INSTANTANEOUS UNBOUNDEDNESS OF THE ENTROPY AND UNIFORM POSITIVITY OF THE TEMPERATURE FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH FAST DECAY DENSITY

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ABSTRACT. This paper concerns the physical behaviors of any solutions to the one dimensional compressible Navier-Stokes equations for viscous and heat conductive gases with constant viscosities and heat conductivity for fast decaying density at far fields only. First, it is shown that the specific entropy becomes not uniformly bounded immediately after the initial time, as long as the initial density decays to vacuum at the far field at the rate not slower than $O\left(\frac{1}{|x|^{\ell_\rho}}\right)$ with $\ell_\rho > 2$. Furthermore, for faster decaying initial density, i.e., $\ell_\rho \geq 4$, a sharper result is discovered that the absolute temperature becomes uniformly positive at each positive time, no matter whether it is uniformly positive or not initially, and consequently the corresponding entropy behaves as $O(-\log(\varrho_0(x)))$ at each positive time, independent of the boundedness of the initial entropy. Such phenomena are in sharp contrast to the case with slowly decaying initial density of the rate no faster than $O(\frac{1}{x^2})$, for which our previous works [34–36] show that the uniform boundedness of the entropy can be propagated for all positive time and thus the temperature decays to zero at the far field. These give a complete answer to the problem concerning the propagation of uniform boundedness of the entropy for the heat conductive ideal gases and, in particular, show that the algebraic decay rate 2 of the initial density at the far field is sharp for the uniform boundedness of the entropy. The tools to prove our main results are based on some scaling transforms, including the Kelvin transform, and a Hopf type lemma for a class of degenerate equations with possible unbounded coefficients.

1. INTRODUCTION

The compressible Navier–Stokes equations for the ideal viscous and heat conductive gases read as

$$\partial_t\rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1}$$

$$\rho(\partial_t u + (u\cdot\nabla)u) - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}\, u + \nabla p = 0, \tag{1.2}$$

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$$c_v\rho(\partial_t\theta + u\cdot\nabla\theta) + p\mathrm{div}\, u - \kappa\Delta\theta = \mathscr{Q}(\nabla u), \tag{1.3}$$

where the unknowns $\rho \geq 0$, $u \in \mathbb{R}^N$, with N the spatial dimension, $\theta \geq 0$, and $p = R\rho\theta$, respectively, represent the density, velocity, temperature, and pressure. Here, R and c_v are positive constants, μ and λ are the viscous coefficients, both assumed to be constants and satisfy the physical constraints $\mu > 0$ and $2\mu + N\lambda > 0$, κ is the heat conductive coefficient, assumed to be a positive constant, and $\mathscr{Q}(\nabla u)$ is a quadratic term of ∇u given as

$$\mathscr{Q}(\nabla u) = \frac{\mu}{2}|\nabla u + (\nabla u)^T|^2 + \lambda(\mathrm{div}\, u)^2.$$

By the Gibbs equation $\theta Ds = De + pD(\frac{1}{\rho})$, where s is the specific entropy and $e = c_v\theta$ is the specific internal energy, it holds that $p = Ae^{\frac{s}{c_v}}\rho^\gamma$ for some positive constant A, where $\gamma - 1 = \frac{R}{c_v}$. It is clear that $\gamma > 1$. In terms of ρ and θ, the specific entropy s can be expressed as

$$s = c_v\left(\log\frac{R}{A} + \log\theta - (\gamma-1)\log\rho\right), \tag{1.4}$$

satisfying

$$\rho(\partial_t s + u\cdot\nabla s) - \frac{\kappa}{c_v}\Delta s = \kappa(\gamma-1)\mathrm{div}\left(\frac{\nabla\rho}{\rho}\right) + \frac{1}{\theta}\left(\mathcal{Q}(\nabla u) + \kappa\frac{|\nabla\theta|^2}{\theta}\right), \tag{1.5}$$

in the region where both ρ and θ are positive.

As the governing system in the gas dynamics, the compressible Navier–Stokes equations have been studied extensively. One of the central concepts in the mathematical theory for the compressible Navier–Stokes equations is the vacuum, which, if occurs, means that the density vanishes at either some interior points or on the boundary or at the far fields. Indeed, the possible presence of vacuum is one of the main difficulties in the theory of global well-posedness of general solutions to the compressible Navier–Stokes equations. Note that the equation (1.5) for the entropy is highly degenerate and singular near the vacuum, it is even more difficult to analyze the dynamic behavior of the entropy in the presence of vacuum. Due to this, most of the mathematical theories developed in the existing literatures on the compressible Navier–Stokes equations in the presence of vacuum are for system (1.1)–(1.3) regardless of the entropy.

There are extensive literatures on the mathematical studies concerning the compressible Navier–Stokes equations (1.1)–(1.3). In the one-dimensional case, the corresponding theory is satisfactory and in particular the global well-posedness has been known for long time. In the absence of vacuum, for which the information of the entropy follows from that of the density and the temperature directly by (1.4), the global well-posedness of strong solutions was established by Kazhikov–Shelukin [24] and Kazhikov [25], which were later extended in the setting of weak solutions, see, e.g., [2, 23, 58, 59]; large time behavior of solutions with general initial data was proved by Li–Liang [32]. In the presence of vacuum, but without considering the entropy, the corresponding global well-posedness were established by the first author of this paper in [29, 30], for both heat conductive and non-heat conductive ideal gases. As shown by Hoff–Smoller [17], for the one-dimensional compressible Navier–Stokes equations, no vacuum can be formed later in finite time from non-vacuum initial data, while such a result remains open in the multidimensional case.

In the multi-dimensional case, the mathematical theory for the compressible Navier–Stokes equations is less complete than that in the one-dimensional case. The breakthrough for the global existence of finite energy weak solutions with general initial data and possible vacuum, to the isentropic compressible Navier–Stokes equations, was achieved by Lions [37, 38]. The results of Lions [37, 38] were later improved by Feireisl–Novotný–Petzeltová [12], Jiang–Zhang [22], and more recently Bresch–Jabin [1]. For the full compressible Navier–Stokes equations, the global existence of variational weak solutions was proved by Feireisl [14], under some assumptions on the equations of states. The uniqueness of weak solutions is still a challenging open problem. If the initial datum is suitably regular, then the compressible Navier–Stokes equations admit a unique local strong or classic solution, see [21, 39, 46, 48, 50, 51, 53] for the case in the absence of vacuum, and [5–7, 15, 18, 31, 49] for the case in the presence of vacuum. However, the corresponding global existence with general initial data may not be expected, due to the recent finite time blow up results by Merle–Rapha'el–Rodnianski–Szeftel [44, 45], where for the three-dimensional isentropic compressible Navier–Stokes equations with spherical symmetry, regular solutions with finite time singularities are constructed for a class of initial data with far field vacuum. Indeed, up to now, global strong or classical solutions are established only under some additional conditions on the initial data: the case with small perturbed initial data around non-vacuum equilibriums was achieved by Matsumura–Nishida [40–43], and later developed in many works, see, e.g., [3, 4, 8–11, 16, 26, 47, 52]; while the case with initial data of small energy but allowing large oscillations and vacuum was proved by Huang–Li–Xin [20] and Li–Xin [33] for the isentropic system, and later generalized to the full system in [19, 28, 54].

It is worth pointing out that there are some significant differences in the mathematical theories for the compressible Navier–Stokes equations between the vacuum and non-vacuum cases and new phenomena may occur depending on the locations and states of vacuum. In the non-vacuum case, the solutions can be establish in both the homogeneous and inhomogeneous spaces depending on the properties of the initial data, and the solution spaces guarantee the uniform boundedness of the entropy. However, these may fail in general in the presence of vacuum. Indeed, in the case that the density has compact support, the solution can be established in the homogeneous spaces, see, e.g., [5–7, 15, 18, 20, 31], but not in the inhomogeneous spaces, see Li–Wang–Xin [27]. Further more, the blowup results of Xin [56] and Xin–Yan [57] imply that the global solutions established in [19, 28, 54] must have unbounded entropy, if initially there is an isolated mass group surrounded by the vacuum region.

However, it is somewhat surprising that if the initial density vanishes only at far fields with a rate no more than $O(\frac{1}{|x|^2})$, then, as for the non-vacuum case, the solutions can be established in both the homogeneous and inhomogeneous spaces, and the entropy can be uniformly bounded, see the recent works by the authors [34–36].

It should be noted that since system (1.1)–(1.3) is already closed, one can indeed establish self-contained mathematical theories for it, as already developed in the previous works mentioned above. However, since the second law of the thermodynamics is not taken in to account, these theories are insufficient from the physical point of view. Therefore, some new theories are needed to provide information for the entropy in the presence of vacuum to meet the physical requirements. However, due to the lack of the expression and high singularity and degeneracy of the governing equation for the entropy near the vacuum region, in spite of its importance, the mathematical analysis of the entropy for the viscous compressible fluids in the presence of vacuum was rarely carried out before. Mathematical studies towards this direction has been initiated in our previous works [34, 35] and further developed in [36], where the propagation of the uniform boundedness of the entropy and the inhomogeneous Sobolev regularities was achieved for the compressible Navier–Stokes equations, with or without heat conductivities, in the presence of vacuum at the far fields, under the crucial condition that the initial density decays to vacuum at the rate no faster than $O(\frac{1}{|x|^2})$.

In this paper, we continue our studies on the dynamic behavior of the entropy in the presence of vacuum. Different from the cases considered in [34–36], where the density decays slowly to the vacuum at far fields, in the current paper, we investigate the case with fast decaying density at the far fields. For simplicity, we study the one-dimensional case in the current paper while leave the multi-dimensional case as future works. It will be shown in this paper that, in sharp contrast to the cases with slowly decaying density in [34–36], the uniform boundedness of the entropy can not be propagated by the compressible Navier–Stokes equations for viscous and heat conductive ideal gases with constant viscosities and heat conductivities, if the initial density decays faster than the order $O(\frac{1}{|x|^{\ell_\rho}})$ at the far fields with $\ell_\rho > 2$. Since the uniform boundedness of the entropy has already been established in [34–36] if the decay rate is less than $O(\frac{1}{|x|^2})$, our results in this paper reveal that the decay rate 2 of the initial density at the far field is sharp for the uniform boundedness of the entropy. Surprisingly, in case that the initial density decays faster than the order $O(\frac{1}{x^4})$, some sharper results can be achieved: the temperature is uniformly positive immediately after the initial time, for any general nonnegative (not identically zero) initial temperature, and, as a result, the entropy tends to infinity at the order $O(-\log(\varrho_0(x)))$ at any positive time.

Consider the Cauchy problem to the one-dimensional compressible Navier–Stokes equations for viscous and heat conductive ideal gases

$$\rho_t + (\rho u)_x = 0, \tag{1.6}$$

$$\rho(u_t + uu_x) - \mu u_{xx} + p_x = 0, \tag{1.7}$$

$$c_v\rho(\theta_t + u\theta_x) + pu_x - \kappa\theta_{xx} = \mu(u_x)^2, \tag{1.8}$$

where $p = R\rho\theta$, subject to the initial condition

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0). \tag{1.9}$$

The main results of this paper will be stated and proved in the Lagrangian coordinates; however, since the velocity of the solutions obtained in this paper have Lipschitz regularities in the spatial variable, the results can be transformed back to those in the Eulerian coordinates.

Define the coordinate transform between the Lagrangian coordinate y and the Eulerian coordinate x as $x = \eta(y, t)$ satisfying

$$\begin{cases} \partial_t\eta(y, t) = u(\eta(y, t), t), \\ \eta(y, 0) = y. \end{cases}$$

Set

$$\varrho(y, t) := \rho(\eta(y, t), t), \quad v(y, t) := u(\eta(y, t), t), \quad \vartheta(y, t) := \theta(\eta(y, t), t),$$

and

$$J := J(y, t) = \eta_y(y, t).$$

Then, it holds that

$$J_t = v_y, \quad J|_{t=0} \equiv 1, \quad J\varrho = \varrho_0,$$

with $\varrho_0 := \rho_0$. We still use s to denote the specific entropy in the Lagrangian coordinates. Then, it follows from (1.4) that

$$s(y, t) = c_v\left(\log\frac{R}{A} + \log\vartheta(y, t) - (\gamma - 1)\log\varrho_0(y) + (\gamma - 1)\log J(y, t)\right), \tag{1.10}$$

for any $y \in \mathbb{R}$ and $t \in [0, \infty)$.

Then, in the Lagrangian coordinates, the system (1.6)–(1.8) becomes

$$J_t = v_y, \tag{1.11}$$

$$\varrho_0 v_t - \mu\left(\frac{v_y}{J}\right)_y + \pi_y = 0, \tag{1.12}$$

$$c_v\varrho_0\vartheta_t + v_y\pi - \kappa\left(\frac{\vartheta_y}{J}\right)_y = \mu\frac{|v_y|^2}{J}, \tag{1.13}$$

where $\pi = R\frac{\varrho_0}{J}\vartheta$. The initial data can be taken as

$$(J, v, \vartheta)|_{t=0} = (1, v_0, \vartheta_0), \tag{1.14}$$

where $v_0 = u_0$ and $\vartheta_0 = \theta_0$.

The following conventions will be used throughout this paper. For $1 \leq q \leq \infty$ and positive integer m, $L^q = L^q(\mathbb{R})$ and $W^{1,q} = W^{m,q}(\mathbb{R})$ denote the standard Lebesgue and Sobolev spaces, respectively, and $H^m = W^{m,2}$. For simplicity, L^q and H^m denote also their N product spaces $(L^q)^N$ and $(H^m)^N$, respectively. $\|u\|_q$ is the L^q norm of u, and $\|(f_1, f_2, \cdots, f_n)\|_X$ is the sum $\sum_{i=1}^N \|f_i\|_X$ or the equivalent norm $\left(\sum_{i=1}^N \|f_i\|_X^2\right)^{\frac{1}{2}}$.

The main results of this paper are the following three theorems. The first one yields the global existence of a solution to the Cauchy problem (1.11)–(1.13), subject to (1.14).

Theorem 1.1. *Let the initial density ϱ_0 be given such that $0 < \varrho_0 \in L^1(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ and*

$$|\varrho_0'| + |\varrho_0''| \leq K_1 \varrho_0 \quad \text{on } \mathbb{R}, \tag{H1}$$

for a positive constant K_1. Assume that (v_0, ϑ_0) satisfies $\vartheta_0 \geq 0$ on $\mathbb{R}$ and

$$(\sqrt{\varrho_0} v_0, \sqrt{\varrho_0} v_0^2, v_0', v_0'', \sqrt{\varrho_0}\vartheta_0, \sqrt{\varrho_0}\vartheta_0', \sqrt{\varrho_0}\vartheta_0'') \in L^2(\mathbb{R}), \quad \frac{G_0'}{\sqrt{\varrho_0}} \in L^2(\mathbb{R}), \tag{1.15}$$

$$\varlimsup_{y \to -\infty} \frac{|v_0'(y)|}{\sqrt{\varrho_0(y)}} + \varlimsup_{y \to +\infty} \frac{|v_0'(y)|}{\sqrt{\varrho_0(y)}} < +\infty, \tag{1.16}$$

where $G_0 := \mu v_0' - R\varrho_0\vartheta_0$.

Then, there is a global solution (J, v, ϑ) to (1.11)–(1.13), subject to (1.14), satisfying $\inf_{(y,t)\in\mathbb{R}\times(0,T)} J > 0$, $\theta \geq 0$, and

$$\begin{aligned}
&\frac{J_y}{\sqrt{\varrho_0}}, J_{yy}, J_t, J_{yt} \in L^\infty(0,T; L^2(\mathbb{R})), \\
&\sqrt{\varrho_0} v, \sqrt{\varrho_0} v^2, v_y, \frac{v_{yy}}{\sqrt{\varrho_0}}, \sqrt{\varrho_0} v_t \in L^\infty(0,T; L^2(\mathbb{R})), \quad v_{yyy}, v_{yt} \in L^2(0,T; L^2(\mathbb{R})), \\
&\sqrt{\varrho_0}\vartheta, \sqrt{\varrho_0}\vartheta_y, \sqrt{\varrho_0}\vartheta_{yy}, \varrho_0^{\frac{3}{2}}\vartheta_t \in L^\infty(0,T; L^2(\mathbb{R})), \quad \vartheta_y \in L^2(0,T; H^2(\mathbb{R})), \\
&\varrho_0\vartheta_t, \varrho_0\vartheta_{yt} \in L^2(0,T; L^2(\mathbb{R})), \quad G_t, \left(\frac{G_y}{\varrho_0}\right)_y \in L^2(0,T; L^2(\mathbb{R})),
\end{aligned}$$

for any positive time T, where $G := \mu\frac{v_y}{J} - R\frac{\varrho_0}{J}\vartheta$.

Remark 1.1. *(i) Condition (H1) allows arbitrary algebraic and even exponential decay rate of ϱ_0 at far fields. Indeed, one can check that functions of the forms $\frac{A}{(1+y^2)^\ell}$ and $e^{-(1+y^2)^\delta}$, with $A, \ell \in (0,\infty)$ and $\delta \in (0, \frac{1}{2}]$, satisfy (H1). Thus, Theorem 1.1 generalizes the global existence result in our previous work [35], where some assumptions on slow decay at far fields on ϱ_0 are assumed.*

(ii) Condition (1.16) is used only to construct suitable approximated initial data for the corresponding initial boundary value problems (which are expected to converge to the Cauchy problem), see Step 1 in the proof of Theorem 1.1.

The second theorem gives the immediate unboundedness of the specific entropy if the algebraic decay rate of the initial density is greater than 2.

Theorem 1.2. *Assume, in addition to the conditions in Theorem 1.1, that*

$$(1+|y|)^{\ell_\rho}\varrho_0(y)\le K_2,\quad \forall y\in\mathbb{R}, \tag{H2}$$

for some positive constants $\ell_\rho\in(2,\infty)$ *and* K_2, *and either* ϑ_0 *is not identically zero or* v_0 *is not identically a constant. Let* (J,v,ϑ) *be a solution to system (1.11)–(1.13), subject to (1.14), satisfying the properties stated in Theorem 1.1. Then, the specific entropy* $s\notin L^\infty(\mathbb{R}\times(0,T))$, *for any positive time* $T\in(0,\infty)$.

