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Cluster algebras arising from cluster tubes

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Dedicated to the memory of Dieter Happel

Abstract

We study the cluster algebras arising from cluster tubes with rank bigger than 1. Cluster tubes are 2-Calabi–Yau triangulated categories that contain no cluster tilting objects, but maximal rigid objects. Fix a certain maximal rigid object T in the cluster tube $\mathcal{C}_n$ of rank n. For any indecomposable rigid object M in $\mathcal{C}_n$, we define an analogous X_M of Caldero–Chapoton's formula (or Palu's cluster character formula) by using the geometric information of M. We show that $X_M, X_{M'}$ satisfy the mutation formula when M, M' form an exchange pair, and that $X_? : M \mapsto X_M$ gives a bijection from the set of indecomposable rigid objects in $\mathcal{C}_n$ to the set of cluster variables of cluster algebra of type C_{n-1}, which induces a bijection between the set of basic maximal rigid objects in $\mathcal{C}_n$ and the set of clusters. This yields a surprising result proved recently by Buan–Marsh–Vatne that the combinatorics of maximal rigid objects in the cluster tube $\mathcal{C}_n$ encodes the combinatorics of the cluster algebra of type B_{n-1}, since the combinatorics of cluster algebras of type B_{n-1} and of type C_{n-1} is the same by a result of Fomin and Zelevinsky. As a consequence, we give a categorification of cluster algebras of type C.

1. *Introduction*

Cluster algebras were introduced around 2000 by Fomin and Zelevinsky [**15**] in order to give an algebraic and combinatorial framework for the canonical basis of quantum groups and for the notion of total positivity for semisimple algebraic groups; see [**13**, **18**] for a nice survey on this topic and its background. Since they were introduced, interesting connections between such algebras and several branches of mathematics have emerged. In the categorification theory of cluster algebras, cluster categories [**1**, **4**, **8**, **9**, **25**] and (stable) module categories over preprojective algebras [**3**, **20**, **21**] play a central role. They all have cluster tilting objects, which model the clusters of the corresponding cluster algebras via Caldero–Chapoton's formula [**7**] in the case of cluster categories or Geiss–Leclerc–Shröer's map [**20**] in the case of preprojective algebras. This motivates the study of arbitrary 2-Calabi–Yau triangulated categories with cluster tilting objects (subcategories). Palu defined a cluster character for any Hom-finite 2-Calabi–Yau triangulated categories that have cluster tilting objects [**31**, **32**] (see also [**19**]). Recently, Plamondon defined cluster characters for Hom-infinite 2-Calabi–Yau triangulated categories with some hypotheses [**33**, **34**].

It was proved in [**3**, **24**] that one can mutate a cluster tilting object $T = \oplus_{i=1}^{n} T_i$ (respectively, maximal rigid object) at any indecomposable direct summand T_i to get a new cluster tilting object $\mu_i(T)$ (respectively, maximal rigid object) via exchange triangles in a 2-Calabi–Yau triangulated category $\mathcal{C}$. To any maximal rigid object T, one can associate an integer matrix A_T by using the exchange triangles (where A_T is the transpose of B_T defined in [**5**]; see Section 2). If A_T and $A_{\mu_i(T)}$ are related by Fomin–Zelevinsky's matrix mutation for any maximal rigid object T and any direct summand T_i, then we say that the maximal rigid objects form a

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cluster structure in $\mathcal{C}$ [**3**, **5**]. In [**5**], Buan–Marsh–Vatne showed that maximal rigid objects without 2-cycles form a cluster structure in $\mathcal{C}$ (see also [**3**]). Therefore, cluster tilting objects and maximal rigid objects are important objects in 2-Calabi–Yau triangulated categories. They have many nice properties; see, for example, [**11**, **23**, **24**, **28**, **29**, **36**–**38**]. Cluster tilting objects are obviously maximal rigid, but the converse is not true; see [**6**] for the first examples. Cluster tubes provide the second examples, in which the quivers of endomorphism algebras of maximal rigid objects contain loops, but no 2-cycles [**5**]. A cluster tube of rank n, denoted by $\mathcal{C}_n$, is, by definition, the orbit category by $\tau^{-1}[1]$ of the derived category of the hereditary abelian category of nilpotent representations of the quiver with underlying graph $\widetilde{A}_{n-1}$ and with cyclic orientation. It is a 2-Calabi–Yau triangulated category [**2**, **25**]. In [**5**], a classification of maximal rigid objects in the cluster tube $\mathcal{C}_n$ is given. The maximal rigid objects are proved to form a cluster structure. Furthermore, they use the geometric description of the exchange graph of the cluster algebra of type B_{n-1} in [**16**] to prove that there is a bijection between the set of indecomposable rigid objects in the cluster tube $\mathcal{C}_n$ and the set of cluster variables of the cluster algebra of type B_{n-1}. Under this bijection, maximal rigid objects go to clusters. Since the cluster combinatorics of the cluster algebra of type C_{n-1} is the same as that of the cluster algebra of type B_{n-1} by [**16**, Proposition 3.15], there is a bijection between the set of indecomposable rigid objects in the cluster tube $\mathcal{C}_n$ and the set of cluster variables of the cluster algebra of type C_{n-1}.

The aim of the paper is to study the cluster algebras arising from cluster tubes. This is the first attempt to answer the well-known question of how to define cluster characters with respect to a maximal rigid object in a 2-Calabi–Yau triangulated category, in which maximal rigid objects may have loops (cf. [**33**, **34**]). We give an analogue of Caldero–Chapoton's formula [**7**] (or Palu's character [**31**]) for cluster tubes. Fix a certain basic maximal rigid object T in the cluster tube $\mathcal{C}_n(n>1)$. Here, A_T denotes the skew-symmetrizable matrix associated with T, which is of type C_{n-1} [**5**] (please see the precise meaning in Section 2). For any indecomposable rigid object M in $\mathcal{C}_n$, with respect to T, we define a Laurent polynomial X_M. We prove that the formula X_M satisfies the mutation formula for cluster variables: that is, if M and M^* are indecomposable rigid objects such that $M\oplus N$ and $M^*\oplus N$, for some rigid object N, are maximal rigid objects in $\mathcal{C}_n$, then $X_M\cdot X_{M^*}=X_E+X'_E$, where E,E' are the middles of the exchange triangles: $M\to E\to M^*\to M[1]$, $M^*\to E'\to M\to M^*[1]$. We note here that the dimension of $\mathrm{Ext}^1(M,M^*)$ can be 2 (see the cases considered before in [**7**, **19**, **31**, **33**, **34**], where the k-dimension of $\mathrm{Ext}^1(M,M^*)$ is always 1). Thus, the map $X_?$ gives a bijection from the set of indecomposable rigid objects in $\mathcal{C}_n$ to the set of cluster variables of the cluster algebra of type C_{n-1}. This gives an explicit bijection parallel with that given by Buan–Marsh–Vatne [**5**] for type B_{n-1} (since there is a natural bijection between type B_{n-1} and type C_{n-1}; see [**16**]). The algebra generated by the X_M, where M runs over all indecomposable rigid objects in $\mathcal{C}_n$ is isomorphic to the cluster algebras of type C_{n-1}. In [**10**], Dupont proved the multiplication formula for cluster characters associated to regular modules over the path algebra of any representation-infinite quiver; Ding and Xu [**12**] also defined an analogous map for cluster tubes and gave multiplication formulas. But their formulas are not the exchange formula for cluster variables on the one hand, and their maps cannot be used to realize the cluster structure of $\mathcal{C}_n$ on the other hand.

The paper is organized as follows: In Section 2, we recall some basics on cluster algebras and 2-Calabi–Yau triangulated categories. In particular, we recall the definition of cluster tubes and basic descriptions on indecomposable rigid objects in cluster tubes from [**5**]. In Section 3, for any positive number $n>1$, fix a basic maximal rigid object T in $\mathcal{C}_n$, we calculate the index of any indecomposable rigid object M with respect to T (defined in [**11**, **31**, **33**, **34**]) and define the analogue X_M of the CC-map or Palu's map for an indecomposable rigid object M with respect to T. This map $X_?$ is called cluster map. Using the structure of the cluster tube $\mathcal{C}_n$ of rank n, we divide the set of indecomposable rigid objects into three disjoint subsets. Using the structure of endomorphism algebras of T [**5**, **36**–**38**], we calculate the explicit formula of X_M according to which subset M belongs to. In Section 4, we prove that X_M, X_{M^*} satisfy the mutation formula when M, M^* form a mutation pair. Using the mutation triangles, we explain that the matrix A_T associated with T is a skew-symmetrizable matrix of type C_{n-1}. We prove that the map $X_?$ gives a bijection between the set of indecomposable rigid objects in $\mathcal{C}_n$ and the set of cluster variables of A_T, which induces a bijection between the set of basic maximal rigid objects and the set of clusters of A_T. It follows that the cluster algebra generated by X_M, where M runs over all indecomposable rigid objects, is isomorphic to the cluster algebra of type C_{n-1}. In the final section, we give an application of the cluster map $X_?$. We prove that the simplicial complex generated by the indecomposable rigid objects in $\mathcal{C}_n$ gives a realization of the cluster complex of the root system of type C_{n-1} defined in [**16**].

2. *Preliminaries*

We recall some basic notions on cluster algebras that can be found in the papers by Fomin and Zelevinsky [**15**–**17**]. The cluster algebras we deal with in this paper are without coefficients.

