Global Well-Posedness of the Three-Dimensional Primitive Equations with Only Horizontal Viscosity and Diffusion

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Abstract

In this paper, we consider the initial boundary value problem of the three-dimensional primitive equations for planetary oceanic and atmospheric dynamics with only horizontal eddy viscosity in the horizontal momentum equations and only horizontal diffusion in the temperature equation. Global well-posedness of the strong solution is established for any H^2 initial data. An N-dimensional logarithmic Sobolev embedding inequality, which bounds the L^∞-norm in terms of the L^q-norms up to a logarithm of the L^p-norm for $p > N$ of the first-order derivatives, and a system version of the classic Grönwall inequality are exploited to establish the required a priori H^2 estimates for global regularity. © 2016 Wiley Periodicals, Inc.

1 Introduction

The primitive equations are derived from the Boussinesq system of incompressible flow and they form a fundamental block in models for planetary oceanic and atmospheric dynamics; see, e.g., Lewandowski [16], Majda [20], Pedlosky [21], Vallis [25], and Washington and Parkinson [26]. Due to their importance, the primitive equations has been studied analytically by many authors; see, e.g., [17, 18, 20, 22, 24] and the references therein.

In this paper, we consider the following version of primitive equations with only horizontal eddy viscosities and only horizontal diffusion due to strong dominant horizontal turbulence mixing:

$$\partial_t v + (v \cdot \nabla_H)v + w\partial_z v + \nabla_H p - \Delta_H v + f_0 k \times v = 0, \tag{1.1}$$

$$\partial_z p + T = 0, \tag{1.2}$$

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$$\nabla_H \cdot v + \partial_z w = 0, \tag{1.3}$$

$$\partial_t T + v \cdot \nabla_H T + w \partial_z T - \Delta_H T = 0, \tag{1.4}$$

where the horizontal velocity $v = (v^1, v^2)$, the vertical velocity w, the temperature T, and the pressure p are the unknowns, and f_0 is the Coriolis parameter. In this paper, we use the notations $\nabla_H = (\partial_x, \partial_y)$ and $\Delta_H = \partial_x^2 + \partial_y^2$ to denote the horizontal gradient and the horizontal Laplacian, respectively.

For the primitive equations with full viscosities and full diffusion, the mathematical analysis was initialed in 1990s by Lions, Temam, and Wang [17–19], where among other issues they established the global existence of weak solutions. The uniqueness of weak solutions for the two-dimensional case was later proved by Bresch, Guillén-González, Masmoudi, and Rodríguez-Bellido [1]; however, the uniqueness of weak solutions for the three-dimensional case is still unclear. Local well-posedness of strong solutions was obtained by Guillén-González, Masmoudi, and Rodríguez-Bellido [12]. Global existence of strong solutions for the two-dimensional case was established by Bresch, Kazhikhov, and Lemoine in [2] and Temam and Ziane in [24], while the three-dimensional case was established in [8]. Global strong solutions for the three-dimensional case were also obtained by Kobelkov [13] later by using a different approach; see also the subsequent articles by Kukavica and Ziane [14, 15].

The systems considered in all the papers [1, 2, 8, 12–15, 17–19, 22, 24] are assumed to have full dissipation, i.e., with both full viscosities and full diffusion. Both physically and mathematically, it is also important and interesting to study the system with partial dissipation, i.e., with only partial viscosities or only partial diffusion. The first result in this direction for the primitive equations was obtained in [9], where the authors considered the system with full viscosities but only vertical diffusion and proved that such a system has a unique global strong solution, provided the local-in-time one exists. As the complement and a generalization of [9], the local and global well-posedness of strong solutions were recently established in [6] with H^2 initial data. As the counterpart of [6], global well-posedness of strong solutions to the primitive equations with full viscosities but only horizontal diffusion was later obtained in [5], still for H^2 initial data. Notably, smooth solutions to the inviscid primitive equation, with or without coupling to the temperature equation, has been shown [4] to blow up in finite time (see also Wong [27]).

Note that in all the papers [1,2,5,6,8,9,12–15,17–19,22,24], no matter whether the systems are considered to have full or partial diffusion in the temperature equation, they are assumed to have full viscosities in the horizontal momentum equations. Physically, in the oceanic and atmospheric dynamics, the horizontal scales are much lager than the vertical one with dominant strong horizontal turbulence mixing that induces horizontal viscosities, i.e., system (1.1)–(1.4).

From the mathematical point of view, there are two obvious difficulties in studying system (1.1)–(1.4). One is that the strongest nonlinear term, i.e., $(\int_{-h}^z \nabla_H \cdot v \, d\xi)\partial_z v$, is quadratic in the first derivatives of the unknowns. This is caused by the lack of the dynamical equation for the vertical component of the velocity. The other one is that, due to the lack of the vertical viscosity in the horizontal momentum equations, one cannot expect any smoothing effect in the vertical direction.

The aim of this paper is to show that strong solutions exist globally for system (1.1)–(1.4), subject to some initial and boundary conditions, for any H^2 initial data. More precisely, we consider the problem in the domain $\Omega_0 = M \times (-h, 0)$, with $M = (0,1) \times (0,1)$, and supplement system (1.1)–(1.4) with the following boundary and initial conditions:

$$v, w \text{ and } T \text{ are periodic in } x \text{ and } y, \tag{1.5}$$

$$(\partial_z v, w)|_{z=-h,0} = (0,0), \quad T|_{z=-h} = 1, \quad T|_{z=0} = 0, \tag{1.6}$$

$$(v, T)|_{t=0} = (v_0, T_0). \tag{1.7}$$

By replacing T and p by $T + \frac{z}{h}$ and $p - \frac{z^2}{2h}$, respectively, then system (1.1)–(1.4) with (1.5)–(1.7) is reduced to

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H p - \Delta_H v + f_0 k \times v = 0, \tag{1.8}$$

$$\partial_z p + T = 0, \tag{1.9}$$

$$\nabla_H \cdot v + \partial_z w = 0, \tag{1.10}$$

$$\partial_t T + v \cdot \nabla_H T + w \left(\partial_z T + \frac{1}{h} \right) - \Delta_H T = 0, \tag{1.11}$$

subject to the boundary and initial conditions

$$v, w, T \text{ are periodic in } x \text{ and } y, \tag{1.12}$$

$$(\partial_z v, w)|_{z=-h,0} = 0, \quad T|_{z=-h,0} = 0, \tag{1.13}$$

$$(v, T)|_{t=0} = (v_0, T_0). \tag{1.14}$$

Here, for simplicity, we still use T_0 to denote the initial temperature in (1.14), though it is obtained by replacing the T_0 in (1.7) by $T_0 - \frac{z}{h}$.

For the same reasons used to in [5, 6], system (1.8)–(1.14) defined on Ω_0 is equivalent to the following system defined on $\Omega := M \times (-h, h)$:

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H p - \Delta_H v + f_0 k \times v = 0, \tag{1.15}$$

$$\partial_z p + T = 0, \tag{1.16}$$

$$\nabla_H \cdot v + \partial_z w = 0, \tag{1.17}$$

$$\partial_t T + v \cdot \nabla_H T + w \left(\partial_z T + \frac{1}{h} \right) - \Delta_H T = 0, \tag{1.18}$$

subject to the boundary and initial conditions

$$v, w, p, \text{ and } T \text{ are periodic in } x, y, z, \tag{1.19}$$

$$v \text{ and } p \text{ are even in } z, \text{ and } w \text{ and } T \text{ are odd in } z, \tag{1.20}$$

$$(v, T)|_{t=0} = (v_0, T_0). \tag{1.21}$$

Note that the restriction on the subdomain Ω_0 of a solution (v, w, p, T) to system (1.15)–(1.21) is a solution to the original system (1.8)–(1.14). Because of this, throughout this paper, we are mainly concerned with the study of system (1.15)–(1.21) defined on Ω, while the well-posedness results for system (1.8)–(1.14) defined on Ω_0 follow as a corollary of those for system (1.15)–(1.21).

One can check that system (1.15)–(1.21) is equivalent to (see [9] for example)

$$\begin{aligned} &\partial_t v - \Delta_H v + (v \cdot \nabla_H) v - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \partial_z v \\ &\quad + f_0 k \times v + \nabla_H \left(p_s(x, y, t) - \int_{-h}^{z} T(x, y, \xi, t) d\xi \right) = 0, \end{aligned} \tag{1.22}$$

$$\nabla_H \cdot \int_{-h}^{h} v(x, y, z, t) dz = 0, \tag{1.23}$$

$$\begin{aligned} &\partial_t T - \Delta_H T + v \cdot \nabla_H T \\ &\quad - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \left(\partial_z T + \frac{1}{h} \right) = 0, \end{aligned} \tag{1.24}$$

subject to the following boundary and initial conditions:

$$v \text{ and } T \text{ are periodic in } x, y, z, \tag{1.25}$$

$$v \text{ and } T \text{ are even and odd in } z, \text{ respectively}, \tag{1.26}$$

$$(v, T)|_{t=0} = (v_0, T_0). \tag{1.27}$$

Before stating our main results, let's introduce some necessary notation and give the definitions of strong solutions. Throughout this paper, for $1 \leq q \leq \infty$, we use $L^q(\Omega), L^q(M)$, and $W^{m,q}(\Omega), W^{m,q}(M)$ to denote the standard Lebesgue and Sobolev spaces, respectively. For $q = 2$, we use H^m instead of $W^{m,2}$. We use $W^{m,q}_{\text{per}}(\Omega)$ and H^m_{per} to denote the spaces of periodic functions in $W^{m,q}(\Omega)$ and $H^m(\Omega)$, respectively. For simplicity, we still use the notation L^p and H^m to denote the N product spaces $(L^p)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_p$ to denote the L^p-norm of u. Moreover, for convenience, we often use $\|(f_1, \ldots, f_n)\|_2^2$ to denote the summation $\sum_{i=1}^n \|f_i\|_2^2$.

DEFINITION 1.1. Given a positive time $\mathcal{T}$, let $v_0 \in H^2(\Omega)$ and $T_0 \in H^2(\Omega)$ be two periodic functions such that they are even and odd in z, respectively. A couple (v, T) is called a *strong solution* to system (1.22)–(1.27) (or equivalently (1.15)–(1.21)) on $\Omega \times (0, \mathcal{T})$ if

(i) v and T are periodic in x, y, z, and they are even and odd in z, respectively;

(ii) v and T have the regularities

$$(v, T) \in L^\infty(0, \mathcal{T}; H^2(\Omega)) \cap C([0, \mathcal{T}]; H^1(\Omega)),$$

$$(\nabla_H v, \nabla_H T) \in L^2(0, \mathcal{T}; H^2(\Omega)), \quad (\partial_t v, \partial_t T) \in L^2(0, \mathcal{T}; H^1(\Omega));$$

(iii) v and T satisfy equations (1.22)–(1.24) a.e. in $\Omega \times (0, \mathcal{T})$ and the initial condition (1.27).

DEFINITION 1.2. A couple (v, T) is called a *global strong solution* to system (1.22)–(1.27) if it is a strong solution on $\Omega \times (0, \mathcal{T})$ for any $\mathcal{T} \in (0, \infty)$.

The main result of this paper is the following global well-posedness result.

THEOREM 1.3. *Suppose that the periodic functions $v_0, T_0 \in H^2(\Omega)$ are even and odd in z, respectively. Then system* (1.22)-(1.27) (*or equivalently* (1.15)–(1.21)) *has a unique global strong solution* (v, T) *that is continuously dependent on the initial data.*

The key issue in proving Theorem 1.3 is establishing the a priori $L^\infty(0, \mathcal{T}; H^2(\Omega))$ estimates on the strong solutions. Our analysis shows that once the $L^\infty(0, \mathcal{T}; H^2(\Omega))$ estimate on $u = \partial_z v$ is obtained, all the required estimates of the other derivatives can be successfully achieved. Unfortunately, due to the lack of the vertical viscosity in the horizontal momentum equations, such an $L^\infty(0, \mathcal{T}; H^2(\Omega))$ estimate cannot be obtained solely without the contribution of the other derivative of the velocity. We observe that in all the arguments of existing articles, the full viscosities play an essential role in obtaining the $L^\infty(0, \mathcal{T}; H^2(\Omega))$ estimate on $u = \partial_z v$, and thus the existing arguments cannot be applied in our case. Still because of the lack of vertical viscosity, one will encounter $\|v\|_\infty^2$, which appears as the coefficients in the higher-order energy inequalities; in other words, the energy inequalities we arrive to are all of the form

$$\frac{d}{dt} f \leq C \|v\|_\infty^2 f + \text{other terms}; \tag{1.28}$$

see Section 4 for details.

Since we do not know whether $\|v\|_\infty^2$ is integrable in time, we cannot obtain the required estimate for f directly from this kind of energy inequalities. Besides, recalling that f will represent quantities involving the H^2-norm of v, though $\|v\|_\infty^2$ can be bounded by f, by the Sobolev embedding inequality, one will still be unable to obtain the global-in-time H^2 estimate for v by the simple application of the standard embedding inequality. Observe that if $\|v\|_\infty^2$ and f have the relationship

$$\|v\|_\infty^2 \leq C \log f,$$

then the energy inequality (1.28) implies the global-in-time estimate for f. To guarantee this relationship, thanks to the logarithmic Sobolev embedding inequality (Lemma 2.2 below), it suffices to prove that the L^q-norms of v grow no faster than $C\sqrt{q}$. By taking advantage of the property that the pressure p depends essentially only on the horizontal spatial variables, and using the Ladyzhenskaya-type inequalities (Lemma 2.1 below) for a class of integrals in three dimensions, one can successfully prove the desired growth of the L^q-norms of v, and thus obtain the a priori H^2 estimates, and hence the global regularity.

It should be pointed out that, due to the anisotropic structure of the momentum equation (1.22) (the advection term $(v \cdot \nabla_H)v$ and the vertical advection term $\left(\int_{-h}^{z} \nabla_H \cdot v\, d\xi\right)\partial_z v$ play different roles), the treatments for different derivatives of the same order will vary. More precisely, when dealing with the derivatives of the same order, the treatment of the vertical derivatives precedes that of the horizontal ones, because the estimates of the horizontal derivatives may depend on those of the vertical ones, see Proposition 4.1 and Proposition 4.2, below, for the details. Accordingly, a system version of the classic Grönwall inequality, Lemma 2.3 below, is exploited to derive the a priori bounds from the energy inequalities. We believe that this system version of the Grönwall inequality is interesting on its own, and in fact it can benefit us when using the energy approach; see Remark 2.4 below.

As a corollary of Theorem 1.3, we have the following theorem, which states the global well-posedness of strong solutions to system (1.8)–(1.14). Strong solutions to system (1.8)–(1.14) are defined similarly as before.

THEOREM 1.4. *Let v_0 and T_0 be two functions defined on Ω_0 such that they are both periodic in x and y. Denote by v_0^{ext} and T_0^{ext} the even and odd extensions in z of v_0 and T_0, respectively. Suppose that $v_0^{\mathrm{ext}}, T_0^{\mathrm{ext}} \in H^2_{\mathrm{per}}(\Omega)$. Then system* (1.8)–(1.14) *has a unique global strong solution* (v, T).

