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Decorated marked surfaces: spherical twists versus braid twists

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Abstract We are interested in the 3-Calabi-Yau categories $\mathcal{D}$ arising from quivers with potential associated to a triangulated marked surface $\mathbf{S}$ (without punctures). We prove that the spherical twist group ST of $\mathcal{D}$ is isomorphic to a subgroup (generated by braid twists) of the mapping class group of the decorated marked surface $\mathbf{S}_\triangle$. Here $\mathbf{S}_\triangle$ is the surface obtained from $\mathbf{S}$ by decorating with a set of points, where the number of points equals the number of triangles in any triangulations of $\mathbf{S}$. For instance, when $\mathbf{S}$ is an annulus, the result implies that the corresponding spaces of stability conditions on $\mathcal{D}$ are contractible.

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1 Introduction

1.1 Calabi-Yau (CY) categories from mirror symmetry

We are interested in a class of 3-Calabi-Yau categories $\mathcal{D}$ arising from (homological) mirror symmetry. These 3-CY categories are not only interesting in mathematics [17], [22], but also in string theory ([8], cf. [2]). On the symplectic geometry side, the category $\mathcal{D}$ (of type A) was first studied by Khovanov and Seidel [17]. They showed that there is a faithful braid group action on $\mathcal{D}$. Moreover, when realizing $\mathcal{D}$ as the subcategory of the derived Fukaya category of the Milnor fibre of a simple singularities of type A, such a braid group is generated by the (higher) Dehn twists along certain Lagrangian spheres. On the algebraic geometry side, Seidel and Thomas [22] studied the mirror counterpart of [17] (also in type A). They showed that $\mathcal{D}$ can be realized as a subcategory of the bounded derived category of coherent sheaves of the mirror variety with a faithful braid group action. Recently, Smith [23] showed that if $\mathcal{D}$ is coming from triangulations of marked surfaces $\mathbf{S}$, then it also can be embedded into some derived Fukaya category. This class of cases is the one we will study. Our focus is on the spherical twist group $\mathrm{ST} \subset \mathrm{Aut}\mathcal{D}$, a subgroup of the auto-equivalence group of $\mathcal{D}$ generated by Khovanov-Seidel-Thomas (KST) spherical twists. The aim is to generalize KST's result, that ST is 'faithful', in the sense that ST is isomorphic to the classical (type A) braid group (and in general, isomorphic to a subgroup of a certain mapping class group). We need to restrict ourselves in the case when marked surfaces are unpunctured. In the twin paper [15], we will make an effort to attack the problem when the marked surfaces are punctured.

Note that the spherical twist group ST acts freely on the space $\mathrm{Stab}^\circ\mathcal{D}$ of Bridgeland's stability condition of $\mathcal{D}$. This is one of our main motivations to study such a group. In fact, Bridgeland-Smith (BS) [2] recently showed that the quotient (orbifold) $\mathrm{Stab}^\circ\mathcal{D}/\mathrm{Aut}^\circ$ is isomorphic to the moduli space $\mathrm{Quad}_\heartsuit(\mathbf{S})$ of meromorphic quadratic differentials with simple zeroes on the marked surfaces $\mathbf{S}$, where $\mathrm{Aut}^\circ\mathcal{D}$ is the extension of the (tagged) mapping class group of $\mathbf{S}$ on top of ST. And one would expect that the faithfulness of spherical twist group actions will imply the simply connectedness of $\mathrm{Stab}^\circ\mathcal{D}$. For instance, this implication holds for the (3-CY) Dynkin case (see [18]); also, such faithfulness (and its implication of simply connectedness) was proved by Brav and Thomas [1] for the 2-CY Dynkin case and by Ishii et al. [10] for the 2-CY affine $\widetilde{A}$ case.

Our main result says that ST is isomorphic to a subgroup of the mapping class group of some surface. As an example, we will show the contractibility of the corresponding $\mathrm{Stab}^\circ\mathcal{D}$ in this paper. In the sequel, we will prove that this result indeed implies the simply connectedness of $\mathrm{Stab}^\circ\mathcal{D}$ for any unpuncutred marked surface $\mathbf{S}$.

1.2 Quivers with potential and categorification of cluster algebras

Quiver mutation was invented by Fomin-Zelevinsky (FZ) around 2000, as the combinatorial aspect of cluster algebras. Later, mutation was developed by Derksen-Weyman-Zelevinsky (DWZ) for quivers with potential.

The first (additive) categorification of cluster algebras (with certain associated acyclic quivers) was due to Buan-Marsh-Reineke-Reiten-Todorov, via representations of the corresponding quivers. Amiot introduced the generalized cluster categories via Ginzburg dg algebras for quivers with potential. In her construction, the cluster category $\mathcal{C}(\Gamma)$ is defined by the following short exact sequence of triangulated categories

$$0 \to \mathcal{D}_{fd}(\Gamma) \to \mathrm{per}\Gamma \xrightarrow{\pi} \mathcal{C}(\Gamma) \to 0, \tag{1.1}$$

where $\Gamma = \Gamma(Q, W)$ is the Ginzburg dg algebra of the quiver with potential and $\mathrm{per}\Gamma$ (resp. $\mathcal{D}_{fd}(\Gamma)$) are the perfect (resp. finite-dimensional) derived category of Γ. Here, $\mathcal{D}_{fd}(\Gamma)$ is the 3-CY category we mentioned above and it also provides a categorification for cluster algebras.

There is an exchange graph associated to each of the categories in (1.1), namely:

- the reachable hearts/t-structures in $\mathcal{D}_{fd}(\Gamma)$ as vertices and simple tilting as edges for the exchange graph $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma))$;
- the reachable silting sets in $\mathrm{per}\Gamma$ as vertices and mutation as edges for the silting exchange graph $\mathrm{SEG}^\circ(\mathrm{per}(\Gamma))$;
- the cluster tilting sets in $\mathcal{C}(\Gamma)$ as vertices and mutation as edges for the cluster exchange graph $\mathrm{CEG}(\mathcal{C}(\Gamma))$.

They play a crucial role in categorifying cluster algebras, understanding quantum dilogarithm identities and computing stability conditions. By simple-projective duality, there is a canonical isomorphism between the first two graphs. Moreover, they are coverings of the third (cf. [14]) by the spherical twist group action we mentioned above.

1.3 Triangulations of marked surfaces

A geometric aspect of cluster theory was explored by Fomin-Shapiro-Thurston (FST). They constructed a quiver $Q_{\mathbf{T}}$ for each (tagged) triangulation $\mathbf{T}$ of a marked surface $\mathbf{S}$ and showed that flipping triangulations corresponds to FZ mutation of quivers. Here, the marked surface $\mathbf{S}$ is a surface with marked points on its boundaries and punctures in its interior. Further, Labardini-Fragoso gave a rigid potential $W_{\mathbf{T}}$ for each FST quiver $Q_{\mathbf{T}}$, which is the unique 'good' (rigid, to be precise) one (cf. [9]), that is compatible with DWZ mutation. Then one can construct the Ginzburg dg algebra $\Gamma_{\mathbf{T}} = \Gamma(Q_{\mathbf{T}}, W_{\mathbf{T}})$ and the associated categories, as in (1.1).

In this paper, we will deal the case when $\mathbf{S}$ is unpunctured and introduce a new surface from $\mathbf{S}$ by decorating it with a set $\triangle$ of points as a topological model for these categories. The number of points in $\triangle$ equals the number of triangles in any triangulation of $\mathbf{S}$. This decorating idea already appeared in various contexts (e.g. Krammer [16] and Gaiotto et al. [8]). In the theory of BS ([2]), these decorating points are simple zeroes of quadratic differentials (cf. Fig. 10); the boundary components of $\mathbf{S}$ are the real blow-up of higher order (≥ 3) poles of quadratic differentials. Further, when considering the mapping class group of $\mathbf{S}_\triangle$, these decorating points are serving as punctures in topology; however, we reserve the terminology 'punctures' for the FST setting of marked surfaces.

Denote such a surface by $\mathbf{S}_\triangle$ and call it the *decorated marked surface*. A triangulation of $\mathbf{S}_\triangle$ is a maximal collection of simple open arcs that divides $\mathbf{S}_\triangle$ into triangles such that each triangle contains exactly one decorating point. One important feature of $\mathbf{S}_\triangle$ is that flipping a triangulation has directions (cf. Sect. 3.2). Then we obtain a list of correspondences, as shown in Table 1 (some of the correspondences will be given in the second part of the paper). Simple closed arcs, i.e. the simple arcs connecting different decorating points, play a crucial role in the construction/proof of these correspondences. In the theory of BS, they should correspond to stable objects (w.r.t. some stability conditions) and saddle connections (w.r.t. some quadratic differentials).

1.4 The project: decorated marked surfaces

This paper initiates a project: **DMS** = decorated marked surfaces. In the first paper, we prove the following theorem.

Theorem 1 *Suppose* $\mathbf{S}$ *is a marked surface without punctures and* $\mathbf{T}$ *a triangulation of its decorated version* $\mathbf{S}_\triangle$. *There is a canonical isomorphism*

$$\iota\colon \mathrm{BT}(\mathbf{T}) \to \mathrm{ST}(\Gamma_{\mathbf{T}}), \tag{1.2}$$

sending the standard generators (i.e. braid twists of the closed arcs η *in the dual* $\mathbf{T}^*$*) to the standard generators (i.e. spherical twists of the corresponding spherical objects* X_η*).*

The topics/plan for the sequels are:

Table 1 Correspondences

Topological side		Categorical side
Braid twists	$\xrightarrow{\cong}$	Spherical twists
Simple closed arcs in $\mathbf{S}_\triangle$ Dual Tri. with Whitehead moves	$\xrightarrow[\text{up to }[1]]{\text{1-1}}$	Spherical obj. in $\mathcal{D}_{fd}(\Gamma_\mathbf{T})$ Hearts with simple tilting
graph dual $\updownarrow$		$\updownarrow$ sim.-proj. dual
Reachable open arcs in $\mathbf{S}_\triangle$ Triangulations with flips	$\xrightarrow{\text{1-1}}$	Reachable ind. in $\operatorname{per}\Gamma_\mathbf{T}$ Silting with mutation
forgetful map $\mathbf{S}_\triangle \downarrow \mathbf{S}$		$\operatorname{per}\Gamma_\mathbf{T} \downarrow \mathcal{C}_\mathbf{S}$ quotient map
Open arcs in $\mathbf{S}$ Triangulations with flips	$\xrightarrow{\text{1-1}}$	Rigid ind. in $\mathcal{C}_\mathbf{S}$ Cluster tilting with mutation

DMS (Part B) We give a geometric realization of silting objects in $\operatorname{per}(\Gamma_\mathbf{T})$, simple-projective duality for $\Gamma_\mathbf{T}$ and Amoit's quotient π in (1.1) that defines cluster categories.
DMS II We prove Conjectures 10.5 and 10.6, that the dimensions of homomorphisms between objects in $\mathcal{D}(\Gamma)$ equals the intersection numbers between the corresponding arcs in $\mathbf{S}_\triangle$. This is a joint work with Yu Zhou.
DMS III We show that there is a unique canonical way to identify $\mathcal{D}(\Gamma_\mathbf{T})$, for any triangulation $\mathbf{T}$ in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$. Thus, one can associate a unique 3-Calabi-Yau category $\mathcal{D}_{fd}(\mathbf{S}_\triangle)$ to $\mathbf{S}_\triangle$. As an application, we show that the spherical twist group $\mathrm{ST}(\mathbf{S}_\triangle)$ acts faithfully on the corresponding space $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\mathbf{S}_\triangle)$ of stability conditions. This is a joint work with Aslak Buan.

We will prove the simply connectedness of $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\mathbf{S}_\triangle)$ by calculating the fundamental group of the space $\mathrm{Quad}(\mathbf{S})$ of quadratic differentials in [15].

2 Preliminaries

2.1 Quivers with potential and Ginzburg algebras

Fix an algebraically closed field $\mathbf{k}$ and all categories are $\mathbf{k}$-linear. Denote by $\Gamma = \Gamma(Q, W)$ the *Ginzburg dg algebra (of degree 3)* associated to a quiver with potential (Q, W), which is constructed as follows (cf. [13]):

- Let Q^3 be the graded quiver whose vertex set is Q_0 and whose arrows are:
  - the arrows in Q_1 with degree 0;
  - an arrow $a^* : j \to i$ with degree -1 for each arrow $a : i \to j$ in Q_1;
  - a loop $e_i^* : i \to i$ with degree -2 for each vertex i in Q_0.
- The underlying graded algebra of $\Gamma(Q, W)$ is the completion of the graded path algebra $\mathbf{k}Q^3$ in the category of graded vector spaces w.r.t. the ideal generated by the arrows of Q^3.
- The differential of $\Gamma(Q, W)$ is the unique continuous linear endomorphism, homogeneous of degree 1, which satisfies the Leibniz rule and takes the following values
  - $\mathrm{d}a = 0$ for any $a \in Q_1$,
  - $\mathrm{d}a^* = \partial_a W$ for any $a \in Q_1$ and
  - $\mathrm{d} \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*]$.

Example 2.1 Let Q be a 3-cycle with edges a, b, c and the potential $W = abc$. Then the (graded) quiver Q^3 is

$$Q^3: \begin{tikzcd} & 2 \arrow[loop above, "f_2"] \arrow[dl, shift left, "b^*"] \arrow[dr, shift left, "a"] & \\ 1 \arrow[ur, shift left, "b"] \arrow[rr, shift left, "c^*"] \arrow[loop left, "f_1"] & & 3 \arrow[ul, shift left, "a^*"] \arrow[ll, shift left, "c"] \arrow[loop right, "f_3"] \end{tikzcd} \tag{2.1}$$

and the (non-trivial) differentials are

$$\begin{gathered} d(a^*) = bc,\ d(b^*) = ca,\ d(c^*) = ab, \\ d(f_1) = cc^* - b^*b,\ d(f_2) = bb^* - a^*a,\ d(f_1) = aa^* - c^*c. \end{gathered} \tag{2.2}$$

In this paper, the quivers with potential we are considering are *rigid* (and hence *non-degenerated*), which basically means that they behave nicely under mutation, in the sense of DWZ. For details about these notions, see, e.g. [13] and [9].

