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Non-uniqueness of admissible weak solutions to compressible Euler systems with source terms

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We consider admissible weak solutions to the compressible Euler system with source terms, which include rotating shallow water system and the Euler system with damping as special examples. In the case of anti-symmetric sources such as rotations, for general piecewise Lipschitz initial densities and some suitably constructed initial momentum, we obtain infinitely many global admissible weak solutions. Furthermore, we construct a class of finite-states admissible weak solutions to the Euler system with anti-symmetric sources. Under the additional smallness assumption on the initial densities, we also obtain multiple global-in-time admissible weak solutions for more general sources including damping. The basic framework are based on the convex integration method developed by De Lellis and Székelyhidi [13,14] for the Euler system. One of the main ingredients of this paper is the construction of specified localized plane wave perturbations which are compatible with a given source term.

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1. Introduction

The well-posedness of the compressible Euler system is one of the central issues in hyperbolic balance laws. The uniqueness of admissible solutions is of fundamental importance. Although there is a quite mature well-posedness theory for one dimensional hyperbolic conservation laws with small BV initial data [12], the problem for multi-dimensional systems is very challenging. Recently, a major breakthrough for the uniqueness problem is made by De Lellis and Székelyhidi in [13,14]. Inspired by the surprising examples in [27] and [28], they developed a convex integration framework and obtained infinitely many bounded weak solutions to the incompressible Euler system. The convex integration methods are later refined to generate even Hölder continuous solutions for the incompressible Euler system, see [2,16,22]. The ideas have also been applied to other systems of PDEs; see [10,23,29,33] and the references therein. We refer to [15] for a general survey on the results in this direction.

Multiple admissible solutions to the compressible Euler system are obtained in [5,7,6,14,18], by an adaptation of the convex integration method. A major consequence is that the admissible weak solutions for the polytropic gases in multidimension are in general non-unique. These solutions, which are called “wild solutions” in the literature, reflect the flexibility of the solution space with low regularity and are quite different in nature from those in one dimensional setting such as [31]. It has been shown that many of the available criteria [7,14], with the exception of vanishing viscosity limit, are not able to single out a unique solution.

It is interesting to investigate the stability mechanisms which may help to rule out the wild solutions. A natural candidate is the lower order dissipation and dispersion, such as damping and rotating forces.

In this paper, we consider the compressible Euler system with source terms as follows

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \mathbf{B}(\rho\mathbf{u}), \end{cases} \tag{1}$$

where $(x,t)\in\Omega\times[0,\infty)$ with $\Omega=\mathbb{R}^n$ or $\mathbb{T}^n$; ρ, $\mathbf{u}$, and p denote the density, velocity, and the pressure of the flows, respectively. Assume that the equation of states satisfies $p(0)=0$ and $p'(\rho)>0$ for $\rho>0$, and $\mathbf{B}$ is an $n\times n$ constant matrix. In particular, the effects of damping and rotating forces are included in this model, where in the case of $n=2$ and non-dimensionalization, the matrix $\mathbf{B}$ has the form $\mathbf{B}=-\mathbf{I}$ and $\mathbf{B}=\mathbf{J}$, respectively, with

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

Most of the previous investigations [5,7,6,14,18] focused on the isentropic Euler system corresponding to $\mathbf{B}=\mathbf{0}$ in the system (1). The existence of infinitely many bounded admissible solutions is first showed by De Lellis and Székelyhidi in [14] for a special class of piecewise constant initial densities. The local-in-time existence of multiple admissible solutions for general smooth initial densities is proved in [5]. When the initial density is close to a constant, the global existence of multiple admissible solutions is obtained in [18]. In these works [5,14,18], the initial momentum are suitably constructed and not explicit. For a class of Riemann initial data connected by shocks, it is proved in [7,6] that the admissible weak solutions for polytropic gases in two-dimension are not unique. The non-uniqueness issue has also been studied in the class of dissipative weak solutions for the Euler–Fourier system [8], the Euler–Korteweg–Poisson system [17], and the Savage–Hutter model [19], respectively, by adaptations of the convex integration method.

The global well-posedness of classical solutions with damping and rotations has been well-known under certain smallness assumptions on the initial data; see for example [4,21,25,35]. This is in sharp contrast with the Euler system whose classical solutions generally develop singularities no matter how smooth and small the initial data are; see for example [26,30]. The main aim of this paper is to investigate whether these lower order dissipations or dispersions can prevent the loss of uniqueness for admissible weak solutions. Our next aim is to construct wild solutions with special structures in order to gain a better understanding for the fine properties of weak solutions.

The weak solutions of the system (1) are the ones which satisfy (1) in the sense of distribution, i.e.,

Definition 1 *(Weak solutions).* A pair $(\rho, \mathbf{m}) \in L^\infty(\mathbb{R}^n \times [0,\infty); (0,\infty) \times \mathbb{R}^n)$ is said to be a bounded weak solution of the system (1) with initial data

$$\rho(x,0) = \rho_0 \quad \text{and} \quad \mathbf{m}(x,0) = \mathbf{m}^\diamond(x), \tag{3}$$

if for any $(\varphi, \psi) \in C_c^\infty(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n)$, it holds that

$$\int_0^\infty \int_{\mathbb{R}^n} (\rho \partial_t \varphi + \mathbf{m} \cdot \nabla \varphi) dx dt = - \int_{\mathbb{R}^n} \rho_0 \varphi(x,0) dx,$$

$$\int_0^\infty \int_{\mathbb{R}^n} \left(\mathbf{m} \cdot \partial_t \psi + \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} : \nabla \psi + p(\rho) \nabla \cdot \psi + \psi \cdot \mathbf{B}\mathbf{m} \right) dx dt = - \int_{\mathbb{R}^n} \mathbf{m}^\diamond \cdot \psi(x,0) dx.$$

It is well-known that weak solutions are in general not unique. Several admissibility criteria to exclude the non-physical solutions have been introduced. An important one is the so called entropy condition. Set

$$\mathcal{I}(\rho) = \rho \int_0^\rho \frac{p(r)}{r^2} dr. \tag{4}$$

It is easy to see that $(\mathcal{I}(\rho)+\frac{|\mathbf{m}|^2}{2\rho},(\mathcal{I}(\rho)+\frac{|\mathbf{m}|^2}{2\rho}+p(\rho))\frac{\mathbf{m}}{\rho})$ is an entropy–entropy flux pair for the system (1) [12]. Thus one can define the admissible weak solutions to the system (1) as follows.

Definition 2 *(Admissible weak solutions).* A bounded weak solution $(\rho,\mathbf{m})$ is called an admissible weak solution to (1) with initial data (3) if for any non-negative test function $\varphi\in C_c^\infty(\mathbb{R}^n\times[0,\infty))$, it holds that

$$\int_0^\infty\int_{\mathbb{R}^n}\left(\mathcal{I}(\rho)+\frac{|\mathbf{m}|^2}{2\rho}\right)\partial_t\varphi+\left(\mathcal{I}(\rho)+\frac{|\mathbf{m}|^2}{2\rho}+p(\rho)\right)\frac{\mathbf{m}}{\rho}\cdot\nabla\varphi+\frac{\mathbf{m}}{\rho}\cdot\mathbf{B}\mathbf{m}\varphi dxdt$$
$$+\int_{\mathbb{R}^n}\left(\mathcal{I}(\rho_0(x))+\frac{|\mathbf{m}^\diamond(x)|^2}{2\rho_0}\right)\varphi(x,0)dx\geq 0, \tag{5}$$

where $\mathcal{I}$ is defined in (4).