Theorem 1.4 follows directly by applying Theorem 1.3 with initial data $(v_0^{\mathrm{ext}}, T_0^{\mathrm{ext}})$ and restricting the solution to the subdomain Ω_0.

The rest of this paper is arranged as follows: in Section 2 we collect some preliminary results that will be used in subsequent sections; in Section 3 we establish the a priori low-order energy estimates, which are independent of the regularization parameter ε for strong solutions to a regularized system, while the ε independent higher-order energy inequalities are given in Section 4. In Section 5, with the aid of the a priori estimates and the higher-order energy inequalities achieved in the previous two sections, we first establish the a priori H^2 estimates for the strong solutions to the regularized system, and then obtain the global well-posedness of strong solutions to system (1.22)–(1.27) (or, equivalently, system (1.15)–(1.21)) by a standard approach. Finally, in the appendix, an N-dimensional logarithmic Sobolev embedding inequality is established.

Throughout this paper, C denotes a general constant that may be different from line to line.

2 Preliminaries

In this section we collect some preliminary results that will be used in the rest of this paper, and we start with the following Ladyzhenskaya-type inequality in three dimensions for a class of integrals, which will be frequently used throughout the paper.

LEMMA 2.1. *The following inequalities hold true*:

$$\int_M \left(\int_{-h}^h |\phi(x,y,z)|\,dz\right)\left(\int_{-h}^h |\varphi(x,y,z)\psi(x,y,z)|\,dz\right)dx\,dy$$
$$\le C\min\Big\{\|\phi\|_2^{1/2}\big(\|\phi\|_2^{1/2}+\|\nabla_H\phi\|_2^{1/2}\big)\|\varphi\|_2\|\psi\|_2^{1/2}\big(\|\psi\|_2^{1/2}+\|\nabla_H\psi\|_2^{1/2}\big),$$
$$\|\phi\|_2\|\varphi\|_2^{1/2}\big(\|\varphi\|_2^{1/2}+\|\nabla_H\varphi\|_2^{1/2}\big)\|\psi\|_2^{1/2}\big(\|\psi\|_2^{1/2}+\|\nabla_H\psi\|_2^{1/2}\big)\Big\},$$

and

$$\int_M \left(\int_{-h}^h |\phi(x,y,z)|\,dz\right)\left(\int_{-h}^h |\varphi(x,y,z)\nabla_H\Psi(x,y,z)|\,dz\right)dx\,dy$$
$$\le C\min\Big\{\|\phi\|_2^{1/2}\big(\|\phi\|_2^{1/2}+\|\nabla_H\phi\|_2^{1/2}\big)\|\varphi\|_2\|\Psi\|_\infty^{1/2}\big\|\nabla_H^2\Psi\big\|_2^{1/2},$$
$$\|\phi\|_2\|\varphi\|_2^{1/2}\big(\|\varphi\|_2^{1/2}+\|\nabla_H\varphi\|_2^{1/2}\big)\|\Psi\|_\infty^{1/2}\big\|\nabla_H^2\Psi\big\|_2^{1/2}\Big\},$$

for every $\phi, \varphi, \psi, \Psi$ such that the right-hand sides make sense and are finite, where C is a positive constant depending only on h. Moreover, if ϕ has the form $\phi = \nabla_H f$ for some function f, then by the Poincáre inequality, the lower-order term $\|\phi\|_2^{1/2}$ in the parentheses can be dropped in the above inequalities, and the same can also be said for φ and ψ.

PROOF. Similar inequalities have been established in [7, 9]. However, for completeness, we sketch the proofs here, and the ideas used here are the same as in those papers. By the Hölder and Minkowski inequalities, we deduce

$$\int_M \left(\int_{-h}^h |\phi(x,y,z)|\,dz\right)\left(\int_{-h}^h |\varphi(x,y,z)\psi(x,y,z)|\,dz\right)dx\,dy$$
$$\le \int_M \left(\int_{-h}^h |\phi(x,y,z)|\,dz\right)\left(\int_{-h}^h |\varphi(x,y,z)|^2\,dz\right)^{1/2}$$
$$\times\left(\int_{-h}^h |\psi(x,y,z)|^2\,dz\right)^{1/2}dx\,dy \le$$

$$
\begin{aligned}
&\le \min\Bigg\{\bigg(\iint_M \bigg|\int_{-h}^{h}|\phi|dz\bigg|^2 dx\,dy\bigg)^{1/2}\bigg(\iint_M \bigg|\int_{-h}^{h}|\varphi|^2\,dz\bigg|^2 dx\,dy\bigg)^{1/4},\\
&\qquad\bigg(\iint_M \bigg|\int_{-h}^{h}|\phi|dz\bigg|^4 dx\,dy\bigg)^{1/4}\bigg(\iint_M\int_{-h}^{h}|\varphi|^2\,dz\,dx\,dy\bigg)^{1/2}\Bigg\}\\
&\quad\times\bigg(\iint_M \bigg|\int_{-h}^{h}|\psi|^2\,dz\bigg|^2 dx\,dy\bigg)^{1/4}\\
&\le C\min\Bigg\{\|\phi\|_2\bigg(\int_{-h}^{h}\|\varphi\|_{4,M}^2\,dz\bigg)^{1/2},\\
&\qquad\int_{-h}^{h}\|\phi\|_{4,M}\,dz\|\varphi\|_2\Bigg\}\bigg(\int_{-h}^{h}\|\psi\|_{4,M}^2\,dz\bigg)^{1/2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\iint_M\bigg(\int_{-h}^{h}|\phi(x,y,z)|dz\bigg)\bigg(\int_{-h}^{h}|\varphi(x,y,z)\nabla_H\Psi(x,y,z)|dz\bigg)dx\,dy\\
&\quad\le C\min\Bigg\{\|\phi\|_2\bigg(\int_{-h}^{h}\|\varphi\|_{4,M}^2\,dz\bigg)^{1/2},\\
&\qquad\int_{-h}^{h}\|\phi\|_{4,M}\,dz\|\varphi\|_2\Bigg\}\bigg(\int_{-h}^{h}\|\nabla_H\Psi\|_{4,M}^2\,dz\bigg)^{1/2}.
\end{aligned}
$$

It follows from the two-dimensional Ladyzhenskaya and Gagliardo-Nirenberg inequalities that

$$
\begin{aligned}
\int_{-h}^{h}\|\phi\|_{4,M}\,dz &\le C\int_{-h}^{h}\|\phi\|_{2,M}^{1/2}\|\phi\|_{H^1(M)}^{1/2}dz\\
&\le C\|\phi\|_2^{1/2}\big(\|\phi\|_2^{1/2}+\|\nabla_H\phi\|_2^{1/2}\big),\\
\int_{-h}^{h}\|\psi\|_{4,M}^2\,dz &\le C\int_{-h}^{h}\|\psi\|_{2,M}\|\psi\|_{H^1(M)}\,dz\\
&\le C\|\psi\|_2(\|\psi\|_2+\|\nabla_H\psi\|_2),\\
\int_{-h}^{h}|\nabla_H\Psi\|_{4,M}dz &\le C\int_{-h}^{h}\|\Psi\|_{\infty,M}\|\Psi\|_{H^2(M)}dz\le C\|\Psi\|_\infty\|\nabla_H^2\Psi\|_2.
\end{aligned}
$$

The conclusions follow from combining the previous five inequalities. □

The following logarithmic Sobolev inequality, which bounds the L^∞-norm in terms of the L^q-norms up to the logarithm of the norms of the high-order derivatives will play an important role in establishing the a priori H^2 estimates later.

Some relevant inequalities can be found in [3, 10, 11], where the two-dimensional case is considered.

LEMMA 2.2. *Let $F \in W^{1,p}(\Omega)$, with $p > 3$, be a periodic function. Then the following inequality holds true*:

$$\|F\|_\infty \le C_{p,\lambda} \max\left\{1, \sup_{r\ge 2} \frac{\|F\|_r}{r^\lambda}\right\} \log^\lambda(\|F\|_{W^{1,p}(\Omega)} + e)$$

for any $\lambda > 0$.

PROOF. Extend F periodically to the whole space. Take $\phi \in C_0^\infty(\mathbb{R}^3)$, a function such that $\phi \equiv 1$ on Ω and $0 \le \phi \le 1$ on $\mathbb{R}^3$. Set $f = F\phi$. By Lemma A.1 (choose $R = 1$ there) in the appendix, it holds that

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le C_{p,\lambda} \max\left\{1, \sup_{r\ge 2} \frac{\|f\|_{L^r(\mathbb{R}^3)}}{r^\lambda}\right\} \log^\lambda(\|f\|_{W^{1,p}(\mathbb{R}^3)} + e).$$

Noticing that

$$\|F\|_\infty \le \|f\|_{L^\infty(\mathbb{R}^3)}, \quad \|f\|_{L^r(\mathbb{R}^3)} \le C\|F\|_r, \quad \|f\|_{W^{1,p}(\mathbb{R}^3)} \le C\|F\|_{W^{1,p}(\Omega)},$$

we deduce

$$\begin{aligned}\|F\|_\infty \le \|f\|_{L^\infty(\mathbb{R}^3)} &\le C_{p,\lambda} \max\left\{1, \sup_{r\ge 2} \frac{\|f\|_{L^r(\mathbb{R}^3)}}{r^\lambda}\right\} \log^\lambda(\|f\|_{W^{1,p}(\mathbb{R}^3)} + e) \\ &\le C_{p,\lambda} \max\left\{1, \sup_{r\ge 2} \frac{\|F\|_r}{r^\lambda}\right\} \log^\lambda(\|F\|_{W^{1,p}(\Omega)} + e),\end{aligned}$$

proving the conclusion. □

The following lemma is a system version of the classic Grönwall inequality.

LEMMA 2.3. *Let $m(t)$, $K(t)$, $A_i(t)$, and $B_i(t)$ be nonnegative functions such that $A_i \ge e$, are absolutely continuous for $i = 1, \dots, n$, $K \in L^1_{\mathrm{loc}}([0,\infty))$, and*

$$m(t) \le K(t) \log \sum_{i=1}^n A_i(t).$$

Given a positive time $\mathcal{T}$, suppose that

$$\frac{d}{dt} A_1(t) + B_1(t) \le m(t) A_1(t), \tag{2.1}$$

$$\frac{d}{dt} A_i(t) + B_i(t) \le m(t) A_i(t) + \zeta A_{i-1}^\alpha(t) B_{i-1}(t), \tag{2.2}$$

for $i = 2, \dots, n$ and for any $t \in (0, \mathcal{T})$ where $\alpha \ge 1$ and $\zeta \ge 1$ are two constants. Then it holds that

$$\sum_{i=1}^n A_i(t) + \sum_{i=1}^n \int_0^t B_i(s)\,ds \le Q(t) \quad \forall t \in [0, \mathcal{T}),$$

where Q is a continuous function on $[0,\infty)$ *that is determined by* $A_i(0)$*,* $i = 1,\dots,n$*, and* K*, given explicitly in equation* (2.4) *below.*

PROOF. Multiplying inequality (2.1) by $\zeta(\alpha+1)A_1^\alpha$ yields

$$\zeta\frac{d}{dt}A_1^{\alpha+1} + \zeta(\alpha+1)A_1^\alpha B_1 \le \zeta(\alpha+1)mA_1^{\alpha+1}.$$

Summing this with (2.2) for $i = 2$ leads to

$$\frac{d}{dt}\big(A_2 + \zeta A_1^{\alpha+1}\big) + \zeta\alpha A_1^\alpha B_1 + B_2 \le (\alpha+1)m(A_2 + \zeta A_1^{\alpha+1}).$$

Set $\mathcal{A}_1 = A_1$, $\mathcal{A}_2 = A_2 + \zeta\mathcal{A}_1^{\alpha+1}$, $\mathcal{B}_1 = B_1$, and $\mathcal{B}_2 = \mathcal{B}_1 + B_2$; then the above inequality gives

$$\frac{d}{dt}\mathcal{A}_2 + \mathcal{B}_2 \le (\alpha+1)m\mathcal{A}_2.$$

Continuing the previous procedure inductively, we obtain

$$\mathcal{A}_i = A_i + \zeta\mathcal{A}_{i-1}^{\alpha+1}, \quad \mathcal{B}_i = B_i + \mathcal{B}_{i-1}, \quad i = 2,\dots,N,$$
$$\frac{d}{dt}\mathcal{A}_i + \mathcal{B}_i \le (\alpha+1)^{i-1}m\mathcal{A}_i, \quad i = 1,\dots,N.$$

In particular, it holds that

$$\frac{d}{dt}\mathcal{A}_N + \mathcal{B}_N \le (\alpha+1)^{N-1}m\mathcal{A}_N. \tag{2.3}$$

By the assumption on $m(t)$, the above inequality implies

$$\frac{d}{dt}\mathcal{A}_N(t) \le (\alpha+1)^{N-1}K(t)\mathcal{A}_N(t)\log\mathcal{A}_N(t).$$

Therefore

$$\frac{d}{dt}\log\mathcal{A}_N(t) \le (\alpha+1)^{N-1}K(t)\log\mathcal{A}_N,$$

from which we obtain

$$\log\mathcal{A}_N \le e^{(\alpha+1)^{N-1}\int_0^t K(s)ds}\log\mathcal{A}_N(0) =: q_0(t)$$

and

$$\mathcal{A}_N(t) \le e^{q_0(t)} =: q_1(t)$$

for $t \in [0,\mathcal{T})$. Note that q_1 is an increasing function on $[0,\infty)$. Thanks to this estimate, it follows from integrating inequality (2.3) with respect to t that

$$\begin{aligned}\mathcal{A}_N(t) + \int_0^t \mathcal{B}_N(s)ds &\le (\alpha+1)^{N-1}\int_0^t m(s)\mathcal{A}_N(s)ds\\ &\le (\alpha+1)^{N-1}\int_0^t \mathcal{A}_N(s)K(s)\log\mathcal{A}_N(s)ds\\ &\le (\alpha+1)^{N-1}\int_0^t K(s)dsq_1(t)q_0(t) = Q(t),\end{aligned}$$

for all $t \in [0, \mathcal{T})$, where

$$Q(t) = (\alpha + 1)^{N-1} \int_0^t K(s) ds q_1(t) q_0(t). \tag{2.4}$$

From this we obtain the conclusion. □

Remark 2.4. Lemma 2.3 indicates that when doing the energy estimates, step by step, the quantities appearing on the left-hand side in the previous steps can be treated freely as if they were a priori bounded, provided the coefficient term does not grow too fast compared to the quantities under consideration (no faster than the logarithm of their summation). This gives us room to handle some hard terms in the current step.

We also need the following Aubin-Lions lemma.

LEMMA 2.5 (Aubin-Lions Lemma; see Simon [23, cor. 4]). *Assume that X, B, and Y are three Banach spaces, with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Then it holds that*

- (i) *If F is a bounded subset of $L^p(0, T; X)$, where $1 \le p < \infty$, and $\frac{\partial F}{\partial t} = \{\frac{\partial f}{\partial t} \mid f \in F\}$ is bounded in $L^1(0, T; Y)$, then F is relatively compact in $L^p(0, T; B)$.*
- (ii) *If F is bounded in $L^\infty(0, T; X)$ and $\frac{\partial F}{\partial t}$ is bounded in $L^r(0, T; Y)$, where $r > 1$, then F is relatively compact in $C([0, T]; B)$.*

Finally, we will use the following global existence result for a regularized system.

