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Translation Lengths of Parabolic Isometries of CAT(0) Spaces and Their Applications

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Abstract In this article, we provide a sufficient and necessary condition on parabolic isometries of positive translation lengths on complete visibility CAT(0) spaces. One of the consequences is that each parabolic isometry of a complete simply connected visibility manifold of nonpositive sectional curvature has zero translation length. Applications on the geometry of open negatively curved manifolds will also be discussed.

Keywords CAT(0)space · Parabolic isometry · Translation length · Nonpositively curved manifold

Mathematics Subject Classification 53C20 · 53C23

1 Introduction

A CAT(0) space is a geodesic metric space whose geodesic triangles are "slimmer" than the corresponding flat triangles in the plane $\mathbb{R}^2$. Typical examples are complete simply connected manifolds of nonpositive sectional curvature, which are proper. Trees, one-dimensional connected graphs without loops, are also CAT(0) spaces. A CAT(0) space may be not proper, i.e., certain closed geodesic ball of finite radius may be not compact. Indeed, a locally infinite tree is the simplest no-proper CAT(0) space. The first part of this paper will focus on certain isometries of complete CAT(0) spaces which is not required to be proper.

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Let γ be an isometry of a complete CAT(0) space M. The *translation length* $|\gamma|$ of γ is defined by

$$|\gamma| := \inf_{p \in M} \operatorname{dist}(\gamma \circ p, p).$$

The isometry γ is said to be *parabolic* if $|\gamma|$ is not achieved in M. In this article, we will focus on parabolic isometries. Let us look at the following two examples. Let $\mathbb{H}^2$ be the upper half plane endowed with the hyperbolic metric and define $\gamma : \mathbb{H}^2 \to \mathbb{H}^2$ to be $\gamma \circ (x, y) = (x + 1, y)$. Let c:[0,1]$\to$ $\mathbb{H}^2$ be the curve with $c(t) = (x + t, y)$. Since $\operatorname{dist}((x, y), \gamma \circ (x, y))$ is less than the length of $c([0, 1])$ that is $\frac{1}{y}$, which goes to zero as y goes to infinity. Thus, γ is parabolic and $|\gamma| = 0$. Similarly, consider the space $\mathbb{R} \times \mathbb{H}^2$ endowed with the product metric, and define $\gamma : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{R} \times \mathbb{H}^2$ to be $\gamma \circ (z, (x, y)) = (z + 1, (x + 1, y))$. Then it is easy to see that γ is parabolic and $|\gamma| = 1$. So parabolic isometries with positive translation lengths may occur in CAT(0) spaces.

A visibility CAT(0) space, whose manifold case was introduced by Eberlein and O'Neill in [12], needs the space to be more curved. In some sense, it means that for any two different points at "infinity," they can be viewed from each other along the space. One can see Definition 2.2 for the precise description. Classical examples for visibility CAT(0) spaces include trees and complete simply connected Riemannian manifolds of uniformly negative sectional curvatures. It is clear that a visibility CAT(0) space cannot contain any totally geodesic flat half plane.

1.1 Translation Lengths of Parabolic Isometries

Bishop and O'Neill [6] proved that *any parabolic isometry of a complete simply connected manifold M of uniformly negative sectional curvature has zero translation length*. One can also see [17] for an alternative proof using the geometry on the horosphere. Buyalo [9] proved that *any parabolic isometry of a complete Gromov hyperbolic CAT(0) space has zero translation length*. It is known that a complete Gromov hyperbolic CAT(0) space is a visibility CAT(0) space (one may see Proposition 2.8 for details). It is natural to ask

Question 1.1 *Does every parabolic isometry of a complete visibility CAT(0) space have zero translation length?*

Before this paper, even for the manifold case the answer to the question above is unknown. Unfortunately, in general the answer to Question 1.1 is negative. We will give a counterexample to this question in Sect. 2. Motivated from this counterexample, by applying a theorem of Karlsson and Margulis in [19] we obtain the following result which characterizes parabolic isometries of positive translation lengths on complete visibility CAT(0) spaces.

Theorem 1.2 *Let M be a complete visibility CAT(0) space. Then a parabolic isometry γ of M satisfies $|\gamma| > 0$ if and only if there exists a γ-invariant infinite-flat-strip $U \times \mathbb{R}$, which is a closed convex subset of M, such that γ acts on $U \times \mathbb{R}$ as*

$$\gamma \circ (x, t) = (\gamma_1 \circ x, t + t_0), \quad \forall (x, t) \in U \times \mathbb{R},$$

where γ_1 *is a parabolic isometry of* U *with* $|\gamma_1| = 0$ *and* $0 \neq t_0 \in \mathbb{R}$.

Recall that a metric space is called *proper* if every closed ball of finite radius is compact. Since a complete proper visibility CAT(0) space does not contain infinite-flat-strips (see Part (1) of Proposition 2.9), Theorem 1.2 implies

Theorem 1.3 *Let* M *be a complete proper visibility CAT(0) space. Then, for any parabolic isometry* γ *of* M*, we have*

$$|\gamma| = 0.$$

Remark 1.1 We call a manifold M tame if M is the interior of some compact manifold $\overline{M}$ with boundary. Phan conjectured in [20] that *let* M *be a tame, finite volume, negatively curved manifold; then* M *is not visible if the fundamental group of* M *contains a parabolic isometry of* $\tilde{M}$ *with positive translation length*. Theorem 1.3 implies this conjecture.

1.2 Negatively Curved Manifolds Without Visibility

First we call a complete manifold of nonpositive sectional curvature a *visibility manifold* if its universal cover is a visibility CAT(0) manifold. In the first paragraph of page 438 of [11], Eberlein conjectured that *a complete open manifold* M *with sectional curvature* $-1 \leq K_M \leq 0$ *and finite volume is a visibility manifold if the universal cover* $\tilde{M}$ *of* M *contains no imbedded flat half planes*. For dimension 2, the result in [10] tells that this conjecture is true. Based on the result of Abresch and Schroeder in [1], Buyalo in [8] showed that certain 4-dimensional manifold M with sectional curvature $-1 \leq K_M < 0$ and finite volume has an end which is not incompressible. And this example is known to experts for the first counterexample to Eberlein's conjecture. In this paper, we will use Theorem 1.3 as a bridge to show that the negatively curved manifolds, constructed by Fujiwara in [15], are also counterexamples to Eberlein's conjecture. More precisely,

Theorem 1.4 *For the manifolds of finite volumes with sectional curvatures in* $[-1, 0)$*, constructed in [15], their fundamental groups contain parabolic isometries of positive translation lengths. In particular, they are not visibility manifolds.*

Recently, Phan in [20] also proved that Fujiwara's example M is not visible by using a different method.

