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Entropy bounded solutions to the one-dimensional compressible Navier-Stokes equations with zero heat conduction and far field vacuum

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A B S T R A C T

The entropy is one of the fundamental states of a fluid and, in the viscous case, the equation that it satisfies is highly singular in the region close to the vacuum. In spite of its importance in the gas dynamics, the mathematical analyses on the behavior of the entropy near the vacuum region, were rarely carried out; in particular, in the presence of vacuum, either at the far field or at some isolated interior points, it was unknown whether the entropy remains its boundedness. The results obtained in this paper indicate that the ideal gases retain their uniform boundedness of the entropy, locally or globally in time, if the vacuum occurs at the far field only and the density decays slowly enough at the far field. Precisely, we consider the Cauchy problem to the one-dimensional full compressible Navier-Stokes equations without heat conduction, and establish the local and global existence and uniqueness of entropy-bounded solutions, in the presence of vacuum at the far field only. It is also shown that, different from the case that with compactly supported initial density, the compressible Navier-Stokes equations, with

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slowly decaying initial density, can propagate the regularities in inhomogeneous Sobolev spaces.

1. Introduction

1.1. The compressible Navier-Stokes equations in Euler coordinates

Let ρ, u, and θ, respectively, be the density, velocity, and temperature of a fluid. Denote by t and x the time and spatial variables. Then, the full compressible Navier-Stokes equations read as

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) = \operatorname{div}\mathbb{T} + \rho f, \tag{1.2}$$

$$\partial_t(\rho E) + \operatorname{div}(\rho u E) + \operatorname{div} q = \operatorname{div}(\mathbb{T}u) + \rho\mathcal{Q}, \tag{1.3}$$

where $E = \frac{|u|^2}{2} + e$ is the specific total energy, $e = e(\rho,\theta)$ is the specific internal energy, $\mathbb{T}$ is the stress tensor, q is the internal energy flux directly related to the transfer of heat, f is the external force, and $\mathcal{Q}$ is the external heat source.

By (1.1)–(1.3), one can obtain the following equation for e:

$$\partial_t(\rho e) + \operatorname{div}(\rho u e) + p\operatorname{div} u + \operatorname{div} q = \mathbb{S} : \nabla u + \rho\mathcal{Q}. \tag{1.4}$$

The stress tensor $\mathbb{T}$ is given by

$$\mathbb{T} = \mathbb{S} - pI, \quad \mathbb{S} = 2\mu\mathbb{D}u + \lambda\operatorname{div} u I, \quad \mathbb{D}u = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

where I is the 3×3 identity matrix, p is the pressure, and μ and λ are viscosity coefficients, satisfying $\mu > 0$ and $2\mu + 3\lambda \geq 0$. In this paper, we consider the ideal gases, and state equations are

$$p = R\rho\theta, \quad e = c_v\theta,$$

for two positive constants R and c_v. Then, it follows from (1.4) that

$$c_v[\partial_t(\rho\theta) + \operatorname{div}(\rho u\theta)] + p\operatorname{div} u + \operatorname{div} q = \mathbb{S} : \nabla u + \rho\mathcal{Q}. \tag{1.5}$$

Recalling the state equations for p and e, by the Gibbs equation $\theta Ds = De + pD(\frac{1}{\rho})$, where s is the specific entropy, one has the following relationship between p and s:

$$p = Ae^{\frac{s}{c_v}}\rho^\gamma,$$

for some positive constant A, where $\gamma - 1 = \frac{R}{c_v}$. Since $R, c_v > 0$, it is clear that $\gamma > 1$. Thanks to this, and using the state equations for p and e again, one can derive from (1.1), (1.2), and (1.5) the following equation for the entropy s:

$$\partial_t(\rho s) + \operatorname{div}(\rho s u) + \operatorname{div}\left(\frac{q}{\theta}\right) = \frac{1}{\theta}\left(\mathbb{S} : \nabla u - \frac{q \cdot \nabla\theta}{\theta}\right) + \rho\frac{\mathcal{Q}}{\theta}.$$

For the internal energy flux q, by the Fourier's law of heat conduction, we assume that $q = -\kappa\nabla\theta$, where $\kappa \geq 0$ is the heat conduction coefficient.

There is an extensive literature on the mathematical analysis of the compressible Navier-Stokes equations. In the absence of vacuum, that is the density is bounded from below by some positive constant, the local well-posedness results were proved by Nash [33], Itaya [16], Vol'pert–Hudjaev [39], Tani [36], Valli [37], and Lukaszewicz [28]. The first global well-posedness result was established by Kazhikhov–Shelukhin [21], where they proved the global well-posedness of strong solutions of the initial boundary value problem to the one-dimensional compressible Navier-Stokes equations, for arbitrary H^1 initial data, and the corresponding result for the Cauchy problem was later proved by Kazhikhov [20]; global well-posedness of weak solutions to the one-dimensional compressible Navier-Stokes equations was proved by Zlotnik–Amosov [43,44] and by Chen–Hoff–Trivisa [1] for the initial boundary value problems, and by Jiang–Zlotnik [19] for the Cauchy problem. Large time behavior of solutions to the one dimensional compressible Navier-Stokes equations with large initial data was recently proved by Li–Liang [24]. For the multi-dimensional case, the global well-posedness of strong solutions were established only for small perturbed initial data around some non-vacuum equilibrium or for spherically symmetric large initial data, see Matsumura–Nishida [29–32], Ponce [34], Valli–Zajaczkowski [38], Deckelnick [8], Jiang [17], Hoff [12], Kobayashi–Shibata [22], Danchin [7], Chen–Miao–Zhang [2], and Chikami–Danchin [3]. One of the major differences between one dimensional case from the multi-dimensional one is that if no vacuum is contained initially, then no vacuum will form later on in finite time, for the one dimensional compressible Navier-Stokes equations, as shown by Hoff-Smoller [13], while the similar result remains unknown for the multi-dimensional case.

In the presence of vacuum, that is the density may vanish on some set, or tends to zero at the far field, the breakthrough was made by Lions [26,27], where he proved the global existence of weak solutions to the isentropic compressible Navier-Stokes equations, with adiabatic constant $\gamma \geq \frac{9}{5}$; the requirement on γ was later relaxed by Feireisl–Novotný–Petzeltová [9] to $\gamma > \frac{3}{2}$, and further by Jiang–Zhang [18] to $\gamma > 1$ but only for the axisymmetric solutions. For the full compressible Navier-Stokes equations, global existence of the variational weak solutions was proved by Feireisl [10,11]; however, due to the assumptions on the constitutive equations made in [10,11], the ideal gases were not included there. Local well-posedness of strong solutions, in the presence of vacuum, was proved first for the isentropic case by Salvi–Straškraba [35], Cho–Choe–Kim [4], and

Cho–Kim [5], and later for the polytropic case by Cho–Kim [6]. It should be noticed that, in [35,4–6], the solutions were established in the homogeneous Sobolev spaces, that is, it is $\sqrt{\rho}u$ rather than u itself that has the $L^\infty(0,T;L^2)$ regularity. Generally, one can not expect that the strong solutions to the compressible Navier-Stokes equations lie in the inhomogeneous Sobolev spaces, if the initial density has compact support. Actually, it was proved recently by Li–Wang–Xin [23] that: neither isentropic nor the full compressible Navier-Stokes equations on $\mathbb{R}$, with $\kappa = 0$ for the full case, has any solution (ρ, u, θ) in the inhomogeneous Sobolev spaces $C^1([0,T]; H^m(\mathbb{R}))$, with $m > 2$, if ρ_0 is compactly supported and some appropriate conditions on the initial data are satisfied; the N-dimensional full compressible Navier-Stokes equations, with positive heat conduction, have no solution (ρ, u, θ), with finite entropy, in the inhomogeneous Sobolev spaces $C^1([0,T]; H^m(\mathbb{R}^N))$, with $m > [\frac{N}{2}] + 2$, if ρ_0 is compactly supported. Global existence of strong and classical solutions to the compressible Navier-Stokes equations, in the presence of initial vacuum, was first proved by Huang–Li–Xin [15], where they established the global well-posedness of strong and classical solutions, with small initial basic energy, to the three-dimensional isentropic compressible Navier-Stokes equations, see Li–Xin [25] for further developments. However, due to the finite in time blow-up results by Xin [41] and Xin–Yan [42], one can not expect the global well-posedness of classical solutions, in either inhomogeneous or homogeneous Sobolev spaces, to the full compressible Navier-Stokes equations in the presence of vacuum. In particular, it was proved in [42] that, for the full compressible Navier-Stokes equations, if initially there is an isolated mass group surrounded by the vacuum region, then for the case $\kappa = 0$, any classical solution must blow-up in finite time, and for the case $\kappa > 0$, any classical solutions, with finite entropy in the vacuum region, must blow-up in finite time. Global existence of strong solutions to the heat conducting full compressible Navier-Stokes equations were obtained by Huang–Li [14] for the case that with non-vacuum far field, and by Wen–Zhu [40] for the case that with vacuum far field. The spaces of the solutions obtained in [14,40] can not exclude the possibility that the entropy is infinite somewhere in the vacuum region, even if it is initially finite; in fact, due to the results in [42], the corresponding entropy in [14,40] must be infinite somewhere in the vacuum region, if initially there is an isolated mass group surrounded by the vacuum region.

By the state equations for the ideal gases, the entropy can be expressed in terms of the density and temperature as

$$s = c_v \left(\log \frac{R}{A} + \log \theta - (\gamma - 1) \log \rho \right),$$

from which it follows that the entropy may develop singularities or even is not well defined in the vacuum region and, consequently, it is impossible to obtain the desired regularities of s merely from those of θ and ρ, in the presence of vacuum. Therefore, though the vacuums are allowed for the solutions established in [6,14,40] by choosing (ρ, u, θ) as the unknowns, no regularities of the entropy s can be implied in the vacuum region there and, due to the result in [42], the entropy of the solutions obtained in

[14,40] must be infinite in the vacuum region. To the best of our knowledge, in the existing literatures, there were no such results that provided the uniform lower or upper bounds of the entropy near the vacuum.

As stated in the previous paragraph, since the entropy can not be even defined at the places where the density vanishes, it may be unreasonable to study the entropy for the full compressible Navier-Stokes equations of the ideal gases, if the vacuum region is an open set; however, when the vacuum occurs only at some isolated interior points or at the far field and if, moreover, the entropy behaves well when the fluid tends to these vacuum points or to the far field, it is still possible to define the entropy there. Therefore, a natural question is what kind of behavior of the entropy, at the vacuum far field or near the isolated interior vacuum points, can be preserved by the ideal gases, when the flow evolves. The aim of this paper is to give some answers to this question and, in particular, as indicated in our main results, the ideal gases can preserve their boundedness of the entropy, locally or globally in time, if the vacuum happens at the far field only.

Another question that we want to address in this paper is: under what kind of assumptions on the initial density, beyond the case that the initial density is uniformly away from the vacuum, the compressible Navier-Stokes equations admit solutions in the inhomogeneous Sobolev spaces. On one hand, recalling the result in [23], for the case that the initial density has a compact support, the compressible Navier-Stokes equations are ill-posed in the inhomogeneous Sobolev spaces; on the other hand, for the case that the initial density is uniformly away from the vacuum, the compressible Navier-Stokes equations are well-posed in the inhomogeneous Sobolev spaces. Comparing these two cases, and regarding the case that the density has compact support as that the density has supper fast decay at the far field, one may ask *whether the fast decay of the density can cause the ill-posedness of the compressible Navier-Stokes equations in the inhomogeneous spaces, or whether the compressible Navier-Stokes equations will be well-posed in the inhomogeneous spaces when the initial density decays slowly at the far field.* We will show in this paper that if the initial density decays slower than $\frac{K_0}{|x|^2}$, for some positive constant K_0, at the far field, then the compressible Navier-Stokes equations are indeed well-posed in the inhomogeneous Sobolev spaces, where K_0 is an arbitrary positive constant. Note that this is consistent with the well-posedness result for the compressible Navier-Stokes equations in the inhomogeneous Sobolev spaces in the absence of vacuum.

In this paper, we consider the one dimensional case, and assume that there are no external forces and heating source, i.e. $f \equiv \mathcal{Q} \equiv 0$, and that there is no heat conduction in the fluids, that is $\kappa = 0$, while the multi-dimensional case and the cases that with heat conduction will be studied in the further works. Under these assumptions, the system considered in this paper is the following one-dimensional compressible Navier-Stokes equations:

$$\rho_t + (\rho u)_x = 0, \tag{1.6}$$

$$\rho(u_t + uu_x) - \mu u_{xx} + p_x = 0, \tag{1.7}$$

$$c_v[(\rho\theta)_t + (\rho u\theta)_x] + pu_x = \mu(u_x)^2. \tag{1.8}$$

Due to $p = R\rho\theta$ and $c_v = \frac{R}{\gamma-1}$, equation (1.8) can be rewritten equivalently as a equation for the pressure p, that is

$$p_t + up_x + \gamma u_x p = \mu(\gamma - 1)(u_x)^2. \tag{1.9}$$

It is more convenient to use (1.9), instead of (1.8), to state and prove the results, in other words, we will use the pressure, instead of the temperature, as one of the unknowns, throughout this paper; however, it should be mentioned that, as we consider the case that the vacuum appears only at the far field, (1.9) is equivalent to (1.8).

The main results of this paper are stated and proved in the Lagrangian coordinates, see Section 1.2; however, since the solutions being established are Lipschitz continuous and the density vanishes only at the far fields, all results can be transformed back in the Euler coordinates accordingly.

1.2. Reformulation in Lagrangian coordinates and main results

Let y be the Lagrangian coordinate, and define the coordinate transform between y and the Euler coordinate x as $x = \eta(y, t)$, where

$$\begin{cases} \partial_t\eta(y,t) = u(\eta(y,t),t), \\ \eta(y,0) = y. \end{cases}$$

Denote

$$\varrho(y,t) := \rho(\eta(y,t),t), \quad v(y,t) := u(\eta(y,t),t), \quad \pi(y,t) := p(\eta(y,t),t),$$

and define

$$J(y,t) = \eta_y(y,t).$$

One can verify easily that

$$J_t = v_y, \quad J|_{t=0} = 1, \quad J\varrho = \varrho_0$$

with ϱ_0 being the initial value of the density. Then, system (1.6), (1.7), and (1.9) transforms to the following one in the Lagrangian coordinate:

$$J_t = v_y, \tag{1.10}$$

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J}\right)_y + \pi_y = 0, \tag{1.11}$$

$$\pi_t + \gamma \frac{v_y}{J}\pi = (\gamma - 1)\mu \left(\frac{v_y}{J}\right)^2, \tag{1.12}$$

where $\mu > 0$ and $\gamma > 1$ are constants.

We will consider the Cauchy problem and, thus, complement system (1.10)–(1.12) with the initial condition

$$(J, v, \pi)|_{t=0} = (J_0, v_0, \pi_0), \tag{1.13}$$

where J_0 has uniform positive lower and upper bounds. It should be pointed out that, by the definition of J, the initial J_0 should be identically one; however, for the aim of extending a local solution (J, v, π) to be a global one, we need the local existence of solutions to system (1.10)–(1.12), with initial J_0 not being identically one. Therefore, in this paper, in the study of the local solutions, the initial J_0 is allowed to be not identically one, but for the global solutions, we always assume that J_0 is identically one.

As will be shown later, the effective viscous flux G defined as

$$G := \mu \frac{v_y}{J} - \pi,$$

which was first introduced by D. Hoff, plays a crucial role in proving the global existence of solutions to system (1.10)–(1.12). By straightforward calculations, it follows from (1.11) and (1.12) that

$$G_t - \frac{\mu}{J}\left(\frac{G_y}{\varrho_0}\right)_y = -\gamma \frac{v_y}{J} G. \tag{1.14}$$

Before stating the main results, we first give some conventions on necessary notations to be used throughout this paper and define strong solutions. For $1 \leq q \leq \infty$, the Lebesgue space $L^q(\mathbb{R})$ consists of all measurable functions f on $\mathbb{R}$ with finite norm $\|f\|_{L^q}$, where

$$\|f\|_{L^q} = \begin{cases} \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty; \\ \operatorname{ess\,sup}_{\mathbb{R}} f, & \text{if } q = \infty. \end{cases}$$

For positive integer m and for $1 \leq q \leq \infty$, $W^{m,q} = W^{m,q}(\mathbb{R})$ is the Sobolev space consisting of all functions on $\mathbb{R}$ whose generalized derivatives up to order m belong to L^q. H^m stands for $W^{m,2}$. For simplicity, we also use the notations L^q and H^m to denote the N product spaces $(L^q)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_q$ to denote the L^q norm of u. For simplicity of presentation, we sometimes use $\|(f_1, f_2, \cdots, f_n)\|_X$ to denote the sum $\sum_{i=1}^N \|f_i\|_X$ or its equivalent norm $\left(\sum_{i=1}^N \|f_i\|_X^2\right)^{\frac{1}{2}}$.

Local and global strong solutions to the problem (1.10)–(1.13), are defined in the following two definitions.

Definition 1.1. Given a positive time $T \in (0, \infty)$. A triple (J, v, π) is called a strong solution to the problem (1.10)–(1.13), on $\mathbb{R} \times (0, T)$, if it has the properties

$$\inf_{y\in\mathbb{R}, t\in(0,T)} J(y,t) > 0, \quad \pi \geq 0 \text{ on } \mathbb{R} \times (0, T),$$

$$J - J_0 \in C([0,T]; L^2), \quad \frac{J_y}{\sqrt{\varrho_0}} \in L^\infty(0,T;L^2), \quad J_t \in L^\infty(0,T;L^2),$$

$$\sqrt{\varrho_0} v \in C([0,T]; L^2), \quad v_y \in L^\infty(0,T;L^2), \quad \left(\sqrt{\varrho_0} v_t, \frac{v_{yy}}{\sqrt{\varrho_0}}\right) \in L^2(0,T;L^2),$$

$$\pi \in C([0,T]; L^2), \quad \frac{\pi_y}{\sqrt{\varrho_0}} \in L^\infty(0,T;L^2), \quad \pi_t \in L^4(0,T;L^2),$$

satisfies equations (1.10)–(1.12), a.e. in $\mathbb{R} \times (0, T)$, and fulfills the initial condition (1.13).

Definition 1.2. A triple (J, v, π) is called a global strong solution to the problem (1.10)–(1.13), if it is a strong solution to the same system on $\mathbb{R} \times (0, T)$, for any positive time $T \in (0, \infty)$.

The main results of this paper are the following two theorems concerning the local and global existence of strong solutions to the problem (1.10)–(1.13).