Remark 1.2. *Theorem 1.2 reveals a completely different phenomenon from that in [34–36], where the initial density decays no faster than* $O(\frac{1}{y^2})$ *at far fields, so that the entropy keeps uniformly bounded. While Theorem 1.2 shows that if the initial density decays faster than* $O\left(\frac{1}{|y|^{\ell_\rho}}\right)$, *with* $\ell_\rho>2$, *at far fields, then the entropy becomes not uniformly bounded immediately after the initial time. Consequently, we have given a complete answer to the problem concerning the propagation of uniform boundedness of entropy for ideal gases in one dimension: the uniform boundedness of the entropy for the ideal gases, in the presence of vacuum at the far fields only in one dimension, can be propagated if and only if the algebraic decay rate of the initial density is not greater than* 2. *In other words, the decay rate* 2 *of the initial density at the far fields is sharp for the uniform boundedness of the entropy in one dimension.*

The main ingredients of the proof of Theorem 1.2 are based on using some scaling transform to transform the far field vacuum to an interior vacuum and applying a Hopf type lemma for a class of linear degenerate elliptic equations with degeneracy in the time variable and possible unbounded coefficients. The scaling transform for the temperature to be used here is

$$f(y,t):=\vartheta(y^{-\beta},t),\quad y\in(0,\infty),t\in[0,\infty),$$

for some suitably chosen $\beta>0$. Similar transform can also be introduced for negative y. Due to the continuity equation (1.6) and the assumption that the initial density reaches vacuum only at the far fields, the density remains positive on any compact interval for all positive time. Thus the equation (1.8) can be regarded a uniform parabolic equation for θ on compact domains. Consequently, the temperature will be positive on any finite interval for any positive time t by the strong maximum principle, and thus f is positive for any positive y and t. By using the properties of ϑ stated in Theorem 1.1, one can verify that $0<f\in C^{2,1}((0,\infty)\times(0,\infty))$. Assuming by contradiction that the entropy is uniformly bounded, one can extend f by zero on the positive time axis, such that $0\le f\in C([0,\infty)\times[0,\infty))$ and reaches zero on the positive time axis only. The temperature equation yields

$$a_0f_t-af_{yy}+bf_y+\tilde{c}f\ge0,\quad \text{in }(0,\infty)\times(0,\infty),$$

which motivates us to apply the Hopf type lema to f at the points on the positive time axis. By choosing β suitably, one can verify that the coefficients a_0 and $\tilde{c}$ are uniformly bounded near the positive time axis; however, the coefficient b contains an unbounded term involving $\frac{1}{y}$. Fortunately, such an unbounded term in b is of "right" sign while the remaining term in b is uniformly bounded for suitably chosen β, so that the Hopf type lemma still holds (see Lemma 4.2). Thus applying the Hopf type lemma to f near the positive time axis leads to a quantitative asymptotic behavior of the temperature at the far field. The contradiction comes from the fact that the asymptotic behavior of the temperature derived from the Hopf type lemma is not consistent with that derived from (H2) and the uniform boundedness of the entropy. This inconsistency implies that the entropy can not be uniformly bounded and thus Theorem 1.2 follows.

The third theorem gives the uniform positivity of the temperature and consequently the asymptotic unboundedness of the entropy, which are sharper results than those in Theorem 1.2, under the stronger assumption that the algebraic decay rate of the initial density at the far field is greater than 4.

Theorem 1.3. *Assume, in addition to the conditions in Theorem 1.1, that*

$$(1+|y|)^4\varrho_0(y) \leq K_3, \quad \forall y \in \mathbb{R}, \tag{H3}$$

for a positive constant K_3, and either ϑ_0 is not identically zero or v_0 is not identically a constant. Let (J, v, ϑ) be a solution to system (1.11)–(1.13), subject to (1.14), satisfying the properties stated in Theorem 1.1.

Then, the following statements hold:

(i) the temperature ϑ satisfies

$$\inf_{y\in\mathbb{R}} \vartheta(y,t) > 0, \quad \forall t \in (0,\infty);$$

(ii) the specific entropy s satisfies

$$R \leq \varliminf_{|y|\to\infty} \frac{s(y,t)}{-\log(\varrho_0(y))} \leq \varlimsup_{|y|\to\infty} \frac{s(y,t)}{-\log(\varrho_0(y))} < \infty, \quad \forall t \in (0,\infty).$$

In particular, s becomes unbounded immediately after the initial time, regardless of whether it is uniformly bounded or not at the initial time.

Remark 1.3. *It is an interesting question to show whether Theorem 1.3 still holds in the case that the algebraic decay rate of ϱ_0 lies between 2 and 4. However, as already shown in Theorem 1.2, in this case, though the uniform positivity of the temperature is not clear, yet the specific entropy becomes not uniformly bounded in any positive time.*

Recall that the temperature is positive on any finite interval for any positive time t. To obtain the positive lower bound for the temperature at any positive time, it suffices to achieve this at far fields. To this end, similar as in the proof of Theorem 1.2, we apply some scaling technique to transform the far field vacuum to an interior vacuum and take advantage of the Hopf type lemma. However, the scaling transform introduced before does not work here directly. Instead, we apply the Kelvin transform to the temperature ϑ and denote by h the transformed temperature, that is,

$$h(y,t) = y\vartheta\left(\frac{1}{y},t\right), \quad \forall y \neq 0, t \in [0,\infty),$$

which satisfies a linear degenerate equation, with all coefficients being uniformly bounded by the assumption (H3). By using the properties of ϑ stated in Theorem 1.1, one can verify that $0 \leq h \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ and more importantly $h(0,t) = 0$, where $\Omega = ((-\infty,0)\cup(0,\infty))\times(0,\infty)$. Note that different from the proof of Theorem 1.2, here the important property that $h(0,t) = 0$ holds without any condition on the entropy. By the Hopf type lemma (Lemma 4.2) and applying the strong maximum principle, we can derive that h behaves linearly near the origin at each positive time and hence obtain the uniformly positive lower bound for the temperature near the far fields. With the aid of the positive lower bound of the temperature, the asymptotic unboundedness of the entropy follows from (1.10) as J has uniform positive lower and upper bounds.

The rest of this paper is arranged as follows: in Section 2, we consider a carefully designed initial-boundary value problem for the system (1.11)–(1.13) and establish a series of a priori estimates on the solution independent of the length of the spatial interval; in Section 3, we obtain the global existence of solutions to the Cauchy problem and thus prove Theorem 1.1 by taking limit of the solutions obtained in Section 2; Section 4 is devoted to the proof of Theorem 1.2; and finally, the proof of Theorem 1.3 is given in Section 5.

Throughout this paper, C will denote a generic positive constant which may vary from place to place.

2. Initial-boundary value problem and a priori estimates

Throughout this section, we consider the initial-boundary value problem to the system (1.11)–(1.13), in $(\alpha,\beta)\times(0,\infty)$, with $-\infty < \alpha < \beta < +\infty$, subject to the initial-boundary conditions:

$$(J,v,\vartheta)|_{t=0} = (1,v_0,\vartheta_0), \tag{2.1}$$

$$(v_y,\vartheta)|_{y=\alpha,\beta} = (0,0). \tag{2.2}$$

The following global well-posedness can be proved in the same way as in [30].

Proposition 2.1. *Let $(\varrho_0,v_0,\vartheta_0) \in H^2((\alpha,\beta))$ be given such that $\varrho_0,\vartheta_0 \geq 0$ on (α,β) and $v_0'(\alpha) = v_0'(\beta) = \vartheta_0(\alpha) = \vartheta_0(\beta) = 0$. Assume that*

$$\mu v_0'' - R(\varrho_0\vartheta_0)' = \sqrt{\varrho_0}g_1, \quad \kappa\vartheta_0'' + \mu(v_0')^2 - Rv_0'\varrho_0\vartheta_0 = \sqrt{\varrho_0}g_2,$$

for two functions $g_1, g_2 \in L^2((\alpha,\beta))$.

Then, there is a unique global solution (J, v, ϑ) *to system (1.11)–(1.13), in* $(\alpha,\beta)\times[0,\infty)$*, subject to (2.1)–(2.2), satisfying* $\inf_{(y,t)\in(\alpha,\beta)\times(0,T)} J>0$, $\vartheta\geq 0$, *and*

$$\begin{aligned}&J\in C([0,T];H^2((\alpha,\beta))),\quad J_t\in L^2(0,T;H^2((\alpha,\beta))),\\&v,\vartheta\in C([0,T];H^2((\alpha,\beta)))\cap L^2(0,T;H^3((\alpha,\beta))),\quad v_t,\vartheta_t\in L^2(0,T;H^1((\alpha,\beta))),\end{aligned}$$

for any $T\in(0,\infty)$.

The rest of this section is devoted to deriving the a priori estimates, independent of α and β, on the unique global solution (J, v, ϑ) stated in Proposition 2.1. Keeping this in mind, in the rest of this section, we will always assume that (J, v, ϑ) is the solution stated in Proposition 2.1.

Throughout this section, for simplicity of notations, the norms $\|\cdot\|_q$ and $\|\cdot\|_{H^1}$ are the corresponding ones on the interval (α,β), that is,

$$\|\cdot\|_q:=\|\cdot\|_{L^q((\alpha,\beta))}\quad\text{and}\quad\|\cdot\|_{H^1}:=\|\cdot\|_{H^1((\alpha,\beta))}.$$

Denote

$$m_0:=\int_\alpha^\beta\varrho_0dy,\quad \mathscr{E}_0:=\int_\alpha^\beta\varrho_0\left(\frac{v_0^2}{2}+c_v\vartheta_0\right)dy.$$

Proposition 2.2. *It holds that*

$$\int_\alpha^\beta\varrho_0\left(\frac{v^2}{2}+c_v\vartheta\right)dy\leq\mathscr{E}_0.$$

Proof. Multiplying (1.12) with v, integrating over (α,β), and by the boundary conditions, one gets by integration by parts that

$$\frac12\frac{d}{dt}\int_\alpha^\beta\varrho_0v^2dy+\mu\int_\alpha^\beta\frac{|v_y|^2}{J}dy-\int_\alpha^\beta v_y\pi dy=0. \tag{2.3}$$

Since $\vartheta\geq0$ in $(\alpha,\beta)\times(0,\infty)$, it is clear that $\vartheta_y(\alpha,t)\geq0$ and $\vartheta_y(\beta,t)\leq0$, for any $t\in(0,\infty)$. As are result, integrating (1.13) over (α,β) and integration by parts yield

$$c_v\frac{d}{dt}\int_\alpha^\beta\varrho_0\vartheta dy+\int_\alpha^\beta v_y\pi dy\leq\mu\int_\alpha^\beta\frac{|v_y|^2}{J}dy. \tag{2.4}$$

Summing (2.3) with (2.4) and integrating with respect to t lead to the conclusion. □

Proposition 2.3. *It holds that*

$$e^{-\frac2\mu\sqrt{2m_0\mathscr{E}_0}}\leq J\leq e^{\frac4\mu\sqrt{2m_0\mathscr{E}_0}}\left(1+\frac R\mu\int_0^t\varrho_0\vartheta d\tau\right),\quad\forall t\in(0,\infty).$$

Proof. Since $v_y|_{y=\alpha}=0$ and $J|_{t=0}=1$, it follows from (1.11) that $J|_{y=\alpha}=1$. Substituting (1.11) into (1.12) yields

$$\varrho_0v_t-\mu(\log J)_{yt}+\pi_y=0,$$

from which, integrating over $(0,t)$ and using $J|_{t=0}=1$, one can get

$$\varrho_0(v-v_0)+\int_0^t \pi_y ds=\mu(\log J)_y.$$

Integrating this over (α,y) and noticing that $J|_{y=\alpha}=1$ and $\pi|_{y=\alpha}=R\frac{\varrho_0}{J}\vartheta|_{y=\alpha}=0$, one gets

$$\int_\alpha^y \varrho_0(v-v_0)dz+\int_0^t \pi ds=\mu\log J,$$

which leads to

$$J=e^{\frac{1}{\mu}\left(\int_\alpha^y \varrho_0(v-v_0)dz+\int_0^t \pi ds\right)}. \tag{2.5}$$

It follows from Proposition 2.2 and the Hölder inequality that

$$\begin{aligned}\int_\alpha^\beta \varrho_0(|v|+|v_0|)dz &\leq \left(\int_\alpha^\beta \varrho_0 dz\right)^{\frac{1}{2}}\left[\left(\int_\alpha^\beta \varrho_0 v^2 dz\right)^{\frac{1}{2}}+\left(\int_\alpha^\beta \varrho_0 v_0^2 dz\right)^{\frac{1}{2}}\right]\\ &\leq 2\sqrt{2m_0\mathscr{E}_0}.\end{aligned} \tag{2.6}$$

With the aid of (2.6) and since $\pi\geq 0$, it follows from (2.5) that

$$J\geq e^{-\frac{1}{\mu}\int_\alpha^\beta \varrho_0(|v|+|v_0|)dz}\geq e^{-\frac{2}{\mu}\sqrt{2m_0\mathscr{E}_0}}. \tag{2.7}$$

Rewrite (2.5) as $Je^{-\frac{1}{\mu}\int_\alpha^y \varrho_0(v-v_0)dz}=e^{\frac{1}{\mu}\int_0^t \pi ds}$. Thus

$$\frac{1}{\mu}J\pi\exp\left\{-\frac{1}{\mu}\int_\alpha^y \varrho_0(v-v_0)dz\right\}=\partial_t(e^{\frac{1}{\mu}\int_0^t \pi ds}).$$

Hence, one gets by noticing $J\pi=R\varrho_0\vartheta$ that

$$\exp\left\{\frac{1}{\mu}\int_0^t \pi ds\right\}=1+\frac{R}{\mu}\int_0^t \varrho_0\vartheta\exp\left\{-\frac{1}{\mu}\int_\alpha^y \varrho_0(v-v_0)dz\right\}ds.$$

Substituting this into (2.5) and using (2.6) lead to

$$\begin{aligned}J &= e^{\frac{1}{\mu}\int_\alpha^y \varrho_0(v-v_0)dz}\left(1+\frac{R}{\mu}\int_0^t \varrho_0\vartheta\exp\left\{-\frac{1}{\mu}\int_\alpha^y \varrho_0(v-v_0)dz\right\}ds\right)\\ &\leq e^{\frac{4}{\mu}\sqrt{2m_0\mathscr{E}_0}}\left(1+\frac{R}{\mu}\int_0^t \varrho_0\vartheta ds\right).\end{aligned}$$

Combining this with (2.7) yields the conclusion. □

In the rest of this section, we will always assumed that C is a general positive constant depending only on $R, c_v, \mu, \kappa, K_1, T$, and the upper bound of $\mathscr{N}_0$, but independent of α and β with $\beta-\alpha\geq 1$, where

$$\mathscr{N}_0:=\|\varrho_0\|_\infty+m_0+\mathscr{E}_0+\left\|\left(\sqrt{\varrho_0}v_0^2, v_0', v_0'', \sqrt{\varrho_0}\vartheta_0, \sqrt{\varrho_0}\vartheta_0', \sqrt{\varrho_0}\vartheta_0'', G_0, \frac{G_0'}{\sqrt{\varrho_0}}\right)\right\|_2. \tag{2.8}$$

Proposition 2.4. *It holds that*

$$\sup_{0\leq t\leq T}\|(\sqrt{\varrho_0}v^2,\sqrt{\varrho_0}\vartheta)\|_2^2+\int_0^T\left(\|\sqrt{\varrho_0}\vartheta\|_\infty^2+\left\|\frac{vv_y}{\sqrt J}\right\|_2^2+\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2\right)dt\leq C.$$

Proof. Set $E=\frac{v^2}{2}+c_v\vartheta$. Then, it follows from (1.12) and (1.13) that

$$\varrho_0E_t-\kappa\left(\frac{\vartheta_y}{J}\right)_y=\left(\mu\frac{vv_y}{J}-R\frac{\varrho_0}{J}\vartheta v\right)_y.$$

Note that $\vartheta_y(\alpha,t)\geq0$ and $\vartheta_y(\beta,t)\leq0$ due to the boundary condition $\vartheta|_{y=\alpha,\beta}=0$ and the fact that $\vartheta\geq0$ in $(\alpha,\beta)\times(0,\infty)$. Multiplying the above equation with E and integration by parts yield

$$\begin{aligned}&\frac12\frac{d}{dt}\|\sqrt{\varrho_0}E\|_2^2+\kappa c_v\int_\alpha^\beta\frac{|\vartheta_y|^2}{J}dy-\kappa E\frac{\vartheta_y}{J}\Big|_{y=\alpha}^\beta\\
\leq&-\int_\alpha^\beta\left(\mu\frac{vv_y}{J}-R\frac{\varrho_0}{J}\vartheta v\right)(vv_y+c_v\vartheta_y)dy-\kappa\int_\alpha^\beta\frac{\vartheta_y}{J}vv_ydy\\
\leq&\frac{\kappa c_v}{2}\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2+C\int_\alpha^\beta\frac1J\left(|vv_y|^2+\varrho_0^2v^2\vartheta^2\right)dy,\end{aligned}$$

and thus, by the Cauchy inequality and that $-\kappa E\frac{\vartheta_y}{J}\Big|_{y=\alpha}^\beta\geq0$, it follows that

$$\frac{d}{dt}\|\sqrt{\varrho_0}E\|_2^2+\kappa c_v\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2\leq A_1\left\|\frac{vv_y}{\sqrt J}\right\|_2^2+A_1\int_\alpha^\beta\frac1J\varrho_0^2v^2\vartheta^2dy,\tag{2.9}$$

for a positive constant A_1 depending only on κ,c_v,μ, and R. Multiplying (1.12) with $4v^3$, using the boundary conditions, and integration by parts, one deduces

$$\begin{aligned}\frac{d}{dt}\int_\alpha^\beta\varrho_0v^4dy+12\mu\left\|\frac{vv_y}{\sqrt J}\right\|_2^2&=12R\int_\alpha^\beta\frac1Jv^2v_y\varrho_0\vartheta dy\\
&\leq6\mu\left\|\frac{vv_y}{\sqrt J}\right\|_2^2+C\int_\alpha^\beta\frac1J\varrho_0^2v^2\vartheta^2dy,\end{aligned}$$

and thus,

$$\frac{d}{dt}\int_\alpha^\beta\varrho_0v^4dy+6\mu\left\|\frac{vv_y}{\sqrt J}\right\|_2^2\leq C\int_\alpha^\beta\frac1J\varrho_0^2v^2\vartheta^2dy.\tag{2.10}$$

Multiplying (2.10) with $\frac{A_1}{3\mu}$ and summing the resultant with (2.9) yield

$$\frac{d}{dt}\left(\|\sqrt{\varrho_0}E\|_2^2+\frac{A_1}{3\mu}\|\sqrt{\varrho_0}v^2\|_2^2\right)+\kappa c_v\left\|\frac{\vartheta_y}{\sqrt J}\right\|_2^2+A_1\left\|\frac{vv_y}{\sqrt J}\right\|_2^2\leq C\int_\alpha^\beta\frac1J\varrho_0^2v^2\vartheta^2dy,$$

from which, by Proposition 2.2 and Proposition 2.3, one gets

$$\frac{d}{dt}\left(\|\sqrt{\varrho_0}E\|_2^2+\frac{A_1}{3\mu}\|\sqrt{\varrho_0}v^2\|_2^2\right)+\kappa c_v\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2+A_1\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2$$
$$\leq C\|\sqrt{\varrho_0}v\|_2^2\|\sqrt{\varrho_0}\vartheta\|_\infty^2\leq C\|\sqrt{\varrho_0}\vartheta\|_\infty^2. \tag{2.11}$$