2.2 The 3-Calabi-Yau categories

A triangulated category $\mathcal{D}$ is called *N-Calabi-Yau* (N-CY) if, for any objects X, X' in $\mathcal{D}$ we have a natural isomorphism

$$\mathfrak{S} : \mathrm{Hom}^\bullet_{\mathcal{D}}(X, X') \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{D}}(X', X)^\vee[N]. \tag{2.3}$$

Note that the graded dual of a graded $\mathbf{k}$-vector space $V = \oplus_{k \in \mathbb{Z}} V_k[k]$ is

$$V^\vee = \bigoplus_{k \in \mathbb{Z}} V_k^*[-k].$$

Further, an object S is *N-spherical* if $\mathrm{Hom}^{\bullet}(S,S)=\mathbf{k}\oplus\mathbf{k}[-N]$ and (2.3) holds functorially for $X=S$ and X' in $\mathcal{D}$. Denote by $\mathcal{D}_{fd}(\Gamma)$ the finite-dimensional derived category of Γ. It is well-known that this is a 3-CY category. We also know that (see, e.g. [12]) $\mathcal{D}_{fd}(\Gamma)$ admits a canonical heart $\mathcal{H}_\Gamma$ generated by simple Γ-modules S_e, for $e\in Q_0$, each of which is 3-spherical. Recall that the *twist functor ϕ of a spherical object S* is defined by

$$\phi_S(X)=\mathrm{Cone}\left(S\otimes\mathrm{Hom}^{\bullet}(S,X)\to X\right) \tag{2.4}$$

with inverse

$$\phi_S^{-1}(X)=\mathrm{Cone}\left(X\to S\otimes\mathrm{Hom}^{\bullet}(X,S)^{\vee}\right)[-1]$$

Denote by $\mathrm{ST}(\Gamma)$ the spherical twist group of $\mathcal{D}_{fd}(\Gamma)$ in $\mathrm{Aut}\mathcal{D}_{fd}(\Gamma)$, generated by $\{\phi_{S_e}\mid e\in Q_0\}$. By [22, Lemma 2.11], we have the formula

$$\phi_{\psi(S)}=\psi\circ\phi_S\circ\psi^{-1} \tag{2.5}$$

for any spherical object S and $\psi\in\mathrm{Aut}\mathcal{D}_{fd}(\Gamma)$.

Denote by $\mathrm{Sph}(\Gamma)$ the set of *reachable* spherical objects in $\mathcal{D}_{fd}(\Gamma)$, that is,

$$\mathrm{Sph}(\Gamma)=\mathrm{ST}(\Gamma)\cdot\mathrm{Sim}\mathcal{H}_\Gamma, \tag{2.6}$$

where $\mathrm{Sim}\mathcal{H}$ denotes the set of simples of an abelian category $\mathcal{H}$.

We have the following observations.

- The the twist functor is well-defined on $\mathrm{Sph}(\Gamma)/[1]$, i.e. $\phi_S=\phi_{S[1]}$.
- Clearly, for any $\phi\in\mathrm{ST}(\Gamma)$ and $X\in\mathrm{Sph}(\Gamma)$, $\phi(X)$ is still in $\mathrm{Sph}(\Gamma)$.
- By (2.5), $\mathrm{ST}(\Gamma)$ is also generated by all ϕ_X for $X\in\mathrm{Sph}(\Gamma)$ (cf. [14]).

Remark 2.2 Two elements ψ and ψ' in $\mathrm{Aut}\mathcal{D}_{fd}(\Gamma)$ are *isotopic*, denote by $\psi\sim\psi'$, if $\psi^{-1}\circ\psi'$ acts trivially on $\mathrm{Sph}(\Gamma)$. In this paper, we will only consider the auto-equivalences up to isotopy, i.e. we will consider $\mathrm{ST}(\Gamma)$ as a subgroup of

$$\mathrm{Aut}^{\circ}\mathcal{D}_{fd}(\Gamma)=\mathrm{Aut}\mathcal{D}_{fd}(\Gamma)/\sim. \tag{2.7}$$

However, we will show in the sequel that: the identity is the only spherical twist which acts trivially on $\mathrm{Sph}(\Gamma)$ in our case.

2.3 Triangulations of marked surfaces

Throughout the paper, **S** denotes a *marked surface* without punctures in the sense of [7], that is, a connected surface with a fixed orientation and a finite set **M** of marked point on the (non-empty) boundary $\partial\mathbf{S}$ satisfying that each connected component of $\partial\mathbf{S}$ contains at least one marked point. Up to homeomorphism, **S** is determined by the following data

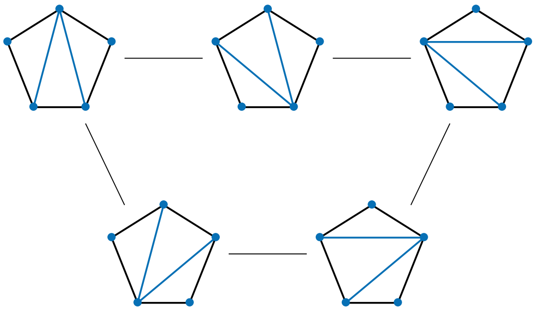

Fig. 1 The exchange graph of triangulations of a pentagon

- the genus g;
- the number $|\partial \mathbf{S}|$ of boundary components;
- the integer partition of $|\mathbf{M}|$ into $|\partial \mathbf{S}|$ parts describing the number of marked points on its boundary.

As in [7, p5], we will exclude the case when there is no triangulation or there is no arcs in the triangulation. In other words, we require $n \geq 1$ in (2.8).

An (open) *arc* in $\mathbf{S}$ is a curve (up to homotopy) that connects two marked points in $\mathbf{M}$, which is neither isotopic to a boundary segment nor to a point. The *intersection number* is defined to be

$$\mathrm{Int}(\gamma_1, \gamma_2) = \min\{|\gamma_1' \cap \gamma_2' \cap (\mathbf{S} - \mathbf{M})| \,\Big|\, \gamma_i \sim \gamma_i'\}.$$

An *(ideal) triangulation* $\mathbb{T}$ of $\mathbf{S}$ is a maximal collection of compatible simple arcs. Here, compatible means any two arcs in $\mathbb{T}$ that do not intersect.

Moreover, it is well-known that any triangulation $\mathbb{T}$ of $\mathbf{S}$ consists of

$$n = 6g + 3|\partial \mathbf{S}| + |\mathbf{M}| - 6 \tag{2.8}$$

(simple) arcs and divides $\mathbf{S}$ into

$$\aleph = \frac{2n + |\mathbf{M}|}{3} \tag{2.9}$$

triangles. Denote by $\mathrm{EG}(\mathbf{S})$ the *exchange graph* of triangulations of $\mathbf{S}$, that is, the unoriented graph whose vertices are triangulation of $\mathbf{S}$ and whose edges correspond to flips (see the lower pictures in Fig. 3 for a flip). It is known that $\mathrm{EG}(\mathbf{S})$ is connected. If $\mathbf{S}$ is an $(n+3)$-gon, then $\mathrm{EG}(\mathbf{S})$ is the associahedron of dimension n (cf. Fig. 1).

Let $\mathbf{S}$ be a marked surface and $\mathbb{T}$ a triangulation of $\mathbf{S}$. Then there is an associated quiver $Q_{\mathbb{T}}$ with a potential $W_{\mathbb{T}}$, constructed as follows (See, e.g. [9] or [20] for the precise definition):

- the vertices of $Q_{\mathbb{T}}$ are (indexed by) the arcs in $\mathbb{T}$;
- for each triangle T in $\mathbb{T}$, there are three arrows between the corresponding vertices as shown in Fig. 2;
- these three arrows form a 3-cycle in $Q_{\mathbb{T}}$ and $W_{\mathbb{T}}$ is the sum of all such 3-cycles.

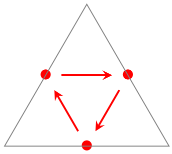

Fig. 2 The (sub-)quiver associated to a triangle (with a potential)

3 Triangulations of decorated marked surfaces

3.1 Decorated marked surfaces

Recall that any triangulation of $\mathbf{S}$ consists of $\aleph$ triangles, where $\aleph$ is given by the formula (2.9).

Definition 3.1 The *decorated marked surface* $\mathbf{S}_\triangle$ is a marked surface $\mathbf{S}$ together with a fixed set $\triangle$ of $\aleph$ 'decorating' points (in the interior of $\mathbf{S}$, where $\aleph$ is defined in (2.9)), which serve as punctures. Moreover,

- An *open arc* in $\mathbf{S}_\triangle$ is (the isotopy class of) a curve in $\mathbf{S}_\triangle - \triangle$ that connects two marked points in $\mathbf{M}$, which is neither isotopic to a boundary segment nor to a point.
- a *closed arc* in $\mathbf{S}_\triangle$ is (the isotopy class of) a curve in $\mathbf{S}_\triangle - \triangle$ that connects different decorating points in $\triangle$. Denote by $\mathrm{CA}(\mathbf{S}_\triangle)$ the set of simple closed arcs.
- An *L-arc* η in $\mathbf{S}_\triangle$ is (the isotopy class of) a curve in $\mathbf{S}_\triangle - \triangle$ such that its endpoints coincide at a decorating point in $\triangle$ and it is not isotopic to a point.
- A *general closed arc* in $\mathbf{S}_\triangle$ is either a closed arc or an L-arc; denote by $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ the set of simple general closed arcs.

The *intersection numbers* between arcs in $\mathbf{S}_\triangle$ are defined as follows:

- For an open arc γ and any arc η, their intersection number is the geometric intersection number in $\mathbf{S}_\triangle - \mathbf{M}$:

$$\mathrm{Int}(\gamma, \eta) = \min\{|\gamma' \cap \eta' \cap (\mathbf{S}_\triangle - \mathbf{M})| \,\Big|\, \gamma' \sim \gamma, \eta' \sim \eta\}.$$

- For two general closed arcs α, β in $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$, their intersection number is an half integer in $\frac{1}{2}\mathbb{Z}$ and defined as follows (following [17]):

$$\mathrm{Int}(\alpha, \beta) = \tfrac{1}{2}\mathrm{Int}_\triangle(\alpha, \beta) + \mathrm{Int}_{\mathbf{S}-\triangle}(\alpha, \beta),$$

where

$$\mathrm{Int}_{\mathbf{S}_\triangle-\triangle}(\alpha, \beta) = \min\{|\alpha' \cap \beta' \cap \mathbf{S}_\triangle - \triangle| \,\Big|\, \alpha' \sim \alpha, \beta' \sim \beta\} \tag{3.1}$$

and

$$\mathrm{Int}_{\triangle}(\alpha, \beta) = \sum_{Z \in \triangle} |\{t \mid \alpha(t) = Z\}| \cdot |\{r \mid \beta(r) = Z\}|.$$

3.2 Triangulations and flips (after Krammer)

Definition 3.2 A *triangulation* $\mathbf{T}$ of $\mathbf{S}_{\triangle}$ is a maximal collection of open arcs such that

- for any $\gamma_1, \gamma_2 \in \mathbf{T}$, $\mathrm{Int}(\gamma_1, \gamma_2) = 0$;
- $\mathbf{T}$ is compatible with $\triangle$ in the sense that the open arcs in $\mathbf{T}$ divide $\mathbf{S}_{\triangle}$ into $\aleph$ triangles, each of which contains exactly one point in $\triangle$.

Let $\mathbf{T}$ be a triangulation of $\mathbf{S}_{\triangle}$ (consisting of n open arcs). The *dual triangulation* $\mathbf{T}^*$ of $\mathbf{T}$ is the collection of n closed arcs in $\mathrm{CA}(\mathbf{S}_{\triangle})$, such that every closed arc only intersects one open arc in $\mathbf{T}$ and with intersection one. See the left picture of Fig. 15 for an example. More precisely, for γ in $\mathbf{T}$, the corresponding closed arc in $\mathbf{T}^*$ is the unique open arc s that is contained in the quadrilateral A with diagonal γ, connecting the two decorating points in A and intersecting γ only once. We will call s and γ the dual of each other, w.r.t. $\mathbf{T}$ (or $\mathbf{T}^*$), cf. left picture of Fig. 15.

There is a canonical map, the *forgetful map*

$$F : \mathbf{S}_{\triangle} \to \mathbf{S},$$

forgetting the decorating points. Clearly, F induces a map from the set of open arcs in $\mathbf{S}_{\triangle}$ to the set of open arcs in $\mathbf{S}$. And the image of a triangulation $\mathbf{T}$ is still a triangulation $\mathbb{T} = F(\mathbf{T})$. The (FST) quiver $Q_{\mathbf{T}}$ associated to $\mathbf{T}$ is defined to be the (FST) quiver $Q_{\mathbb{T}}$ associated to $\mathbb{T} = F(\mathbf{T})$. We proceed to introduce the notion of forward/backward flip of triangulations (after [16] and cf. [15]).

Definition 3.3 Let γ be an open arc in a triangulation $\mathbf{T}$ of $\mathbf{S}_{\triangle}$. The arc $\gamma^{\sharp} = \gamma^{\sharp}(\mathbf{T})$ is the arc obtained from γ by anticlockwise moving its endpoints along the quadrilateral in $\mathbf{T}$ whose diagonal is γ (cf. upper pictures of Fig. 3), to the next marked points. The *forward flip* of a triangulation $\mathbf{T}$ of $\mathbf{S}_{\triangle}$ at $\gamma \in \mathbf{T}$ is the triangulation $\mathbf{T}^{\sharp}_{\gamma}$ obtained from $\mathbf{T}$ by replacing the arc γ with $\gamma^{\sharp}$. Similarly, we can define arc $\gamma^{\flat} = \gamma^{\flat}(\mathbf{T})$ to be the arc obtained from γ by clockwise moving its endpoints, and the *backward flip* $\mathbf{T}^{\flat}_{\gamma}$ of $\mathbf{T}$ at $\gamma \in \mathbf{T}$ is the triangulation $\mathbf{T}^{\flat}_{\gamma}$ obtained from $\mathbf{T}$ by replacing the arc γ with $\gamma^{\flat}$.

Clearly, these two flips are inverse operations. Also note that under the forgetful map F, a forward/backward flip in $\mathbf{S}_{\triangle}$ becomes a normal flip (without direction) of $\mathbf{S}$, cf. Fig. 3, which is an involution.

Definition 3.4 The exchange graph $\mathrm{EG}(\mathbf{S}_{\triangle})$ is the oriented graph whose vertices are triangulations of $\mathbf{S}_{\triangle}$ and whose edges correspond to forward flips between them.

Remark 3.5 Recall that $\pi_1\mathrm{EG}(\mathbf{S})$ is generated by squares and pentagons ([7, Theorem 3.10]). By [16], forward flips also satisfy the square and pentagon relations (cf. Fig. 4). We believe that $\pi_1\mathrm{EG}(\mathbf{S}_{\triangle})$ is also generated by squares and pentagons.