In fact, the inequality (5) is exactly the energy inequality for the system (1), and each strong solution to the Cauchy problem (1) and (3) satisfies (5) with equality. Furthermore, if there is a strong solution for the problem (1) and (3), then any admissible weak solutions must coincide with it, see [12].

Now we are ready to state our main results. We start with the case in which $\mathbf{B}$ satisfies the following nonnegative condition

$$\xi^T\mathbf{B}\xi\geq 0 \text{ for any } \xi\in\mathbb{R}^n,\ |\xi|=1. \tag{6}$$

The typical examples for $\mathbf{B}$ include the zero matrix and antisymmetric matrices, such as in the compressible Euler system and the rotating shallow water system, etc. When the initial density is a piecewise constant function, the following results hold.

Theorem 1. *Suppose that $\mathbf{B}$ satisfies the condition (6). Assume that $\Omega=\mathbb{R}^n$ or $\mathbb{T}^n$. Let ρ_0 be a piecewise constant positive function in Ω, in the sense that there are a family of at most countably many mutually disjoint open sets $\{\Omega_i\}$ with the n-dimensional Hausdorff measure $\mathcal{H}^n(\Omega\setminus(\cup_i\Omega_i))=0$ and positive constants $\{\bar\rho_i\}$ with $0<\inf_i\bar\rho_i\leq\sup_i\bar\rho_i<\infty$, such that*

$$\rho_0(x)\equiv\bar\rho_i\qquad \textit{for } x\in\Omega_i.$$

Then there exists an $\mathbf{m}^\diamond\in L^\infty(\mathbb{R}^n)$ such that there are infinitely many global bounded admissible weak solutions $(\rho,\mathbf{m})$ to the Cauchy problem (1) and (3).

Furthermore, either of the following two cases holds:

1. *$(\rho,\mathbf{m})(x,t)=(\bar\rho_i,0)$ for a.e. $(x,t)\in\Omega_i\times[0,\infty)$;*
2. *$(\rho,\mathbf{m})$ has exactly N_n^* states in $\Omega_i\times[0,\infty)$, where $N_n^*=\frac{n(n+3)}{2}$.*

There exists at least one i such that the above second case holds in $\Omega_i \times [0, \infty)$.

Several remarks are in order.

Remark 1. As long as a piecewise constant function satisfies the Rankine–Hugoniot conditions for the compressible Euler system, it is a weak solution for the system (1) associated with $\mathbf{B} = 0$. However, if a piecewise constant function is a weak solution of the rotating shallow water system, it satisfies not only the Rankine–Hugoniot conditions, but also the algebraic system $\mathbf{Bm} = 0$. It is quite surprising that Theorem 1 shows that we can still obtain families of non-trivial finite-states solutions in the class of bounded admissible solutions. This also implies that in the case $\mathbf{B} = \mathbf{J}$ the finite-states weak solution $(\rho, \mathbf{m})$ cannot be continuous at any point in $\Omega_i \times [0, \infty)$ where the second case in Theorem 1 holds.

Remark 2. Theorem 1 is also inspired by the study on the problem of finding deformations with finitely many gradients in non-convex calculus of variations (see [1]).

Remark 3. For the class of Riemann initial data considered in [7,6], we can also show the existence of infinitely many admissible finite-states solutions to (1) with $\mathbf{B} = 0$.

If the initial density is a general piecewise Lipschitz continuous function, then we have the following results.

Theorem 2. *Suppose that $\mathbf{B}$ satisfies the condition (6). Let ρ_0 be a piecewise Lipschitz function in $\mathbb{R}^n$, then there exists an $\mathbf{m}^\diamond \in L^\infty(\mathbb{R}^n)$ such that there are infinitely many global bounded admissible weak solutions $(\rho, \mathbf{m})$ to the Cauchy problem (1) and (3).*

Furthermore, there exists a $T_* > 0$ such that each of these solutions $(\rho, \mathbf{m})$ reduces to locally finite-states when $t > T_*$, i.e. for a.e. $(x, t) \in \mathbb{R}^n \times (T_*, \infty)$, there exists a neighborhood $\mathcal{N}$ of (x, t) such that $(\rho, \mathbf{m})$ has exactly N_n^* states in $\mathcal{N}$.

Remark 4. The densities of the solutions in Theorem 2 in general develop discontinuities with complicated geometry even if they are smooth initially. Theorem 2 removes the key small-oscillation assumptions on the initial density in [18], where the densities remain smooth for all time.

When $\mathbf{B}$ is a general matrix, set

$$\beta = -\min\left[0, \inf\{\xi^T \mathbf{B} \xi : \xi \in \mathbb{R}^n, |\xi| = 1\}\right]. \tag{7}$$

Our main results on the non-uniqueness of admissible weak solutions to the Euler system with general source term are as follows.

Theorem 3. *Suppose that $0 < \check{\rho} < \hat{\rho}$ are two positive constants. There exists a positive constant $\bar{\varepsilon}$ depending on $\check{\rho}, \hat{\rho}$ and β. Given $\varepsilon \in (0, \bar{\varepsilon})$, if ρ_0 satisfies*

$$0 < \check{\rho} \leq \rho_0 \leq \hat{\rho}$$

and

$$\|(\rho_0 - \rho^\sharp, \nabla\rho_0)\|_{L^\infty(\mathbb{T}^n)} \leq \varepsilon \quad \text{where} \quad \rho^\sharp = \frac{1}{|\mathbb{T}^n|}\int_{\mathbb{T}^n} \rho_0(x)dx, \tag{8}$$

then there exists an $\mathbf{m}^\diamond \in L^\infty(\mathbb{T}^n)$ *such that the Cauchy problem* (1) *and* (3) *has infinitely many global-in-time bounded admissible weak solutions which also satisfy*

$$\|(\rho - \rho^\sharp, \mathbf{m})\|_{L^\infty(\mathbb{T}^n)} \leq \kappa e^{-\beta t}, \tag{9}$$

where κ *is a constant which depends on* ε *and tends to zero as* ε *goes to zero.*

Remark 5. Theorem 3 is also true for the Cauchy problem in $\mathbb{R}^n$ provided that there exists a constant $\bar{\rho}$ such that $\rho_0 - \bar{\rho}$ decays sufficiently fast at infinity.

Remark 6. Since the admissible weak solutions obtained in Theorem 3 satisfy the local energy inequality (5), the total energy of these weak solutions is non-increasing. Recall that a weak solution to the Cauchy problem (1) and (3) is said to be dissipative if its total energy is non-increasing. Therefore, the admissible criterion (5) is more restrictive than the dissipative criterion. In fact, Theorem 3 holds without the smallness assumption (8) in the class of dissipative weak solutions. For the recent studies on uniqueness of dissipative weak solutions, see [17,19].

We now make some comments on the analysis in this paper. We adapt the convex integration framework in [5,14] and reformulate the system (1) as a linear system $\mathscr{L}$ coupled with nonlinear pointwise constrains. Subsolutions are defined by relaxing the constrain sets. In order to obtain finite-states solutions we consider more general relaxed sets than those in [5,14]. The key for the convex integration scheme [14,34] is to analyze the wave cone $\Lambda_{\mathscr{L}}$ which generates localized plane waves of the linear system $\mathscr{L}$. Since the plane waves of $\mathscr{L}$ are different when the source terms appear, the results for the Euler system with $\mathbf{B} = \mathbf{0}$ in [13,14] do not apply directly. Our key observation is that in the high frequency regime, the plane wave solutions for the homogeneous system $\mathscr{L}_0$ are good approximations for $\mathscr{L}$. After correcting the errors by solving divergence equations [16], we find an elementary method to generate localized plane wave solutions to $\mathscr{L}$ for any constant matrix $\mathbf{B}$. In the case that $\mathbf{B} = \mathbf{0}$, our method also gives an alternative construction for localized plane waves from the ones in [13,14].