Since $\vartheta|_{y=\alpha}=0$, it follows from Proposition 2.3, the Hölder and Young inequalities, and (H1) that

$$\begin{aligned}
\varrho_0\vartheta^2 &= \int_\alpha^y(\varrho_0\vartheta^2)_y dz=\int_\alpha^y(\varrho_0'\vartheta^2+2\varrho_0\vartheta\vartheta_y)dz\\
&\leq \int_\alpha^\beta\left(K_1\varrho_0\vartheta^2+2\varrho_0\vartheta\frac{\vartheta_y}{\sqrt{J}}\sqrt{J}\right)dz\\
&\leq K_1\|\sqrt{\varrho_0}\vartheta\|_2^2+2\|\varrho_0\vartheta\|_1^{\frac12}\|\varrho_0\|_\infty^{\frac14}\|\sqrt{\varrho_0}\vartheta\|_\infty^{\frac12}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2\|J\|_\infty^{\frac12}\\
&\leq K_1\|\sqrt{\varrho_0}\vartheta\|_2^2+C\|\sqrt{\varrho_0}\vartheta\|_\infty^{\frac12}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2\left(1+\int_0^t\|\varrho_0\vartheta\|_\infty d\tau\right)^{\frac12}\\
&\leq K_1\|\sqrt{\varrho_0}\vartheta\|_2^2+C\|\sqrt{\varrho_0}\vartheta\|_\infty^{\frac12}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2\left[1+\left(\int_0^t\|\varrho_0\vartheta\|_\infty^2 d\tau\right)^{\frac14}\right]\\
&\leq \frac12\left(\|\sqrt{\varrho_0}\vartheta\|_\infty^2+\epsilon\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2\right)+C\left(1+\|\sqrt{\varrho_0}\vartheta\|_2^2+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau\right),
\end{aligned}$$

and thus

$$\|\sqrt{\varrho_0}\vartheta\|_\infty^2\leq\epsilon\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2+C_\epsilon\left(1+\|\sqrt{\varrho_0}\vartheta\|_2^2+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau\right) \tag{2.12}$$

for any $\epsilon>0$. Choosing ϵ sufficiently small and plugging (2.12) into (2.11) yield

$$\frac{d}{dt}\left(\|\sqrt{\varrho_0}E\|_2^2+\frac{A_1}{3\mu}\|\sqrt{\varrho_0}v^2\|_2^2\right)+A_1\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2+\frac{\kappa c_v}{2}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2$$
$$\leq C\left(1+\|\sqrt{\varrho_0}\vartheta\|_2^2+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau\right). \tag{2.13}$$

Combining (2.12) with (2.13) leads to

$$\frac{d}{dt}\left(\|\sqrt{\varrho_0}E\|_2^2+\frac{A_1}{3\mu}\|\sqrt{\varrho_0}v^2\|_2^2+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau\right)+A_1\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2+\frac{\kappa}{2}\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2$$
$$\leq C\left(1+\|\sqrt{\varrho_0}E\|_2^2+\int_0^t\|\sqrt{\varrho_0}\vartheta\|_\infty^2 d\tau\right),$$

which, together with the Grönwall inequality, implies that

$$\sup_{0\leq t\leq T}\|\sqrt{\varrho_0}E\|_2^2+\int_0^T\left(\|\sqrt{\varrho_0}\vartheta\|_\infty^2+\left\|\frac{vv_y}{\sqrt{J}}\right\|_2^2+\left\|\frac{\vartheta_y}{\sqrt{J}}\right\|_2^2\right)dt\leq C.$$

This completes the proof of the conclusion. □

Corollary 2.1. *There are two positive constants $\underline{C}$ and $\overline{C}$, such that*

$$\underline{C}\leq J\leq\overline{C}\quad on\ (\alpha,\beta)\times(0,T),\qquad\int_0^T\|v_y\|_2^2dt\leq C.$$

Proof. The lower bound of J follows directly from Proposition 2.3 while the upper bound of J follows from combining Proposition 2.3 and Proposition 2.4. Testing (1.12) with v and integrating by parts yield

$$\begin{aligned}\frac12\frac{d}{dt}\|\sqrt{\varrho_0}v\|_2^2+\mu\left\|\frac{v_y}{\sqrt{J}}\right\|_2^2 &= R\int_\alpha^\beta\frac{\varrho_0}{J}\vartheta v_ydy\\ \leq C\left\|\frac{v_y}{\sqrt{J}}\right\|_2\|\sqrt{\varrho_0}\vartheta\|_2 &\leq \frac{\mu}{2}\left\|\frac{v_y}{\sqrt{J}}\right\|_2^2+C\|\sqrt{\varrho_0}\vartheta\|_2^2,\end{aligned}$$

where the lower bound of J was used, and thus

$$\frac{d}{dt}\|\sqrt{\varrho_0}v\|_2^2+\mu\left\|\frac{v_y}{\sqrt{J}}\right\|_2^2\leq C\|\sqrt{\varrho_0}\vartheta\|_2^2.$$

The second conclusion follows from this, the upper bound of J just proved, and Proposition 2.4. □

In the rest of this section, we always assume that $\beta-\alpha\geq1$. We will use the following elementary inequality.

Lemma 2.1. *It holds that*

$$\|f\|_{L^p((\alpha,\beta))}\leq C(\|f\|_{L^2((\alpha,\beta))}+\|f\|_{L^2((\alpha,\beta))}^{\frac12+\frac1p}\|f'\|_{L^2((\alpha,\beta))}^{\frac12-\frac1p}),\quad p\in[2,\infty],$$

for any $f\in H^1((\alpha,\beta))$, and for a positive constant C depending only on p.

Proof. This can be proved by scaling the corresponding inequality in (α,β) to that in $(0,1)$, applying the Gagliardo-Nirenberg inequality for functions in $H^1((0,1))$, and using the condition $\beta-\alpha\geq1$. Since the proof is straightforward, and thus is omitted here. □

Let G be the effective viscous flux, i.e.,

$$G:=\mu\frac{v_y}{J}-\pi=\mu\frac{v_y}{J}-R\frac{\varrho_0\vartheta}{J}.$$

Then, it holds that

$$G_t - \frac{\mu}{J}\left(\frac{G_y}{\varrho_0}\right)_y = -\frac{\kappa(\gamma-1)}{J}\left(\frac{\vartheta_y}{J}\right)_y - \gamma\frac{v_y}{J}G \tag{2.14}$$

and

$$G|_{y=\alpha,\beta} = 0. \tag{2.15}$$

Proposition 2.5. *It holds that*

$$\sup_{0\le t\le T}\|G\|_2^2 + \int_0^T\left(\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + \|G\|_\infty^4\right)dt \le C(1+\|G_0\|_2^2).$$

Proof. Testing (2.14) with JG, using (1.11), (2.15), Lemma 2.1, Corollary 2.1, and the Young inequality, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 \\
=\ & \kappa(\gamma-1)\int_\alpha^\beta\frac{\vartheta_y G_y}{J}dy + \left(\frac{1}{2}-\gamma\right)\int_\alpha^\beta v_y G^2 dy \\
\le\ & C\left(\|\vartheta_y\|_2\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 + \|v_y\|_2\|G\|_4^2\right) \\
\le\ & C\left[\|\vartheta_y\|_2\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 + \|v_y\|_2\left(\|G\|_2^2 + \|G\|_2^{\frac{3}{2}}\|G_y\|_2^{\frac{1}{2}}\right)\right] \\
\le\ & \frac{\mu}{2}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 + C[\|\vartheta_y\|_2^2 + (1+\|v_y\|_2^2)\|G\|_2^2],
\end{aligned}$$

that is,

$$\frac{d}{dt}\|\sqrt{J}G\|_2^2 + \mu\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2 \le C[\|\vartheta_y\|_2^2 + (1+\|v_y\|_2^2)\|G\|_2^2].$$

Thanks to this and the Grönwall inequality, the desired conclusion, except the estimate on $\int_0^T\|G\|_\infty^4 dt$, follows from Proposition 2.4 and Corollary 2.1. While the estimate for $\int_0^T\|G\|_\infty^4 dt$ follows from Corollary 2.1, Lemma 2.1, and the estimate just proved. ☐

Proposition 2.6. *It holds that*

$$\sup_{0\le t\le T}\left(\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2 + \|v_y\|_2^2 + \|J_t\|_2^2\right) + \int_0^T\left(\|\sqrt{\varrho_0}v_t\|_2^2 + \left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2\right)dt \le C.$$

Proof. Note that $v_y = \frac{1}{\mu}(JG + R\varrho_0\vartheta)$ and $\sqrt{\varrho_0}v_t = \frac{G_y}{\sqrt{\varrho_0}}$. It follows from Proposition 2.4, Proposition 2.5, and Corollary 2.1 that

$$\sup_{0\leq t\leq T}\|v_y\|_2^2 + \int_0^T \|\sqrt{\varrho_0}v_t\|_2^2 dt \leq C,$$

which by (1.11) implies

$$\sup_{0\leq t\leq T}\|J_t\|_2^2 \leq C.$$

Direct calculations yield

$$J_{yt} = \frac{1}{\mu}(JG_y + J_yG + R\varrho_0'\vartheta + R\varrho_0\vartheta_y).$$

Taking the inner product of the above with $\frac{J_y}{\varrho_0}$, one obtains from Proposition 2.4, Corollary 2.1, and (H1) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2 &\leq C\int_\alpha^\beta \left(|J|\left|\frac{G_y}{\sqrt{\varrho_0}}\right| + \left|\frac{J_y}{\sqrt{\varrho_0}}\right||G| + \sqrt{\varrho_0}\vartheta + \sqrt{\varrho_0}|\vartheta_y|\right)\frac{|J_y|}{\sqrt{\varrho_0}}dy \\ &\leq C\left(\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 + \|G\|_\infty\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2 + \|\sqrt{\varrho_0}\vartheta\|_2 + \|\vartheta_y\|_2\right)\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2 \\ &\leq C(1+\|G\|_\infty)\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2 + C\left(1 + \|\vartheta_y\|_2^2 + \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2\right),\end{aligned}$$

which, together with the Grönwall inequality, Proposition 2.4, Corollary 2.1, and Proposition 2.5, yields

$$\sup_{0\leq t\leq T}\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2 \leq C. \tag{2.16}$$

Since

$$v_{yy} = \frac{1}{\mu}(J_yG + JG_y + R\varrho_0'\vartheta + R\varrho_0\vartheta_y), \tag{2.17}$$

it follows from (2.16), Corollary 2.1, Propositions 2.4, Proposition 2.5, and (H1) that

$$\int_0^T \left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2 dt \leq C\int_0^T\left(\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2\|G\|_\infty^2 + \left\|\left(\frac{G_y}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}\vartheta, \vartheta_y\right)\right\|_2^2\right)dt \leq C.$$

This completes the proof. □

Proposition 2.7. *It holds that*

$$\sup_{0\leq t\leq T}\|\sqrt{\varrho_0}\vartheta_y\|_2^2 + \int_0^T(\|\varrho_0\vartheta_t\|_2^2 + \|\vartheta_{yy}\|_2^2)dt \leq C.$$

Proof. Rewrite (1.13) as

$$c_v\varrho_0\vartheta_t-\kappa\left(\frac{\vartheta_y}{J}\right)_y=v_yG. \tag{2.18}$$

Note that $\vartheta_t|_{y=\alpha,\beta}=0$. Taking the inner product of the above equation with $\varrho_0\vartheta_t$ yields

$$\kappa\int_\alpha^\beta\frac{\vartheta_y}{J}(\varrho_0\vartheta_{yt}+\varrho_0'\vartheta_t)dy+c_v\|\varrho_0\vartheta_t\|_2^2=\int_\alpha^\beta v_yG\varrho_0\vartheta_tdy. \tag{2.19}$$

It follows from (1.11) that

$$\int_\alpha^\beta\frac{\vartheta_y}{J}\varrho_0\vartheta_{yt}dy=\frac12\frac{d}{dt}\left\|\sqrt{\frac{\varrho_0}{J}}\vartheta_y\right\|_2^2+\frac12\int_\alpha^\beta\frac{v_y}{J^2}\varrho_0|\vartheta_y|^2dy.$$

Substituting this into (2.19) and using (H1) and Corollary 2.1, one gets

$$\begin{aligned}&\frac{\kappa}{2}\frac{d}{dt}\left\|\sqrt{\frac{\varrho_0}{J}}\vartheta_y\right\|_2^2+c_v\|\varrho_0\vartheta_t\|_2^2\\
=\ &\int_\alpha^\beta\left(v_yG\varrho_0\vartheta_t-\frac{\kappa}{2}\frac{v_y}{J^2}\varrho_0|\vartheta_y|^2-\kappa\frac{\vartheta_y}{J}\varrho_0'\vartheta_t\right)dy\\
\leq\ &\int_\alpha^\beta\left(|v_y||G|\varrho_0|\vartheta_t|+\frac{\kappa}{2}\frac{|v_y|}{J^2}\varrho_0|\vartheta_y|^2+\kappa K_1\frac{|\vartheta_y|}{J}\varrho_0|\vartheta_t|\right)dy\\
\leq\ &\frac{c_v}{2}\|\varrho_0\vartheta_t\|_2^2+C(\|G\|_\infty^2\|v_y\|_2^2+\|v_y\|_\infty\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\|\vartheta_y\|_2^2),\end{aligned}$$

which implies

$$\begin{aligned}&\kappa\frac{d}{dt}\left\|\sqrt{\frac{\varrho_0}{J}}\vartheta_y\right\|_2^2+c_v\|\varrho_0\vartheta_t\|_2^2\\
\leq\ &C\left[\|G\|_\infty^2\|v_y\|_2^2+(\|G\|_\infty+\|\varrho_0\vartheta\|_\infty)\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\|\vartheta_y\|_2^2\right].\end{aligned}$$

It follows from this, the Grönwall inequality, Propositions 2.4–2.6, and Corollary 2.1 that

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\int_0^T\|\varrho_0\vartheta_t\|_2^2dt\\
\leq\ &Ce^{C\int_0^T(\|G\|_\infty+\|\varrho_0\vartheta\|_\infty)dt}\left(\|\sqrt{\varrho_0}\vartheta_0'\|_2^2+\int_0^T(\|G\|_\infty^2\|v_y\|_2^2+\|\vartheta_y\|_2^2)dt\right)\leq C.\end{aligned} \tag{2.20}$$

Direct calculations and using (2.18) yield

$$\kappa\vartheta_{yy}=\kappa\left(\frac{\vartheta_y}{J}\right)_yJ+\kappa\frac{\vartheta_y}{J}J_y=J(c_v\varrho_0\vartheta_t-v_yG)+\kappa\frac{\vartheta_y}{J}J_y.$$

It follows from this, (2.20), Propositions 2.5–2.6, Corollary 2.1, and Lemma 2.1 that

$$\begin{aligned}\int_0^T \|\vartheta_{yy}\|_2^2 dt &\leq C\int_0^T \left(\|\varrho_0\vartheta_t\|_2^2+\|v_y\|_2^2\|G\|_\infty^2+\|\vartheta_y\|_\infty^2\|J_y\|_2^2\right)dt\\ &\leq C+C\int_0^T\|\vartheta_y\|_\infty^2 dt\leq C+C\int_0^T\|\vartheta_y\|_2(\|\vartheta_y\|_2+\|\vartheta_{yy}\|_2)dt\\ &\leq \frac{1}{2}\int_0^T\|\vartheta_{yy}\|_2^2dt+C,\end{aligned}$$

and thus $\int_0^T\|\vartheta_{yy}\|_2^2dt\leq C$. This completes the proof. □

Proposition 2.8. *It holds that*

$$\sup_{0\leq t\leq T}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+\int_0^T\left(\|G_t\|_2^2+\left\|\left(\frac{G_y}{\varrho_0}\right)_y\right\|_2^2\right)dt\leq C\left(1+\left\|\frac{G_0'}{\sqrt{\varrho_0}}\right\|_2^2\right).$$

Proof. Combining (2.14) with (2.18) yields

$$G_t-\frac{\mu}{J}\left(\frac{G_y}{\varrho_0}\right)_y = -\frac{R}{J}\varrho_0\vartheta_t-\frac{v_y}{J}G.$$

Note that $G_t|_{y=\alpha,\beta}=0$. Multiplying the above with JG_t, integrating by parts, and using Corollary 2.1 yield

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+\|\sqrt{J}G_t\|_2^2&=-\int_\alpha^\beta(R\varrho_0\vartheta_t+v_yG)\,G_tdy\\ &\leq\frac{1}{2}\|\sqrt{J}G_t\|_2^2+C(\|\varrho_0\vartheta_t\|_2^2+\|v_y\|_2^2\|G\|_\infty^2),\end{aligned}$$

from which, by Propositions 2.5–2.7, the conclusion follows. □

Proposition 2.9. *It holds that*

$$\sup_{0\leq t\leq T}\left(\left\|\varrho_0^{\frac{3}{2}}\vartheta_t\right\|_2^2+\|\sqrt{\varrho_0}\vartheta_{yy}\|_2^2\right)+\int_0^T\|\varrho_0\vartheta_{yt}\|_2^2dt\leq C.$$

Proof. Note that $v_y=\frac{1}{\mu}(JG+R\varrho_0\vartheta)$ and

$$v_{yt}=\frac{1}{\mu}(JG_t+v_yG+R\varrho_0\vartheta_t). \tag{2.21}$$

It follows from (1.11), (2.18), and direct calculations that

$$c_v\varrho_0\vartheta_{tt}-\kappa\left(\frac{\vartheta_{yt}}{J}\right)_y=-\kappa\left(\frac{v_y\vartheta_y}{J^2}\right)_y+\frac{v_y}{\mu}G^2+\frac{1}{\mu}(2JG+R\varrho_0\vartheta)G_t+\frac{R}{\mu}\varrho_0\vartheta_tG.$$

Note that $\vartheta_t|_{y=\alpha,\beta}=0$. Multiplying the above equation with $\varrho_0^2\vartheta_t$ and integrating by parts yield

$$\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\left\|\varrho_0^{\frac{3}{2}}\vartheta_t\right\|_2^2+\kappa\int_\alpha^\beta\frac{\vartheta_{yt}}{J}(\varrho_0^2\vartheta_{yt}+2\varrho_0\varrho_0'\vartheta_t)dy\\
=\;&\frac{1}{\mu}\int_\alpha^\beta[v_yG^2+(2JG+R\varrho_0\vartheta)G_t+R\varrho_0\vartheta_tG]\varrho_0^2\vartheta_tdy\\
&+\kappa\int_\alpha^\beta\frac{v_y\vartheta_y}{J^2}(\varrho_0^2\vartheta_{yt}+2\varrho_0\varrho_0'\vartheta_t)dy.
\end{aligned}$$