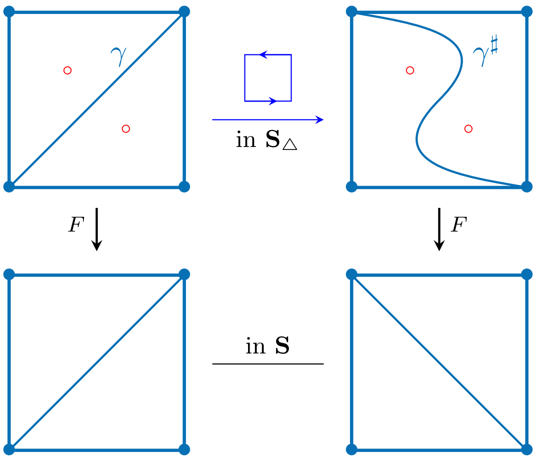

Fig. 3 The flip

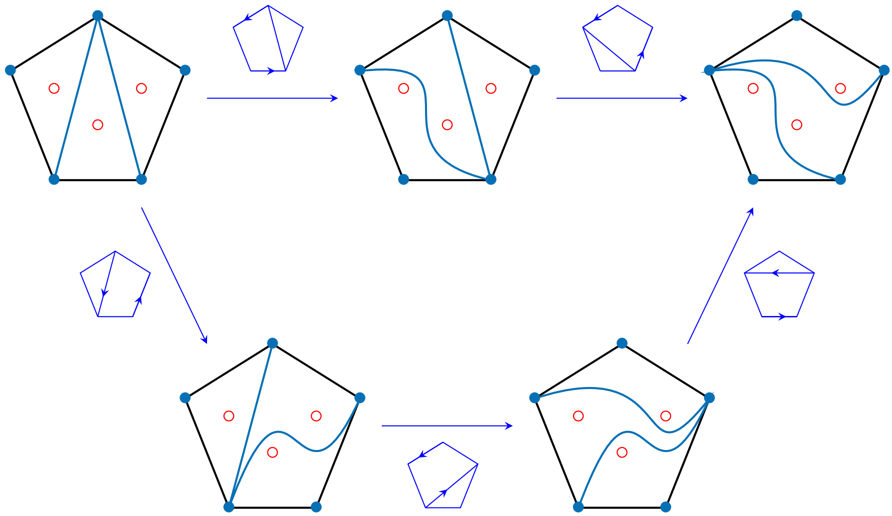

Fig. 4 The pentagon relation for forward flips

3.3 The braid twists

The mapping class group $\mathrm{MCG}(\mathbf{S}_\triangle)$ is the group of isotopy classes of (orientation preserving) homeomorphisms of $\mathbf{S}_\triangle$, where all homeomorphisms and isotopies are required to: (i) fix $\partial\mathbf{S}_\triangle(\supset \mathbf{M})$ pointwise; (ii) fix the decorating points set $\triangle$ (but allow to permutate points in it). Note that the mapping class group $\mathrm{MCG}(\mathbf{S})$ of $\mathbf{S}$ will require only the first condition and thus there is a canonical map

$$F_*\colon \mathrm{MCG}(\mathbf{S}_\triangle) \twoheadrightarrow \mathrm{MCG}(\mathbf{S}) \tag{3.2}$$

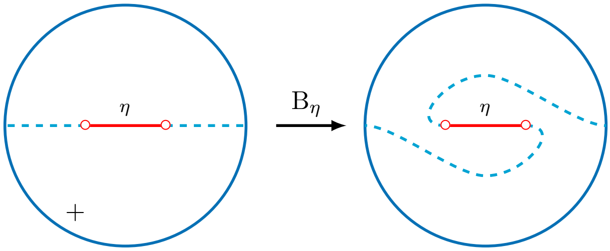

Fig. 5 The Braid twist

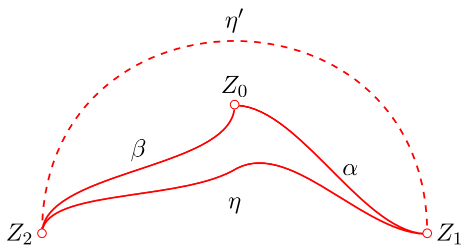

Fig. 6 Intersecting one half

induced by the forgetful map F. As $\mathrm{MCG}(\mathbf{S})$ is generated by Dehn twists along simple closed curves (which misses the decorating points), F_* is clearly surjective.

For any closed arc $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, there is the (positive) *braid twist* $\mathrm{B}_\eta \in \mathrm{MCG}(\mathbf{S}_\triangle)$ along η, which is shown in Fig. 5.

Further, there is the following well-known formula

$$\mathrm{B}_{\Psi(\eta)} = \Psi \circ \mathrm{B}_\eta \circ \Psi^{-1}, \tag{3.3}$$

for any $\Psi \in \mathrm{MCG}(\mathbf{S}_\triangle)$.

Definition 3.6 The *braid twist group* $\mathrm{BT}(\mathbf{S}_\triangle)$ of the decorated marked surface $\mathbf{S}_\triangle$ is the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by the braid twists B_η for $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$.

Example 3.7 If $\mathrm{Int}(\alpha, \beta) = \frac{1}{2}$, there is a closed arc η (cf. Fig. 6) such that

$$\eta = \mathrm{B}_\alpha(\beta) = \mathrm{B}_\beta^{-1}(\alpha), \quad \alpha = \mathrm{B}_\beta(\eta) = \mathrm{B}_\eta^{-1}(\beta), \quad \beta = \mathrm{B}_\eta(\alpha) = \mathrm{B}_\alpha^{-1}(\eta). \tag{3.4}$$

Note that η is the closed arc such that the interior of the triangle bounded by α, β, η is contractible. In fact, there is exactly one more such closed arc (dashed arc in Fig. 6), namely

$$\eta' = \mathrm{B}_\alpha^{-1}(\beta) = \mathrm{B}_\beta(\alpha),$$

satisfying the triangle bounded by these three arcs is contractible.

We have the following straightforward observation:

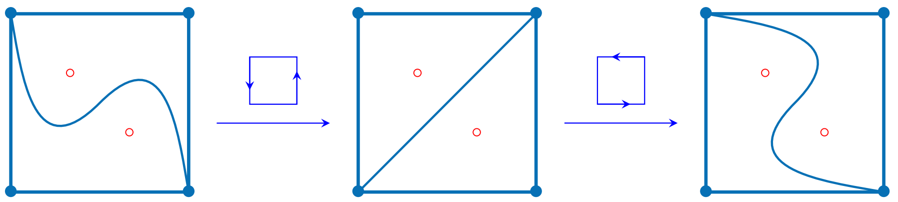

Fig. 7 Composition of forward flips as a negative braid twist

Lemma 3.8 *Let γ be a open arc in $\mathbf{T}$ and s be its dual in $\mathbf{T}^*$. Then in the triangulation $\mathbf{T}_\gamma^\sharp$, the dual of $\gamma^\sharp$ is still s. Moreover, let $\mathbf{T}_\gamma^\sharp$ and $\mathbf{T}_\gamma^\flat$ be the two flips of $\mathbf{T}$ at γ. Then*

$$\gamma^\flat = \mathrm{B}_s(\gamma^\sharp) \quad \textit{and} \quad \mathbf{T}_\gamma^\flat = \mathrm{B}_s(\mathbf{T}_\gamma^\sharp).$$

Proof The first claim follows from the upper pictures in Fig. 3 and the equations follow from Fig. 7. □

As a consequence, we obtain a map between exchange graphs.

Lemma 3.9 *As graphs, we have the following surjective map induced by the forgetful map F:*

$$F_*\colon \mathrm{EG}(\mathbf{S}_\triangle)/\mathrm{BT}(\mathbf{S}_\triangle) \twoheadrightarrow \mathrm{EG}(\mathbf{S}). \tag{3.5}$$

Proof Recall that there is a canonical surjection $F_*\colon \mathrm{MCG}(\mathbf{S}_\triangle) \twoheadrightarrow \mathrm{MCG}(\mathbf{S})$ in (3.2). By definition, it is straightforward to see that

$$\mathrm{BT}(\mathbf{S}_\triangle) \subset \ker F_*. \tag{3.6}$$

Thus, F induces a quotient map $F_*\colon \mathrm{EG}(\mathbf{S}_\triangle)/\mathrm{BT}(\mathbf{S}_\triangle) \to \mathrm{EG}(\mathbf{S})$ between sets. Next, the F_* preserves the edges (cf. Fig. 3), in the sense that the forward and backward flips of a triangulation $\mathbf{T}$ at some closed γ both become the flip of $\mathbb{T} = F(\mathbf{T})$ at $F(\gamma)$. Thus, F_* is a map between graphs. Finally, by definition, $\mathrm{EG}(\mathbf{S}_\triangle)$ is an oriented (n, n)-regular graph (that is, every vertex has n arrows in and n arrow out) and $\mathrm{EG}(\mathbf{S})$ is an unoriented n-regular graph. Therefore we deduce that F is surjective. □

Remark 3.10 In fact, if we take any connected component $\mathrm{EG}^\chi(\mathbf{S}_\triangle)$ of $\mathrm{EG}(\mathbf{S}_\triangle)$, then F_* induces an isomorphism

$$F_*\colon \mathrm{EG}^\chi(\mathbf{S}_\triangle)/\mathrm{BT}(\mathbf{S}_\triangle) \cong \mathrm{EG}(\mathbf{S})$$

since $\mathrm{EG}(\mathbf{S})$ is connected and both graphs are n-regular.

3.4 The initial triangulation

Remark 3.11 For technical reasons, we will exclude two special cases for the moment:

I). an annulus with one marked point on each of its boundary components;
II). a torus with only boundary component and one marked point.

These cases will be discussed independently in Sect. 7.

Lemma 3.12 *There exists a triangulation* $\mathbf{T}$ *of* $\mathbf{S}_\triangle$ *such that any two triangles share at most one edge. In other words, the quiver* $Q_{\mathbf{T}}$ *has no double arrows.*

Proof The second statement, which is equivalent to the first one, follows from [9, Proposition 7.13], noticing that we have excluded the two special cases (a torus with one marked point and an annulus with two marked points). □

Notations 3.13 *We will fix a triangulation* $\mathbf{T}_0$ *such that its image* $\mathbb{T}_0 = F(\mathbf{T}_0)$ *(a triangulation of* $\mathbf{S}$*) satisfies the condition in Lemma 3.12. Let*

$$\mathbf{T}_0 = \{\gamma_1, \ldots, \gamma_n\} \quad \textit{and} \quad \mathbf{T}_0^* = \{s_1, \ldots, s_n\},$$

where s_i *is the dual of* γ_i *w.r.t.* $\mathbf{T}_0$. *Denote by* $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ *the connected component of* $\mathrm{EG}(\mathbf{S}_\triangle)$ *that contains* $\mathbf{T}_0$.

We say a curve is in a *minimal position* w.r.t. $\mathbf{T}_0$, if it has minimal intersections with (arcs in) $\mathbf{T}_0$. Let $\mathrm{Int}(\mathbf{T}_0, \eta) = \sum_{i=1}^n \mathrm{Int}(\gamma_i, \eta)$. Then a representative η is in a minimal position if it intersects $\mathbf{T}_0$ exactly $\mathrm{Int}(\mathbf{T}_0, \eta)$ times.

We will repeatedly use induction on $\mathrm{Int}(\mathbf{T}_0, \eta)$ later. The next lemma is the basic idea of those inductions.

Lemma 3.14 *Suppose that a general closed arc* η *in* $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ *is not a closed arc* s *in* $\mathbf{T}_0^*$. *Then there are two closed arcs* α, β *in* $\mathrm{CA}(\mathbf{S}_\triangle)$ *such that*

1°. $\mathrm{Int}(\mathbf{T}_0, \eta) = \mathrm{Int}(\mathbf{T}_0, \alpha) + \mathrm{Int}(\mathbf{T}_0, \beta)$ *and*
2°. α, β, η *form a contractible triangle in* $\mathbf{S}_\triangle$.

In the case when $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, 2° *is equivalent to*

$\widetilde{2}°$ $\mathrm{Int}(\alpha, \beta) = \frac{1}{2}$ *and* $\eta = \mathrm{B}_\alpha(\beta)$.

Proof Recall that we require that any two triangles in $\mathbf{T}_0$ share at most one edge. Thus if η only intersects two triangles of $\mathbf{T}_0$, then $\eta = s_j \in \mathbf{T}_0$ for some j which we will exclude. Now suppose that η intersects at least three triangles of $\mathbf{T}_0$. Then one of the decorating points in these triangles is not an endpoint of η. Denote the triangle by Λ_0 with the decorating point Z_0 inside. Choose a representative of η, also denoted by η, when there is no confusion, such that it is in a minimal position w.r.t. $\mathbf{T}_0$. One can draw a line segment l from Z_0 to some point Y of η within Λ_0 such that l doesn't intersect η except at the endpoints (cf. Fig. 8).

Let Z_1 and Z_2 be the endpoints of η such that l is in the left side when we pass from Z_2 to Z_1. Consider two closed arcs α and β which are isotopic to $l \cup \eta\ |_{Z_1 Y}$ and $l \cup \eta\ |_{Z_2 Y}$ respectively (cf. Fig. 9). Clearly, 2° is satisfied. Since η is in a minimal position (w.r.t. $\mathbf{T}_0$), so are α and β. Thus 1° is also satisfied.

Moreover, η is one of $\mathrm{B}_\alpha(\beta)$ and $\mathrm{B}_\alpha^{-1}(\beta)$ when the endpoints of η do not coincide (i.e. $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$). Thus, by choosing α and β in some order we will obtain $\widetilde{2}°$ as required. □

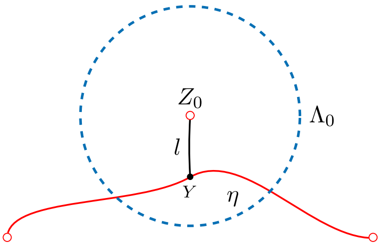

Fig. 8 The line segment l

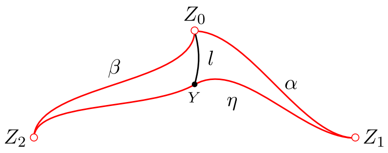

Fig. 9 Decomposing η

4 On the braid twist groups

4.1 Generators

Recall that we have the braid twist group for $\mathbf{S}_\triangle$. Now we define the braid twist group for $\mathbf{T}_0$.

Definition 4.1 Let $\mathbf{T}$ be a triangulation of $\mathbf{S}_\triangle$. The *braid twist group* $\mathrm{BT}(\mathbf{T})$ of the triangulation $\mathbf{T}$ is the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by the braid twists B_s for the closed arcs s in $\mathbf{T}^*$.

In fact, these two groups are the same.

Lemma 4.2 $\mathrm{BT}(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{T}_0)$.

Proof Use induction on $\mathrm{Int}(\mathbf{T}_0, \eta)$ to show that B_η is in $\mathrm{BT}(\mathbf{T}_0)$. If so, then $\mathrm{BT}(\mathbf{S}_\triangle) \subset \mathrm{BT}(\mathbf{T}_0)$. Clearly, $\mathrm{BT}(\mathbf{S}_\triangle) \supset \mathrm{BT}(\mathbf{T}_0)$ and therefore the lemma follows.

If $\mathrm{Int}(\mathbf{T}_0, \eta) = 1$, then $\eta \in \mathbf{T}_0^*$ and the claim is trivial. Suppose that the claim holds for any η' with $\mathrm{Int}(\mathbf{T}_0, \eta') < m$. Consider a closed arc $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$ with $\mathrm{Int}(\mathbf{T}_0, \eta) = m$. Applying Lemma 3.14, we have $\eta = \mathrm{B}_\alpha(\beta)$ for some α, β. By the inductive assumption, B_α and B_β are in $\mathrm{BT}(\mathbf{T}_0)$. By formula (3.3), we have

$$\mathrm{B}_\eta = \mathrm{B}_{\mathrm{B}_\alpha(\beta)} = \mathrm{B}_\alpha \circ \mathrm{B}_\beta \circ \mathrm{B}_\alpha^{-1} \in \mathrm{BT}(\mathbf{T}_0),$$

which completes the proof. □

Proposition 4.3 $\mathrm{BT}(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{T})$ *for any* $\mathbf{T} \in \mathrm{EG}(\mathbf{S}_\triangle)$.