PROPOSITION 2.6. *Suppose that the periodic functions $v_0, T_0 \in H^2(\Omega)$ are even and odd in z, respectively. Then for any $\varepsilon > 0$, there is a unique global strong solution (v, T) to the following system*:

$$\begin{aligned} &\partial_t v + (v \cdot \nabla_H) v - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \partial_z v - \Delta_H v - \varepsilon \partial_z^2 v \\ &\quad + f_0 k \times v + \nabla_H \left(p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi \right) = 0, \end{aligned} \tag{2.5}$$

$$\int_{-h}^h \nabla_H \cdot v(x, y, z, t) dz = 0, \tag{2.6}$$

$$\begin{aligned} &\partial_t T + v \cdot \nabla_H T - \left(\int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi \right) \left(\partial_z T + \frac{1}{h} \right) \\ &\quad - \Delta_H T - \varepsilon \partial_z^2 T = 0, \end{aligned} \tag{2.7}$$

subject to the boundary and initial conditions (1.25)–(1.27)*, such that*

$$(v, T) \in L^\infty_{\mathrm{loc}}([0, \infty); H^2(\Omega)) \cap C([0, \infty); H^1(\Omega)) \cap L^2_{\mathrm{loc}}([0, \infty); H^3(\Omega)),$$
$$(\partial_t v, \partial_t T) \in L^2_{\mathrm{loc}}([0, \infty); H^1(\Omega)).$$

PROOF. The proof can be given in the same way as in [6] (see proposition 2.1 there), and thus we omit it here. □

3 Low-Order Energy Estimates

In this section, we work on the low-order energy estimates on the strong solution to system (2.5)–(2.7), subject to the boundary and initial conditions (1.25)–(1.27). In particular, we prove that the growth of the L^q-norms of v is not faster than $C\sqrt{q}$ for a constant C independent of q.

PROPOSITION 3.1. *Let* (v, T) *be the global strong solution to system* (2.5)–(2.7)*, subject to the boundary and initial conditions* (1.25)–(1.27). *Then for any* $\mathcal{T} \in (0, \infty)$*, we have the following*:

(i) *Basic energy estimates*:

$$\sup_{0\le t\le\mathcal{T}} \|(v, T)\|_2^2 + \int_0^{\mathcal{T}} \|(\nabla_H v, \nabla_H T, \sqrt{\varepsilon}\partial_z v, \sqrt{\varepsilon}\partial_z T)\|_2^2\, dt \le K_0(\mathcal{T}),$$

where $K_0(\mathcal{T}) = Ce^{\mathcal{T}}(\|v_0\|_2^2 + \|T_0\|_2^2)$ *for a positive constant* C *depending only on* h.

(ii) L^q *estimate on* T*, with* $2 \le q \le \infty$:

$$\sup_{0\le t\le\mathcal{T}} \|T^*\|_q \le \|T_0^*\|_q,$$

where $T^* = T + \frac{z}{h}$ *and* $T_0^* = T_0 + \frac{z}{h}$.

(iii) L^q *estimate on* v*, with* $q \in [4, \infty)$:

$$\sup_{0\le t\le\mathcal{T}} \|v\|_q \le K_1(\mathcal{T}) e^{C\|T_0\|_q^2 \mathcal{T}} (1 + \|v_0\|_q)\sqrt{q},$$

where K_1*, a continuously increasing function, is determined by* $\|v_0\|_2$*,* $\|T_0\|_2$*,* $\|v_0\|_4$*, and* $\|T_0\|_4$.

PROOF.

(i) If we multiply equations (2.5) and (2.7) by v and T, respectively, sum the resulting equations, and integrate over Ω, then it follows from integrating by parts and using (2.6) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|v|^2 + |T|^2)\,dx\,dy\,dz \\
&\quad + \int_\Omega \big(|\nabla_H v|^2 + \varepsilon|\partial_z v|^2 + |\nabla_H T|^2 + \varepsilon|\partial_z T|^2\big)\,dx\,dy\,dz \\
&\quad = -\int_\Omega \left[\left(\int_{-h}^z T\,d\xi\right)\nabla_H \cdot v - \frac{1}{h}\left(\int_{-h}^z \nabla_H \cdot v\,d\xi\right)T\right] dx\,dy\,dz \\
&\quad \le C\|T\|_2\|\nabla_H v\|_2 \le \frac{1}{2}\|\nabla_H v\|_2^2 + C\|T\|_2^2.
\end{aligned}$$

Thus

$$\frac{d}{dt}\|(v,T)\|_2^2 + \|(\nabla_H v, \nabla_H T, \sqrt{\varepsilon}\partial_z v, \sqrt{\varepsilon}\partial_z T)\|_2^2 \le C\|T\|_2^2,$$

from which, by the Grönwall inequality, one obtains (i).

(ii) Recalling the definition of T^*, using equation (2.7), one can easily check that T^* satisfies

$$\partial_t T^* + v\cdot\nabla_H T^* - \left(\int_{-h}^z \nabla_H\cdot v(x,y,\xi,t)\,d\xi\right)\partial_z T^* - \Delta_H T^* - \varepsilon\partial_z^2 T^* = 0.$$

By multiplying the above equation by $|T^*|^{q-2}T^*$, with $2 \le q < \infty$ and integrating over Ω, it follows from integration by parts and using the divergence-free condition (2.6) that

$$\frac{1}{q}\frac{d}{dt}\|T^*\|_q^q \le 0,$$

which implies the conclusion for $2 \le q < \infty$. The case that $q = \infty$ follows by taking $q \to \infty$ and using the fact that $\|T^*\|_q \to \|T^*\|_\infty$ as $q \to \infty$.

(iii) Let $4 \le q < \infty$. The $L^\infty(0,\mathcal{T};L^q(\Omega))$ estimate on v is proved in two steps: a rough estimate and then a more refined estimate. As we shall see below, the latter is based on the former.

Step 1. *The rough* $L^\infty(0,\mathcal{T};L^q(\Omega))$ *estimate on* v. By multiplying equation (2.5) by $|v|^{q-2}v$ and integrating the resulting equation over Ω, then it follows from integrating by parts that

$$\begin{aligned}\text{(3.1)}\quad \frac{1}{q}\frac{d}{dt}\int_\Omega |v|^q\,dx\,dy\,dz + \int_\Omega |v|^{q-2}\big(|\nabla_H v|^2 + (q-2)\big|\nabla_H|v|\big|^2 \\ + \varepsilon|\partial_z v|^2 + (q-2)\varepsilon\big|\partial_z|v|\big|^2\big)dx\,dy\,dz = I_1 + I_2,\end{aligned}$$

where

$$I_1 := \int_\Omega \nabla_H\left(\int_{-h}^z T\,d\xi\right)\cdot|v|^{q-2}v\,dx\,dy\,dz,$$

$$I_2 := -\int_\Omega \nabla_H p_s(x,y,t)\cdot|v|^{q-2}v\,dx\,dy\,dz.$$

An estimate for I_1 is given as follows. By the Hölder and Young inequalities and using (ii), we deduce

$$\begin{aligned}I_1 &= \int_\Omega \nabla_H\left(\int_{-h}^z T\,d\xi\right)\cdot|v|^{q-2}v\,dx\,dy\,dz \\ &= -\int_\Omega\left(\int_{-h}^z T\,d\xi\right)(|v|^{q-2}\nabla_H\cdot v + (q-2)|v|^{q-3}v\cdot\nabla_H|v|)dx\,dy\,dz \le\end{aligned}$$

$$\leq \int_\Omega \left|\int_{-h}^z T\,d\xi\right| |v|^{q-2}(|\nabla_H v| + (q-2)\big|\nabla_H|v|\big|)dx\,dy\,dz$$

$$\leq C\|T\|_q\|v\|_q^{\frac{q}{2}-1}\left(\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2 + (q-2)\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2\right)$$

$$\leq \frac{1}{8}\left(\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2 + (q-2)\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2\right) + Cq\|T\|_q^2\|v\|_q^{q-2}$$

$$\leq \frac{1}{8}\left(\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2 + (q-2)\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2\right) + Cq(1+\|T_0\|_q^2)\|v\|_q^{q-2}, \tag{3.2}$$

where the constant C is independent of $q \in [4,\infty)$.

By applying the operator $\frac{1}{2h}\int_{-h}^h \mathrm{div}_H(\cdot)dz$ to equation (2.5) and using (2.6), it follows from integrating by parts that

$$\begin{aligned}-\Delta_H p_s(x,y,t) &= \mathrm{div}_H\,\mathrm{div}_H\left(\frac{1}{2h}\int_{-h}^h v\otimes v\,dz\right) + \mathrm{div}_H\left(\frac{1}{2h}\int_{-h}^h f_0 k\times v\,dz\right)\\ &\quad - \Delta_H\left(\frac{1}{2h}\int_{-h}^h\left(\int_{-h}^z T\,d\xi - \int_M\int_{-h}^z T\,d\xi\,dx\,dy\right)dz\right).\end{aligned}$$

Note that by assuming that $\int_M p_s(x,y,t)dx\,dy = 0$, $p_s(x,y,t)$ can be chosen in a unique way. Set

$$p_s^0 = \frac{1}{2h}\int_{-h}^h\left(\int_{-h}^z T\,d\xi - \int_M\int_{-h}^z T\,d\xi\,dx\,dy\right)dz,$$

and decompose p_s as $p_s = p_s^0 + p_s^1 + p_s^2$, with

$$\begin{cases}-\Delta_H p_s^1 = \mathrm{div}_H\,\mathrm{div}_H\left(\frac{1}{2h}\int_{-h}^h (v\otimes v)dz\right) & \text{in } M,\\ \int_M p_s^1\,dx\,dy = 0, & p_s^1 \text{ is periodic,}\end{cases}$$

and

$$\begin{cases}-\Delta_H p_s^2 = \mathrm{div}_H\left(\frac{1}{2h}\int_{-h}^h f_0 k\times v\,dz\right) & \text{in } M,\\ \int_M p_s^2\,dx\,dy = 0, & p_s^2 \text{ is periodic.}\end{cases}$$

Then by the elliptic estimates, we have

$$\|p_s^1\|_{q,M} \leq C_q\left\|\int_{-h}^h (v\otimes v)dz\right\|_{q,M} \leq C_q\int_{-h}^h \|v\|_{2q,M}^2\,dz \tag{3.3}$$

for all $q \in (1,\infty)$, and

$$\|\nabla_H p_s^1\|_{2,M} \leq C\left\|\mathrm{div}_H\left(\int_{-h}^h v\otimes v\,dz\right)\right\|_{2,M} \leq C\||v||\nabla_H v|\|_2, \tag{3.4}$$

$$\|\nabla_H p_s^2\|_{2,M} \leq C\left\|\int_{-h}^h k\times v\,dz\right\|_{2,M} \leq C\|v\|_2. \tag{3.5}$$

By setting

$$I_{2i} := -\int_{\Omega} \nabla_H p_s^i(x,y,t)\cdot |v|^{q-2}v\,dx\,dy\,dz, \quad i=0,1,2,$$

then it is obvious that $I_2 = I_{20} + I_{21} + I_{22}$. We estimate I_{2i}, $i = 0,1,2$, as follows. For I_{20}, a similar argument similar to that for (3.2) yields

$$\begin{aligned} I_{20} \le{}& \frac{1}{8}\big(\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2 + (q-2)\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2\big)\\ &+ Cq(1+\|T_0\|_q^2)\|v\|_q^{q-2}, \end{aligned} \tag{3.6}$$

with constant C independent of q. For I_{22}, by the Hölder, Minkowski, Ladyzhenskaya, and Young inequalities, we deduce

$$\begin{aligned} I_{22} ={}& -\int_{\Omega} \nabla_H p_s^2(x,y,t)\cdot |v|^{q-2}v\,dx\,dy\,dz\\ \le{}& \left(\int_M |\nabla_H p_s^2(x,y,t)|^2\,dx\,dy\right)^{1/2}\left[\int_M\left(\int_{-h}^h |v|^{q-1}dz\right)^2 dx\,dy\right]^{1/2}\\ \le{}& C\|\nabla_H p_s^2\|_{2,M}\int_{-h}^h \|v\|_{2(q-1),M}^{q-1}\,dz\\ \le{}& C\|\nabla_H p_s^2\|_{2,M}\int_{-h}^h \|v\|_{2,M}^{\frac{1}{q-1}}\|v\|_{2q,M}^{\frac{q(q-2)}{q-1}}\,dz\\ \le{}& C\|\nabla_H p_s^2\|_{2,M}\int_{-h}^h \|v\|_{2,M}^{\frac{1}{q-1}}\,\big\||v|^{\frac{q}{2}}\big\|_{2,M}^{\frac{q-2}{q-1}}\,\big\||v|^{\frac{q}{2}}\big\|_{H^1(M)}^{\frac{q-2}{q-1}}dz\\ \le{}& C\|\nabla_H p_s^2\|_{2,M}\|v\|_2^{\frac{1}{q-1}}\left[\|v\|_q^{\frac{q(q-2)}{q-1}} + \big\||v|^{\frac{q}{2}}\big\|_2^{\frac{q-2}{q-1}}\left(\frac{q}{2}\left\||v|^{\frac{q}{2}-1}\nabla_H|v|\right\|_2\right)^{\frac{q-2}{q-1}}\right]\\ \le{}& \frac{q-2}{8}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2 + C\|\nabla_H p_s^2\|_{2,M}^{\frac{2(q-1)}{q}}\|v\|_2^{\frac{2}{q}}\big\||v|^{\frac{q}{2}}\big\|_2^{\frac{2(q-2)}{q}}q^{\frac{q-2}{q}}\\ &+ C\|\nabla_H p_s^2\|_{2,M}\|v\|_2^{\frac{1}{q-1}}\|v\|_q^{\frac{q(q-2)}{q-1}}\\ \le{}& \frac{q-2}{8}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2 + Cq(1+\|v\|_2^2)(1+\|\nabla_H p_s^2\|_{2,M}^2)\|v\|_q^{q-2}\\ &+ C(1+\|v\|_2^2)(1+\|\nabla_H p_s^2\|_{2,M}^2)\|v\|_q^{\frac{q(q-2)}{q-1}},\\ \le{}& C(1+\|v\|_2^2)(1+\|\nabla_H p_s^2\|_{2,M}^2)\big(q\|v\|_q^{q-2} + \|v\|_q^{\frac{q(q-2)}{q-1}}\big)\\ &+ \frac{q-2}{8}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2, \end{aligned} \tag{3.7}$$

with the constant C independent of $q \in [4,\infty)$. Recalling the elliptic estimate (3.5), the above inequality gives

$$I_{22} \le \frac{q-2}{8}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2 + C\big(1+\|v\|_2^2\big)^2\big(q\|v\|_q^{q-2} + \|v\|_q^{\frac{q(q-2)}{q-1}}\big). \tag{3.8}$$