To obtain weak solutions, we use a convex integration scheme to iterate subsolutions. The scheme is inspired by [9,34]. One of the key ingredients of this paper is that we get some suitable relaxed sets with $\frac{n(n+3)}{2}$ extreme points by a careful perturbation argument using the Carathéodory's theorem for convex sets, which plays an important role in constructing finite-states weak solutions. When the initial densities have large variations, we construct non-smooth subsolution ansatz with complicated geometry building on the smooth ansatz in [18]. This allows us to remove the key small-oscillation assumption on initial densities in [18], and is the key to prove Theorem 2. The major idea to prove Theorem 3 is to construct a strict subsolution ansatz employing the acoustic potential in [18].

The rest of the paper is organized as follows. Some notations in this paper are summarized at the end of this section. The reformulations of the problem and the concepts of subsolutions are introduced in Section 2. We analyze the localized plane waves in Section 3. In Section 4, weak solutions are obtained by iterating the subsolutions using the localized plane waves. In Section 5, suitable subsolutions ansatz with various initial data are constructed and the main results are proved.

Notations

The following notations are used in the rest of the paper. The bold letters in lower and upper cases are used to denote vectors and matrices, respectively. Set $\mathbb{R}_+ = (0, \infty)$, $\bar{\mathbb{R}}_+ = [0, \infty)$, and S^{n-1} to be the unit sphere of $\mathbb{R}^n$. Denote by $\mathbb{S}_0^{n\times n}$ the vector space consisting of symmetric trace-free $n \times n$ matrices. Let

$$N_n = \dim(\mathbb{R}^n \times \mathbb{S}_0^{n\times n}) = \frac{n(n+3)}{2} - 1, \quad \text{and} \quad N_n^* = \frac{n(n+3)}{2}. \tag{10}$$

Given a set $K \subset \mathbb{R}^n \times \mathbb{S}_0^{n\times n}$, denote by conv K the convex hull of K, and int conv K the interior of conv K. The space $C^k(\mathfrak{D}; \mathfrak{R})$ is the set of functions mapping from $\mathfrak{D}$ to $\mathfrak{R}$ with continuous derivatives up to order k (k can be infinity), and $C_c^k(\mathfrak{D}; \mathfrak{R})$ is a subset of $C^k(\mathfrak{D}; \mathfrak{R})$ for functions with compact support. Suppose $f \in C^0(\mathfrak{D}; \mathfrak{R})$, let

$$\text{image}(f) = \{\mathfrak{r} \in \mathfrak{R} : \mathfrak{r} = f(\mathfrak{d}) \text{ for some } \mathfrak{d} \in \mathfrak{D}\} \tag{11}$$

denote the image of f. A function $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ is said to be in $C_{loc}(\bar{\mathbb{R}}_+; L^\infty_{w*}(\mathbb{R}^n))$, provided that the map $t \mapsto \int_{\mathbb{R}^n} f(x,t)\phi(x)dx$ is continuous on $\bar{\mathbb{R}}_+$ for any function $\phi \in L^1(\mathbb{R}^n)$. A sequence f_n converges to f in $C_{loc}(\bar{\mathbb{R}}_+; L^\infty_{w*}(\mathbb{R}^n))$, if for any $\phi \in L^1(\mathbb{R}^n)$ and any compact subset $J \subset \bar{\mathbb{R}}_+$,

$$\int_{\mathbb{R}^n} f_n(x,t)\phi(x)dx \to \int_{\mathbb{R}^n} f(x,t)\phi(x)dx \quad \text{uniformly on } J. \tag{12}$$

Denote by

$$\mathcal{L}(\mathfrak{D}) = \{f | f \in L^1(\mathfrak{D}), \int_{\mathfrak{D}} f = 0\}.$$

Let $\mathrm{tr}(\mathbf{A})$ denote the trace of the matrix $\mathbf{A}$. Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be $\mathbf{A} \le \mathbf{B}$ (or $\mathbf{A} < \mathbf{B}$) if for any $\xi \in S^{n-1}$, one has

$$\xi^T \mathbf{A} \xi \le \xi^T \mathbf{B} \xi \quad (\text{or } \xi^T \mathbf{A} \xi < \xi^T \mathbf{B} \xi).$$

Suppose that A, $B \in Y$ which is a linear space, then $[A, B]$ is called the line segment connecting A and B, i.e.,

$$[A, B] = \{\theta A + (1-\theta) B | \theta \in [0, 1]\}.$$

Suppose M, N are two subsets in a vector space Y. Define

$$M + N = \{y \in Y : y = y' + y'', y' \in M, y'' \in N\}.$$

Given a space–time set $\mathcal{D} \subset \mathbb{R}^{n+1}$, then $\mathcal{D}_t = \{x | (x, t) \in \mathcal{D}\}$ is the projection of $\mathcal{D}$ on $\mathbb{R}^n$ and $\mathcal{D}^t = \mathcal{D} \setminus (\overline{\mathcal{D}_t} \times \{t\})$. Let $\bar{z} = (\bar{x}, \bar{t})$ be a point in $\mathbb{R}^{n+1}$ and denote by $\mathscr{Q}_r(\bar{x})$ and $Q_r(\bar{z}) = Q_r(\bar{x}, \bar{t})$ the open space cube and space–time cube, respectively, i.e.

$$\mathscr{Q}_r(\bar{x}) = \{x \in \mathbb{R}^n : |x_i - \bar{x}_i| < \frac{r}{2}, i = 1, \cdots, n\} \tag{13}$$

and

$$Q_r(\bar{z}) = \{z \in \mathbb{R} \times \mathbb{R}^n : |t - \bar{t}| < \frac{r}{2}, |x_i - \bar{x}_i| < \frac{r}{2}, i = 1, \cdots, n\}. \tag{14}$$

For simplicity, denote $Q_r = Q_r(0)$. Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Given a set $\mathcal{O} \subset \mathbb{R}^k$, we also use $|\mathcal{O}|$ to denote $\mathcal{H}^k(\mathcal{O})$ when there is no ambiguity.

2. Subsolutions

In this section, the system (1) is reformulated as a differential inclusion in the spirit of De Lellis and Székelyhidi [5,14]. Then subsolutions as relaxed inclusions are introduced. However, the formulation here allows for more general constrain sets compared with [5,14], which will be employed to obtain finite-states weak solutions. Our formulation is inspired by [9,11,33].