1.3 Negatively Curved Manifolds with Zero Axioms

In [12], Eberlein and O'Neill introduced the zero axiom which says that there does not exist a gap between asymptotic rays and strongly asymptotic rays (see the definition in

Sect. 5), which holds on complete simply connected manifolds of uniformly negative curvatures.

The structure of manifolds with pinched negative sectional curvature and finite volume is well studied by Eberlein and Schroeder in [11,22]. In Sect. 5, we will prove the following result using Theorem 1.3 and the arguments in [11,22].

Theorem 1.5 *Let M be a complete n-dimensional visibility manifold satisfying the zero axiom, the sectional curvature $-1 \leq K_M \leq 0$, and the volume $\mathrm{Vol}(M) < \infty$. Let $\tilde{M}(\infty)$ be the ideal boundary of the universal cover $\tilde{M}$ of M. Let $\pi_1(M)$ be the fundamental group of M which acts on $\tilde{M}$ by isometries. Then,*

(1) for any $x \in \tilde{M}(\infty)$, Γ_x is almost nilpotent;
(2) if Γ_x contains a parabolic isometry of $\tilde{M}$, then the rank of Γ_x is $n-1$;
(3) the maximal almost nilpotent subgroups of $\pi_1(M)$ are precisely the nonidentity stability groups Γ_x, $x \in \tilde{M}(\infty)$,

where $\Gamma_x := \{\alpha \in \pi_1(M) : \alpha(x) = x\}$.

In [13], Farb conjectures that *the moduli space $\mathbb{M}_{S_g}$ of closed surface S_g $(g \geq 2)$ admits no complete, finite volume Riemannian metric with sectional curvature* $-1 \leq K(\mathbb{M}6_{S_g}) \leq 0$. The last result in this article is the following which says that the Farb conjecture is true if we assume that $g \geq 3$ and the universal cover satisfies the zero axiom. More precisely,

Theorem 1.6 *The moduli space $\mathbb{M}_{S_g}$ of closed surface S_g $(g \geq 3)$ admits no complete Riemannian metric with sectional curvature $-1 \leq K(\mathbb{M}_{S_g}) \leq 0$ such that the universal cover $\mathrm{Teich}(S_g)$ of $\mathbb{M}_{S_g}$ satisfies the zero axiom.*

There is no finite volume condition in Theorem 1.6.

For the geometry and topology of open Riemannian manifold of nonpositive sectional curvature, one can refer to the recent nice survey of Belegradek [2] for more details.

1.4 Plan of the Paper

In Sect. 2, we will demonstrate some necessary backgrounds and prove some basic properties on CAT(0) spaces, which will be applied in subsequent sections. Section 3 will establish Theorem 1.2. Theorem 1.4 is proved in Sect. 4. In Sect. 5, we will prove Theorems 1.5 and 1.6. In the last section, we will provide an acknowledgment.

2 Notations and Preliminaries

2.1 CAT(0) Spaces

A CAT(0) space is a geodesic metric space in which each geodesic triangle is no fatter than a triangle in the Euclidean plane with the same edge lengths. More precisely,

Definition 2.1 Let M be a geodesic metric space. For any $a, b, c \in M$, three geodesics $[a, b]$, $[b, c]$, $[c, a]$ form a geodesic triangle Δ. Let $\overline{\Delta}(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be a triangle in the Euclidean plane with the same edge lengths as Δ. Let p, q be points on $[a, b]$ and $[a, c]$, and let $\overline{p}, \overline{q}$ be points on $[\overline{a}, \overline{b}]$ and $[\overline{a}, \overline{c}]$, respectively, such that $\mathrm{dist}_M(a, p) = \mathrm{dist}_{\mathbb{R}^2}(\overline{a}, \overline{p})$, $\mathrm{dist}_M(a, q) = \mathrm{dist}_{\mathbb{R}^2}(\overline{a}, \overline{q})$. We call M a **CAT(0)** space if for all Δ the inequality $\mathrm{dist}_M(p, q) \leq \mathrm{dist}_{\mathbb{R}^2}(\overline{p}, \overline{q})$ holds.

Let M be a complete CAT(0) space. The ideal boundary, denoted by $M(\infty)$, consists of asymptotic rays. For each point $p \in M$ and $x \in M(\infty)$, since the distance function between geodesics is convex, there exists a unique geodesic ray c which represents x and starts from p. We write $c(+\infty) = x$. It is clear that flat planes are CAT(0) spaces. The following definition will exclude CAT(0) spaces with flat sectors.

Definition 2.2 A complete CAT(0) space M is called a *visibility CAT(0) space* if, for any $x \neq y \in M(\infty)$, there exists a geodesic line $c: (-\infty, +\infty) \to M$ such that $c(-\infty) = x$ and $c(+\infty) = y$.

Although a complete CAT(0) space may be singular, the definition of CAT(0) spaces can guarantee that the notation of the angle, like the smooth case, still makes sense (see [5]). Given two points x, y in $M(\infty)$, let $\angle_p(x, y)$ denote the angle at p between the unique geodesic rays which issue from p and lie in the classes x and y, respectively. The angular metric is defined to be $\angle(x, y) := \sup_{p \in M} \angle_p(x, y)$. Then, $\angle(x, y) = 0$ if and only if $x = y$. On a complete visibility CAT(0) space M, for any $x \neq y \in M(\infty)$, $\angle(x, y) = \pi$. So the angular metric gives a discrete topology on the ideal boundary of a complete visibility CAT(0) space.

The following lemma will be used in the next section, which gives us a way to compute the angular metric.

Lemma 2.3 (Proposition 9.8 of chapter II.9 in [5]) *Let M be a complete CAT(0) space with a basepoint p. Let $x, y \in M(\infty)$ and c, c' be two geodesic rays with $c(0) = c'(0) = p$, $c(+\infty) = x$, and $c'(+\infty) = y$. Then*

$$2 \sin\left(\frac{\angle(x, y)}{2}\right) = \lim_{t \to +\infty} \frac{\mathrm{dist}(c(t), c'(t))}{t}.$$

2.2 Product

Let X_1 and X_2 be two metric spaces. The product $X = X_1 \times X_2$ has a natural metric which is called the product metric. Let γ_i be an isometry of X_i $(i = 1, 2)$. It is obvious that $\gamma = (\gamma_1, \gamma_2)$ is an isometry of X under the natural action. The following lemma tells when the converse is true.