Let $\mathcal{F} = \mathbb{Q}(x_1, x_2, \ldots, x_n)$ be the field of rational functions in indeterminates $x_1, x_2, \ldots, x_n$. Set $\underline{x} = \{x_1, x_2, \ldots, x_n\}$. Let $A = (a_{ij})$ be an $n \times n$ skew-symmetrizable integer matrix. For any $k \in \{1, 2, \ldots, n\}$, the mutation $\mu_k(A)$ of A in direction k is, by definition, an integer matrix $A' = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} -a_{ij} & \text{if } i = k \text{ or } j = k, \\ a_{ij} + \dfrac{|a_{ik}|a_{kj} + a_{ik}|a_{kj}|}{2} & \text{otherwise.} \end{cases}$$

We see that A' is a skew-symmetrizable matrix too. A seed is a pair $(\underline{u}, A)$, where $\underline{u} = \{u_1, u_2, \ldots, u_n\}$ is a transcendence base of $\mathcal{F}$, and A is an $n \times n$ skew-symmetrizable integer matrix. A mutation $\mu_k(\underline{u}, A)$ of a seed $(\underline{u}, A)$ in direction k is a new seed $(\underline{u'}, A')$, where $A' = \mu_k(A)$ and $\underline{u'} = (\underline{u} \setminus \{u_k\}) \bigcup \{u'_k\}$, where u'_k is defined in the following mutation formula:

$$u_k u'_k = \prod_{a_{ik}>0} u_i^{a_{ik}} + \prod_{a_{ik}<0} u_i^{-a_{ik}}.$$

The cluster algebra $\mathcal{A}_A$ associated to the skew-symmetrizable matrix A is by definition the subalgebra of $\mathcal{F}$ generated by all u_i in $\underline{u}$ such that $(\underline{u}, A')$ is obtained from $(\underline{x}, A)$ by mutations for some A'. Such $\underline{u} = (u_1, u_2, \ldots, u_n)$ is called a cluster of the cluster algebra $\mathcal{A}_A$ or simply of the matrix A, and any u_i is called a cluster variable. The seed $(\underline{x}, A)$ is called an initial seed. The set of all cluster variables is denoted by χ_A. If the set χ_A is finite, then the cluster algebra $\mathcal{A}_A$ is called of finite type. For any skew-symmetrizable integer matrix A, one can define the Cartan part C_A of A as follows: $C_A = (c_{ij})_{n\times n}$, where

$$c_{ij} = \begin{cases} -|a_{ij}| & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

It was proved by Fomin and Zelevinsky [**17**] that cluster algebras are of finite type if and only if there is a seed $(\underline{u}, A')$ obtained from the initial seed $(\underline{x}, A)$ by mutations such that the Cartan part $C_{A'}$ of A' is of finite type. In this case, the type of the Cartan matrix C_A is called the type of the cluster algebra $\mathcal{A}_A$. For example, if

$$C_{A'} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -2 & 2 & -1 & \cdots & 0 \\ & & \cdots & \cdots & \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & & \cdots & -1 & 2 \end{pmatrix},$$

then the cluster algebra is called of type C_n.

Now we recall some basics on 2-Calabi–Yau triangulated categories. Fix an algebraically closed field k. A triangulated category $\mathcal{C}$ is called k-linear, provided all Hom-spaces in $\mathcal{C}$ are k-spaces and the compositions of maps are k-linear. The k-linear triangulated categories in this paper will be assumed Hom-finite and Krull–Remak–Schmidt, that is, $\dim_k \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty$ for any two objects X and Y in $\mathcal{C}$, and every object decomposes into a finite direct sum of objects having local endomorphism rings. We fix some notation. For an object M in $\mathcal{C}_n$, denote by $\mathrm{add}\, M$ the subcategory of $\mathcal{C}_n$ consisting of (finite) direct sums of direct summands of M. For two subcategories $\mathcal{D}_1, \mathcal{D}_2$ of $\mathcal{C}_n$, denote by $\mathcal{D}_1 * \mathcal{D}_2$ the full subcategory of $\mathcal{C}_n$ consisting of object E such that there is a triangle $D_1 \to E \to D_2 \to D_1[1]$, where $D_i \in \mathcal{D}_i$, for $i = 1, 2$.

A k-linear triangulated category $\mathcal{C}$ is called 2-Calabi–Yau if there is a functorial isomorphism $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{C}}(Y, X[2])$ for any objects $X, Y \in \mathcal{C}$, where $D = \mathrm{Hom}_k(-, k)$. The main examples of 2-Calabi–Yau triangulated categories from representation theory of algebras are the cluster categories of abelian hereditary categories with tilting objects [**4**, **25**], the Hom-finite generalized cluster categories of algebras with global dimension of at most 2 [**1**], the stable categories of Cohen–Macaulay modules [**3**], cluster tubes [**2**] and some others, please see the survey [**26**].

Cluster tilting objects are defined first in cluster categories [**4**], which are generalized to arbitrary 2-Calabi–Yau triangulated categories by Keller and Reiten [**28**].

Definition 2.1. Let T be an object of a 2-Calabi–Yau triangulated category $\mathcal{C}$. We have the following notion:

(i) T is called basic if any two indecomposable summands of T are not isomorphic;
(ii) T is rigid, provided $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$;
(iii) T is maximal rigid, provided T is rigid and is maximal with respect to this property; that is, if $\mathrm{Ext}^1_{\mathcal{C}}(T \oplus M, T \oplus M) = 0$, then $M \in \mathrm{add}\, T$;
(iv) T is cluster-tilting, provided, for any $M \in \mathcal{C}$, $M \in \mathrm{add}\, T$ if and only if $\mathrm{Ext}^1_{C}(M, T) = 0$.

From the definition, any cluster tilting object is maximal rigid, but the converse is not true. It was proved in [**5**] that the cluster tube $\mathcal{C}_n$ of rank n ($n > 1$) has no cluster tilting objects, but maximal rigid objects. See [**6**] for more such examples. The 2-Calabi–Yau triangulated categories with cluster tilting objects are important for the categorification of cluster algebras of skew-symmetric matrices; see the survey [**27**, **35**] and the references therein.

Fix a basic maximal rigid object $T = T_1 \oplus \cdots \oplus T_n$ with all T_j indecomposable. For an $i \in \{1, \ldots, n\}$, write $\bar{T} = \oplus_{j \neq i} T_j$. Then there are two nonsplit triangles:

$$T_i \xrightarrow{f_i} E_i \xrightarrow{g_i} T_i^* \longrightarrow T_i[1],$$
$$T_i^* \xrightarrow{f_i'} E_i' \xrightarrow{g_i'} T_i \longrightarrow T_i^*[1],$$

where f_i and f_i' are minimal left $\bar{T}$-approximations, and g_i and g_i' are minimal right $\bar{T}$-approximations. Furthermore, T_i^* is indecomposable and $\bar{T} \oplus T_i^*$ is maximal rigid [**3**, **24**]. Define the mutation of maximal rigid object T in direction i to be $\mu_i(T) = \bar{T} \oplus T_i$. It is easy to see that $\mu_i \circ \mu_i(T) = T$. The two triangles above are called exchange triangles. We define an integer matrix $A_T = (a_{ij})$ as follows:

$$a_{ij} = \alpha_{ij} - \alpha_{ij}',$$

where α_{ij} denotes the multiplicity of T_i as a direct summand of E_j and α_{ij}' denotes the multiplicity of T_i as a direct summand of E_j'. Note that $a_{ii} = 0$. Our definition of the matrix A_T associated to T is the transpose of the matrix B_T defined in [**5**]. When the endomorphism

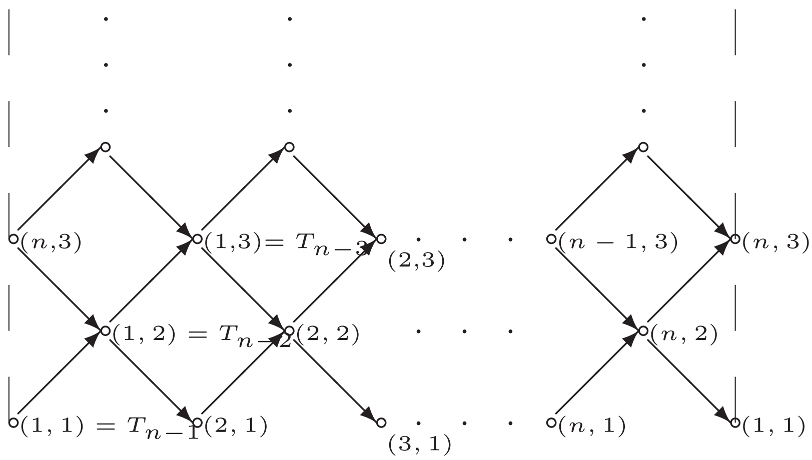

FIGURE 1. *The tube of rank n.*

algebra of T contains no loops or 2-cycles, A_T, B_T are skew-symmetric matrices, and $A_T = -B_T$. In general, the matrices A_T, B_T are sign-skew-symmetric (see Lemma 1.2 in [**5**]).

REMARK 2.2. We use our definition of the matrix A_T associated to T to replace the matrix B_T defined in [**5**] since we will use the mutation formula $u_k u'_k = \prod_{a_{ik}>0} u_i^{a_{ik}} + \prod_{a_{ik}<0} u_i^{-a_{ik}}$ for the definition of cluster algebras, where the index a_{ik} comes from the kth column of the matrix A (see [**15**]).

Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with maximal rigid objects. Suppose that, for all maximal rigid objects, E_i and E'_i have no common direct summands for any $i \in \{1, \ldots, n\}$. Then $\mu_i(A_T) = A_{\mu_i(T)}$ (equivalent to $\mu_i(B_T) = B_{\mu_i(T)}$ proved in [**5**]). In this case, one can say that the maximal rigid objects form a cluster structure in $\mathcal{C}$ [**5**].