Theorem 1.1 *(Local well-posedness). Let $\mu > 0$ and $\gamma > 1$ be constants. Assume that $(\varrho_0, J_0, v_0, \pi_0)$ satisfies*

$$\inf_{y\in(-r,r)} \varrho_0(y) > 0, \quad \forall r \in (0, \infty), \quad \varrho_0 \leq \bar{\varrho} \text{ on } \mathbb{R}, \tag{H1}$$

$$\left(\sqrt{\varrho_0} v_0, v_0', \pi_0, \frac{\pi_0'}{\sqrt{\varrho_0}}\right) \in L^2, \quad \pi_0 \geq 0 \text{ on } \mathbb{R}, \tag{H2}$$

$$\underline{J} \leq J_0 \leq \bar{J} \text{ on } \mathbb{R}, \quad \frac{J_0'}{\sqrt{\varrho_0}} \in L^2,$$

for positive constants $\bar{\varrho}$, $\underline{J}$, and $\bar{J}$, and denote $G_0 := \mu v_0' - \pi_0$.

The following two hold:

(i) *There is a positive time T depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|v_0'\|_2, \|\pi_0\|_2$, and $\|\pi_0\|_\infty$, such that system (1.10)–(1.12), subject to the initial condition (1.13), has a unique strong solution (J, v, π), on $\mathbb{R} \times (0, T)$.*

(ii) *Assume in addition that*

$$\left|\left(\frac{1}{\sqrt{\varrho_0}}\right)'(y)\right| \leq \frac{K_0}{2}, \quad \forall y \in \mathbb{R}, \quad \varrho_0^{-\frac{\delta}{2}} G_0 \in L^2, \tag{H3}$$

for two positive constants δ and K_0.

Then, (J, v, π) *has the additional regularities*

$$\varrho_0^{-\frac{\delta}{2}} G \in L^\infty(0,T;L^2) \cap L^4(0,T;L^\infty), \quad \varrho_0^{-\frac{\delta+1}{2}} G_y \in L^2(0,T;L^2), \tag{1.15}$$

where $G := \mu \frac{v_y}{J} - \pi$ *is the effective viscous flux, and*

$$\begin{cases} v \in L^\infty(0,T;H^1), & \text{if } v_0 \in H^1 \text{ and } \delta \geq 1, \\ \vartheta \in L^\infty(0,T;H^1), & \text{if } \vartheta_0 \in H^1, \dfrac{J_0'}{\varrho_0} \in L^2, \text{ and } \delta \geq 1, \\ s \in L^\infty(0,T;L^\infty), & \text{if } s_0 \in L^\infty \text{ and } \delta \geq \gamma, \end{cases} \tag{1.16}$$

where $\vartheta := \frac{\pi}{R\varrho}$ *and* $s := c_v \log\left(\frac{\pi}{A\varrho^\gamma}\right)$*, respectively, are the corresponding temperature and entropy, with* $\varrho := \frac{\varrho_0}{J}$ *being the density, and* $\vartheta_0 := \frac{\pi_0}{R\varrho_0}$ *and* $s_0 := c_v \log\left(\frac{\pi_0}{A\varrho_0^\gamma}\right)$*, respectively are the initial temperature and entropy.*

Remark 1.1. Basically, the condition $\left|\left(\frac{1}{\sqrt{\varrho_0}}\right)'\right| \leq \frac{K_0}{2}$ or equivalently $|\varrho_0'| \leq K_0 \varrho_0^{\frac{3}{2}}$ on $\mathbb{R}$ means that ϱ_0 decays no faster than $\frac{K}{y^2}$ at the far field. Indeed, if

$$\varrho_0(y) = \frac{K_\varrho}{\langle y \rangle^{\ell_\varrho}}, \quad 0 < K_\varrho < \infty, 0 \leq \ell_\varrho < \infty, \quad \text{where } \langle y \rangle = (1+y^2)^{\frac{1}{2}},$$

then

$$\left|\left(\frac{1}{\sqrt{\varrho_0}}\right)'\right| \leq \frac{K_0}{2} \text{ on } \mathbb{R} \quad \Longleftrightarrow \quad \ell_\varrho \leq 2.$$

Remark 1.2. Choose

$$\varrho_0(y) = \frac{K_\varrho}{\langle y \rangle^{\ell_\varrho}}, \quad J_0 \equiv 1, \quad v_0 \in C_c^\infty, \quad \pi_0 = A e^{\frac{1}{c_v}} \varrho_0^\gamma,$$

where K_ϱ and ℓ_ϱ are positive numbers.

(i) If $\frac{1}{2\gamma-1} < \ell_\varrho \leq 2$, then (ϱ_0, v_0, π_0) satisfies conditions (H1), (H2), and (H3), with $\delta = 1$. Therefore, by Theorem 1.1, there is a unique local strong solution (J, v, π), with v being in the inhomogeneous Sobolev space $L^\infty(0,T;H^1)$; if moreover that $\gamma > \frac{5}{4}$ and $\frac{1}{2(\gamma-1)} < \ell_\varrho \leq 2$, then $\vartheta_0 \in H^1$, and, consequently, the temperature ϑ also lies in the inhomogeneous Sobolev space $L^\infty(0,T;H^1)$. Note that this does not contradict to the ill-posedness results for the compressible Navier-Stokes equations in [23], as the initial density there is assumed to be compactly supported.

(ii) If $\frac{1}{\gamma} < \ell_\varrho \leq 2$, then (ϱ_0, v_0, π_0) satisfies conditions (H1), (H2), (H3), with $\delta = \gamma$, and $s_0 \equiv 1$. Therefore, by Theorem 1.1, there is a unique local strong solution

(J, v, π), and the corresponding entropy s is uniformly bounded on $\mathbb{R} \times (0, T)$. To the best of our knowledge, this is the first time that the boundedness of the entropy is achieved, in the presence of vacuum at the far field, for the compressible Navier-Stokes equations.

(iii) Combining (i) with (ii) shows that if $\gamma > \frac{5}{4}$ and $\max\left\{\frac{1}{\gamma}, \frac{1}{2(\gamma-1)}\right\} < \ell_\varrho \le 2$, there is a unique local strong solution (J, v, π), with the properties that the corresponding entropy is uniformly bounded, and the velocity and the corresponding temperature lie in the inhomogeneous space $L^\infty(0, T; H^1)$.

Remark 1.3.

(i) The assumption $\inf_{y\in(-R,R)} \varrho_0(y) > 0$, for all $R \in (0, \infty)$, is used for the boundedness of the entropy and the regularities of the velocity and temperature in the inhomogeneous Sobolev spaces, but it is not needed for the local well-posedness (in the homogeneous Sobolev spaces).

(ii) The compressible Navier-Stokes equations propagate the regularities in the homogeneous Sobolev spaces, see [4–6], but not in the inhomogeneous Sobolev spaces (in particular, the L^2 regularity of v can not be propagated), see [23], if the initial density has a compact support. While (ii) of Theorem 1.1 shows that, if the initial density decays slowly to the vacuum at the far field, then the regularities in the homogeneous Sobolev spaces, in particular, the L^2 regularity of v, can be also propagated by the compressible Navier-Stokes equations.

(iii) The result in (ii) of Theorem 1.1 also indicates that the uniform boundedness of the entropy can be propagated by the compressible Navier-Stokes equations, if the initial density decays slowly to the vacuum at the far field.

Remark 1.4. It follows from the definition of strong solutions that

$$\sqrt{\varrho_0} v \in L^\infty(0, T; L^2), \quad v_y \in L^2(0, T; H^1),$$

which implies $v \in L^2(0, T; W^{1,\infty}(\mathbb{R}))$. Define the Euler coordinate as

$$x = \eta(y, t), \quad \partial_t \eta(y, t) = v(y, t), \quad \eta(y, 0) = y.$$

Note that

$$\partial_t \eta_y = v_y = \partial_t J, \quad \eta_y(y, 0) = 1 = J(y, 0).$$

Thus, $\eta_y \equiv J$ on $\mathbb{R} \times (0, T)$. Since J has uniform positive lower and upper bounds on $\mathbb{R} \times (0, T)$, for any fixed $t \in (0, T)$, η is reversible in y. Therefore, one can define the density ρ, velocity u, and pressure p, in the Euler coordinate as

$$\rho(x, t) = \varrho(y, t), \quad u(x, t) = v(y, t), \quad p(x, t) = \pi(y, t),$$

where $\varrho(y,t) := \frac{\varrho_0(y)}{J(y,t)}$. It can be checked that (ρ, u, p) has appropriate regularities, in particular $u \in L^1(0,T; W^{1,\infty}(\mathbb{R}))$, and it is a solution to system (1.6), (1.7), and (1.9), subject to the initial data (ϱ_0, v_0, π_0); while the uniqueness in the Euler coordinate can be proven by transforming it to the Lagrangian coordinate, as $u \in L^1(0,T; W^{1,\infty}(\mathbb{R}))$, and applying the uniqueness result stated in Proposition 3.1.

Next, we have the following global well-posedness.

Theorem 1.2 *(Global well-posedness). Assume that (H1)–(H2) hold, and that*

$$\varrho_0 \in L^1, \quad \pi_0 \in L^1, \quad \varrho_0(y) \geq \frac{A_0}{(1+|y|)^2}, \quad \forall y \in \mathbb{R}, \tag{H4}$$

for some positive constant A_0.

Then, the following two hold:

(i) *There is a unique global strong solution (J, v, π) to system (1.10)–(1.12), subject to the initial condition $(J, v, \pi)|_{t=0} = (1, v_0, \pi_0)$. Moreover, it holds that*

$$\int_{\mathbb{R}} \left(\frac{1}{2}\varrho_0(y) v^2(y,t) + \frac{1}{\gamma-1} J(y,t)\pi(y,t) \right) dy = \mathcal{E}_0,$$

$$\int_{\mathbb{R}} \varrho_0(y) v(y,t) dy = m_0, \quad \inf_{y\in\mathbb{R}} J(y,t) \geq c_0,$$

for any $t \in [0,\infty)$, where

$$\mathcal{E}_0 := \int_{\mathbb{R}} \left(\frac{\varrho_0 v_0^2}{2} + \frac{\pi_0}{\gamma-1} \right) dy, \quad m_0 = \int_{\mathbb{R}} \varrho_0 v_0 dy, \quad c_0 = e^{-\frac{2\sqrt{2}}{\mu}\sqrt{\mathcal{E}_0}\|\varrho_0\|_1}.$$

(ii) *Assume further that (H3) holds, for two positive constants δ and K_0. Then, (1.15) and (1.16) hold for any $T \in (0,\infty)$.*

Remark 1.5. If the initial data has more regularities, then the corresponding solution (ϱ, v, π) in Theorem 1.2 can be classical ones and, consequently, we obtain the global existence of classical solutions to the compressible Navier-Stokes equations without heat conduction, in the presence of vacuum at far fields. To the best of our knowledge, this is the first result on the global existence of strong solutions to the compressible Navier-Stokes equations without heat conduction, for arbitrary large initial data, in the presence of far field vacuum. Note that this global existence result does not contradict to the finite time blow-up results in [42], as the assumption of having initial isolated mass group there is excluded in our case.

Remark 1.6. The following assumption in (H4)

$$\varrho_0(y) \geq \frac{A_0}{(1+|y|)^2}, \quad \forall y \in \mathbb{R}$$

can be removed. In fact, noticing that, essentially, the role that this assumption played in the proof of Theorem 1.2 is to justify integration by parts of some integrals defined on the whole line, so that one can get the basic energy inequality and the estimates on G, see Proposition 4.1, Proposition 4.4, and Proposition 4.6. Alternatively, to get the desired basic energy inequality and the estimates on G, one can approximate the Cauchy problem by a sequence of initial-boundary value problems, while for the initial-boundary value problems, the integration by parts to the corresponding integrals can be carried out without the above assumption.

Remark 1.7. Choose

$$\varrho_0(y) = \frac{K_\varrho}{\langle y \rangle^{\ell_\varrho}}, \quad J_0 \equiv 1, \quad v_0 \in C_c^\infty, \quad \pi_0 = Ae^{\frac{1}{c_v}}\varrho_0^\gamma,$$

where K_ϱ and ℓ_ϱ are positive numbers.

(i) If $1 < \ell_\varrho \leq 2$, then (ϱ_0, v_0, π_0) satisfies assumptions (H1), (H2), (H3), with $\delta = 1$, and (H4). Therefore, by Theorem 1.2, there is a unique global strong solution (J, v, π), with v being in the inhomogeneous Sobolev space $L^\infty(0, T; H^1)$; if moreover that $\gamma > \frac{5}{4}$ and $\max\left\{1, \frac{1}{2(\gamma-1)}\right\} < \ell_\varrho \leq 2$, then $\vartheta_0 \in H^1$, and, consequently, the temperature ϑ also lies in the inhomogeneous Sobolev space $L^\infty(0, T; H^1)$.

(ii) If $\max\left\{1, \frac{1}{\gamma}\right\} < \ell_\varrho \leq 2$, then (ϱ_0, v_0, π_0) satisfies conditions (H1), (H2), (H3), with $\delta = \gamma$, (H4), and $s_0 \equiv 1$. Therefore, by Theorem 1.2, there is a unique strong solution (J, v, π), and the corresponding entropy s is uniformly bounded on $\mathbb{R} \times (0, T)$.

(iii) Consequently, if $\gamma > \frac{5}{4}$ and $\max\left\{1, \frac{1}{\gamma}, \frac{1}{2(\gamma-1)}\right\} < \ell_\varrho \leq 2$, then there is a unique global strong solution (J, v, π), with the properties that the corresponding entropy is uniformly bounded and that the velocity and the corresponding temperature lie in the inhomogeneous space $L^\infty(0, T; H^1)$.

Remark 1.8. Same as in Remark 1.4, one can obtain the corresponding global existence of solutions in the Euler coordinates to the compressible Navier-Stokes equations (1.6), (1.7), and (1.9), subject to the initial data (ϱ_0, v_0, π_0).

The rest of this paper is arranged as follows: in Section 2, we consider the system in the absence of vacuum, and carry out some a priori estimates, which are independent of the positive lower bound of the density; the proof of Theorem 1.1 is given in Section 3, while that of Theorem 1.2 is given in the last section.

2. Local existence in the absence of vacuum

In this section, we study the Cauchy problem (1.10)–(1.13), in the absence of vacuum, that is, the density ϱ_0 is assumed to have a positive lower bound. We focus on some a priori estimates of the solutions which are independent of the positive lower bound of the density ϱ_0.

Then the following local existence result holds:

Proposition 2.1. *Given a function ϱ_0 satisfying $\underline{\varrho} \le \varrho_0 \le \bar{\varrho}$ on $\mathbb{R}$, for two positive constants $\underline{\varrho}$ and $\bar{\varrho}$. Let $\mathscr{N}_0$ be a positive constant such that $\frac{1}{\underline{J}} + \bar{J} + \|J_0'\|_2 + \|(v_0, \pi_0)\|_{H^1} \le \mathscr{N}_0$. Assume that the initial data (J_0, v_0, π_0) satisfies*

$$\underline{J} \le J_0 \le \bar{J} \text{ on } \mathbb{R}, \quad J_0' \in L^2, \quad v_0 \in H^1, \quad 0 \le \pi_0 \in H^1,$$

for two positive constants $\underline{J}$ and $\bar{J}$.

Then, there is a unique local strong solution (J, v, π) to system (1.10)–(1.12), subject to the initial condition (1.13), on $\mathbb{R} \times (0, T)$, satisfying

$$\begin{aligned}
&\frac{3}{4}\underline{J} \le J \le \frac{5}{4}\bar{J}, \quad \pi \ge 0, \text{ on } \mathbb{R} \times [0, T], \\
&J - J_0 \in C([0, T]; H^1), \quad J_t \in L^\infty(0, T; L^2), \\
&v \in C([0, T]; H^1) \cap L^2(0, T; H^2), \quad v_t \in L^2(0, T; L^2), \\
&\pi \in C([0, T]; H^1), \quad \pi_t \in L^2(0, T; L^2),
\end{aligned}$$

where T is a positive constant depending only on $\mu, \gamma, \underline{\varrho}, \bar{\varrho}$, and $\mathscr{N}_0$.

Proof. This will be proved by the fixed point argument. Let M and T be two positive constants with T being suitably small, to be determined by the quantity $\mathscr{N}_0$. Set $X_{M,T} := \left\{v \big| \|v\|_{L^\infty(0,T;H^1) \cap L^2(0,T;H^2)} \le M\right\}$. Given $v \in X_{M,T}$, define J and π, successively, as the unique solutions to the following two ordinary differential equations:

$$J_t = v_y,$$

and

$$\pi_t + \gamma \frac{v_y}{J} \pi = (\gamma - 1)\mu \left(\frac{v_y}{J}\right)^2,$$

with initial data $J|_{t=0} = J_0$ and $\pi|_{t=0} = \pi_0$, respectively.