Proof First, if $\mathbf{T}_1$ and $\mathbf{T}_2$ are related by a flip, then their dual graphs are related by a Whitehead move, with respect to the corresponding closed arc η (which is unchanged under the flip), see Fig. 10. Notice that the changes of closed arcs in $\mathbf{T}_i^*$ are given by

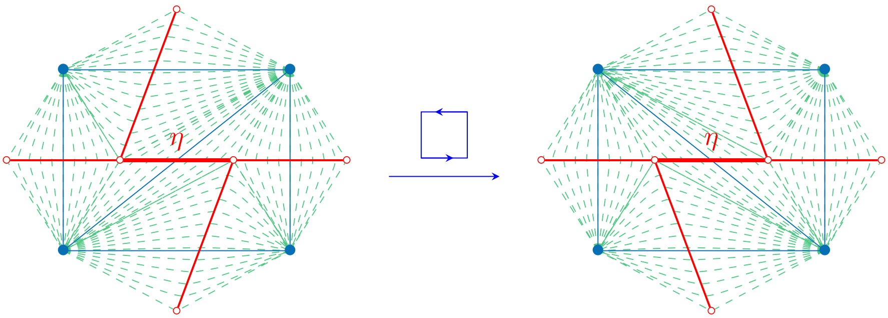

Fig. 10 The Whitehead move, as the flip of the dual triangulations (*red*)

the braid twist $\mathrm{B}_\eta^{\pm 1}$. Then by (3.3) it is straightforward to see that $\mathrm{BT}(\mathbf{T}_1) = \mathrm{BT}(\mathbf{T}_2)$. By Lemma 4.2, the proposition holds for any $\mathbf{T} \in \mathrm{EG}^\circ(\mathbf{S}_\triangle)$.

As for $\mathbf{T}$ in other connected components of $\mathrm{EG}(\mathbf{S}_\triangle)$, we can always find one triangulation in that component satisfying the condition in Lemma 3.12. Then Lemma 4.2 would apply to that triangulation and thus the proposition holds for any $\mathbf{T} \in \mathrm{EG}(\mathbf{S}_\triangle)$. □

Besides, the closed arcs are 'reachable', in the following sense.

Proposition 4.4 *Let* $\mathbf{T} \in \mathrm{EG}(\mathbf{S}_\triangle)$. *For any* $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, *there exists* $s \in \mathbf{T}^*$ *and* $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ *such that* $\eta = b(s)$, *i.e.*

$$\mathrm{CA}(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{S}_\triangle) \cdot \mathbf{T}^*.$$

Proof Consider the case when $\mathbf{T} = \mathbf{T}_0$ first. Then this follows easily by induction on $\mathrm{Int}(\mathbf{T}_0, \eta)$, using Lemma 3.14. Second, by the Whitehead move (cf. Fig. 10), if $\mathbf{T}_1$ and $\mathbf{T}_2$ are related by a flip, then

$$\mathrm{BT}(\mathbf{S}_\triangle) \cdot \mathbf{T}_1^* = \mathrm{BT}(\mathbf{S}_\triangle) \cdot \mathbf{T}_2^*.$$

Therefore the proposition holds for $\mathbf{T} \in \mathrm{EG}^\circ(\mathbf{S}_\triangle)$. Finally, as in the last paragraph of the proof of Proposition 4.3, we deduce that the proposition holds for any $\mathbf{T} \in \mathrm{EG}(\mathbf{S}_\triangle)$. □

4.2 Centers

We recall the definition of the braid groups (a.k.a. Artin groups) of type A and $\widetilde{A}$.

Definition 4.5 Suppose that Q is a quiver or a diagram as in (4.1):

$$\begin{tikzcd}[arrows=-]
A_n: & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n & \\
\widetilde{A}_n: & 1 \arrow[r] \arrow[drr] & 2 \arrow[r] & \cdots & \arrow[r] & n\text{-}1 \,. \\
& & & 0 \arrow[urr] & &
\end{tikzcd} \tag{4.1}$$

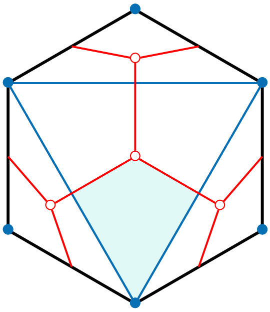

Fig. 11 The full dual of a triangulation

Denote by $\mathrm{Br}(Q)$ the braid group associated to Q, generated by $\mathbf{b} = \{b_i \mid i \in Q_0\}$ subject to the relations

$$\begin{cases} b_j b_i b_j = b_i b_j b_i, & \text{there is exactly one arrow between } i \text{ and } j \text{ in } Q, \\ b_i b_j = b_j b_i, & \text{otherwise.} \end{cases}$$

Let Z_0^{BT} be the center of $\mathrm{BT}(\mathbf{S}_\triangle)$ and $\mathrm{BT}_*(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{S}_\triangle)/Z_0^{\mathrm{BT}}$.

- If $\mathbf{S}_\triangle$ is a polygon, then $\mathrm{BT}(\mathbf{S}_\triangle) \cong \mathrm{Br}(A_n)$ and Z_0^{BT} is the infinite cyclic group generated by $\mathrm{D}_{\partial \mathbf{S}_\triangle}$.
- If $\mathbf{S}_\triangle$ is an annulus, then $\mathrm{BT}(\mathbf{S}_\triangle) \cong \mathrm{Br}(\widetilde{A_n})$ and $Z_0^{\mathrm{BT}} = 1$ ([5]).

We will show that $Z_0^{\mathrm{BT}} = 1$ holds for all other cases. Denote the boundary components of $\mathbf{S}_\triangle$ by ∂_j, $1 \le j \le |\partial \mathbf{S}|$.

Lemma 4.6 *By cutting along the (initial) closed arcs in* $\mathbf{T}_0^*$, $\mathbf{S}_\triangle$ *will be divided into* m *annuli* $\mathbf{A}_i$, $1 \le i \le m$, *such that* ∂_i *is a boundary component of* $\mathbf{A}_i$.

Proof For each boundary segment $\gamma \subset \partial \mathbf{S}_\triangle$ that is in a triangle T in $\mathbf{T}_0$ with decorating point Z, denote by γ^* its dual, which is the simple arc in T (unique up to isotopy) connecting Z and the midpoint of γ. Call the union of $\mathbf{T}_0^*$ and the arcs γ^* as above (for all segments γ in $\partial \mathbf{S}_\triangle$) the full dual of $\mathbf{T}_0$. Denote it by $\widehat{\mathbf{T}_0^*}$, see Fig. 11 for example. □

Then the surface $\mathbf{S}_\triangle - \widehat{\mathbf{T}_0^*}$ obtained from $\mathbf{S}_\triangle$ by cutting along all arcs in $\widehat{\mathbf{T}_0^*}$ satisfies the following:

- it consists of $|\mathbf{M}|$ connected components, each of which contains exactly one marked point in $\mathbf{M}$;
- each component is a disk, since it can be obtained by gluing many quadrilaterals (cf. the shaded area in Fig. 11) along some segment containing the marked point in that component.

Further, by gluing back along the arcs dual to boundary segments in $\mathbf{S}_\triangle$, we deduce that the surface $\mathbf{S}_\triangle - \mathbf{T}_0^*$ obtained from $\mathbf{S}_\triangle$ by cutting along all arcs in $\mathbf{T}_0^*$ satisfies the following:

- it consists of $|\partial|$ connected components;
- each component $\mathbf{A}_i$ is an annulus, such that one of the boundary components of $\mathbf{A}_i$ is a boundary component of $\mathbf{S}_\triangle$.

Thus the lemma follows. □

Proposition 4.7 *If $\mathbf{S}_\triangle$ is neither a polygon nor an annulus, then* $Z_0^{\mathrm{BT}} = 1$.

Proof Denote by $\mathrm{D}(\partial\mathbf{S}_\triangle)$ the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by the Dehn twist $\{\mathrm{D}_{\partial_i}\}$ of its boundary components.

We claim that $Z_0^{\mathrm{BT}} \subset \mathrm{D}(\partial\mathbf{S}_\triangle)$. Let $z \in Z_0^{\mathrm{BT}}$. Then $z \circ \mathrm{B}_\eta = \mathrm{B}_\eta \circ z$ for any $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$. Hence by (3.3) we have

$$\mathrm{B}_{z(\eta)} = z \circ \mathrm{B}_\eta \circ z^{-1} = \mathrm{B}_\eta. \tag{4.2}$$

Thus $z(\eta) = \eta$ for any $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, which in particular implies that z preserves $\triangle$ pointwise (note that $|\triangle| = \aleph \geq 3$ in our situation) and $\mathbf{T}_0^*$. By Lemma 4.6, cutting along closed arcs in $\mathbf{T}_0^*$ divides $\mathbf{S}_\triangle$ into m annuli $\mathbf{A}_i$, such that ∂_i is a boundary component of $\mathbf{A}_i$. Since z preserves all such closed arcs, it can be realized as composition of some element $z_i \in \mathrm{MCG}(\mathbf{A}_i)$ (where the order of the composition does not matter since they commute with each other). Note that $\mathrm{MCG}(\mathbf{A}_i)$ is generated by D_{∂_i}, which implies $z \in \mathrm{D}(\partial\mathbf{S}_\triangle)$. Thus the claim holds.

There is also the subgroup $\mathrm{D}(\partial\mathbf{S})$ of $\mathrm{MCG}(\mathbf{S})$ generated by the Dehn twist along its boundary components and the obvious induced map $F_*\left(\mathrm{D}(\partial\mathbf{S}_\triangle)\right) = \mathrm{D}(\partial\mathbf{S})$, which sends D_{∂_i} to D_{∂_i}. Since $\mathbf{S}$ is not a polygon, $\{\mathrm{D}_{\partial_i}\}$ are non-trivial in both $\mathrm{MCG}(\mathbf{S}_\triangle)$ and $\mathrm{MCG}(\mathbf{S})$. Further, since $\mathbf{S}$ is not an annulus, $\{\mathrm{D}_{\partial_i}\}$ are distinct (and commute with each other). Therefore, $F_* \colon \mathrm{D}(\partial\mathbf{S}_\triangle) \to \mathrm{D}(\partial\mathbf{S})$ is an isomorphism.

Now combining $Z_0^{\mathrm{BT}} \subset \mathrm{D}(\partial\mathbf{S}_\triangle)$ and (3.6), we deduce that $F_*(Z_0^{\mathrm{BT}}) = 1$ in $\mathrm{MCG}(\mathbf{S})$ and hence $Z_0^{\mathrm{BT}} = 1$ in $\mathrm{MCG}(\mathbf{S}_\triangle)$. □

5 From closed arcs to perfect objects

5.1 The Koszul dual and minimal model

Let $\Gamma_{\mathbf{T}} = \Gamma(Q_{\mathbf{T}}, W_{\mathbf{T}})$ be the Ginzburg dg algebra obtained from a triangulation $\mathbf{T}$. Recall that there is a canonical heart $\mathcal{H}_{\mathbf{T}}$ in $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ and let

$$S_{\mathbf{T}} = \bigoplus_{S \in \mathrm{Sim}\,\mathcal{H}_{\mathbf{T}}} S$$

be the direct sum of the simples in $\mathcal{H}_{\mathbf{T}}$. Consider the (dg) endomorphism algebra $\mathfrak{E}_{\mathbf{T}} = \mathrm{RHom}(S_{\mathbf{T}}, S_{\mathbf{T}})$. By [12], we have the following derived equivalence:

$$\mathcal{D}_{fd}(\Gamma_{\mathbf{T}}) \underset{?\otimes^{\mathbf{L}}_{\mathfrak{E}_{\mathbf{T}}} S_{\mathbf{T}}}{\overset{\mathrm{RHom}_{\Gamma_{\mathbf{T}}}(S_{\mathbf{T}},?)}{\rightleftarrows}} \mathrm{per}\,\mathfrak{E}_{\mathbf{T}}, \tag{5.1}$$

In particular, $\{S\}_{S\in\mathcal{H}_\Gamma}$ in $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ become (indecomposable) projectives in $\operatorname{per}\mathfrak{E}_{\mathbf{T}}$. By [11, Sect. A.15], the multiplications in the A_∞-structure of the homology of $\mathfrak{E}_{\mathbf{T}}$ are induced from differentials of $\Gamma_{\mathbf{T}}$. In particular, when $m \geq 3$, the m-multiplications are induced from the $(m+1)$-cycle in the potential $W_{\mathbf{T}}$, which vanish in our case (since we only have 3-cycles in the potential). This means that $\mathfrak{E}_{\mathbf{T}}$ is formal and hence is quasi-isomorphic to its homology (the minimal model), denoted by

$$\mathcal{E}_{\mathbf{T}} = \operatorname{Hom}^\bullet(S_{\mathbf{T}}, S_{\mathbf{T}}). \tag{5.2}$$

which is just a graded algebra. We will identify $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ with $\operatorname{per}\mathcal{E}_{\mathbf{T}}$ when there is no confusion. Recall that the Ext quiver $\mathcal{Q}(\mathcal{H})$ of a finite heart $\mathcal{H}$ is the (positively) graded quiver whose vertices are the simples of $\mathcal{H}$ and whose graded edges correspond to a basis of $\operatorname{End}^{>0}(\bigoplus_{S\in\operatorname{Sim}\mathcal{H}} S)$.

Example 5.1 The Ext quiver of the canonical heart (in the corresponding 3-CY category) of the quiver with potential in Example 2.1 is shown as follows.

$$\begin{tikzcd}
& S_2 \arrow[loop above, "3"] \arrow[dl, shift left, "2"] \arrow[dr, shift left, "1"] & \\
S_1 \arrow[ur, shift left, "1"] \arrow[rr, shift left, "2"] \arrow[loop left, "3"] & & S_3 \arrow[ul, shift left, "2"] \arrow[ll, shift left, "1"] \arrow[loop right, "3"]
\end{tikzcd} \tag{5.3}$$

Moreover, the differentials in (2.2) become the following multiplications:

$$\begin{aligned} \operatorname{Hom}^1(S_{i-1}, S_i) \otimes \operatorname{Hom}^1(S_i, S_{i+1}) &\cong \operatorname{Hom}^2(S_{i-1}, S_{i+1}), \\ \operatorname{Hom}^k(S_i, S_{i+k}) \otimes \operatorname{Hom}^{3-k}(S_{i+k}, S_i) &\cong \operatorname{Hom}^3(S_i, S_i), \end{aligned} \tag{5.4}$$

for $i = 1, 2, 3$ and $k = 0, 1, 2, 3$, where $S_{i+3} = S_i$ for $i \in \mathbb{Z}$.

Notations 5.2 *Recall that we have fixed an initial triangulation* $\mathbf{T}_0$.