In what follows, we will focus on cluster tubes, special 2-Calabi–Yau triangulated categories. We will denote the tube of rank n by $\mathcal{T}_n$, where n is always assumed to be greater than 1. One realization of this category is the category of finite-dimensional nilpotent representations over k of the cyclic quiver $\overrightarrow{\Delta}_n$ with n vertices such that arrows are going from i to $i+1$ (taken modulo n). It is a k-linear hereditary abelian category which is Hom-finite, that is, $\dim\mathrm{Hom}_{\mathcal{T}_n}(X, Y) < \infty$ for any $X, Y \in \mathcal{T}_n$. Each indecomposable representation is uniserial, that is, it has a unique composition series, and hence is determined by its socle and its length up to isomorphism. We denote by (a, b) in $\mathcal{T}_n$ the unique indecomposable object with socle $(a, 1)$ and quasi-length b, where $(a, 1)$ is the simple representation at vertex a, $a \in \{1, \ldots, n\}$ (see Figure 1). For convenience, $(a, 0)$ denotes a zero object. The tube $\mathcal{T}_n$ has Auslander–Reiten sequences, and the Auslander–Reiten translation τ is an automorphism of $\mathcal{T}_n$:

$$\tau(a, b) = (a - 1, b).$$

The cluster tube of rank n is defined in [**2**], as the orbit category

$$\mathcal{C}_n := D^b(\mathcal{T}_n)/\tau^{-1}[1],$$

where [1] is the shift functor of $D^b(\mathcal{T}_n)$. This category is a 2-Calabi–Yau triangulated category such that the projection $\pi : D^b(\mathcal{T}_n) \to \mathcal{C}_n$ is a triangle functor [**2**, **25**]. The cluster tube $\mathcal{C}_n$ has Auslander–Reiten triangles induced from the ones in $D^b(\mathcal{T}_n)$. It is easy to see that indecomposable objects in $\mathcal{T}_n$ are also indecomposable in $\mathcal{C}_n$ (via the composition of the

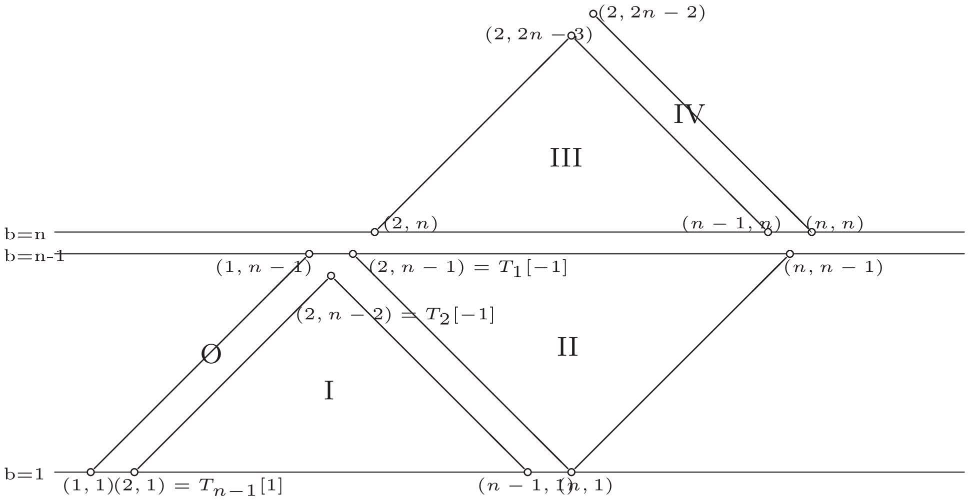

FIGURE 2. *Five disjoint subsets of $\mathcal{D}$.*

inclusion functor $\mathcal{T}_n \hookrightarrow D^b(\mathcal{T}_n)$ with the projection $\pi : D^b(\mathcal{T}_n) \to \mathcal{C}_n$), and all indecomposable objects in $\mathcal{C}_n$ are of this form. So we use the same (a, b) to denote the indecomposable object in $\mathcal{C}_n$ induced from the object (a, b) in $\mathcal{T}_n$.

By the definition of $\mathcal{C}_n$, for two objects $X, Y \in \mathcal{T}_n$,

$$\mathrm{Hom}_{\mathcal{C}_n}(X, Y) \cong \mathrm{Hom}_{\mathcal{T}_n}(X, Y) \oplus D\mathrm{Hom}_{\mathcal{T}_n}(Y, \tau^2 X).$$

As in [**5**], the maps from $\mathrm{Hom}_{\mathcal{T}_n}(X, Y)$ are called $\mathcal{T}$-maps and the maps from $D\mathrm{Hom}_{\mathcal{T}_n}(Y, \tau^2 X)$ are called $\mathcal{D}$-maps. Any map from X to Y in $\mathcal{C}_n$ can be written as the sum of a $\mathcal{T}$-map and a $\mathcal{D}$-map. One knows that the composition of two $\mathcal{T}$-maps is also a $\mathcal{T}$-map, the composition of a $\mathcal{T}$-map and a $\mathcal{D}$-map is a $\mathcal{D}$-map, and the composition of two $\mathcal{D}$-maps is zero.

The indecomposable rigid objects are classified in [**5**].

PROPOSITION 2.3. *The object (a, b) is rigid if and only if $b \leqslant n - 1$.*

Define by $T_i = (1, n - i)$, $i = 1, \ldots, n - 1$. It is easy to see that $\oplus_{i=1}^{n-1} T_i$ is a maximal rigid object in $\mathcal{C}_n$. We will use T to denote this maximal rigid object throughout the paper. Let $\mathcal{D} = \mathrm{add}\, T[-1] * \mathrm{add}\, T$. Following [**36**, **37**], the set of indecomposable objects in $\mathcal{D}[1]$ is the set of indecomposable objects (a, b) satisfying either (1) (a, b) is rigid, or (2) $n \leqslant b \leqslant 2n - 2$ and $a + b \leqslant 2n - 1$. We divide $\mathcal{D}$ into five subsets (see Figure 2):

$$\begin{aligned}
O &= \{(a, b) \mid a = 1, b \leqslant n - 1\},\\
\mathrm{I} &= \{(a, b) \mid 2 \leqslant a \leqslant n - 1, a + b \leqslant n\},\\
\mathrm{II} &= \{(a, b) \mid a + b \geqslant n + 1, b \leqslant n - 1\},\\
\mathrm{III} &= \{(a, b) \mid a + b \leqslant 2n - 1, a \neq 1, b \geqslant n\},\\
\mathrm{IV} &= \{(a, b) \mid a + b = 2n, a \neq 1, b \geqslant n\}.
\end{aligned}$$

Let $f_1^{(a,b)}$ be a nonzero $\mathcal{T}$-map from $T_1[-1]$ to $(a, b) \in \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$ and let g_1^1 be a nonzero $\mathcal{D}$-map from $T_1[-1]$ to itself. Write

$$g_1^{(a,b)} = f_1^{(a,b)} g_1^1.$$

The triangle involving g_1^1 is

$$(3, 2n-2) \longrightarrow T_1[-1] \xrightarrow{g_1^1} T_1[-1] \xrightarrow{f} (2, 2n-2),$$

where $f = \lambda f_1^{(2,2n-1)}$ for some nonzero $\lambda \in k$. We know that $\alpha f = 0$ for any $\mathcal{T}$-map α in $\mathrm{Hom}_{\mathcal{C}_n}((2, 2n-2), (a, b))$ for $(a, b) \in \mathrm{II} \cup \mathrm{III}$, so $f_1^{(a,b)}$ cannot factor through f, and then $g_1^{(a,b)} = f_1^{(a,b)} g_1^1 \neq 0$. Computing the dimension of $\mathrm{Hom}_{\mathcal{C}_n}(T_1[-1], (a, b))$, we have the following obvious fact.

LEMMA 2.4. *For any indecomposable object (a, b) in $\mathcal{D}$, we have that*

$$\dim \mathrm{Hom}_{\mathcal{C}_n}(T_1[-1], (a, b)) = \begin{cases} 0 & \textit{if } (a, b) \in O \textit{ or } \mathrm{I}\ ; \\ 2 & \textit{if } (a, b) \in \mathrm{II} \textit{ or } \mathrm{III}; \\ 1 & \textit{if } (a, b) \in \mathrm{IV}. \end{cases}$$

Moreover, $f_1^{(a,b)}, g_1^{(a,b)}$ form a basis of $\mathrm{Hom}_{\mathcal{C}_n}(T_1[-1], (a, b))$ for $(a, b) \in \mathrm{II}$ or III; $f_1^{(a,b)}$ forms a basis of $\mathrm{Hom}_{\mathcal{C}_n}(T_1[-1], (a, b))$ for $(a, b) \in \mathrm{IV}$.

Let t_i^{i-1} be a nonzero $\mathcal{T}$-map from $T_i[-1]$ to $T_{i-1}[-1]$ and id_i be the identity from $T_i[-1]$ to itself. We write

$$f_i^{(a,b)} = f_k^{(a,b)} \mathrm{id}_k t_{k+1}^k \cdots t_{i-1}^{i-2} t_i^{i-1}$$

and

$$g_i^{(a,b)} = g_1^{(a,b)} f_i^1,$$

where $k = 1$ if $(a, b) \in \mathrm{II}$; $k = n - a - b + 2$ and $f_k^{(a,b)}$ is a nonzero $\mathcal{T}$-map from $T_k[-1]$ to (a, b) if $(a, b) \in \mathrm{I}$. Then we know that $f_i^{(a,b)}$ (respectively, $g_i^{(a,b)}$) is a nonzero $\mathcal{T}$-map (respectively, $\mathcal{D}$-map) if there exist nonzero $\mathcal{T}$-maps (respectively, $\mathcal{D}$-maps) from $T_i[-1]$ to (a, b), since any $\mathcal{D}$-map factors through the ray starting T_i [**5**, Lemma 2.2]. By our setting, we know that $f_i^{(a,b)} \neq f_{i'}^{(a,b)}$ if $i \neq i'$. Computing the dimension of $\mathrm{Hom}_{\mathcal{C}_n}(T_i[-1], (a, b))$, we have the following obvious fact.