We claim that J and π defined as above have the properties stated in the proposition. Note that

$$\pi = e^{-\gamma \int_0^t \frac{v_y}{J} ds} \pi_0 + \mu(\gamma - 1) \int_0^t e^{-\gamma \int_s^t \frac{v_y}{J} d\tau} \left(\frac{v_y}{J}\right)^2 ds. \tag{2.1}$$

Thus, the assumptions $\mu > 0$, $\gamma > 1$, and $\pi_0 \geq 0$ imply $\pi \geq 0$. Since $v_y \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$, so $J - J_0 = \int_0^t v_y ds \in C([0,T];H^1)$ and $J_t = v_y \in L^\infty(0,T;L^2)$. By Gagliardo-Nirenberg and Hölder inequalities,

$$\|J - J_0\|_\infty(t) = \left\| \int_0^t v_y ds \right\|_\infty \leq C_* \int_0^t \|v_y\|_2^{\frac{1}{2}} \|v_{yy}\|_2^{\frac{1}{2}} ds$$

$$\leq C_* \left(\sup_{0 \leq s \leq t} \|v_y\|_2 \right)^{\frac{1}{2}} \left(\int_0^t \|v_{yy}\|_2^2 ds \right)^{\frac{1}{4}} t^{\frac{3}{4}} \leq C_* M T^{\frac{3}{4}} \leq \frac{\underline{J}}{4},$$

as long as $0 \leq t \leq T \leq (\frac{\underline{J}}{4C_* M})^{\frac{4}{3}}$, and, thus, $\frac{3}{4}\underline{J} \leq J \leq \frac{5}{4}\bar{J}$ on $\mathbb{R} \times [0,T]$. Thanks to this and that $v_y \in L^2(0,T;H^1) \hookrightarrow L^2(0,T;L^\infty)$ and $J_y \in L^\infty(0,T;L^2)$, one obtains that $\frac{v_y}{J} \in L^2(0,T;L^2)$ and, by the Hölder inequality, $\left(\frac{v_y}{J}\right)_y = \frac{v_{yy}}{J} - \frac{v_y J_y}{J^2} \in L^2(0,T;L^2)$. Therefore, $\frac{v_y}{J} \in L^2(0,T;H^1)$ and further $L^2(0,T;L^\infty)$ by the embedding. Using these, recalling (2.1), and noticing that

$$\pi_y = \mu(\gamma - 1) \int_0^t e^{-\gamma \int_s^t \frac{v_y}{J} d\tau} \left[2\frac{v_y}{J} \left(\frac{v_y}{J}\right)_y - \gamma \left(\frac{v_y}{J}\right)^2 \int_s^t \left(\frac{v_y}{J}\right)_y d\tau \right] ds$$

$$+ e^{-\gamma \int_0^t \frac{v_y}{J} ds} \left(\pi_0' - \gamma \int_0^t \left(\frac{v_y}{J}\right)_y ds \right),$$

one can get easily by the Hölder inequality that $\pi \in L^\infty(0,T;H^1)$ and further that $\pi \in L^\infty(0,T;L^\infty)$ by the embedding. Recalling that $\frac{v_y}{J} \in L^2(0,T;L^\infty \cap H^1)$ and $\pi \in L^\infty(0,T;H^1 \cap L^\infty)$, one obtains that $\pi_t = (\gamma - 1)\mu \left(\frac{v_y}{J}\right)^2 - \gamma \frac{v_y}{J} \pi \in L^2(0,T;L^2)$ and

$$\pi_{yt} = 2\mu(\gamma - 1)\frac{v_y}{J}\left(\frac{v_y}{J}\right)_y - \gamma \left(\pi_y \frac{v_y}{J} + \pi \left(\frac{v_y}{J}\right)_y \right) \in L^1(0,T;L^2),$$

in other words, $\pi_t \in L^2(0,T;L^2) \cap L^1(0,T;H^1)$. This and $\pi \in L^\infty(0,T;H^1)$ lead to that $\pi \in C([0,T];H^1)$ and $\pi_t \in L^2(0,T;L^2)$.

Take $T \leq (\frac{\underline{J}}{4C_* M})^{\frac{4}{3}}$. Then, as stated in the previous paragraph, it holds that $\frac{3}{4}\underline{J} \leq J \leq \frac{5}{4}\bar{J}$ on $\mathbb{R} \times (0,T)$. Let V be unique solution to the following uniform parabolic equation

$$V_t - \frac{\mu}{\varrho_0}\left(\frac{V_y}{J}\right)_y = -\frac{\pi_y}{\varrho_0}, \quad \text{in } \mathbb{R} \times (0,T),$$

subject to the initial data $V|_{t=0} = v_0$. Then, the classic theory for parabolic equations shows that $V \in L^\infty(0,T;H^1) \cap L^2(0,T;H^1)$ and $V_t \in L^2(0,T;L^2)$. Define a mapping $\mathcal{M} : v \mapsto V$, with V defined as above. By standard energy estimates, one can show that

$\mathcal{M}$ is a contracting mapping on $X_{M,T}$, for some M and T depending only on $\mu, \gamma, \underline{\varrho}, \bar{\varrho}$, and $\mathscr{N}_0$. Consequently, by the contracting mapping principle, there is a unique fixed point, denoted by v, to $\mathcal{M}$ in $X_{M,T}$. Then (J, v, π), with (J, π) defined in the way as above, is a desired solution to system (1.10)–(1.12), subject to (1.13), on $\mathbb{R} \times (0, T)$. Since the proof is lengthy but standard, the details are omitted here. □

By Proposition 2.1, there is a positive time T_1, such that the problem (1.10)–(1.13) has a unique solution (J, v, π), on the time interval $(0, T_1)$, satisfying

$$\begin{cases} \frac{3}{4}\underline{J} \leq J \leq \frac{5}{4}\bar{J}, \quad \pi \geq 0, \quad \text{on } \mathbb{R} \times [0, T_1], \\ J - J_0 \in C([0, T_1]; H^1), \quad J_t \in L^\infty(0, T_1; L^2), \\ v \in C([0, T_1]; H^1) \cap L^2(0, T_1; H^2), \quad v_t \in L^2(0, T_1; L^2), \\ \pi \in C([0, T_1]; H^1), \quad \pi_t \in L^2(0, T_1; L^2). \end{cases}$$

Starting from the time T_1, noticing that $(J, v, \pi)|_{t=T_1}$ satisfies the conditions on the initial data stated in Proposition 2.1, one can extend the solution (J, v, π) forward in time to another time $T_2 = T_1 + t_1$, for some positive time t_1 depending only on $\mu, \gamma, \underline{\varrho}, \bar{\varrho}$, and the upper bound of $\ell(T_1)$, where, for simplicity of notations, we have denoted

$$\ell(t) := \left(\left(\inf_{y \in \mathbb{R}} J \right)^{-1} + \|J\|_\infty + \|J_y\|_2 + \|v\|_{H^1} + \|\pi\|_{H^1} \right)(t), \tag{2.2}$$

such that (J, v, π) is the unique solution to the problem (1.10)–(1.13) on time internal $(0, T_2)$, and that it enjoys the same regularities as above in the time interval $(0, T_2)$, and $(\frac{3}{4})^2\underline{J} \leq J \leq (\frac{5}{4})^2\bar{J}$ on $\mathbb{R} \times [0, T_2]$. Continuing this procedure, one obtains two sequences of positive numbers $\{t_j\}_{j=1}^\infty$ and $\{T_j\}_{J=1}^\infty$, with t_j depending only on $\mu, \gamma, \underline{\varrho}, \bar{\varrho}$, and the upper bound of $\ell(T_j)$, and $T_{j+1} = T_j + t_j$, such that the solution (J, v, π) can be extended to time intervals $(0, T_j)$, satisfying

$$\begin{cases} (\frac{3}{4})^j\underline{J} \leq J \leq (\frac{5}{4})^j\bar{J}, \quad \pi \geq 0, \quad \text{on } \mathbb{R} \times [0, T_j], \\ J - J_0 \in C([0, T_j]; H^1), \quad J_t \in L^\infty(0, T_j; L^2), \\ v \in C([0, T_j]; H^1) \cap L^2(0, T_j; H^2), \quad v_t \in L^2(0, T_j; L^2), \\ \pi \in C([0, T_j]; H^1), \quad \pi_t \in L^2(0, T_j; L^2), \end{cases}$$

for $j = 1, 2, \cdots$. Set the maximal existing time T_∞ as

$$T_\infty = T_1 + \sum_{j=1}^\infty t_j.$$

Then, the solution (J, v, π) can be extended to the time interval $(0, T_\infty)$, such that

$$\begin{cases} 0 < \inf_{y\in\mathbb{R},t\in[0,T]} J(y,t) \leq \sup_{y\in\mathbb{R},t\in[0,T]} J(y,t) < \infty, \\ \pi \geq 0 \quad \text{on } \mathbb{R}\times[0,T], \\ J - J_0 \in C([0,T];H^1), \quad J_t \in L^\infty(0,T;L^2), \\ v \in C([0,T];H^1)\cap L^2(0,T;H^2), \quad v_t \in L^2(0,T;L^2), \\ \pi \in C([0,T];H^1), \quad \pi_t \in L^2(0,T;L^2), \end{cases} \tag{2.3}$$

for any $T \in (0, T_\infty)$. Moreover, if $T_\infty < \infty$, it must have $\lim_{j\to\infty} t_j = 0$ and, consequently, one has

$$\overline{\lim_{j\to\infty}}\, \ell(T_j) = \infty, \tag{2.4}$$

where $\ell(t)$ is defined by (2.2). Otherwise, if (2.4) is not true, then $\ell(T_j) \leq \mathscr{N}_0$ for some positive constant $\mathscr{N}_0$; in this case, noticing that the existence time t_j provided in Proposition 2.1 depends only on $\mu, \gamma, \underline{\varrho}, \bar{\varrho}$, and the upper bound of $\ell(T_j)$, one can choose $t_j > 0$ to be independent of j, contradicting to the fact that $\lim_{j\to\infty} t_j = 0$.

Thanks to the statements in the above paragraph, in the rest of this section, we always assume that (J, v, π) is the unique solution to (1.10)–(1.13) and that it has been extended, in the same way as above, to the maximal existing time interval $(0, T_\infty)$, where the maximal time T_∞ is constructed in the same way as above.

To obtain the a priori estimates on (J, v, π), we define a positive time

$$T_s := \sup\left\{ T \in (0, T_\infty) \,\middle|\, \frac{\underline{J}}{2} \leq J \leq 2\bar{J} \text{ on } \mathbb{R}\times[0,T] \right\}. \tag{2.5}$$

We start with the following estimate on G:

Proposition 2.2. *There is a positive constant $t_*^1 = t_*^1(\gamma, \mu, \bar{\varrho}, \underline{J}, \|\sqrt{J_0}G_0\|_2)$, such that*

$$\sup_{0\leq t\leq T_*^1} \|\sqrt{J}G\|_2^2 + \mu \int_0^{T_*^1} \left\| \frac{G_y}{\sqrt{\varrho_0}} \right\|_2^2 dt \leq 3(1 + \|\sqrt{J_0}G_0\|_2^2),$$

$$\int_0^{T_*^1} \|G\|_\infty^4 dt \leq C(\mu, \bar{\varrho}, \underline{J}, \|\sqrt{J_0}G_0\|_2),$$

where $G_0 := \mu v_0' - \pi_0$, $T_*^1 := \min\{1, t_*^1, T_s\}$, and T_s is defined by (2.5).

Proof. Multiplying (1.14) by JG, and integrating the resultant over $\mathbb{R}$, one gets by integration by parts that

$$\int_{\mathbb{R}} JGG_t dy + \mu \int_{\mathbb{R}} \frac{(G_y)^2}{\varrho_0} dy = -\gamma \int_{\mathbb{R}} v_y G^2 dy. \tag{2.6}$$

Then (1.10) shows

$$\int_{\mathbb{R}} JGG_t dy = \frac{1}{2}\left(\frac{d}{dt}\int_{\mathbb{R}} JG^2 dy - \int_{\mathbb{R}} v_y G^2 dy\right),$$

which, together with (2.6), yields

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 = \left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}} v_y G^2 dy. \tag{2.7}$$

It follows from $v_y = \frac{J}{\mu}(G+\pi)$ that

$$\begin{aligned}\left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}} v_y G^2 dy &= \frac{1-2\gamma}{2\mu}\int_{\mathbb{R}} J(G+\pi)G^2 dy \\ &\le \frac{1-2\gamma}{2\mu}\int_{\mathbb{R}} JG^3 dy \le \frac{\gamma}{\mu}\|G\|_\infty\|\sqrt{J}G\|_2^2,\end{aligned}$$

where $\pi \geq 0$ and $\gamma > 1$ have been used. Therefore, it follows from (2.7) that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 \le \frac{\gamma}{\mu}\|G\|_\infty\|\sqrt{J}G\|_2^2. \tag{2.8}$$

By the Gagliardo-Nirenberg inequality $\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}$, one has

$$\|G\|_\infty \le C\|G\|_2^{\frac{1}{2}}\|G_y\|_2^{\frac{1}{2}} \le C(\bar{\varrho},\underline{J})\|\sqrt{J}G\|_2^{\frac{1}{2}}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^{\frac{1}{2}}. \tag{2.9}$$

Combining (2.8) and (2.9), one obtains from the Young inequality that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 &\le C(\gamma,\mu,\bar{\rho},\underline{J})\|\sqrt{J}G\|_2^{\frac{5}{2}}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^{\frac{1}{2}} \\ &\le \frac{\mu}{2}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + C(\gamma,\mu,\bar{\rho},\underline{J})(1+\|\sqrt{J}G\|_2^2)^2\end{aligned}$$

and, thus,

$$\frac{d}{dt}(1+\|\sqrt{J}G\|_2^2) + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 \le C_1(\gamma,\mu,\bar{\rho},\underline{J})(1+\|\sqrt{J}G\|_2^2)^2. \tag{2.10}$$

Solving (2.10) yields

$$
\begin{aligned}
-(1+\|\sqrt{J}G\|_2^2)^{-1}(t) &\leq -(1+\|\sqrt{J}_0G_0\|_2^2)^{-1} + C_1(\gamma,\mu,\bar{\varrho},\underline{J})\,t \\
&\leq -\frac{1}{2}(1+\|\sqrt{J_0}G_0\|_2^2)^{-1},
\end{aligned}
$$

for any $t\in[0,T_1^*)$, where

$$
T_*^1 := \min\{1, t_*^1, T_s\}, \quad t_*^1 := \frac{1}{2(1+\|\sqrt{J_0}G_0\|_2^2)C_1(\gamma,\mu,\bar{\varrho},\underline{J})}. \tag{2.11}
$$

Therefore, we have

$$
\sup_{0\leq t<T_*^1}(1+\|\sqrt{J}G\|_2^2) \leq 2(1+\|\sqrt{J_0}G_0\|_2^2),
$$

and further from (2.10) that

$$
\sup_{0\leq t<T_*^1}\|\sqrt{J}G\|_2^2 + \mu\int_0^{T_*^1}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 dt \leq 3(1+\|\sqrt{J_0}G_0\|_2^2).
$$

The estimate $\int_0^{T_*^1}\|G\|_\infty^4 dt \leq C(\mu,\bar{\varrho},\underline{J},\|G_0\|_2)$ follows from the above estimate and (2.9). The proof is complete. $\square$

Based on Proposition 2.2, we can derive the following estimates on (J, v, π).

Proposition 2.3.

(i) *Let T_*^1 be as in Proposition 2.2. Then, it holds that*

$$
\sup_{0\leq t\leq T_*^1}\left\|\left(\pi, \frac{\pi_y}{\sqrt{\varrho_0}}, J-J_0, J_t, \frac{J_y}{\sqrt{\varrho_0}}, v_y\right)\right\|_2 \leq C,
$$

$$
\int_0^{T_*^1}\left(\|\pi_t\|_2^4 + \left\|\left(\sqrt{\varrho_0}v_t, \frac{v_{yy}}{\sqrt{\varrho_0}}\right)\right\|_2^2\right) dt \leq C,
$$

and

$$
\sup_{0\leq t\leq T_*^1}\|\sqrt{\varrho_0}v\|_{L^2((-R,R))} \leq \|v_0\|_{L^2((-R,R))} + C,
$$

for a positive constant C depending only on $\gamma,\mu,\bar{\varrho},\underline{J},\bar{J},\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2,\|G_0\|_2,\|\pi_0\|_2$, and $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, but independent of $\underline{\varrho}$.

(ii) *There is a positive constant* t_*^2 *depending only on* $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|G_0\|_2$*, and* $\|\pi_0\|_\infty$*, but independent of* $\underline{\varrho}$*, such that*

$$\frac{3}{4}\underline{J} \le J(y,t) \le \frac{5}{4}\bar{J}, \qquad on\ \mathbb{R}\times[0,T_*^2),$$

where $T_*^2 := \min\{T_*^1, t_*^2\} = \min\{1, t_*^1, t_*^2, T_s\}$*, with* T_s *defined by (2.5).*

Proof. (i) Equation (1.12) can be rewritten in terms of G as

$$\pi_t + \frac{1}{\mu}\left(\pi + \frac{2-\gamma}{2}G\right)^2 = \frac{\gamma^2}{4\mu}G^2, \tag{2.12}$$

from which, one obtains

$$0 \le \pi(y,t) \le \pi_0(y) + \frac{\gamma^2}{4\mu}\int_0^t G^2(y,\tau)d\tau.$$

Thanks to the above, it follows from Proposition 2.2 and the Hölder inequality that

$$\sup_{0\le t\le T_*^1}\|\pi\|_2 \le \|\pi_0\|_2 + \frac{\gamma^2}{4\mu}\int_0^{T_*^1}\|G\|_\infty\|G\|_2 dt \le \|\pi_0\|_2 + C\left(\gamma,\mu,\bar{\varrho},\underline{J},\bar{J},\|G_0\|_2\right) \tag{2.13}$$

and

$$\sup_{0\le t\le T_*^1}\|\pi\|_\infty \le \|\pi_0\|_\infty + \frac{\gamma^2}{4\mu}\int_0^{T_*^1}\|G\|_\infty^2 dt \le \|\pi_0\|_\infty + C\left(\gamma,\mu,\bar{\varrho},\underline{J},\bar{J},\|G_0\|_2\right). \tag{2.14}$$

Hence, it follows from (2.12) and Proposition 2.2 that

$$\begin{aligned}\int_0^{T_*^1}\|\pi_t\|_2^4 dt &\le C\int_0^{T_*^1}(\|G\|_\infty^4 + \|\pi\|_\infty^4)(\|G\|_2^4 + \|\pi\|_2^4)dt \\ &\le C(\gamma,\mu,\bar{\rho},\underline{J},\bar{J},\|G_0\|_2,\|\pi_0\|_2,\|\pi_0\|_\infty).\end{aligned} \tag{2.15}$$

Differentiating equation (2.12) with respect to y yields

$$\pi_{yt} + \frac{2}{\mu}\left(\pi + \frac{2-\gamma}{2}G\right)\left(\pi_y + \frac{2-\gamma}{2}G_y\right) = \frac{\gamma^2}{2\mu}GG_y.$$

Multiplying the above equation by $\frac{\pi_y}{\varrho_0}$ and integrating over $\mathbb{R}$, it follows from the Hölder and Cauchy inequalities that

$$\frac{d}{dt}\left\|\frac{\pi_y}{\sqrt{\varrho_0}}\right\|_2^2 \le C(\gamma,\mu)(\|\pi\|_\infty+\|G\|_\infty)\left(\left\|\frac{\pi_y}{\sqrt{\varrho_0}}\right\|_2+\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2\right)\left\|\frac{\pi_y}{\sqrt{\varrho_0}}\right\|_2$$
$$\le C(\gamma,\mu)\left[\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+(1+\|\pi\|_\infty^2+\|G\|_\infty^2)\left\|\frac{\pi_y}{\sqrt{\varrho_0}}\right\|_2^2\right].$$

It follows from this, Proposition 2.2, (2.14), and the Gronwall inequality that

$$\sup_{0\le t\le T_*^1}\left\|\frac{\pi_y}{\sqrt{\varrho_0}}\right\|_2^2 \le C\left(\gamma,\mu,\bar\varrho,\underline{J},\bar J,\|G_0\|_2,\|\pi_0\|_\infty,\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2\right). \tag{2.16}$$

Recalling the definition of G, one can rewrite the equation for J as

$$J_t=\frac{J}{\mu}(G+\pi). \tag{2.17}$$

Thus, we deduce

$$\|J-J_0\|_2+\|J_t\|_2=\left\|\int_0^t J_t d\tau\right\|_2+\|J_t\|_2$$
$$\le \frac{2\bar J}{\mu}\left(\int_0^t(\|G\|_2+\|\pi\|_2)d\tau+\|G\|_2+\|\pi\|_2\right),$$

which, together with Proposition 2.2 and (2.13), shows that

$$\sup_{0\le t\le T_*^1}(\|J-J_0\|_2+\|J_t\|_2)\le C(\gamma,\mu,\bar\rho,\underline{J},\bar J,\|G_0\|_2,\|\pi_0\|_2). \tag{2.18}$$

Solving the ordinary differential equation (2.17) yields

$$J(y,t)=\exp\left\{\frac{1}{\mu}\int_0^t(G+\pi)d\tau\right\}J_0(y)$$

and, thus,

$$J_y=\left(\frac{1}{\mu}\int_0^t(G_y+\pi_y)d\tau J_0+J_0'\right)\exp\left\{\frac{1}{\mu}\int_0^t(G+\pi)d\tau\right\},$$

from which, applying Propositions 2.2, and using (2.14) and (2.16), one obtains

$$\sup_{0\le t\le T_*^1}\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2 \le \left(\frac{\bar J}{\mu}\int_0^{T_*^1}\left\|\left(\frac{G_y}{\sqrt{\varrho_0}},\frac{\pi_y}{\sqrt{\varrho_0}}\right)\right\|_2 dt+\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2\right) \times\exp\left\{\frac{1}{\mu}\int_0^{T_*^1}\|(G,\pi)\|_\infty dt\right\}\le C, \tag{2.19}$$

for a positive constant C depending only on $\gamma,\mu,\bar\varrho,\underline{J},\bar J,\|\frac{J_0'}{\sqrt{J_0}}\|_2,\|G_0\|_2,\|\pi_0\|_2$, and $\|\frac{\pi_0'}{\sqrt{\varrho_0}}\|_2$, but independent of $\underline{\varrho}$.