- *We will write* Γ_0 *for* $\Gamma_{\mathbf{T}_0}$ *and similar for* $\mathcal{E}_{\mathbf{T}_0}$, $\mathcal{H}_{\mathbf{T}_0}$ *and so on.*
- *Let the* $\Gamma^i = e_i\Gamma_0$ *be the indecomposable projective summands of* Γ_0.
- *Let* $S_1, \ldots, S_n$ *be the simples in* $\mathcal{H}_0$ *which correspond to the projectives above. Under the derived equivalence (5.1), the* S_i *become the (projective) summands of the silting objects* $\mathcal{E}_0$ *in* $\operatorname{per}\mathcal{E}_0 \cong \mathcal{D}_{fd}(\Gamma_0)$.

5.2 The string model

Definition 5.3 A dg $\mathcal{E}_0$-module X is *minimal perfect* if its underlying graded module (denoted by $|X|$) is of the form

$$|X| = \bigoplus_{k=1}^{l} X_k, \tag{5.5}$$

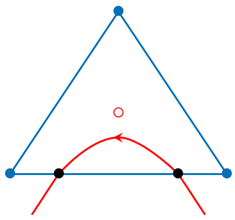

Fig. 12 A digon intersected by some γ and η

where each X_k is a finite direct sum of shifted copies of direct summands of $\mathcal{E}_0$ (i.e. copies of S_j) whose differential, as a degree 1 map from X to itself, is a strictly upper triangular matrix whose entries are in the ideal of $\mathcal{E}_0$ generated by the arrows in $\mathcal{Q}(\mathcal{H}_0)$.

Let η be a general closed arc in $\mathbf{S}_\triangle$ such that it is in a minimal position w.r.t. $\mathbf{T}_0$. This is equivalent to saying that there is no digon shown as in Fig. 12. One can associate a minimal perfect dg $\mathcal{E}_0$ module X_η as follows

- its underlying graded module $|X_\eta|$ has the form as in (5.5).
- Let the endpoints of η be Z and Z'. Suppose that from Z to Z', η intersects $\mathbf{T}_0$ at $V_1, \dots, V_m$ accordingly, where V_i is in the arc $\gamma_{j_i} \in \mathbf{T}_0$ for $1 \le i \le m$ and some $1 \le j_i \le n$ (cf. Fig. 13). Note that since when choose η in a minimal position w.r.t. $\mathbf{T}_0$, j_i are independent of the choice of η (only depend the isotopy class of η).
- Each line segment $V_i V_{i+1}$ in η induces a graded arrow a_i between V_i and V_{i+1} (clockwise within the corresponding triangle). See Fig. 14 for how an edge a in the Ext quiver $\mathcal{Q}(\mathcal{H}_0)$ induces such a graded arrow a between V and W respectively. Then we obtain a string H_η, whose vertices are V_i's and whose (graded) arrows are those induced arrows.

$$H_\eta : \quad V_1 \overset{a_1}{\text{——}} V_2 \overset{a_2}{\text{——}} \cdots \text{——} V_{m-1} \overset{a_{m-1}}{\text{——}} V_m$$

- Each intersection V_i corresponds to a summand $S_{j_i}[\delta_i]$ in some X_{δ_i} for some integer δ_i. So we have

$$\bigoplus_{k=1}^{l} X_k = \bigoplus_{i=1}^{m} S_{j_i}[\delta_i].$$

- For the arrow a_i, we have two cases:

  1°. If its orientation is $V_i \to V_{i+1}$, then the degrees δ_i satisfy

$$\delta_{i+1} = \delta_i + \deg a_i - 1. \tag{5.6}$$

  Moreover, the map $S_{j_i} \to S_{j_{i+1}}[\deg a_i]$ corresponding to a_i induces a degree 1 map

$$d_{a_i} : S_{j_i}[\delta_i] \xrightarrow{1} S_{j_{i+1}}[\delta_{i+1}].$$

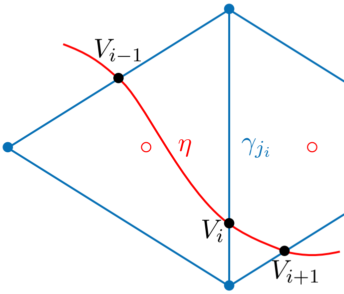

Fig. 13 The intersections between η and $\mathbf{T}_0$

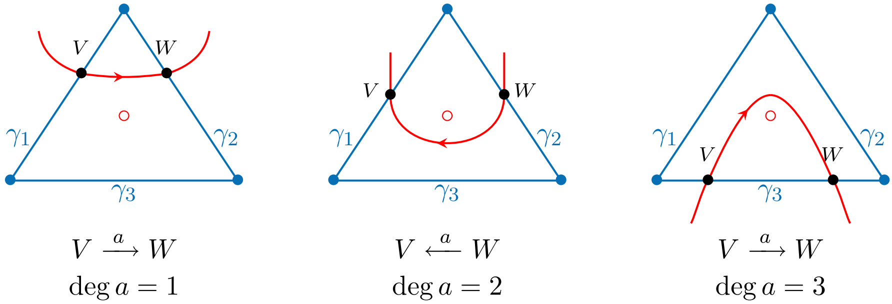

Fig. 14 Inducing graded arrows

2°. If its orientation is $V_i \leftarrow V_{i+1}$, then the degrees δ_i satisfy

$$\delta_i = \delta_{i+1} + \deg a_i - 1. \tag{5.7}$$

Moreover, the map $S_{j_{i+1}} \to S_{j_i}[\deg a_i]$ corresponding to a_i induces a degree 1 map

$$d_{a_i} : S_{j_{i+1}}[\delta_{i+1}] \xrightarrow{1} S_{j_i}[\delta_i].$$

- Finally, the differential d_η of X_η is given by the degree 1 map

$$\mathrm{d}_\eta = \sum_{i=1}^{m-1} d_{a_i}.$$

Lemma 5.4 *The complex X_η above is well-defined.*

Proof We only need to check $\mathrm{d}_\eta^2 = 0$, i.e. $d_{a_{i+1}} d_{a_i} = 0$ and $d_{a_i} d_{a_{i+1}} = 0$ for any i (when they make sense in d_η^2).

On one hand, since η is in a minimal position w.r.t. $\mathbf{T}_0$ and any two triangles in $\mathbf{T}_0$ share at most one edge, we deduce that for any i, V_{i-1}, V_i and V_{i+1} are not in a single triangle of $\mathbf{T}_0$.

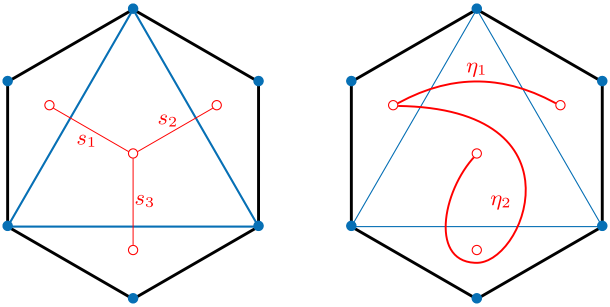

Fig. 15 An initial triangulation and two closed arcs in a 6-gon

On the other hand, if $d_{a_{i+1}} d_{a_i} \neq 0$, then there is a non-zero multiplication in

$$\begin{aligned}&\mathrm{Hom}^\bullet(S_{j_{i-1}}[\delta_{i-1}], S_{j_i}[\delta_i])\\&\quad\otimes \mathrm{Hom}^\bullet(S_{j_i}[\delta_i], S_{j_{i+1}}[\delta_{i+1}]) \to \mathrm{Hom}^\bullet(S_{j_{i-1}}[\delta_{i-1}], S_{j_{i+1}}[\delta_{i+1}]).\end{aligned}$$

As such a multiplication is induced from terms in the potential ([11, Sect. A.15]), which are 3-cycles, we deduce that V_{i_1}, V_i and V_{i+1} are in a single triangle of $\mathbf{T}_0$. This contradicts the fact mentioned above. The case when $d_{a_i} d_{a_{i+1}} = 0$ is similar.

Now we deduce that $\mathrm{d}_\eta^2 = 0$ as required. □

Remark 5.5 As we are flexible about the choice of δ_1 and t_1, X_η is well-defined up to shifts. In other words, we obtain a map

$$\begin{aligned}\widetilde{X}\colon \overline{\mathrm{CA}}(\mathbf{S}_\triangle) &\to \mathrm{per}\mathcal{E}_0/[1],\\ \eta &\mapsto \widetilde{X}(\eta).\end{aligned} \tag{5.8}$$

We will use the convention that X_η will be a representative in the shift orbits $\widetilde{X}(\eta)$ and the $X[\mathbb{Z}]$ denotes the shift orbit that contains X.

Example 5.6 By construction, $\widetilde{X}(s_i) = S_i[\mathbb{Z}]$, where the s_i are the 'initial' closed arcs in $\mathbf{T}_0^*$ and S_i are the simples in the canonical heart $\mathcal{H}_0$. Let us have a look at some non-trivial case. Take an initial triangulation of a 6-gon as shown in the left picture in Fig. 15.

The Ext-quiver of $\mathcal{H}_0$ is as shown in Example 5.1. Then we have

$$\widetilde{X}(\eta_1) = \mathrm{Cone}(S_1 \to S_2[1])[\mathbb{Z}], \quad \widetilde{X}(\eta_2) = \mathrm{Cone}(X \to S_3[3])[\mathbb{Z}],$$

where

$$X = \mathrm{Cone}(S_1[-2] \to S_3).$$

Here, the maps in the Cone are the unique maps (up to scaling) between the corresponding objects.

Lemma 5.7 *Let X_η be a complex associated to a closed arc η as above. Then*

$$\dim \operatorname{Hom}^\bullet(\Gamma^i, X_\eta) = \operatorname{Int}(\gamma_i, \eta), \tag{5.9}$$

$$\dim \operatorname{Hom}^\bullet(\Gamma_0, X_\eta) = \sum_{i=1}^{n} \dim \operatorname{Hom}^\bullet(\Gamma^i, X_\eta) = \operatorname{Int}(\mathbf{T}_0, \eta). \tag{5.10}$$

Proof First, the projective-simple duality implies

$$\operatorname{Hom}^j(\Gamma^i, S_k[l]) = \delta_{ik} \cdot \delta_{jl} \cdot \mathbf{k}, \quad 1 \le i, k \le n;\ \forall j, l \in \mathbb{Z}. \tag{5.11}$$

Second, the differential d_η is generated by the morphisms $\varsigma\colon S_i \to S_j[\delta]$ in (5.2), which satisfy $\operatorname{Hom}^\bullet(\Gamma^i, \varsigma) = 0, \quad 1 \le i \le n$. Thus the lemma follows. □

In particular, we have the following immediate consequence as the s_i are the only closed arcs that intersect once with $\mathbf{T}$.

Corollary 5.8 *If $X_\eta[\mathbb{Z}] = S_i[\mathbb{Z}]$ for some initial closed arc $s_i \in \mathbf{T}_0^*$, then $\eta = s_i$.*

We will prove the following key proposition in Sect. 9.

Proposition 5.9 *Let η_1 and η_2 be two general closed arcs in $\overline{\operatorname{CA}}(\mathbf{S}_\triangle)$. Choose any representative X_k in $\widetilde{X}(\eta_k) = X_k[\mathbb{Z}]$. Then we have*

1°. If η_k is a closed arc, i.e. is in $\operatorname{CA}(\mathbf{S}_\triangle)$, then X_{η_k} is in $\operatorname{Sph}(\Gamma_0)$.
2°. If $\operatorname{Int}(\eta_1, \eta_2) = 0$, then

$$\operatorname{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = 0. \tag{5.12}$$

3°. If $\operatorname{Int}(\eta_1, \eta_2) = \frac{1}{2}$, then

$$\dim \operatorname{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = 1. \tag{5.13}$$

An immediate consequence of Proposition 5.9 and Proposition 9.2 is as follows.

Corollary 5.10 *Let $\alpha, \beta \in \operatorname{CA}(\mathbf{S}_\triangle)$ with $\operatorname{Int}(\alpha, \beta) = \frac{1}{2}$ and $\eta = \mathrm{B}_\alpha(\beta)$. Then*

$$\widetilde{X}(\eta) = \phi_{\widetilde{X}(\alpha)}(\widetilde{X}(\beta)). \tag{5.14}$$

6 Braid twists versus spherical twists

6.1 Two twist group actions

We start with a generalization of Corollary 5.10.

Lemma 6.1 *For any $s \in \mathbf{T}_0^*$ and $\eta \in \operatorname{CA}(\mathbf{S}_\triangle)$, we have*

$$\phi^{\varepsilon}_{\widetilde{X}(s)}\left(\widetilde{X}(\eta)\right) = \widetilde{X}\left(\mathrm{B}_s^{\varepsilon}(\eta)\right), \tag{6.1}$$

where $\varepsilon \in \{\pm 1\}$.

Proof Without loss of generality, we only deal the case for $\varepsilon = 1$. Use induction on $\mathrm{Int}(\mathbf{T}_0, \eta)$ starting with the trivial case when $\mathrm{Int}(\mathbf{T}_0, \eta) = 1$, or equivalently, $\eta \in \mathbf{T}_0^*$. Now, for the inductive step, consider η with $\mathrm{Int}(\mathbf{T}_0, \eta) = m$ while the lemma holds for any η' with $\mathrm{Int}(\mathbf{T}_0, \eta) < m$. Applying Lemma 3.14, we have $\eta = \mathrm{B}_\alpha(\beta)$ for some α, β with $\mathrm{Int}(\alpha, \beta) = \frac{1}{2}$. Twisted by B_s, we have $\mathrm{Int}(\mathrm{B}_s(\alpha), \mathrm{B}_s(\beta)) = \frac{1}{2}$ and $\mathrm{B}_s(\eta) = \mathrm{B}_{\mathrm{B}_s(\alpha)}(\mathrm{B}_s(\beta))$. By (5.14), we have

$$\widetilde{X}\left(\mathrm{B}_s(\eta)\right) = \phi_{\widetilde{X}(\mathrm{B}_s(\alpha))}\left(\widetilde{X}\left(\mathrm{B}_s(\beta)\right)\right). \tag{6.2}$$

By the inductive assumption,

$$\phi_{\widetilde{X}(s)}\left(\widetilde{X}(\alpha)\right) = \widetilde{X}\left(\mathrm{B}_s(\alpha)\right), \quad \phi_{\widetilde{X}(s)}\left(\widetilde{X}(\beta)\right) = \widetilde{X}\left(\mathrm{B}_s(\beta)\right). \tag{6.3}$$

So

$$\begin{aligned}
\phi_{\widetilde{X}(s)}\left(\widetilde{X}(\eta)\right) &= \phi_{\widetilde{X}(s)}\left(\phi_{\widetilde{X}(\alpha)}(\widetilde{X}(\beta))\right) \\
&= \phi_{\widetilde{X}(s)} \circ \phi_{\widetilde{X}(\alpha)} \circ \phi^{-1}_{\widetilde{X}(s)}\left(\phi_{\widetilde{X}(s)}(\widetilde{X}(\beta))\right) \\
&= \phi_{\phi_{\widetilde{X}(s)}(\widetilde{X}(\alpha))}\left(\phi_{\widetilde{X}(s)}(\widetilde{X}(\beta))\right) \\
&= \phi_{\widetilde{X}(\mathrm{B}_s(\alpha))}\left(\widetilde{X}\left(\mathrm{B}_s(\beta)\right)\right) \\
&= \widetilde{X}\left(\mathrm{B}_s(\eta)\right),
\end{aligned}$$

where the first equality follows from (5.14), the third equality follows from (2.5), the fourth equality follows from (6.3) and the last equality follows from (6.2), which completes the proof. □

Remark 6.2 Let $Z_0^{\mathrm{ST}} = \mathrm{ST}(\Gamma_0) \cap \mathbb{Z}[1]$ and

$$\mathrm{ST}_*(\Gamma_0) = \mathrm{ST}(\Gamma_0)/Z_0^{\mathrm{ST}} \subset \mathrm{Aut}^\circ\mathcal{D}_{fd}(\Gamma_0)/\mathbb{Z}[1].$$

Note that $\mathrm{ST}_*(\Gamma_0)$ also acts on $\mathrm{Sph}(\Gamma_0)/[1]$. By [3, Theorem 4.4], $Z_0^{\mathrm{ST}} = 1$ unless $\mathbf{S}$ is a polygon, in which case, $Z_0^{\mathrm{ST}} = \mathbb{Z}[n+3]$.

