$
$$R^{H_{Q,m}}(v, Jv, v, Jv) + R^{H_{Q,m}}(Kv, Iv, Kv, Iv) + 2 \cdot R^{H_{Q,m}}(v, Jv, Kv, Iv).$$

Since I is an isometry, we have

$$\begin{aligned} R^{H_{Q,m}}(Kv, Iv, Kv, Iv) &= R^{H_{Q,m}}(IKv, IIv, IKv, IIv) \\ &= R^{H_{Q,m}}(-Jv, -v, -Jv, -v) \\ &= R^{H_{Q,m}}(v, Jv, v, Jv). \end{aligned}$$

Similarly,

$$\begin{aligned} R^{H_{Q,m}}(v,Jv,Kv,Iv) &= R^{H_{Q,m}}(v,Jv,IKv,IIv) \\ &= R^{H_{Q,m}}(v,Jv,-Jv,-v) \\ &= -R^{H_{Q,m}}(v,Jv,v,Jv). \end{aligned}$$

Combining the terms above, we have

$$Q^{H_{Q,m}}(v\wedge Jv+Kv\wedge Iv, v\wedge Jv+Kv\wedge Iv)=0.$$

Since f is a totally geodesic immersion,

$$Q^{T_H(1)}(v\wedge Jv+Kv\wedge Iv, v\wedge Jv+Kv\wedge Iv)=0.$$

On the other hand, by Theorem 1.1, there exists $E\in\wedge^2 TT_H(1)$ such that

$$v\wedge Jv+Kv\wedge Iv=E-\mathbf{J}\circ E.$$

Hence,

$$\begin{aligned} \mathbf{J}\circ(v\wedge Jv+Kv\wedge Iv) &= \mathbf{J}\circ(E-\mathbf{J}\circ E) \\ &= \mathbf{J}\circ E-\mathbf{J}\circ\mathbf{J}\circ E \\ &= \mathbf{J}\circ E-E \\ &= -(v\wedge Jv+Kv\wedge Iv). \end{aligned} \tag{6.1}$$

On the other hand, since $\mathbf{J}$ is the same as J in $H_{Q,m}$, we also have

$$\begin{aligned} \mathbf{J}\circ(v\wedge Jv+Kv\wedge Iv) &= (Jv\wedge JJv+JKv\wedge JIv) \\ = \quad & Jv\wedge(-v)+Iv\wedge(-Kv)=v\wedge Jv+Kv\wedge Iv. \end{aligned} \tag{6.2}$$

From equations (6.1) and (6.2) we get

$$v\wedge Jv+Kv\wedge Iv=0$$

which is a contradiction since l_Q is spanned over $\mathbb{R}$ by v, Iv, Jv and Kv.

In the case of the Cayley hyperbolic plane $H_{O,2}=F_4^{20}/SO(9)$, the argument is similar by replacing a quaternionic line by a Cayley line [**Cha72**]. □

Remark 6.1. *Since $T_H(1)$ has negative sectional curvature, any symmetric space of rank ≥ 2 can not be totally geodesically immersed in $T_H(1)$. The argument in the proof of Theorem 1.6 shows that the rank one symmetric spaces $Sp(m,1)$ and F_4^{-20} also can not be totally geodesically immersed in $T_H(1)$. It would be interesting to study whether the remaining two non-compact rank one symmetric spaces $\mathbb{H}^n$ and $\mathbb{CH}^n$ can be totally geodesically immersed in $T_H(1)$ (or $Teich(S)$).*

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(Z. H.) Department of Mathematics, The City University of New York, Staten Island, NY 10314, USA

The Graduate Center, The City University of New York, 365 Fifth Ave., New York, NY 10016, USA

E-mail address: `zheng.huang@csi.cuny.edu`

(Y. W.) Yau Mathematical Sciences Center, Tsinghua University, Haidian District, Beijing 100084, China

E-mail address: `yunhui_wu@mail.tsinghua.edu.cn`