Recalling the definition of G, and noticing that $\varrho_0 v_t=G_y$, one gets from Proposition 2.2 and (2.13) that

$$\sup_{0\le t\le T_*^1}\|v_y\|_2^2=\sup_{0\le t\le T_*^1}\left\|\frac{J}{\mu}(G+\pi)\right\|_2^2\le C(\gamma,\mu,\bar\varrho,\underline{J},\bar J,\|G_0\|_2,\|\pi_0\|_2) \tag{2.20}$$

and

$$\int_0^{T_*^1}\|\sqrt{\varrho_0}v_t\|_2^2dt=\int_0^{T_*^1}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2dt\le C(\mu,\bar\varrho,\underline{J},\bar J,\|G_0\|_2). \tag{2.21}$$

Therefore, it holds that

$$\begin{aligned}\sup_{0\le t\le T_*^1}\|\sqrt{\varrho_0}v\|_{L^2((-R,R))}&=\sup_{0\le t\le T_*^1}\left\|\sqrt{\varrho_0}v_0+\int_0^t\sqrt{\varrho_0}v_td\tau\right\|_{L^2((-R,R))}\\&\le\|\sqrt{\varrho_0}v_0\|_{L^2((-R,R))}+\int_0^{T_*^1}\|\sqrt{\varrho_0}v_t\|_2d\tau\\&\le\|\sqrt{\varrho_0}v_0\|_{L^2((-R,R))}+C(\mu,\bar\varrho,\underline{J},\bar J,\|G_0\|_2),\end{aligned}$$

for any $0<R\le\infty$. Since

$$v_{yy}=\left[\frac{J}{\mu}(G+\pi)\right]_y=\frac{J_y}{\mu}(G+\pi)+\frac{J}{\mu}(G_y+\pi_y),$$

it follows from Proposition 2.2, (2.14), (2.16), (2.19), and the Hölder inequality that

$$\int_0^{T_0}\left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2dt\le C\int_0^{T_0}\left(\left\|\left(\frac{G_y}{\sqrt{\varrho_0}},\frac{\pi_y}{\sqrt{\varrho_0}}\right)\right\|_2^2+\|(G,\pi)\|_\infty^2\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2\right)dt\le C, \tag{2.22}$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|G_0\|_2, \|\pi_0\|_2$, and $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, but independent of $\underline{\varrho}$.

(ii) Due to Proposition 2.2 and (2.14), one gets from the Hölder inequality that

$$\begin{aligned}\frac{1}{\mu}\int_0^t \|J(G+\pi)\|_\infty d\tau &\le \frac{2\bar{J}}{\mu}\int_0^t (\|G\|_\infty + \|\pi\|_\infty)d\tau \\ &\le Ct + \frac{2}{\mu}\left(\int_0^t \|G\|_\infty^4 d\tau\right)^{\frac{1}{4}} t^{\frac{3}{4}} \le C_2 t^{\frac{3}{4}} \le \frac{\underline{J}}{4},\end{aligned}$$

for any $t \in [0, T_*^2)$, where

$$T_*^2 := \min\{T_*^1, t_*^2\} = \min\{t_*^1, t_*^2, T_s\}, \quad t_*^2 := \left(\frac{\underline{J}}{4C_2}\right)^{\frac{4}{3}},$$

for a positive constant C_2 depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|G_0\|_2$, and $\|\pi_0\|_\infty$, but independent of $\underline{\varrho}$. Consequently, it follows from (2.17) that

$$|J - J_0| = \left|\int_0^t J_t d\tau\right| = \left|\int_0^t \frac{J}{\mu}(G+\pi)d\tau\right| \le \frac{1}{\mu}\int_0^t \|J(G+\pi)\|_\infty d\tau \le \frac{\underline{J}}{4},$$

which implies

$$\frac{3}{4}\underline{J} \le J_0 - \frac{\underline{J}}{4} \le J \le J_0 + \frac{\underline{J}}{4} \le \bar{J} + \frac{\bar{J}}{4} = \frac{5}{4}\bar{J} \quad \text{on } \mathbb{R}\times[0, T_2^*),$$

which proves (ii). $\square$

Thanks to the estimates stated in Proposition 2.3, one can estimate the lower bound of the time T_s as stated in the following proposition:

Proposition 2.4. *Let T_s be defined by (2.5), t_*^1 and T_*^1 be the constants stated in Proposition 2.2, and t_*^2 and T_*^2 the constants in Proposition 2.3. Then, it holds that $T_s > T_*^2$ and, consequently, $T_*^1 \ge T_*^2 \ge \min\{1, t_*^1, t_*^2\}$.*

Proof. Assume, by contradiction, that $T_s \le T_*^2$. Recall that $T_*^2 = \min\{1, t_*^1, t_*^2, T_s\}$, which shows that $T_*^2 \le T_s$ and, therefore, $T_s = T_*^2$.

If $T_s < T_\infty$, then $T_*^2 = T_s < T_\infty$. Since (2.3) holds for any $T < T_\infty$, so $(J - J_0, v, \pi) \in C([0, T_\infty); H^1(\mathbb{R}))$, which, by the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, yields that $J \in C([0, T_\infty); L^\infty(\mathbb{R}))$. Thanks to this and by (ii) of Proposition 2.3, there is a positive time $T_* \in (T_*^2, T_\infty)$, such that $\frac{1}{2}\underline{J} \le J \le \frac{3}{2}\bar{J}$ on $\mathbb{R}\times[0, T_*]$. By the definition of T_s, then $T_* \le T_s = T_*^2$, which contradicts to $T_* \in (T_*^2, T_\infty)$.

If $T_s = T_\infty$, then $T_s = T_*^2 = T_\infty$. By Proposition 2.3, and noticing that $T_*^2 \le T_*^1$, one has

$$\sup_{0\le t<T_*^2}(\|J_y\|_2 + \|\sqrt{\varrho_0}v\|_2^2 + \|v_y\|_2^2 + \|\pi\|_{H^1}) < \infty$$

and $\frac{3}{4}\underline{J} \le J \le \frac{5}{4}\bar{J}$ on $\mathbb{R}\times[0,T_*^2)$. Therefore, due to $\varrho_0 \ge \underline{\varrho} > 0$, one has $\overline{\lim}_{t\to T_\infty}\ell(t) = \overline{\lim}_{t\to T_*^2}\ell(t) < \infty$, which contradicts to (2.4).

Therefore, we have shown that $T_s > T_*^2$. This, together with the definition $T_*^2 = \min\{1, t_*^1, t_*^2, T_s\}$, shows $T_*^2 = \min\{1, t_*^1, t_*^2\}$. This proves the conclusion. □

Then Propositions 2.1–2.4 yield the following:

Corollary 2.1. *Given a function ϱ_0 satisfying $\underline{\varrho} \le \varrho_0 \le \bar{\varrho}$ on $\mathbb{R}$, for two positive constants $\underline{\varrho}$ and $\bar{\varrho}$. Assume that the initial data (J_0, v_0, π_0) satisfies*

$$\underline{J} \le J_0 \le \bar{J} \text{ on } \mathbb{R}, \quad J_0' \in L^2, \quad v_0 \in L^1_{loc}, \quad v_0' \in L^2, \quad 0 \le \pi_0 \in H^1,$$

for two positive constants $\underline{J}$ and $\bar{J}$.

Then, there is a positive time T_0 depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|v_0'\|_2$, $\|\pi_0\|_2$, and $\|\pi_0\|_\infty$, but independent of $\underline{\varrho}$, such that the problem (1.10)–(1.13) has a unique solution (J, v, π) on $\mathbb{R}\times[0,T_0]$, satisfying

$$\pi \ge 0, \quad \frac{3}{4}\underline{J} \le J \le \frac{5}{4}\bar{J}, \quad \text{on } \mathbb{R}\times[0,T_0],$$

$$\sup_{0\le t\le T_0}\left\|\left(\pi, \frac{\pi_y}{\sqrt{\varrho_0}}, J - J_0, J_t, \frac{J_y}{\sqrt{\varrho_0}}, v_y\right)\right\|_2 \le C,$$

$$\int_0^{T_0}\left(\|\pi_t\|_2^4 + \left\|\left(\sqrt{\varrho_0}v_t, \frac{v_{yy}}{\sqrt{\varrho_0}}\right)\right\|_2^2\right)dt \le C,$$

$$\sup_{0\le t\le T_0}\|\sqrt{\varrho_0}v\|_{L^2((-R,R))} \le \|\sqrt{\varrho_0}v_0\|_{L^2((-R,R))} + C,$$

for any $0 < R \le \infty$, where C is a positive constant depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}$, $\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2, \|v_0'\|_2, \|\pi_0\|_2$, and $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, but independent of $\underline{\varrho}$.

Proof. The Hölder inequality yields

$$|v_0(y)| = \left|v_0(0) + \int_0^y v_0'(z)dz\right| \le |v_0(0)| + \|v_0'\|_2\sqrt{|y|}, \quad \forall y\in\mathbb{R}. \tag{2.23}$$

Choose a function $0 \le \phi \in C_c^\infty((-2,2))$, with $\phi \equiv 1$ on $(-1,1)$, $0 \le \phi \le 1$ on $(-2,2)$, and $|\phi'| \le 2$ on $\mathbb{R}$. For any positive integer n, set $\phi_n(\cdot) = \phi(\frac{\cdot}{n})$ and $v_{0n} = v_0\phi_n$. Thanks to (2.23), noticing that $supp\,\phi_n \subseteq (-2n,-n)\cup(n,2n)$, and that

$$v_{0n}' = v_0'\phi_n + v_0\phi_n' = v_0'\phi_n + \frac{v_0}{n}\phi'\left(\frac{\cdot}{n}\right),$$

one has

$$\begin{aligned}
\|v_{0n}'\|_2 &\le \|v_0'\|_2 + \frac{1}{n}\left\|v_0\phi'\left(\frac{\cdot}{n}\right)\right\|_2 \\
&= \|v_0'\|_2 + \frac{1}{n}\left(\int\limits_{n<|y|<2n} |v_0|^2\left|\phi'\left(\frac{y}{n}\right)\right|^2 dy\right)^{\frac{1}{2}} \\
&\le \|v_0'\|_2 + \frac{2}{n}\left[\int\limits_{n<|y|<2n} (|v_0(0)| + \sqrt{|y|}\|v_0'\|_2)^2 dy\right]^{\frac{1}{2}} \\
&\le \|v_0'\|_2 + \frac{2}{n}(|v_0(0)| + \sqrt{2n}\|v_0'\|_2)\sqrt{2n} \\
&= 5\|v_0'\|_2 + \frac{2\sqrt{2}}{\sqrt{n}}|v_0(0)| \le 5\|v_0'\|_2 + 1,
\end{aligned} \tag{2.24}$$

for any $n \ge 8|v_0(0)|^2$.

Consider the system (1.10)–(1.12), subject to the initial condition

$$(J, v, \pi)|_{t=0} = (J_0, v_{0n}, \pi_0). \tag{2.25}$$

Due to (2.24), it holds that

$$\|G_{0n}\|_2 \le \mu\|v_{0n}'\|_2 + \|\pi_0\|_2 \le \mu(5\|v_0'\|_2 + 1) + \|\pi_0\|_2,$$

for any $n \ge 8|v_0(0)|^2$, where $G_{0n} := \mu v_{0n}' - \pi_0$. Thanks to this and Propositions 2.1–2.4, there is a positive time T_0 depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|v_0'\|_2, \|\pi_0\|_2$, and $\|\pi_0\|_\infty$, but independent of $\underline{\varrho}$, such that system (1.10)–(1.12), subject to (2.25), has a unique solution (J_n, v_n, π_n), on $\mathbb{R} \times (0, T_0)$, satisfying

$$\pi_n \ge 0, \quad \frac{3}{4}\underline{J} \le J_n \le \frac{5}{4}\bar{J}, \quad \text{on } \mathbb{R} \times [0, T_0], \tag{2.26}$$

$$\sup_{0\le t\le T_0}\left\|\left(\pi_n, \frac{\partial_y\pi_n}{\sqrt{\varrho_0}}, J_n - J_0, \partial_t J_n, \frac{\partial_y J_n}{\sqrt{\varrho_0}}, \partial_y v_n\right)\right\|_2 \le C, \tag{2.27}$$

$$\int_0^{T_0}\left(\|\partial_t\pi_n\|_2^4 + \left\|\left(\sqrt{\varrho_0}\partial_t v_n, \frac{\partial_y^2 v_n}{\sqrt{\varrho_0}}\right)\right\|_2^2\right) dt \le C, \tag{2.28}$$

$$\sup_{0\le t\le T_0}\|\sqrt{\varrho_0}v_n\|_{L^2((-R,R))} \le \|\sqrt{\varrho_0}v_0\|_{L^2((-R,R))} + C, \tag{2.29}$$

for any $n \geq 8|v_0(0)|^2$, and for any $0 < R \leq \infty$, where C is a positive constant depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}$, $\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2$, $\|v_0'\|_2, \|\pi_0\|_2$, and $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, but independent of $\underline{\varrho}$ and $n \geq 8|v_0(0)|^2$.

With the above a priori estimates in hand, one can apply the Banach-Alaoglu theorem, Cantor's diagonal arguments, the Aubin-Lions lemma, and make use of the weakly lower semi-continuity of the norms, to show that there is a subsequence, still denoted by (J_n, v_n, π_n), and a triple (J, v, π), which satisfies the same a priori estimates as above, such that (J_n, v_n, π_n) converges, strongly, weakly, and weak-* in appropriate spaces, to (J, v, π), and (J, v, π) is a solution to the problem (1.10)–(1.13). All these can be proved similarly as in the proof of (i) of Theorem 1.1 to be given in the next section; in particular, noticing that (2.26)–(2.29) are the analogies of (3.8)–(3.11), one can show that the analogous weak, weak-*, and strong convergences of (3.12)–(3.18) and (3.22)–(3.24) hold for (J_n, v_n, π_n) and (J, v, π), that is, we have the following convergences which enable us to pass the limit as $n \to \infty$ to show that (J, v, π) is a desired solution:

$$\begin{aligned}
J_n - J_0 \to J - J_0, &\quad \text{weak-* in } L^\infty(0,T;H^1),\\
\partial_t J_n \to J_t, &\quad \text{weak-* in } L^\infty(0,T;L^2),\\
v_n \to v, &\quad \text{weakly in } L^2(0,T;H^2((-R,R))),\\
\partial_t v_n \to v_t, &\quad \text{weakly in } L^2(0,T;L^2((-R,R))),\\
\partial_y v_n \to v_y, &\quad \text{weak-* in } L^\infty(0,T;L^2) \text{ and weakly in } L^2(0,T;H^1),\\
\pi_n \to \pi, &\quad \text{weak-* in } L^\infty(0,T;H^1),\\
\partial_t \pi_n \to \pi_t, &\quad \text{weakly in } L^4(0,T;L^2),
\end{aligned}$$

and

$$\begin{gathered}
J_n \to J, \quad \text{strongly in } C([0,T];L^\infty((-r,r))),\\
v_n \to v, \quad \text{strongly in } C([0,T];L^2((-R,R))) \cap L^2(0,T;H^1((-r,r))),\\
\pi_n \to \pi, \quad \text{strongly in } C([0,T];L^\infty((-r,r))),
\end{gathered}$$

for any $r \in (0,\infty)$. Since the proof is very similar to that of (i) of Theorem 1.1, we omit the details here. While the uniqueness is guaranteed by Proposition 3.1, in the next section. $\square$

As the end of this section, we give some more estimates on G stated in the next proposition, which will be the key to obtain the boundedness of the entropy.

Proposition 2.5. *In addition to the assumptions in Corollary 2.1, we assume that*

$$\left|\left(\frac{1}{\sqrt{\varrho_0}}\right)'(y)\right| \leq \frac{K_0}{2}, \quad \forall y \in \mathbb{R},$$

for some positive constant K_0. Let T_0 be the positive constant in Corollary 2.1 and (J, v, π) the solution stated in Corollary 2.1.