As for I_{21}, recalling the elliptic estimate (3.3), and using Lemma 2.6 and the Hölder, Ladyzhenskaya, and Young inequalities, we deduce

$$
\begin{aligned}
I_{21} &= -\int_\Omega p_s^1(x,y,t)\nabla_H\cdot(|v|^{q-2}v)\,dx\,dy\,dz \\
&\le C_q\int_M |p_s^1(x,y,t)|\left(\int_{-h}^h |\nabla_H v|^2|v|^{q-2}dz\right)^{1/2}\left(\int_{-h}^h |v|^{q-2}dz\right)^{1/2}dx\,dy \\
&\le C_q\|p_s^1\|_{\frac{4q}{q+2},M}\left\||v|^{\frac q2-1}\nabla_H v\right\|_2\left[\int_M\left(\int_{-h}^h |v|^{q-2}dz\right)^{\frac{2q}{q-2}}dx\,dy\right]^{\frac{q-2}{4q}} \\
&\le C_q\|p_s^1\|_{\frac{4q}{q+2},M}\left\||v|^{\frac q2-1}\nabla_H v\right\|_2\left[\int_{-h}^h\left(\int_M |v|^{2q}dx\,dy\right)^{\frac{q-2}{2q}}dz\right]^{1/2} \\
&\le C_q\left(\int_{-h}^h \|v\|^2_{\frac{8q}{q+2},M}\,dz\right)\left\||v|^{\frac q2-1}\nabla_H v\right\|_2\left(\int_{-h}^h \left\||v|^{\frac q2}\right\|_{4,M}^{\frac{2(q-2)}{q}}dz\right)^{1/2} \\
&\le C_q\left(\int_{-h}^h \|v\|_{4,M}\|v\|_{2q,M}dz\right)\left\||v|^{\frac q2-1}\nabla_H v\right\|_2\left(\int_{-h}^h \left\||v|^{\frac q2}\right\|_{4,M}^{\frac{2(q-2)}{q}}dz\right)^{1/2} \\
&\le C_q\left(\int_{-h}^h \|v\|_{2,M}^{1/2}\|v\|_{H^1(M)}^{1/2}\left\||v|^{\frac q2}\right\|_{2,M}^{\frac1q}\left\||v|^{\frac q2}\right\|_{H^1(M)}^{\frac1q}dz\right) \\
&\quad\times\left\||v|^{\frac q2-1}\nabla_H v\right\|_2\left(\int_{-h}^h \left\||v|^{\frac q2}\right\|_{2,M}^{\frac{q-2}{q}}\left\||v|^{\frac q2}\right\|_{H^1(M)}^{\frac{q-2}{q}}dz\right)^{1/2} \\
&\le C_q\|v\|_2^{1/2}\left(\|v\|_2^{1/2}+\|\nabla_H v\|_2^{1/2}\right)\left\||v|^{\frac q2-1}\nabla_H v\right\|_2\left\||v|^{\frac q2}\right\|_2^{1/2} \\
&\quad\times\left(\left\||v|^{\frac q2}\right\|_2^{1/2}+\left\|\nabla_H|v|^{\frac q2}\right\|_2^{1/2}\right) \\
&\le \frac18\left\||v|^{\frac q2-1}\nabla_H v\right\|_2^2+C_q[\|v\|_2^2(\|v\|_2^2+\|\nabla_H v\|_2^2)+1]\|v\|_q^q \\
&\le \frac18\left\||v|^{\frac q2-1}\nabla_H v\right\|_2^2+C_q(1+\|v\|_2^2)(1+\|v\|_2^2+\|\nabla_H v\|_2^2)\|v\|_q^q.
\end{aligned}
$$

With the aid of the above estimate, as well as (3.2), (3.6), and (3.8), it follows from the Young inequality that

$$
\begin{aligned}
I_1+I_2 &= I_1+I_{20}+I_{21}+I_{22} \\
&\le \frac38\left((q-2)\left\||v|^{\frac q2-1}\nabla_H|v|\right\|_2^2+\left\||v|^{\frac q2-1}\nabla_H v\right\|_2^2\right) \\
&\quad+Cq\left(1+\|T_0\|_q^2\right)\|v\|_q^{q-2}+Cq\left(1+\|v\|_2^2\right)^2\left(1+\|v\|_q^q\right) \\
&\quad+C_q\left(1+\|v\|_2^2\right)\left(1+\|v\|_2^2+\|\nabla_H v\|_2^2\right)\|v\|_q^q\le
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{3}{8}\big((q-2)\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2+\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2\big)\\
&\quad +C_q\big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2+\|\nabla_H v\|_2^2+\|T_0\|_q^2\big)\big(1+\|v\|_q^q\big).
\end{aligned}
$$

Substituting this into (3.1), one obtains

$$
\begin{aligned}
&\frac{d}{dt}\|v\|_q^q+\frac{5q}{8}\big(\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2+\varepsilon\big\||v|^{\frac{q}{2}-1}\partial_z v\big\|_2^2\big)\leq\\
&\qquad\qquad C_q\big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2+\|\nabla_H v\|_2^2+\|T_0\|_q^2\big)\big(1+\|v\|_q^q\big),
\end{aligned}
$$

from which, using (i), one obtains

$$
\begin{aligned}
&\sup_{0\leq t\leq \mathcal{T}}\|v\|_q^q+\int_0^{\mathcal{T}}\big(\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2+\varepsilon\big\||v|^{\frac{q}{2}-1}\partial_z v\big\|_2^2\big)dz\\
&\quad\leq \exp\left\{C_q\int_0^{\mathcal{T}}\big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2+\|\nabla_H v\|_2^2+\|T_0\|_q^2\big)ds\right\}\big(1+\|v_0\|_q^q\big)\\
&\quad\leq \exp\big\{C_q e^{2\mathcal{T}}(\mathcal{T}+1)\big(1+\|v_0\|_2^2+\|T_0\|_2^2+\|T_0\|_q^2\big)^2\big\}\big(1+\|v_0\|_q^q\big),
\end{aligned}
$$

in particular,

$$
\sup_{0\leq t\leq \mathcal{T}}\|v\|_4^4+\int_0^{\mathcal{T}}\big\||v|\nabla_H v\big\|_2^2\,dz\leq K_1'(\mathcal{T}), \tag{3.9}
$$

where

$$
K_1'(\mathcal{T})=e^{Ce^{2\mathcal{T}}(\mathcal{T}+1)(1+\|v_0\|_2^2+\|T_0\|_2^2+\|T_0\|_4^2)^2}\big(1+\|v_0\|_4^4\big)
$$

for a positive constant C depending only on h.

Step 2. *The refined $L^\infty(0,\mathcal{T};L^q(\Omega))$ estimate on* v. Noticing that all the constants C in the estimates for I_1, I_{20}, and I_{22} are independent of $q\in[4,\infty)$, it suffices to give a refined estimate for I_{21}. Recalling the elliptic estimate (3.4), a similar argument to that for (3.7) yields

$$
\begin{aligned}
I_{21}&\leq \int_M|\nabla_H p_s^1(x,y,t)|\left(\int_{-h}^h|v|^{q-1}dz\right)dx\,dy\\
&\leq \frac{q-2}{8}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2\\
&\quad +C\big(1+\|v\|_2^2\big)\big(1+\|\nabla_H p_s^1\|_{2,M}^2\big)\big(q\|v\|_q^{q-2}+\|v\|_q^{\frac{q(q-2)}{q-1}}\big)\\
&\leq C\big(1+\|v\|_2^2\big)\big(1+\||v||\nabla_H v|\|_2^2\big)\big(q\|v\|_q^{q-2}+\|v\|_q^{\frac{q(q-2)}{q-1}}\big)\\
&\quad +\frac{q-2}{8}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2,
\end{aligned}
$$

where the constant C is independent of $q \in [4, \infty)$. Combining this with (3.2), (3.6), and (3.8), one obtains

$$\begin{aligned} I_1 + I_2 &= I_1 + I_{20} + I_{21} + I_{22} \\ &\le \frac{q-2}{2}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2 + \frac{1}{4}\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2 \\ &\quad + Cq\big[1 + \|T_0\|_q^2 + \big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2 + \||v||\nabla_H v|\|_2^2\big)\big]\|v\|_q^{q-2} \\ &\quad + C\big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2 + \||v||\nabla_H v|\|_2^2\big)\|v\|_q^{\frac{q(q-2)}{q-1}} \\ &\le C\big[\|T_0\|_q^2 + \big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2 + \||v||\nabla_H v|\|_2^2\big)\big]\big(q\|v\|_q^{q-2} + \|v\|_q^{\frac{q(q-2)}{q-1}}\big) \\ &\quad + \frac{q-2}{2}\big\||v|^{\frac{q}{2}-1}\nabla_H|v|\big\|_2^2 + \frac{1}{4}\big\||v|^{\frac{q}{2}-1}\nabla_H v\big\|_2^2. \end{aligned}$$

Substituting this into (3.1) and then using the Young inequality gives

$$\begin{aligned} &\frac{d}{dt}\big(q + 1 + \|v\|_q^2\big) \\ &\quad\le C\big[\|T_0\|_q^2 + \big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2 + \||v||\nabla_H v|\|_2^2\big)\big]\big(q + \|v\|_q^{\frac{q-2}{q-1}}\big) \\ &\quad\le C\big[\|T_0\|_q^2 + \big(1+\|v\|_2^2\big)\big(1+\|v\|_2^2 + \||v||\nabla_H v|\|_2^2\big)\big]\big(q + 1 + \|v\|_q^2\big). \end{aligned}$$

By recalling the estimate in (i) and (3.9) and applying the Grönwall inequality, it follows from the above inequality that

$$\begin{aligned} &(q+1) + \sup_{0\le t\le \mathcal{T}} \|v\|_q^2 \\ &\quad\le e^{C\int_0^{\mathcal{T}}[\|T_0\|_q^2 + (1+\|v\|_2^2)(1+\|v\|_2^2 + \||v||\nabla_H v|\|_2^2)]dt}\big(q + 1 + \|v_0\|_q^2\big) \\ &\quad\le 2e^{C[\|T_0\|_q^2\mathcal{T} + (1+K_0(\mathcal{T}))^2\mathcal{T} + (1+K_0(\mathcal{T}))K_1'(\mathcal{T})]}(1+\|v_0\|_q)^2 q \\ &\quad=: K_1''(\mathcal{T})e^{C\|T_0\|_q^2\mathcal{T}}(1+\|v_0\|_q)^2 q, \end{aligned}$$

where

$$K_1''(\mathcal{T}) = 2e^{C[(1+K_0(\mathcal{T}))^2\mathcal{T} + (1+K_0(\mathcal{T}))K_1'(\mathcal{T})]}$$

for a constant C that depends only on h, where $K_0(\mathcal{T})$ and $K_1'(\mathcal{T})$ are given as before. Therefore, one obtains

$$\sup_{0\le t\le \mathcal{T}} \|v\|_q \le K_1(\mathcal{T})e^{C\|T_0\|_q^2\mathcal{T}}(1+\|v_0\|_q)\sqrt{q},$$

where $K_1(\mathcal{T})$ is given by

$$K_1(\mathcal{T}) = \sqrt{K_1''(\mathcal{T})}, \tag{3.10}$$

with $K_1''(\mathcal{T})$ being given as before. This proves (iii) and completes the proof of Proposition 3.1. □

4 High-Order Energy Inequalities

In this section, we establish the energy inequalities for the derivatives up to second order of the strong solutions to system (2.5)–(2.7), subject to the boundary and initial conditions (1.25)–(1.27). As we stated in the introduction and will also see below, the treatment of the different derivatives varies: we always work on the vertical derivatives first and then on the horizontal ones.

We first deal with energy inequalities for the first-order derivatives, which are described by the following proposition.

PROPOSITION 4.1. *Let* (v, T) *be the global strong solution of system* (2.5)–(2.7)*, subject to the boundary and initial conditions* (1.25)–(1.27)*. Then for any* $\mathcal{T} \in (0, \infty)$*, we have the following*:

(i) $L^\infty(0, \mathcal{T}; L^q(\Omega))$, $q \in [2, \infty)$*, estimate of* $u := \partial_z v$:

$$\frac{d}{dt}\|u\|_q^q + \int_\Omega |u|^{q-2}\big(|\nabla_H u|^2 + \varepsilon|\partial_z u|^2\big)dx\,dy\,dz \le C_q\big(\|v\|_\infty^2 + 1\big)\big(\|u\|_q^q + 1\big);$$

(ii) $L^\infty(0, \mathcal{T}; L^2(\Omega))$ *estimate on* $\partial_z T$:

$$\frac{d}{dt}\|\partial_z T\|_2^2 + \big\|\big(\nabla_H \partial_z T, \sqrt{\varepsilon}\partial_z^2 T\big)\big\|_2^2 \le C\big(1 + \|v\|_\infty^2\big)\|\partial_z T\|_2^2 + C\|(\nabla_H v, u, \nabla_H u)\|_2^2;$$

(iii) $L^\infty(0, \mathcal{T}; L^2(\Omega))$ *estimate on* $\nabla_H v$:

$$\frac{d}{dt}\|\nabla_H v\|_2^2 + \|(\Delta_H v, \sqrt{\varepsilon}\nabla_H \partial_z v)\|_2^2 \le C\|v\|_\infty^2\|\nabla_H v\|_2^2 + C\big(\|\nabla_H T\|_2^2 + \|u\|_4^8\big);$$

(iv) $L^\infty(0, \mathcal{T}; L^2(\Omega))$ *estimate on* $\nabla_H T$:

$$\frac{d}{dt}\|\nabla_H T\|_2^2 + \|(\Delta_H T, \sqrt{\varepsilon}\nabla_H \partial_z T)\|_2^2 \le C\big(1 + \|(\partial_z T, \nabla_H v)\|_2^2\big)^2\big(1 + \|(\nabla_H \partial_z T, \Delta_H v)\|_2^2\big),$$

where C *is a positive constant depending only on* h*,* $\mathcal{T}$*, and the initial data* (*the constant* C_q *in* (i) *also depends on* q).

PROOF.

(i) Differentiating equation (2.5) with respect to z, one can easily check that $u := \partial_z v$ satisfies

$$\partial_t u + (v \cdot \nabla_H)u - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)\,d\xi\right)\partial_z u - \Delta_H u - \varepsilon\partial_z^2 u + (u \cdot \nabla_H)v - (\nabla_H \cdot v)u + f_0 k \times u - \nabla_H T = 0. \tag{4.1}$$

By multiplying the above equation by $|u|^{q-2}u$ and integrating over Ω, it follows from integration by parts that

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega |u|^q\,dx\,dy\,dz \\
&\quad + \int_\Omega |u|^{q-2}\big(|\nabla_H u|^2 + \varepsilon|\partial_z u|^2 + (q-2)\varepsilon|\partial_z|u||^2\big)dx\,dy\,dz \\
&\qquad = \int_\Omega ((\nabla_H \cdot v)u - (u \cdot \nabla_H)v + \nabla_H T) \cdot |u|^{q-2}u\,dx\,dy\,dz \\
&\qquad \le C_q \int_\Omega (|v||u|^{q-1}|\nabla_H u| + |T||u|^{q-2}|\nabla_H u|)dx\,dy\,dz \\
&\qquad \le \frac{1}{2}\int_\Omega |u|^{q-2}|\nabla u|^2\,dx\,dy\,dz + C_q\int_\Omega (|v|^2|u|^q + |T|^2|u|^{q-2})dx\,dy\,dz.
\end{aligned}$$

Recalling that $\sup_{0\le t\le \mathcal{T}}\|T\|_\infty \le C$, which is guaranteed by Proposition 3.1(ii), we have

$$\begin{aligned}
&\frac{d}{dt}\|u\|_q^q + \int_\Omega |u|^{q-2}(|\nabla_H u|^2 + \varepsilon|\partial_z u|^2)dx\,dy\,dz \\
&\qquad \le C_q\big(\|v\|_\infty^2 + \|T\|_\infty^2\big)\big(\|u\|_q^q + 1\big) \le C_q\big(\|v\|_\infty^2 + 1\big)\big(\|u\|_q^q + 1\big),
\end{aligned}$$

proving (i).