Then, by Corollary 2.1 and (H1), one deduces

$$\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+\kappa\left\|\frac{\varrho_0}{\sqrt{J}}\vartheta_{yt}\right\|_2^2\\
\leq\;&C\int_\alpha^\beta[\varrho_0^2|\vartheta_t||\vartheta_{yt}|+|v_y||\vartheta_y|(\varrho_0^2|\vartheta_{yt}|+\varrho_0^2|\vartheta_t|)]dy\\
&+C\int_\alpha^\beta[|v_y|G^2+(|G|+\varrho_0\vartheta)|G_t|+\varrho_0|\vartheta_t||G|]\varrho_0^2|\vartheta_t|dy\\
\leq\;&\frac{\kappa}{2}\left\|\frac{\varrho_0}{\sqrt{J}}\vartheta_{yt}\right\|_2^2+C(\|\varrho_0\vartheta_t\|_2^2+\|v_y\|_\infty^2\|\varrho_0\vartheta_y\|_2^2)+C\|G\|_\infty^2(\|v_y\|_2^2+\|\varrho_0^2\vartheta_t\|_2^2)\\
&+C\|G_t\|_2^2+C(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2)\|\varrho_0^2\vartheta_t\|_2^2+C\|G\|_\infty\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2,
\end{aligned}$$

from which, by Propositions 2.6–2.7 and $v_y=\frac{1}{\mu}(JG+R\varrho_0\vartheta)$, one obtains

$$\begin{aligned}
&c_v\frac{d}{dt}\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+\kappa\left\|\frac{\varrho_0}{\sqrt{J}}\vartheta_{yt}\right\|_2^2\\
\leq\;&C(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2+1)\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+C(\|v_y\|_\infty^2+\|G\|_\infty^2+\|G_t\|_2^2+\|\varrho_0\vartheta_t\|_2^2)\\
\leq\;&C(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2+1)\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+C(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2+\|G_t\|_2^2+\|\varrho_0\vartheta_t\|_2^2).
\end{aligned}$$

Applying the Grönwall inequality to the above, one can get by Propositions 2.4–2.5 and 2.7–2.8, and Corollary 2.1 that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+\int_0^T\|\varrho_0\vartheta_{yt}\|_2^2dt\\
\leq\;&Ce^{C\int_0^T(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2)dt}\left\|\varrho_0^{\frac{3}{2}}\vartheta_t\right\|_2^2\Big|_{t=0}\\
&+Ce^{C\int_0^T(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2)dt}\int_0^T(\|G\|_\infty^2+\|\varrho_0\vartheta\|_\infty^2+\|G_t\|_2^2+\|\varrho_0\vartheta_t\|_2^2)dt\\
\leq\;&C(1+\|\sqrt{\varrho_0}\vartheta_0''\|_2^2+\|\sqrt{\varrho_0}v_0'G_0\|_2^2),
\end{aligned}\tag{2.22}$$

where the fact that $\varrho_0^{\frac{3}{2}}\vartheta_t|_{t=0}=\frac{\sqrt{\varrho_0}}{c_v}(\kappa\vartheta_0''+v_0'G_0)$ has been used, which follows from (2.18). Therefore, noticing that Lemma 2.1 implies

$$\|\sqrt{\varrho_0}v_0'G_0\|_2^2\leq C\|v_0'\|_\infty^2\|G_0\|_2^2\leq C\|v_0'\|_{H^1}^2\|G_0\|_2^2,$$

one gets from (2.22) that

$$\sup_{0\leq t\leq T}\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+\int_0^T\|\varrho_0\vartheta_{yt}\|_2^2dt\leq C(1+\|\sqrt{\varrho_0}\vartheta_0''\|_2^2). \tag{2.23}$$

Note that

$$\vartheta_{yy}=J\left(\frac{\vartheta_y}{J}\right)_y+\frac{\vartheta_y}{J}J_y=\frac{1}{\kappa}(c_v\varrho_0\vartheta_t-v_yG)J+\frac{1}{J}\vartheta_yJ_y.$$

It follows from this, (2.23), Proposition 2.6, and Corollary 2.1 that

$$\begin{aligned}\|\sqrt{\varrho_0}\vartheta_{yy}\|_2^2&\leq C(\|\varrho_0^{\frac{3}{2}}\vartheta_t\|_2^2+\|v_y\|_2^2\|G\|_\infty^2+\|\sqrt{\varrho_0}\vartheta_y\|_\infty^2\|J_y\|_2^2)\\&\leq C(1+\|G\|_\infty^2+\|\sqrt{\varrho_0}\vartheta_y\|_\infty^2).\end{aligned}\tag{2.24}$$

It remains to estimate $\|G\|_\infty^2$ and $\|\sqrt{\varrho_0}\vartheta_y\|_\infty^2$ as follows. Note that Lemma 2.1, Proposition 2.5, and Proposition 2.8 imply that

$$\|G\|_\infty^2\leq C\|G\|_2(\|G\|_2+\|G_y\|_2)\leq C. \tag{2.25}$$

By Lemma 2.1 and (H1), and Proposition 2.7, it holds that

$$\begin{aligned}\|\sqrt{\varrho_0}\vartheta_y\|_\infty^2&\leq C\|\sqrt{\varrho_0}\vartheta_y\|_2\left(\|\sqrt{\varrho_0}\vartheta_y\|_2+\|\sqrt{\varrho_0}\vartheta_{yy}\|_2+\left\|\frac{\varrho_0'}{\sqrt{\varrho_0}}\vartheta_y\right\|_2\right)\\&\leq C(1+\|\sqrt{\varrho_0}\vartheta_{yy}\|_2).\end{aligned}\tag{2.26}$$

Plugging (2.25) and (2.26) into (2.24) and using the Cauchy inequality yield

$$\|\sqrt{\varrho_0}\vartheta_{yy}\|_2^2\leq C(1+\|\sqrt{\varrho_0}\vartheta_{yy}\|_2)\leq\frac{\|\sqrt{\varrho_0}\vartheta_{yy}\|_2^2}{2}+C,$$

which gives $\|\sqrt{\varrho_0}\vartheta_{yy}\|_2^2\leq C$. This completes the proof. □

Proposition 2.10. *It holds that*

$$\sup_{0\leq t\leq T}\left(\|\sqrt{\varrho_0}v_t\|_2^2+\left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2\right)+\int_0^T(\|v_{yt}\|_2^2+\|v_{yyy}\|_2^2+\|J_{yy}\|_2^2)dt\leq C.$$

Proof. The estimate for $\sqrt{\varrho_0}v_t$ follows directly from Proposition 2.8 since $\sqrt{\varrho_0}v_t=\frac{G_y}{\sqrt{\varrho_0}}$. It follows from (H1), (2.17), (2.21), (2.25), Corollary 2.1, and Propositions 2.4–2.8 that

$$\left\|\frac{v_{yy}}{\sqrt{\varrho_0}}\right\|_2^2\leq C\left(\left\|\frac{J_y}{\sqrt{\varrho_0}}\right\|_2^2\|G\|_\infty^2+\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2^2+\|\sqrt{\varrho_0}\vartheta_y\|_2^2+\|\sqrt{\varrho_0}\vartheta\|_2^2\right)\leq C,$$

$$\int_0^T \|v_{yt}\|_2^2 dt \le C \int_0^T (\|v_y\|_2^2 \|G\|_\infty^2 + \|G_t\|_2^2 + \|\varrho_0 \vartheta_t\|_2^2) dt \le C.$$

Noticing that

$$v_{yyy} = \frac{1}{\mu}(J_{yy}G + 2J_yG_y + JG_{yy} + R\varrho_0''\vartheta + 2R\varrho_0'\vartheta_y + R\varrho_0\vartheta_{yy}).$$

one can get from (H1), (2.25), Corollary 2.1, and Propositions 2.4–2.5 and 2.7–2.8 that

$$\begin{aligned}\int_0^t \|v_{yyy}\|_2^2 d\tau \quad &\le \quad C \int_0^t (\|J_{yy}\|_2^2 \|G\|_\infty^2 + \|J_y\|_\infty^2 \|G_y\|_2^2 + \|G_{yy}\|_2^2 \\ &\quad + \|\varrho_0 \vartheta\|_2^2 + \|\varrho_0 \vartheta_y\|_2^2 + \|\varrho_0 \vartheta_{yy}\|_2^2) d\tau \\ &\le \quad C \int_0^t (\|J_{yy}\|_2^2 + \|J_y\|_\infty^2 + \|G_{yy}\|_2^2) d\tau + C, \qquad (2.27)\end{aligned}$$

where $\|G\|_\infty^2 \le C(\|G\|_2^2 + \|G_y\|_2^2)$ guaranteed by Lemma 2.1 wa used. Next, $\|J_y\|_\infty^2$ and $\|G_{yy}\|_2^2$ can be estimated as follows. Lemma 2.1 and Proposition 2.6 imply that

$$\|J_y\|_\infty^2 \le C(\|J_y\|_2^2 + \|J_y\|_2 \|J_{yy}\|_2) \le C(1 + \|J_{yy}\|_2^2). \qquad (2.28)$$

While (H1) and Proposition 2.8 yield

$$\begin{aligned}\int_0^T \|G_{yy}\|_2^2 dt \quad &\le \quad \int_0^T \left(\left\| \varrho_0 \left(\frac{G_y}{\varrho_0} \right)_y \right\|_2 + \left\| \varrho_0' \frac{G_y}{\varrho_0} \right\|_2 \right)^2 dt \\ &\le \quad C \int_0^T \left(\left\| \left(\frac{G_y}{\varrho_0} \right)_y \right\|_2^2 + \|G_y\|_2^2 \right) dt \le C.\end{aligned}$$

It follows from this, (2.27), and (2.28) that

$$\int_0^t \|v_{yyy}\|_2^2 d\tau \le C \left(1 + \int_0^t \|J_{yy}\|_2^2 d\tau \right). \qquad (2.29)$$

Since $J_{yy} = \int_0^t v_{yyy} d\tau$, one has

$$\int_0^t \|J_{yy}\|_2^2 d\tau \le \int_0^t \left\| \int_0^\tau v_{yyy} d\tau' \right\|_2^2 d\tau \le C \int_0^t \left(\int_0^\tau \|v_{yyy}\|_2^2 d\tau' \right) d\tau. \qquad (2.30)$$

Plugging this into (2.29) leads to

$$\int_0^t \|v_{yyy}\|_2^2 d\tau \le C + C \int_0^t \left(\int_0^\tau \|v_{yyy}\|_2^2 d\tau' \right) d\tau,$$

which implies $\int_0^T \|v_{yyy}\|_2^2 dt \le Ce^T \le C$ by the Grönwall inequality. This, together with (2.30), shows that $\int_0^t \|J_{yy}\|_2^2 d\tau \le C$. This completes the proof. □

Proposition 2.11. *It holds that*

$$\sup_{0\le t\le T}(\|J_{yy}\|_2^2+\|J_{yt}\|_2^2)\le C.$$

Proof. This follows from Proposition 2.10 by using $J_{yy}=\int_0^t v_{yyy}d\tau$ and $J_{yt}=v_{yy}$. □

Proposition 2.12. *It holds that*

$$\int_0^T\|\vartheta_{yyy}\|_2^2dt\le C.$$

Proof. By Lemma 2.1 and Propositions 2.5, 2.6, 2.8, 2.10, and 2.11, one has

$$\|J_y\|_\infty+\|J_{yy}\|_2+\|v_y\|_\infty+\|G\|_\infty\le C. \tag{2.31}$$

It follows from (2.18) that

$$\begin{aligned}\vartheta_{yyy} &= \frac{c_v}{\kappa}(\varrho_0'J\vartheta_t+2\varrho_0J_y\vartheta_t+\varrho_0J\vartheta_{yt})+\frac{\vartheta_y}{J}J_{yy}\\ &\quad -\frac{1}{\kappa}(2J_yv_yG+Jv_{yy}G+Jv_yG_y).\end{aligned}$$

Then, by Corollary 2.1, (H1), (2.31), and Proposition 2.11, one deduces

$$\begin{aligned}\int_0^T\|\vartheta_{yyy}\|_2^2dt &\le C\int_0^T(\|\varrho_0\vartheta_t\|_2^2+\|J_y\|_\infty^2\|\varrho_0\vartheta_t\|_2^2+\|\varrho_0\vartheta_{yt}\|_2^2+\|\vartheta_y\|_\infty^2\|J_{yy}\|_2^2\\ &\quad +\|J_y\|_\infty^2\|v_y\|_2^2\|G\|_\infty^2+\|v_{yy}\|_2^2\|G\|_\infty^2+\|v_y\|_\infty^2\|G_y\|_2^2)dt\\ &\le C\int_0^T(\|\varrho_0\vartheta_t\|_2^2+\|\varrho_0\vartheta_{yt}\|_2^2+\|\vartheta_y\|_2^2\\ &\quad +\|\vartheta_{yy}\|_2^2+\|v_y\|_2^2+\|v_{yy}\|_2^2+\|G_y\|_2^2)dt,\end{aligned}$$

where $\|\vartheta_y\|_\infty^2\le C(\|\vartheta_y\|_2^2+\|\vartheta_{yy}\|_2^2)$ guaranteed by Lemma 2.1 was used, from which, by Corollary 2.1 and Propositions 2.4–2.10, it follows $\int_0^T\|\vartheta_{yyy}\|_2^2dt\le C$. This proves the conclusion. □

As a consequence of Propositions 2.2–2.12 and Corollary 2.1, one has:

Corollary 2.2. *Let (J,v,ϑ) be the unique global solution stated in Proposition 2.1 to system (1.11)–(1.13), subject to (2.1)–(2.2), and $\mathscr{N}_0$ be given by (2.8). Then, for any $T\in[0,\infty)$, it holds that*

$$\inf_{(\alpha,\beta)\times(0,T)}J\ge \underline{C}_T,\quad \sup_{0\le t\le T}\left\|\left(\frac{J_y}{\sqrt{\varrho_0}},J_{yy},J_t,J_{yt}\right)\right\|_{L^2((\alpha,\beta))}^2\le C_T,$$

$$\sup_{0\le t\le T}\left\|\left(\sqrt{\varrho_0}v,\sqrt{\varrho_0}v^2,v_y,\frac{v_{yy}}{\sqrt{\varrho_0}},\sqrt{\varrho_0}v_t\right)\right\|_{L^2((\alpha,\beta))}^2 +\int_0^T\|(v_{yyy},v_{yt})\|_{L^2((\alpha,\beta))}^2dt\le C_T,$$

$$\sup_{0\leq t\leq T}\left(\|\varrho_0\vartheta\|_{L^1((\alpha,\beta))}+\left\|\left(\sqrt{\varrho_0}\vartheta,\sqrt{\varrho_0}\vartheta_y,\sqrt{\varrho_0}\vartheta_{yy},\varrho_0^{\frac{3}{2}}\vartheta_t\right)\right\|^2_{L^2((\alpha,\beta))}\right)$$
$$+\int_0^T\left(\|\vartheta_y\|^2_{H^2((\alpha,\beta))}+\|(\varrho_0\vartheta_t,\varrho_0\vartheta_{yt})\|^2_{L^2((\alpha,\beta))}\right)dt\leq C_T,$$
$$\sup_{0\leq t\leq T}\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|^2_{L^2((\alpha,\beta))}+\int_0^T\left(\left\|\left(\frac{G_y}{\varrho_0}\right)_y\right\|^2_{L^2((\alpha,\beta))}+\|G_t\|^2_{L^2((\alpha,\beta))}\right)dt\leq C_T,$$

where $\underline{C}_T$ and C_T are positive constants depending only on $R, c_v, \mu, \kappa, K_1, T$, and the upper bound of $\mathscr{N}_0$, but independent of α and β with $\beta-\alpha\geq 1$.

3. Global existence of solutions: proof of Theorem 1.1

Proof of Theorem 1.1. The proof is given in three steps as follows.

Step 1. Approximations of the initial data. By the assumption (1.16), there are two sequences $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$, with $\lim_{n\to\infty}\alpha_n=-\infty$ and $\lim_{n\to\infty}\beta_n=\infty$, and a positive constant M_0, such that

$$\left|\frac{v_0'(\alpha_n)}{\sqrt{\varrho_0(\alpha_n)}}\right|+\left|\frac{v_0'(\beta_n)}{\sqrt{\varrho_0(\beta_n)}}\right|\leq M_0,\quad\forall n\geq 1. \tag{3.1}$$

Set $I_n=(\alpha_n-1,\beta_n+1)$. For each n, choose $0\leq\chi_n\in C_0^\infty(I_n)$, satisfying

$$\chi\equiv 1\text{ on }[\alpha_n,\beta_n],\quad 0\leq\chi_n\leq 1\text{ and }|\chi_n'|+|\chi_n''|\leq C_0\text{ on }I_n, \tag{3.2}$$

for a positive constant C_0 independent of n. Define v_{0n} and ϑ_{0n} as

$$\vartheta_{0n}=\vartheta_0\chi_n,$$

and

$$v_{0n}=\begin{cases} v_0(\alpha_n)+\frac{2}{\pi}v_0'(\alpha_n)\sin\left(\frac{\pi}{2}(y-\alpha_n)\right), & y\in[\alpha_n-1,\alpha_n],\\ v_0(y), & y\in[\alpha_n,\beta_n],\\ v_0(\beta_n)+\frac{2}{\pi}v_0'(\beta_n)\sin\left(\frac{\pi}{2}(y-\beta_n)\right), & y\in[\beta_n,\beta_n+1].\end{cases}$$

It can be checked easily that

$$v_{0n}'(\alpha_n-1)=v_{0n}'(\beta_n+1)=\vartheta_{0n}(\alpha_n-1)=\vartheta_{0n}(\beta_n+1)=0. \tag{3.3}$$

Noticing that

$$v_{0n}(\alpha_n)=v_0(\alpha_n),\quad v_{0n}'(\alpha_n)=v_0'(\alpha_n),\quad v_{0n}(\beta_n)=v_0(\beta_n),\quad v_{0n}'(\beta_n)=v_0'(\beta_n),$$

and since $v_0\in H^2_{loc}(\mathbb{R})$ and $0\leq\vartheta_0\in H^2_{loc}(\mathbb{R})$, one has

$$v_{0n}\in H^2(I_n),\quad 0\leq\vartheta_{0n}\in H^2(I_n). \tag{3.4}$$

Due to $0\leq\chi_n\leq 1$, it is clear that

$$\|\sqrt{\varrho_0}\vartheta_{0n}\|_{L^2(I_n)}\leq\|\sqrt{\varrho_0}\vartheta_0\|_2. \tag{3.5}$$

For any $y \in [\alpha_n - 1, \alpha_n)$, the definition of v_{0n} implies that

$$|v_{0n}(y) - v_0(y)| \leq |v_0(\alpha_n) - v_0(y)| + \frac{2}{\pi}|v_0'(\alpha_n)| \leq 2\|v_0'\|_\infty.$$

Similarly, it holds that $|v_{0n}(y) - v_0(y)| \leq 2\|v_0'\|_\infty$, for any $y \in (\beta_n, \beta_n + 1]$. As a result, one has

$$|v_{0n}(y) - v_0(y)| \leq 2\|v_0'\|_\infty, \quad \forall y \in I_n. \tag{3.6}$$

Hence

$$\begin{aligned} \|\sqrt{\varrho_0} v_{0n}\|_{L^2(I_n)} &\leq \|\sqrt{\varrho_0}(v_{0n} - v_0)\|_{L^2(I_n)} + \|\sqrt{\varrho_0} v_0\|_{L^2(I_n)} \\ &= 2\|v_0'\|_\infty \|\varrho_0\|_1^{\frac{1}{2}} + \|\sqrt{\varrho_0} v_0\|_2, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|\sqrt{\varrho_0}|v_{0n}|^2\|_{L^2(I_n)} &\leq 2\left\|\sqrt{\varrho_0}(|v_0|^2 + |v_0 - v_{0n}|^2)\right\|_{L^2(I_n)} \\ &\leq 2\|\sqrt{\varrho_0}|v_0|^2\|_2 + 8\|v_0'\|_\infty^2 \|\varrho_0\|_1^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

It follows from (3.2) and direct calculations that

$$\|\sqrt{\varrho_0}\vartheta_{0n}'\|_{L^2(I_n)} \leq \|\sqrt{\varrho_0}\vartheta_0'\|_2 + C_0\|\sqrt{\varrho_0}\vartheta_0\|_2, \tag{3.9}$$

$$\|\sqrt{\varrho_0}\vartheta_{0n}''\|_{L^2(I_n)} \leq \|\sqrt{\varrho_0}\vartheta_0''\|_2 + 2C_0(\|\sqrt{\varrho_0}\vartheta_0'\|_2 + \|\sqrt{\varrho_0}\vartheta_0\|_2). \tag{3.10}$$

By direct calculations, one gets by the Sobolev inequality that

$$\begin{aligned} \|v_{0n}'\|_{H^1(I_n)} &\leq \|v_0'\|_{H^1} + C(|v_0'(\alpha_n)| + |v_0'(\beta_n)|) \\ &\leq \|v_0'\|_{H^1((\alpha_n,\beta_n))} + \|v_0'\|_{H^1((\alpha_n-1,\alpha_n)\cup(\beta_n,\beta_n+1))} \\ &\leq C\|v_0'\|_{H^1}^2, \end{aligned} \tag{3.11}$$

for a positive constant C independent of n.