Then, for any $\delta \in (0, \infty)$, there is a positive constant C depending only on $\gamma, \mu, \bar{\varrho}$, $\underline{J}$, $\bar{J}$, K_0, $\left\|\frac{J_0'}{\sqrt{\varrho_0}}\right\|_2, \|v_0'\|_2, \|\pi_0\|_2, \left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, and $\left\|\varrho_0^{-\frac{\delta}{2}} G_0\right\|_2$, but independent of $\underline{\varrho}$, such that

$$\sup_{0 \le t \le T_0} \left\| \frac{G}{\varrho_0^{\frac{\delta}{2}}} \right\|_2^2 + \int_0^{T_0} \left\| \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \right\|_2^2 dt \le C.$$

Proof. Multiplying (1.14) by $\frac{JG}{\varrho_0^\delta}$ and integrating over $\mathbb{R}$, one gets from integration by parts that

$$\int_{\mathbb{R}} \frac{JG}{\varrho_0^\delta} G_t dy + \mu \int_{\mathbb{R}} \frac{G_y}{\varrho_0} \left(\frac{G}{\varrho_0^\delta} \right)_y dy = -\gamma \int_{\mathbb{R}} v_y \frac{G^2}{\varrho_0^\delta} dy. \tag{2.30}$$

Direct calculations yield

$$\begin{aligned} \int_{\mathbb{R}} \frac{G_y}{\varrho_0} \left(\frac{G}{\varrho_0^\delta} \right)_y dy &= \int_{\mathbb{R}} \frac{G_y}{\varrho_0} \left(\frac{G_y}{\varrho_0^\delta} - \delta \frac{\varrho_0'}{\varrho_0} \frac{G}{\varrho_0^\delta} \right) dy \\ &= \left\| \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \right\|_2^2 - \delta \int_{\mathbb{R}} \frac{\varrho_0'}{\varrho_0} \frac{G G_y}{\varrho_0^{\delta+1}} dy. \end{aligned} \tag{2.31}$$

It follows from (1.10) that

$$\begin{aligned} \int_{\mathbb{R}} \frac{JG}{\varrho_0^\delta} G_t dy &= \frac{1}{2} \left(\frac{d}{dt} \int_{\mathbb{R}} \frac{JG^2}{\varrho_0^\delta} dy - \int_{\mathbb{R}} J_t \frac{G^2}{\varrho_0^\delta} dy \right) \\ &= \frac{1}{2} \left(\frac{d}{dt} \left\| \sqrt{\frac{J}{\varrho_0^\delta}} G \right\|_2^2 - \int_{\mathbb{R}} v_y \frac{G^2}{\varrho_0^\delta} dy \right). \end{aligned} \tag{2.32}$$

Plugging (2.31) and (2.32) into (2.30) yields

$$\frac{1}{2} \frac{d}{dt} \left\| \sqrt{\frac{J}{\varrho_0^\delta}} G \right\|_2^2 + \mu \left\| \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \right\|_2^2 = \left(\frac{1}{2} - \gamma \right) \int_{\mathbb{R}} v_y \frac{G^2}{\varrho_0^\delta} dy + \delta\mu \int_{\mathbb{R}} \frac{\varrho_0'}{\varrho_0} \frac{G G_y}{\varrho_0^{\delta+1}} dy. \tag{2.33}$$

Due to the assumption $\left| \left(\frac{1}{\sqrt{\varrho_0}} \right)' \right| \le \frac{K_0}{2}$, or equivalently $|\varrho_0'| \le K_0 \varrho_0^{\frac{3}{2}}$, it follows from the Cauchy inequality that

$$\int_{\mathbb{R}} \frac{\varrho_0'}{\varrho_0} \frac{GG_y}{\varrho_0^{\delta+1}} dy \le K_0 \int_{\mathbb{R}} \left| \frac{G}{\varrho_0^{\frac{\delta}{2}}} \right| \left| \frac{G_y}{\varrho^{\frac{\delta+1}{2}}} \right| dy \le \frac{1}{2\delta} \left\| \frac{G_y}{\varrho^{\frac{\delta+1}{2}}} \right\|_2^2 + \frac{\delta K_0^2}{2} \left\| \frac{G}{\varrho_0^{\frac{\delta}{2}}} \right\|_2^2. \tag{2.34}$$

Using the definition of G leads to

$$\begin{aligned}\left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}} v_y \frac{G^2}{\varrho_0^\delta} dy &= \left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}} \frac{J}{\mu}(G+\pi)\frac{G^2}{\varrho_0^\delta} dy \\ &\le \frac{1-2\gamma}{2\mu}\int_{\mathbb{R}} \frac{JG^3}{\varrho_0^\delta} dy \le \frac{\gamma}{\mu}\|G\|_\infty \left\| \sqrt{\frac{J}{\varrho_0^\delta}} G \right\|_2^2, \end{aligned} \tag{2.35}$$

here, $\gamma > 1$ and $\pi \ge 0$ have been used. Plugging (2.34)–(2.35) into (2.33) yields that

$$\frac{d}{dt} \left\| \sqrt{\frac{J}{\varrho_0^\delta}} G \right\|_2^2 + \mu \left\| \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \right\|_2^2 \le C(\gamma,\mu,\delta,K_0)(1+\|G\|_\infty) \left(\left\| \sqrt{\frac{J}{\varrho_0^\delta}} G \right\|_2^2 + 1 \right),$$

from which, by Corollary 2.1 and the Gronwall inequality, one obtains

$$\sup_{0\le t\le T_0} \left\| \frac{G}{\varrho_0^{\frac{\delta}{2}}} \right\|_2^2 + \int_0^{T_0} \left\| \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \right\|_2^2 dt \le C,$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}$, $\underline{J}$, $\bar{J}$, K_0, $\|\frac{J_0'}{\sqrt{\varrho_0}}\|_2$, $\|v_0'\|_2$, $\|\pi_0\|_2$, $\|\frac{\pi_0'}{\sqrt{\varrho_0}}\|_2$, and $\left\|\varrho_0^{-\frac{\delta}{2}} G_0\right\|_2$, but independent of $\underline{\varrho}$. □

3. Local existence in the presence of far field vacuum

In this section, we prove the local existence and uniqueness of strong solutions to the Cauchy problem (1.10)–(1.13), in the presence of vacuum at far fields.

We start with the uniqueness of the solutions.

Proposition 3.1. *Let ϱ_0 be given such that $\inf_{y\in(-r,r)} \varrho_0(y) > 0$, for any $r \in (0,\infty)$, and $\varrho_0 \le \bar{\varrho}$ on $\mathbb{R}$ for a positive constant $\bar{\varrho}$. Let (J_1, v_1, π_1) and (J_2, v_2, π_2) be two solutions to system (1.10)–(1.12), subject to the same initial data, on $\mathbb{R} \times (0,T)$, satisfying $c_0 \le J_i \le C_0$ on $\mathbb{R} \times (0,T)$ for two positive numbers c_0 and C_0, and*

$$\pi_i \in L^2(0,T;L^\infty), \quad (\partial_t J_i, \partial_t \pi_i) \in L^1_{loc}(\mathbb{R}\times[0,T)),$$
$$(\sqrt{\varrho_0} v, \partial_y J_i, \partial_y \pi_i) \in L^\infty(0,T;L^2), \quad (\sqrt{\varrho_0}\partial_t v_i, \partial_y v_i, \partial_y^2 v_i) \in L^2(0,T;L^2),$$

for $i = 1, 2$. Then $(J_1, v_1, \pi_1) \equiv (J_2, v_2, \pi_2)$ on $\mathbb{R} \times [0,T]$.

Proof. Define (J, v, π) as

$$J = J_1 - J_2, \quad v = v_1 - v_2, \quad \pi = \pi_1 - \pi_2.$$

Then, (J, v, π) satisfies

$$\partial_t J = \partial_y v, \tag{3.1}$$

$$\varrho_0 \partial_t v - \mu \partial_y \left(\frac{\partial_y v}{J_1} \right) + \partial_y \left(\pi + \frac{\alpha}{J_1} J \right) = 0, \tag{3.2}$$

$$\partial_t \pi + \gamma \beta \pi = \chi \left(\partial_y v - \alpha J \right), \tag{3.3}$$

where $\alpha = \alpha(y,t), \beta = \beta(y,t)$, and $\chi = \chi(y,t)$ are given as

$$\alpha(y,t) = \frac{\partial_y v_2}{J_2}, \quad \beta(y,t) = \frac{\partial_y v_1}{J_1}, \quad \chi(y,t) = \left[(\gamma - 1)\mu \left(\frac{\partial_y v_1}{J_1^2} + \frac{\partial_y v_2}{J_1 J_2} \right) - \gamma \frac{\pi_2}{J_1} \right].$$

Due to the regularities of $(J_i, v_i, \pi_i), i = 1, 2$, it holds that

$$(\alpha, \beta) \in L^2(0,T; L^2), \quad (\alpha, \beta, \chi) \in L^2(0,T; L^\infty). \tag{3.4}$$

Choose a function $\eta \in C_c^\infty((-2,2))$, with $\eta \equiv 1$ on $(-1,1)$, and $0 \leq \eta \leq 1$ on $(-2,2)$. For each $r \geq 1$, we set $\eta_r(y) = \eta(\frac{y}{r})$, for $y \in \mathbb{R}$. Multiplying equations (3.1), (3.2), and (3.3), respectively, by $J\eta_r^2$, $v\eta_r^2$, and $\pi\eta_r^2$, summing the resultants up, and integrating over $\mathbb{R}$, one gets from integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (J^2 + \varrho_0 v^2 + \pi^2)\eta_r^2 dy + \mu \int_{\mathbb{R}} \frac{(\partial_y v)^2}{J_1}\eta_r^2 dy \\
= &\int_{\mathbb{R}} \left[\partial_y v J + \left(\pi + \frac{\alpha}{J_1} J \right) \partial_y v + \chi(\partial_y v - \alpha J)\pi - \gamma \beta \pi^2 \right] \eta_r^2 dy \\
&+2 \int_{\mathbb{R}} \left(\pi + \frac{\alpha}{J_1} J - \mu \frac{\partial_y v}{J_1} \right) v \eta_r \eta_r' dy.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\partial_y v J + \left(\pi + \frac{\alpha}{J_1} J \right) \partial_y v + \chi(\partial_y v - \alpha J)\pi - \gamma \beta \pi^2 \right] \eta_r^2 dy \\
\leq &\frac{\mu}{2} \int_{\mathbb{R}} \frac{(v_y)^2}{J_1}\eta_r^2 dy + C \int_{\mathbb{R}} (J^2 + \pi^2 + \alpha^2 J^2 + \chi^2 \pi^2 + |\beta|\pi^2)\eta_r^2 dy \\
\leq &\frac{\mu}{2} \int_{\mathbb{R}} \frac{(v_y)^2}{J_1}\eta_r^2 dy + C(1 + \|(\alpha, \beta, \chi)\|_\infty^2) \int_{\mathbb{R}} (J^2 + \pi^2)\eta_r^2 dy,
\end{aligned}$$

so

$$\frac{d}{dt}\int_{\mathbb{R}}(J^2+\varrho_0 v^2+\pi^2)\eta_r^2 dy \le C(1+\|(\alpha,\beta,\chi)\|_\infty^2)\int_{\mathbb{R}}(J^2+\pi^2)\eta_r^2 dy + C\int_{\mathbb{R}}(|\pi|+|\alpha||J|+|\partial_y v|)|v||\eta_r'|dy.$$

It follows from this, the Gronwall inequality, and (3.4) that

$$\sup_{0\le t\le T}\int_{\mathbb{R}}(J^2+\varrho_0 v^2+\pi^2)\eta_r^2 dy \le C\int_0^T\int_{\mathbb{R}}(|\pi|+|\alpha||J|+|\partial_y v|)|v||\eta_r'|dydt =: Q_r.$$

It suffices to show that Q_r tends to zero as $r\to\infty$.

Note that

$$v(y,t)=v(z,t)+\int_z^y \partial_y v(y',t)dy', \quad 0<z<1<y<\infty.$$

Integrating the above identity with respect to z over the interval $(0,1)$, and denoting $D:=\sup_{y\in(-1,1)}\frac{1}{\varrho_0(y)}$, one obtains by the Hölder inequality that

$$\begin{aligned}|v(y,t)| &= \left|\int_0^1 v(z,t)dz+\int_0^1\int_z^y \partial_y v(y',t)dy'dz\right| \\ &\le \int_0^1|v|dz+\int_0^y|\partial_y v|dy' \le \frac{1}{\sqrt{D}}\int_0^1|\sqrt{\varrho_0}v|dz+\int_0^y|\partial_y v|dy' \\ &\le \frac{1}{\sqrt{D}}\|\sqrt{\varrho_0}v\|_2+\sqrt{y}\|\partial_y v\|_2 \le C\sqrt{y}, \quad \forall y\ge 1.\end{aligned}$$

In the same way, one obtains that $|v(y,t)|\le C\sqrt{|y|}$, for any $y\le -1$ and, thus,

$$|v(y,t)|\le C\sqrt{|y|}, \quad \forall y\in(-\infty,-1]\cup[1,\infty). \tag{3.5}$$

Note that (3.1) implies that

$$\sup_{0\le t\le T}\|J\|_2\le\int_0^T\|\partial_y v\|_2 d\tau<\infty,$$

and (3.3) yields that

$$\pi = e^{-\gamma \int_0^t \beta d\tau} \int_0^t e^{\gamma \int_0^\tau \beta d\tau'} \chi(\partial_y v - \alpha J) d\tau.$$

It follows from this, (3.4), and the Cauchy inequality that

$$\begin{aligned}\sup_{0\le t\le T} \|\pi\|_2 &\le e^{\gamma \int_0^T \|\beta\|_\infty dt} \int_0^T \|\chi\|_\infty (\|\partial_y v\|_2 + \|\alpha\|_\infty \|J\|_2) dt \\ &\le C \int_0^T (\|\chi\|_\infty^2 + \|\partial_y v\|_2^2 + \|\alpha\|_\infty^2 + 1) dt < \infty. \end{aligned} \tag{3.6}$$

Thanks to (3.5) and (3.6), noticing that $supp\, \eta_r' \subseteq (-2r, -r) \cup (r, 2r)$, and $|\eta_r'| \le \frac{C}{r}$, one obtains by the Hölder inequality that

$$\begin{aligned} Q_r &= C \int_0^T \int_{\mathbb{R}} (|\pi| + |\alpha||J| + |\partial_y v|)|v||\eta_r'| dy dt \\ &= \int_0^T \int_{r<|y|<2r} (|\pi| + |\alpha||J| + |\partial_y v|)|v||\eta_r'| dy dt \\ &\le \frac{C}{\sqrt{r}} \int_0^T \int_{r<|y|<2r} (|\pi| + |\alpha| + |\partial_y v|) dy dt \\ &\le C \left(\int_0^T \int_{r<|y|<2r} (|\pi|^2 + |\alpha|^2 + |\partial_y v|^2) dy dt \right)^{\frac{1}{2}}, \end{aligned}$$

for any $r \ge 1$, which, together with the fact that $\pi, \alpha, \partial_y v \in L^2(\mathbb{R} \times (0,T))$, implies

$$Q_r \to 0, \quad \text{as } r \to \infty.$$

The proof is complete. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. (i) Since the uniqueness is a direct corollary of Proposition 3.1, it remains to prove the existence. For any positive number $\varepsilon \in (0,1)$, set $\varrho_{0\varepsilon}(y) = \varrho_0(y) + \varepsilon$, for $y \in \mathbb{R}$. It is clear that $\varepsilon \leq \varrho_{0\varepsilon}(y) \leq \bar{\varrho} + 1$, for $y \in \mathbb{R}$. Consider the following approximate system of (1.10)–(1.12):

$$\begin{cases} J_t = v_y, \\ \varrho_{0\varepsilon} v_t - \mu \left(\frac{v_y}{J}\right)_y + \pi_y = 0, \\ \pi_t + \gamma \frac{v_y}{J} \pi = (\gamma - 1)\mu \left(\frac{v_y}{J}\right)^2. \end{cases} \tag{3.7}$$

By Corollary 2.1, there is a positive constant T depending only on $\gamma, \mu, \bar{\varrho}, \underline{J}, \bar{J}, \|v_0'\|_2$, $\|\pi_0\|_2$, and $\|\pi_0\|_\infty$, but independent of ε, such that system (3.7), subject to (1.13), has a unique solution $(J_\varepsilon, v_\varepsilon, \pi_\varepsilon)$, satisfying

$$\pi_\varepsilon \geq 0, \quad \frac{3}{4}\underline{J} \leq J_\varepsilon \leq \frac{5}{4}\bar{J}, \quad \text{on } \mathbb{R} \times [0, T], \tag{3.8}$$

$$\sup_{0 \leq t \leq T} \left\| \left(J_\varepsilon - J_0, \frac{\partial_y J_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}, \partial_t J_\varepsilon, \partial_y v_\varepsilon, \pi_\varepsilon, \frac{\partial_y \pi_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}} \right) \right\|_2^2 \leq C, \tag{3.9}$$

$$\sup_{0 \leq t \leq T} \|\sqrt{\varrho_{0\varepsilon}} v_\varepsilon\|_{L^2((-r,r))} \leq \|\sqrt{\varrho_{0\varepsilon}} v_0\|_{L^2((-r,r))} + C, \tag{3.10}$$

for any $0 < r < \infty$, and

$$\int_0^T \left(\left\| \left(\frac{\partial_y^2 v_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}, \sqrt{\varrho_{0\varepsilon}} \partial_t v_\varepsilon, \right) \right\|_2^2 + \|\partial_t \pi_\varepsilon\|_2^4 \right) dt \leq C, \tag{3.11}$$

for a positive constant C independent of ε.

Due to the a priori estimates (3.9)–(3.11), recalling that $\inf_{y \in (-r,r)} \varrho_0(y) > 0$ for any $r > 0$ and $\varrho_{0\varepsilon} \leq \bar{\varrho} + 1$, one can see that $J_\varepsilon - J_0$ and π_ε are uniformly bounded in $L^\infty(0, T; H^1)$, $\partial_y v_\varepsilon$ is uniformly bounded in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, and v_ε is uniformly bounded in $L^2(0, T; H^2((-r, r))$ for any $r > 0$. Therefore, by the Banach-Alaoglu theorem and a Cantor's diagonal argument, there is a subsequence, still denoted by $(J_\varepsilon, v_\varepsilon, \pi_\varepsilon)$, and a triple (J, v, π), such that

$$J_\varepsilon - J_0 \to J - J_0, \quad \text{weak-* in } L^\infty(0, T; H^1), \tag{3.12}$$

$$\partial_t J_\varepsilon \to J_t, \quad \text{weak-* in } L^\infty(0, T; L^2), \tag{3.13}$$

$$v_\varepsilon \to v, \quad \text{weakly in } L^2(0, T; H^2((-r, r))), \tag{3.14}$$

$$\partial_t v_\varepsilon \to v_t, \quad \text{weakly in } L^2(0, T; L^2((-r, r))), \tag{3.15}$$

$$\partial_y v_\varepsilon \to v_y, \quad \text{weak-* in } L^\infty(0, T; L^2) \text{ and weakly in } L^2(0, T; H^1), \tag{3.16}$$

$$\pi_\varepsilon \to \pi, \quad \text{weak-* in } L^\infty(0, T; H^1), \tag{3.17}$$

$$\partial_t \pi_\varepsilon \to \pi_t, \quad \text{weakly in } L^4(0, T; L^2), \tag{3.18}$$

for any $r \in (0, \infty)$, and

$$J - J_0 \in L^\infty(0,T;H^1), \quad J_t \in L^\infty(0,T;L^2), \tag{3.19}$$

$$v_y \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \tag{3.20}$$

$$\pi \in L^\infty(0,T;H^1), \quad \pi_t \in L^4(0,T;L^2). \tag{3.21}$$

Thanks to (3.12)–(3.13), (3.14)–(3.15), and (3.17)–(3.18), noticing that $H^1((-r,r))$ and $H^2((-r,r))$ compactly embed into $L^\infty((-r,r))$ and $H^1((-r,r))$, respectively, for any $r>0$, one can obtain from the Aubin-Lions compactness lemma and Cantor's diagonal argument that there is a subsequence, still denoted by $(J_\varepsilon, v_\varepsilon, \pi_\varepsilon)$, such that

$$J_\varepsilon \to J, \quad \text{strongly in } C([0,T];L^\infty((-r,r))), \tag{3.22}$$

$$v_\varepsilon \to v, \quad \text{strongly in } C([0,T];L^2((-r,r))) \cap L^2(0,T;H^1((-r,r))), \tag{3.23}$$

$$\pi_\varepsilon \to \pi, \quad \text{strongly in } C([0,T];L^\infty((-r,r))), \tag{3.24}$$

for any $r \in (0,\infty)$.