(ii) By multiplying equation (2.7) by $-\partial_z^2 T$ and integrating over Ω, it follows from integrating by parts and using $\sup_{0\le t\le \mathcal{T}}\|T\|_\infty \le C$ again that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_z T|^2\,dx\,dy\,dz + \int_\Omega \big(|\nabla_H \partial_z T|^2 + \varepsilon\big|\partial_z^2 T\big|^2\big)dx\,dy\,dz \\
&\qquad = \int_\Omega \left[(v \cdot \nabla_H)T - \left(\int_{-h}^{z}\nabla_H \cdot v\,d\xi\right)\left(\partial_z T + \frac{1}{h}\right)\right]\partial_z^2 T\,dx\,dy\,dz \\
&\qquad = -\int_\Omega \big[u \cdot \nabla_H T - (\nabla_H \cdot v)\partial_z T - h^{-1}(\nabla_H \cdot v)\big]\partial_z T\,dx\,dy\,dz =
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega [\nabla_H \cdot (u \partial_z T) T - v \cdot \nabla_H |\partial_z T|^2 + h^{-1} (\nabla_H \cdot v) \partial_z T] dx\, dy\, dz \\
&\leq C \int_\Omega \big(|\nabla_H u| |\partial_z T| |T| + |u| |\nabla_H \partial_z T| |T| \\
&\qquad\qquad + |v| |\partial_z T| |\nabla_H \partial_z T| + |\nabla_H v| |\partial_z T| \big) dx\, dy\, dz \\
&\leq \frac{1}{2} \|\nabla_H \partial_z T\|_2^2 + C \|(\nabla_H u, u, \partial_z T, \nabla_H v)\|_2^2 + C \|v\|_\infty^2 \|\partial_z T\|_2^2 \\
&\leq \frac{1}{2} \|\nabla_H \partial_z T\|_2^2 + C \|(\nabla_H u, u, \nabla_H v)\|_2^2 + C(1 + \|v\|_\infty^2) \|\partial_z T\|_2^2,
\end{aligned}
$$

from which one obtains (ii).

(iii) By multiplying equation (2.5) by $-\Delta_H v$ and integrating over Ω, then it follows from integrating by parts and the Cauchy inequality that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla_H v|^2 \, dx\, dy\, dz + \int_\Omega \big(|\Delta_H v|^2 + \varepsilon |\nabla_H \partial_z v|^2 \big) dx\, dy\, dz \\
&\quad = \int_\Omega \left[(v \cdot \nabla_H) v - \left(\int_{-h}^z \nabla_H \cdot v \, d\xi \right) u - \nabla_H \left(\int_{-h}^z T \, d\xi \right) \right] \cdot \Delta_H v \, dx\, dy\, dz \\
&\quad \leq C \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |u| |\Delta_H v| dz \right) dx\, dy \\
&\qquad + C (\|v\|_\infty \|\nabla_H v\|_2 \|\Delta_H v\|_2 + \|\nabla_H T\|_2 \|\Delta_H v\|_2) \\
&\quad \leq C \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |u| |\Delta_H v| dz \right) dx\, dy \\
&\qquad + \frac{1}{4} \|\Delta_H v\|_2^2 + C \|v\|_\infty^2 \|\nabla_H v\|_2^2 + C \|\nabla_H T\|_2^2.
\end{aligned} \tag{4.2}
$$

It follows from Lemma 2.6 and the Hölder, Gagliardo-Nirenberg, and Young inequalities that

$$
\begin{aligned}
&C \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |u| |\Delta_H v| dz \right) dx\, dy \\
&\quad \leq C \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |u|^2 dz \right)^{1/2} \left(\int_{-h}^h |\Delta_H v|^2 dz \right)^{1/2} dx\, dy \\
&\quad \leq C \left[\int_M \left(\int_{-h}^h |\nabla_H v| dz \right)^4 dx\, dy \right]^{1/4} \left[\int_M \left(\int_{-h}^h |u|^2 dz \right)^2 dx\, dy \right]^{1/4} \|\Delta_H v\|_2 \leq
\end{aligned}
$$

$$
\begin{aligned}
&\le C\left(\int_{-h}^{h}\|\nabla_H v\|_{4,M}\,dz\right)\left[\int_{-h}^{h}\left(\int_M |u|^4 dz\right)^{1/2} dx\,dy\right]^{1/2}\|\Delta_H v\|_2\\
&\le C\left(\int_{-h}^{h}\|v\|_{2,M}^{1/4}\|\nabla_H^2 v\|_{2,M}^{\frac{3}{4}}\,dz\right)\|u\|_4\|\Delta_H v\|_2\\
&\le C\|v\|_2^{1/4}\|u\|_4\|\Delta_H v\|_2^{\frac{7}{4}} \le \frac{1}{4}\|\Delta_H v\|_2^2 + C\|v\|_2^2\|u\|_4^8. \qquad (4.3)
\end{aligned}
$$

Substituting this into (4.2) and recalling that $\sup_{0\le t\le \mathcal{T}}\|v\|_2^2 \le C$ is guaranteed by Proposition 3.1(i), one obtains

$$
\frac{d}{dt}\|\nabla_H v\|_2^2 + \|(\Delta_H v, \sqrt{\varepsilon}\nabla_H \partial_z v)\|_2^2 \le C\|v\|_\infty^2\|\nabla_H v\|_2^2 + C(\|\nabla_H T\|_2^2 + \|u\|_4^8),
$$

proving (iii).

(iv) By multiplying equation (2.7) by $-\Delta_H T$ and integrating over Ω, then it follows from integrating by parts and the Cauchy inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_H T|^2\,dx\,dy\,dz + \int_\Omega \left(|\Delta_H T|^2 + \varepsilon|\nabla_H \partial_z T|^2\right)dx\,dy\,dz\\
&\quad = \int_\Omega \left[v\cdot\nabla_H T - \left(\int_{-h}^{z}\nabla_H\cdot v\,d\xi\right)\left(\partial_z T + \frac{1}{h}\right)\right]\Delta_H T\,dx\,dy\,dz\\
&\quad \le C(\|v\|_\infty\|\nabla_H T\|_2\|\Delta_H T\|_2 + \|\nabla_H v\|_2\|\Delta_H T\|_2)\\
&\qquad - \int_\Omega \left(\int_{-h}^{z}\nabla_H\cdot v\,d\xi\right)\partial_z T\Delta_H T\,dx\,dy\,dz\\
&\quad \le \frac{1}{4}\|\Delta_H T\|_2^2 + C\|v\|_\infty^2\|\nabla_H T\|_2^2 + C\|\nabla_H v\|_2^2 \qquad (4.4)\\
&\qquad - \int_\Omega \left(\int_{-h}^{z}\nabla_H\cdot v\,d\xi\right)\partial_z T\Delta_H T\,dx\,dy\,dz.
\end{aligned}
$$

It follows from integrating by parts, Lemma 2.1, and Proposition 3.1(ii) that

$$
\begin{aligned}
&-\int_\Omega \left(\int_{-h}^{z}\nabla_H\cdot v\,d\xi\right)\partial_z T\Delta_H T\,dx\,dy\,dz\\
&\quad = \int_\Omega \left(\int_{-h}^{z}\nabla_H\nabla_H\cdot v\,d\xi\,\partial_z T + \int_{-h}^{z}\nabla_H\cdot v\,d\xi\,\nabla_H\partial_z T\right)\cdot\nabla_H T\,dx\,dy\,dz \le
\end{aligned}
$$

$$\begin{aligned}
&\le C\int_M \left(\int_{-h}^h |\nabla_H^2 v|dz\right)\left(\int_{-h}^h |\partial_z T||\nabla_H T|dz\right)dx\,dy \\
&\quad + C\int_M \left(\int_{-h}^h |\nabla_H v|dz\right)\left(\int_{-h}^h |\nabla_H \partial_z T||\nabla_H T|dz\right)dx\,dy \\
&\le C\|\nabla_H^2 v\|_2\|\partial_z T\|_2^{1/2}\big(\|\partial_z T\|_2^{1/2}+\|\nabla_H\partial_z T\|_2^{1/2}\big)\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{1/2} \\
&\quad + C\|\nabla_H v\|_2^{1/2}\|\nabla_H^2 v\|_2^{1/2}\|\nabla_H\partial_z T\|_2\|T\|_\infty^{1/2}\|\nabla_H^2 T\|_2^{1/2} \\
&\le \frac{1}{4}\|\nabla_H^2 T\|_2^2 + C\big[\|\nabla_H^2 v\|_2^2 + \|\partial_z T\|_2^2\big(\|\partial_z T\|_2^2+\|\nabla_H\partial_z T\|_2^2\big) \\
&\qquad\qquad + \|\nabla_H\partial_z T\|_2^2 + \|\nabla_H v\|_2^2\|\nabla_H^2 v\|_2^2\big] \\
&\le \frac{1}{4}\|\Delta_H T\|_2^2 + C\big(1+\|(\partial_z T,\nabla_H v)\|_2^2\big)^2\big(1+\|(\nabla_H\partial_z T,\Delta_H v)\|_2^2\big).
\end{aligned}$$

Substituting this into (4.4) yields

$$\begin{aligned}
&\frac{d}{dt}\|\nabla_H T\|_2^2 + \|(\Delta_H T,\sqrt{\varepsilon}\nabla_H\partial_z T)\|_2^2 \le \\
&\qquad\qquad C\big(1+\|(\partial_z t,\nabla_h v)\|_2^2\big)^2\big(1+\|(\nabla_h\partial_z t,\delta_h v)\|_2^2\big),
\end{aligned}$$

proving (iv). □

Now we consider the energy inequalities for the second-order derivatives. We have the following proposition.

PROPOSITION 4.2. *Let (v,T) be the global strong solution of system* (2.5)–(2.7) *subject to the boundary and initial conditions* (1.25)–(1.27)*. Then for any $\mathcal{T}\in(0,\infty)$, we have the following*:

(i) *$L^\infty(0,\mathcal{T};L^2(\Omega))$ estimate on $\partial_z u$*:

$$\begin{aligned}
&\frac{d}{dt}\|\partial_z u\|_2^2 + \big\|\big(\nabla_H\partial_z u,\sqrt{\varepsilon}\partial_z^2 u\big)\big\|_2^2 \le \\
&\qquad\qquad C\big(1+\|v\|_\infty^2\big)\|\partial_z u\|_2^2 + C\big(\||u||\nabla_H u|\|_2^2+\|\nabla_H\partial_z T\|_2^2\big);
\end{aligned}$$

(ii) *$L^\infty(0,\mathcal{T};L^2(\Omega))$ estimate on $\partial_z^2 T$*:

$$\begin{aligned}
&\frac{d}{dt}\|\partial_z^2 T\|_2^2 + \big\|\big(\nabla_H\partial_z^2 T,\sqrt{\varepsilon}\partial_z^3 T\big)\big\|_2^2 \\
&\quad \le C\big(1+\|v\|_\infty^2\big)\|\partial_z^2 T\|_2^2 + C\big(1+\|(\partial_z u,\partial_z T)\|_2^2\big)^2 \\
&\qquad \times\big(1+\|(\nabla_H u,\nabla_H\partial_z u,\nabla_H\partial_z T)\|_2^2\big);
\end{aligned} \tag{4.5}$$

(iii) $L^\infty(0,\mathcal{T};L^2(\Omega))$ *estimate on* $\nabla_H u$:

$$\frac{d}{dt}\|\nabla_H u\|_2^2+\|(\Delta_H u,\sqrt{\varepsilon}\nabla_H\partial_z u)\|_2^2 \le C\|v\|_\infty^2\|\nabla_H u\|_2^2+C\|(u,\nabla_H v,\partial_z u,\nabla_H T)\|_2^2 \times\|(\nabla_H u,\Delta_H v,\nabla_H\partial_z u)\|_2^2;$$

(iv) $L^\infty(0,\mathcal{T};L^2(\Omega))$ *estimate on* $\nabla_H\partial_z T$:

$$\frac{d}{dt}\|\nabla_H\partial_z T\|_2^2+\|(\Delta_H\partial_z T,\nabla_H\partial_z^2 T)\|_2^2 \le C\|v\|_\infty^2\|\nabla_H\partial_z T\|_2^2+C\big(1+\big\|\big(\nabla_H v,\nabla_H T,\partial_z u,\partial_z^2 T\big)\big\|_2^2\big)^2 \times\big(1+\big\|\big(\Delta_H v,\Delta_H T,\nabla_H\partial_z u,\nabla_H\partial_z^2 T\big)\big\|_2^2\big);$$

(v) $L^\infty(0,\mathcal{T};L^2(\Omega))$ *estimate on* $\Delta_H v$ *and* $\Delta_H T$:

$$\frac{d}{dt}\|(\Delta_H v,\Delta_H T)\|_2^2+\|(\nabla_H\Delta_H v,\nabla_H\Delta_H T,\sqrt{\varepsilon}\partial_z\Delta_H v,\sqrt{\varepsilon}\partial_z\Delta_H T)\|_2^2 \le C\big(1+\|v\|_\infty^2\big)\|(\Delta_H v,\Delta_H T)\|_2^2 + C\big(1+\big\|\big(\nabla_H v,\nabla_H T,\nabla_H u,\partial_z^2 T,\nabla_H\partial_z T\big)\big\|_2^2+\|u\|_4^4\big)^3 \times\big(1+\|(\Delta_H v,\Delta_H T,\Delta_H u,\Delta_H\partial_z T)\|_2^2\big),$$

where C *is a positive constant depending only on* h, $\mathcal{T}$, *and the initial data.*

PROOF.