Lemma 2.4 (Proposition 5.3 of chapter I.5 in [5]) *Let $X = X_1 \times X_2$. Then an isometry γ on X decomposes as (γ_1, γ_2), with γ_i be an isometry of X_i $(i = 1, 2)$, if and only if, for every $x_1 \in X_1$, there exists a point denoted $\gamma_1 \circ x_1 \in X_1$ such that $\gamma \circ (\{x_1\} \times X_2) = \{\gamma_1 \circ x_1\} \times X_2$.*

The following product decomposition theorem will be applied several times in this article.

Proposition 2.5 (Theorem 2.14 of chapter II.2 in [5]) *Assume that M is a complete CAT(0) space. Let $c : \mathbb{R} \to M$ be a geodesic line and P_c be the set of geodesic lines which are parallel to c. Then P_c is isometric to the product $P'_c \times \mathbb{R}$, where P'_c is a closed convex subset in M.*

2.3 Isometries on CAT(0) Spaces

Let M be a complete CAT(0) space. An isometry γ of M is a map $\gamma : M \to M$ which satisfies $\text{dist}(\gamma \circ p, \gamma \circ q) = \text{dist}(p, q)$, for all $p, q \in M$. The set of isometries on a metric space is a group. An isometry can be classified as elliptic, hyperbolic, or parabolic. An isometry is called *elliptic* if it has a fixed point in M. The classical Cartan Fixed Point Theorem (see [4,5]) says that an isometry γ of a complete CAT(0) space is elliptic provided that γ has a bounded orbit. An isometry γ is called *hyperbolic* if there exists a geodesic line $c : (-\infty, +\infty) \to M$ such that γ acts on the line $c(\mathbb{R})$ by a non-trivial translation. Each non-trivial element in the fundamental group of a closed nonpositively curved Riemannian manifold acts on its universal cover as a hyperbolic isometry. If an isometry is neither elliptic nor hyperbolic, then we call it to be *parabolic*. Recall that in the introduction we also say that an isometry γ is parabolic if the translation length $|\gamma|$ is not obtained in M. Actually, these two definitions are equivalent. One may see [5] or [4] for more details.

The following lemma is well known which gives us a new viewpoint for the translation length. One can refer to [4] for the proof.

Lemma 2.6 *Let M be a complete CAT(0) space and γ be a parabolic isometry of M. Then*

(1) $|\gamma| = \lim_{n\to+\infty} \frac{\text{dist}(\gamma^n \circ p, p)}{n}, \ \forall p \in M;$
(2) $|\gamma^2| = 2 \cdot |\gamma|.$

Part(2) follows directly from Part(1).

We recall a theorem of Karlsson and Margulis which is crucial for this article. Use the same notations as in [19]. We set

$$X = \{one \ \ point\} = \{x\}, \ \ L = identity\ map, \ \ \omega(x) = \gamma$$

and

$$D = M,$$

where M is a complete CAT(0) space and γ is a parabolic isometry of M. Then $u(n, x)$ in Eq. (2.1) in [19] is equal to $\gamma^n \circ x$. The Lemma above tells that A in Theorem 2.1 of [19] is equal to $|\gamma|$. Then the following result is a special case of Theorem 2.1 in [19].

Theorem 2.7 (Karlsson–Margulis) *Let M be a complete CAT(0) space with a base point $p \in M$ and γ be a parabolic isometry with $|\gamma| > 0$. Then there exist a unique $x_0 \in Fix(\gamma)$ and a geodesic ray $c : \mathbb{R}^{\geq 0} \to M$ such that $c(0) = p$, $c(+\infty) = x_0$, and*

$$\lim_{n\to+\infty} \frac{\text{dist}(\gamma^n \circ p, c(|\gamma| \cdot n))}{n} = 0.$$

2.4 Visibility CAT(0) Spaces Containing Infinite-Flat-Strips

Let us firstly look at the following two examples, which are motivated by Example 8.28 in [5]. We are grateful to Tushar Das for the discussions on these examples.

Example 1 We are going to construct a complete unbounded CAT(0) space M such that $M(\infty)$ is empty and there exists a parabolic isometry on M.

Let H be the Hilbert space $l^2(\mathbb{Z}) := \{(x_n);\ \sum_{-\infty}^{\infty} x_n^2 < \infty\}$, σ be the right shift map of H, and σ^{-1} be the left shift map of H, that is $\sigma(x)$ is a sequence whose n-th entry is x_{n+1}, where $x = (x_n) \in H$. Let $\delta \in H$ be the point whose only non-zero entry is $\delta_0 = 1$. Define $\gamma : H \to H$ by

$$\gamma(x) = \sigma(x) + \delta.$$

On page 276 of [5], it is shown that γ is a parabolic isometry which does not fix any point of the visual boundary $H(\infty)$ of H.

Let 0 denote the point whose entries are all zeros. We consider the orbit $\{\gamma^n(0)\}_{n\in\mathbb{Z}}$. A direct computation implies that if $n \geq 0$,

$$\gamma^n(0) = \sum_{i=0}^{n} \sigma^i(\delta). \tag{1}$$

So $\gamma^n(0)_i = 1$ for all $0 \leq i \leq n$, and otherwise $\gamma^n(0)_i = 0$.

Similarly, if $n < 0$,

$$\gamma^n(0) = -\sum_{i=1}^{-n} (\sigma^{-1})^i(\delta). \tag{2}$$

So $\gamma^n(0)_i = -1$ for all $n \leq i < 0$, and otherwise $\gamma^n(0)_i = 0$.

Let M be the closed convex hull $ch(\{\gamma^n(0)\}_{n\in\mathbb{Z}})$ of the orbit $\{\gamma^n(0)\}_{n\in\mathbb{Z}}$. It is clear that M is a complete unbounded CAT(0) space and γ acts on M as a parabolic isometry. Indeed, the isometry γ satisfies

$$|\gamma| = 0$$

which cannot be achieved in M.