It follows from (3.8), (3.22), and (3.24) that $\frac{3}{4}\underline{J} \le J \le \frac{5}{4}\bar{J}$ and $\pi \ge 0$ on $\mathbb{R} \times [0,T]$. Since $(J_\varepsilon, v_\varepsilon, \pi_\varepsilon)|_{t=0} = (J_0, v_0, \pi_0)$, so (3.22)–(3.24) imply that $(J, v, \pi)|_{t=0} = (J_0, v_0, \pi_0)$. Due to $\frac{3}{4}\underline{J} \le J_\varepsilon, J \le \frac{5}{4}\bar{J}$ on $\mathbb{R} \times [0,T]$, one can verify easily from (3.22) that $\frac{1}{J_\varepsilon} \to \frac{1}{J}$ strongly in $C([0,T];L^\infty((-r,r)))$ for any $r>0$. By the aid of this and using (3.23)–(3.24), one can see that all the nonlinear terms in (3.7), i.e., $\frac{\partial_y v_\varepsilon}{J_\varepsilon}$, $\frac{\partial_y v_\varepsilon}{J_\varepsilon}\pi_\varepsilon$, and $(\frac{\partial_y v_\varepsilon}{J_\varepsilon})^2$ converge strongly in $L^2(0,T;L^2((-r,r)))$, $L^2(0,T;L^2((-r,r)))$, and $L^1(0,T;L^1((-r,r)))$ to $\frac{v_y}{J}$, $\frac{v_y}{J}\pi$, and $(\frac{v_y}{J})^2$, respectively, for any $r>0$. Therefore, one can take the limit as $\varepsilon \to 0$ in (3.7) to show that (J,v,π) satisfies equations (1.10)–(1.12), in the sense of distribution. Furthermore, due to the regularities of (J,v,π), which are going to be proved in the next paragraph, one can further show that (J,v,π) satisfies equations (1.10)–(1.12), a.e., in $\mathbb{R} \times (0,T)$. Therefore, (J,v,π) is a solution to the problem (1.10)–(1.13).

One can now verify that (J,v,π) has the regularities stated in Definition 1.1. Indeed, besides those in (3.19)–(3.21), the other desired regularities of (J,v,π) can be verified as follows. First, it follows from (3.19), (3.21), and

$$Y := \{f | f \in L^\infty(0,T;L^2), f' \in L^1(0,T;L^2)\} \hookrightarrow C([0,T];L^2),$$

that $(J - J_0, \pi) \in C([0,T];L^2)$. Next, noticing that $\sqrt{\varrho_{0\varepsilon}} \to \sqrt{\varrho_0}$ and $\frac{1}{\sqrt{\varrho_{0\varepsilon}}} \to \frac{1}{\sqrt{\varrho_0}}$, in $L^\infty((-r,r))$, for any $r \in (0,\infty)$, one gets from (3.12), (3.14)–(3.15), and (3.17) that

$$\left(\frac{\partial_y J_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}, \frac{\partial_y \pi_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}\right) \to \left(\frac{J_y}{\sqrt{\varrho_0}}, \frac{\pi_y}{\sqrt{\varrho_0}}\right), \quad \text{weak-* in } L^\infty(0,T;L^2((-r,r))),$$

$$\left(\frac{\partial_y^2 v_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}, \sqrt{\varrho_{0\varepsilon}}\partial_t v_\varepsilon\right) \to \left(\frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}\partial_t v\right), \quad \text{weakly in } L^2((-r,r) \times (0,T)),$$

for any $r \in (0, \infty)$. Consequently, it follows from the weakly lower semi-continuity of the norms, (3.9), and (3.11) that

$$\left\|\left(\frac{J_y}{\sqrt{\varrho_0}}, \frac{\pi_y}{\sqrt{\varrho_0}}\right)\right\|_{L^\infty(0,T;L^2((-r,r)))} \leq \varliminf_{\varepsilon\to 0}\left\|\left(\frac{\partial_y J_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}, \frac{\partial_y \pi_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}\right)\right\|_{L^\infty(0,T;L^2((-r,r)))} \leq C,$$

$$\left\|\left(\frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}\partial_t v\right)\right\|_{L^2(0,T;L^2((-r,r)))} \leq \varliminf_{\varepsilon\to 0}\left\|\left(\frac{\partial_y^2 v_\varepsilon}{\sqrt{\varrho_{0\varepsilon}}}, \sqrt{\varrho_{0\varepsilon}}\partial_t v_\varepsilon\right)\right\|_{L^2(0,T;L^2((-r,r)))} \leq C,$$

for a positive constant C independent of r. Therefore,

$$\left(\frac{J_y}{\sqrt{\varrho_0}}, \frac{\pi_y}{\sqrt{\varrho_0}}\right) \in L^\infty(0,T;L^2), \quad \left(\sqrt{\varrho_0}v_t, \frac{v_{yy}}{\sqrt{\varrho_0}}\right) \in L^2(0,T;L^2).$$

And finally, $\sqrt{\varrho_0}v_t \in L^2(0,T;L^2)$ implies $\sqrt{\varrho_0}v \in C([0,T];L^2)$. Therefore, (J, v, π) meet the required regularities in Definition 1.1.

(ii) We first prove the regularities of G, i.e., (1.15). Let $\varrho_{0\varepsilon}, (v_\varepsilon, J_\varepsilon, \pi_\varepsilon)$, and T be the same as in (i). Then, $\frac{3}{4}\underline{J} \leq J_\varepsilon \leq \frac{5}{4}\bar{J}$ on $\mathbb{R} \times (0,T)$, and (3.12)–(3.18) and (3.22)–(3.24) hold. It is clear that

$$\left|\left(\frac{1}{\sqrt{\varrho_{0\varepsilon}}}\right)'\right| = \frac{1}{2}\left|\frac{\varrho_{0\varepsilon}'}{\varrho_{0\varepsilon}^{\frac{3}{2}}}\right| = \frac{1}{2}\left|\frac{\varrho_0'}{\varrho_{0\varepsilon}^{\frac{3}{2}}}\right| \leq \frac{1}{2}\left|\frac{\varrho_0'}{\varrho_0^{\frac{3}{2}}}\right| = \left|\left(\frac{1}{\sqrt{\varrho_0}}\right)'\right| \leq \frac{K_0}{2}, \quad \forall y \in \mathbb{R}.$$

Therefore, one can apply Proposition 2.5 to get

$$\sup_{0\leq t\leq T}\left\|\frac{G_\varepsilon}{\varrho_{0\varepsilon}^{\frac{\delta}{2}}}\right\|_2^2 + \int_0^T \left\|\frac{\partial_y G_\varepsilon}{\varrho_{0\varepsilon}^{\frac{\delta+1}{2}}}\right\|_2^2 dt \leq C, \tag{3.25}$$

for a positive constant C independent of ε, where $G_\varepsilon := \mu\frac{\partial_y v_\varepsilon}{J_\varepsilon} - \pi_\varepsilon$.

Recalling $\frac{3}{4}\underline{J} \leq J_\varepsilon \leq \frac{5}{4}\bar{J}$, and using (3.12), (3.16)–(3.17), and (3.22)–(3.23), one can show that

$$G_\varepsilon \to G, \quad \text{weak-* in } L^\infty(0,T;L^2((-r,r))),$$
$$\partial_y G_\varepsilon \to G_y, \quad \text{weakly in } L^2(0,T;L^2((-r,r))),$$

for any $r \in (0,\infty)$. Therefore, noticing that $\frac{1}{\varrho_{0\varepsilon}} \to \frac{1}{\varrho_0}$ in $L^\infty((-r,r))$, for any $r \in (0,\infty)$, it is easily to verify that

$$\frac{G_\varepsilon}{\varrho_{0\varepsilon}^{\frac{\delta}{2}}} \to \frac{G}{\varrho_0^{\frac{\delta}{2}}}, \quad \text{weak-* in } L^\infty(0,T;L^2((-r,r))), \tag{3.26}$$

$$\frac{\partial_y G_\varepsilon}{\varrho_{0\varepsilon}^{\frac{\delta+1}{2}}} \to \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}}, \quad \text{weakly in } L^2(0,T;L^2((-r,r))), \tag{3.27}$$

for any $r \in (0,\infty)$.

Due to (3.26), (3.27), and the weakly lower semi-continuity of the norms, it follows from (3.25) that $\frac{G}{\varrho_0^{\frac{\delta}{2}}}$ and $\frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}}$, respectively, are bounded in $L^\infty(0,T;L^2((-r,r)))$ and $L^2(0,T;L^2((-r,r)))$, uniformly in $r \in (0,\infty)$, and, consequently, it holds that

$$\frac{G}{\varrho_0^{\frac{\delta}{2}}} \in L^\infty(0,T;L^2), \quad \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \in L^2(0,T;L^2).$$

Thanks to these regularities of G, it follows from the Gagliardo-Nirenber inequality $\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}$, for $f \in H^1(\mathbb{R})$, and the assumption $\left|\left(\frac{1}{\sqrt{\varrho_0}}\right)'\right| \le \frac{K_0}{2}$, or equivalently $|\varrho_0'| \le K_0\rho_0^{\frac{3}{2}}$, that

$$\begin{aligned}
\int_0^T \left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_\infty^4 dt &\le C\int_0^T \left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 \left\|\left(\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right)_y\right\|_2^2 dt \\
&= C\int_0^{T_0} \left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 \left\|\frac{G_y}{\varrho_0^{\frac{\delta}{2}}} - \frac{\delta}{2}\frac{\varrho_0'}{\varrho_0}\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 dt \\
&\le C\int_0^T \left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 \left(\left\|\frac{G_y}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 + \delta^2K_0^2\left\|\frac{G}{\varrho_0^{\frac{\delta-1}{2}}}\right\|_2^2\right) dt \\
&\le C\int_0^T \left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 \left(\left\|\frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}}\right\|_2^2 + \left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2\right) dt < \infty
\end{aligned}$$

and, thus, $\frac{G}{\varrho_0^{\frac{\delta}{2}}} \in L^4(0,T;L^\infty)$.

Next, we show that $v \in L^\infty(0,T;H^1)$, under the assumption that $\delta \ge 1$ and $v_0 \in H^1$. Since $\frac{G_y}{\varrho_0} \in L^2(0,T;L^2)$ and $\varrho_0 v_t = G_y$, it is straightforward that $v_t \in L^2(0,T;L^2)$. This and $v_0 \in L^2$ imply that $v \in L^\infty(0,T;L^2)$. Thus, $v \in L^\infty(0,T;H^1)$.

We now prove $\vartheta \in L^\infty(0,T;H^1)$, under the assumption that $\delta \ge 1$, $\vartheta_0 \in H^1$, and $\frac{\pi_0}{\varrho_0} \in H^1$. In this case, one has

$$\frac{G}{\sqrt{\varrho_0}} \in L^\infty(0,T;L^2) \cap L^4(0,T;L^\infty), \quad \frac{G_y}{\varrho_0} \in L^2(0,T;L^2).$$

It follows from the assumption $|(\frac{1}{\sqrt{\varrho_0}})'| \le \frac{K_0}{2}$ that $\frac{\pi_0'}{\varrho_0} \in L^2$.

Recalling that

$$\pi_t + \frac{1}{\mu}\left(\pi + \frac{2-\gamma}{2}G\right)^2 = \frac{\gamma^2}{4\mu}G^2, \tag{3.28}$$

one obtains

$$\sup_{0\le t\le T}\left\|\frac{\pi}{\varrho_0}\right\|_2 \le \left\|\frac{\pi_0}{\varrho_0}\right\|_2 + \frac{\gamma^2}{4\mu}\int_0^T \left\|\frac{G}{\sqrt{\varrho_0}}\right\|_\infty \left\|\frac{G}{\sqrt{\varrho_0}}\right\|_2 dt < \infty$$

and, thus, $\frac{\pi}{\varrho_0} \in L^\infty(0,T;L^2)$. This and the fact that $J \in L^\infty(0,T;L^\infty)$ imply

$$\vartheta := \frac{\pi}{R\varrho} = \frac{J\pi}{R\varrho_0} \in L^\infty(0,T;L^2), \quad \text{where } \varrho := \frac{\varrho_0}{J}.$$

Differentiating (3.28) in y, multiplying the resultant by $\frac{\pi_y}{\varrho_0^2}$, and integrating over $\mathbb{R}$, one gets from the Hölder and Cauchy inequalities, and (2.14) that

$$\begin{aligned}\frac{d}{dt}\left\|\frac{\pi_y}{\varrho_0}\right\|_2^2 &\le C(\gamma,\mu)(\|\pi\|_\infty + \|G\|_\infty)\left(\left\|\frac{\pi_y}{\varrho_0}\right\|_2 + \left\|\frac{G_y}{\varrho_0}\right\|_2\right)\left\|\frac{\pi_y}{\varrho_0}\right\|_2 \\ &\le C(\gamma,\mu)\left(\left\|\frac{G_y}{\varrho_0}\right\|_2^2 + (1+\|\pi\|_\infty^2 + \|G\|_\infty^2)\left\|\frac{\pi_y}{\varrho_0}\right\|_2^2\right).\end{aligned}$$

It follows from this and the Gronwall inequality that $\sup_{0\le t\le T}\|\frac{\pi_y}{\sqrt{\varrho_0}}\|_2^2 < \infty$ and, thus, $\frac{\pi_y}{\varrho_0} \in L^\infty(0,T;L^2)$.

Rewrite equation (1.10) in terms of G as $J_t = \frac{J}{\pi}(G+\pi)$, from which, one obtains $J(y,t) = e^{\frac{1}{\mu}\int_0^t (G+\pi)d\tau} J_0(y)$ and, thus,

$$J_y = \left(\frac{1}{\mu}\int_0^t (G_y + \pi_y)dJ_0 + J_0'\right)\tau \exp\left\{\frac{1}{\mu}\int_0^t (G+\pi)d\tau\right\}.$$

Hence,

$$\sup_{0\le t\le T}\left\|\frac{J_y}{\varrho_0}\right\|_2 \le \left(\frac{\bar{J}}{\mu}\int_0^{T_*^1}\left\|\left(\frac{G_y}{\varrho_0}, \frac{\pi_y}{\varrho_0}\right)\right\|_2 dt + \left\|\frac{J_0'}{\varrho_0}\right\|_2\right) e^{\frac{1}{\mu}\int_0^T \|(G,\pi)\|_\infty dt} < \infty,$$

that is, $\frac{J_y}{\varrho_0} \in L^\infty(0,T;L^2)$.

Thanks to the regularities $\left(\frac{\pi}{\varrho_0}, \frac{\pi_y}{\varrho_0}, \frac{J_y}{\varrho_0}\right) \in L^\infty(0,T;L^2)$, noticing that $(J,\pi) \in L^\infty(0,T;L^\infty)$, and recalling the assumption $|(\frac{1}{\sqrt{\varrho_0}})'| \le \frac{K_0}{2}$, we have

$$\vartheta_y = \frac{1}{R}\left(\frac{J\pi_y}{\varrho_0} + \frac{J_y\pi}{\varrho_0} - \frac{\varrho_0' J\pi}{\varrho_0^2}\right) \in L^\infty(0,T;L^2).$$

It remains to prove that $s \in L^\infty(0,T;L^\infty)$, under the assumption that $s_0 \in L^\infty$ and $\delta \geq \gamma$. To this end, by the definition of s, it suffices to verify that $\frac{\pi}{\varrho^\gamma} = \frac{J^\gamma \pi}{\varrho_0^\gamma}$ has uniform positive lower and upper bounds on $\mathbb{R}\times(0,T)$. Since $\delta \geq \gamma$, it follows from (1.15) that $\frac{G}{\varrho_0^{\frac{\gamma}{2}}} \in L^4(0,T;L^\infty)$. To show the boundedness from above of $\frac{J^\gamma\pi}{\varrho_0^\gamma}$, due to $J \in [\frac34 \underline{J}, \frac54 \bar{J}]$ on $\mathbb{R}\times[0,T]$, one needs only to verify that for $\frac{\pi}{\varrho_0^\gamma}$. (3.28) implies that

$$\sup_{0\leq t\leq T}\left\|\frac{\pi}{\varrho_0^\gamma}\right\|_\infty \leq \left\|\frac{\pi_0}{\varrho_0^\gamma}\right\|_\infty + \frac{\gamma^2}{4\mu}\int_0^T \left\|\frac{G}{\varrho_0^{\frac{\gamma}{2}}}\right\|_\infty^2 dt < \infty.$$

Thus, $\frac{\pi}{\varrho^\gamma}$ has a uniform upper bound on $\mathbb{R}\times(0,T)$. Concerning the uniform positive lower bound, one obtains from (1.10) and (1.12) that

$$(J^\gamma\pi)_t = (\gamma-1)\mu J^{\gamma-2}(v_y)^2 \geq 0.$$

So $J^\gamma(y,t)\pi(y,t) \geq J_0^\gamma(y)\pi_0(y) = \pi_0(y)$, which leads to

$$\inf_{y\in\mathbb{R},t\in[0,T]} \frac{J^\gamma(y,t)\pi(y,t)}{\varrho_0^\gamma(y)} \geq \inf_{y\in\mathbb{R}} \frac{\pi_0(y)}{\varrho_0^\gamma(y)} > 0.$$

Hence, $\frac{\pi}{\varrho^\gamma}$ has a uniform positive lower bound on $\mathbb{R}\times(0,T)$. □

4. Global existence in the presence of far field vacuum

This section is devoted to establishing the global existence of strong solutions to the Cauchy problem (1.10)–(1.13), which proves Theorem 1.2. Throughout this section, it is always assumed that $J_0 \equiv 1$.