(i) Differentiating equation (4.1) with respect to z, one can easily check that $\partial_z u$ satisfies

$$\partial_t\partial_z u+(v\cdot\nabla_H)\partial_z u-\left(\int_{-h}^z\nabla_H\cdot v(x,y,\xi,t)\,d\xi\right)\partial_z^2 u - \Delta_H\partial_z u-\varepsilon\partial_z^3 u+2(u\cdot\nabla_H)u-(\nabla_H\cdot u)u+(\partial_z u\cdot\nabla_H)v - 2(\nabla_H\cdot v)\partial_z u+f_0k\times\partial_z u-\nabla_H\partial_z T=0.$$

By multiplying the above equation by $\partial_z u$ and integrating over Ω, it follows from integrating by parts and the Cauchy inequality that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\partial_z u|^2\,dx\,dy\,dz+\int_\Omega\big(|\nabla_H\partial_z u|^2+\varepsilon|\partial_z^2 u|^2\big)dx\,dy\,dz = \int_\Omega\big[(\nabla_H\cdot u)u-2(u\cdot\nabla_H)u+2(\nabla_H\cdot v)\partial_z u-(\partial_z u\cdot\nabla_H)v+\nabla_H\partial_z T\big] \cdot\partial_z u\,dx\,dy\,dz=$$

$$= \int_\Omega \{[(\nabla_H \cdot u)u - 2(u \cdot \nabla_H)u + \nabla_H \partial_z T] \cdot \partial_z u - 2v \cdot \nabla_H |\partial_z u|^2 + \nabla_H \cdot (\partial_z u \otimes \partial_z u) \cdot v\} dx\,dy\,dz$$

$$\leq C \int_\Omega (|u||\nabla_H u||\partial_z u| + |v||\partial_z u||\nabla_H \partial_z u| + |\nabla_H \partial_z T||\partial_z u|) dx\,dy\,dz$$

$$\leq C(\||u|\nabla_H u\|_2 \|\partial_z u\|_2 + \|v\|_\infty \|\partial_z u\|_2 \|\nabla_H \partial_z u\|_2 + \|\nabla_H \partial_z T\|_2 \|\partial_z u\|_2)$$

$$\leq \frac{1}{2}\|\nabla_H \partial_z u\|_2^2 + C\big(\||u|\nabla_H u\|_2^2 + \|\nabla_H \partial_z T\|_2^2\big) + C\big(1 + \|v\|_\infty^2\big)\|\partial_z u\|_2^2,$$

from which one obtains (i).

(ii) Differentiating equation (2.7) with respect to z yields

$$\partial_t \partial_z T + v \cdot \nabla_H \partial_z T - \left(\int_{-h}^z \nabla_H \cdot v\, d\xi\right)\partial_z^2 T + u \cdot \nabla_H T - (\nabla_H \cdot v)\left(\partial_z T + \frac{1}{h}\right) - \Delta_H \partial_z T - \varepsilon \partial_z^3 T = 0. \tag{4.6}$$

By multiplying the above equation by $-\partial_z^3 T$, integrating over Ω, and using the facts

$$|\nabla_H \partial_z T(x, y, z, t)| \leq \int_{-h}^h \big|\nabla_H \partial_z^2 T(x, y, \xi, t)\big| d\xi,$$

$$|u(x, y, z, t)| \leq \int_{-h}^h |\partial_z u(x, y, \xi, t)| d\xi,$$

it follows from integrating by parts, Lemma 2.1, Proposition 3.1, and the Young inequality that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big|\partial_z^2 T\big|^2 dx\,dy\,dz + \int_\Omega \big(\big|\nabla_H \partial_z^2 T\big|^2 + \varepsilon\big|\partial_z^3 T\big|^2\big) dx\,dy\,dz$$

$$= \int_\Omega \bigg[v \cdot \nabla_H \partial_z T - \int_{-h}^z \nabla_H \cdot v\, d\xi\, \partial_z^2 T + u \cdot \nabla_H T - \nabla_H \cdot v(\partial_z T + h^{-1})\bigg]\partial_z^3 T\, dx\,dy\,dz$$

$$= -\int_\Omega \big[2u \cdot \nabla_H \partial_z T - 2\nabla_H \cdot v \partial_z^2 T + \partial_z u \cdot \nabla_H T - \nabla_H \cdot u(\partial_z T + h^{-1})\big]\partial_z^2 T\, dx\,dy\,dz =$$

$$
\begin{aligned}
&= \int_\Omega \big[3\partial_z u \cdot \nabla_H \partial_z T \partial_z T + 2u \cdot \nabla_H \partial_z^2 T \partial_z T - 4v \cdot \nabla_H \partial_z^2 T \partial_z T \\
&\qquad + \big(\nabla_H \cdot \partial_z u \partial_z^2 T + \partial_z u \cdot \nabla_H \partial_z^2 T\big) T + h^{-1} \nabla_H \cdot u \partial_z^2 T\big] dx\, dy\, dz \\
&\le C \int_M \left(\int_{-h}^h \left|\nabla_H \partial_z^2 T\right| dz\right)\left(\int_{-h}^h |\partial_z u||\partial_z T| dz\right) dx\, dy \\
&\quad + C \int_M \left(\int_{-h}^h |\partial_z u| dz\right)\left(\int_{-h}^h |\partial_z T|\left|\nabla_H \partial_z^2 T\right| dz\right) dx\, dy \\
&\quad + C\big(\|v\|_\infty \left\|\partial_z^2 T\right\|_2 \left\|\nabla_H \partial_z^2 T\right\|_2 \\
&\quad + \|\nabla_H \partial_z u\|_2 \left\|\partial_z^2 T\right\|_2 + \|\partial_z u\|_2 \left\|\nabla_H \partial_z^2 T\right\|_2 + \|\nabla_H u\|_2 \left\|\partial_z^2 T\right\|_2\big) \\
&\le C \left\|\nabla_H \partial_z^2 T\right\|_2 \|\partial_z u\|_2^{1/2} \big(\|\partial_z u\|_2^{1/2} + \|\nabla_H \partial_z u\|_2^{1/2}\big) \\
&\quad \times \|\partial_z T\|_2^{1/2} \big(\|\partial_z T\|_2^{1/2} + \|\nabla_H \partial_z T\|_2^{1/2}\big) \\
&\quad + C\big(\|v\|_\infty \left\|\partial_z^2 T\right\|_2 \left\|\nabla_H \partial_z^2 T\right\|_2 + \|\nabla_H \partial_z u\|_2 \left\|\partial_z^2 T\right\|_2 \\
&\quad + \|\partial_z u\|_2 \left\|\nabla_H \partial_z^2 T\right\|_2 + \|\nabla_H u\|_2 \left\|\partial_z^2 T\right\|_2\big) \\
&\le \frac{1}{2} \left\|\nabla_H \partial_z^2 T\right\|_2^2 + C\big[\|\partial_z u\|_2^2 \big(\|\partial_z u\|_2^2 + \|\nabla_H \partial_z u\|_2^2\big) \\
&\qquad\qquad + \|\partial_z T\|_2^2 \big(\|\partial_z T\|_2^2 + \|\nabla_H \partial_z T\|_2^2\big)\big] \\
&\quad + C\big(\|v\|_\infty^2 + 1\big) \left\|\partial_z^2 T\right\|_2^2 + C\big(\|\nabla_H u\|_2^2 + \|\nabla_H \partial_z u\|_2^2 + \|\partial_z u\|_2^2\big) \\
&\le \frac{1}{2} \left\|\nabla_H \partial_z^2 T\right\|_2^2 + C\big(\|v\|_\infty^2 + 1\big) \left\|\partial_z^2 T\right\|_2^2 + C\big(1 + \|\partial_z u\|_2^2 + \|\partial_z T\|_2^2\big)^2 \\
&\quad \times \big(1 + \|\nabla_H u\|_2^2 + \|\nabla_H \partial_z u\|_2^2 + \|\nabla_H \partial_z T\|_2^2\big),
\end{aligned}
$$

from which one obtains (ii).

(iii) By multiplying equation (4.1) by $-\Delta_H u$, integrating over Ω, and using the fact that $|u(x, y, z, t)| \le \int_{-h}^h |\partial_z u(x, y, \xi, t)| d\xi$, it follows from integrating by parts, Lemma 2.1, and the Young inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_H u|^2\,dx\,dy\,dz + \int_\Omega \big(|\Delta_H u|^2 + \varepsilon|\nabla_H \partial_z u|^2\big)dx\,dy\,dz\\
&\quad = \int_\Omega \bigg[(v\cdot\nabla_H)u - \int_{-h}^{z} \nabla_H\cdot v\,d\xi\,\partial_z u\\
&\qquad\qquad + (u\cdot\nabla_H)v - (\nabla_H\cdot v)u - \nabla_H T\bigg]\cdot \Delta_H u\,dx\,dy\,dz\\
&\quad \le \int_\Omega \bigg[|v||\nabla_H u||\Delta_H u| + \bigg(\int_{-h}^{h}|\nabla_H v|dz\bigg)|\partial_z u||\Delta_H u|\\
&\qquad\qquad + |u||\nabla_H v||\Delta_H u| + |\nabla_H T||\Delta_H u|\bigg]dx\,dy\,dz\\
&\quad \le \|v\|_\infty\|\nabla_H u\|_2\|\Delta_H u\|_2\\
&\qquad + C\int_M \bigg(\int_{-h}^{h}|\nabla_H v|dz\bigg)\bigg(\int_{-h}^{h}|\partial_z u||\Delta_H u|dz\bigg)dx\,dy\\
&\qquad + C\int_M \bigg(\int_{-h}^{h}|\partial_z u|dz\bigg)\bigg(\int_{-h}^{h}|\nabla_H v||\Delta_H u|dz\bigg)dx\,dy + \|\nabla_H T\|_2\|\Delta_H u\|_2\\
&\quad \le \|v\|_\infty\|\nabla_H u\|_2\|\Delta_H u\|_2\\
&\qquad + C\|\nabla_H v\|_2^{1/2}\|\nabla_H^2 v\|_2^{1/2}\|\partial_z u\|_2^{1/2}\big(\|\partial_z u\|_2^{1/2} + \|\nabla_H\partial_z u\|_2^{1/2}\big)\|\Delta_H u\|_2\\
&\qquad + C\|\partial_z u\|_2^{1/2}\big(\|\partial_z u\|_2^{1/2} + \|\nabla_H\partial_z u\|_2^{1/2}\big)\|\nabla_H v\|_2^{1/2}\big\|\nabla_H^2 v\big\|_2^{1/2}\|\Delta_H u\|_2\\
&\qquad + \|\nabla_H T\|_2\|\Delta_H u\|_2\\
&\quad \le \frac{1}{2}\|\Delta_H u\|_2^2 + C\|v\|_\infty^2\|\nabla_H u\|_2^2 + C\big[\|\partial_z u\|_2^2\big(\|\partial_z u\|_2^2 + \|\nabla_H\partial_z u\|_2^2\big)\\
&\qquad + \|\nabla_H v\|_2^2\|\Delta_H v\|_2^2 + \|\nabla_H T\|_2^2\big]\\
&\quad \le \frac{1}{2}\|\Delta_H u\|_2^2 + C\|v\|_\infty^2\|\nabla_H u\|_2^2\\
&\qquad + C\big(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2 + \|\partial_z u\|_2^2\big)^2\big(1 + \|\Delta_H v\|_2^2 + \|\nabla_H\partial_z u\|_2^2\big),
\end{aligned}
$$

from which one obtains (iii).

(iv) By multiplying equation (4.6) by $-\Delta_H \partial_z T$, integrating the resulting equation over Ω, and using the facts that

$$|u(x, y, z, t)| \le \int_{-h}^{h} |\partial_z u(x, y, \xi, t)| d\xi,$$

$$|\partial_z T(x, y, z, t)| \le \int_{-h}^{h} \left|\partial_z^2 T(x, y, \xi, t)\right| d\xi,$$

it follows from integration by parts, Lemma 2.1, and the Young inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_H \partial_z T|^2\, dx\, dy\, dz + \int_\Omega \left(|\Delta_H \partial_z T|^2 + \varepsilon\left|\nabla_H \partial_z^2 T\right|^2\right) dx\, dy\, dz \\
&= \int_\Omega \Big[v \cdot \nabla_H \partial_z T - \int_{-h}^{z} \nabla_H \cdot v\, d\xi\, \partial_z^2 T + u \cdot \nabla_H T \\
&\qquad - \nabla_H \cdot v\left(\partial_z T + h^{-1}\right)\Big] \Delta_H \partial_z T\, dx\, dy\, dz \\
&\le \int_\Omega \Big[|v||\nabla_H \partial_z T| + \left(\int_{-h}^{h} |\nabla_H v| dz\right)\left|\partial_z^2 T\right| + |u||\nabla_H T| \\
&\qquad + |\nabla_H v||\partial_z T| + |\nabla_H v|\Big] |\Delta_H \partial_z T| dx\, dy\, dz \\
&\le \|v\|_\infty \|\nabla_H \partial_z T\|_2 \|\Delta_H \partial_z T\|_2 \\
&\quad + \int_\Omega \Big[\left(\int_{-h}^{h} |\nabla_H v| dz\right)\left|\partial_z^2 T\right| + \left(\int_{-h}^{h} |\partial_z u| dz\right)|\nabla_H T| \\
&\qquad + |\nabla_H v|\left(\int_{-h}^{h}\left|\partial_z^2 T\right| dz\right)\Big] |\Delta_H \partial_z T| dx\, dy\, dz + \|\nabla_H v\|_2 \|\Delta_H \partial_z T\|_2 \\
&\le \|v\|_\infty \|\nabla_H \partial_z T\|_2 \|\Delta_H \partial_z T\|_2 + \|\nabla_H v\|_2 \|\Delta_H \partial_z T\|_2 \\
&\quad + \int_M \left(\int_{-h}^{h} |\nabla_H v| dz\right)\left(\int_{-h}^{h}\left|\partial_z^2 T\right| |\Delta_H \partial_z T| dz\right) dx\, dy \\
&\quad + \int_M \left(\int_{-h}^{h} |\partial_z u| dz\right)\left(\int_{-h}^{h} |\nabla_H T||\Delta_H \partial_z T| dz\right) dx\, dy \\
&\quad + \int_M \left(\int_{-h}^{h}\left|\partial_z^2 T\right| dz\right)\left(\int_{-h}^{h} |\nabla_H v||\Delta_H \partial_z T| dz\right) dx\, dy \le
\end{aligned}$$

$$
\begin{aligned}
&\le C[\|v\|_\infty \|\nabla_H \partial_z T\|_2 \|\Delta_H \partial_z T\|_2 + \|\nabla_H v\|_2 \|\Delta_H \partial_z T\|_2 \\
&\quad + \|\nabla_H v\|_2^{1/2} \|\nabla_H^2 v\|_2^{1/2} \|\partial_z^2 T\|_2^{1/2} \big(\|\partial_z^2 T\|_2^{1/2} + \|\nabla_H \partial_z^2 T\|_2^{1/2}\big) \|\Delta_H \partial_z T\|_2 \\
&\quad + \|\partial_z u\|_2^{1/2} \big(\|\partial_z u\|_2^{1/2} + \|\nabla_H \partial_z u\|_2^{1/2}\big) \|\nabla_H T\|_2^{1/2} \|\nabla_H^2 T\|_2^{1/2} \|\Delta_H \partial_z T\|_2] \\
&\le \frac{1}{2} \|\Delta_H \partial_z T\|_2^2 + C\big[\|v\|_\infty^2 \|\nabla_H \partial_z T\|_2^2 + \|\nabla_H v\|_2^2 + \|\nabla_H v\|_2^2 \|\Delta_H v\|_2^2 \\
&\qquad\qquad + \|\nabla_H T\|_2^2 \|\Delta_H T\|_2^2 + \|\partial_z^2 T\|_2^2 \big(\|\partial_z^2 T\|_2^2 + \|\nabla_H \partial_z^2 T\|_2^2\big) \\
&\qquad\qquad + \|\partial_z u\|_2^2 \big(\|\partial_z u\|_2^2 + \|\nabla_H \partial_z u\|_2^2\big)\big] \\
&\le \frac{1}{2} \|\Delta_H \partial_z T\|_2^2 + C\big[\|v\|_\infty^2 \|\nabla_H \partial_z T\|_2^2 \\
&\quad + \big(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2 + \|\partial_z^2 T\|_2^2 + \|\partial_z u\|_2^2\big)^2 \\
&\quad \times \big(1 + \|\Delta_H v\|_2^2 + \|\Delta_H T\|_2^2 + \|\nabla_H \partial_z^2 T\|_2^2 + \|\nabla_H \partial_z u\|_2^2\big)\big],
\end{aligned}
$$

from which one obtains (iv).