Claim: *the visual boundary $M(\infty)$ is empty.*

Proof of Claim Set

$$A := \{(x_n);\ \ -1 \leq x_m \leq 0\ \ for\ \ m < 0,\ \ 0 \leq x_n \leq 1\ \ for\ \ n \geq 0\}.$$

From Eqs. (1) and (2), we know that

$$ch(\{\gamma^n(0)\}_{n\in\mathbb{Z}}) \subset A.$$

If $M(\infty)$ is not empty, let $z \in M(\infty)$ which can be represented by a geodesic ray $c : [0, \infty) \to M$ such that $c(0) = 0, c(\infty) = z$. Therefore, $c(t) = t \cdot x$ for some non-zero $x = (x_n) \in M$ which is a contradiction since $t \cdot x \notin A$ when t is large enough. □

Example 2 (Counterexample to Question 1.1) Let M be the CAT(0) space in example 1. Consider the product space

$$N := M \times \mathbb{R}$$

which is endowed with the product metric.

Since the visual boundary of N consists of two points which can be joined by a geodesic line, N is a complete visibility CAT(0) space. Consider the isometry $\gamma_0 : N \to N$ which is defined as $\gamma_0 \circ (m, t) = (\gamma \circ m, t + 1)$. A direct computation implies that γ_0 is a parabolic isometry on N with $|\gamma_0| = \sqrt{|\gamma|^2 + 1^2} = 1 > 0$.

Let M be a complete CAT(0) space. We call M has *an infinite-flat-strip* if there exists a totally geodesic convex subset $U \times \mathbb{R} \subseteq M$, where U is unbounded. From Proposition 2.5, we can always assume that U is closed convex. Example 2 tells that a complete visibility CAT(0) space may contain an infinite-flat-strip.

Recall that a metric space M is called *Gromov hyperbolic* if there exists a $\delta > 0$ such that every geodesic triangle is δ-thin, where a *δ-thin geodesic triangle* means that each of its sides is contained in the δ-neighborhood of the union of the other two sides. An $\mathbb{R}$-tree is a Gromov hyperbolic space which holds for any $\delta > 0$. For more details one can see [5,16]. The following result will be used later. One may see Proposition 1.4 in chapter III.H of [5] for the proper case.

Proposition 2.8 (Proposition 10.1 in [9]) *Every complete Gromov hyperbolic CAT(0) space is a visibility CAT(0) space.*

We enclose the section by the following result, which tells that a lot of standard CAT(0) spaces have no infinite-flat-strips.

Proposition 2.9 *(1) A complete proper visibility CAT(0) space does not contain any infinite-flat-strips.*
(2) A complete Gromov hyperbolic CAT(0) space does not contain any infinite-flat-strips.

Proof Proof of Part (1). If not. Then we can assume that M is a complete proper visibility CAT(0) space which contains an infinite-flat-strip $U \times \mathbb{R}$. By the definition of infinite-flat-strips, we know that U is unbounded. Let $c : (-\infty, +\infty) \to M$ be the geodesic line $x_0 \times \mathbb{R}$ where $x_0 \in U$. And let P_c be the set of geodesic lines which are parallel to c. By Proposition 2.5, P_c is isometric to the product $P'_c \times \mathbb{R}$, where P'_c is a closed convex subset in M. For any $u \in U$, the line $u \times \mathbb{R} \subset U \times \mathbb{R}$ is parallel to $c(\mathbb{R})$. Thus, $U \subseteq P'_c$. Since U is unbounded, P'_c is also unbounded. Thus, P'_c a complete proper unbounded geodesic space. The Arzelà Ascoli theorem would guarantee that there exists a geodesic ray $d : [0, +\infty) \to P'_c$. Hence, M contains a flat half plane $[0, +\infty) \times \mathbb{R}$ in M which is impossible because M is a visibility space.

Proof of Part (2). Assume not. Then we can assume that M is a complete Gromov hyperbolic CAT(0) space which contains an infinite-flat-strip $U \times \mathbb{R}$ where U is unbounded. Let $\delta > 0$ be the number such that every geodesic triangle in M is δ-thin. Let $c : (-\infty, +\infty) \to M$ be the geodesic line $x_0 \times \mathbb{R}$ where $x_0 \in U$. And let P_c be the set of geodesic lines which are parallel to c. By Proposition 2.5, P_c is isometric to the product $P'_c \times \mathbb{R}$, where P'_c is a closed convex subset in M. Hence $U \subseteq P'_c$. Since U is unbounded, we can find a flat strip $[0, k] \times \mathbb{R}$ with width k, where k is an arbitrary positive number. If we choose k to be large enough, then we can find a geodesic triangle Δ in $[0, k] \times \mathbb{R}$ such that Δ is not δ-thin, which is a contradiction. □

3 Proofs of Theorem 1.2

Before proving Theorem 1.2, let us control the size of the fixed points of parabolic isometries.

Proposition 3.1 *Let M be a complete CAT(0) space and γ be an isometry on M. If $|\gamma| > 0$, then*

$$\#\{x \in M(\infty) : \gamma \circ x = x\} \geq 2.$$

Proof Since $|\gamma| > 0$, γ is either hyperbolic or parabolic.

If γ is hyperbolic, let $c : (-\infty, +\infty) \to M$ be an axis for γ. The conclusion follows from the fact that $\{c(-\infty), c(+\infty)\}$ belongs to $\{x \in M(\infty) : \gamma \circ x = x\}$.

If γ is parabolic. Since $|\gamma| > 0$, Theorem 2.7 tells that there exists a unique fixed point $x \in M(\infty)$ such that for every $p \in M$ and every geodesic ray c:$[0,+\infty) \to M$ with $c(0) = p$ and $c(+\infty) = x$ we have

$$\lim_{n \to +\infty} \frac{\operatorname{dist}(\gamma^n \circ p, c(n|\gamma|))}{n} = 0.$$

Since $|\gamma^{-1}| = |\gamma| > 0$, by Theorem 2.7, again we know that γ^{-1} has a unique fixed point $y \in M(\infty)$ such that for every $p \in M$ and every geodesic ray $c' : [0, +\infty) \to M$

with $c'(0) = p$ and $c'(+\infty) = y$ we have

$$\lim_{n\to+\infty} \frac{\text{dist}(\gamma^{-n} \circ p, c'(n|\gamma|))}{n} = 0.$$

By the triangle inequality,

$$\lim_{n\to+\infty} \frac{\text{dist}(c(n|\gamma|), c'(n|\gamma|))}{n} = \lim_{n\to+\infty} \frac{\text{dist}(\gamma^{n} \circ p, \gamma^{-n} \circ p)}{n}.$$