As preparations, several a priori estimates are stated in the next propositions. We start with the basic energy identity of a strong solution to the problem (1.10)–(1.13).

Proposition 4.1. *Given a positive time T, and let (J,v,π) be a strong solution to the problem (1.10)–(1.13), on $\mathbb{R}\times(0,T)$, with (ϱ_0,v_0,π_0) satisfying (H1), (H2), and (H4). Then, it holds that*

$$\int_{\mathbb{R}}\left(\frac{\varrho_0 v^2}{2} + \frac{J\pi}{\gamma-1}\right)dy = \int_{\mathbb{R}}\left(\frac{\varrho_0 v_0^2}{2} + \frac{\pi_0}{\gamma-1}\right)dy, \quad t\in[0,T].$$

Proof. Fix a nonnegative function $\eta \in C_c^\infty((-2,2))$ such that $\eta \equiv 1$ on $[-1,1]$ and $0 \leq \eta \leq 1$ on $(-2,2)$. For each $r \in (0,\infty)$, define a function η_r as $\eta_r(\cdot) = \eta(\frac{\cdot}{r})$. Multiplying (1.11) and (1.12), respectively, by $v\eta_r^2$ and $J\eta_r^2$, and integrating the resultants over $\mathbb{R}$, one gets from integration by parts that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \varrho_0 v^2\eta_r^2 dy + \mu \int_{\mathbb{R}} \frac{v_y^2\eta_r^2}{J} dy = \int_{\mathbb{R}} \pi v_y \eta_r^2 dy + 2\int_{\mathbb{R}} \left(\pi v - \mu \frac{v_y v}{J}\right) \eta_r \eta_r' dy, \tag{4.1}$$

and

$$\int_{\mathbb{R}} J\pi_t \eta_r^2 dy + \gamma \int_{\mathbb{R}} \pi v_y \eta_r^2 dy = (\gamma - 1)\mu \int_{\mathbb{R}} \frac{v_y^2 \eta_r^2}{J} dy. \tag{4.2}$$

(1.10) implies that

$$\int_{\mathbb{R}} J\pi_t \eta_r^2 dy = \frac{d}{dt}\int_{\mathbb{R}} J\pi\eta_r^2 dy - \int_{\mathbb{R}} J_t \pi \eta_r^2 dy = \frac{d}{dt}\int_{\mathbb{R}} J\pi\eta_r^2 dy - \int_{\mathbb{R}} v_y \pi \eta_r^2 dy,$$

which, plugged in to (4.2), yields

$$\frac{d}{dt}\int_{\mathbb{R}} J\pi\eta_r^2 dy + (\gamma - 1)\int_{\mathbb{R}} v_y \pi \eta_r^2 dy = (\gamma - 1)\mu \int_{\mathbb{R}} \frac{v_y^2\eta_r^2}{J} dy.$$

This, together with (4.1), yields

$$\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{\varrho_0 v^2}{2} + \frac{J\pi}{\gamma - 1}\right) \eta_r^2 dy = 2\int_{\mathbb{R}} \left(\pi - \mu \frac{v_y}{J}\right) v \eta_r \eta_r' dy,$$

which implies

$$\begin{aligned}\int_{\mathbb{R}} \left(\frac{\varrho_0 v^2}{2} + \frac{J\pi}{\gamma - 1}\right) \eta_r^2 dy &= 2\int_0^t \int_{\mathbb{R}} \left(\pi - \mu \frac{v_y}{J}\right) v\eta_r\eta_r' dy d\tau \\ &\quad + \int_{\mathbb{R}} \left(\frac{\varrho_0 v_0^2}{2} + \frac{\pi_0}{\gamma - 1}\right) \eta_r^2 dy, \end{aligned} \tag{4.3}$$

for any $t \in [0, T]$. Due to $|\eta_r'| \le \frac{\|\eta'\|_\infty}{r}$ and the assumption $\varrho_0(y) \ge \frac{A_0}{(1+|y|)^2}$, one deduces

$$|\eta_r'(y)| \le \frac{\|\eta'\|_\infty}{r} \le \frac{2\|\eta'\|_\infty}{|y|} \le \frac{4\|\eta'\|_\infty}{|y| + 1} \le \frac{4\|\eta'\|_\infty}{\sqrt{A_0}} \sqrt{\varrho_0(y)}, \quad \forall\ 1 \le r < |y| < 2r.$$

It follows from this and $supp\, \eta_r' \subseteq (-2r, r) \cup (r, 2r)$ that

$$|\eta_r'(y)| \le M_0 \sqrt{\varrho_0(y)}, \quad \forall y \in \mathbb{R}, \qquad \text{where } M_0 = \frac{4\|\eta'\|_\infty}{\sqrt{A_0}}, \tag{4.4}$$

for any $r \ge 1$. Therefore, denoting $\delta_T = \inf_{y\in\mathbb{R}, t\in[0,T]} J(y, t)$, and noticing that $\sqrt{\varrho_0} v\pi$, $\sqrt{\varrho_0} v v_y \in L^1(\mathbb{R} \times (0, T))$, one gets that

$$\left|\int_0^t\int_{\mathbb{R}}\left(\pi-\mu\frac{v_y}{J}\right)v\eta_r\eta_r'dyd\tau\right|=\left|\int_0^t\int_{r<|y|<2r}\left(\pi-\mu\frac{v_y}{J}\right)v\eta_r\eta_r'dyd\tau\right|$$
$$\leq M_0\int_0^t\int_{r<|y|<2r}\left(|\pi|+\frac{\mu}{\delta_T}|v_y|\right)|\sqrt{\varrho_0}v|dyd\tau\to 0,\quad \text{as } r\to\infty,$$

for any $t\in[0,T]$. Thanks to this, one can take the limit as $r\to\infty$ in (4.3) to get the conclusion. □

The next proposition yields the uniform positive lower bound of J.

Proposition 4.2. *Under the same assumptions as in Proposition 4.1, it holds that*

$$\inf_{y\in\mathbb{R}}J(y,t)\geq e^{-\frac{2\sqrt{2}}{\mu}\sqrt{\mathcal{E}_0}\|\varrho_0\|_1},\quad t\in[0,T],$$

where $\mathcal{E}_0:=\int_{\mathbb{R}}\left(\frac{\varrho_0v_0^2}{2}+\frac{\pi_0}{\gamma-1}\right)dy$.

Proof. Due to (1.10), one can rewrite (1.11) as

$$\varrho_0v_t-\mu(\log J)_{yt}+\pi_y=0.$$

Thus,

$$\varrho_0v-\mu(\log J)_y+\int_0^t\pi_yd\tau=\varrho_0v_0.$$

Integrating the above identity over the interval (z_0,z) yields

$$J(z,t)=J(z_0,t)\exp\left\{\frac{1}{\mu}\int_{z_0}^z\varrho_0(v-v_0)dz\right\}\exp\left\{\frac{1}{\mu}\int_0^t(\pi-\pi(z_0,\tau))d\tau\right\},\tag{4.5}$$

for any $z,z_0\in\mathbb{R}$ and $t\in(0,T_\infty)$. By Proposition 4.1, it follows from the Hölder inequality that

$$\left|\int_{z_0}^z\varrho_0(v-v_0)dy\right|\leq\|\varrho_0\|_1(\|\sqrt{\varrho_0}v\|_2+\|\sqrt{\varrho_0}v_0\|_2)\leq 2\sqrt{2\mathcal{E}_0}\|\varrho_0\|_1.$$

It then follows from this, $\pi\geq 0$, and (4.5) that

$$J(z,t) \geq \exp\left\{-\frac{2\sqrt{2}}{\mu}\sqrt{\mathcal{E}_0}\|\varrho_0\|_1\right\}\exp\left\{-\frac{1}{\mu}\int_0^t \pi(z_0,\tau)d\tau\right\}J(z_0,t), \tag{4.6}$$

for any $z, z_0 \in \mathbb{R}$ and $t \in (0, T_\infty)$.

Since $v_y \in L^2(0,T;H^1)$, it is clear that, for any fixed $t \in [0,T]$, $\int_0^t v_y d\tau \in H^1$. Thanks to this, and using the fact that $f(y) \to 0$, as $y \to \infty$, for any $f \in H^1$, one obtains from equation (1.10) that

$$J(y,t) = 1 + \int_0^t v_y(y,\tau)d\tau \to 1, \quad \text{as } y \to \infty, \tag{4.7}$$

for any $t \in [0,T]$. By the Sobolev embedding inequality, it follows from $v_y \in L^2(0,T;H^1)$ that $v_y \in L^2(0,T;L^\infty)$. Therefore, since $v_y \in L^\infty(0,T;L^2)$, it follows from the Hölder inequality that

$$(v_y)^2 \in L^2(0,T;L^2), \quad v_y v_{yy} \in L^1(0,T;L^2),$$

which imply $\int_0^t (v_y)^2 d\tau \in H^1$, for any fixed $t \in [0,T]$. Thanks to this, and using again the fact that $f(y) \to 0$, as $y \to \infty$, for any $f \in H^1$, one has

$$\lim_{y\to\infty}\int_0^t |v_y|(y,\tau)d\tau \leq \sqrt{t}\left(\lim_{y\to\infty}\int_0^t (v_y)^2(y,\tau)d\tau\right)^{\frac{1}{2}} = 0,$$

for any $t \in [0,T]$. Therefore, denoting $\delta_T = \inf_{y\in\mathbb{R},t\in(0,T)} J(y,t)$, and solving equation (1.12), one obtains that

$$\begin{aligned}\pi(y,t) &= \exp\left\{-\gamma\int_0^t \frac{v_y}{J}d\tau\right\}\left(\pi_0(y) + (\gamma-1)\mu\int_0^t e^{\gamma\int_0^\tau \frac{v_y}{J}d\tau'}\left(\frac{v_y}{J}\right)^2 d\tau\right)\\ &\leq e^{\frac{\gamma}{\delta_T}\int_0^t |v_y|d\tau}\left(\pi_0(y) + \frac{(\gamma-1)\mu}{\delta_T^2}\int_0^t v_y^2 d\tau\right) \to 0, \quad \text{as } y\to\infty,\end{aligned} \tag{4.8}$$

for any $t \in [0,T]$.

Due to (4.7) and (4.8), by taking $z_0 \to \infty$, one obtains from (4.6) that

$$\inf_{y\in\mathbb{R}} J(y,t) \geq e^{-\frac{2\sqrt{2}}{\mu}\sqrt{\mathcal{E}_0}\|\varrho_0\|_1},$$

for any $t \in [0,T]$, so the conclusion follows. □

The next proposition gives the conservation of momentum.

Proposition 4.3. *Under the same assumptions as in Proposition 4.1, it holds that*

$$\int_{\mathbb{R}} \varrho_0(y)v(y,t)dy = \int_{\mathbb{R}} \varrho_0(y)v_0(y)dy, \quad \forall t \in [0,T].$$

Proof. Let η and η_r be the same functions as in the proof of Proposition 4.1. Multiplying equation (1.11) by η_r, and integrating the resultant over $\mathbb{R}$, one gets by integration by parts that

$$\frac{d}{dt}\int_{\mathbb{R}} \varrho_0 v\eta_r dy = \int_{\mathbb{R}} \left(\pi - \mu\frac{v_y}{J}\right)\eta_r' dy,$$

which implies that

$$\int_{\mathbb{R}} \varrho_0 v\eta_r dy = \int_{\mathbb{R}} \varrho_0 v_0\eta_r dy + \int_0^t\int_{\mathbb{R}} \left(\pi - \mu\frac{v_y}{J}\right)\eta_r' dyd\tau. \tag{4.9}$$

We claim that

$$\lim_{r\to\infty}\int_0^t\int_{\mathbb{R}} \left(\pi - \mu\frac{v_y}{J}\right)\eta_r' dyd\tau = 0. \tag{4.10}$$

It follows from this claim, $\varrho_0 v \in L^1(\mathbb{R})$, and $\varrho_0 v_0 \in L^1(\mathbb{R})$ that the desired conclusion holds by taking the limit $r \to \infty$ in (4.9).

It remains to verify (4.10). To this end, recalling $supp\,\eta_r \subseteq (-2r,-r)\cup(r,2r)$ and (4.4), by Proposition 4.2, and noticing that $\sqrt{\varrho_0}\pi \in L^1(\mathbb{R}\times(0,T))$ and $\sqrt{\varrho_0}v_y \in L^1(\mathbb{R}\times(0,T))$, one can get

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}} \pi\eta_r' dyd\tau\right| &= \left|\int_0^t\int_{r<|y|<2r} \pi\eta_r' dyd\tau\right| \\ &\le M_0\int_0^t\int_{r<|y|<2r} \sqrt{\varrho_0}\pi dyd\tau \to 0, \quad \text{as } r\to\infty,\end{aligned}$$

and

$$\left|\int_0^t\int_{\mathbb{R}}\frac{v_y}{J}\eta_r'dyd\tau\right| = \left|\int_0^t\int_{r<|y|<2r}\frac{v_y}{J}\eta_r'dyd\tau\right|$$
$$\leq \frac{M_0}{c_0}\int_0^t\int_{r<|y|<2r}\sqrt{\varrho_0}|v_y|dyd\tau \to 0, \quad \text{as } r\to\infty,$$

where $c_0 = e^{-\frac{2\sqrt{2}}{\mu}\sqrt{\mathcal{E}_0}\|\varrho_0\|_1}$. Therefore, (4.10) holds and, consequently, the conclusion follows. □

The following proposition on the global in time a priori estimates for the effective viscous flux G is the key for proving the global existence of strong solutions.

Proposition 4.4. *Under the same assumptions as in Proposition 4.1, it holds that*

$$\sup_{0\leq t\leq T}\|G\|_2^2+\int_0^T\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2dt+\left(\int_0^T\|G\|_\infty^4dt\right)^{\frac{1}{2}}\leq Ce^{CT}\|G_0\|_2^2,$$

for a positive constant C depending only on $\gamma,\mu,\bar{\varrho},\mathcal{E}_0$, and $\|\varrho_0\|_1$.

Proof. Let η and η_r be the same functions as in the proof of Proposition 4.1. Note that G satisfies (1.14). Multiplying (1.14) by $JG\eta_r^2$ and integrating the resultant over $\mathbb{R}$, one gets by integration by parts that

$$\int_{\mathbb{R}}JGG_t\eta_r^2dy+\mu\int_{\mathbb{R}}\frac{(G_y)^2\eta_r^2}{\varrho_0}dy=-\gamma\int_{\mathbb{R}}v_yG^2\eta_r^2dy-2\mu\int_{\mathbb{R}}\frac{G_y}{\varrho_0}G\eta_r\eta_r'dy. \tag{4.11}$$

On the other hand, (1.10) implies that

$$\int_{\mathbb{R}}JGG_t\eta_r^2dy=\frac{1}{2}\left(\frac{d}{dt}\int_{\mathbb{R}}JG^2\eta_r^2dy-\int_{\mathbb{R}}v_yG^2\eta_r^2dy\right).$$

Plugging the above into (4.11) and integrating by parts show that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}JG^2\eta_r^2dy+\mu\int_{\mathbb{R}}\frac{(G_y)^2\eta_r^2}{\varrho_0}dy=\left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}}v_yG^2\eta_r^2dy-2\mu\int_{\mathbb{R}}\frac{GG_y}{\varrho_0}\eta_r\eta_r'dy$$
$$=(2\gamma-1)\int_{\mathbb{R}}vGG_y\eta_r^2dy+\int_{\mathbb{R}}\left[(2\gamma-1)vG^2-2\mu\frac{GG_y}{\varrho_0}\right]\eta_r\eta_r'dy,$$

which implies that

$$\int_{\mathbb{R}} JG^2\eta_r^2 dy + 2\mu\int_0^t\int_{\mathbb{R}}\frac{(G_y)^2\eta_r^2}{\varrho_0}dyd\tau$$

$$= 2\int_0^t\int_{\mathbb{R}}\left((2\gamma-1)vG^2 - 2\mu\frac{G_y}{\varrho_0}G\right)\eta_r\eta_r' dyd\tau$$

$$+(4\gamma-4)\int_0^t\int_{\mathbb{R}} vGG_y\eta_r^2 dyd\tau + \int_{\mathbb{R}} G_0^2\eta_r^2 dy, \tag{4.12}$$

for any $t\in[0,T]$.

We are going to show that

$$\lim_{r\to\infty}\int_0^t\int_{\mathbb{R}} vGG_y\eta_r^2 dyd\tau = \int_0^t\int_{\mathbb{R}} vGG_y dyd\tau, \tag{4.13}$$

$$\lim_{r\to\infty}\int_0^t\int_{\mathbb{R}} vG^2\eta_r\eta_r' dyd\tau = \lim_{r\to\infty}\int_0^t\int_{\mathbb{R}}\frac{G_y}{\varrho_0}G\eta_r\eta_r' dyd\tau = 0, \tag{4.14}$$

for any $t\in[0,T]$. For (4.13), it suffices to show $vGG_y\in L^1(\mathbb{R}\times(0,T))$. By the regularities of (J,v,π), one can check that

$$G\in L^\infty(0,T;L^2), \quad \frac{G_y}{\sqrt{\varrho_0}}\in L^2(0,T;L^2),$$

and further $G\in L^2(0,T;L^\infty)$ by the Sobolev embedding. As a result, by the Hölder inequality, we have $vGG_y=\sqrt{\varrho_0}vG\frac{G_y}{\sqrt{\varrho_0}}\in L^1(0,T;L^1)$ and, thus, (4.13) holds.

Observing that $\sqrt{\varrho_0}vG^2\in L^1(0,T;L^1)$ and $\frac{G_yG}{\sqrt{\varrho_0}}\in L^1(0,T;L^1)$, and recalling $supp\,\eta_r\subset(-2r,r)\cup(r,2r)$ and (4.4), we have

$$\left|\int_0^t\int_{\mathbb{R}} vG^2\eta_r\eta_r' dyd\tau\right| = \left|\int_0^t\int_{r<|y|<2r} vG^2\eta_r\eta_r' dyd\tau\right|$$

$$\le M_0\int_0^t\int_{r<|y|<2r}\sqrt{\varrho_0}|v|G^2 dyd\tau\to 0, \quad \text{as } r\to\infty,$$

and

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{G_y}{\varrho_0}G\eta_r\eta_r'dyd\tau\right| &= \left|\int_0^t\int_{r<|y|<2r}\frac{G_y}{\varrho_0}G\eta_r\eta_r'dyd\tau\right|\\ &\le M_0\int_0^t\int_{r<|y|<2r}\frac{|G_y||G|}{\sqrt{\varrho_0}}dyd\tau\to 0,\quad \text{as } r\to\infty,\end{aligned}$$

for any $t\in[0,T]$. Therefore, (4.14) holds.