Recall that the initial triangulation consists of closed arcs s_i, whose braid twists $b_i = \mathrm{B}_{s_i}$ generate $\mathrm{BT}(\mathbf{T}_0) = \mathrm{BT}(\mathbf{S}_\triangle)$ by Lemma 4.2. Moreover, the canonical heart $\mathcal{H}_0$ in $\mathcal{D}_{fd}(\Gamma_0)$ has simples S_i satisfying $S_i[\mathbb{Z}] = \widetilde{X}(s_i)$, whose spherical twists $\phi_i = \phi_{S_i}$ generate $\mathrm{BT}(\mathbf{S}_\triangle)$.

Proposition 6.3 *There is a canonical group homomorphism*

$$\iota\colon \mathrm{BT}(\mathbf{T}_0) \to \mathrm{ST}_*(\Gamma_0), \tag{6.4}$$

sending the generator b_i to the generator ϕ_i.

Proof Consider first the case when $\mathbf{S}$ is not a polygon. We only need to prove that, if

$$b = b_{i_1}^{\varepsilon_1} \circ \cdots \circ b_{i_k}^{\varepsilon_k} \tag{6.5}$$

equals 1 in $\mathrm{MCG}(\mathbf{S}_\triangle)$, for some $i_j \in \{1, \ldots, n\}$, $\varepsilon_j \in \{\pm 1\}$, $1 \le j \le k$ and $k \in \mathbb{N}$, then

$$\phi = \phi_{i_1}^{\varepsilon_1} \circ \cdots \circ \phi_{i_l}^{\varepsilon_l} \tag{6.6}$$

equals 1 in $\mathrm{Aut}^\circ \mathcal{D}_{fd}(\Gamma_0)$.

First, $b = 1$ implies $b(s_i) = s_i$ for any $1 \le i \le n$. By (repeatedly using) Lemma 6.1, we have

$$\widetilde{X}\,(b(s_i)) = \phi\left(\widetilde{X}(s_i)\right).$$

Thus, $S_i[\mathbb{Z}] = \widetilde{X}\,(s_i) = \phi\,(S_i[\mathbb{Z}])$, i.e. $\phi(S_i) = S_i[t_i]$ for some integer t_i. Since ϕ is an equivalence, we deduce that all t_i must be the same. Therefore $\phi = [t]$ for some integer t. However, we have $\phi \in Z_0^{\mathrm{ST}} = 1$ in this case, which implies $t = 0$ and $\phi = 1$ in $\mathrm{Aut}^\circ \mathcal{D}_{fd}(\Gamma_0)$, as required.

In the case when $\mathbf{S}_\triangle$ is a polygon, $b = 1$ still implies $\phi = [t]$ for some $t \in \mathbb{Z}$ and thus the proposition holds too. $\square$

A consequence of the existence of ι is that the braid twist group actions $\mathrm{BT}(\mathbf{S}_\triangle)$ on $\mathrm{CA}(\mathbf{S}_\triangle)$ are compatible with the spherical twist group actions $\mathrm{ST}_*(\Gamma_0)$ on $\mathrm{Sph}(\Gamma_0)/[1]$, under the map $\widetilde{X}$ in (5.8). More precisely, we have the commutative diagram below, where the commutativity is in the sense of (6.9) in the following corollary.

$$\begin{tikzcd} \mathrm{BT}(\mathbf{S}_\triangle) \arrow[r, "\iota"] \arrow[d, phantom, "\circlearrowright"] & \mathrm{ST}_*(\Gamma_0) \arrow[d, phantom, "\circlearrowright"] \\ \mathrm{CA}(\mathbf{S}_\triangle) \arrow[r, "\widetilde{X}"] & \mathrm{Sph}(\Gamma_0)/[1] \end{tikzcd} \tag{6.7}$$

Corollary 6.4 *For any $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ and $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, we have*

$$\iota(\mathrm{B}_\eta^\varepsilon) = \phi_{\widetilde{X}(\eta)}^\varepsilon, \quad \varepsilon \in \{\pm 1\} \tag{6.8}$$

$$\widetilde{X}\,(b(\eta)) = \iota(b)\left(\widetilde{X}(\eta)\right). \tag{6.9}$$

Proof Again, we will only deal with the case when $\varepsilon = 1$. By Proposition 4.4, $\eta = b(s_j)$ for some $s_j \in \mathbf{T}^*$ and $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ with the form (6.5). Let ϕ be as in (6.6) and by (repeatedly using) (6.1), we have

$$\widetilde{X}(\eta) = \widetilde{X}(b(s_j)) = \phi(\widetilde{X}(s_j)) = \phi(S_j).$$

Then using formulae (3.3), (2.5) and the equality above we have

$$\begin{aligned} \iota(\mathrm{B}_\eta) &= \iota(\mathrm{B}_{b(s)}) \\ &= \iota\left(b_{i_1}^{\varepsilon_1} \circ \cdots \circ b_{i_k}^{\varepsilon_k} \circ \mathrm{B}_{s_j} \circ b_{i_1}^{-\varepsilon_1} \circ \cdots \circ b_{i_k}^{-\varepsilon_k}\right) \end{aligned}$$

$$\begin{aligned}&= \iota(b_{i_1}^{\varepsilon_1}) \circ \cdots \circ \iota(b_{i_k}^{\varepsilon_k}) \circ \iota(b_j) \circ \iota(b_{i_1}^{-\varepsilon_1}) \circ \cdots \circ \iota(b_{i_k}^{-\varepsilon_k})\\ &= \phi_{i_1}^{\varepsilon_1} \circ \cdots \circ \phi_{i_k}^{\varepsilon_k} \circ \phi_j \circ \phi_{i_1}^{-\varepsilon_1} \circ \cdots \circ \phi_{i_k}^{-\varepsilon_k}\\ &= \phi \circ \phi_j \circ \phi^{-1}\\ &= \phi_{\phi(S_j)} = \phi_{\widetilde{X}(\eta)},\end{aligned}$$

i.e. (6.8). A similar calculation gives (6.9), as the generalization of (6.1). □

When specifying $b = \mathrm{B}_s^{\varepsilon}$ in (6.9) and using (6.8), we see that (6.1) holds for any $s, \eta \in \mathrm{CA}(\mathbf{S}_\triangle)$.

Corollary 6.5 (6.1) *holds for any* $s, \eta \in \mathrm{CA}(\mathbf{S}_\triangle)$.

Now, we are ready to prove the main theorem of this paper.

6.2 The main result

We start to show that $\widetilde{X}$ is bijective.

Theorem 6.6 *The map* $\widetilde{X}$ *in* (5.8) *induces a bijection*

$$\widetilde{X} \colon \mathrm{CA}(\mathbf{S}_\triangle) \xrightarrow{\mathit{1\text{-}1}} \mathrm{Sph}(\Gamma_0)/[1].$$

Proof First we prove the injectivity. Suppose $\widetilde{X}(\eta) = \widetilde{X}(\eta')$ for $\eta, \eta' \in \mathrm{CA}(\mathbf{S}_\triangle)$. Let $\eta = b(s_i)$ for some $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ and initial closed arc $s_i \in \mathbb{T}_0^*$. Then by (6.9) we have

$$S_i[\mathbb{Z}] = \widetilde{X}(s_i) = \widetilde{X}(b^{-1}(\eta)) = \iota(b)^{-1}\left(\widetilde{X}(\eta)\right) = \iota(b)^{-1}\left(\widetilde{X}(\eta')\right) = \widetilde{X}(b^{-1}(\eta')).$$

By Corollary 5.8, $s_i = b^{-1}(\eta')$ or $\eta = \eta'$ as required.

Second, we prove the surjectivity. Let η be a closed arc in $\mathrm{CA}(\mathbf{S}_\triangle)$ and $\widetilde{X}(\eta) = X_\eta[\mathbb{Z}]$ for some representative X_η. We only need to show that X_η is in $\mathrm{Sph}(\Gamma_0)$. Use induction on $I = \mathrm{Int}(\mathbf{T}_0, \eta)$. If $I = 1$, then η is some $s_i \in \mathbf{T}_0$ and $X_\eta = S_i[\delta]$ for some integer δ, which is in $\mathrm{Sph}(\Gamma_0)$. Now suppose that the claim is true for $I \le r$ for some $r \ge 1$ and consider the case when $I = r+1$. Apply Lemma 3.14, we find α and β with $\mathrm{Int}(\alpha, \beta) = \frac{1}{2}$ and (3.4). By Corollary 5.10, we have representatives X_α and X_β with (9.3). By the inductive assumption, we know that X_α and X_β are in $\mathrm{Sph}(\Gamma_0)$. On the other hand, we have $\phi_{X_\alpha} \in \mathrm{ST}(\Gamma_0)$ by (2.5) and the theorem follows from (2.6). □

We proceed to show that the bijectivity above implies isomorphism between twisted groups.

Theorem 6.7 *Let* $\mathbf{S}$ *be an unpunctured marked surface and* $\mathbf{T}_0$ *a triangulation of* $\mathbf{S}_\triangle$ *such that the corresponding FST'quiver has no double arrows. Then there is a canonical isomorphism*

$$\iota \colon \mathrm{BT}(\mathbf{T}_0) \to \mathrm{ST}(\Gamma_0), \tag{6.10}$$

sending the generator b_i to the generator ϕ_i, where Γ_0 is the Ginzburg dg algebra associated to $\mathbf{T}_0$.

Proof When $\mathbf{S}$ is a polygon, this follows from [17] and [22]. Now suppose $\mathbf{S}$ is not a polygon. We first prove the case for the initial triangulation $\mathbf{T}_0$ (whose FST quiver has no double arrows). Then $\mathrm{ST}_*(\Gamma_0) = \mathrm{ST}(\Gamma_0)$. In this case, we have the surjective homomorphism ι in (6.4) and only need to show that it is injective.

Let $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ with $\iota(b) = 1$ in $\mathrm{ST}(\Gamma_0)$. By (6.9), we have

$$\widetilde{X}\,(b(\eta)) = \iota(b)\left(\widetilde{X}(\eta)\right) = \widetilde{X}(\eta),$$

which implies $b(\eta) = \eta$ by Theorem 6.6, for any closed arc η. By (4.2), this implies $b \circ \mathrm{B}_\eta = \mathrm{B}_\eta \circ b$ and thus b is the center Z_0^{BT} of $\mathrm{BT}(\mathbf{S}_\triangle)$. But $Z_0^{\mathrm{BT}} = 1$ in this case. So $b = 1$ and ι is injective. □

Remark 6.8 We can generalize Theorem 6.7 to any triangulations $\mathbf{T} \in \mathrm{EG}^\circ(\mathbf{S}_\triangle)$, i.e. as Theorem 1. This follows by a standard induction, on the number of flips from $\mathbf{T}_0$ to $\mathbf{T}$; so we only need to prove the case when $\mathbf{T}$ is a flip of $\mathbf{T}_0$.

On one hand, $\mathbf{T}^*$ and $\mathbf{T}_0^*$ are related by a Whitehead move as in Fig. 10. Thus, the standard generators of $\mathrm{BT}(\mathbf{T})$ are conjugates of standard generators of $\mathrm{BT}(\mathbf{T}_0)$. It is straightforward to write down the formula of the conjugates. On the other hand, this is also true for $\mathrm{ST}(\Gamma_{\mathbf{T}})$ and $\mathrm{ST}(\Gamma_0)$. Namely,

- by [13], there is a (canonical) derived equivalence

$$\Psi\colon \mathcal{D}_{fd}(\Gamma_{\mathbf{T}}) \cong \mathcal{D}_{fd}(\Gamma_0),$$

  such that the canonical heart $\mathcal{H}_{\Gamma_{\mathbf{T}}}$ becomes a tilt $\mathcal{H}'$ (cf. [14, Definition 3.7]) of the canonical heart $\mathcal{H}_0$;
- [[14], Proposition 5.4] provides a formula for how simples change under tilting (i.e. each simple in $\mathcal{H}'$ is a twist or a shift of some simple in $\mathcal{H}$);
- then we deduce that under the induced isomorphism $\Psi_*\colon \mathrm{ST}(\Gamma_{\mathbf{T}}) \cong \mathrm{ST}(\Gamma_0)$, the standard generators of $\mathrm{ST}(\Gamma_{\mathbf{T}})$ become the conjugates of the standard generators of $\mathrm{ST}(\Gamma_0)$.

By comparing the two formulae of the conjugates, we deduce that (6.10) implies (1.2).

We will use the same trick again in Sect. 7 to prove the special cases in Remark 3.11, which completes the generalization from Theorem 6.7 to Theorem 1.

7 Special cases

In this section, we first deal with the two special cases in Remark 3.11. Then we discuss the affine $\widetilde{A}$ case in more detail.

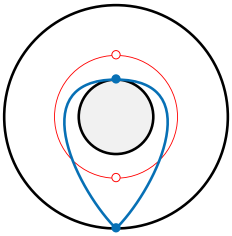

Fig. 16 The Kronecker case

7.1 The Kronecker case

We first discuss the special case I) in Remark 3.11. Note that in case I), all triangulations of $\mathbf{S}$ or $\mathbf{S}_\triangle$ look the same, cf. Fig. 16. Choose any triangulation $\mathbf{T}_0$ of $\mathbf{S}_\triangle$ as the initial triangulation. Keep all the notations as above.

The dynamic of proof here is the reverse compared with the previous cases: we will show the relation between the twist groups first; then the relations between closed arcs and spherical objects.

First, we claim that (6.10) also holds in this case.

Proposition 7.1 *Let $\mathbf{S}$ be an annulus with two marked points and $\mathbf{T}_0$ a triangulation of $\mathbf{S}_\triangle$. There is a canonical isomorphism*

$$\iota\colon \mathrm{BT}(\mathbf{T}_0) \to \mathrm{ST}(\Gamma_0), \tag{7.1}$$

sending the generator b_i to the generator ϕ_i, where Γ_0 is the Ginzburg dg algebra associated to $\mathbf{T}_0$.