Since γ is an isometry, by Lemmas 2.3 and 2.6, we have

$$2|\gamma| \sin(\frac{\angle(c(+\infty), c'(+\infty))}{2}) = |\gamma^2|.$$

From Lemma 2.6, we know that

$$2|\gamma| \sin(\frac{\angle(c(+\infty), c'(+\infty))}{2}) = 2|\gamma|.$$

Since $|\gamma| \neq 0$, $\angle(c(+\infty), c'(+\infty)) = \pi \neq 0$. That is $\angle(x, y) \neq 0$. Since $x, y \in Fix(\gamma)$, we have

$$\#\{x \in M(\infty) : \gamma \circ x = x\} \geq 2.$$

□

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 "⇐". Since γ acts on $U \times \mathbb{R}$ as

$$\gamma \circ (x, t) = (\gamma_1 \circ x, t + t_0), \quad \forall (x, t) \in U \times \mathbb{R},$$

the translation length $|\gamma|$ of γ satisfies that

$$|\gamma| = \sqrt{|\gamma_1|^2 + t_0^2}.$$

Thus, $|\gamma| = |t_0|$ because $|\gamma_1| = 0$. Since $|\gamma_1|$ is a parabolic isometry of U, the translation length $|\gamma|$ can not be obtained in $U \times \mathbb{R}$. That is, γ is parabolic on $U \times \mathbb{R}$. Since $U \times \mathbb{R}$ is totally geodesic in M, Part (4) of Proposition 6.2 on page 229 of [5] tells that γ is a parabolic isometry of M. It is clear that $|\gamma|_{U\times R} = |t_0| > 0$. By Part (4) of Proposition 6.2 on page 229 of [5] again we have that

$$|\gamma| = |t_0| > 0.$$

" ⇒". Since $|\gamma| > 0$, from Proposition 3.1 we know that

$$\#\{x \in M(\infty) : \gamma \circ x = x\} \geq 2.$$

Let $x \neq y \in Fix(\gamma)$. Since M is a visibility CAT(0) space, there exists a geodesic line $c : \mathbb{R} \to M$ such that $c(-\infty) = x$ and $c(+\infty) = y$. Let P_c be the set of geodesic lines which are parallel to c. By Proposition 2.5, we know that P_c is isometric to the product $P'_c \times \mathbb{R}$, where $P'_c \times \{0\}$, still denoted by P'_c, is a closed convex subset in M. We let $U = P'_c$.

Since $c(-\infty), c(+\infty) \in Fix(\gamma)$, $\gamma \circ (c(\mathbb{R}))$ is also geodesic line which is parallel to $c(\mathbb{R})$. In particular, $P_c = U \times \mathbb{R}$ is a γ-invariant subset in M. From Lemma 2.4, we know that γ splits as (γ_1, γ_2), where γ_1 is an isometry on U and γ_2 is an isometry on $\mathbb{R}$. Since $U \times \{0\}$, still denoted by U, is closed convex in M, the subspace U is also a complete CAT(0) space. First we claim that P_c *is an infinite-flat-strip.*

Proof of Claim If U is bounded, the classical Cartan Fixed Point Theorem gives that there exists $x_1 \in U$ such that $\gamma_1 \circ x_1 = x_1$. Since γ_2 acts on $\mathbb{R}$ by isometry, it is either elliptic or hyperbolic.

Case 1 γ_2 *is elliptic.*

There exists $x_2 \in \mathbb{R}$ such that $\gamma_2 \circ x_2 = x_2$. In particular, $\gamma = (\gamma_1, \gamma_2)$ fixes the point (x_1, x_2), which means that γ is elliptic, which contradicts the assumption that γ is parabolic.

Case 2 γ_2 *is hyperbolic.*

Since $\gamma_1 \circ x_1 = x_1$, $\gamma = (\gamma_1, \gamma_2)$ acts on the line $x_1 \times \mathbb{R}$ as a translation. From the definition of hyperbolic isometries, $x_1 \times \mathbb{R}$ is an axis for γ. In particular, γ is hyperbolic, which also contradicts our assumption that γ is parabolic.

It remains to show that γ_1 *is parabolic with* $|\gamma_1| = 0$ *and* $|\gamma_2| = |\gamma| > 0$.

First we show that $|\gamma_1| = 0$. If not, that is $|\gamma_1| > 0$. From Proposition 3.1, we know that

$$\#\{x \in U(\infty) : \gamma_1 \circ x = x\} \geq 2.$$

In particular $U(\infty)$ is not empty. That is, there exists a geodesic ray $c : [0, \infty) \to U$. Thus, M contains a flat half plane $c([0, \infty)) \times \mathbb{R}$ which is a contradiction since M is a visibility CAT(0) space.

The conclusion $|\gamma_2| = |\gamma| > 0$ follows from $|\gamma| = \sqrt{|\gamma_1|^2 + |\gamma_2|^2}$ and $|\gamma_1| = 0$.

Since γ is a parabolic isometry of M and γ_2 is a hyperbolic isometry of $\mathbb{R}$, the isometry γ_1 is parabolic on U. □

4 Proof of Theorem 1.4

First let us recall Fujiwara's example. We use the same notations as in [15].

Let V be a 3-dimensional closed hyperbolic manifold and S be a simple closed geodesic in V with length $a > 0$. Let $\sigma > 0$ be small enough. Then the σ-neighborhood $N_\sigma(S)$ of S is $S \times S^1 \times [0, \sigma)$ where the subset $S \times S^1 \times \{0\}$ degenerates into the core closed geodesic S. We introduce polar coordinates (ω, θ, r) on $N_\sigma(S)$. The hyperbolic metric of V on a σ-neighborhood $N_\sigma(S)$ of V is given by

$$g_V = \cosh^2(r)d\omega^2 + \sinh^2(r)d\theta^2 + dr^2 \quad (0 \leq \theta \leq 2\pi,\ \ 0 \leq r \leq \sigma).$$

Let $M = V - S$ and g be the metric on M as follows; the metric $g|_{M-N_\sigma(S)}$ is the hyperbolic metric and restricted on $N_\sigma(S) - S$:

$$g = \cosh^2(r)d\omega^2 + \sinh^2(r)d\theta^2 + f^2(r)dr^2 \quad (0 \le \theta \le 2\pi,\ \ 0 \le r \le \sigma),$$

where $f(r)$ is a bump function which satisfies that $f|_{[\frac{\sigma}{2},\sigma]} \equiv 1$, $f|_{[0,\frac{\sigma}{4}]} = \frac{1}{\sqrt{r}\sinh r}$, and so on (one can see the Lemma in [15] for more details). It is shown in [15] that (M, g) has finite volume and sectional curvature $-1 \le K_M < 0$.