Due to (4.13) and (4.14), by taking $r\to\infty$, one obtains from (4.12) that

$$\int_{\mathbb{R}}JG^2dy+2\mu\int_0^t\int_{\mathbb{R}}\frac{(G_y)^2}{\varrho_0}dyd\tau=(4\gamma-2)\int_0^t\int_{\mathbb{R}}vGG_ydyd\tau+\int_{\mathbb{R}}G_0^2dy,$$

which and Proposition 4.2 yield

$$\|G\|_2^2(t)+\int_0^t\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2d\tau\le C\left(\int_0^t\int_{\mathbb{R}}|v||G||G_y|dyd\tau+\|G_0\|_2^2\right), \tag{4.15}$$

for any $t\in[0,T]$, where C is a positive constant depending only on $\gamma,\mu,\mathcal{E}_0$, and $\|\varrho_0\|_1$. It follows from the Hölder, Yong, and Gagliardo-Nirenberg inequalities, and Proposition 4.1 that

$$\begin{aligned}&C\int_{\mathbb{R}}|v||G||G_y|dy=C\int_{\mathbb{R}}\sqrt{\varrho_0}|v||G|\frac{|G_y}{\sqrt{\varrho_0}}dy\\ &\le C\|\sqrt{\varrho_0}v\|_2\|G\|_\infty\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2\le C\|\sqrt{\varrho_0}v\|_2\|G\|_2\|G_y\|_2^{\frac12}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2\\ &\le C\|G\|_2^{\frac12}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^{\frac32}\le\frac12\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+C\|G\|_2^2,\end{aligned}$$

for a positive constant C depending only on $\gamma,\mu,\bar{\varrho},\mathcal{E}_0$, and $\|\varrho_0\|_1$. Plugging the above estimate into (4.15) yields

$$\|G\|_2^2(t)+\int_0^t\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2d\tau\le C\left(\int_0^t\|G\|_2^2d\tau+\|G_0\|_2^2\right),$$

for any $t\in[0,T]$, for a positive constant C depending only on $\gamma,\mu,\bar{\varrho},\mathcal{E}_0$, and $\|\varrho_0\|_1$. Then the Gronwall inequality shows that

$$\sup_{0\le t\le T}\|G\|_2^2+\int_0^T\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2dt\le Ce^{CT}\|G_0\|_2^2$$

and, consequently, by the Gagliardo-Nirenberg inequality, one has

$$\int_0^T \|G\|_\infty^4 dt \le C\int_0^T \|G\|_2^2\|G_y\|_2^2 dt \le Ce^{CT}\|G_0\|_2^4,$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \mathcal{E}_0$, and $\|\varrho_0\|_1$. $\square$

Based on Proposition 4.4, similar to Proposition 2.3, one can obtain the following a priori estimates.

Proposition 4.5. *Under the same assumptions as in Proposition 4.1, it holds that*

$$\sup_{0\le t\le T}\left\|\left(J-1, \frac{J_y}{\sqrt{\varrho_0}}, J_t, v_y, \pi, \frac{\pi_y}{\sqrt{\varrho_0}}\right)\right\|_2 \le C,$$

and

$$\int_0^T \left(\|\pi_t\|_2^4 + \left\|\left(\frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}v_t\right)\right\|_2^2\right) dt \le C,$$

for a positive constant C depending only on $\gamma, \mu, \bar{\varrho}, \mathcal{E}_0$, $\|\varrho_0\|_1$, $\|\pi_0\|_2$, $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, and T, and the constant C, viewing as a function of T, is continuously increasing with respect to $T \in [0,\infty)$.

Proof. The proof is almost the same as that for Proposition 2.3, and the only difference is that the role played by Proposition 2.2 there is now played by Proposition 4.4, and in addition here we have to estimate the upper bound of J on $[0,T]$. Therefore, we only sketch the proof here.

Repeating the arguments as those for (2.13), (2.15), (2.16), (2.19), and (2.21), but applying Proposition 4.4, instead of applying Proposition 2.2 there, one obtains

$$\sup_{0\le t\le T}\left\|\left(\frac{J_y}{\sqrt{\varrho_0}}, \pi, \frac{\pi_y}{\sqrt{\varrho_0}}\right)\right\|_2 + \int_0^T (\|\pi_t\|_2^4 + \|\sqrt{\varrho_0}v_t\|_2^2)dt \le C, \tag{4.16}$$

here and throughout the proof of this proposition, C is a positive constant depending only on $\gamma, \mu, \bar{\varrho}, \mathcal{E}_0$, $\|\varrho_0\|_1$, $\|\pi_0\|_2$, $\|\frac{\pi_0'}{\sqrt{\varrho_0}}\|_2$, and T, and it is continuously increasing with respect to $T \in [0,\infty)$. It follows from (4.16) and the Sobolev embedding inequality that $\sup_{0\le t\le T}\|\pi\|_\infty \le C$. Since $J(y,t) = e^{\frac{1}{\mu}\int_0^t (G+\pi)d\tau}$, it follows from Proposition 4.4 and (4.16) that $\sup_{0\le t\le T}\|J\|_\infty \le C$. Thanks to (4.16) and $\sup_{0\le t\le T}(\|\pi\|_\infty + \|J\|_\infty) \le C$, repeating the arguments as those for (2.18), (2.20), and (2.22), but applying Proposition 4.4, instead of applying Proposition 2.2 there, one can get

$$\sup_{0\le t\le T}\|(J-1,J_t,v_y)\|_2^2+\int_0^T\left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2dt\le C.$$

Thus, Proposition 4.5 is proved. □

The next proposition will be the key to show the global in time boundedness of the entropy.

Proposition 4.6. *Let the assumption in Proposition 4.1 hold. Assume in addition that (H3) holds and* $\frac{G}{\varrho_0^{\frac{\delta}{2}}}\in L^2(0,T;L^2)$. *Then, it holds that*

$$\sup_{0\le t\le T}\left\|\frac{G}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2+\int_0^T\left\|\frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}}\right\|_2^2dt\le C\left\|\frac{G_0}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2,$$

for a positive constant C *depending only on* $\gamma,\mu,\delta,K_0,\bar\varrho,\mathcal{E}_0$, $\|\varrho_0\|_1$, $\|\pi_0\|_2$, $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, *and* T, *and the constant* C, *viewing as a function of* T, *is continuously increasing with respect to* $T\in[0,\infty)$.

Proof. Let η_r be the same function as in the proof of Proposition 4.1. Multiplying equation (1.14) by $\frac{JG\eta_r^2}{\varrho_0^\delta}$, and integrating the resultant over $\mathbb{R}$, one obtains through integration by parts that

$$\int_{\mathbb{R}}\frac{JG\eta_r^2}{\varrho_0^\delta}G_tdy+\mu\int_{\mathbb{R}}\frac{G_y}{\varrho_0}\left(\frac{G\eta_r^2}{\varrho_0^\delta}\right)_ydy=-\gamma\int_{\mathbb{R}}v_y\frac{G^2\eta_r^2}{\varrho_0^\gamma}dy. \tag{4.17}$$

Direct calculations yield

$$\int_{\mathbb{R}}\frac{G_y}{\varrho_0}\left(\frac{G\eta_r^2}{\varrho_0^\delta}\right)_ydy=\int_{\mathbb{R}}\frac{G_y}{\varrho_0}\left[\left(\frac{G_y}{\varrho_0^\delta}-\delta\frac{\varrho_0'}{\varrho_0}\frac{G}{\varrho_0^\delta}\right)\eta_r^2+2\frac{G}{\varrho_0^\delta}\eta_r\eta_r'\right]dy. \tag{4.18}$$

Using equation (1.10), one can get

$$\int_{\mathbb{R}}\frac{JG\eta_r^2}{\varrho_0^\delta}G_tdy=\frac12\left(\frac{d}{dt}\int_{\mathbb{R}}\frac{JG^2\eta_r^2}{\varrho_0^\delta}dy-\int_{\mathbb{R}}v_y\frac{G^2\eta_r^2}{\varrho_0^\delta}dy\right). \tag{4.19}$$

Plugging (4.18) and (4.19) into (4.17), one can get from the assumption $|(\frac{1}{\sqrt{\varrho_0}})'|\le\frac{K_0}{2}$ and the Cauchy inequality that

$$\frac12\frac{d}{dt}\int_{\mathbb{R}}\frac{JG^2\eta_r^2}{\varrho_0^\delta}dy+\mu\int_{\mathbb{R}}\frac{G_y^2\eta_r^2}{\varrho_0^{\delta+1}}dy$$

$$
\begin{aligned}
&= \left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}} v_y \frac{G^2\eta_r^2}{\varrho_0^\delta}dy + \mu\int_{\mathbb{R}}\left(\delta\frac{\varrho_0'}{\varrho_0}\eta_r^2 - 2\eta_r\eta_r'\right)\frac{GG_y}{\varrho_0^{\delta+1}}dy \\
&\le \gamma\int_{\mathbb{R}} |v_y|\frac{G^2\eta_r^2}{\varrho_0^\delta}dy + \mu\int_{\mathbb{R}}(\delta K_0\sqrt{\varrho_0}\eta_r^2 + 2\eta_r|\eta_r'|)\frac{GG_y}{\varrho_0^{\delta+1}}dy \\
&\le \frac{\mu}{2}\int_{\mathbb{R}}\frac{G_y^2\eta_r^2}{\varrho_0^{\delta+1}}dy + C\int_{\mathbb{R}}\left[(|v_y|+1)\frac{G^2\eta_r^2}{\varrho_0^\delta} + \frac{G^2}{\varrho_0^{\delta+1}}|\eta_r'|^2\right]dy, \qquad (4.20)
\end{aligned}
$$

for a positive constant C depending only on γ, μ, δ, and K_0. It follows from $supp\,\eta_r \subseteq (-2r,r)\cup(r,2r)$ and (4.4) that

$$
\int_{\mathbb{R}} \frac{G^2}{\varrho_0^{\delta+1}}|\eta_r'|^2 dy \le M_0^2 \int_{r<|y|<2r} \frac{G^2}{\varrho_0^\delta}dy, \quad r \ge 1.
$$

This, together with Proposition 4.2 and (4.20), implies that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\sqrt{\frac{J}{\varrho_0^\delta}}G\eta_r\right\|_2^2 + \mu\left\|\frac{G_y\eta_r}{\varrho_0^{\frac{\delta+1}{2}}}\right\|_2^2 \\
&\le C(\|v_y\|_\infty + 1)\left\|\sqrt{\frac{J}{\varrho_0^\delta}}G\eta_r\right\|_2^2 + CM_0^2\int_{r<|y|<2r}\frac{G^2}{\varrho_0^\delta}dy, \qquad (4.21)
\end{aligned}
$$

for $r \ge 1$, and for a positive constant C depending only on $\gamma, \mu, \delta, K_0, \mathcal{E}_0$, and $\|\varrho_0\|_1$. By Proposition 4.5, it follows from the Sobolev embedding inequality that $\int_0^T \|v_y\|_\infty dt \le C$, for a positive constant C depending only on $\gamma, \mu, \bar\varrho, \mathcal{E}_0$, $\|\varrho_0\|_1$, $\|\pi_0\|_2$, $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, and T, and C is continuously increasing with respect to $T \in [0,\infty)$. Thanks to this, by applying the Gronwall inequality to (4.21), we obtain

$$
\sup_{0\le t\le T}\left\|\sqrt{\frac{J}{\varrho_0^\delta}}G\eta_r\right\|_2^2 + \int_0^T\left\|\frac{G_y\eta_r}{\varrho_0^{\frac{\delta+1}{2}}}\right\|_2^2 dt \le C\left(\left\|\frac{G_0}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2 + M_0^2\int_0^T\int_{r<|y|<2r}\frac{G^2}{\varrho_0^\delta}dydt\right),
$$

for $r \ge 1$. Note that the assumption $\frac{G}{\varrho_0^{\frac{\delta}{2}}} \in L^2(0,T;L^2))$ implies that the last term of the right hand side of the above inequality tends to zero as $r \to \infty$. Thus, we obtain by taking $r \to \infty$ in the inequality above that

$$
\sup_{0\le t\le T}\left\|\sqrt{\frac{J}{\varrho_0^\delta}}G\right\|_2^2 + \int_0^T\left\|\frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}}\right\|_2^2 dt \le C\left\|\frac{G_0}{\varrho_0^{\frac{\delta}{2}}}\right\|_2^2,
$$

for a positive constant C depending only on $\gamma, \mu, \delta, K_0, \bar{\varrho}, \mathcal{E}_0$, $\|\varrho_0\|_1$, $\|\pi_0\|_2$, $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, and T, and C is continuously increasing with respect to $T \in [0, \infty)$. This, together with Proposition 4.2, yields the desired conclusion. $\square$

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2. (i) By (i) of Theorem 1.1, there is a strong solution (J, v, π) to the problem (1.10)–(1.13), on $\mathbb{R} \times (0, T_0)$, for some positive time T_0. Extend this local strong solution to the maximal time of existence T_*. Then, (J, v, π) is a strong solution to the problem (1.10)–(1.13), one $\mathbb{R} \times (0, T)$, for any $T \in (0, T_*)$. Denote $\mathcal{E}_0 = \int_{\mathbb{R}} \left(\frac{\varrho_0 v_0^2}{2} + \frac{\pi_0}{\gamma-1}\right) dy$. By Propositions 4.1–4.3, we have

$$
\begin{gathered}
\left[\int_{\mathbb{R}} \left(\frac{\varrho_0 v^2}{2} + \frac{J\pi}{\gamma - 1}\right) dy\right](t) = \mathcal{E}_0, \\
\inf_{y \in \mathbb{R}} J(y, t) \geq \exp\left\{-\frac{2\sqrt{2}}{\mu}\sqrt{\mathcal{E}_0}\|\varrho_0\|_1\right\}, \\
\left(\int_{\mathbb{R}} \varrho_0 v dy\right)(t) = \int_{\mathbb{R}} \varrho_0 v_0 dy,
\end{gathered}
$$

for any $t \in [0, T_*)$. The conclusion will follow if $T_* = \infty$.

Assume, by contradiction, that $T_* < \infty$. By Proposition 4.5, it follows that

$$
\begin{gathered}
\sup_{0 \leq t \leq T} \left\|\left(J - 1, \frac{J_y}{\sqrt{\varrho_0}}, J_t, v_y, \pi, \frac{\pi_y}{\sqrt{\varrho_0}}\right)\right\|_2 \leq C, \\
\int_0^T \left(\|\pi_t\|_2^4 + \left\|\left(\frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0} v_t\right)\right\|_2^2\right) dt \leq C,
\end{gathered}
$$

for any $T \in [0, T_*)$, where C is a positive constant depending only on γ, μ, $\bar{\varrho}$, $\mathcal{E}_0$, $\|\varrho_0\|_1$, $\|\pi_0\|_2$, $\left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, and T, and this constant C, viewing as a function of T, is continuously increasing with respect to $T \in [0, \infty)$. Since T_* is a finite positive number, the positive constants in the above are actually independent of $T \in (0, T_*)$. Thanks to the above estimates, one can obtain that

$$
\inf_{y \in \mathbb{R}} J(y, T_1) > 0, \quad J(\cdot, T_1) \in L^\infty, \quad \left(\frac{\partial_y J}{\sqrt{\varrho_0}}, \sqrt{\varrho_0} v, \partial_y v, \pi, \frac{\partial_y \pi}{\sqrt{\varrho_0}}\right)(\cdot, T_1) \in L^2,
$$

and

$$
\frac{1}{\inf_{y \in \mathbb{R}} J(y, T_1)} + \left(\|J\|_\infty + \|G\|_2 + \|\pi\|_\infty\right)(T_1) \leq C_0,
$$

for a positive constant C_0 independent of $T_1 \in (0, T_*)$. Therefore, by Theorem 1.1, there is a positive time t_0, such that, starting from time $T_* - \frac{t_0}{2}$, one can extend the strong solution (J, v, π) uniquely to another time $T_* - \frac{t_0}{2} + t_0 = T_* + \frac{t_0}{2} > T_*$, which contradicts to the definition of T_*. Therefore, $T_* = \infty$ and, thus, one obtains a unique global strong solution to the problem (1.10)–(1.13).

(ii) We prove only that (1.15) holds for any finite $T \in (0, \infty)$, while the validity of (1.16) follows from (1.15) and (i) by exactly the same arguments as in the proof of Theorem 1.1. By (ii) of Theorem 1.1, there is a positive time T such that (1.15) holds. Denote by T_ℓ the maximal time such that (1.15) holds for any $T \in (0, T_\ell)$. In order to verify that (1.15) holds for any finite $T \in (0, \infty)$, it suffices to show that $T_\ell = \infty$. Assume, by contradiction, that $T_\ell < \infty$. By Proposition 4.6, the following estimate holds

$$\sup_{0 \le t \le T} \left\| \frac{G}{\varrho_0^{\frac{\delta}{2}}} \right\|_2^2 + \int_0^T \left\| \frac{G_y}{\varrho_0^{\frac{\delta+1}{2}}} \right\|_2^2 dt \le C \left\| \frac{G_0}{\varrho_0^{\frac{\delta}{2}}} \right\|_2^2,$$

for any $T \in (0, T_\ell)$, where C is a positive constant depending only on $\gamma, \mu, \delta, K_0, \bar{\varrho}, \mathcal{E}_0$, $\|\varrho_0\|_1, \|\pi_0\|_2, \left\|\frac{\pi_0'}{\sqrt{\varrho_0}}\right\|_2$, and T, and this constant C, viewing as a function of T, is continuously increasing with respect to $T \in [0, \infty)$. Since T_ℓ is a positive finite number, the constant C above is actually independent of $T \in (0, T_\ell)$. It follows from this fact that

$$\left. \frac{G(\cdot, T_\ell)}{\varrho_0^{\frac{\delta}{2}}(\cdot)} \right|_{t = T_\ell} \in L^2.$$

With the aid of this, taking T_ℓ as the initial time, by Theorem 1.1, one can see that (1.15) holds for some other time $T_\ell' > T_\ell$, which contradicts to the definition of T_ℓ. Therefore, $T_\ell = \infty$, in other words, (1.15) holds for any finite time $T \in (0, \infty)$. □

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