Given $\rho > 0$ and $q \ge 0$, set

$$K_{\rho,q} = \{(\mathbf{m}, \mathbf{U}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n} : \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} - \mathbf{U} = qI\}.$$

By taking traces, it is easy to see that

$$K_{\rho,q} = \{(\mathbf{m}, \mathbf{U}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n} : |\mathbf{m}|^2 = n\rho q, \mathbf{U} = \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} - qI\}. \tag{15}$$

Therefore, $K_{\rho,q}$ is bounded in $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$. We denote

$$\mathfrak{C}(\rho, q) = \sup_{w\in \text{conv } K_{\rho,q}} |w| < \infty. \tag{16}$$

Observe that a bounded weak solution $(\rho, \mathbf{m})$ to the system (1) is equivalent to a bounded quadruple $(\rho, \mathbf{m}, \mathbf{U}, q)$ satisfying the differential equations

$$\partial_t \rho + \nabla \cdot \mathbf{m} = 0, \tag{17}$$

$$\partial_t \mathbf{m} + \nabla \cdot \mathbf{U} + \nabla (p(\rho) + q) = \mathbf{Bm}, \tag{18}$$

in the sense of distribution and the constrains

$$(\mathbf{m}, \mathbf{U})(x,t) \in K_{\rho(x,t),q(x,t)} \quad a.e. \text{ in } \mathbb{R}^n \times \bar{\mathbb{R}}_+. \tag{19}$$

The subsolutions are defined by relaxing the constrains as follows.

Definition 3. The quadruple $(\rho, \mathbf{m}, \mathbf{U}, q) \in L^\infty(\mathbb{R}^n \times \bar{\mathbb{R}}_+; \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}_0^{n\times n} \times \bar{\mathbb{R}}_+)$ is called a subsolution of the system (1) if it solves (17) and (18) in the sense of distribution and satisfies the relaxed constrains

$$(\mathbf{m}, \mathbf{U})(x,t) \in \text{conv } K_{\rho(x,t),q(x,t)} \quad a.e. \text{ in } \mathbb{R}^n \times \bar{\mathbb{R}}_+. \tag{20}$$

Furthermore, given a family of compact sets $K_{(x,t)} \subset K_{\rho(x,t),q(x,t)}$ such that the map $(x,t) \mapsto K_{(x,t)}$ is continuous in a space–time open set $\mathcal{D}$ in the Hausdorff distance ([3, Definition 7.3.1.]), a subsolution $(\rho, \mathbf{m}, \mathbf{U}, q)$ is said to be strict in $\mathcal{D}$ with constrain sets $K_{(x,t)}$ if

$$(\rho, \mathbf{m}, \mathbf{U}, q) \in C_{loc}(\bar{\mathbb{R}}_+; L^\infty_{w*}(\mathbb{R}^n)), \quad (\rho, \mathbf{m}, \mathbf{U}, q)\big|_{\mathcal{D}} \in C^0(\mathcal{D}),$$

and

$$(\mathbf{m}, \mathbf{U})(x,t) \in \text{int conv } K_{(x,t)}, \;\; \rho(x,t) > 0, \text{ and } q(x,t) > 0 \text{ for } (x,t) \in \mathcal{D}. \tag{21}$$

Notice that the strict subsolutions defined in [5,14] correspond to the cases that the constrain sets $K_{(x,t)} = K_{\rho(x,t),q(x,t)}$ in Definition 3.

The following lemma, appeared in [14], plays an important role in constructing strict subsolutions, and implies in particular that $(0,0) \in \text{int conv } K_{\bar\rho,\bar q}$, for $\bar\rho, \bar q > 0$.

Lemma 1. *The convex hull of $K_{\bar\rho,\bar q}$ is characterized as:*

$$conv\ K_{\bar\rho,\bar q} = \{(\bar{\boldsymbol{n}}, \bar{\mathbf{V}}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n} : \bar{\boldsymbol{n}} \otimes \bar{\boldsymbol{n}} - \bar\rho \bar{\mathbf{V}} \le \bar\rho \bar q \mathbf{I}\}. \tag{22}$$

Consequently,

$$int\ conv\ K_{\bar\rho,\bar q} = \{(\bar{\boldsymbol{n}}, \bar{\mathbf{V}}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n} : \bar{\boldsymbol{n}} \otimes \bar{\boldsymbol{n}} - \bar\rho \bar{\mathbf{V}} < \bar\rho \bar q \mathbf{I}\}. \tag{23}$$

Proof. In the case of $K_{1,1/n}$, the characterization (22) is obtained in [14]. The general case follows from the linear transform

$$(\mathbf{m},\mathbf{U}) \mapsto \left(\frac{\mathbf{m}}{\sqrt{n\rho q}}, \frac{1}{nq}\mathbf{U}\right), \qquad \text{for } (\mathbf{m},\mathbf{U}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n}, \tag{24}$$

which maps $K_{\rho,q}$ one-to-one to $K_{1,1/n}$, and preserves convex hulls. □

In the rest of this section, we exploit some fine properties of convex sets in finite dimensional space, which will be employed for constructing finite-states weak solutions. First, one has the following elementary result.

Lemma 2. *Suppose that K is a compact set in $\mathbb{R}^s$. If $\mathcal{C}$ is a compact subset of int conv K, then there exists a finite subset $\tilde{K}$ of K such that $\mathcal{C} \subset$ int conv $\tilde{K}$.*

Proof. For $w \in \mathcal{C}$, there exists a simplex $\mathcal{S}$ such that $w \in \text{int } \mathcal{S}$ and $\mathcal{S} \subset \text{int conv } K$. Let $\{\mathcal{S}_i\}$ be a finite sequence of simplexes which cover the compact set $\mathcal{C}$. It follows from Carathéodory's theorem that each vertex $\mathcal{W}_{i,j}$ of $\mathcal{S}_i$ can be represented as convex combinations of at most $s+1$ extreme points $\{\mathcal{V}_{i,j,k}\}_{k=1}^{s+1}$ of conv K. Since K is compact, the extreme points of conv K must belong to K. Let $\tilde{K}$ be the collection of these extreme points $\{\mathcal{V}_{i,j,k}\}$. Clearly that $\mathcal{C} \subset \cup_i \text{int } \mathcal{S}_i \subset \text{int conv } \tilde{K}$. □

In the case that $\mathcal{C}$ consists of a single point, the following sharp result holds.

Lemma 3. *Let $\bar{w} \in$ int conv $K_{\bar{\rho},\bar{q}}$. Then there exists a finite set of N_n^* points $\mathcal{K} \subset K_{\bar{\rho},\bar{q}}$ such that $\bar{w} \in$ int conv $\mathcal{K}$, where $N_n^* = N_n + 1 = \frac{n(n+3)}{2}$ as defined in (10).*

Proof. It follows from Lemma 2 that there exists a finite set $F = \{w_i\}_{i=1}^N \subset K_{\bar{\rho},\bar{q}}$ such that $\bar{w} \in \text{int conv } F$. Using Carathéodory's theorem yields that there is a subset $\{w_{i_j}\}_{j=1}^{N_n^*}$ with $\bar{w} \in \text{conv } \{w_{i_j}\}_{j=1}^{N_n^*}$. However, it may happen that $\bar{w} \notin \text{int conv } \{w_{i_j}\}_{j=1}^{N_n^*}$. The strategy is to avoid the degenerate case by perturbing the vertices slightly. An N_n-set $G = \{v_j\}_{j=1}^{N_n} \subset K_{\bar{\rho},\bar{q}}$ is said to be non-degenerate if

$$\{v_j - \bar{w}\}_{j=1}^{N_n} \text{ is a set of linearly independent vectors in } \mathbb{R}^n \times \mathbb{S}_0^{n\times n}. \tag{25}$$

The number of distinct N_n-subsets of F is $J = \binom{N}{N_n} = \frac{N!}{N_n!(N-N_n)!}$.