Proof of Theorem 1.4 First, we prove the result when the dimension of M is 3.

From the definition of g, we know that for any fixed positive number $c_0 \in (0, 2\pi)$, the surface $\theta = c_0$, corresponding to the submanifold $S \times \{c_0\} \times (0, \sigma)$ in M, is totally geodesic. Indeed, the surface $\theta = c_0$ is the set of fixed points of an isometric reflection.

The metric g, restricted to $\theta = c_0$, is

$$g_{\theta=c_0} = \cosh^2(r)d\omega^2 + f^2(r)dr^2 \quad (0 \le r \le \sigma).$$

We denote $M|_{\theta=c_0}$ by $S \times (0, \sigma)$. The universal cover of $S \times (0, \sigma)$ is $\mathbb{R} \times (0, \sigma)$. Let ϕ be the generator of the fundamental group of $S \times (0, \sigma)$. Since the length of S is a, it is not hard to see that, for all $(\omega, r) \in \mathbb{R} \times (0, \sigma)$, we have

$$\phi \circ (\omega, r) = (\omega + a, r) \ \ (0 \le r \le \sigma).$$

Claim: *ϕ is a parabolic isometry with positive translation length.*

Proof of Claim First, we consider the curve $c(t) : [0, 1] \to \mathbb{R} \times (0, \sigma)$ defined by $c(t) = (\omega + t \cdot a, r)$. Then we have

$$\begin{aligned}|\phi| \le \ell(c([0, 1])) &= \int_0^1 \sqrt{\cosh^2(c_2(t)) \cdot c_1'(t)^2} dt \\ &= \cosh(r) \cdot \int_0^1 |c_1'(t)| dt = a \cdot cosh(r).\end{aligned}$$

Since r is arbitrary, letting $r \to 0$ we get $|\phi| \le a$.

Secondly, let $c(t) = (c_1(t), c_2(t)) : [0, 1] \to \mathbb{R} \times (0, \sigma)$ be any smooth curve joining (ω, r) and $(\omega + a, r)$, so that in particular $c_1(0) = \omega$ and $c_1(1) = \omega + a$. The length of $c([0, 1])$ is

$$\begin{aligned}\ell(c([0, 1])) &= \int_0^1 \sqrt{\cosh^2(c_2(t)) \cdot c_1'(t)^2 + f^2(c_2(t)) \cdot c_2'(t)^2} dt \\ &\ge \int_0^1 |\cosh(c_2(t)) \cdot c_1'(t)| dt > \int_0^1 |c_1'(t)| dt \\ &\ge (c_1(1) - c_1(0)) = a > 0.\end{aligned}$$

Since $c(t)$ is arbitrary, $|\phi| \ge a > 0$. Hence $|\phi| = a > 0$. $|\phi|$ cannot be attained in $\mathbb{R} \times (0, \sigma)$ since $\ell(c([0, 1])) > a$ for any curve joining (ω, r) and $(\omega + a, r)$, so ϕ is parabolic.

Hence, ϕ restricted to $\mathbb{R} \times (0, \sigma)$ is a parabolic isometry with positive translation length. Since $\theta = c_0$ is totally geodesic in M, $\mathbb{R} \times (0, \sigma)$ is totally geodesic in the universal covering of M. So ϕ is also a parabolic isometry with positive translation length in the universal covering of M.

From Theorem 1.3, we know that M is not a visibility manifold.

For the case that the dimension of M is greater than 3, from the construction in [15] the closed geodesic S in the argument above is replaced by a totally geodesic closed manifold W of codimension 2 in M. Recall that W is also hyperbolic. Let γ be the shortest closed geodesic in W. We lift $W \times (0, \sigma)$ onto its universal covering space and let $\mathbb{R}$ be the geodesic line which projects into the closed geodesic γ. Then applying a similar argument to $\mathbb{R} \times (0, \sigma)$ as in the 3 dimension case, one can show that the deck transformation w.r.t. γ is a parabolic isometry of the universal cover $\tilde{M}$ of M, which has positive translation length. Then by Theorem 1.3 we know that M is not a visibility manifold. □

Theorem 1.4 tells that the fundamental group of a negatively curved Riemannian manifold with finite volume may contain parabolic isometries of positive translation lengths if the dimension of the manifold is greater than or equal to 3. However, the following result tells that parabolic isometries with positive translation lengths do not exist in the fundamental group of nonpositively curved surface with finite volume. More precisely,

Theorem 4.1 *Let M be a complete two-dimensional Riemannian manifold with non-positive Gauss curvature. If the fundamental group $\pi_1(M)$ of M contains a parabolic isometry ϕ with translation length $|\phi| > 0$, then we have*

$$\mathrm{Vol}(M) = \infty.$$

Proof Since $\pi_1(M)$ contains a parabolic isometry, M is non-compact. Suppose that $\mathrm{Vol}(M) < \infty$. Then M is not flat since there does not exist a non-compact flat surface of finite area. By Corollary 3.2 of [11], the universal cover $\tilde{M}$ of M is a visibility CAT(0) manifold. Since ϕ is parabolic, by Theorem 1.3 we know that $|\phi| = 0$, which is a contradiction. □

Example 3 Consider the upper half plane H^2 endowed with a metric $ds^2 := (dx^2 + dy^2) + \frac{dx^2+dy^2}{y^2}$. Since ds^2 is the sum of one complete metric and another metric, (H^2, ds^2) is complete. The curvature formula tells that the sectional curvature of (H^2, ds^2) at (x, y) is given by

$$K(x, y) = -\frac{1}{2(1+\frac{1}{y^2})} \times \frac{\partial}{\partial y}\left(\frac{-\frac{2}{y^3}}{1+\frac{1}{y^2}}\right) = \frac{-(1+3y^2)}{(1+y^2)^3}.$$

The formula above clearly implies that the sectional curvature satisfies $-1 \leq K_{H^2} < 0$.

Let $\phi : H^2 \to H^2$ defined by $(x, y) \mapsto (x+1, y)$. A direct computation implies that ϕ is a parabolic isometry with $|\phi| = 1 > 0$. By Theorem 1.3, we know that (H^2, ds^2) is a complete 2-dimensional negatively curved surface which is not a visibility manifold. Moreover, Theorem 4.1 tells us that (H^2, ds^2) cannot cover any surface of finite volume.