LEMMA 2.5. *For any indecomposable rigid object (a, b), that is, $(a, b) \in O \cup \mathrm{I} \cup \mathrm{II}$, we have that*

$$\dim \mathrm{Hom}_{\mathcal{C}_n}(T_i[-1], (a, b)) = \begin{cases} 2 & \textit{if } 1 \leqslant i \leqslant n - a + 1,\ (a, b) \in \mathrm{II}; \\ 1 & \textit{if } n - a + 2 \leqslant i \leqslant 2n - a - b,\ (a, b) \in \mathrm{II}; \\ & \quad \textit{or } n - a - b + 2 \leqslant i \leqslant n - a + 1,\ (a, b) \in \mathrm{I}; \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, a basis of $\mathrm{Hom}_{\mathcal{C}_n}(T_i[-1], (a, b))$ is $\{f_i^{(a,b)}, g_i^{(a,b)}\}$ for $1 \leqslant i \leqslant n - a + 1$, $(a, b) \in \mathrm{II}$; $\{g_i^{(a,b)}\}$ for $n - a + 2 \leqslant i \leqslant 2n - a - b$, $(a, b) \in \mathrm{II}$; and $\{f_i^{(a,b)}\}$ for $n - a - b + 2 \leqslant i \leqslant n - a + 1$, $(a, b) \in \mathrm{I}$, respectively.

3. *Index and the cluster map*

We use the same notation as in the above section. In this section, we will define the cluster map $X_?$ from the set of indecomposable rigid objects in $\mathcal{C}_n$ to $\mathcal{F} = \mathbb{Q}(x_1, x_2, \ldots, x_n)$ by using the geometric information of the indecomposable rigid objects. We will give an explicit expression of X_M as a Laurent polynomial of $x_1, \ldots, x_{n-1}$ according to which subset M belongs to (recall that the set of indecomposable rigid objects in $\mathcal{C}_n$ is divided into three disjoint subsets O, I and II; see Figure 2).

Let $K_0^{\text{split}}(T)$ be the split-Grothendieck group of $\operatorname{add} T$, that is, the free abelian group with a basis consisting of isomorphism classes $[T_1], \ldots, [T_{n-1}]$ of indecomposable direct summands of T.

For an object X of $\mathcal{D}[1] = \operatorname{add} T * \operatorname{add} T[1]$, there exists a triangle

$$T''_X \longrightarrow T'_X \xrightarrow{f} X \longrightarrow T''_X[1],$$

where $T''_X, T'_X \in \operatorname{add} T$. It follows that f is a right $\operatorname{add} T$-approximation. We define the index

$$\operatorname{ind}_T(X) = [T'_X] - [T''_X] \in K_0^{\text{split}}(T)$$

as in [**11**, **31**, **33**, **34**]. It was proved in [**38**] that any rigid object belongs to $\mathcal{D}[1]$ (also in $\mathcal{D}$). The next lemma tells us how to get the right $\operatorname{add} T$-approximation of any indecomposable rigid object in $\mathcal{C}_n$.

LEMMA 3.1. *For any indecomposable rigid object (a, b) in $\mathcal{T}_n$, every right $\operatorname{add} T$-approximation of (a, b) in $\mathcal{T}_n$ is a right $\operatorname{add} T$-approximation of (a, b) in $\mathcal{C}_n$.*

Proof. If there are no nonzero $\mathcal{D}$-maps in $\operatorname{Hom}_{\mathcal{C}_n}(T, (a, b))$, then a right $\operatorname{add} T$-approximation of (a, b) in $\mathcal{T}_n$ is a right $\operatorname{add} T$-approximation of (a, b) in $\mathcal{C}_n$. Now suppose there are nonzero $\mathcal{D}$-maps in $\operatorname{Hom}_{\mathcal{C}_n}(T, (a, b))$; then, by Lemmas 2.4 and 2.5, anyone of them is a sum of some maps $f g_1^1 f_i^1$, where f is a $\mathcal{T}$-map from T_i to (a, b). Therefore, every $\mathcal{D}$-map in $\operatorname{Hom}_{\mathcal{C}_n}(T, (a, b))$ factors through $\mathcal{T}$-maps from T to (a, b). Thus, we have the assertion. $\square$

We calculate the index of any indecomposable rigid object (a, b) with respect to T.

LEMMA 3.2. *For any indecomposable rigid object (a, b) in $\mathcal{C}_n$,*

$$\operatorname{ind}_T(a, b) = \begin{cases} [T_1] - [T_{n-a+1}] - [T_{2n-a-b}] & \text{if } a + b \geqslant n + 1, \\ [T_{n-a-b+1}] - [T_{n-a+1}] & \text{if } a + b \leqslant n. \end{cases}$$

Proof. For $a + b \geqslant n + 1$, there is a minimal right $\operatorname{add} T$-approximation $f : T_1 \to (a, b)$ in $\mathcal{T}_n$. Then $\ker(f) = (1, a - 1)$ and $\operatorname{coker}(\tau^{-1} f) = (1, a + b - n)$. We get a triangle

$$C \longrightarrow T_1 \xrightarrow{f} (a, b) \longrightarrow C[1]$$

in $\mathcal{C}_n$ and an exact sequence

$$0 \longrightarrow \operatorname{coker}(\tau^{-1} f) \longrightarrow C \longrightarrow \ker(f) \longrightarrow 0$$

in $\mathcal{T}_n$ (cf. [**37**]). Since $\operatorname{Ext}^1_{\mathcal{T}_n}((1, a - 1), (1, a + b - n)) = 0$, then $C \cong (1, a - 1) \oplus (1, a + b - n)$. Hence $\operatorname{ind}_T(a, b) = [T_1] - [T_{n-a+1}] - [T_{2n-a-b}]$.

For $a + b \leqslant n$, there is a minimal right $\operatorname{add} T$-approximation $f : T_{n-a-b+1} \to (a, b)$ in $\mathcal{T}_n$. Then $\ker(f) = (1, a - 1)$ and $\operatorname{coker}(\tau^{-1} f) = 0$. Hence $\operatorname{ind}_T(a, b) = [T_{n-a-b+1}] - [T_{n-a+1}]$. $\square$

Let $B = \operatorname{End}_{\mathcal{C}_n}(T[-1])$; then $F := \operatorname{Hom}_{\mathcal{C}_n}(T[-1], -)\colon \mathcal{D}/\operatorname{add} T \to \operatorname{mod} B$ is an equivalence of abelian categories [**24**, **37**, **38**], where $\operatorname{mod} B$ denotes the category of finite-dimensional right B-modules. The maps id_i, $1 \leqslant i \leqslant n - 1$, t_i^{i-1}, $2 \leqslant i \leqslant n - 1$, and g_1^1 form a set of generators of B.

Definition 3.3. For any indecomposable rigid object M in $\mathcal{C}_n$, we define

$$X_M = x^{\mathrm{ind}_T M} \sum_{e\in\mathbb{N}^{n-1}} \chi(\mathrm{Gr}_e(FM))x^{-\iota(e)},$$

where the sum takes over all dimension vectors e such that there exists an object Y in $\mathcal{D}\cap\mathcal{D}[1]$ with $e = \underline{\dim} FY$, where $\iota(e) = \mathrm{ind}_T Y + \mathrm{ind}_T Y[1]$ and $\chi(\mathrm{Gr}_e FM)$ is the Euler characteristic of the quiver Grassmannian of dimension vector e of FM (see [**27**]), where $x^{\sum_{i=1}^{n-1} a_i[T_i]} = \prod_{i=1}^{n-1} x_i^{a_i}$.

The definition of X_M can be extended to rigid objects: for any rigid object $N = \bigoplus_{i=1}^m M_i$ in $\mathcal{C}_n$ with M_i indecomposable, we define

$$X_N = \prod_{i=1}^m X_{M_i}.$$

In [**33**, **34**], a similar definition is given with respect to a fixed rigid object T in a 2-Calabi–Yau triangulated category with infinite Hom-spaces. The definition there needs an additional assumption that any finite-dimensional B-module can be lifted through F to an object in $\mathcal{D}\cap\mathcal{D}[1]$. This assumption is not satisfied in our situation. For example, the simple B-module S_1 corresponding to T_1 cannot lift to any object in $\mathcal{D}\cap\mathcal{D}[1]$ by the functor F (see the dimension formula in Lemma 2.4). So the definition in [**33**, **34**] cannot apply to our case. In our definition, we omit the B-submodules that cannot be lifted through F to $\mathcal{D}\cap\mathcal{D}[1]$.

The following lemma points out that, for an indecomposable rigid object M, if a B-submodule of FM cannot be lifted to $\mathcal{D}\cap\mathcal{D}[1]$, then neither can other B-submodules of FM with the same dimension be lifted to $\mathcal{D}\cap\mathcal{D}[1]$. So the Euler characteristic of $\mathrm{Gr}_e(FM)$ is well defined.

Lemma 3.4. *Let M be an indecomposable rigid object in $\mathcal{C}_n$ and Y be an object in $\mathcal{D}$ such that FY is a B-submodule of FM. Then $Y\in\mathcal{D}\cap\mathcal{D}[1]$ if and only if* $\dim\mathrm{Hom}_{\mathcal{C}_n}(T_1[-1], Y) \neq 1$.

Proof. For an object Y in $\mathcal{D}$, it is easy to see that $Y\in\mathcal{D}\cap\mathcal{D}[1]$ if and only if all indecomposable summands of Y are in $O\cup\mathrm{I}\cup\mathrm{II}\cup\mathrm{III}$; see Figure 2. Since $\dim\mathrm{Hom}(T_1[-1],(a,b)) \leqslant 2$, it follows that $\dim\mathrm{Hom}(T_1[-1], Y) = 2$ implies that $\dim\mathrm{Hom}(T_1[-1],(a,b)) = 2$. In this case, $f_1^{(a,b)}$ is a generator of FM as a B-module by Lemma 2.5 and $f_1^{(a,b)}\in FY$. So we have that $FY\cong FM$. Then $Y\cong M\oplus T'$, where $T'\in\mathrm{add}\,T$. Hence $Y\in\mathcal{D}\cap\mathcal{D}[1]$. When $\dim\mathrm{Hom}(T_1[-1], Y) = 1$, Y has some summand in IV by the dimension formula in Lemma 2.4, then $Y\notin\mathcal{D}\cap\mathcal{D}[1]$. When $\dim\mathrm{Hom}(T_1[-1], Y) = 0$, all summands of Y are in O or I; then $Y\in\mathcal{D}\cap\mathcal{D}[1]$. Thus, the statement holds. □

We determine the submodules of $F(a,b)$ for any indecomposable rigid object (a,b). Put

$$\mathfrak{X}(a,b) = \{(a,k) \mid 0\leqslant k\leqslant \min\{n-a,b\}\}\subset \mathrm{I}$$

and

$$\mathfrak{Y}(a,b) = \{(a+b-n+1,l) \mid 0\leqslant l\leqslant 2n-a-b-1\}\subset \mathrm{I}.$$

We denote by $\mathrm{Sub}(a,b)$ the set of submodules of $F(a,b)$, the first terms of whose dimension vectors are not 1.