We claim that for each integer k with $0 \le k \le J$, there exists a set $F_k = \{w_i^{(k)}\}_{i=1}^N \subset K_{\bar{\rho},\bar{q}}$ satisfying the following two properties:

1. there are at least k non-degenerate N_n-subsets of F_k;
2. it holds that $\bar{w} \in \text{int conv } F_k$.

We prove this claim by mathematical induction. For $k = 0$, the set $F_0 = F$ clearly satisfies the above two properties. Suppose that we have obtained $F_0, \cdots, F_j$ which satisfy the above two properties. Let $G_1, \cdots, G_l$ be all the non-degenerate N_n-subsets of F_j. If $l > j$, then we can define $F_{j+1} = F_j$, which satisfies the above two properties with $k = j + 1$. If $l \le j$, then the existence of F_j implies that the number of non-degenerate N_n-subsets of F_j is exactly j. Since $j < \binom{N}{N_n}$, there is a degenerate N_n-subset $G = \{v_i\}_{i=1}^{N_n}$ of F_j, i.e. the vectors $\{v_i - \bar{w}\}_{i=1}^{N_n}$ are linearly dependent. Consider a perturbation $\tilde{G} = \{\tilde{v}_i\}_{i=1}^{N_n} \subset K_{\bar{\rho},\bar{q}}$ of G. It follows from a continuity argument that there exists a $\delta_0 > 0$ such that $\bar{w} \in \text{int conv}\,((F_j \setminus G) \cup \tilde{G})$ and that the N_n-subsets $\tilde{G}_s = (G_s \setminus G) \cup \{\tilde{v}_i | \text{if } v_i \in G_s\}$ are non-degenerate for $1 \le s \le j$, whenever

$$\max_{i=1,\cdots,N_n} |\tilde{v}_i - v_i| < \delta_0. \tag{26}$$

Using Lemma 4 below yields that there exists a set $\tilde{G}$ which is non-degenerate and satisfies (26). Set $F_{j+1} = (F_j \setminus G) \cup \tilde{G}$. It is easy to see that $F_{j+1} \subset K_{\bar{\rho},\bar{q}}$ and satisfies the two properties for $k = j + 1$. After at most $\binom{N}{N_n}$ steps, we obtain $\tilde{F} = F_J$ such that $\bar{w} \in \text{int conv}\,\tilde{F}$ and all the N_n-subsets of $\tilde{F}$ are non-degenerate.

It follows from Carathéodory's theorem that there exists an N_n^*-subset $\mathcal{K}$ of $\tilde{F}$ such that $\bar{w} \in \text{conv}\,\mathcal{K}$. Furthermore, since any N_n-subset of $\tilde{F}$ is non-degenerate, one has $\bar{w} \in \text{int conv}\,\mathcal{K}$. □

Lemma 4. *Suppose $\bar{\rho} > 0$ and $\bar{q} > 0$. Given $w_0 \in K_{\bar{\rho},\bar{q}}$, for any given $\delta > 0$, the set*

$$\mathcal{N}_\delta(w_0) := \{w \in K_{\bar{\rho},\bar{q}} : |w - w_0| < \delta\}$$

does not lie in any hyperplane $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{S}_0^{n\times n}$.

Proof. First consider the case that $\bar{\rho} = 1, \bar{q} = 1/n$, i.e. $K_{\bar{\rho},\bar{q}} = K_{1,1/n}$. Recall that the affine hull of a set $M \subset \mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ is the intersection of all affine sets containing it. Given a subset $M \subset \mathbb{R}^n \times \mathbb{S}_0^{n\times n}$, denote the dimension of the affine hull of M in $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ by $\text{affdim}(M)$. For $w \in K_{1,1/n}$, let

$$\mathfrak{A}(w) = \inf_{\delta>0}(\text{affdim}(\mathcal{N}_\delta(w))).$$

It is easy to see that $K_{1,1/n}$ is diffeomorphic to S^{n-1} via the map

$$\xi \mapsto (\xi, \xi \otimes \xi - \frac{1}{n}\mathbf{I}) \in K_{1,1/n} \qquad \text{for } \xi \in S^{n-1}.$$

Let $T_w(K_{1,1/n})$ denote the tangent space of $K_{1,1/n}$ at w. It is easy to see that

$$n - 1 = \dim(T_w(K_{1,1/n})) \le \mathfrak{A}(w) \le N_n, \qquad \text{for } w \in K_{1,1/n}.$$

We claim that $\mathfrak{A}(w)$ equals to some positive constant N_0 for all $w \in K_{1,1/n}$. To see this, consider the action of $SO(n)$ in $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ inspired by [9]. It is easy to see that for any $\mathbf{R} \in SO(n)$, the linear transform on $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ defined by

$$\mathscr{R}(\mathbf{m}, \mathbf{U}) = (\mathbf{Rm}, \mathbf{RUR}^t) \quad \text{for } (\mathbf{m}, \mathbf{U}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n},$$

preserves affine hulls and transforms $K_{1,1/n}$ into itself. Therefore, one has $\mathfrak{A}(\mathscr{R}w) = \mathfrak{A}(w)$ for $w \in K_{1,1/n}$. Suppose that w_1 and w_2 are any two points of $K_{1,1/n}$ which are of the forms

$$w_i = (\xi_i, \xi_i \otimes \xi_i - \frac{1}{n}\mathbf{I}),\ \xi_i \in S^{n-1},\ i = 1, 2.$$

There exists a rotation $\mathbf{R} \in SO(n)$ such that $\xi_2 = \mathbf{R}\xi_1$. It follows that $w_2 = \mathscr{R}w_1$ and thus $\mathfrak{A}(w_2) = \mathfrak{A}(w_1)$.

We now show that $N_0 = N_n$. Suppose not, then one has $N_0 < N_n$. Fix some $w_0 \in K_{1,1/n}$. Then there exist a $\delta_0 > 0$ and an affine subspace L_0 of dimension N_0 such that $\mathcal{N}_{\delta_0}(w_0) \subset L_0$. Recall from Lemma 1 that $(0, 0) \in \text{int conv } K_{1,1/n}$, hence the affine dimension of $K_{1,1/n}$ is N_n, which is larger than $\dim(L_0) = N_0$. It follows that the number

$$\bar{\delta} = \sup\{\delta > 0 : \mathcal{N}_\delta(w_0) \subset L_0\}$$

is finite. From the definition of $\bar{\delta}$ and the compactness of $\partial\mathcal{N}_{\bar{\delta}}(w_0) \subset K_{1,1/n}$, there exists a $w_1 \in \partial\mathcal{N}_{\bar{\delta}}(w_0)$ satisfying that $\mathcal{N}_\delta(w_1)$ is not contained in L_0 for any $\delta > 0$. Since $\mathfrak{A}(w_1) = N_0$, there exist a $\delta_1 > 0$ and an affine subspace L_1 of dimension N_0 such that $\mathcal{N}_{\delta_1}(w_1) \subset L_1$. It is easy to see that $L_0 \neq L_1$ and thus $\dim(L_0 \cap L_1) < N_0$. Observe that for $\delta < \bar{\delta}$ which is sufficiently close to $\bar{\delta}$, the intersection $\mathcal{N}_\delta(w_0) \cap \mathcal{N}_{\delta_1}(w_1)$ is a nonempty open set of $K_{1,1/n}$. Therefore, for $w \in \mathcal{N}_\delta(w_0) \cap \mathcal{N}_{\delta_1}(w_1) \subset L_0 \cap L_1$, it holds that $\mathfrak{A}(w) \le \dim(L_0 \cap L_1) < N_0$. This contradicts with the fact that $\mathfrak{A}(\cdot)$ is constant on $K_{1,1/n}$. It follows that $N_0 = N_n$ and thus the lemma in the case for $(\bar{\rho}, \bar{q}) = (1, 1/n)$ is proved.