Set $G_{0n} = \mu v_{0n}' - R\varrho_0\vartheta_{0n}$. Combining (3.5) with (3.11) leads to

$$\begin{aligned} \|G_{0n}\|_{L^2(I_n)} &\leq \mu\|v_{0n}'\|_{L^2(I_n)} + R\|\varrho_0\|_\infty^{\frac{1}{2}}\|\sqrt{\varrho_0}\vartheta_{0n}\|_{L^2(I_n)} \\ &\leq C(\|v_0'\|_{H^1} + \|\varrho_0\|_\infty^{\frac{1}{2}}\|\sqrt{\varrho_0}\vartheta_0\|_2), \end{aligned} \tag{3.12}$$

for a positive constant C independent of n.

For $y \in (\beta_n, \beta_n + 1)$, one has

$$\frac{\varrho_0(\beta_n)}{\varrho_0(y)} = 1 + \int_{\beta_n}^y k(z)\frac{\varrho_0(\beta_n)}{\varrho_0(z)}dz, \quad \text{where } k(z) = -\frac{\varrho_0'(z)}{\varrho_0(z)}. \tag{3.13}$$

By (H1), it holds that $|k(z)| \leq K_1$, for any $z \in \mathbb{R}$. Set

$$f(y) = 1 + \int_{\beta_n}^y k(z)\frac{\varrho_0(\beta_n)}{\varrho_0(z)}dz, \quad \forall y \in (\beta_n, \beta_n + 1).$$

Then, it follows from (3.13) that

$$f'(y) = k(y)\frac{\varrho_0(\beta_n)}{\varrho_0(y)} = k(y)f(y),$$

and thus

$$f(y) = e^{\int_{\beta_n}^y k(z)dz} f(\beta_n) = e^{\int_{\beta_n}^y k(z)dz} \leq e^{K_1}, \quad \forall y \in (\beta_n, \beta_n + 1).$$

It follows from this and (3.13) that

$$\frac{\varrho_0(\beta_n)}{\varrho_0(y)} = f(y) \leq e^{K_1}, \quad \forall y \in (\beta_n, \beta_n + 1). \tag{3.14}$$

Similarly, one has

$$\frac{\varrho_0(\alpha_n)}{\varrho_0(y)} \leq e^{K_1}, \quad \forall y \in (\alpha_n - 1, \alpha_n). \tag{3.15}$$

Recall that $G_{0n} = \mu v_{0n}' - R\varrho_0\vartheta_{0n}$. Then, direct calculations yield

$$\frac{G_{0n}'}{\sqrt{\varrho_0}} = \begin{cases} -\frac{\pi}{2}\mu\frac{v_0'(\alpha_n)}{\sqrt{\varrho_0}}\sin\left(\frac{\pi}{2}(y-\alpha_n)\right) - R\left(\sqrt{\varrho_0}\vartheta_{0n}' + \frac{\varrho_0'}{\sqrt{\varrho_0}}\vartheta_{0n}\right), & y \in (\alpha_n - 1, \alpha_n), \\ \frac{G_0'}{\sqrt{\varrho_0}}, & y \in (\alpha_n, \beta_n), \\ -\frac{\pi}{2}\mu\frac{v_0'(\beta_n)}{\sqrt{\varrho_0}}\sin\left(\frac{\pi}{2}(y-\beta_n)\right) - R\left(\sqrt{\varrho_0}\vartheta_{0n}' + \frac{\varrho_0'}{\sqrt{\varrho_0}}\vartheta_{0n}\right), & y \in (\beta_n, \beta_n + 1). \end{cases}$$

It follows from (3.1) and (3.14)–(3.15) that

$$\begin{aligned} \left|\frac{v_0'(\alpha_n)}{\sqrt{\varrho_0(y)}}\right| + \left|\frac{v_0'(\beta_n)}{\sqrt{\varrho_0(y)}}\right| &= \left|\frac{v_0'(\alpha_n)}{\sqrt{\varrho_0(\alpha_n)}}\sqrt{\frac{\varrho_0(\alpha_n)}{\varrho_0(y)}}\right| + \left|\frac{v_0'(\beta_n)}{\sqrt{\varrho_0(\beta_n)}}\sqrt{\frac{\varrho_0(\beta_n)}{\varrho_0(y)}}\right| \\ &\leq 2M_0e^{\frac{K_1}{2}}, \quad \forall y \in (\alpha_n - 1, \alpha_n) \cup (\beta_n, \beta_n + 1). \end{aligned}$$

This together with (H1) yields

$$\left|\frac{G_{0n}'(y)}{\sqrt{\varrho_0(y)}}\right| \leq \pi\mu M_0 e^{\frac{K_1}{2}} + R\left(\sqrt{\varrho_0}|\vartheta_{0n}'| + K_1\sqrt{\varrho_0}\vartheta_{0n}\right),$$

for any $y \in (\alpha_n - 1, \alpha_n) \cup (\beta_n, \beta_n + 1)$. Due to this and that $\frac{G_{0n}'}{\sqrt{\varrho_0}} = \frac{G_0'}{\sqrt{\varrho_0}}$ on (α_n, β_n), it follows from (3.9) that

$$\begin{aligned} \left\|\frac{G_{0n}'}{\sqrt{\varrho_0}}\right\|_{L^2(I_n)} &\leq \left\|\frac{G_0'}{\sqrt{\varrho_0}}\right\|_2 + 2\mu\pi M_0e^{K_1/2} + R\left(\|\sqrt{\varrho_0}\vartheta_{0n}'\|_{L^2(I_n)} + K_1\|\sqrt{\varrho_0}\vartheta_{0n}\|_{L^2(I_n)}\right) \\ &\leq \left\|\frac{G_0'}{\sqrt{\varrho_0}}\right\|_2 + C\left(\|\sqrt{\varrho_0}\vartheta_0'\|_2 + \|\sqrt{\varrho_0}\vartheta_0\|_2 + 1\right), \end{aligned} \tag{3.16}$$

for a positive constant C independent of n.

Step 2. Solutions to the system in $I_n \times (0,\infty)$ and a priori estimates.

For each positive integer n, let (v_{0n}, ϑ_{0n}) be the initial data constructed as before. Consider the initial-boundary value problem to the system (1.11)–(1.13) in $(\alpha_n - 1, \beta_n + 1) \times (0, \infty)$, subject to

$$(J, v, \vartheta)|_{t=0} = (1, v_{0n}, \vartheta_{0n}), \quad (v_y, \vartheta)|_{y=a_n-1,\beta_n+1} = (0,0). \tag{3.17}$$

Thanks to (3.3) and (3.4), and noticing that $\inf_{y\in I_n} \varrho_0 > 0$, one can verify that the initial datum (v_{0n}, ϑ_{0n}) satisfies all the assumptions in Proposition 2.1, for each fixed n. Thus, there is a unique global strong solution (J_n, v_n, ϑ_n) to (1.11)–(1.13) with (3.17). Moreover, due to (3.5), (3.7)–(3.12), and (3.16), it follows from Corollary 2.2 that

$$\inf_{I_n\times(0,T)} J_n \geq \underline{C}_T, \quad \vartheta_n(y,t) \geq 0, \tag{3.18}$$

$$\sup_{0\leq t\leq T} \left\| \left(\frac{\partial_y J_n}{\sqrt{\varrho_0}}, \partial_y^2 J_n, \partial_t J_n, \partial_{yt} J_n \right) \right\|_{L^2(I_n)}^2 \leq C_T, \tag{3.19}$$

$$\sup_{0\leq t\leq T} \left\| \left(\sqrt{\varrho_0} v_n, \sqrt{\varrho_0} v_n^2, \partial_y v_n, \frac{\partial_y^2 v_n}{\sqrt{\varrho_0}}, \sqrt{\varrho_0}\partial_t v_n \right) \right\|_{L^2(I_n)}^2 + \int_0^T \|(\partial_y^3 v_n, \partial_{yt}^2 v_n)\|_{L^2(I_n)}^2 dt \leq C_T, \tag{3.20}$$

$$\sup_{0\leq t\leq T} \left(\|\varrho_0 \vartheta_n\|_{L^1(I_n)} + \left\| \left(\sqrt{\varrho_0}\vartheta_n, \sqrt{\varrho_0}\partial_y\vartheta_n, \sqrt{\varrho_0}\partial_y^2\vartheta_n, \varrho_0^{\frac{3}{2}}\partial_t\vartheta_n \right) \right\|_{L^2(I_n)}^2 \right) + \int_0^T \left(\|\partial_y\vartheta_n\|_{H^2(I_n)}^2 + \|(\varrho_0\partial_t\vartheta_n, \varrho_0\partial_{yt}\vartheta_n)\|_{L^2(I_n)}^2 \right) dt \leq C_T, \tag{3.21}$$

and

$$\sup_{0\leq t\leq T} \left\| \frac{\partial_y G_n}{\sqrt{\varrho_0}} \right\|_{L^2(I_n)}^2 + \int_0^T \left(\left\| \left(\frac{\partial_y G_n}{\varrho_0} \right)_y \right\|_{L^2(I_n)}^2 + \|\partial_t G_n\|_{L^2(I_n)}^2 \right) dt \leq C_T, \tag{3.22}$$

for any positive time T, where $G_n = \mu\frac{\partial_y v_n}{J_n} - R\frac{\varrho_0}{J_n}\vartheta_n$, and $\underline{C}_T$ and C_T are positive constants independent of n.

Step 3. Convergence and existence.

Thanks to the a priori estimates (3.18)–(3.22) and $\inf_{(-k,k)} \varrho_0(y) > 0$ for any $k \in \mathbb{N}$, the following estimate holds

$$\|(J_n, v_n, \vartheta_n)\|_{L^\infty(0,T;H^2((-k,k)))} + \|(v_n, \vartheta_n)\|_{L^2(0,T;H^3((-k,k)))} + \|\partial_t J_n\|_{L^\infty(0,T;H^1((-k,k)))} + \|(\partial_t v_n, \partial_t\vartheta_n)\|_{L^\infty(0,T;L^2((-k,k)))\cap L^2(0,T;H^1((-k,k)))} \leq C_{k,T}, \quad \forall k \in \mathbb{N},$$

for a positive constant $C_{k,T}$ independent of n. Due to this and the Cantor's diagonal argument, there is a subsequence, still denoted by (J_n, v_n, ϑ_n), and (J, v, ϑ), such that

$$(J_n, v_n, \vartheta_n) \overset{*}{\rightharpoonup} (J, v, \vartheta), \quad \text{in } L^\infty(0,T;H^2((-k,k))), \tag{3.23}$$

$$(v_n,\vartheta_n)\rightharpoonup(v,\vartheta),\quad \text{in } L^2(0,T;H^3((-k,k))), \tag{3.24}$$

$$\partial_t J_n \overset{*}{\rightharpoonup} J_t,\quad \text{in } L^\infty(0,T;H^1((-k,k))), \tag{3.25}$$

$$(\partial_t v_n,\partial_t\vartheta_n)\overset{*}{\rightharpoonup}(v_t,\vartheta_t),\quad \text{in } L^\infty(0,T;L^2((-k,k))), \tag{3.26}$$

$$(\partial_t v_n,\partial_t\vartheta_n)\rightharpoonup(v_t,\vartheta_t),\quad \text{in } L^2(0,T;H^1((-k,k))), \tag{3.27}$$

for any $k\in\mathbb{N}$. Moreover, since $H^3((-k,k))\hookrightarrow\hookrightarrow C^2([-k,k])$ and $H^2((-k,k))\hookrightarrow\hookrightarrow C^1([-k,k])$, it follows from the Aubin–Lions lemma that

$$(J_n,v_n,\vartheta_n)\to(J,v,\vartheta),\quad \text{in } C([0,T];C^1([-k,k])), \tag{3.28}$$

$$(v_n,\vartheta_n)\to(v,\vartheta),\quad \text{in } L^2(0,T;C^2([-k,k])), \tag{3.29}$$

for any $k\in\mathbb{N}$. Thanks to these and by (3.18), one has

$$\inf_{(y,t)\in\mathbb{R}\times(0,T)}J(y,t)\geq\underline{C}_T,\quad \frac{1}{J_n}\to\frac{1}{J}\ \text{in } C([0,T];C^1([-k,k])), \tag{3.30}$$

for any $k\in\mathbb{N}$.

Thanks to (3.23)–(3.30) and noticing that $(v_{0n},\vartheta_{0n})\to(v_0,\vartheta_0)$ in $H^2((-L,L))$ for any $L>0$, one can take the limit as $n\to\infty$ to show that (J,v,ϑ) is a solution to the Cauchy problem to the system (1.11)–(1.13) subject to $(J,v,\vartheta)|_{t=0}=(1,v_0,\vartheta_0)$. The desired regularities of (J,v,ϑ) stated in Theorem 1.1 follow from the a priori estimates (3.18)–(3.22) and convergence (3.23)–(3.29) by the weakly lower semi-continuity of norms. This proves Theorem 1.1. □

4. A Hopf type lemma and unboundedness of the entropy

In this section, we prove the unboundedness of the entropy immediately after the initial time, i.e. Theorem 1.2. As stated in the Introduction, this is based on some suitable scaling transform and a Hopf type lemma for a class of general linear degenerate equations. So, we first establish a Hopf type lemma in the first subsection and then present the proof of Theorem 1.2 in the second subsection. The Hopf type lemma has its own independent interests and will also be applied to prove the uniform positivity of the temperature in the next section.

4.1. **A Hopf type lemma.** Since the results in this subsection hold in any dimension, we use the following notations. Denote by $x=(x_1,x_2,\cdots,x_n)$ and t the spatial and time variables respectively and $P=(x,t)$ a point in $\mathbb{R}^{n+1}$. For $P_0=(x_0,t_0)\in\mathbb{R}^{n+1}$ and $r>0$, denote

$$\mathcal{B}_r(P_0):=\left\{(x,t)\in\mathbb{R}^{n+1}\middle||x-x_0|^2+(t-t_0)^2<r^2\right\}.$$

Let $(a_{ij})_{n\times n},a_0,b=(b_1,b_2,\cdots,b_n)$, and c be given functions satisfying suitable properties to be specified later. Consider the operator

$$\mathscr{L}\varphi=-a_{ij}\partial_{ij}\varphi+a_0\partial_t\varphi+b\cdot\nabla\varphi+c\varphi.$$

Note that here a_0 is not required to have fixed sign and this linear operator can be regarded only as a linear degenerate elliptic operator in the space and time variables with degeneracy occurring in the time direction.

Lemma 4.1. *Let $\mathcal{O}$ be a domain in $\mathbb{R}^{n+1}$. Assume that $a_{ij}, a_0, b,$ and c are finitely valued functions in $\mathcal{O}$ with $c \geq 0$, and the matrix $(a_{ij})_{n\times n}$ is nonnegative definite in $\mathcal{O}$. Then, for any $\varphi \in C^{2,1}(\mathcal{O}) \cap C(\overline{\mathcal{O}})$, satisfying*

$$\mathscr{L}\varphi > 0 \text{ in } \mathcal{O}, \quad \text{and} \quad \varphi|_{\partial\mathcal{O}} \geq 0,$$

it holds that $\varphi > 0$ in $\mathcal{O}$. Here $C^{2,1}(\mathcal{O})$ denotes the space of all function f satisfying $f, \partial_t f, \nabla f, \nabla^2 f \in C(\mathcal{O})$.

Proof. First, we claim that $\varphi \geq 0$ in $\mathcal{O}$. Otherwise, since $\varphi \geq 0$ on $\partial\mathcal{O}$ and $\varphi \in C(\overline{\mathcal{O}})$, there is $P_0 \in \mathcal{O}$, such that $\varphi(P_0) = \min_{\overline{\mathcal{O}}} \varphi < 0$. Since $\varphi \in C^{2,1}(\mathcal{O})$, it is clear that $\partial_t \varphi(P_0) = \nabla\varphi(P_0) = 0$ and $\nabla^2\varphi(P_0)$ is nonnegative definite. As a result

$$(\mathscr{L}\varphi)(P_0) = -a_{ij}(P_0)\partial_{ij}\varphi(P_0) + c(P_0)\varphi(P_0) \leq 0,$$

which contradicts to the assumption. Therefore, the claim holds. Next, we show that $\varphi > 0$ in $\mathcal{O}$. Otherwise, there is $P_0^* \in \mathcal{O}$, such that $\varphi(P_0^*) = 0$. Then, $\varphi(P_0^*) = \min_{\overline{\mathcal{O}}} \varphi = 0$, from which, similar as before, one has $(\mathscr{L}\varphi)(P_0^*) \leq 0$, contradicting to the assumption. Thus, $\varphi > 0$ in $\mathcal{O}$, which proves the conclusion. ☐

Lemma 4.2 (Hopf type lemma). *Given $P_0 = (x_0, t_0)$, $r > 0$, $P_* = (x_*, t_*) \in \partial\mathcal{B}_r(P_0)$, $x_* \neq x_0$, and set $P_0^* = (x_0^*, t_0^*)$, with $x_0^* = \frac{x_0 + x_*}{2}$ and $t_0^* = \frac{t_0 + t_*}{2}$. Assume that there are positive constants $\lambda, \Lambda, \delta_*$, and C_*, with $\delta_* < \frac{|x_0 - x_*|}{4}$, such that*

$$\begin{cases} \lambda|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \\ (t - t_0^*)a_0(x,t) + (x - x_0^*) \cdot b(x,t) \geq -C_*, \qquad \forall (x,t) \in \mathcal{B}_{\frac{r}{2}}(P_0^*) \cap \mathcal{B}_{\delta_*}(P_*). \\ 0 \leq c(x,t)\sqrt{|x - x_*|^2 + (t - t_*)^2} \leq C_*, \end{cases}$$

Let $\varphi \in C^{2,1}(\mathcal{B}_r(P_0)) \cap C(\overline{\mathcal{B}_r}(P_0))$ satisfy

$$\mathscr{L}\varphi \geq 0, \quad \varphi > \varphi(P_*), \quad \text{in } \mathcal{B}_{\frac{r}{2}}(P_0^*) \cap \mathcal{B}_{\delta_*}(P_*), \quad \varphi(P_*) \leq 0.$$

Then, it holds that

$$\overline{\lim_{\ell\to 0^+}} \frac{\varphi(P_*) - \varphi(P_* - \ell n_*)}{\ell} < 0,$$

where $n_* = \frac{P_* - P_0}{r}$ is the unit outward normal vector to $\partial\mathcal{B}_r(P_0)$ at P_*.