Lemma 3.5. *Let (a,b) be an indecomposable rigid object in $\mathcal{D}$. Then*

$$Sub(a,b) = \begin{cases} \{FX \mid X\in\mathfrak{X}(a,b)\} & \text{if } (a,b)\in\mathrm{I};\\ \{FX\oplus FY \mid X\in\mathfrak{X}(a,b), Y\in\mathfrak{Y}(a,b)\}\cup\{(a,b)\} & \text{if } (a,b)\in\mathrm{II}.\end{cases}$$

For every dimension vector $e \neq \dim F(a,b)$*, whose first term is not 1,* $\chi(\mathrm{Gr}_e F(a,b))$ *is equal to the number of elements in the set* $sub_e(a,b)$*, where*

$$sub_e(a,b) = \begin{cases} \{X \in \mathfrak{X}(a,b) \mid \dim FX = e\} & \text{if } (a,b) \in \mathrm{I}; \\ \{(X,Y) \in \mathfrak{X}(a,b) \times \mathfrak{Y}(a,b) \mid \dim(FX \oplus FY) = e\} & \text{if } (a,b) \in \mathrm{II}. \end{cases}$$

Proof. (i) The case $(a,b) \in \mathrm{I}$. By Lemma 2.5, a basis of $F(a,b)$ is

$$\{f_i^{(a,b)} \mid n-a-b+2 \leqslant i \leqslant n-a+1\},$$

which satisfies the conditions

$$f_i^{(a,b)}\mathrm{id}_j = \begin{cases} f_i^{(a,b)} & \text{if } j = i, \\ 0 & \text{if } j \neq i; \end{cases}$$

$$f_i^{(a,b)}t_j^{j-1} = \begin{cases} f_{i+1}^{(a,b)} & \text{if } j = i+1 \leqslant n-a+1, \\ 0 & \text{otherwise}; \end{cases}$$

$$f_i^{(a,b)}g_1^1 = 0.$$

For every nonzero submodule S of $F(a,b)$, there exists the minimal i_0, $n-a-b+2 \leqslant i_0 \leqslant n-a+2$ such that

$$f_{i_0}^{(a,b)} + \lambda_1 f_{i_0+1}^{(a,b)} + \cdots + \lambda_{n-a-i_0+1} f_{n-a+1}^{(a,b)} \in S.$$

By multiplying with id_{i_0}, we have that the first term $f_{i_0}^{(a,b)}$ is in S. Let $k = n-a-i_0+2$; then $0 \leqslant k \leqslant b$ and a basis of S is

$$\{f_i^{(a,b)} \mid n-a-k+2 \leqslant i \leqslant n-a+1\}.$$

Consider the module $F(a,k)$ whose basis is

$$\{f_i^{(a,k)} \mid n-a-k+2 \leqslant i \leqslant n-a+1\},$$

which satisfies the conditions

$$f_i^{(a,k)}\mathrm{id}_{i'} = \begin{cases} f_i^{(a,k)} & \text{if } i' = i, \\ 0 & \text{if } j \neq i; \end{cases}$$

$$f_i^{(a,k)}t_{i'}^{i'-1} = \begin{cases} f_{i+1}^{(a,k)} & \text{if } i' = i+1 \leqslant n-a+1, \\ 0 & \text{otherwise}; \end{cases}$$

$$f_i^{(a,k)}g_1^1 = 0.$$

So there is a natural isomorphism of modules:

$$S \longrightarrow F(a, n-a-k+2),$$
$$f_i^{(a,b)} \longmapsto f_i^{(a,k)}.$$

Clearly, every dimension vector corresponds to one submodule and $\chi(\mathrm{Gr}_e F(a,b)) = 1$. We complete the proof in this case.

(ii) The case $(a,b) \in \mathrm{II}$. In this case, by Lemma 2.5, there is a basis

$$\{f_i^{(a,b)}, g_j^{(a,b)} \mid 1 \leqslant i \leqslant n-a+1,\ 1 \leqslant j \leqslant 2n-a-b\}$$

of $F(a,b)$ and

$$f_i^{(a,b)}\mathrm{id}_{i'} = \begin{cases} f_i^{(a,b)} & \text{if } i'=i,\\ 0 & \text{if } i'\neq i;\end{cases}$$

$$f_i^{(a,b)}t_{i'}^{i'-1} = \begin{cases} f_{i+1}^{(a,b)} & \text{if } i'=i+1\leqslant n-a+1,\\ 0 & \text{otherwise};\end{cases}$$

$$f_i^{(a,b)}g_1^1 = \begin{cases} g_1^{(a,b)} & \text{if } i=1,\\ 0 & \text{if } i\neq 1;\end{cases}$$

$$g_j^{(a,b)}\mathrm{id}_{j'} = \begin{cases} g_j^{(a,b)} & \text{if } j'=j,\\ 0 & \text{if } j'\neq j;\end{cases}$$

$$g_j^{(a,b)}t_{j'}^{j'-1} = \begin{cases} g_{j+1}^{(a,b)} & \text{if } j'=j+1\leqslant 2n-a-b,\\ 0 & \text{otherwise};\end{cases}$$

$$g_j^{(a,b)}g_1^1 = 0.$$

For a submodule $S\in\mathrm{Sub}(a,b)$, if the first term of dimension vector of S is not 0, then $f_1^{(a,b)}\in S$. Since $f_1^{(a,b)}$ is a generator of $F(a,b)$, $S=F(a,b)$. If $f_1^{(a,b)}, g_1^{(a,b)}\notin S$, by a similar discussion as above, there is a minimal i_0, $2\leqslant i_0\leqslant n-a+2$ (respectively, a minimal j_0, $2\leqslant j_0\leqslant 2n-a-b+1$) such that $f_{i_0}^{(a,b)}$ is in S (respectively, $\lambda f_{j_0}^{(a,b)}+g_{j_0}^{(a,b)}$ is in S for some $\lambda\in k$). Let $k=n-a-i_0+2$, $l=2n-a-b-j_0+1$; then $0\leqslant k\leqslant n-a$, $0\leqslant l\leqslant 2n-a-b-1$, and a basis of S is

$$\left\{ f_i^{(a,b)}, \lambda f_j^{(a,b)}+g_j^{(a,b)} \,\middle|\, \begin{matrix} n-a-k+2\leqslant i\leqslant n-a+1,\\ 2n-a-b-l+1\leqslant j\leqslant 2n-a-b\end{matrix}\right\}.$$

Consider the module $F(a,k)\oplus F(a+b-n+1,l)$ whose basis is

$$\left\{ f_i^{(a,k)}, f_j^{(a+b-n+1,l)} \,\middle|\, \begin{matrix} n-a-k+2\leqslant i\leqslant n-a+1,\\ 2n-a-b-l+1\leqslant j\leqslant 2n-a-b\end{matrix}\right\},$$

which satisfies the conditions

$$f_i^{(a,k)}\mathrm{id}_{i'} = \begin{cases} f_i^{(a,k)} & \text{if } i'=i,\\ 0 & \text{if } i'\neq i;\end{cases}$$

$$f_i^{(a,k)}t_{i'}^{i'-1} = \begin{cases} f_{i+1}^{(a,k)} & \text{if } i'=i+1\leqslant n-a+1,\\ 0 & \text{otherwise};\end{cases}$$

$$f_i^{(a,k)}g_1^1 = 0;$$

$$f_j^{(a+b-n+1,l)}\mathrm{id}_{j'} = \begin{cases} f_j^{(a+b-n+1,l)} & \text{if } j'=j,\\ 0 & \text{if } j'\neq j;\end{cases}$$

$$f_j^{(a+b-n+1,l)}t_{j'}^{j'-1} = \begin{cases} f_{j+1}^{(a+b-n+1,l)} & \text{if } j'=j+1\leqslant 2n-a-b,\\ 0 & \text{otherwise};\end{cases}$$

$$f_j^{(a+b-n+1,l)}g_1^1 = 0.$$

So there is a natural isomorphism of modules:

$$\begin{aligned} S &\longrightarrow F(a, n-a-k+2), \\ f_i^{(a,b)} &\longmapsto f_i^{(a,k)}, \\ \lambda f_j^{(a,b)} + g_j^{(a,b)} &\longmapsto f_j^{(a+b-n+1,l)}. \end{aligned}$$

Using the same notation as given above, to get the formula of the Euler character, we only need to prove that the Euler character of the Grassmannian associated to the submodule S above with all values of λ chosen to be 1, since any λ offer the same submodule up to isomorphism and χ is additive with respect to disjoint unions. If $i_0 \geqslant j_0$, then the associated Grassmannian contains only one point, so its Euler character is 1; if $i_0 < j_0$, then the associated Grassmannian is $\mathbb{P}^1 \setminus \{\text{a point}\}$, so its Euler character is also 1. Thus, we complete the proof. $\square$