The general case for $K_{\bar{\rho},\bar{q}}$ follows from the linear transform (24) which maps $K_{\bar{\rho},\bar{q}}$ one-to-one to $K_{1,1/n}$ and preserves hyperplanes. This finishes the proof of the lemma. □

3. Localized plane waves

In this section, localized plane waves are constructed, which form the building blocks for the iteration scheme to obtain weak solutions.

Define the linear operator

$$\mathscr{L}(\boldsymbol{n}, \mathbf{V}) = (\nabla \cdot \boldsymbol{n}, \partial_t \boldsymbol{n} + \nabla \cdot \mathbf{V} - \mathbf{B}\boldsymbol{n})^t. \tag{27}$$

Observe that if $(\rho, \mathbf{m}, \mathbf{U}, q)$ solves (17) and (18) and $(\boldsymbol{n}, \mathbf{V})$ satisfies

$$\mathscr{L}(\boldsymbol{n}, \mathbf{V}) = 0, \tag{28}$$

then $(\rho, \mathbf{m}+\boldsymbol{n}, \mathbf{U}+\mathbf{V}, q)$ also solves (17) and (18). The convex integration scheme will employ localized plane waves of $\mathscr{L}$ as corrections for subsolutions. It needs to be shown that the wave cone $\Lambda_{\mathscr{L}}$ defined below (according to [33]) is suitably large.

Definition 4. The wave cone $\Lambda_{\mathscr{L}}$ associated with $\mathscr{L}$ is a subset of $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$, to which there exists a constant $C>0$ such that for any $(\bar{\mathbf{n}}, \bar{\mathbf{V}}) \in \Lambda_{\mathscr{L}}$ there exists a sequence $(\tilde{\boldsymbol{n}}_k, \tilde{\mathbf{V}}_k) \in C_c^\infty(Q_1; \mathbb{R}^n \times \mathbb{S}_0^{n\times n})$ solving $\mathscr{L}(\tilde{\boldsymbol{n}}_k, \tilde{\mathbf{V}}_k) = 0$ such that

- $\mathrm{dist}((\tilde{\boldsymbol{n}}_k, \tilde{\mathbf{V}}_k), [-(\bar{\boldsymbol{n}}, \bar{\mathbf{V}}), (\bar{\boldsymbol{n}}, \bar{\mathbf{V}})]) \to 0$ uniformly in Q_1 as $k\to\infty$,
- $(\tilde{\boldsymbol{n}}_k, \tilde{\mathbf{V}}_k) \to 0$ in the sense of distribution as $k\to\infty$,
- $\int_{Q_1} |(\tilde{\boldsymbol{n}}_k, \tilde{\mathbf{V}}_k)|^2(x,t)dxdt \ge C|(\bar{\boldsymbol{n}}, \bar{\mathbf{V}})|^2$ for all $k\in\mathbb{N}$.

The functions $(\tilde{\boldsymbol{n}}_k, \tilde{\mathbf{V}}_k)$ are called localized plane waves of $\mathscr{L}$ associated with $(\bar{\boldsymbol{n}}, \bar{\mathbf{V}})$.

Let $\mathscr{L}_0$ denote the homogeneous operator corresponding to $\mathbf{B}=0$ and $\Lambda_{\mathscr{L}_0}$ the associated wave cone. It is proved in [14] that for any w_1, $w_2 \in K_{\bar\rho,\bar q}$ with $\bar\rho>0$ and $\bar q>0$, if $w_1 \neq w_2$, then one has

$$w_1 - w_2 \in \Lambda_{\mathscr{L}_0}. \tag{29}$$

The proof of (29) employs the key property that w_1 and w_2 is connected by a plane wave solution to $\mathscr{L}_0(\boldsymbol{n}, \mathbf{V}) = 0$ as follows.

Let Λ denote the subset of $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ such that for $(\bar{\boldsymbol{n}}, \bar{\mathbf{V}}) \in \Lambda$ there exists a $(\tau,\xi) \in \mathbb{R}\times S^{n-1}$ such that $(\bar{\boldsymbol{n}}, \bar{\mathbf{V}})h(\tau t + \xi\cdot x)$ is a plane wave solution to $\mathscr{L}_0(\boldsymbol{n}, \mathbf{V}) = 0$ for any smooth function h.

Lemma 5. *For any w' and $w'' \in K_{\bar\rho,\bar q}$ with $w' \neq w''$, one has $w'-w'' \in \Lambda$.*

Proof. For $(\bar{\boldsymbol{n}}', \bar{\mathbf{V}}')$, $(\bar{\boldsymbol{n}}'', \bar{\mathbf{V}}'') \in K_{\bar\rho,\bar q}$, set $(\bar{\boldsymbol{n}}, \bar{\mathbf{V}}) = (\bar{\boldsymbol{n}}', \bar{\mathbf{V}}') - (\bar{\boldsymbol{n}}'', \bar{\mathbf{V}}'')$. Let $\xi \in S^{n-1}$ satisfy $\xi\cdot\bar{\boldsymbol{n}} = 0$ so that $\xi\cdot\bar{\boldsymbol{n}}' = \xi\cdot\bar{\boldsymbol{n}}''$. Denote $\tau = -(1/\bar\rho)\xi\cdot\bar{\boldsymbol{n}}'$. Then one has

$$\tau\bar{\boldsymbol{n}} + \bar{\mathbf{V}}\cdot\xi = \tau\bar{\boldsymbol{n}} + \frac{1}{\bar\rho}(\bar{\boldsymbol{n}}'\otimes\bar{\boldsymbol{n}}' - \bar{\boldsymbol{n}}''\otimes\bar{\boldsymbol{n}}'')\cdot\xi = (\tau + \frac{1}{\bar\rho}\xi\cdot\bar{\boldsymbol{n}}')(\bar{\boldsymbol{n}}' - \bar{\boldsymbol{n}}'') = 0.$$

This finishes the proof of the lemma. □

However, when the source $\mathbf{B}$ is non-zero, for $w_1, w_2 \in K_{\bar\rho,\bar q}$ there may not be plane wave solutions to $\mathscr{L}$ with profiles $w_1 - w_2$. It seems initially unclear whether the property (29) is still true for $\Lambda_{\mathscr{L}}$ in the presence of the source terms. Our key observation is that: in the high-frequency regime, localized plane waves with sources can be constructed as perturbations of the plane waves of $\mathscr{L}_0$. This is stated as follows.