Proof. Set

$$\mathscr{D} = \mathcal{B}_{\frac{r}{2}}(P_0^*) \cap \mathcal{B}_{\delta_*}(P_*).$$

It suffices to consider the case that $\varphi(P_*) = 0$. Otherwise, one may consider $\Phi := \varphi - \varphi(P_*)$, which reduces to the case considered, due to

$$\mathscr{L}\Phi = \mathscr{L}\varphi - \mathscr{L}(\varphi(P_*)) = \mathscr{L}\varphi - c\varphi(P_*) \geq \mathscr{L}\varphi \geq 0 \quad \text{in } \mathscr{D},$$

as $\varphi(P_*) \le 0$ and $c \ge 0$ in $\mathscr{D}$. It is clear that $\mathcal{B}_{\frac{r}{2}}(P_0^*) \subset \mathcal{B}_r(P_0)$. By assumption, it holds that

$$\varphi(P) > \varphi(P_*) = 0, \quad \forall P \in \overline{\mathscr{D}} \setminus \{P_*\}. \tag{4.1}$$

Define

$$\phi(x,t) = e^{-\zeta(|x-x_0^*|^2+(t-t_0^*)^2)} - e^{-\frac{r^2}{4}\zeta} = e^{-\zeta|P-P_0^*|^2} - e^{-\frac{r^2}{4}\zeta},$$

where $\zeta > 0$ is a constant to be determined. Then, it follows from direct calculations that

$$\begin{aligned}\mathscr{L}\phi = &- e^{-\zeta|P-P_0^*|^2}\Big[4(x-x_0^*)^T A(x-x_0^*)\zeta^2 - 2trA\zeta \\ &+ 2((t-t_0^*)a_0 + (x-x_0^*)\cdot b)\zeta + c\left(e^{\zeta(|P-P_0^*|^2-\frac{r^2}{4})} - 1\right)\Big],\end{aligned} \tag{4.2}$$

where $A = (a_{ij})_{n\times n}$ and $trA = a_{ii}$. Note that the assumptions imply

$$\begin{aligned}&4(x-x_0^*)^T A(x-x_0^*)\zeta^2 - 2trA\zeta + 2((t-t_0^*)a_0 + (x-x_0^*)\cdot b)\zeta \\ \ge &4\lambda|x-x_0^*|^2\zeta^2 - 2n\Lambda\zeta - 2C_*\zeta \ge \frac{\lambda}{4}|x_0-x_*|^2\zeta^2 - (2n\Lambda + 2C_*)\zeta,\end{aligned} \tag{4.3}$$

for any $(x,t) \in \mathscr{D}$, due to $trA \le n\Lambda$ and

$$|x-x_0^*| \ge |x_0^* - x_*| - |x_* - x| \ge \frac{|x_0-x_*|}{2} - \delta_* \ge \frac{|x_0-x_*|}{4}, \quad \forall (x,t) \in \mathscr{D}.$$

Note that $|P-P_0^*| < \frac{r}{2}$ for any $P \in \mathscr{D}$. It follows from the mean value theorem and the triangular inequality that

$$\begin{aligned}&\left|e^{\zeta(|P-P_0^*|^2-\frac{r^2}{4})} - 1\right| = e^{\tau\zeta(|P-P_0^*|^2-\frac{r^2}{4})}\left||P-P_0^*|^2 - \frac{r^2}{4}\right|\zeta \\ \le &\left||P-P_0^*| - \frac{r}{2}\right|\left||P-P_0^*| + \frac{r}{2}\right|\zeta \le r\zeta\big||P-P_0^*| - |P_0^*-P_*|\big| \le r\zeta|P-P_*|,\end{aligned}$$

for any $P \in \mathscr{D}$, where $\tau \in (0,1)$. This, together with the assumptions, yields

$$\left|c\left(e^{\zeta(|P-P_0^*|^2-\frac{r^2}{4})} - 1\right)\right| \le cr|P-P_*|\zeta \le C_* r\zeta, \quad \forall P \in \mathscr{D}. \tag{4.4}$$

Combining (4.3) with (4.4) leads to

$$\begin{aligned}&4(x-x_0^*)^T A(x-x_0^*)\zeta^2 - 2trA\zeta \\ &+2((t-t_0^*)a_0 + (x-x_0^*)\cdot b)\zeta + c\left(e^{\zeta(|P-P_0^*|^2-\frac{r^2}{4})} - 1\right) \\ \ge\ &\frac{\lambda}{4}|x_0-x_*|^2\zeta^2 - (2n\Lambda + 2C_* + rC_*)\zeta > 0, \quad \forall P \in \mathscr{D},\end{aligned} \tag{4.5}$$

if $\zeta > \zeta_0 := \frac{8n\Lambda+8c_*+4rC_*}{\lambda|x_0-x_*|^2}$. Choose $\zeta = 2\zeta_0$. Then, it follows from (4.2) and (4.5) that

$$\mathscr{L}\phi < 0 \quad \text{in } \mathscr{D}. \tag{4.6}$$

It follows from (4.1) that

$$\varphi \geq 0 = \phi \text{ on } \partial\mathcal{B}_{\frac{r}{2}}(P_0^*) \cap \mathcal{B}_{\delta_*}(P_*), \qquad \inf_{\partial\mathcal{B}_{\delta_*}(P_*)\cap\mathcal{B}_{\frac{r}{2}}(P_0^*)} \varphi > 0.$$

Therefore, for $\varepsilon > 0$ sufficiently small, it follows from the assumptions and (4.6) that

$$\mathscr{L}\varphi \geq 0 > \mathscr{L}(\varepsilon\phi) \text{ in } \mathscr{D}, \quad \varphi \geq \varepsilon\phi \text{ on } \partial\mathscr{D},$$

and thus

$$\mathscr{L}(\varphi - \varepsilon\phi) > 0 \text{ in } \mathscr{D}, \quad \varphi - \varepsilon\phi \geq 0 \text{ on } \partial\mathscr{D}.$$

With the aid of this, noticing that $\varphi - \varepsilon\phi \in C^{2,1}(\mathscr{D}) \cap C(\overline{\mathscr{D}})$, and applying Lemma 4.1, one gets

$$\varphi > \varepsilon\phi \quad \text{in } \mathscr{D}.$$

Therefore, for $\ell > 0$ sufficiently small, one has

$$\varphi(P_*) - \varphi(P_* - \ell n_*) = -\varphi(P_* - \ell n_*) < -\varepsilon\phi(P_* - \ell n_*) = \varepsilon(\phi(P_*) - \phi(P_* - \ell n_*))$$

and thus

$$\begin{aligned} \overline{\lim_{\ell\to 0^+}} \frac{\varphi(P_*) - \varphi(P_* - \ell n_*)}{\ell} &\leq \varepsilon \overline{\lim_{\ell\to 0^+}} \frac{\phi(P_*) - \phi(P_* - \ell n_*)}{\ell} \\ &= \varepsilon\partial_{n_*}\phi(P_*) = -\varepsilon\zeta r e^{-\frac{r^2}{4}\zeta} < 0. \end{aligned}$$

This proves the conclusion. □

As a direct consequence of Lemma 4.2, the following corollary holds.

Corollary 4.1. *Given $P_0 = (x_0, t_0)$, $r > 0$, $P_* = (x_*, t_*) \in \partial\mathcal{B}_r(P_0)$, $x_* \neq x_0$. Assume that $a_0, b, c \in L^\infty(\mathcal{B}_r(P_0))$, $c \geq 0$ in $\mathcal{B}_r(P_0)$, and*

$$\lambda|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, (x,t) \in \mathcal{B}_r(P_0),$$

for some positive constants λ and Λ. Let $\varphi \in C^{2,1}(\mathcal{B}_r(P_0)) \cap C(\overline{\mathcal{B}_r}(P_0))$ satisfy

$$\mathscr{L}\varphi \geq 0, \quad \varphi > \varphi(P_*), \quad \text{in } \mathcal{B}_r(P_0), \quad \varphi(P_*) \leq 0.$$

Then, it holds that

$$\overline{\lim_{\ell\to 0^+}} \frac{\varphi(P_*) - \varphi(P_* - \ell n_*)}{\ell} < 0,$$

where $n_* = \frac{P_* - P_0}{r}$ is the unit outward normal vector to $\partial\mathcal{B}_r(P_0)$ at P_*.

4.2. **Unboundedness of the entropy.** This subsection is devoted to proving Theorem 1.2. We start with the following embedding lemma, which is used to verify the Hölder regularity of J_y required in the proof of Theorem 1.2.

Lemma 4.3. *Let $L > 0$ be a positive number. Then, the following embedding inequality holds*

$$\|f\|_{C^{\frac{1}{2},\frac{1}{4}}([-L,L]\times[0,T])} \leq C(\|f\|_{L^\infty(0,T;H^1((-L,L)))} + \|\partial_t f\|_{L^2(0,T;L^2((-L,L)))}),$$

for any function $f \in L^\infty(0,T;H^1((-L,L)))$ such that $\partial_t f \in L^2(0,T;L^2((-L,L)))$, where C is an absolute positive constant.

Proof. For any $t, \tau \in [0,T]$, one deduces by Lemma 2.1, the Minkovski, Hölder, and Cauchy inequalities that

$$\begin{aligned}
& \|f(\cdot,t) - f(\cdot,\tau)\|_{L^\infty((-L,L))} \\
\leq\ & C\|f(\cdot,t) - f(\cdot,\tau)\|^{\frac{1}{2}}_{L^2((-L,L))}\|f(\cdot,t) - f(\cdot,\tau)\|^{\frac{1}{2}}_{H^1((-L,L))} \\
\leq\ & C\|f\|^{\frac{1}{2}}_{L^\infty(0,T;H^1((-L,L)))}\left\|\int_\tau^t \partial_t f(\cdot,s)ds\right\|^{\frac{1}{2}}_{L^2((-L,L))} \\
\leq\ & C\|f\|^{\frac{1}{2}}_{L^\infty(0,T;H^1((-L,L)))}\left(\int_\tau^t \|\partial_t f\|_{L^2((-L,L))}ds\right)^{\frac{1}{2}} \\
\leq\ & C(\|f\|_{L^\infty(0,T;H^1((-L,L)))} + \|\partial_t f\|_{L^2(0,T;L^2((-L,L)))})|t-\tau|^{\frac{1}{4}},
\end{aligned}$$

for an absolute positive constant C. For any $x, y \in [-L, L]$ and $t \in [0,T]$, it follows from the Hölder inequality that

$$\begin{aligned}
& |f(x,t) - f(y,t)| \leq \left|\int_x^y \partial_x f(z,t)dz\right| \\
\leq\ & \left|\int_{-L}^L |\partial_x f|^2 dx\right|^{\frac{1}{2}} |y-x|^{\frac{1}{2}} \leq \|f\|_{L^\infty(0,T;H^1((-L,L)))}|y-x|^{\frac{1}{2}}.
\end{aligned}$$

Therefore, for any $x, y \in [-L, L]$ and $t, \tau \in [0,T]$, it holds that

$$\begin{aligned}
& |f(x,t) - f(y,\tau)| \leq |f(x,t) - f(y,t)| + |f(y,t) - f(y,\tau)| \\
\leq\ & C(\|f\|_{L^\infty(0,T;H^1((-L,L)))} + \|\partial_t f\|_{L^2(0,T;L^2((-L,L)))})(|t-\tau|^{\frac{1}{4}} + |y-x|^{\frac{1}{2}}).
\end{aligned}$$

This leads to the conclusion. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is dived into five steps as follows.

Step 1. Regularities and pointwise positivity of ϑ. For $L > 0$, denote by $W^{2,1}_2((-L,L)\times(0,T))$ the space of all functions $f \in L^2(0,T;H^2((-L,L)))$ satisfying

$\partial_t f \in L^2(0,T;L^2((-L,L)))$. Recall the embedding that $W_2^{2,1}([-L,L]\times(0,T)) \hookrightarrow C^{\frac12,\frac14}([-L,L]\times[0,T])$ (Theorem 4.1 of [55]). Note that

$$(v_y,\vartheta_y)\in W_2^{2,1}((-L,L)\times(0,T)),\quad J,J_y\in L^\infty(0,T;H^1((-L,L))),$$

and $J_t, J_{yt}\in L^\infty(0,T;L^2((-L,L)))$. Hence, it follows from the Sobolev embedding and Lemma 4.3 that

$$(J,J_y,v_y,\vartheta_y)\in C^{\frac12,\frac14}([-L,L]\times[0,T]),\quad \forall L>0. \tag{4.7}$$

Rewrite (1.13) as

$$c_v\varrho_0\vartheta_t-\frac{\kappa}{J}\vartheta_{yy}+\kappa\frac{J_y}{J^2}\vartheta_y+R\varrho_0\frac{v_y}{J}\vartheta=\frac{\mu}{J}|v_y|^2. \tag{4.8}$$

Since J is uniformly positive on $\mathbb{R}\times(0,T)$, it can be checked that all the coefficients in (4.8), i.e., ϱ_0, $\frac1J$, $\frac{J_y}{J^2}$, $\varrho_0\frac{v_y}{J}$, and $\frac{|v_y|^2}{J}$, belong to $C^{\frac12,\frac14}([-L,L]\times[0,T])$. Thanks to these and the fact that $\varrho_0(y)>0$ for all $y\in\mathbb{R}$, it follows from the classic Schauder theory on interior regularities for uniform parabolic equations that $\vartheta\in C^{2+\frac12,1+\frac14}((-L,L)\times(0,T))$. On the other hand, by the embedding theorem, it follows from the regularities of ϑ that $\vartheta\in C([-L,L]\times[0,T])$. Therefore, it holds that

$$\vartheta\in C^{2,1}((-L,L)\times(0,T))\cap C([-L,L]\times[0,T]).$$

Note that $\vartheta\not\equiv0$ on $\mathbb{R}\times(0,T)$. Otherwise, noticing that $\vartheta\in C([0,T];L^2(-L,L))$ for any $L>0$, one has $\vartheta_0\equiv0$; furthermore, it follows from (1.13) that $v_y\equiv0$ on $\mathbb{R}\times(0,T)$, from which, since $v\in C([0,T];L^2((-L,L)))$ for any $L>0$, one has $v_0\equiv$ Const. This contradicts to the assumptions. Therefore, one has $\vartheta\not\equiv0$ on $\mathbb{R}\times(0,T)$ and $\vartheta\geq0$. Thanks to this and by the strong maximum principle, one gets

$$0<\vartheta\in C^{2,1}(\mathbb{R}\times(0,T))\cap C(\mathbb{R}\times[0,T]). \tag{4.9}$$

Step 2. Asymptotic behavior of J_y. Note that

$$J_{yt}=\frac1\mu(GJ_y+JG_y+R\varrho_0'\vartheta+R\varrho_0\vartheta_y)$$

which implies

$$J_y=\frac1\mu\int_0^t e^{\frac1\mu\int_s^tGd\tau}(JG_y+R\varrho_0'\vartheta+R\varrho_0\vartheta_y)ds.$$

Therefore

$$\left|\frac{J_y(y,t)}{\varrho_0(y)}\right|\leq\frac1\mu e^{\frac1\mu\int_0^t\|G\|_\infty d\tau}\int_0^t\left(J\left|\frac{G_y}{\varrho_0}\right|+R\left|\frac{\varrho_0'}{\varrho_0}\right|\vartheta+R|\vartheta_y|\right)ds. \tag{4.10}$$

For any $y\geq0$, it follows that

$$\left|\frac{G_y(y,t)}{\varrho_0(y)}\right| = \left|\int_0^1\frac{G_y}{\varrho_0}dz+\int_0^1\int_z^y\left(\frac{G_y}{\varrho_0}\right)_y(z',t)dz'dz\right|$$

$$
\begin{aligned}
&\le \int_0^1 \left|\frac{G_y}{\varrho_0}\right| dz + \int_0^{y+1} \left|\left(\frac{G_y}{\varrho_0}\right)_y\right| dz \\
&\le \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 (t) + \sqrt{y+1}\left\|\left(\frac{G_y}{\varrho_0}\right)_y\right\|_2 (t)
\end{aligned}
$$

and

$$
\begin{aligned}
\vartheta(y,t) &= \int_0^1 \vartheta(z,t)dz + \int_0^1\int_z^y \vartheta_y(z',t)dz'dz \\
&\le \int_0^1 \frac{\sqrt{\varrho_0}\vartheta}{\sqrt{\varrho_0}}dz + \int_0^{y+1} |\vartheta_y|dz \le \frac{\|\sqrt{\varrho_0}\vartheta\|_2(t)}{\sqrt{\delta_0}} + \sqrt{y+1}\|\vartheta_y\|_2(t), \quad (4.11)
\end{aligned}
$$

where $\delta_0 := \inf_{[-1,1]} \varrho_0 > 0$. Similar estimates hold also for $y < 0$ and thus it holds for any $y \in \mathbb{R}$ that

$$
\left|\frac{G_y(y,t)}{\varrho_0(y)}\right| \le \left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 (t) + \sqrt{|y|+1}\left\|\left(\frac{G_y}{\varrho_0}\right)_y\right\|_2 (t) \quad (4.12)
$$

and

$$
\vartheta(y,t) \le \frac{\|\sqrt{\varrho_0}\vartheta\|_2(t)}{\sqrt{\delta_0}} + \sqrt{|y|+1}\|\vartheta_y\|_2(t). \quad (4.13)
$$

Substituting (4.12)–(4.13) into (4.10) and using (H1), one can get by the Hölder and Sobolev inequalities that

$$
\begin{aligned}
\left|\frac{J_y(y,t)}{\varrho_0(y)}\right| &\le Ce^{C\int_0^t \|G\|_\infty ds}\int_0^t \left(\left\|\frac{G_y}{\sqrt{\varrho_0}}\right\|_2 + \sqrt{|y|+1}\left\|\left(\frac{G_y}{\varrho_0}\right)_y\right\|_2\right)ds \\
&\quad + Ce^{C\int_0^t \|G\|_\infty ds}\int_0^t \left(\|\sqrt{\varrho_0}\vartheta\|_2 + \|\vartheta_y\|_2\sqrt{|y|+1} + \|\vartheta_y\|_{H^1}\right)ds \\
&\le C\sqrt{t}\sqrt{|y|+1}\left[\int_0^t\left(\left\|\left(\frac{G_y}{\sqrt{\varrho_0}}, \left(\frac{G_y}{\varrho_0}\right)_y\right)\right\|_2^2 + \|\vartheta_y\|_{H^1}^2\right)ds\right]^{\frac{1}{2}} \\
&\quad + Ct\sqrt{|y|+1}\|\sqrt{\varrho_0}\vartheta\|_{L^\infty(0,T;L^2)} \\
&\le C_1\sqrt{|y|+1},
\end{aligned}
$$

that is,

$$
\left|\frac{J_y(y,t)}{\varrho_0(y)}\right| \le C_1\sqrt{|y|+1}, \quad \forall y \in \mathbb{R}, t \in [0,T], \quad (4.14)
$$

where the regularities of (ϑ, G) have been used.