Before giving the explicit formula of $X_{(a,b)}$, we need to show why $\iota(e)$ does not depend on the choice of Y. Define

$$\iota(Y) = \mathrm{ind}_T Y + \mathrm{ind}_T Y[1].$$

Clearly, $\iota(Y_1 \oplus Y_2) = \iota(Y_1) + \iota(Y_2)$ and $\iota(T') = 0$ for any $T' \in \operatorname{add} T$. If there are two submodules $FY \cong FY'$, then $Y \oplus T_1 \cong Y' \oplus T_2$ with $T_1, T_2 \in \operatorname{add} T$. So $\iota(Y) = \iota(Y')$. If there are two nonisomorphic submodules FY, FY' with the same dimension, by Lemma 3.5 and the discussion above, we can write $Y = (a, k) \oplus (a + b - n + 1, l)$ and $Y' = (a, k') \oplus (a + b - n + 1, l')$. Then

$$\dim FY = \sum_{i=n-a-k+2}^{n-a+1} e_i + \sum_{j=2n-a-b-l+1}^{2n-a-b} e_j = \dim FY' = \sum_{i=n-a-k'+2}^{n-a+1} e_i + \sum_{j=2n-a-b-l'+1}^{2n-a-b} e_j,$$

where e_i is the dimension vector of the simple module of B corresponding to T_i. Note that $k = k'$ implies $l = l'$ and FY is not isomorphic to FY', so $k \neq k'$. Without loss of generality, we assume that $k > k'$. Then $k' = l - n + b + 1$, $l' = k + n - b - 1$. By Lemma 3.2, we have that

$$\iota((a, k) \oplus (a + b - n + 1, l)) = [T_{n-a-k+1}] - [T_{n-a+1}] + [T_{2n-a-b-l}] - [T_{2n-a-b}],$$

and

$$\iota((a, k') \oplus (a + b - n + 1, l')) = [T_{2n-a-b-l}] - [T_{n-a+1}] + [T_{n-a-k+1}] - [T_{2n-a-b}].$$

So $\iota(Y) = \iota(Y')$. Combining with Lemma 3.5, we have a new form of Definition 3.3.

Lemma 3.6. *For any indecomposable rigid object (a, b) in $\mathcal{C}_n$, we have that*

$$X_{(a,b)} = \begin{cases} x^{\mathrm{ind}_T(a,b)} & \text{if } (a,b) \in O; \\ x^{\mathrm{ind}_T(a,b)} \displaystyle\sum_{X \in \mathfrak{X}(a,b)} x^{-\iota(X)} & \text{if } (a,b) \in \mathrm{I}; \\ x^{\mathrm{ind}_T(a,b)} \left(x^{-\iota(a,b)} + \displaystyle\sum_{X \in \mathfrak{X}(a,b), Y \in \mathfrak{Y}(a,b)} x^{-\iota(X \oplus Y)} \right) & \text{if } (a,b) \in \mathrm{II}. \end{cases}$$

Now we give the explicit expression of $X_{(a,b)}$ for each indecomposable rigid object (a, b). In the following expression of x_m, when $m = n$, x_m is taken to be 1. In this setting, the first formula in Lemma 3.2 also holds in the case of $a + b = n$.

THEOREM 3.7. *For any indecomposable rigid object (a,b), we have that*

$$X_{(a,b)} = \begin{cases} x_{n-b} & \text{if } (a,b) \in O; \\ \displaystyle\sum_{k=0}^{b} \frac{x_{n-a-b+1}x_{n-a+2}}{x_{n-a-k+1}x_{n-a-k+2}} & \text{if } (a,b) \in \mathrm{I}; \\ x_1 x_{K+2} x_{L+2}\left(\dfrac{1}{x_1^2} + \left(\displaystyle\sum_{k=0}^{K} \frac{1}{x_{K-k+1}x_{K-k+2}}\right)\left(\displaystyle\sum_{l=0}^{L} \frac{1}{x_{L-l+1}x_{L-l+2}}\right)\right) & \text{if } (a,b) \in \mathrm{II}; \end{cases}$$

where $K = n-a$, $L = 2n-a-b-1$.

Proof.

(i) The case of $(a,b) \in O$. We have $X_{(a,b)} = x^{\mathrm{ind}_T(a,b)} = x_{n-b}$.

(ii) The case of $(a,b) \in \mathrm{I}$. By Lemma 3.6, we have that

$$\begin{aligned} X_{(a,b)} &= x^{\mathrm{ind}_T(a,b)} \sum_{X \in \mathfrak{X}(a,b)} x^{-\iota(X)} \\ &= x^{\mathrm{ind}_T(a,b)} \sum_{k=0}^{b} x^{-\iota(a,k)} \\ &= \frac{x_{n-a-b+1}}{x_{n-a+1}} \sum_{k=0}^{b} \frac{x_{n-a+1}x_{n-a+2}}{x_{n-a-k+1}x_{n-a-k+2}} \\ &= \sum_{k=0}^{b} \frac{x_{n-a-b+1}x_{n-a+2}}{x_{n-a-k+1}x_{n-a-k+2}}. \end{aligned}$$

(iii) The case of $(a,b) \in \mathrm{II}$. By Lemma 3.6, we have that

$$\begin{aligned} X_{(a,b)} &= x^{\mathrm{ind}_T(a,b)} \left(x^{-\iota(a,b)} + \sum_{X \in \mathfrak{X}(a,b), Y \in \mathfrak{Y}(a,b)} x^{-\iota(X \oplus Y)} \right) \\ &= x^{\mathrm{ind}_T(a,b)} \left(x^{-\iota(a,b)} + \left(\sum_{k=0}^{b} x^{-\iota(a,k)} \right) \left(\sum_{l=0}^{2n-a-b-l} x^{-\iota(a+b-n+1,l)} \right) \right) \\ &= x_1 x_{K+2} x_{L+2} \left(\frac{1}{x_1^2} + \left(\sum_{k=0}^{K} \frac{1}{x_{K-k+1}x_{K-k+2}} \right) \left(\sum_{l=0}^{L} \frac{1}{x_{L-l+1}x_{L-l+2}} \right) \right). \end{aligned}$$

□

4. *Mutation relations*

By [**5**, Proposition 2.6], every maximal rigid object in $\mathcal{C}_n$ is in some wing of $(a, n-1)$ and there is a natural bijection between the set of maximal rigid objects in the wing of $(a, n-1)$, and the set of tilting modules over the path algebra $k\vec{A}_{n-1}$ of the linear quiver of type A_{n-1}. Hence, using the complete description of all tilting modules of quivers of type A in [**22**], we can get all maximal rigid objects by the following induction starting with an chosen object $(a, n-1)$: for each chosen object (a,b), one choose two objects $(a, h-1)$ and $(a+h, b-h)$ until all the new objects are in the bottom of the tube; see Figure 3; take the direct sum of all chosen objects.

Under this construction of maximal rigid objects, we have the following fact.

LEMMA 4.1. *Let R be a maximal rigid object in $\mathcal{C}_n$ and (a,b) be an indecomposable summand of R. Then there are $(a,h-1)$ and $(a+h,b-h)$ in* add R *for some h, $1 \leqslant h \leqslant b$*

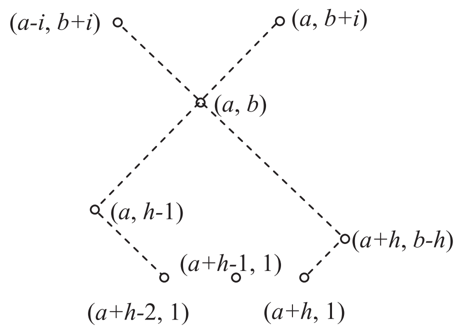

Figure 3. *The construction of maximal rigid objects.*

and every indecomposable summand of R in the wing of (a,b) that is not isomorphic to (a,b) is in the wing of $(a,h-1)$ or in the wing of $(a+h,b-h)$. When $b \neq n-1$, there exists an i, $1 \leqslant i \leqslant n-b-1$, such that either $(a,b+i)$ or $(a-i,b+i)$ is in $\operatorname{add} R$*; see Figure 3.*

By this lemma, we can give all exchange triangles in $\mathcal{C}_n$.

Lemma 4.2. *Given two basic maximal rigid objects $T' \oplus \bar{T}$ and $T'' \oplus \bar{T}$ in $\mathcal{C}_n$ such that both T' and T'' are indecomposable. Then $\dim \operatorname{Ext}^1_{\mathcal{C}_n}(T',T'')=1$ or 2. Moreover, if $\dim \operatorname{Ext}^1_{\mathcal{C}_n}(T',T)=2$, then the exchange triangles have the following form:*

$$(a,n-1) \longrightarrow (a+h,n-h-1)\oplus(a+h,n-h-1) \longrightarrow (a+h,n-1) \longrightarrow (a,n-1)[1],$$
$$(a+h,n-1) \longrightarrow (a,h-1)\oplus(a,h-1) \longrightarrow (a,n-1) \longrightarrow (a+h,n-1)[1],$$

where $1 \leqslant a \leqslant n$, $1 \leqslant h \leqslant n-1$. If $\dim \operatorname{Ext}^1_{\mathcal{C}_n}(T',T)=1$, then the exchange triangles have the following form:

$$(a,b) \longrightarrow (a,b+i)\oplus(a+h,b-h) \longrightarrow (a+h,b-h+i) \longrightarrow (a,b)[1],$$
$$(a+h,b-h+i) \longrightarrow (a+b+1,i-1)\oplus(a,h-1) \longrightarrow (a,b) \longrightarrow (a+h,b-h+i)[1],$$

where $1 \leqslant a \leqslant n$, $b \leqslant n-2$, $1 \leqslant h \leqslant b$, $1 \leqslant i \leqslant n-b-1$.

Proof. Let $(a,n-1)\oplus\bar{R}$ be a basic maximal rigid object, $1 \leqslant a \leqslant n$. Then, by Lemma 4.1, there is an h, $1 \leqslant h \leqslant n-1$, such that $(a,h-1)$ and $(a+h,n-h-1)$ are in $\operatorname{add}\bar{R}$ and other indecomposable summands of $\bar{R}$ are in the wing of $(a,h-1)$ or in the wing of $(a+h,n-h-1)$; see Figure 4. Now $(a+h,n-1)\oplus\bar{R}$ is a basic maximal rigid object.