Lemma 6 *(Localized plane waves). Suppose that $w, w_1, w_2 \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ satisfy*

$$w = \mu_1 w_1 + \mu_2 w_2,\ \mu_1+\mu_2 = 1,\ \mu_i > 0, \quad \text{and} \ \ \bar w = w_2 - w_1 \in \Lambda.$$

Given a bounded open space–time set $\mathcal{O}$ and $\varepsilon > 0$, there exists a $\tilde{w} \in C_c^\infty(\mathcal{O}; \mathbb{R}^n \times \mathbb{S}_0^{n\times n})$ satisfying the following properties

1. $\tilde{w}(\cdot, t) \in \mathcal{L}(\mathbb{R}^n)$ *and* $\tilde{w}$ *solves the equation* (28), *i.e.*

$$\int \tilde{w}(x,t)dx = 0 \qquad \text{and} \qquad \mathscr{L}\tilde{w} = 0; \tag{30}$$

2. $dist(w + \tilde{w}(x,t), [w_1, w_2]) < \varepsilon$ *for* $(x,t) \in \mathcal{O}$;
3. *there exist two disjoint open subsets* $\mathcal{O}_i \subset \mathcal{O}$ *such that for* $i = 1, 2$,

$$|w + \tilde{w}(x,t) - w_i| < \varepsilon \text{ for } (x,t) \in \mathcal{O}_i; \quad |\mathcal{H}^{n+1}(\mathcal{O}_i) - \mu_i \mathcal{H}^{n+1}(\mathcal{O})| < \varepsilon. \tag{31}$$

The following characterization of $\Lambda_{\mathscr{L}}$ is a consequence of Lemma 6.

Corollary 1. $\Lambda \subset \Lambda_{\mathscr{L}}$.

Proof of Corollary 1. Let $\bar{w} = (\bar{\boldsymbol{n}}, \bar{\mathbf{V}}) \in \Lambda$. For each $k \in \mathbb{N}$, applying Lemma 6 to $w = 0 = \frac{1}{2}(-\bar{w}) + \frac{1}{2}\bar{w}$ with $\varepsilon = 2^{-k(n+1)}\frac{1}{k}$ and $\mathcal{O} = Q_{2^{-k}}$ yields that there exists $v_k \in C_c^\infty(Q_{2^{-k}})$ satisfying (30) and

$$\mathrm{dist}(v_k, [-\bar{w}, \bar{w}]) < \frac{2^{-k(n+1)}}{k}.$$

Furthermore, there exist $\mathcal{O}_i^{(k)}$ $(i = 1,\ 2)$ such that

$$|v_i(z) - (-1)^i \bar{w}| < \frac{2^{-k(n+1)}}{k} \text{ for } z \in \mathcal{O}_i^{(k)} \text{ and } \left|\mathcal{H}^{n+1}(\mathcal{O}_i^{(k)}) - \frac{1}{2^{k(n+1)+1}}\right| < \frac{2^{-k(n+1)}}{k}. \tag{32}$$

Let $\{Q_{2^{-k}}(z^{(j,k)})\}_{j=1}^{2^{k(n+1)}}$ be the decomposition of Q_1 into mutually disjoint cubes of length 2^{-k}. Denote

$$\tilde{w}_k = \sum_{j=1}^{2^{k(n+1)}} a_j^{(k)} \qquad \text{where } a_j^{(k)}(z) = v_k(z - z^{(j,k)}) \in C_c^\infty(Q_{2^{-k}}(z^{(j,k)})).$$

It is easy to see that $\mathscr{L}\tilde{w}_k = 0$ and $\mathrm{dist}(\tilde{w}_k, [-\bar{w}, \bar{w}]) \to 0$ as $k \to \infty$. Since $a_j^{(k)}(\cdot, t) \in \mathcal{L}(\mathbb{R}^n)$ and $\mathrm{diam}(\mathrm{supp}\, a_j^{(k)}) \le \sqrt{(n+1)}2^{-k} \to 0$, one has $\tilde{w}_k \to 0$ in the sense of distribution. Notice that

$$\int_{Q_1} |\tilde{w}_k|^2 = \sum_{j=1}^{2^{k(n+1)}} \int |a_j^{(k)}|^2 = 2^{k(n+1)} \int_{Q_{2^{-k}}} |v_k|^2 dxdt.$$

It follows from (32) that one has

$$
\begin{aligned}
&2^{k(n+1)} \int\limits_{Q_{2^{-k}}} |v_k|^2 dxdt - |\bar{w}|^2 = \frac{1}{|Q_{2^{-k}}|} \int\limits_{Q_{2^{-k}}} |v_k|^2 - |\bar{w}|^2 dxdt \\
&= \frac{1}{|Q_{2^{-k}}|} \left(\sum_{i=1}^{2} \int_{\mathcal{O}_i} |v_k|^2 - |\bar{w}|^2 dxdt + \int\limits_{Q_{2^{-k}} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)} |v_k|^2 - |\bar{w}|^2 dxdt \right) \\
&\le C \frac{1}{k}.
\end{aligned}
$$

Therefore, for sufficiently large k, one has

$$
\int\limits_{Q_1} |\tilde{w}_k|^2 dxdt \ge \frac{1}{2} |\bar{w}|^2.
$$

This shows that $\bar{w} \in \Lambda_{\mathscr{L}}$ and finishes the proof of the corollary. $\square$

The rest of the section is devoted to the proof of Lemma 6.

To correct the errors from the source terms, we recall the linear operators which solve divergence equations in $C^\infty(\mathbb{T}^n; \mathbb{S}_0^{n\times n})$ [16] and adapt them to the whole spaces.

Lemma 7. *(See [16, Proposition 5.1].) There exists a linear operator $\mathcal{R}_{\mathbb{T}^n}$ from $C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ to $C^\infty(\mathbb{T}^n; \mathbb{S}_0^{n\times n})$ such that for any $\mathbf{f} \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$, it holds that*

$$
\nabla \cdot \mathcal{R}_{\mathbb{T}^n}[\mathbf{f}] = \mathbf{f} - \fint_{\mathbb{T}^n} \mathbf{f},
$$

and

$$
\|\mathcal{R}_{\mathbb{T}^n}[\mathbf{f}]\|_{C^{1,\alpha}(\mathbb{T}^n)} \le C(n,\alpha) \|\mathbf{f}\|_{C^\alpha(\mathbb{T}^n)}.
$$

In order to prove Lemma 6, we also need the following elementary lemma.

Lemma 8. *For any $\mathbf{f} \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, there exists an $\mathcal{R}[\mathbf{f}] \in C^\infty(\mathbb{R}^n; \mathbb{S}_0^{n\times n})$ satisfying*

$$
\nabla \cdot \mathcal{R}[\mathbf{f}] = \mathbf{f}.
$$

Furthermore, $\mathcal{R}$ satisfies the following properties:

1. *$\mathcal{R}$ is a linear operator from $C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n; \mathbb{S}_0^{n\times n})$;*
2. *$\mathcal{R}[\Delta^2 \mathbf{f}]$ is a linear combination of third order derivatives of $\mathbf{f}$;*

3. *supp* $\mathcal{R}[\Delta^2\mathbf{f}] \subset$ *supp* $\mathbf{f}$ *and* $\mathcal{R}[\Delta^2\mathbf{f}] \in \mathcal{L}(\mathbb{R}^n)$*, i.e.,*

$$\int_{\mathbb{R}^n} \mathcal{R}[\Delta^2\mathbf{f}]dx = 0;$$

4. *there exists a constant* $0 < \alpha < 1$ *such that*

$$\|\mathcal{R}[\mathbf{f}]\|_{C^\alpha(\mathbb{R}^n)} \leq C\max(\|\mathbf{f}\|_{L^1(\mathbb{R}^n)}, \|\mathbf{f}\|_{L^\infty(\mathbb{R}^n)}),$$

where the constants C *and* α *depend only on the dimension* n*.*

Proof. Given $\mathbf{f} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, define $\mathbf{g} = \Delta^{-1}\mathbf{f}$ where $\Delta^{-1}u = \mathcal{N} * u$ with the Newtonian potential $\mathcal{N}$ defined by

$$\mathcal{N}(x) = \begin{cases} \dfrac{1}{\Gamma(n)}\dfrac{1}{|x|^{n-2}} & \text{for } n \geq 3, \\ \dfrac{1}{2\pi}\ln|x| & \text{for } n = 2. \end{cases} \tag{33}$$