Proof Consider an annulus $\mathbf{S}'_\triangle$ with triangulation $\mathbf{T}'_0$ (cf. left picture in Fig. 17), whose FST quiver is the affine quiver Q' of type $\widetilde{A_{1,2}}$:

$$\begin{tikzcd} & 3' \arrow[dl] & \\ 2' & & 1' \arrow[ul] \arrow[ll] \end{tikzcd}$$

We can choose another triangulation $\mathbf{T}'$, as shown in the right picture in Fig. 17, whose FST quiver is

$$\begin{tikzcd} & 3 \arrow[dr] & \\ 2 \arrow[ur] & & 1. \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

By Remark 6.8, we have (1.2) for $\mathbf{T}'$. On the other hand, we have the following two facts:

- the subcategory $\mathcal{D}_0$ of $\mathcal{D}_{fd}(\Gamma')$ generated by X'_1 and X'_2 is equivalent to the 3-CY category for a Kronecker quiver, where X'_i is the spherical object corresponding to s'_i;

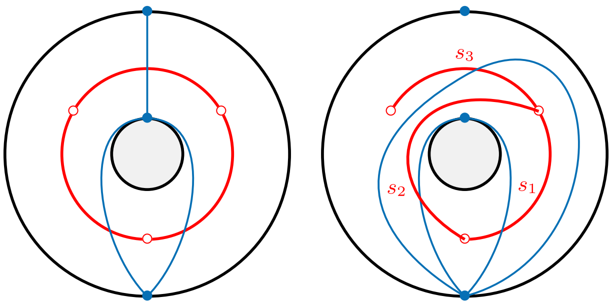

Fig. 17 $\widetilde{A_{1,2}}$ case: $\mathbf{T}'$ on the *left* and $\mathbf{T}'_0$ on the *right*

- there is a subsurface $\mathbf{Y}_\triangle$ of $\mathbf{S}'_\triangle$, with inherited triangulation from $\mathbf{T}'$ (whose dual consists of s'_1 and s'_2), that is isomorphic to any triangulation of an annulus with two marked points.

Therefore, by identifying $\mathcal{D}_{fd}(\Gamma_0)$ with $\mathcal{D}_0$ and $\mathbf{S}_\triangle$ with $\mathbf{Y}_\triangle$, we have

$$\mathrm{ST}(\Gamma_0) \cong \langle \phi_{X'_1}, \phi_{X'_2} \rangle \cong \langle \mathrm{B}_{s'_1}, \mathrm{B}_{s'_2} \rangle \cong \mathrm{BT}(\mathbf{T}_0),$$

which implies the proposition. $\square$

7.2 The one marked point torus case

In this section, we give the analogue of Proposition 7.1 for the special case II) in Remark 3.11. The proof is almost the same, by considering a torus with one boundary component and two marked points on it for instance.

Proposition 7.2 *Let* $\mathbf{S}$ *be a torus with one marked point and* $\mathbf{T}_0$ *a triangulation of* $\mathbf{S}_\triangle$. *There is a canonical isomorphism*

$$\iota\colon \mathrm{BT}(\mathbf{T}_0) \to \mathrm{ST}(\Gamma_0), \tag{7.2}$$

sending the generator b_i *to the generator* ϕ_i, *where* Γ_0 *is the Ginzburg dg algebra associated to* $\mathbf{T}_0$.

7.3 Example: annulus case

When $\mathbf{S}$ is an annulus, Theorem 6.7, (together with Proposition 7.1) can be stated as follows.

Theorem 7.3 *Let* $\mathbf{S}$ *be an annulus and* $\mathbf{T}$ *be a triangulation of* $\mathbf{S}_\triangle$ *with associated Ginzburg dg algebra* $\Gamma_\mathbf{T}$. *Suppose there are* p *and* q *marked points on the two boundary components of* $\mathbf{S}$, *respectively. Then the spherical twist group* $\mathrm{ST}(\Gamma_\mathbf{T})$ *is (canonically) isomorphic to the braid group* $\mathrm{Br}(\widetilde{A_{p,q}})$ *of affine* $\widetilde{A_{p,q}}$.

Proof The case $p = q = 1$ is Proposition 7.1, noticing that the braid group $\mathrm{Br}(\widetilde{A_{1,1}})$ is a rank 2 free group. The other case follows from Theorem 6.7, noticing that $\mathrm{BT}(\mathbf{S}_\triangle)$ is (canonically) isomorphic to $\mathrm{Br}(\widetilde{A_{p,q}})$ by the geometric description of the affine braid group in [5]. □

8 On the space of stability conditions

8.1 Stability conditions

First recall Bridgeland's notion of stability conditions.

Definition 8.1 (cf. [2]) A *stability condition* $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism (*the central charge*) $Z : K(\mathcal{D}) \to \mathbb{C}$ and full additive subcategories $\mathcal{P}(\varphi) \subset \mathcal{D}$ for each $\varphi \in \mathbb{R}$, satisfying the following axioms:

- if $0 \neq E \in \mathcal{P}(\varphi)$ then $Z(E) = m(E)\exp(\varphi\pi\mathbf{i})$ for some $m(E) \in \mathbb{R}_{>0}$;
- $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1]$, for all $\varphi \in \mathbb{R}$;
- if $\varphi_1 > \varphi_2$ and $A_i \in \mathcal{P}(\varphi_i)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$;
- for each nonzero object $E \in \mathcal{D}$ there is a finite sequence of real numbers

$$\varphi_1 > \varphi_2 > \ldots > \varphi_m$$

and a collection of triangles (the Harder-Narashimhan filtration)

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \ldots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_m \arrow[ul, dashed] &
\end{tikzcd} \tag{8.1}$$

with $A_j \in \mathcal{P}(\varphi_j)$ for all j.

Let I be an interval in $\mathbb{R}$ and define $\mathcal{P}(I)$ to be the subcategory generated by $\{\mathcal{P}(\varphi) \mid \varphi \in I\}$. The heart of a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is $\mathcal{P}[0, 1)$.

An important result by Bridgeland is that all stability conditions on a triangulated category $\mathcal{D}$ form a space $\mathrm{Stab}(\mathcal{D})$ that has the structure of a complex manifold. We are interested in the stability conditions on the 3-CY category $\mathcal{D}_{fd}(\Gamma)$ for a Ginzburg dg algebra Γ arising from quivers with potential. Note that for the stability conditions on $\mathcal{D}_{fd}(\Gamma)$ whose heart is the canonical heart $\mathcal{H}_\Gamma$ form a half open half closed n-cell $U(\mathcal{H}_\Gamma)$ in $\mathrm{Stab}\mathcal{D}_{fd}(\Gamma)$ (see [18]). Denote by $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma)$ the connected component of $\mathrm{Stab}\mathcal{D}_{fd}(\Gamma)$ that contains $U(\mathcal{H}_\Gamma)$.

8.2 Quadratic differentials

Recall that $\mathbf{S}$ is a marked surface with initial triangulation $\mathbf{T}_0$, associated Ginzburg dg algebra Γ_0 and $\mathrm{Aut}^\circ\mathcal{D}_{fd}(\Gamma_0)$ is defined as in (2.7). Denote by $\mathrm{Quad}_\heartsuit(\mathbf{S})$ is the moduli space of quadratic differentials on $\mathbf{S}$, in the sense of [2, Sect. 6]. The main result there is as follows.

Theorem 8.2 [2, Theorem 1.2] *As complex manifolds,* $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma_0)/\mathrm{Aut}^\circ \cong \mathrm{Quad}_\heartsuit(\mathbf{S})$.

For our purpose, we prefer to deal with the space $\mathrm{Quad}(\mathbf{S})$ of quadratic differentials on a fixed marked surface $\mathbf{S}$ instead of the moduli space. These two spaces of quadratic differentials differ by the symmetry of the marked mapping class group $\mathrm{MMCG}(\mathbf{S})$, i.e.

$$\mathrm{Quad}_\heartsuit(\mathbf{S}) = \mathrm{Quad}(\mathbf{S})/\mathrm{MMCG}(\mathbf{S}).$$

Here, $\mathrm{MMCG}(\mathbf{S})$ of a marked surface $\mathbf{S}$ is the group of isotopy classes of (orientation preserving) homeomorphisms of $\mathbf{S}$, where all homeomorphisms and isotopies are required to fix the set $\mathbf{M}$ of marked points as a set.

By [2, Theorem 9.9], there is the short exact sequence

$$1 \to \mathrm{ST}(\Gamma_0) \to \mathrm{Aut}^\circ\mathcal{D}_{fd}(\Gamma_0) \to \mathrm{MMCG}(\mathbf{S}) \to 1 \tag{8.2}$$

and the theorem above can be alternatively stated as: $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma_0)/\mathrm{ST} \cong \mathrm{Quad}(\mathbf{S})$. Thus there is a short exact sequence

$$1 \to \pi_1\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma_0) \to \pi_1\mathrm{Quad}(\mathbf{S}) \xrightarrow{\pi} \mathrm{ST}(\Gamma_0) \to 1. \tag{8.3}$$

8.3 On the contractibility

In this subsection, let $\mathbf{S}$ be an annulus with p and q marked points on its boundary components respectively.

Suppose first $p \neq q$. It is straightforward to calculate $\mathrm{MMCG}(\mathbf{S})$ in this case: it is generated by the two rotations along the two boundary components. More precisely, $\mathrm{MCG}(\mathbf{S})$ is the infinite cyclic group generated by the Dehn twist D_C along the only (up to isotopy) non-trivial simple closed curve in $\mathbf{S}$. The two rotations are the p-th and q-th roots of D_C, denoted by r_0 and r_1, respectively. Then $\mathrm{MMCG}(\mathbf{S})$ is the abelian group with generators r_0 and r_1 and with relation $r_0^p = r_1^q$, which fits into the following short exact sequence

$$1 \to \mathbb{Z}\langle r_0\rangle \to \mathrm{MMCG}(\mathbf{S}) \to \mathbb{Z}_q\langle r_1\rangle \to 1.$$

Besides $\underline{\xi} = r_0 \cdot r_1$ is the universal rotation that corresponds to [1].

Next, as shown in [2, Sect. 12.3],

$$\mathrm{Quad}_\heartsuit(\mathbf{S}) \cong \mathrm{Conf}^n(\mathbb{C}^*)/\mathbb{Z}_q, \tag{8.4}$$

where $\mathrm{Conf}^n(\mathbb{C}^*)$ denotes the configuration space of n distinct points in $\mathbb{C}^*$ and $\mathbb{Z}_q$ acts by multiplication by a q-th root of unity. By the description of $\mathrm{Br}(\widetilde{A_{p,q}})$ in [5], there is short exact sequence

$$1 \to \mathrm{Br}(\widetilde{A_{p,q}}) \to \pi_1\mathrm{Conf}^n(\mathbb{C}^*) \to \mathbb{Z} \to 1. \tag{8.5}$$

As $\mathrm{Quad}_{\heartsuit}(\mathbf{S})$ consists of differentials of the form

$$\Theta(z) = \prod_{i=1}^{n}(z - z_i)\frac{\mathrm{d}z^{\otimes 2}}{z^{p+2}}, \quad z_i \in \mathbb{C}^*, \quad z_i \neq z_j$$

and considered modulo the action of $\mathbb{C}$ rescaling z. Note that z_i corresponds to the decorating points in $\mathbf{S}_\triangle$, the rotation r_q becomes the $\mathbb{Z}_q$ symmetry at the origin and the rotation r_p becomes the $\mathbb{Z}_p$ symmetry at the infinity.

Thus, combining the short exact sequences above and the calculation of fundamental groups of spaces in (8.4), we have the commutative diagram (8.6), which implies the dashed short exact sequence.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d, dashed] & & \\
 & \mathrm{Br}(\widetilde{A_{p,q}}) \arrow[r, equal] \arrow[d] & \mathrm{Br}(\widetilde{A_{p,q}}) \arrow[r] \arrow[d, dashed] & 1 \arrow[d] & \\
1 \arrow[r] & \pi_1\,\mathrm{Conf}^n(\mathbb{C}^*) \arrow[r] \arrow[d] & \pi_1\,\mathrm{Quad}_{\heartsuit}(\mathbf{S}) \arrow[r] \arrow[d, dashed] & \mathbb{Z}_q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathrm{MMCG}(\mathbf{S}) \arrow[r] \arrow[d, dashed] & \mathbb{Z}_q \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd} \tag{8.6}$$

Therefore we have $\pi_1\mathrm{Quad}(\mathbf{S}) = \mathrm{Br}(\widetilde{A_{p,q}})$ and hence $\pi_1\mathrm{Quad}(\mathbf{S}) \cong \mathrm{ST}(\Gamma_0)$ by Theorem 7.3. Further, by examining the generators, we deduce that the surjective map π in (8.3) gives the isomorphism above. Thus, $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma_0)$ is simply connected.

In the case when $p = q$, $\mathrm{MMCG}(\mathbf{S}_\triangle)$ contains one more $\mathbb{Z}_2$ symmetry. In the same way, we will have $\pi_1\mathrm{Quad}(\mathbf{S}) = \mathrm{Br}(\widetilde{A_{p,q}})$ and simply connectedness.

Theorem 8.3 *Let* $\mathbf{S}$ *be an annulus (without punctures) and* $\mathcal{D}_{fd}(\Gamma_0)$ *be the 3-CY category associated to some triangulation of* $\mathbf{S}$. *Then* $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma_0)$ *is the universal cover of* $\mathrm{Conf}^n(\mathbb{C}^*)$.

By [4, Theorem 2.7], the universal cover of $\mathrm{Conf}^n(\mathbb{C}^*)$ is contractible. So we have:

Corollary 8.4 $\mathrm{Stab}^\circ\mathcal{D}_{fd}(\Gamma_0)$ *is contractible.*

9 Proof of Proposition 5.9

9.1 Preparation

See [21, Appendix A] for the details of homological algebra calculations for the string model in § 5.2. The key results are the following two.

Proposition 9.1 [21, Corollary A.9] *Let η_1, η_2 be two closed arcs in $\mathbf{S}_\triangle$ that share an endpoint. Fix orientations of them and suppose that they share the starting endpoint $Z \in \triangle$. Then there is a unique non-zero homomorphism $\zeta_{12}^Z \in \mathrm{Hom}^\bullet(X_\alpha, X_\beta)$ induced by Z. Moreover, suppose there is another closed arc η_3 starting at Z, such that η_1, η_3, η_2 are in a clockwise order at Z. Then ζ_{12}^Z is the composition of ζ_{13}^Z with ζ_{23}^Z.*

Proposition 9.2 [21, Proposition A.11] *Let α, β and η be three closed arcs in $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ such that at least one of them is in $\mathrm{CA}(\mathbf{S}_\triangle)$. Moreover, we require that α, η, β are in a clockwise order to form a contractible triangle in $\mathbf{S}_\triangle$. Then there are representatives $X_?$ in $\widetilde{X}(?)$ for $? = \alpha, \beta, \eta$ such that there is a non-trivial triangle*

$$X_\beta \to X_\eta \to X_\alpha \to X_\beta[1], \tag{9.1}$$

where the homomorphisms are of the form in Proposition 9.1.

Remark 9.3 Note that, in the setting of Lemma 3.14, the line segment l, from Z_0 to some point Y in η (cf. Fig. 8), plays an important role. We will say η decomposes into α and β w.r.t. l.