Consider a nonzero $\mathcal{T}$-map $f:(a+h,n-1)\to(a,n-1)[1]=(a-1,n-1)$, we have a triangle in $\mathcal{C}_n$:

$$C \longrightarrow (a+h,n-1) \xrightarrow{f} (a,n-1)[1] \longrightarrow C[1]$$

with an exact sequence

$$0 \longrightarrow \operatorname{coker}(\tau^{-1}f) \longrightarrow C \longrightarrow \ker(f) \longrightarrow 0$$

in $\mathcal{T}_n$ (cf. [**37**]). Since $\ker(f)=(a+h,n-1-h)$, $\operatorname{coker}(\tau^{-1}f)=(a+h,n-1-h)$, and $(a+h,n-1-h)$ is rigid in $\mathcal{T}_n$, we have that $C\cong(a+h,n-1-h)\oplus(a+h,n-1-h)$. Thus, we get a triangle in $\mathcal{C}_n$:

$$(a,n-1) \longrightarrow (a+h,n-1-h)\oplus(a+h,n-1-h) \longrightarrow (a+h,n-1) \xrightarrow{f} (a,n-1)[1]$$

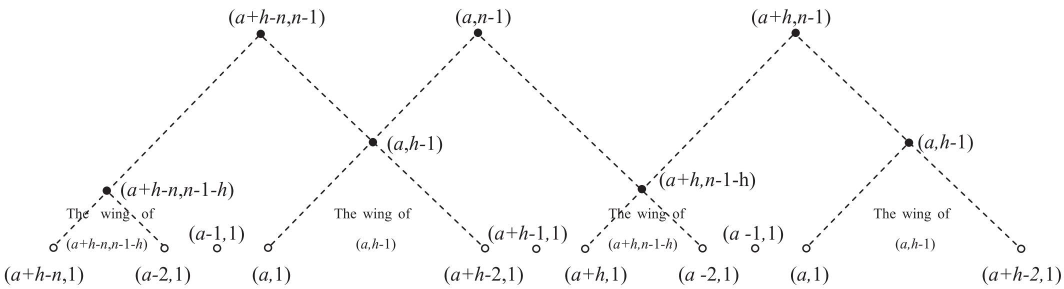

FIGURE 4. *Mutations of maximal rigid objects as in the first part of the proof of Lemma* 4.2.

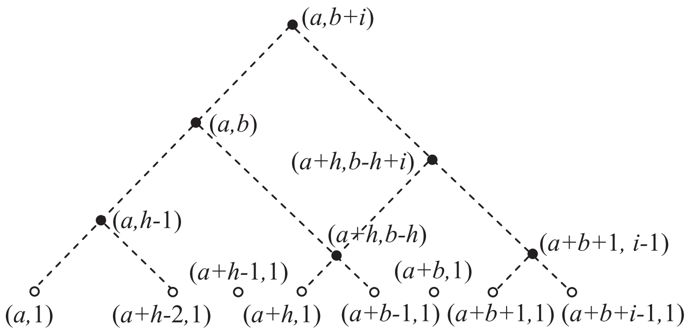

FIGURE 5. *Mutations of maximal rigid objects as in the second part of the proof of Lemma* 4.2.

and it is the exchange triangle starting at $(a, n-1)$ associated to $\bar{R}$ since $(a+h, n-1)$ is another complement of $\bar{R}$ and $(a+h, n-1-h) \in \operatorname{add} \bar{R}$. The exchange triangle starting at $(a+h, n-1-h)$ associated to $\bar{R}$ can be obtained similarly. Clearly, $\dim \operatorname{Ext}^1_{\mathcal{C}_n}((a, n-1), (a+h, n-1)) = 2$.

Let $(a, b) \oplus \bar{R}$ be a basic maximal rigid object with $1 \leqslant a \leqslant n$, $b \leqslant n-2$. By Lemma 4.1, we can assume that there is the minimum i, $1 \leqslant i \leqslant n-b-1$, such that $(a, b+i) \in \bar{R}$. By the construction of maximal objects, $(a+b+1, i-1) \in \bar{R}$; see Figure 5. Then $(a+h, b-h+i) \oplus \bar{R}$ is a basic maximal rigid object. Clearly, $\dim \operatorname{Ext}^1_{\mathcal{C}_n}((a, b), (a+h, b-h+i)) = 1$.

There is an obvious nonsplit triangle in $\mathcal{C}_n$:

$$(a, b) \longrightarrow (a, b+i) \oplus (a+h, b-h) \longrightarrow (a+h, b-h+i) \longrightarrow (a, b)[1],$$

which is the exchange triangle starting at (a, b) associated to $\bar{R}$. To get the other exchange triangle, we consider a nonzero $\mathcal{T}$-map $f : (a, b) \to (a+h, b-h+i)[1] = (a+h-1, b-h+i)$. We have that $\ker(f) = (a, h-1)$ and $\operatorname{coker}(\tau^{-1} f) = (a+b+1, i-1)$ in $\mathcal{T}_n$. Since $\operatorname{Ext}^1_{\mathcal{T}_n}((a, h-1), (a+b+1, i-1)) = 0$, we get a nonsplit triangle in $\mathcal{C}_n$:

$$(a+h, b-h+i) \longrightarrow (a+b+1, i-1) \oplus (a, h-1) \longrightarrow (a, b) \longrightarrow (a+h, b-h+i)[1],$$

which is what we need. □

The main result in this section is that the cluster map defined in Section 3 satisfies the cluster structure of $\mathcal{C}_n$.

THEOREM 4.3. *Given two basic maximal rigid objects $T' \oplus \bar{T}$ and $T'' \oplus \bar{T}$ in $\mathcal{C}_n$ such that both T' and T'' are indecomposable. Then $X_{T'}X_{T''} = X_E + X_{E'}$, where $T' \to E \to T'' \to T'[1]$ and $T'' \to E' \to T' \to T''[1]$ are the exchange triangles.*

Proof. Since both the formulas of cluster maps in Theorem 3.7 and the exchange triangles in Lemma 4.2 depend on the positions of objects, we can check that $X_{T'}X_{T''} = X_E + X_{E'}$ case by case. We omit the details here. $\square$

For the fixed maximal rigid object $T = \oplus_{i=1}^{n-1} T_i$, where $T_i = (1, n-i)$, from Lemma 4.1 (or [5, Proposition 3.4]), we have that the matrix A_T associated to T is

$$A_T = \begin{pmatrix} 0 & 1 & & & & \\ -2 & 0 & 1 & & & \\ & -1 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

Its Cartan part is of type C_{n-1}.

Let $\operatorname{indr} \mathcal{C}_n$ be the set of indecomposable rigid objects in $\mathcal{C}_n$, and χ_{A_T} be the set of cluster variables of the skew-symmetrizable matrix A_T associated to T. Then we have the following corollary.

COROLLARY 4.4. *The map $X_? : \operatorname{indr} \mathcal{C}_n \to \chi_{A_T} : M \mapsto X_M$ is a bijection. This bijection induces a bijection between the set of isoclasses of basic maximal rigid objects in $\mathcal{C}_n$ and the set of clusters of type C_{n-1}. Furthermore, the algebra generated by all X_M, where M runs through $\operatorname{indr} \mathcal{C}_n$, is isomorphic to the cluster algebra of type C_{n-1}.*

Proof. From Theorem 3.7, we have that $X_{T_i} = x_i$. Then $X_?$ sends the couple $(\{T_1, \ldots, T_{n-1}\}, A_T)$ to the initial seed $(\{x_1, \ldots, x_{n-1}\}, A_T)$ of the cluster algebra $\mathcal{A}_{A_T}$. By Lemma 4.2, we have that the exchange graph of maximal rigid objects in $\mathcal{C}_n$ is connected. Then, by Theorem 4.3 and Theorem 1.1 in [5], $X_?(\operatorname{indr} \mathcal{C}_n) \subset \chi_{A_T}$, $X_? : \operatorname{indr} \mathcal{C}_n \to \chi_{A_T}$ is surjective and $X_?$ induces a map sending basic maximal rigid objects to clusters. It follows from Theorem 3.7 that the denominators of X_M are different for all indecomposable rigid objects. Then $X_? : \operatorname{indr} \mathcal{C}_n \to \chi_{A_T}$ is injective. Thus, the statements hold. $\square$

5. *Cluster complex of type C*

In this section, we use the results proved in Sections 3 and 4 to prove that the combinatorics of indecomposable rigid objects in $\mathcal{C}_n$ encodes the cluster combinatorics of the root system of type C and type B. Cluster complexes were defined in [16] for finite root systems. They were realized by quiver representations via decorated representations [30], and later via cluster categories of the corresponding quivers [4, 39]. Combining with the geometric description of the cluster complex of the root system of type B [16], Buan–Marsh–Vatne [5] give a realization of this cluster complex via cluster tubes.

We recall the cluster complex associated to any finite root system from [16]. Let Φ be any finite root system with simple roots $\alpha_1, \ldots, \alpha_n$ and $\Phi_{\geqslant -1}$ be the set of almost positive roots in Φ, that is, the union of positive roots with negative simple roots. Fomin and Zelevinsky define a function

$$(- \mid\mid -) : \Phi_{\geqslant -1} \times \Phi_{\geqslant -1} \longrightarrow \mathbb{Z}_{\geqslant 0}$$

called the compatibility degree. A pair of roots α, β in $\Phi_{\geqslant -1}$ are compatible if $(\alpha \mid\mid \beta) = 0$. The cluster complex $\Delta(\Phi)$ associated to Φ is a simplicial complex, the set of vertices is $\Phi_{\geqslant -1}$ and the simplices are mutually compatible subsets of $\Phi_{\geqslant -1}$. This combinatorial object has many interesting properties and applications; we refer the reader to the survey [**14**] for further reading.