Step 3. A scaling transform. Let $T > 0$ be any arbitrary given constant. Assume by contradiction that $s \in L^\infty(\mathbb{R}\times(0,T))$. Since $\vartheta = \frac{A}{R}e^{\frac{s}{c_v}}\left(\frac{\varrho_0}{J}\right)^{\gamma-1}$ and J has uniform positive lower and upper bounds, it follows that

$$0 \le \vartheta(y,t) \le C_T \varrho_0^{\gamma-1}(y), \quad \forall (y,t) \in \mathbb{R} \times [0,T]. \tag{4.15}$$

Let $\beta > 0$ to be determined later and introduce a scaling transform as

$$f(y,t) := \vartheta(y^{-\beta}, t), \quad y \in (0,\infty), t \ge 0.$$

Then, direct calculations yield

$$\begin{aligned}
&\vartheta(y,t) = f(y^{-\frac{1}{\beta}}, t),\\
&\vartheta_t(y,t) = f_t(y^{-\frac{1}{\beta}}, t), \quad \vartheta_y(y,t) = -\frac{1}{\beta} y^{-(1+\frac{1}{\beta})} f_y(y^{-\frac{1}{\beta}}, t),\\
&\vartheta_{yy}(y,t) = \frac{1}{\beta^2} y^{-(2+\frac{2}{\beta})} f_{yy}(y^{-\frac{1}{\beta}}, t) + \frac{\beta+1}{\beta^2} y^{-(2+\frac{1}{\beta})} f_y(y^{-\frac{1}{\beta}}, t),
\end{aligned}$$

for any $(y,t) \in (0,\infty) \times (0,\infty)$. Besides, one deduces from (4.8) that

$$\begin{aligned}
&c_v \varrho_0(y^{-\beta}) y^{-(2+2\beta)} J(y^{-\beta}, t) f_t(y,t) - \frac{\kappa}{\beta^2} f_{yy}(y,t)\\
&- \left(\frac{\kappa(\beta+1)}{\beta^2} y^{-1} + \frac{\kappa}{\beta} y^{-(1+\beta)} \frac{J_y(y^{-\beta}, t)}{J(y^{-\beta}, t)} \right) f_y(y,t)\\
&+ R\varrho_0(y^{-\beta}) y^{-(2\beta+2)} v_y(y^{-\beta}, t) f(y,t) \ge 0,
\end{aligned} \tag{4.16}$$

for all $(y,t) \in (0,\infty) \times (0,\infty)$.

Step 4. Verifying conditions of Hopf type lemma. Let M_T be a positive constant to be determined later and define

$$F(y,t) = e^{-M_T t} f(y,t), \quad y \in (0,\infty), t \in [0,\infty).$$

Due to (4.9), it is clear that

$$0 < F \in C^{2,1}((0,\infty) \times (0,\infty)) \cap C((0,\infty) \times [0,\infty)).$$

Moreover, it follows from (4.15) and (H2) that

$$\begin{aligned}
F(y,t) &= e^{-M_T t} \vartheta(y^{-\beta}, t) \le C_T e^{-M_T t} \varrho_0^{\gamma-1}(y^{-\beta})\\
&\le C_T K_2^{\gamma-1} e^{-M_T t} (1 + y^{-\beta})^{-(\gamma-1)\ell_\rho} \le C_T K_2^{\gamma-1} e^{-M_T t} y^{(\gamma-1)\beta\ell_\rho},
\end{aligned}$$

for an $y \in (0,\infty)$ and $t \in [0,\infty)$. Thus, one can define $F(0,t) = 0$ for $t \in [0,\infty)$, such that F is well defined on $[0,\infty) \times [0,\infty)$, satisfying

$$\begin{cases} F \in C^{2,1}((0,\infty) \times (0,\infty)) \cap C([0,\infty) \times [0,\infty)),\\ F > 0 \text{ in } (0,\infty) \times (0,\infty), \quad F(0,t) = 0, \quad \forall t \in [0,\infty). \end{cases} \tag{4.17}$$

It follows from (4.16) that

$$a_0(y,t) F_t(y,t) - a F_{yy}(y,t) + b(y,t) F_y(y,t) + c(y,t) F(y,t) \ge 0, \tag{4.18}$$

in $(0,\infty) \times (0,\infty)$, where

$$a_0(y,t) = c_v \varrho_0(y^{-\beta}) y^{-(2+2\beta)} J(y^{-\beta}, t), \qquad a = \frac{\kappa}{\beta^2},$$

$$
\begin{aligned}
b(y,t) &= -\left(\frac{\kappa(\beta+1)}{\beta^2}y^{-1}+\frac{\kappa}{\beta}y^{-(1+\beta)}\frac{J_y(y^{-\beta},t)}{J(y^{-\beta},t)}\right),\\
c(y,t) &= \varrho_0(y^{-\beta})y^{-(2+2\beta)}\Big(c_vM_TJ(y^{-\beta},t)+Rv_y(y^{-\beta},t)\Big).
\end{aligned}
$$

Take arbitrary $t_0\in(0,T)$, $0<y_0<\min\{\frac{1}{2},t_0\}$, and set

$$
P_0=(y_0,t_0),\quad r=y_0,\quad P_*=(0,t_0)=:(y_*,t_*),\quad \delta_*=\frac{y_0}{8},
$$
$$
P_0^*=\left(\frac{y_0}{2},t_0\right)=:(y_0^*,t_0^*),\quad \mathscr{D}=\mathcal{B}_{\delta_*}(P_*)\cap\mathcal{B}_{\frac{r}{2}}(P_0^*).
$$

Then,

$$
P_*\in\partial\mathcal{B}_r(P_0),\quad \mathscr{D}=\mathcal{B}_{\frac{y_0}{8}}((0,t_0))\cap\mathcal{B}_{\frac{y_0}{2}}((\tfrac{y_0}{2},t_0)).
$$

For any $(y,t)\in\mathscr{D}$, due to $0<y<\frac{y_0}{8}<\frac{1}{16}$ and $\frac{t_0}{2}<t<\frac{3}{2}t_0$, one deduces

$$
\begin{aligned}
&\quad (t-t_0^*)a_0(y,t)+(y-y_0^*)b(y,t)\\
&= c_v(t-t_0)\varrho_0(y^{-\beta})y^{-(2+2\beta)}J(y^{-\beta},t)\\
&\quad -\left(y-\frac{y_0}{2}\right)\left(\frac{\kappa(\beta+1)}{\beta^2}y^{-1}+\frac{\kappa}{\beta}y^{-(1+\beta)}\frac{J_y(y^{-\beta},t)}{J(y^{-\beta},t)}\right)\\
&\geq -c_vt_0\varrho_0(y^{-\beta})y^{-(2+2\beta)}J(y^{-\beta},t)-\frac{\kappa y_0}{2\beta}y^{-(1+\beta)}\frac{|J_y(y^{-\beta},t)|}{J(y^{-\beta},t)}\\
&\geq -c_vt_0\bar{j}_T\varrho_0(y^{-\beta})y^{-(2+2\beta)}-\frac{\kappa}{\underline{j}_T\beta}y^{-(1+\beta)}|J_y(y^{-\beta},t)|, \qquad (4.19)
\end{aligned}
$$

where

$$
\bar{j}_T:=\sup_{(y,t)\in\mathbb{R}\times[0,T]}J(y,t),\quad \underline{j}_T:=\inf_{(y,t)\in\mathbb{R}\times[0,T]}J(y,t). \qquad (4.20)
$$

Set $M_T:=\frac{R\|v_y\|_{L^\infty(\mathbb{R}\times(0,T))}}{c_v\underline{j}_T}$. Then,

$$
c_vM_TJ(y^{-\beta},t)+Rv_y(y^{-\beta},t)\geq c_vM_T\underline{j}_T-R\|v_y\|_{L^\infty(\mathbb{R}\times(0,T))}=0
$$

and

$$
\begin{aligned}
c_vM_TJ(y^{-\beta},t)+Rv_y(y^{-\beta},t) &\leq c_vM_T\bar{j}_T+R\|v_y\|_{L^\infty(\mathbb{R}\times(0,T))}\\
&= c_vM_T(\bar{j}_T+\underline{j}_T).
\end{aligned}
$$

Thus, for any $(y,t)\in\mathscr{D}$, since $\sqrt{|y-y_*|^2+|t-t_*|^2}\leq\delta_*\leq\frac{1}{16}$, it holds that

$$
0\leq c(y,t)\sqrt{|y-y_*|^2+|t-t_*|^2}\leq c_vM_T(\bar{j}_T+\underline{j}_T)\varrho_0(y^{-\beta})y^{-(2+2\beta)}. \qquad (4.21)
$$

For any $(y,t)\in\mathscr{D}$, since $0<y<1$, it follows from (4.14) and (H2) that

$$
\varrho_0(y^{-\beta})y^{-(2+2\beta)}\leq K_2(1+y^{-\beta})^{-\ell_\rho}y^{-(2+2\beta)}\leq K_2y^{(\ell_\rho-2)\beta-2}\leq K_2 \qquad (4.22)
$$

and

$$
y^{-(1+\beta)}|J_y(y^{-\beta},t)| \leq C_1y^{-(1+\beta)}\varrho_0(y^{-\beta})\sqrt{1+y^{-\beta}}\leq C_1K_2y^{-(1+\beta)}(1+y^{-\beta})^{-\ell_\rho+\frac{1}{2}}
$$

$$\leq \; C_1K_2y^{(\ell_\rho-\frac{3}{2})\beta-1}\leq C_1K_2, \tag{4.23}$$

as long as $\beta\geq\max\left\{\frac{2}{\ell_\rho-2},\frac{2}{2\ell_\rho-3}\right\}=\frac{2}{\ell_\rho-2}$.

Due to (4.22) and (4.23), it follows from (4.19) and (4.21) that

$$(t-t_0^*)a_0(y,t)+(y-y_0^*)b(y,t)\geq-(C_1+1)K_2\left(c_vt_0\bar{j}_T+\frac{\kappa}{\underline{j}_T\beta}\right), \tag{4.24}$$

$$0\leq c(y,t)\sqrt{|y-y_*|^2+|t-t_*|^2}\leq c_vM_T(\bar{j}_T+\underline{j}_T)K_2, \tag{4.25}$$

for any $(y,t)\in\mathscr{D}$, as long as $\beta\geq\frac{2}{\ell_\rho-2}$.

Step 5. Unboundedness of entropy. Choose

$$\beta=\beta_0:=\max\left\{\frac{2}{(\gamma-1)\ell_\rho},\frac{2}{\ell_\rho-2}\right\}.$$

Due to (4.17), (4.18), (4.24), and (4.25), it follows from Lemma 4.2 that

$$\overline{\lim_{\ell\to0^+}}\frac{F(P_*)-F(P_*-n_*\ell)}{\ell}=-\underline{\lim_{\ell\to0^+}}\frac{F(P_*-n_*\ell)}{\ell}=-2\varepsilon_2,$$

for some positive constant ε_2, where we recall $P_*=(0,t_0)$, $n_*=\frac{P_*-P_0}{r}=(-1,0)$, and thus $P_*-n_*\ell=(\ell,t_0)$. Thus, there is a positive number ℓ_0, such that

$$F(y,t_0)=e^{-M_Tt_0}\vartheta(y^{-\beta_0},t_0)\geq\varepsilon_2y,\quad\forall y\in(0,\ell_0),$$

that is

$$\vartheta(y,t_0)\geq\varepsilon_2e^{M_Tt_0}y^{-\frac{1}{\beta_0}},\quad\forall y\in\left(\ell_0^{-\frac{1}{\beta_0}},\infty\right).$$

On the other hand, it follows from (4.15) and (H2) that

$$\vartheta(y,t)\leq C_TK_2^{\gamma-1}(1+y)^{-\ell_\rho(\gamma-1)}\leq C_TK_2^{\gamma-1}y^{-\ell_\rho(\gamma-1)},\quad\forall y>0.$$

Combing the previous two inequalities leads to

$$y^{(\gamma-1)\ell_\rho-\frac{1}{\beta_0}}\leq C_TK_2^{\gamma-1}\varepsilon_2^{-1}e^{-M_Tt_0},\quad\forall y\in(\ell_0^{-\frac{1}{\beta_0}},\infty),$$

which is impossible when $y\to\infty$, as $(\gamma-1)\ell_\rho-\frac{1}{\beta_0}\geq\frac{\gamma-1}{2}\ell_\rho>0$. This contradiction leads to the desired conclusion that $s\notin L^\infty(\mathbb{R}\times(0,T))$. □

5. Uniform positivity of ϑ and asymptotic unboundedness of s

In this section, we prove the uniform positivity of the temperature and asymptotic unboundedness of the entropy, under the condition that the initial density decays at the far field not slower than $O(\frac{1}{x^4})$, which yields the proof of Theorem 1.3.

Proof of Theorem 1.3. We need only to prove (i), while the conclusion (ii) follows from (i), (1.10), and (4.13), as J has uniformly positive lower and upper bounds at each time $t \in (0,\infty)$.

Let h be the Kelvin transform of ϑ defined as

$$h(y,t) = y\vartheta\left(\frac{1}{y},t\right), \quad \forall y \neq 0, t \in [0,T]. \tag{5.1}$$

Then (4.9) implies

$$h \in C^{2,1}((\mathbb{R}_+ \cup \mathbb{R}_-) \times (0,T)) \cap C((\mathbb{R}_+ \cup \mathbb{R}_-) \times [0,T]). \tag{5.2}$$

Note that

$$\vartheta(y,t) = yh\left(\frac{1}{y},t\right), \quad \vartheta_t(y,t) = yh_t\left(\frac{1}{y},t\right),$$
$$\vartheta_y(y,t) = h\left(\frac{1}{y},t\right) - \frac{1}{y}h_y\left(\frac{1}{y},t\right), \quad \vartheta_{yy}(y,t) = \frac{1}{y^3}h_{yy}\left(\frac{1}{y},t\right),$$

for any $y \neq 0$ and $t \in (0,T)$. It follows from these and (4.8) that

$$c_v\varrho_0\left(\frac{1}{y}\right)\frac{1}{y^4}h_t\left(y,t\right) - \frac{\kappa}{J\left(\frac{1}{y},t\right)}h_{yy}\left(y,t\right) - \kappa\frac{J_y\left(\frac{1}{y},t\right)}{J^2\left(\frac{1}{y},t\right)}\frac{1}{y^2}h_y\left(y,t\right)$$
$$+\left(R\frac{v_y\left(\frac{1}{y},t\right)}{J\left(\frac{1}{y},t\right)}\frac{\varrho_0\left(\frac{1}{y}\right)}{y^4} + \kappa\frac{J_y\left(\frac{1}{y},t\right)}{J^2\left(\frac{1}{y},t\right)}\frac{1}{y^3}\right)h(y,t) = \mu\frac{\left|v_y\left(\frac{1}{y},t\right)\right|^2}{y^3J\left(\frac{1}{y},t\right)},$$

for $y \neq 0$. Define a_0, a, b, and $\tilde{c}$ as

$$a_0 := c_v\varrho_0\left(\frac{1}{y}\right)\frac{1}{y^4}, \quad a := \frac{\kappa}{J\left(\frac{1}{y},t\right)}, \quad b := -\kappa\frac{J_y\left(\frac{1}{y},t\right)}{J^2\left(\frac{1}{y},t\right)}\frac{1}{y^2},$$
$$\tilde{c} := R\frac{v_y\left(\frac{1}{y},t\right)}{J\left(\frac{1}{y},t\right)}\frac{\varrho_0\left(\frac{1}{y}\right)}{y^4} + \kappa\frac{J_y\left(\frac{1}{y},t\right)}{J^2\left(\frac{1}{y},t\right)}\frac{1}{y^3}, \quad \forall y \neq 0, t \in [0,T].$$

Then, it holds that

$$\begin{cases} a_0h_t - ah_{yy} + bh_y + \tilde{c}h \geq 0, & \text{in } Q_T^+, \\ a_0h_t - ah_{yy} + bh_y + \tilde{c}h \leq 0, & \text{in } Q_T^-, \end{cases} \tag{5.3}$$

where

$$Q_T^+ := \mathbb{R}_+ \times (0,T), \quad Q_T^- := \mathbb{R}_- \times (0,T).$$

Properties of a_0, a, b, and $\tilde{c}$ are analyzed as follows. It follows from (4.7) and the regularities of ϱ_0 and J that

$$a_0 \in C(\mathbb{R}^+ \cup \mathbb{R}_-), \quad a, b, \tilde{c} \in C(Q_T^+ \cup Q_T^-). \tag{5.4}$$

For a_0, it follows from (H3) that

$$0 \leq a_0(y) \leq \frac{c_v K_3}{(|y|+1)^4} \leq c_v K_3, \quad \forall y \neq 0. \tag{5.5}$$

For a, it holds that

$$\lambda_T \leq a(y,t) \leq \Lambda_T, \quad \forall y \neq 0, t \in [0,T], \tag{5.6}$$

where $\lambda_T = \frac{\kappa}{\overline{j}_T}$ and $\Lambda_T = \frac{\kappa}{\underline{j}_T}$, with $\underline{j}_T$ and $\overline{j}_T$ given by (4.20).