(v) By applying the operator ∇_H to equations (2.5) and (2.7), multiplying the resulting equations by $-\nabla_H \Delta_H v$ and $-\nabla_H \Delta_H T$, respectively, and noticing that

$$
|\nabla_H v(x, y, z, t)| \le \frac{1}{2h} \int_{-h}^{h} |\nabla_H v(x, y, \xi, t)| d\xi + \int_{-h}^{h} |\nabla_H u(x, y, \xi, t)| d\xi,
$$

it follows from integration by parts that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_\Omega (|\Delta_H v|^2 + |\Delta_H T|^2) dx\, dy\, dz \\
&\quad + \int_\Omega (|\nabla_H \Delta_H v|^2 + |\nabla_H \Delta_H T|^2 + \varepsilon |\partial_z \Delta_H v|^2 + \varepsilon |\partial_z \Delta_H T|^2) dx\, dy\, dz \\
&\qquad = \int_\Omega \nabla_H \bigg((v \cdot \nabla_H) v - \bigg(\int_{-h}^{z} \nabla_H \cdot v\, d\xi \bigg) \partial_z v - \nabla_H \bigg(\int_{-h}^{z} T\, d\xi \bigg) \bigg) \\
&\qquad\qquad\qquad \cdot \nabla_H \Delta_H v\, dx\, dy\, dz \\
&\qquad\quad + \int_\Omega \nabla_H \bigg(v \cdot \nabla_H T - \bigg(\int_{-h}^{z} \nabla_H \cdot v\, d\xi \bigg) \bigg(\partial_z T + \frac{1}{h} \bigg) \bigg) \\
&\qquad\qquad\qquad \cdot \nabla_H \Delta_H T\, dx\, dy\, dz \le
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_\Omega \Big[|v|\big(|\nabla_H^2 v||\nabla_H \Delta_H v| + |\nabla_H^2 T||\nabla_H \Delta_H T|\big) \\
&\qquad + |\nabla_H v|(|\nabla_H v||\nabla_H \Delta_H v| + |\nabla_H T||\nabla_H \Delta_H T|) \\
&\qquad + \left(\int_{-h}^h |\nabla_H v| dz\right)(|\nabla_H u||\nabla_H \Delta_H v| + |\nabla_H \partial_z T||\nabla_H \Delta_H T|) \\
&\qquad + \left(\int_{-h}^h |\nabla_H^2 v| dz\right)(|u||\nabla_H \Delta_H v| + (|\partial_z T| + 1)|\nabla_H \Delta_H T|) \\
&\qquad + \left(\int_{-h}^h |\nabla_H^2 T| dz\right)|\nabla_H \Delta_H v|\Big] dx\, dy\, dz \\
&\le C\big[\|v\|_\infty\big(\|\nabla_H^2 v\|_2\|\nabla_H \Delta_H v\|_2 + \|\nabla_H^2 T\|_2\|\nabla_H \Delta_H T\|_2\big) \\
&\quad + \|\nabla_H^2 v\|_2\|\nabla_H \Delta_H T\|_2 + \|\nabla_H^2 T\|_2\|\nabla_H \Delta_H v\|_2\big] \\
&\quad + C\iint_M \int_{-h}^h (|\nabla_H v| + |\nabla_H u|) dz \\
&\quad \times \int_{-h}^h (|\nabla_H v||\nabla_H \Delta_H v| + |\nabla_H T||\nabla_H \Delta_H T|) dz\, dx\, dy \\
&\quad + C\iint_M \int_{-h}^h |\nabla_H v| dz \\
&\quad \times \int_{-h}^h (|\nabla_H u||\nabla_H \Delta_H v| + |\nabla_H \partial_z T||\nabla_H \Delta_H T|) dz\, dx\, dy \\
&\quad + C\iint_M \int_{-h}^h |\nabla_H^2 v| dz \int_{-h}^h (|u||\nabla_H \Delta_H v| + |\partial_z T||\nabla_H \Delta_H T|) dz\, dx\, dy.
\end{aligned} \tag{4.7}
$$

By Lemma 2.1, one has

$$
\begin{aligned}
&C\iint_M \int_{-h}^h (|\nabla_H v| + |\nabla_H u|) dz \int_{-h}^h (|\nabla_H v||\nabla_H \Delta_H v| + |\nabla_H T||\nabla_H \Delta_H T|) dz\, dx\, dy \\
&\quad + C\iint_M \int_{-h}^h |\nabla_H v| dz \int_{-h}^h (|\nabla_H u||\nabla_H \Delta_H v| + |\nabla_H \partial_z T||\nabla_H \Delta_H T|) dz\, dx\, dy \le
\end{aligned}
$$

$$
\begin{aligned}
&\le C\big(\|\nabla_H v\|_2^{1/2}+\|\nabla_H u\|_2^{1/2}\big)\big(\big\|\nabla_H^2 v\big\|_2^{1/2}+\|\nabla_H^2 u\|_2^{1/2}\big)\\
&\quad\times\big(\|\nabla_H v\|_2^{1/2}+\|\nabla_H T\|_2^{1/2}\big)\big(\|\nabla_H^2 v\|_2^{1/2}+\big\|\nabla_H^2 T\big\|_2^{1/2}\big)\\
&\quad\times(\|\nabla_H\Delta_H v\|_2+\|\nabla_H\Delta_H T\|_2)\\
&\quad+C\|\nabla_H v\|_2^{1/2}\big\|\nabla_H^2 v\big\|_2^{1/2}\big(\|\nabla_H u\|_2^{1/2}+\|\nabla_H\partial_z T\|_2^{1/2}\big)\\
&\quad\times\big(\big\|\nabla_H^2 u\big\|_2^{1/2}+\big\|\nabla_H^2\partial_z T\big\|_2^{1/2}\big)\big(\|\nabla_H\Delta_H v\|_2+\|\nabla_H\Delta_H T\|_2\big)\\
&\le\frac18\big(\|\nabla_H\Delta_H v\|_2^2+\|\nabla_H\Delta_H T\|_2^2\big)\\
&\quad+C\big(\|\nabla_H v\|_2^2+\|\nabla_H T\|_2^2+\|\nabla_H u\|_2^2\big)\big(\|\Delta_H v\|_2^2+\|\Delta_H T\|_2^2+\|\Delta_H u\|_2^2\big)\\
&\quad+C\big[\|\nabla_H v\|_2^2\|\Delta_H v\|_2^2\\
&\qquad+\big(\|\nabla_H u\|_2^2+\|\nabla_H\partial_z T\|_2^2\big)\big(\|\Delta_H u\|_2^2+\|\Delta_H\partial_z T\|_2^2\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&\le\frac18\big(\|\nabla_H\Delta_H v\|_2^2+\|\nabla_H\Delta_H T\|_2^2\big)\\
&\quad+C\big(\|\nabla_H v\|_2^2+\|\nabla_H T\|_2^2+\|\nabla_H u\|_2^2+\|\nabla_H\partial_z T\|_2^2\big)\\
&\quad\times(\|\Delta_H v\|_2^2+\|\Delta_H T\|_2^2+\|\Delta_H u\|_2^2+\|\Delta_H\partial_z T\|_2^2).
\end{aligned}
\tag{4.8}
$$

A similar argument to that for (4.3) yields

$$
\iint_M\int_{-h}^{h}\big|\nabla_H^2 v\big|dz\int_{-h}^{h}|u||\nabla_H\Delta_H v|dz\,dx\,dy\le\frac18\|\nabla_H\Delta_H v\|_2^2+C\|\nabla_H v\|_2^2\|u\|_4^8.
\tag{4.9}
$$

It follows from integration by parts and Proposition 3.1(ii) that

$$
\begin{aligned}
\int_\Omega|\partial_z T|^4dx\,dy\,dz&=-\int_\Omega T\partial_z(|\partial_z T|^2\partial_z T)dx\,dy\,dz\\
&=-3\int_\Omega T|\partial_z T|^2\partial_z^2 T\,dx\,dy\,dz\\
&\le 3\|T\|_\infty\big\|\partial_z^2 T\big\|_2\|\partial_z T\|_4^2\le C\big\|\partial_z^2 T\big\|_2\|\partial_z T\|_4^2,
\end{aligned}
$$

and thus

$$
\|\partial_z T\|_4^2\le C\big\|\partial_z^2 T\big\|_2.
$$

By the aid of this inequality, the same argument as that for (4.3) yields

$$
\begin{aligned}
&\iint_M \int_{-h}^h \left|\nabla_H^2 v\right| dz \int_{-h}^h |\partial_z T||\nabla_H \Delta_H T| dz\, dx\, dy \\
&\quad \le \frac{1}{8}\|\nabla_H \Delta_H T\|_2^2 + C\|\nabla_H v\|_2^2 \|\partial_z T\|_4^8 \\
&\quad \le \frac{1}{8}\|\nabla_H \Delta_H T\|_2^2 + C\|\nabla_H v\|_2^2 \left\|\partial_z^2 T\right\|_2^4.
\end{aligned}
\tag{4.10}
$$

Substituting (4.8)–(4.10) into (4.7) and using the Young inequality, one obtains

$$
\begin{aligned}
&\frac{d}{dt}\|(\Delta_H v, \Delta_H T)\|_2^2 + \|(\nabla_H \Delta_H v, \nabla_H \Delta_H T, \sqrt{\varepsilon}\partial_z \Delta_H v, \sqrt{\varepsilon}\partial_z \Delta_H T)\|_2^2 \\
&\quad \le C\big(\|v\|_\infty^2 + 1\big)\big(\|\Delta_H v\|_2^2 + \|\Delta_H T\|_2^2\big) \\
&\qquad + C\|\nabla_H v\|_2^2\big(\|u\|_4^8 + \left\|\partial_z^2 T\right\|_2^4\big) \\
&\qquad + C\big(\|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2 + \|\nabla_H u\|_2^2 + \|\nabla_H \partial_z T\|_2^2\big) \\
&\qquad \times \big(\|\Delta_H v\|_2^2 + \|\Delta_H T\|_2^2 + \|\Delta_H u\|_2^2 + \|\Delta_H \partial_z T\|_2^2\big) \\
&\quad \le C(\|v\|_\infty^2 + 1)\big(\|\Delta_H v\|_2^2 + \|\Delta_H T\|_2^2\big) \\
&\qquad + C\big(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2 + \|\nabla_H u\|_2^2 + \|\nabla_H \partial_z T\|_2^2 \\
&\qquad\qquad + \|u\|_4^4 + \left\|\partial_z^2 T\right\|_2^2\big)^3 \\
&\qquad \times \big(1 + \|\Delta_H v\|_2^2 + \|\Delta_H T\|_2^2 + \|\Delta_H u\|_2^2 + \|\Delta_H \partial_z T\|_2^2\big),
\end{aligned}
$$

proving (v). □

5 A Priori H^2 Estimates and Global Well-Posedness

In this section, based on the a priori low-order energy estimates established in Section 3 and the high-order energy inequalities established in Section 4, we can apply the logarithmic Sobolev embedding inequality (Lemma 2.2) and the system version of the Grönwall inequality (Lemma 2.3) to obtain the a priori H^2 estimate for strong solutions to system (2.5)–(2.7), subject to the boundary and initial conditions (1.25)–(1.27), and further establish the global well-posedness of strong solutions to system (1.22)–(1.27), or equivalently to system (1.15)–(1.21).

We first focus on the a priori H^2 bounds for the strong solutions to the regularized system (2.5)–(2.7) subject to the boundary and initial conditions (1.25)–(1.27).

PROPOSITION 5.1. *Let a positive time $\mathcal{T}$ be given. Let (v, T) be the strong solution of system* (2.5)–(2.7) *on $\Omega \times (0, \mathcal{T})$ subject to the boundary and initial conditions*

(1.25)–(1.27). *Then we have*

$$\sup_{0\le t\le\mathcal{T}}\|(v,T)\|_{H^2(\Omega)}^2 + \int_0^{\mathcal{T}}\big(\|(\partial_t v,\partial_t T)\|_{H^1(\Omega)}^2 + \|(\nabla_H v,\nabla_H T,\sqrt{\varepsilon}\partial_z v,\sqrt{\varepsilon}\partial_z T)\|_{H^2(\Omega)}^2\big)\,dt \le C(h,\mathcal{T},v_0,T_0),$$

where $C(h,\mathcal{T},v_0,T_0)$ *denotes a positive constant depending only on h,* $\mathcal{T}$*, and the initial data.*

PROOF. Define nonnegative functions a_i and b_i, $i = 1,2,\dots,9$, as follows:

$$\begin{aligned}
&a_1 = \|u\|_2^2 + \|u\|_4^4, && b_1 = \|(\nabla_H u,\sqrt{\varepsilon}\partial_z u,|u|\nabla_H u)\|_2^2,\\
&a_2 = \|\partial_z T\|_2^2, && b_2 = \big\|\big(\nabla_H\partial_z T,\sqrt{\varepsilon}\partial_z^2 T\big)\big\|_2^2,\\
&a_3 = \|\nabla_H v\|_2^2, && b_3 = \|(\Delta_H v,\sqrt{\varepsilon}\nabla_H\partial_z v)\|_2^2,\\
&a_4 = \|\nabla_H T\|_2^2, && b_4 = \|(\Delta_H T,\sqrt{\varepsilon}\nabla_H\partial_z T)\|_2^2,\\
&a_5 = \|\partial_z u\|_2^2, && b_5 = \big\|\big(\nabla_H\partial_z u,\sqrt{\varepsilon}\partial_z^2 u\big)\big\|_2^2,\\
&a_6 = \big\|\partial_z^2 T\big\|_2^2, && b_6 = \big\|\big(\nabla_H\partial_z^2 T,\sqrt{\varepsilon}\partial_z^3 T\big)\big\|_2^2,\\
&a_7 = \|\nabla_H u\|_2^2, && b_7 = \|(\Delta_H u,\sqrt{\varepsilon}\nabla_H\partial_z u)\|_2^2,\\
&a_8 = \|\nabla_H\partial_z T\|_2^2, && b_8 = \big\|\big(\Delta_H\partial_z T,\sqrt{\varepsilon}\nabla_H\partial_z^2 T\big)\big\|_2^2,\\
&a_9 = \|(\Delta_H,\Delta_H T)\|_2^2, && b_9 = \|(\nabla_H\Delta_H v,\nabla_H\Delta_H T,\sqrt{\varepsilon}\partial_z\Delta_H v,\sqrt{\varepsilon}\partial_z\Delta_H T)\|_2^2.
\end{aligned}$$