For the cluster tube $\mathcal{C}_n$, we call a set of indecomposable objects a rigid subset, provided the direct sum of all indecomposable objects in this set is rigid. Now we define a simplicial complex associated to the cluster tube $\mathcal{C}_n$. We always assume $n > 1$ throughout this section.

Definition 5.1. Let $\mathcal{C}_n$ be the cluster tube of rank n. The cluster complex $\Delta(\mathcal{C}_n)$ associated to $\mathcal{C}_n$ is a simplicial complex whose vertices are the isoclasses of indecomposable rigid objects and whose simplices are the isoclasses of rigid subsets of $\mathcal{C}_n$.

Now fix a root system Φ^C of type C_{n-1}. In [**15**], Fomin–Zelevinsky give a bijection from the set of cluster variables of the cluster algebras of type C_{n-1} to $\Phi^C_{\geqslant -1}$ (they gave this bijection for all finite root systems). Under this bijection, when a cluster variable y is expressed as

$$y = \frac{P(x)}{x^{\alpha}},$$

where P is a polynomial that is not divisible by x_i for every i, the corresponding almost positive root is α. Therefore, combining this bijection with the bijective map $X_?$ from $\operatorname{indr}\mathcal{C}_n$ to χ_{A_T} in Section 3, we have a bijection from $\Phi^C_{\geqslant -1}$ to $\operatorname{indr}\mathcal{C}_n$. This map is denoted by M_T. So, for any $\alpha \in \Phi^C_{\geqslant -1}$, we denote by $M_T(\alpha)$ the object in $\mathcal{C}_n$ corresponding to α under this bijection.

Theorem 5.2. *Let Φ^C be the root system of type C_{n-1}. Then the map M_T induces an isomorphism from the cluster complex $\Delta(\Phi^C)$ to the cluster complex $\Delta(\mathcal{C}_n)$, which sends vertices to vertices, and simplices to simplices.*

To prove the theorem, we need some preparation:

Definition 5.3. For any two almost positive roots $\alpha, \beta \in \Phi^C_{\geqslant -1}$, we define the T-compatibility degree $(\alpha \mid\mid \beta)_T$ of α, β by

$$(\alpha \mid\mid \beta)_T = \frac{\dim \operatorname{Ext}^1(M_T(\alpha), M_T(\beta))}{\dim \operatorname{End}(M_T(\alpha))}.$$

As in [**16**], let σ_i be the permutation of $\Phi^C_{\geqslant -1}$ defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j, j \neq i, \\ s_i(\alpha) & \text{otherwise,} \end{cases}$$

where s_i is the Coxeter generator of the Weyl group of Φ^C corresponding to i. We denote by R the Coxeter element $\sigma_1 \cdots \sigma_{n-1}$ in the Coxeter group (cf. [**39**]). The function $(- \mid\mid -)$ is determined by the following two properties [**16**]:

$$(-\alpha_i \mid\mid \beta) = \max([\beta : \alpha_i], 0),$$
$$(R\alpha \mid\mid R\beta) = (\alpha \mid\mid \beta),$$

where $[\beta : \alpha_i]$ denotes the coefficient of α_i in the expansion of β in the basis $\{\alpha_1, \ldots, \alpha_n\}$. We will prove that the function $(- \,||\, -)_T$ satisfies the same properties.

LEMMA 5.4. *For any $\alpha \in \Phi^C_{\geqslant -1}$, $M_T(R\alpha) = \tau M_T(\alpha)$.*

Proof. Owing to $\operatorname{indr} \mathcal{C}_n = O \cup \mathrm{I} \cup \mathrm{II}$ (please see the definitions of $O, \mathrm{I}, \mathrm{II}$ and Figure 2 in Section 2), we divide the proof into several cases according to the position of the object $M_T(\alpha)$.

1. When $M_T(\alpha) = (a, b) \in O$, then $\alpha = -\alpha_{n-b}$. We have

$$R\alpha = \sigma_1 \cdots \sigma_{n-1}(\alpha) = \sum_{i=1}^{n-b} \alpha_i.$$

Hence $M_T(R\alpha) = (n, n-b) = \tau(1, n-b) = \tau M_T(\alpha)$.

2. When $M_T(\alpha) = (a, b) \in \mathrm{I}$, then $\alpha = \sum_{i=n-a-b+2}^{n-a+1} \alpha_i$. If $a \neq 2$, then

$$R\alpha = \sum_{i=n-a-b+3}^{n-a+2} \alpha_i.$$

Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$. If $a = 2$, then

$$R\alpha = -\alpha_{n-b}.$$

Hence $M_T(R\alpha) = (1, b) = \tau(2, b) = \tau M_T(\alpha)$.

3. When $M_T(\alpha) = (2, n-1) \in \mathrm{II}$, then $\alpha = \alpha_1 + 2\sum_{i=2}^{n-1} \alpha_i$. We have

$$R\alpha = -\alpha_1.$$

Hence $M_T(R\alpha) = (1, n-1) = \tau(2, n-1) = \tau M_T(\alpha)$.

4. When $M_T(\alpha) = (a, n-1) \in \mathrm{II}$ with $3 \leqslant a \leqslant n$, then $\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+1} \alpha_i$. We have

$$R\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+2} \alpha_i.$$

Hence $M_T(R\alpha) = (a-1, n-1) = \tau(a, n-1) = \tau M_T(\alpha)$.

5. When $M_T(\alpha) = (a, b) \in \mathrm{II}$ with $a + b = n + 1$ and $2 \leqslant b \leqslant n-2$, then $\alpha = \alpha_1 + 2\sum_{i=1}^{n-a+1} \alpha_i + \sum_{i=n-a+2}^{2n-a-b} \alpha_i$. It follows that

$$R\alpha = \sum_{i=2}^{n-a+2} \alpha_i.$$

Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$.

6. When $M_T(\alpha) = (n, 1) \in \mathrm{II}$, then $\alpha = \sum_{i=1}^{n-1} \alpha_i$. It follows that

$$R\alpha = \alpha_2.$$

Hence $M_T(R\alpha) = (n-1, 1) = \tau(n, 1) = \tau M_T(\alpha)$.

7. When $M_T(\alpha) = (a, b) \in \mathrm{II}$ with $a = n$ and $2 \leqslant b \leqslant n-2$, then $\alpha = \sum_{i=1}^{n-b} \alpha_i$. It follows that

$$R\alpha = \alpha_1 + 2\alpha_2 + \sum_{i=3}^{n-b+1} \alpha_i.$$

Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$.

8. When $M_T(\alpha) = (a, b) \in \mathrm{II}$ with $a < n, 1 < b < n-1$ and $a + b > n + 1$, then $\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+1} \alpha_i + \sum_{i=n-a+2}^{2n-a-b} \alpha_i$. It follows that

$$R\alpha = \alpha_1 + 2\sum_{i=2}^{n-a+2} \alpha_i + \sum_{i=n-a+3}^{2n-a-b+1} \alpha_i.$$

Hence $M_T(R\alpha) = (a-1, b) = \tau(a, b) = \tau M_T(\alpha)$.

The proof of this lemma is completed. $\square$

Using the dimension formulas in the proof of Theorem 3.7, we have the following fact:

LEMMA 5.5. *For any positive root β and any i, $[\beta : \alpha_i] = \dim \mathrm{Hom}(T_i[-1], M_T(\beta))/\dim \mathrm{End}(T_i)$.*

LEMMA 5.6. *The T-compatibility degree satisfies the following conditions*:

$$(-\alpha_i \mid\mid \beta)_T = \max([\beta : \alpha_i], 0), \tag{1}$$

$$(R\alpha \mid\mid R\beta)_T = (\alpha \mid\mid \beta)_T, \tag{2}$$

for any $\alpha, \beta \in \Phi^C_{\geqslant -1}$, with any $1 \leqslant i \leqslant n-1$.

Proof. By the definition,

$$(-\alpha_i \mid\mid \beta)_T = \frac{\dim \mathrm{Ext}^1(T_i, M_T(\beta))}{\dim \mathrm{End}(T_i)} = \frac{\dim \mathrm{Hom}(T_i[-1], M_T(\beta))}{\dim \mathrm{End}(T_i)}.$$

Then, by Lemma 5.5 it equals $[\beta : \alpha_i]$ if β is a positive root, or 0 otherwise. This proves that (1) holds. From Lemma 5.4, we have

$$(R\alpha \mid\mid R\beta)_T = \frac{\dim \mathrm{Ext}^1_{\mathcal{C}}(M_T(R\alpha), M_T(R\beta))}{\dim \mathrm{End}_{\mathcal{C}}(M_T(R\alpha))} = \frac{\dim \mathrm{Ext}^1_{\mathcal{C}}(\tau M_T(\alpha), \tau M_T(\beta))}{\dim \mathrm{End}_{\mathcal{C}}(\tau M_T(\alpha))} = (\alpha \mid\mid \beta)_T.$$

This proves that (2) holds. $\square$

Proof of Theorem 5.2. By Lemmas 5.5, 5.6, the compatibility degree $(-||-)_T$ is the same as $(-||-)$ in [**16**]. It follows that α, β are compatible if and only if $M_T(\alpha), M_T(\beta)$ form a rigid subset. Therefore, M_T induces the desired bijection from $\Delta(\Phi^C)$ to $\Delta(\mathcal{C}_n)$. $\square$

Let Φ^B be the root system of type B_{n-1}. Then $\Phi^B_{\geqslant -1}$ is the dual of $\Phi^C_{\geqslant -1}$ via $\alpha \mapsto \alpha^\vee$. So $(\alpha \mid\mid \beta) = (\beta^\vee \mid\mid \alpha^\vee)$ [**16**, Proposition 3.15]. Then we have the following corollary (cf. [**5**, Theorem 3.5]).

COROLLARY 5.7. *Let Φ^B be the root system of type B_{n-1}. Then the cluster complex $\Delta(\Phi^B)$ of Φ of type B_{n-1} is isomorphic to $\Delta(\mathcal{C}_n)$.*

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