5 Zero Axiom

In [12], Eberlein and O'Neill first introduced the so-called zero axiom. Recall that M satisfies the *zero axiom* if for any two rays $r : [0, +\infty) \to M$ and $\sigma : [0, +\infty) \to M$ with $r(+\infty) = \sigma(+\infty)$ in $M(\infty)$ we have

$$\lim_{t \to +\infty} \text{dist}(r(t), \sigma(\mathbb{R}^{\geq 0})) = 0.$$

A typical example of CAT(0) manifold satisfying the zero axiom is a complete simply connected Riemannian manifold whose sectional curvature is uniformly negative.

Proposition 5.1 *Let M be a complete CAT(0) space satisfying the zero axiom, γ be an infinite ordered isometry of M, and $Fix(\gamma)$ be the subset in $M(\infty)$ fixed by γ (i.e., $\gamma(x) = x,\ \forall x \in Fix(\gamma)$). Then, for any geodesic ray $r : [0, +\infty] \to M$ with $r(+\infty) \in Fix(\gamma)$, we have*

$$\lim_{t \to +\infty} \text{dist}(\gamma \circ r(t), r(t)) = |\gamma|.$$

Proof Let $\{p_i\}_{i \geq 1}$ be a sequence in M such that $\lim_{i \to +\infty} \text{dist}(\gamma \circ p_i, p_i) = |\gamma|$ and $r_i : [0, +\infty) \to M$ be a sequence of rays in M with $r_i(0) = p_i$ and $r_i(+\infty) = r(+\infty)$. Since M satisfies the zero axiom, for each i there exist $t_i, s_i > 0$ such that

$$\text{dist}(r_i(s_i), r(t_i)) < \frac{1}{i}. \tag{3}$$

By the triangle inequality, we have

$$\text{dist}(\gamma \circ r(t_i), r(t_i)) \leq \text{dist}(\gamma \circ r_i(s_i), r_i(s_i)) + 2 \times \text{dist}(r_i(s_i), r(t_i)). \tag{4}$$

Since $r_i(+\infty) = r(+\infty) \in Fix(\gamma)$ and the distance function between two rays in M is convex (see [5]), $\text{dist}(\gamma \circ r_i(t), r_i(t))$ is a decreasing function. In particular, we have

$$\text{dist}(\gamma \circ r_i(s_i), r_i(s_i)) \leq \text{dist}(\gamma \circ r_i(0), r_i(0)) = \text{dist}(\gamma \circ p_i, p_i). \tag{5}$$

Equations (3), (4), and (5) lead to

$$\text{dist}(\gamma \circ r(t_i), r(t_i)) \leq \text{dist}(\gamma \circ p_i, p_i) + \frac{2}{i}.$$

Taking the limit,

$$\lim_{i\to+\infty} \operatorname{dist}(\gamma \circ r(t_i), r(t_i)) \leq |\gamma|.$$

From the definition, we also know that

$$\lim_{i\to+\infty} \operatorname{dist}(\gamma \circ r(t_i), r(t_i)) \geq |\gamma|.$$

Hence,

$$\lim_{i\to+\infty} \operatorname{dist}(\gamma \circ r(t_i), r(t_i)) = |\gamma|.$$

Since $r(+\infty) \in Fix(\gamma)$, $\operatorname{dist}(\gamma \circ r(t), r(t))$ is decreasing. Therefore,

$$\lim_{t\to+\infty} \operatorname{dist}(\gamma \circ r(t), r(t)) = \lim_{i\to+\infty} \operatorname{dist}(\gamma \circ r(t_i), r(t_i)) = |\gamma|.$$

□

Let M be a complete open n-dimensional visibility manifold satisfying the sectional curvature $-1 \leq K_M \leq 0$ and the volume $\operatorname{Vol}(M) < \infty$. In [11], Eberlein showed that each end is a so-called Riemannian collared, which is the cross product of a compact quotient space of a horosphere and a ray. The compact quotient is said to be a *cross section*. The fundamental group of a cross section is a finitely generated subgroup of the fundamental group of M, which consisting of parabolic isometries fixing a common boundary point. If the curvature of M is pinched by two negative numbers, the Gromov–Margulis Lemma can be applied to show that this subgroup is almost nilpotent. For the case that M is a visibility manifold and satisfies the zero axiom, Theorem 1.3 and Proposition 5.1 can guarantee that the argument of Eberlein above still works in our setting, which is sufficient to prove Theorem 1.5. We use the same notations as the ones in [11,22].

Proof of Theorem 1.5 Proof of Part (1). If Γ_x contains a semisimple isometry of $\tilde{M}$, then it is easy to see that Γ_x is an infinite cyclic group (see [12]).

If Γ_x contains only parabolic elements, by Lemma 3.3 in [11], there exists $\phi \in \pi_1(M, p)$ such that $x = \phi(x_i)$ (see the definition of x_i in Lemma 3.3 in [11]). From Lemma 3.1g in [11], we know that Γ_{x_i} is finitely generated. The fact that Γ_x has a finite generating set follows from $\Gamma_x = \phi\Gamma_{x_i}\phi^{-1}$. Assume that $\Gamma_x =< \psi_1, \dots, \psi_k >$. Since Γ_x contains only parabolic elements and M is a visibility manifold, from Theorem 1.3 we know that the translation length of ψ_i $|\psi_i| = 0$ for all $1 \leq i \leq k$. Now let $r : [0, +\infty) \to \tilde{M}$ be a geodesic ray in $\tilde{M}$ with $r(+\infty) = x$. Proposition 5.1 tells us that $\lim_{t\to+\infty} \operatorname{dist}(\psi_i \circ r(t), r(t)) = 0$ for each $1 \leq i \leq k$. Let $\epsilon(n)$ be the Margulis constant for M (see [4]). For all i and suitable large t, we have $\operatorname{dist}(\psi_i \circ r(t), r(t)) < \epsilon(n)$. Then the conclusion that $\Gamma_x =< \psi_1, \dots, \psi_k >$ is almost nilpotent follows from the Gromov–Margulis Lemma.