It follows from (4.14) and (H3) that

$$|J_y(y,t)| \leq C_1 \varrho_0(y)\sqrt{|y|+1} \leq C_1 K_3 (|y|+1)^{-\frac{7}{2}}, \quad \forall y \in \mathbb{R}, t \in [0,T]. \tag{5.7}$$

This implies that

$$|b(y,t)| \leq \frac{\kappa}{\underline{j}_T^2}\frac{1}{y^2}\left|J_y\left(\frac{1}{y},t\right)\right| \leq \frac{\kappa}{\underline{j}_T^2}\frac{1}{y^2} C_1 K_3 \left(\frac{1}{|y|}+1\right)^{-\frac{7}{2}} \leq \frac{C_1 K_3 \kappa}{\underline{j}_T^2}, \tag{5.8}$$

for any $y \neq 0$ and $t \in [0,T]$. By (H3) and (5.7), one deduces

$$\begin{aligned} |\tilde{c}(y,t)| &\leq \frac{R}{\underline{j}_T}\frac{1}{y^4}\frac{K_3}{\left(1+\frac{1}{|y|}\right)^4}\|v_y\|_\infty(t) + \frac{\kappa}{\underline{j}_T^2}\frac{1}{|y|^3} C_1 K_3 \left(1+\frac{1}{|y|}\right)^{-\frac{7}{2}} \\ &\leq \frac{RK_3}{\underline{j}_T}\|v_y\|_\infty(t) + \frac{\kappa}{\underline{j}_T^2} C_1 K_3 \leq C(\|v_y\|_{L^\infty(0,T;H^1)}+1), \end{aligned} \tag{5.9}$$

for any $y \neq 0$ and $t \in [0,T]$.

Set

$$N_T = \frac{2}{c_v}\left(\frac{R}{\underline{j}_T}\|v_y\|_{L^\infty(\mathbb{R}\times(0,T))} + \frac{\sqrt{2}\kappa C_1}{\underline{j}_T^2}\right)$$

and define

$$H(y,t) = e^{-N_T t} h(y,t), \quad \forall y \neq 0, t \in [0,T]. \tag{5.10}$$

Due to (5.2), it is clear that

$$H \in C^{2,1}((\mathbb{R}_+ \cup \mathbb{R}_-) \times (0,T)) \cap C((\mathbb{R}_+ \cup \mathbb{R}_-) \times [0,T]). \tag{5.11}$$

Since $\vartheta > 0$, it follows from (5.1) and (5.10) that

$$H > 0 \text{ in } Q_T^+ \quad \text{ and } \quad H < 0 \text{ in } Q_T^-. \tag{5.12}$$

By (4.13) and recalling the definitions of h and H, one deduces

$$\begin{aligned} |H(y,t)| &\leq |y|\vartheta\left(\frac{1}{y},t\right) \leq C(\|\sqrt{\varrho_0}\vartheta\|_{L^\infty(0,T;L^2)} + \|\vartheta_y\|_{L^\infty(0,T;L^2)})|y|\sqrt{1+\frac{1}{|y|}} \\ &\leq C\sqrt{y^2+|y|}, \quad \forall y \neq 0, t \in [0,T]. \end{aligned}$$

Thanks to this, it holds that

$$\lim_{(y,\tau)\to(0,t)} H(y,t) = 0, \quad \forall t \in [0,T]. \tag{5.13}$$

It follows from direct calculations and (5.3) that

$$\begin{cases} a_0 H_t - aH_{yy} + bH_y + cH \geq 0, & \text{in } Q_T^+, \\ a_0 H_t - aH_{yy} + bH_y + cH \leq 0, & \text{in } Q_T^-, \end{cases} \tag{5.14}$$

where

$$\begin{aligned} c(y,t) &:= \tilde{c}(y,t) + N_T a_0(y,t) \\ &= \left(c_v N_T + R \frac{v_y\left(\frac{1}{y},t\right)}{J\left(\frac{1}{y},t\right)} + \frac{\kappa}{J^2\left(\frac{1}{y},t\right)} \frac{J_y\left(\frac{1}{y},t\right)}{\varrho_0\left(\frac{1}{y}\right)} y \right) \varrho_0\left(\frac{1}{y}\right) \frac{1}{y^4}. \end{aligned}$$

For any $y \in [-1,0) \cap (0,1]$ and $t \in [0,T]$, it follows from (4.14) that

$$\begin{aligned} c(y,t) &\geq \left(c_v N_T - \frac{R}{\underline{j}_T} \|v_y\|_\infty(t) - \frac{\kappa}{\underline{j}_T^2} |y| C_1 \sqrt{1 + \frac{1}{|y|}} \right) \varrho_0\left(\frac{1}{y}\right) \frac{1}{y^4} \\ &= \left(c_v N_T - \frac{R}{\underline{j}_T} \|v_y\|_\infty(t) - \frac{\kappa}{\underline{j}_T^2} C_1 \sqrt{|y|^2 + |y|} \right) \varrho_0\left(\frac{1}{y}\right) \frac{1}{y^4} \\ &\geq \left(c_v N_T - \frac{R}{\underline{j}_T} \|v_y\|_\infty(t) - \frac{\sqrt{2}\kappa C_1}{\underline{j}_T^2} \right) \varrho_0\left(\frac{1}{y}\right) \frac{1}{y^4} \\ &\geq \left(\frac{R}{\underline{j}_T} \|v_y\|_\infty(t) + \frac{\sqrt{2}\kappa C_1}{\underline{j}_T^2} \right) \varrho_0\left(\frac{1}{y}\right) \frac{1}{y^4} \end{aligned}$$

and thus

$$c(y,t) \geq 0, \quad \forall y \in [-1,0) \cup (0,1], t \in [0,T]. \tag{5.15}$$

Define

$$\widetilde{H}(y,t) = \begin{cases} H(y,t), & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ -H(y,t), & \text{if } y < 0, \end{cases} \tag{5.16}$$

for all $t \in [0,T]$. Denote

$$\Omega_- := (-1,0) \times (0,T), \quad \Omega_+ := (0,1) \times (0,T),$$
$$\Omega := \Omega_+ \cup \Omega_-, \quad \Gamma := \{0\} \times [0,T].$$

Then, it follows from (5.11)–(5.14) that

$$\widetilde{H} \in C^{2,1}(\Omega) \cap C(\overline{\Omega}), \quad \widetilde{H} > 0 \text{ in } \Omega, \quad \widetilde{H}|_\Gamma = 0, \tag{5.17}$$

$$\mathscr{L}\widetilde{H} = a_0 \widetilde{H}_t - a\widetilde{H}_{yy} + b\widetilde{H}_y + c\widetilde{H} \geq 0 \quad \text{in } \Omega, \tag{5.18}$$

with a_0, a, b, and c satisfying

$$\begin{cases} a_0 \in C((-1,1)\setminus\{0\}) \cap L^\infty((-1,1)\setminus\{0\}), \\ a,b,c \in C(\Omega)\cap L^\infty(\Omega), \quad \lambda_T \le a \le \Lambda_T, \quad c \ge 0 \text{ in } \Omega, \end{cases} \tag{5.19}$$

which follows from (5.4)–(5.6), (5.8)–(5.9), and (5.15).

For arbitrary $t_0 \in (0,T)$, set

$$P_* = (0,t_0), \quad P_0 = (y_0,t_0), \quad r = y_0 = \min\left\{\frac{1}{2}, t_0, T-t_0\right\}.$$

Then, it is clear that $n_* := \frac{P_*-P_0}{r} = (-1,0)$. Let $\mathcal{B}_r$ be the space-time ball of radius r and centered at P_0. Thanks to (5.17), (5.18), and (5.19), it is clear that $\widetilde{H}$ satisfies all the conditions in Corollary 4.1, and thus Corollary 4.1 implies

$$\overline{\lim_{\ell\to 0^+}} \frac{\widetilde{H}(P_*) - \widetilde{H}(P_* - \ell n_*)}{\ell} = \overline{\lim_{\ell\to 0^+}} \frac{-\widetilde{H}(\ell,t_0)}{\ell} = -2\varepsilon_0,$$

with a positive constant ε_0. Thus, $\underline{\lim}_{\ell\to 0^+} \frac{\widetilde{H}(\ell,t_0)}{\ell} = 2\varepsilon_0$, which yields that

$$\widetilde{H}(y,t_0) \ge \varepsilon_0 y, \quad \forall y \in (0,\ell_0), \tag{5.20}$$

for some positive constant ℓ_0. Then, by the definition of $\widetilde{H}$, one derives

$$\widetilde{H}(y,t_0) = e^{-N_T t_0} h(y,t_0) = e^{-N_T t_0} y \vartheta\left(\frac{1}{y}, t_0\right) \ge \varepsilon_0 y, \quad \forall y \in (0,\ell_0) \tag{5.21}$$

and thus,

$$\vartheta(y,t_0) \ge \varepsilon_0 e^{N_T t_0} \ge \varepsilon_0, \quad \forall y \in \left(\frac{1}{\ell_0}, \infty\right). \tag{5.22}$$

Similarly, there are positive constants ε_1 and ℓ_1 such that

$$\vartheta(y,t_0) \ge \varepsilon_1, \quad \forall y \in \left(-\infty, -\frac{1}{\ell_1}\right).$$

Combining this with (5.22) and recalling that $0 < \vartheta \in C(\mathbb{R}\times[0,T])$, one has

$$\inf_{y\in\mathbb{R}} \vartheta(y,t_0) = \min\left\{\varepsilon_0, \vartheta_1, \inf_{y\in\left[-\frac{1}{\ell_1}, \frac{1}{\ell_0}\right]} \vartheta(y,t_0)\right\} > 0.$$

This yields the desired conclusion, and the proof of Theorem 1.3 is completed. $\square$

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References

[1] Bresch, D.; Jabin, P.-E: *Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor*, Ann. Math., **(2) 188** (2018), no. 2, 577–684.

[2] Chen, G.-Q.; Hoff, D.; Trivisa, K.: *Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data*, Comm. Partial Differential Equations, **25** (2000), 2233–2257.

[3] Chen, Q.; Miao, C.; Zhang, Z.: *Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity*, Communications on Pure and Applied Mathematics, **63** (2010), 1173–1224.

[4] Chikami, N.; Danchin, R.: *On the well-posedness of the full compressible Naiver-Stokes system in critical Besov spaces*, J. Differential Equations, **258** (2015), 3435–3467.

[5] Cho, Y.; Choe, H. J.; Kim, H.: *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl., **83** (2004), 243–275.

[6] Cho, Y.; Kim, H.: *On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities*, Manuscripta Math., **120** (2006), 91–129.

[7] Cho, Y.; Kim, H.: *Existence results for viscous polytropic fluids with vacuum*, J. Differential Equations, **228** (2006), 377–411.

[8] Danchin, R.: *Global existence in critical spaces for flows of compressible viscous and heat-conductive gases*, Arch. Ration. Mech. Anal., **160** (2001), 1–39.

[9] Danchin, R.; Xu, J.: *Optimal decay estimates in the critical Lp framework for flows of compressible viscous and heat-conductive gases*, J. Math. Fluid Mech., **20** (2018), no. 4, 1641–1665.

[10] Deckelnick, K.: *Decay estimates for the compressible Navier-Stokes equations in unbounded domains*, Math. Z., **209** (1992), 115–130.

[11] Fang, D.; Zhang, T.; Zi, R.: *Global solutions to the isentropic compressible Navier-Stokes equations with a class of large initial data*, SIAM J. Math. Anal., **50** (2018), no. 5, 4983–5026.

[12] Feireisl, E.; Novotný, A.; Petzeltová, H.: *On the existence of globally defined weak solutions to the Navier-Stokes equations*, J. Math. Fluid Mech., **3** (2001), 358–392.

[13] Feireisl, E.: *On the motion of a viscous, compressible, and heat conducting fluid*, Indiana Univ. Math. J., **53** (2004), 1705–1738.

[14] Feireisl, E.: *Dynamics of viscous compressible fluids*, Oxford Lecture Series in Mathematics and its Applications, 26. Oxford University Press, Oxford, 2004. xii+212 pp.

[15] Gong, H.; Li, J.; Liu, X.; Zhang, X.: *Local well-posedness of isentropic compressible Navier-Stokes equations with vacuum*, Commun. Math. Sci. **18** (2020), no. 7, 1891–1909.

[16] Hoff, D.: *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids*, Arch. Rational Mech. Anal., **139** (1997), 303–354.

[17] Hoff, D.; Smoller, J.: *Non-formation of vacuum states for compressible Navier-Stokes equations*, Comm. Math. Phys., **216** (2001), 255–276.

[18] Huang, X.: *On local strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with vacuum*, Sci. China Math. **64** (2021), no. 8, 1771–1788.

[19] Huang, X.; Li, J.: *Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations*, Arch. Ration. Mech. Anal., **227** (2018), no. 3, 995–1059.

[20] Huang, X.; Li, J.; Xin, Z.: *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations*, Comm. Pure Appl. Math., 65 (2012), 549–585.

[21] Itaya, N.: *On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluids*, Kodai Math. Sem. Rep., **23** (1971), 60–120.

[22] Jiang, S.; Zhang, P.: *Axisymmetric solutions of the 3D Navier-Stokes equations for compressible isentropic fluids*, J. Math. Pures Appl., **82** (2003), 949–973.

[23] Jiang, S.; Zlotnik, A.: *Global well-posedness of the Cauchy problem for the equations of a one-dimensional viscous heat-conducting gas with Lebesgue initial data*, Proc. Roy. Soc. Edinburgh Sect. A, **134** (2004), 939–960.

[24] Kazhikhov, A. V.: *Cauchy problem for viscous gas equations*, Siberian Math. J., **23** (1982), 44–49.

[25] Kazhikhov, A. V.; Shelukhin, V. V.: *Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech., **41** (1977), 273–282.

[26] Kobayashi, T.; Shibata, Y.: *Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in* $\mathbb{R}^3$, Commun. Math. Phys., **200** (1999), 621–659.

[27] Li, H.; Wang, Y.; Xin, Z.: *Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier-Stokes equations*, Arch. Rational Mech. Anal., **232** (2019), 557–590

[28] Li, J.: *Global small solutions of heat conductive compressible Navier-Stokes equations with vacuum: smallness on scaling invariant quantity*, Arch. Ration. Mech. Anal. **237** (2020), no. 2, 899–919.

[29] Li, J.: *Global well-posedness of the one-dimensional compressible Navier-Stokes equations with constant heat conductivity and nonnegative density*, SIAM J. Math. Anal. **51** (2019), no. 5, 3666–3693.

[30] Li, J.: *Global well-posedness of non-heat conductive compressible Navier-Stokes equations in 1D*, Nonlinearity **33** (2020), no. 5, 2181–2210.

[31] Li, J.; Zheng, Y.: *Local existence and uniqueness of heat conductive compressible Navier-Stokes equations in the presence of vacuum and without initial compatibility conditions,* arXiv:2108.10783

[32] Li, J.; Liang, Z.: *Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier-Stokes system in unbounded domains with large data*, Arch. Rational Mech. Anal., **220** (2016), 1195–1208.

[33] Li, J.; Xin, Z.: *Global well-posedness and large time asymptotic Bbehavior of classical solutions to the compressible Navier-Stokes equations with vacuum*, Ann. PDE (2019) 5: 7.

[34] Li, J.; Xin, Z.: *Entropy bounded solutions to the one-dimensional compressible Navier-Stokes equations with zero heat conduction and far field vacuum*, Adv. Math. **361** (2020), 106923, 50 pp.

[35] Li, J.; Xin, Z.: *Entropy-bounded solutions to the one-dimensional heat conductive compressible Navier-Stokes equations with far field vacuum*, Comm. Pure Appl. Math. **75** (2022), 2393–2445.

[36] Li, J.; Xin, Z.: *Propagation of uniform boundedness of entropy and inhomogeneous regularities for viscous and heat conductive gases with far field vacuum in three dimensions*, Sci China Math, **65** (2022), https://doi.org/10.1007/ s11425-022-2047-0

[37] Lions, P. L.: *Existence globale de solutions pour les équations de Navier-Stokes compressibles isentropiques*, C. R. Acad. Sci. Paris Sér. I Math., **316** (1993), 1335–1340.

[38] Lions, P. L.: *Mathematical Topics in Fluid Mechanics*, Vol. 2, Clarendon, Oxford, 1998.

[39] Lukaszewicz, G.: *An existence theorem for compressible viscous and heat conducting fluids*, Math. Methods Appl. Sci., **6** (1984), 234–247.

[40] Matsumura, A.; Nishida, T.: *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ., **20** (1980),

67–104.

[41] Matsumura, A.; Nishida, T.: *The initial boundary value problem for the equations of motion of compressible viscous and heat-conductive fluid*, Preprint University of Wisconsin, MRC Technical Summary Report no. 2237 (1981).

[42] Matsumura, A.; Nishida, T.: *Initial-boundary value problems for the equations of motion of general fluids*, Computing methods in applied sciences and engineering, V (Versailles, 1981), 389–406, North-Holland, Amsterdam, 1982.

[43] Matsumura, A.; Nishida, T.: *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Commun. Math. Phys., **89** (1983), 445–464.

[44] Merle, F.; Rapha'el, P.; Rodnianski, I.; Szeftel, J.: *On the implosion of a compressible fluid I: smooth self-similar inviscid profiles*, Ann. of Math., (2) **196** (2022), 567–778.

[45] Merle, F.; Rapha'el, P.; Rodnianski, I.; Szeftel, J.: *On the implosion of a compressible fluid II: singularity formation*, Ann. of Math., (2) **196** (2022), 779–889.

[46] Nash, J.: *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. Fr., **90** (1962), 487–497.

[47] Ponce, G.: *Global existence of small solutions to a class of nonlinear evolution equations*, Nonlinear Anal., **9** (1985), 399–418.

[48] Serrin, J.: *On the uniqueness of compressible fluid motions*, Arch. Rational Mech. Anal., **3** (1959), 271–288.

[49] Salvi, R.; Straškraba, I.: *Global existence for viscous compressible fluids and their behavior as* $t \to \infty$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **40** (1993), 17–51.

[50] Tani, A.: *On the first initial-boundary value problem of compressible viscous fluid motion*, Publ. Res. Inst. Math. Sci., **13** (1977), 193–253.

[51] Valli, A.: *An existence theorem for compressible viscous fluids*, Ann. Mat. Pura Appl., **130** (1982), 197–213; **132** (1982), 399–400.

[52] Valli, A.; Zajaczkowski, W. M.: *Navier-Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case*, Commun. Math. Phys., **103** (1986), 259–296.

[53] Vol'pert, A. I., Hudjaev, S. I.: *On the Cauchy problem for composite systems of nonlinear differential equations*, Math. USSR-Sb, **16** (1972), 517–544 [previously in Mat. Sb. (N.S.), **87** (1972), 504–528(in Russian)].

[54] Wen, H.; Zhu, C.: *Global solutions to the three-dimensional full compressible Navier-Stokes equations with vacuum at infinity in some classes of large data*, SIAM J. Math. Anal., **49** (2017), 162–221.

[55] Wu, Z.; Yin, J.; Wang, C.: Elliptic & Parabolic Equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. xvi+408 pp.

[56] Xin, Z.: *Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density*, Comm. Pure Appl. Math., **51** (1998), 229–240.

[57] Xin, Z.; Yan, W.: *On blowup of classical solutions to the compressible Navier-Stokes equations*, Comm. Math. Phys., **321** (2013), 529–541.

[58] Zlotnik, A. A.; Amosov, A. A.: *On stability of generalized solutions to the equations of one-dimensional motion of a viscous heat-conducting gas*, Siberian Math. J., **38** (1997), 663–684.

[59] Zlotnik, A. A.; Amosov, A. A.: *Stability of generalized solutions to equations of one-dimensional motion of viscous heat conducting gases*, Math. Notes, **63** (1998), 736–746.

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