By Proposition 4.1 and Proposition 4.2, we have

$$\begin{aligned}
&\frac{d}{dt}a_1 + b_1 \le C\big(1+\|v\|_\infty^2\big)a_1,\\
&\frac{d}{dt}a_2 + b_2 \le C\big(1+\|v\|_\infty^2\big)a_2 + C\|\nabla_H v\|_2^2 + C(a_1+b_1),\\
&\frac{d}{dt}a_3 + b_3 \le C\|v\|_\infty^2 a_3 + C\|\nabla_H T\|_2^2 + Ca_1^2,\\
&\frac{d}{dt}a_4 + b_4 \le C(1+a_2+a_3)^2(1+b_2+b_3),\\
&\frac{d}{dt}a_5 + b_5 \le C\big(1+\|v\|_\infty^2\big)a_5 + C(b_1+b_2),\\
&\frac{d}{dt}a_6 + b_6 \le C\big(1+\|v\|_\infty^2\big)a_6 + C(1+a_2+a_5)^2(1+b_1+b_2+b_5),\\
&\frac{d}{dt}a_7 + b_7 \le C\|v\|_\infty^2 a_7 + C(a_1+a_3+a_4+a_5)(b_1+b_3+b_5),\\
&\frac{d}{dt}a_8 + b_8 \le C\|v\|_\infty^2 a_8 + C(1+a_3+a_4+a_5+a_6)^2\\
&\qquad\qquad \times(1+b_3+b_4+b_5+b_6),
\end{aligned}$$

$$\frac{d}{dt}a_9 + b_9 \le C\big(1 + \|v\|_\infty^2\big)a_9 + C(1 + a_1 + a_3 + a_4 + a_6 + a_7 + a_8) \times (1 + b_3 + b_4 + b_7 + b_8).$$

Set $A_1(t) = a_1(t) + e$ and $A_i(t) = A_{i-1}(t) + a_i(t)$, $i = 2, 3, \dots, n$. Then one can easily check from the above inequalities that

$$\frac{d}{dt}A_1(t) \le m(t)A_1(t), \tag{5.1}$$

$$\frac{d}{dt}A_i(t) + B_i(t) \le m(t)A_i(t) + \zeta A_{i-1}^3 B_{i-1}, \quad i = 2, 3, \dots, 9, \tag{5.2}$$

where

$$m(t) = C\big(1 + \|v\|_\infty^2(t) + \|\nabla_H v\|_2^2(t) + \|\nabla_H T\|_2^2(t)\big)$$

for a positive constant C.

Recalling the definitions of a_i, A_i, $i = 1, 2, \dots, 9$, one can easily check

$$\sum_{i=1}^9 A_i \ge \sum_{i=1}^9 a_i + e \ge C\|(\nabla v, \nabla T)\|_{H^1}^2 + e.$$

And thus, by Proposition 3.1 and the Sobolev and Poincaré inequalities, one obtains

$$\sum_{i=1}^9 A_i \ge C\|(\nabla v, \nabla T)\|_{H^1}^2 + e \ge C\|(v, T)\|_{H^2}^2 + e \ge C\big(\|(v, T)\|_{W^{1,6}(\Omega)}^2 + e\big).$$

By the aid of this, it follows from Lemma 2.2 and Proposition 3.1 that

$$\begin{aligned} m(t) &= C\big(1 + \|v\|_\infty^2 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2\big) \\ &\le C\big(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2\big)\big(1 + \|v\|_\infty^2\big) \\ &\le C\big(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2\big) \\ &\quad \times \left[1 + \left(\max\left\{1, \sup_{r\ge 2}\frac{\|v\|_r}{\sqrt{r}}\right\} \log^{1/2}(\|v\|_{W^{1,6}} + e)\right)^2\right] \\ &\le C\big(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2\big)\log(\|v\|_{W^{1,6}} + e) \\ &\le K(t)\log\sum_{i=1}^9 A_i(t), \end{aligned} \tag{5.3}$$

where $K(t) = C(1 + \|\nabla_H v\|_2^2 + \|\nabla_H T\|_2^2)$. By Proposition 3.1, one has $K \in L^1_{\mathrm{loc}}([0, \infty))$.

On account of (5.1)–(5.3), one can apply Lemma 2.3 to conclude that

$$\sum_{i=1}^{9} A_i(t) + \sum_{i=1}^{9} \int_0^t B_i(s)ds \le Q(t) \tag{5.4}$$

for any $t \in (0, \mathcal{T})$, where Q is the corresponding continuous function on $[0, \infty)$, specified in (2.4), determined by the initial data.

Recalling the definitions of A_i and B_i, one can easily check that

$$\sum_{i=1}^{9} A_i \ge C\|(\nabla v, \nabla T)\|_{H^1}^2, \quad \sum_{i=1}^{9} B_i \ge C\|(\nabla_H v, \nabla_H T, \sqrt{\varepsilon}\partial_z v, \sqrt{\varepsilon}\partial_z T)\|_{H^2}^2.$$

By the aid of the above, it follows from (5.4) and Proposition 3.1 that

$$\sup_{0\le t\le \mathcal{T}} \|(v, T)\|_{H^2}^2 + \int_0^{\mathcal{T}} \|(\nabla_H v, \nabla_H T, \sqrt{\varepsilon}\partial_z v, \sqrt{\varepsilon}\partial_z T)\|_{H^2}^2 \, dt \le C$$

for a positive constant C depending only on h, $\mathcal{T}$, and the initial data and is independent of ε.

Thanks to the estimates we have just proved, one can use the same argument as in the last paragraph of the proof of proposition 3.1 in [6] to obtain the corresponding estimates on $\partial_t v$ and $\partial_t T$, and thus we omit the details here. This completes the proof. □

After establishing the a priori H^2 estimate as stated in Proposition 5.1, we are now ready to prove the global well-posedness of strong solutions to system (1.22)–(1.27).

PROOF OF THEOREM 1.3. By Propositions 2.6 and 5.1, for any $\varepsilon > 0$ there is a unique global strong solution $(v_\varepsilon, T_\varepsilon)$ to system (2.5)–(2.7) subject to the boundary and initial conditions (1.25)–(1.27) such that for any $\mathcal{T} > 0$,

$$\begin{aligned}
&\sup_{0\le t\le \mathcal{T}} \|(v_\varepsilon, T_\varepsilon)\|_{H^2(\Omega)}^2 \\
&\quad + \int_0^{\mathcal{T}} \big(\|(\partial_t v_\varepsilon, \partial_t T_\varepsilon)\|_{H^1(\Omega)}^2 + \|(\nabla_H v_\varepsilon, \nabla_H T_\varepsilon, \sqrt{\varepsilon}\partial_z v_\varepsilon, \sqrt{\varepsilon}\partial_z T_\varepsilon)\|_{H^2(\Omega)}^2\big) dt \\
&\quad \le C(h, \mathcal{T}, v_0, T_0),
\end{aligned}$$

where C is independent of ε.

On account of these estimates, applying Lemma 2.5, there is a subsequence, still denoted by $\{(v_\varepsilon, T_\varepsilon)\}$, and (v, T) such that

$$\begin{aligned}
(v_\varepsilon, T_\varepsilon) &\to (v, T) && \text{in } C([0, \mathcal{T}]; H^1(\Omega)), \\
(\nabla_H v_\varepsilon, \nabla_H T_\varepsilon) &\to (\nabla_H v, \nabla_H T) && \text{in } L^2(0, \mathcal{T}; H^1(\Omega)), \\
(v_\varepsilon, T_\varepsilon) &\overset{*}{\rightharpoonup} (v, T) && \text{in } L^\infty(0, \mathcal{T}; H^2(\Omega)),
\end{aligned}$$

$$(\nabla_H v_\varepsilon, \nabla_H T_\varepsilon) \rightharpoonup (\nabla_H v, \nabla_H T) \quad \text{in } L^2(0, \mathcal{T}; H^2(\Omega)),$$
$$(\partial_t v_\varepsilon, \partial_t T_\varepsilon) \rightharpoonup (\partial_t v, \partial_t T) \quad \text{in } L^2(0, \mathcal{T}; H^1(\Omega)),$$

where $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ are the weak and weak-$*$ convergence, respectively. Thanks to this convergence, one can easily show that (v, T) is a strong solution to system (1.22)–(1.27), or equivalently to system (1.15)–(1.21).

The continuous dependence on the initial data, in particular the uniqueness, are a straightforward corollary of proposition 2.4 in [5]. This completes the proof. □

Appendix: A Logarithmic Sobolev Embedding Inequality

In this appendix, we establish a logarithmic Sobolev embedding inequality for any function $f \in W^{1,p}(\mathbb{R}^N)$ with $p > N \geq 2$. Similar inequalities have been established in [3, 10] for the two-dimensional case. We follow here the ideas of the proof presented in [3].

LEMMA A.1. *Let $p > N \geq 2$. Then for any $F \in W^{1,p}(\mathbb{R}^N)$, we have*

$$\|F\|_\infty \leq C_{N,p,\lambda} \max\left\{1, \sup_{r\geq 2} \frac{\|F\|_r}{r^\lambda R^{N/r}}\right\} \log^\lambda\left(e + \frac{\|F\|_p}{R^{N/p}} + \frac{\|\nabla F\|_p}{R^{N/p-1}}\right)$$

for any R, $\lambda > 0$, and some constant $C_{N,p,\lambda} > 0$.

PROOF. We only give the details of the proof for the case of spatial dimension $N \geq 3$; the case for $N = 2$ can be given similarly (see, e.g., [3]). Without loss of generality, we can suppose that $|F(0)| = \|F\|_\infty$. Denote by B_r the ball in $\mathbb{R}^N$ centered at the origin. Let $\phi \in C_0^\infty(B_1)$, with $\phi \equiv 1$ on $B_{1/2}$, $0 \leq \phi \leq 1$ on B_1, and set $f = F\phi$.

Taking α and β as

$$\alpha = \frac{2Np}{p-N}, \qquad \beta = \frac{2Np}{2Np - N - p},$$

one can easily check that

$$\alpha > 2N, \quad 0 < (N-1)\beta < N, \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{p} = 1. \tag{A.1}$$

Recall that f can be represented in terms of Δf by the Newtonian potential. By the aid of (A.1), for any $q \geq 2$, we have

$$\begin{aligned} |f(0)|^q &= C_N \left| \int_{\mathbb{R}^N} \frac{1}{|x|^{N-2}} \Delta(|f|^q) dx \right| \\ &= C_N \left| \int_{B_1} \nabla\left(\frac{1}{|x|^{N-2}}\right) \cdot \nabla\big(|f|^q\big) dx \right| \leq \end{aligned}$$

$$
\begin{aligned}
&\le C_N(N-2)q\int_{B_1}\frac{|f|^{q-1}|\nabla f|}{|x|^{N-1}}\,dx\\
&\le C_N(N-2)q\|f\|_{(q-1)\alpha}^{q-1}\|\nabla f\|_p\left(\int_{B_1}\frac{dx}{|x|^{\beta(N-1)}}\right)^{1/\beta}\\
&\le C_{N,p}q\|f\|_{(q-1)\alpha}^{q-1}\|\nabla f\|_p.
\end{aligned}
$$

From the above inequality, for any $q\ge 2$, noticing that $q^{1/q}\le C$ and $(q-1)\alpha\ge 2$, we deduce

$$
\begin{aligned}
|f(0)| &\le C_{N,p}\|f\|_{(q-1)\alpha}^{1-\frac{1}{q}}\|\nabla f\|_p^{\frac{1}{q}}\\
&= C_{N,p}\left[\frac{\|f\|_{(q-1)\alpha}}{((q-1)\alpha)^\lambda}\right]^{1-\frac{1}{q}}[(q-1)\alpha]^{\lambda(1-\frac{1}{q})}\|\nabla f\|_p^{\frac{1}{q}}\\
&\le C_{N,p,\lambda}\left[\frac{\|f\|_{(q-1)\alpha}}{((q-1)\alpha)^\lambda}\right]^{1-\frac{1}{q}}q^\lambda\|\nabla f\|_p^{\frac{1}{q}}\\
&\le C_{N,p,\lambda}\max\left\{1,\sup_{r\ge 2}\frac{\|f\|_r}{r^\lambda}\right\}q^\lambda\|\nabla f\|_p^{\frac{1}{q}},
\end{aligned}
$$

and thus

$$
|f(0)|\le C_{N,p,\lambda}\max\left\{1,\sup_{r\ge 2}\frac{\|f\|_r}{r^\lambda}\right\}\inf_{q\ge 2}\left(q^\lambda(\|\nabla f\|_p+e^{4\lambda})^{\frac{1}{q}}\right).
$$

One can check that

$$
\log(\|\nabla f\|_p+e^{4\lambda})\le\max\{1,4\lambda\}\log(\|\nabla f\|_p+e)
$$

and

$$
\inf_{q\ge 3}\left(q^\lambda(\|\nabla f\|_p+e^{4\lambda})^{\frac{1}{q}}\right)=\left(\frac{e}{\lambda}\right)^\lambda\log^\lambda(\|\nabla f\|_p+e^{4\lambda}).
$$

Therefore, we have

$$
|f(0)|\le C_{N,p,\lambda}\max\left\{1,\sup_{r\ge 2}\frac{\|f\|_r}{r^\lambda}\right\}\log^\lambda(\|\nabla f\|_p+e).
$$

This implies

$$
\begin{aligned}
\|F\|_\infty=|f(0)| &\le C_{N,p,\lambda}\max\left\{1,\sup_{r\ge 2}\frac{\|f\|_r}{r^\lambda}\right\}\log^\lambda(\|\nabla f\|_p+e)\\
&\le C_{N,p,\lambda}\max\left\{1,\sup_{r\ge 2}\frac{\|F\|_r}{r^\lambda}\right\}\log^\lambda(\|\nabla F\|_p+\|F\|_p+e)\\
&\le C_{N,p,\lambda}\max\left\{1,\sup_{r\ge 2}\frac{\|F\|_r}{r^\lambda}\right\}\log^\lambda(\|F\|_{W^{1,p}}+e). \qquad \text{(A.2)}
\end{aligned}
$$

For any $R > 0$, define function F_R as

$$F_R(x) = F(Rx) \quad \text{for } x \in \mathbb{R}^N.$$

One can easily check that

$$\|F_R\|_p = R^{-N/p}\|F\|_p, \quad \|\nabla F_R\|_p = R^{1-N/p}\|\nabla F\|_p,$$

for any $p \in (0, \infty)$. By the aid of the above, it follows from (A.2) that

$$\begin{aligned}\|F\|_\infty = \|F_R\|_\infty &\le C_{N,p,\lambda} \max\left\{1, \sup_{r\ge 2} \frac{\|F_R\|_r}{r^\lambda}\right\} \log^\lambda(\|F_R\|_{W^{1,p}} + e)\\ &= C_{N,p,\lambda} \max\left\{1, \sup_{r\ge 2} \frac{\|F\|_r}{r^\lambda R^{N/r}}\right\} \log^\lambda\left(e + \frac{\|F\|_p}{R^{N/p}} + \frac{\|\nabla F\|_p}{R^{N/p-1}}\right),\end{aligned}$$

proving the conclusion. □

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