Proof of Part (2). Let N be the nilpotent subgroup of Γ_x of finite index. It is sufficient to show that the rank of N is $n-1$. Since M has nonpositive sectional curvature, $\pi_1(M, p)$ is torsion-free. From Part (1), we know that N is a finitely generated torsion-free nilpotent group. By a theorem of Malcev (see Theorem II. 2.18 in [21]), we have that N is isomorphic to a lattice of a simply connected nilpotent Lie group whose dimension is the same as the rank of N. By Lemma 3.1g in [11], we know that N operates on a horosphere of $\tilde{M}$ which is homeomorphic to $\mathbb{R}^{n-1}$ with compact quotient. The conclusion that the rank of N is $n-1$ follows from the fact that any simply connected nilpotent Lie group of rank d is homeomorphic to $\mathbb{R}^d$.

Proof of Part (3). Let N' be a maximal almost nilpotent subgroup of $\pi_1(M, p)$. By Lemma 3.1b in [11], there exists a point $z \in \tilde{M}(\infty)$ such that $N' \subset \Gamma_z$. From Part (1), we know that Γ_z is almost nilpotent. Since N' is a maximal almost nilpotent subgroup, $N' = \Gamma_z$. □

Before proving Theorem 1.6, let us make some preparations. Let S_g be a hyperbolic surface with genus g. It is well known that the completion $\overline{\mathrm{Teich}(S_g)}$ of the Teichmüller space, endowed with the Weil–Petersson metric, is a CAT(0) space, and $\mathrm{Mod}(S_g)$ acts on $\overline{\mathrm{Teich}(S)}$ by isometries. The Dehn twists here behave as elliptic isometries whose fixed points are products of lower-dimensional Teichmüller spaces. Actually, Bridson in [7] proved the following more general result using Theorem 2.7 (Karlsson–Margulis).

Theorem 5.2 (Bridson) *Whenever* $\mathrm{Mod}(S_g)$ $(g \geq 3)$ *acts by isometries on a complete CAT(0) space M, then each Dehn twist* $\tau \in \mathrm{Mod}(S_g)$ *has* $|\tau| = 0$.

A group G acting on a metric space X is said to act *properly discontinuously* if for each compact subset $K \subset X$, the set $K \cap gK$ is nonempty for only finitely many g in G. The following corollary is a direct result of Theorem 5.2.

Corollary 5.3 *Whenever* $\mathrm{Mod}(S_g)$ $(g \geq 3)$ *acts properly discontinuously on a complete CAT(0) space M by isometries, each Dehn twist* $\tau \in \mathrm{Mod}(S_g)$ *acts as a parabolic isometry with* $|\tau| = 0$.

Proof If not, by Theorem 5.2 τ is elliptic, so τ has a fixed point $x_0 \in M$ which contradicts the assumption that the action is properly discontinuous, since every Dehn twist has infinite order. □

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6 Let σ be a non-separate simple closed curve. Since $g \geq 3$, we can find two intersecting simple closed curves $\sigma_1, \sigma_2 \subset (S_g - \sigma)$ such that the group generated by the two Dehn twists τ_{σ_1} and τ_{σ_2} is a free group of rank ≥ 2 (see [14]).

Let τ_σ be the Dehn twist on σ in $\mathrm{Mod}(S_g)$. We define the centralizer $N(\tau_\sigma)$ of τ_σ in the following way:

$$N(\tau_\sigma) := \{\alpha \in \mathrm{Mod}(S_g) : \alpha \circ \tau_\sigma = \tau_\sigma \circ \alpha\}.$$

Thus, $< \tau_{\sigma_1}, \tau_{\sigma_2} > \subset N(\tau_\sigma)$ since $\sigma_1, \sigma_2 \subset (S_g - \sigma)$.

We argue it by getting a contradiction. Assume that Teich(S_g) admits a complete Mod(S_g)-invariant Riemannian metric ds^2 such that $-1 \leq K_{(\text{Teich}(S_g), ds^2)} \leq 0$ and (Teich$(S_g), ds^2$) satisfies the zero axiom. Then first by Corollary 5.3, the Dehn twist τ_σ would act as a parabolic isometry on the (Teich$(S_g), ds^2$). By Lemma 7.3 on page 87 of [4], we know that there exists a point $x \in \text{Teich}(S_g)(\infty)$ such that $N(\tau_\sigma)$ fixes x, that is for any $\alpha \in N(\tau_\sigma)$, $\alpha(x) = x$. In particular, $< \tau_{\sigma_1}, \tau_{\sigma_2} >$ fixes x since $< \tau_{\sigma_1}, \tau_{\sigma_2} > \subset N(\tau_\sigma)$.

Let $r : [0, +\infty) \to (\text{Teich}(S_g), ds^2)$ be a geodesic ray in M with $r(+\infty) = x$. Since $g \geq 3$, by Corollary 5.3 we know that the translation length of any Dehn twist is zero. Since (Teich$(S_g), ds^2$) satisfies the zero axiom, by Proposition 5.1 we have $\lim_{t\to+\infty} \text{dist}(\tau_{\sigma_1} \circ r(t), r(t)) = \lim_{t\to+\infty} \text{dist}(\tau_{\sigma_2} \circ r(t), r(t)) = 0$. Hence, for any $\epsilon > 0$ we can find t_0 such that

$$\text{dist}(\tau_{\sigma_1} \circ r(t_0), r(t_0)) < \epsilon, \qquad \text{dist}(\tau_{\sigma_2} \circ r(t_0), r(t_0)) < \epsilon.$$

Choose ϵ small enough so that ϵ is smaller than the Margulis constant for (Teich$(S_g), ds^2$). After applying the Gromov–Margulis Lemma (see [4]) at the point $r(t_0)$, we have that the group $\langle \tau_{\sigma_1}, \tau_{\sigma_2} \rangle$ is a finitely generated subgroup of an almost nilpotent group. Thus, the group $\langle \tau_{\sigma_1}, \tau_{\sigma_2} \rangle$ is also almost nilpotent, which contradicts the fact that $\langle \tau_{\sigma_1}, \tau_{\sigma_2} \rangle$ is a free group of rank ≥ 2. □

Remark 5.1 Since Theorem 5.2 of Bridson is also true for $g \geq 3$ and $n \geq 0$, Theorem 1.6 also holds for the moduli space $\mathbb{M}_{g,n}$ of surface of genus g with n punctures if $g \geq 3$. In [3,18], it was proved that the mapping class group Mod$(S_{g,n})$ cannot act properly discontinuously on any complete simply connected Riemannian manifold with pinched negative sectional curvature when $3g + n \geq 5$. Since a complete simply connected Riemannian manifold, whose sectional curvature is bounded above by a negative number, satisfies the zero axiom, Theorem 1.6 generalizes these results for the cases $g \geq 3$. One may also see [23] for related results.

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