Also note that the condition $\mathrm{Int}(\eta_1, \eta_2) = \frac{1}{2}$ in 3° forces that η_1, η_2 are closed arcs in $\mathrm{CA}(\mathbf{S}_\triangle)$.

9.2 The first induction

Use double induction, the first on

$$I = \mathrm{Int}(\mathbf{T}_0, \eta_1) + \mathrm{Int}(\mathbf{T}_0, \eta_2). \tag{9.2}$$

The starting case is when $I = 2$. Then both η_1 and η_2 are in $\mathbf{T}_0^*$, since the only general closed arcs that have exactly one intersection with $\mathbf{T}_0$ are the arcs in $\mathbf{T}_0^*$. It is straightforward to check the proposition in this case. Now suppose that the proposition holds for any (η_1, η_2) with $I \le r$ and consider the case when $I = r + 1$.

First, let us prove 1° for X_η (where $\eta = \eta_1$ or $\eta = \eta_2$ in $\mathrm{CA}(\mathbf{S}_\triangle)$). Apply Lemma 3.14 to decompose η into α and β in $\mathrm{CA}(\mathbf{S}_\triangle)$ with $\mathrm{Int}(\alpha, \beta) = \frac{1}{2}$ (cf. Fig. 8). Then by Proposition 9.2, there is a non-trivial triangle (9.1) By the inductive assumption, the proposition holds for α and β. Then $\alpha, \beta \in \mathrm{CA}(\mathbf{S}_\triangle)$ implies $X_\alpha, X_\beta \in \mathrm{Sph}(\Gamma_0)$ and $\mathrm{Int}(\alpha, \beta) = \frac{1}{2}$ implies

$$\dim \mathrm{Hom}^\bullet(X_\alpha, X_\beta) = 1.$$

Hence (9.1) implies

$$X_\eta = \phi_{X_\alpha}(X_\beta) = \phi_{X_\beta}^{-1}(X_\alpha), \tag{9.3}$$

and thus X_η is also in $\mathrm{Sph}(\Gamma_0)$.

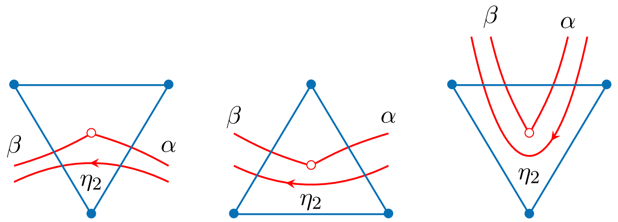

Fig. 18 The three cases for possible position of α and β

9.3 The second induction

Next, we prove 2° and 3°. Use the second induction on

$$\min\{\mathrm{Int}(\mathbf{T}_0, \eta_1), \mathrm{Int}(\mathbf{T}_0, \eta_2)\}.$$

Without loss of generality, suppose that

$$\mathrm{Int}(\mathbf{T}_0, \eta_1) \leq \mathrm{Int}(\mathbf{T}_0, \eta_2). \tag{9.4}$$

The starting case is when $\mathrm{Int}(\mathbf{T}_0, \eta_1) = 1$, which implies that $\eta_1 = s_i$ for some i. Note that we have $\mathrm{Int}(\mathbf{T}_0, \eta_2) > 1$. Applying Lemma 3.14 to decompose η_2 into α and β, w.r.t. some decorating point Z_0. As above, we get a non-trivial triangle (9.1) by Proposition 9.2. There are two cases.

Case i If Z_0 is not an endpoint η_1.

Then the inductive assumption holds for (η_1, α) and (η_1, β). For 2°, we have $\mathrm{Int}(\eta_1, \alpha) = 0 = \mathrm{Int}(\eta_1, \beta)$ and hence

$$\mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha) = 0 = \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta). \tag{9.5}$$

Applying $\mathrm{Hom}(X_{\eta_1}, ?)$ to triangle (9.1), we obtain (5.12). For 3°, we have

$$\{\mathrm{Int}(\eta_1, \alpha), \mathrm{Int}(\eta_1, \beta)\} = \{\tfrac{1}{2}, 0\},$$

and hence one of $\mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha)$ and $\mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta)$ is zero while the other one has dimension one. Applying $\mathrm{Hom}(X_{\eta_1}, ?)$ to triangle (9.1), we obtain (5.13).

Case ii If Z_0 is an endpoint η_1.

For 2°, we have $\mathrm{Int}(\eta_1, \alpha) = \frac{1}{2} = \mathrm{Int}(\eta_1, \beta)$ and thus (by the inductive assumption)

$$\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha) = 1 = \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta). \tag{9.6}$$

There are three cases (as shown in Fig. 18) for the possible positions of α and β in the triangle Λ_0 that contains Z_0. Since η_1 does not intersect η_2, the line segments of η_1, α, β

in Λ_0 are in a clockwise order. By Proposition 9.1, when applying $\mathrm{Hom}(X_{\eta_1}, ?)$ to the triangle (9.1), there will be an isomorphism

$$\mathrm{Hom}^t(X_{\eta_1}, \alpha) \xrightarrow{\simeq} \mathrm{Hom}^t(X_{\eta_1}, \beta[1]) \tag{9.7}$$

in the long exact sequence for some $t \in \mathbb{Z}$, which implies (5.12) by (9.6).

For 3°, without loss of generality, suppose that α and η_1 do not intersect in $\mathbf{S}_\triangle - \triangle$ but share both endpoints, and $\mathrm{Int}(\eta_1, \beta) = \frac{1}{2}$. Then $\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta) = 1$. If α is in $\mathbf{T}_0^*$, then we have $\alpha = s_i = \eta_1$. Applying the inductive assumption to (η_1, β), we have $\eta_2 = \mathrm{B}_{\eta_1}(\beta)$ and $X_{\eta_2} = \phi_{X_{\eta_1}}(X_\beta)$. Then

$$\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta) = 1,$$

as required. Otherwise, apply Lemma 3.14 to decompose α into closed arcs α' and β'. By applying the inductive assumption to (η_1, α') and (η_1, β'), we deduce that

$$\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_{\alpha'}) = 1 = \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_{\beta'}).$$

and hence $\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha)$ is 0 or 2. Moreover, Proposition 9.1 implies an isomorphism between a subspace of $\mathrm{Hom}^\bullet(X_{\eta_1}, \alpha)$ and $\mathrm{Hom}^t(X_{\eta_1}, \beta[1])$ (cf. (9.7)), which implies

$$\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) \leq \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha) + \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta) - 2 = 1.$$

One the other hand,

$$\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) \equiv \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha) + \dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta) \equiv 1(\mathrm{mod} 2).$$

Therefore (5.13) holds as required.

9.4 Inductive step of the second induction

To finish the proof, we only need to show that if 2° and 3° hold for $I \leq r$ or $I = r+1$ with $\mathrm{Int}(\mathbf{T}_0, \eta_1) \leq r_1$, then they hold for $I = r+1$ with $\mathrm{Int}(\mathbf{T}_0, \eta_1) = r_1 + 1$ (recall that I is defined in (9.2) and we assume (9.4)).

Apply Lemma 3.14 to decompose $\eta = \eta_1$ into α, β w.r.t. some decorating point Z_0 and some line segment l (see Fig. 8).

Case i The line segment l does not intersect η_2 in $\mathbf{S}_\triangle - \triangle$.

Then neither α nor β intersect η_2 in $\mathbf{S}_\triangle - \triangle$. Since η_1 and η_2 don't share two endpoints, without loss of generality, suppose that the common endpoint of η_1 and β is not an endpoint of η_2. Consider

$$\eta_1' = \mathrm{B}_\beta(\eta_1) = \alpha \quad \text{and} \quad \eta_2' = \mathrm{B}_\beta(\eta_2).$$

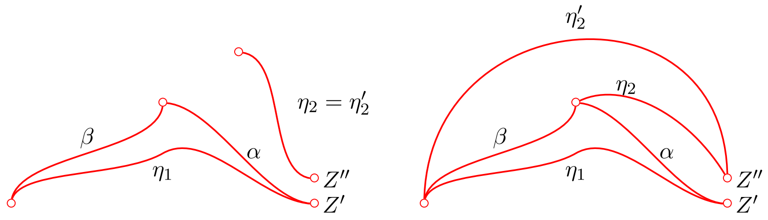

Fig. 19 $\eta_1' = \alpha$ and η_2'

See Fig. 19 for the two possibilities, where Z' and Z'' could coincide.

As in (9.3), we have

$$X_{\eta_1} = \phi_{X_\beta}^{-1}(X_\alpha) = \phi_{X_\beta}^{-1}(X_{\eta_1'}) \quad \text{and} \quad X_{\eta_2} = \phi_{X_\beta}^{-1}(X_{\eta_2'}),$$

which implies

$$\mathrm{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) \simeq \mathrm{Hom}^\bullet(X_{\eta_1'}, X_{\eta_2'}) \tag{9.8}$$

Moreover, we have

$$\begin{aligned} \mathrm{Int}(\mathbf{T}_0, \eta_1') &= \mathrm{Int}(\mathbf{T}_0, \alpha) = \mathrm{Int}(\mathbf{T}_0, \eta_1) - \mathrm{Int}(\mathbf{T}_0, \beta); \\ \mathrm{Int}(\mathbf{T}_0, \eta_2') &= \mathrm{Int}(\mathbf{T}_0, \mathrm{B}_\beta(\eta_2)) \le \mathrm{Int}(\mathbf{T}_0, \eta_2) + \mathrm{Int}(\mathbf{T}_0, \beta). \end{aligned}$$

Thus 2° or 3° hold for (η_1', η_2') by the inductive assumption, which implies that they also hold for (η_1, η_2) by (9.8).

Case ii The line segment l intersects η_2.

Let Y' be their nearest intersection to Z_0. Then we can decompose η_2 to α and β, using the line segment $l' = YZ_0 (\subset l)$ as in Lemma 3.14. There is a small difference here, that Z_0 might be an endpoint of η_2, so α and β are in $\overline{\mathrm{CA}}(\mathbf{S}_\triangle)$ (i.e. they might be L-arc instead of closed arc). Since Z_0 is not an endpoint of η_1, we deduce that

$$\tfrac{1}{2} \ge \mathrm{Int}(\eta_1, \eta_2) = \mathrm{Int}(\eta_1, \alpha) + \mathrm{Int}(\eta_1, \beta).$$

As $\mathrm{Int}(\mathbf{T}_0, \alpha) + \mathrm{Int}(\mathbf{T}_0, \beta) = \mathrm{Int}(\mathbf{T}_0, \eta_2)$, the inductive assumption applies to (η_1, α) and (η_1, β). Then $\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\alpha)$ and $\dim \mathrm{Hom}^\bullet(X_{\eta_1}, X_\beta)$ are both zero (for 2°) and are $\{0, 1\}$ for 3°. Either way, we will have $\mathrm{Hom}^\bullet(X_{\eta_1}, X_{\eta_2}) = 2\mathrm{Int}(\eta_1, \eta_2)$ as required.

10 Further studies

10.1 Algebraic twist group of quivers with potential

Let (Q, W) be a rigid quiver with potential such that there is no double arrow in Q and W is the sum of some cycles in Q.

Definition 10.1 The *algebraic twist group* $\mathrm{AT}(Q, W)$ of such a quiver with potential (Q, W) is the group generated by $\{t_i \mid i \in Q_0\}$ subject to the relations

1°. $t_i t_j = t_j t_i$ if there is no arrow between i and j in Q,
2°. $t_i t_j t_i = t_j t_i t_j$ if there is exactly one arrow between i and j in Q,
3°. $R_i = R_j$ for any i, j (cyclic relation), if there is a cycle $Y : 1 \to 2 \to \cdots \to m \to 1$ in Q (or a term in W by definition), where $R_i = t_i t_{i+1} \cdots t_{2m+i-3}$ with convention $k = m + k$ here.

First, we show that any cyclic relations in Definition 10.1, that correspond to the same cycle Y, are equivalent to each other.

Lemma 10.2 *Let $m \geq 3$ and suppose that $t_1, t_2, \cdots, t_m$ satisfy the relations*

$$\begin{cases} t_j t_i t_j = t_i t_j t_i, & |j - i| = 1 \text{ or } \{i, j\} = \{1, m\}, \\ t_i t_j = t_j t_i, & \text{otherwise}. \end{cases} \tag{10.1}$$

Let $k = m + k$ and $R_i = t_i t_{i+1} \cdots t_{2m+i-3}$. Then the relation $R_1 = R_2$ is equivalent to $R_1 = R_i$ for any $3 \leq i \leq m$.

Proof By the relations in (10.1), it is straightforward to check the following

$$\begin{aligned} t_i R_1 &= R_1 t_{i-2}, \ i = 2, \cdots, m-1. \\ t_i R_{i+1} &= R_i t_{i-2}, \ i = 3, \cdots, m. \end{aligned}$$

Then we have

$$\begin{aligned} R_1 = R_i &\iff R_1 t_{i-2} = R_i t_{i-2} \\ &\iff t_1 R_1 = t_i R_{i+1} \\ &\iff R_1 = R_{i+1} \end{aligned}$$

for any $i = 2, \cdots, m-1$, which implies the lemma. □

A consequence of Lemma 10.2 is

$$R_i = R_j \iff R_k = R_l$$

provided $i \neq j$ and $k \neq l$.

The following result was originally in [15] for type A and D, which is also independently obtained by Grant-Marsh for all Dynkin types.

Proposition 10.3 *If (Q, W) is mutation-equivalent to a Dynkin diagram $\underline{Q}$, then the algebraic twist group $\mathrm{AT}(Q, W)$ is isomorphic to the corresponding braid group $\mathrm{Br}(\underline{Q})$.*

Remark 10.4 We believe that the proposition above also holds for the affine Dynkin case, as long as Q does not have double arrows. The point is, one should be able to define an algebraic twist group for a (good) quiver with potential, which provides a presentation of the corresponding spherical twist group (or/and Dehn twist group). □

10.2 Intersection formulae

Interpreting the intersection formulae between open (resp. closed) arcs as dimension of Hom (resp. Ext) play a crucial role in many proofs (e.g. in [17] and in [20]). We have the following conjecture.

Conjecture 10.5 *Let* $\alpha, \beta \in \mathrm{CA}(\mathbf{S}_\triangle)$. *We have*

$$\dim \mathrm{Hom}^\bullet(\widetilde{X}(\alpha), \widetilde{X}(\beta)) = 2\mathrm{Int}(\alpha, \beta). \tag{10.2}$$

Moreover, we have another conjectured formula.

Conjecture 10.6 *Denote by* $\mathrm{OA}^\circ(\mathbf{S}_\triangle)$ *the set of open arcs that appear in triangulations in* $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$. *Then there is a map* $\rho\colon \mathrm{OA}^\circ(\mathbf{S}_\triangle) \to \mathrm{per}\Gamma_0$, *such that any* $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$,

$$\dim \mathrm{Hom}^\bullet(\rho(\gamma), \widetilde{X}(\eta)) = \mathrm{Int}(\gamma, \eta). \tag{10.3}$$

We will prove these two intersection formulae in [21].

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