Set

$$\mathcal{R}[\mathbf{f}] = \frac{n-2}{2(n-1)}[\nabla\mathbb{P}\mathbf{g} + (\nabla\mathbb{P}\mathbf{g})^t] + \frac{n}{2(n-1)}[\nabla\mathbf{g} + (\nabla\mathbf{g})^t] - \frac{1}{n-1}(\nabla\cdot\mathbf{g})\mathbf{I}, \tag{34}$$

where

$$\mathbb{P}\mathbf{g} := \mathbf{g} - \nabla\Delta^{-1}\nabla\cdot\mathbf{g}.$$

Then one can verify the properties of $\mathcal{R}$ by straightforward computations. First, one has

$$\begin{aligned} \nabla\cdot\mathcal{R}[\mathbf{f}] &= \frac{n-2}{2(n-1)}[\Delta\mathbb{P}\mathbf{g} + \nabla\text{div}\mathbb{P}\mathbf{g}] + \frac{n}{2(n-1)}[\Delta\mathbf{g} + \nabla\text{div}\mathbf{g}] - \frac{1}{n-1}(\nabla\text{div}\mathbf{g}) \\ &= \frac{n-2}{2(n-1)}[\Delta\mathbf{g} - \nabla\text{div}\mathbf{g} + 0] + \frac{n}{2(n-1)}[\Delta\mathbf{g} + \nabla\text{div}\mathbf{g}] - \frac{1}{n-1}(\nabla\text{div}\mathbf{g}) \\ &= \Delta\mathbf{g} = \Delta\Delta^{-1}\mathbf{f} = \mathbf{f}. \end{aligned}$$

It is clear that $\mathcal{R}[\mathbf{f}]$ is a symmetric matrix and linear in $\mathbf{f}$. Furthermore,

$$\begin{aligned} \text{tr}\mathcal{R}[\mathbf{f}] &= \frac{n-2}{2(n-1)}[2\nabla\cdot\mathbb{P}\mathbf{g}] + \frac{n}{2(n-1)}[2\nabla\cdot\mathbf{g}] - \frac{n}{n-1}(\nabla\cdot\mathbf{g}) \\ &= 0 + \frac{n}{n-1}(\nabla\cdot\mathbf{g}) - \frac{n}{n-1}(\nabla\cdot\mathbf{g}) = 0. \end{aligned}$$

Therefore, $\mathcal{R}$ maps $C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n;\mathbb{S}_0^{n\times n})$. This proves the existence of $\mathcal{R}$ and the property (1) in Lemma 8.

Since

$$\mathbb{P}\Delta\mathbf{f} = \Delta\mathbf{f} - \nabla\mathrm{div}\mathbf{f},$$

one has

$$\mathcal{R}[\Delta^2\mathbf{f}] = \frac{n-2}{2(n-1)}[\nabla(\Delta\mathbf{f} - \nabla\mathrm{div}\mathbf{f}) + (\nabla(\Delta\mathbf{f} - \nabla\mathrm{div}\mathbf{f}))^t] \\ + \frac{n}{2(n-1)}[\nabla\Delta\mathbf{f} + (\nabla\Delta\mathbf{f})^t] - \frac{1}{n-1}(\Delta\mathrm{div}\mathbf{f}). \tag{35}$$

This implies the second property of $\mathcal{R}$ in Lemma 8.

It follows from (35) that $\mathrm{supp}\,\mathcal{R}[\Delta^2\mathbf{f}] \subset \mathrm{supp}\,\mathbf{f}$. If $\mathbf{f} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then taking integral for (35) on $\mathbb{R}^n$ yields

$$\int_{\mathbb{R}^n} \mathcal{R}[\Delta^2\mathbf{f}]dx = 0.$$

Hence one has the third property of $\mathcal{R}$ in Lemma 8.

Note that

$$\|\mathbf{f}\|_{L^p(\mathbb{R}^n)} \le \max(\|\mathbf{f}\|_{L^1(\mathbb{R}^n)}, \|\mathbf{f}\|_{L^\infty(\mathbb{R}^n)}) \quad \text{for } 1 \le p \le \infty.$$

Thus the fourth property of $\mathcal{R}$ in Lemma 8 is then an easy consequence of the boundedness of the Riesz operators and Sobolev embeddings [32]. The proof of the lemma is completed. □

Now we are in position to prove Lemma 6.

Proof of Lemma 6. Suppose that δ is a small positive constant to be determined later. Let $\phi \in C_c^\infty(\mathcal{O})$ be a smooth cut-off function satisfying

$$0 \le \phi \le 1 \quad \text{and} \quad \mathcal{H}^{n+1}(\{(x,t) \in \mathcal{O} : \phi(x,t) \neq 1\}) < \delta. \tag{36}$$

Set

$$h(s) = \begin{cases} -\mu_2, & s \in (0, \mu_1], \\ \mu_1, & s \in (\mu_1, 1], \end{cases}$$

and extend h as a periodic function with period 1. Notice that $\int_0^1 h(s)ds = 0$. Let h_0 be a 1-periodic smooth approximation of h satisfying

$$-\mu_2 \le h_0 \le \mu_1,\ \mathcal{H}^1(\{s \in [0,1] : h(s) \neq h_0(s)\}) < \delta,\ \int_0^1 h_0(s)ds = 0. \tag{37}$$

For $k = 0, 1, \cdots$, define

$$\tilde{h}_{k+1}(s) = \int_0^s h_k(\sigma)d\sigma, \quad h_{k+1}(s) = \tilde{h}_{k+1}(s) - \int_0^1 \tilde{h}_{k+1}(\sigma)d\sigma.$$

Thus for any $k \geq 0$, one has

$$\frac{d^j h_k}{ds^j} = h_{k-j} \quad \text{for } 0 \leq j \leq k, \quad \int_0^1 h_k(s)ds = 0,$$

and

$$\|h_k\|_{L^\infty} \leq \|h_{k-1}\|_{L^\infty} \leq \cdots \leq \|h_0\|_{L^\infty}.$$

Denote $\bar{w} = (\bar{\boldsymbol{n}}, \bar{\mathbf{V}})$ and let $(\tau, \xi) \in \mathbb{R} \times S^{n-1}$ be associated with $\bar{w} \in \Lambda$. Set

$$\boldsymbol{n}' = \lambda^{-6}\Delta^3[\bar{\boldsymbol{n}}h_6(\lambda\tau t + \lambda\xi \cdot x)\phi], \quad \mathbf{V}' = \lambda^{-6}\Delta^3[\bar{\mathbf{V}}h_6(\lambda\tau t + \lambda\xi \cdot x)\phi], \tag{38}$$

and

$$\boldsymbol{n}'' = -\nabla\Delta^{-1}\nabla \cdot \boldsymbol{n}', \quad \mathbf{V}'' = \mathcal{R}[\mathbf{B}(\boldsymbol{n}' + \boldsymbol{n}'') - \partial_t(\boldsymbol{n}' + \boldsymbol{n}'') - \nabla \cdot \mathbf{V}'],$$

where $\Delta^{-1}\mathbf{u} := \mathcal{N} * \mathbf{u}$ and $\mathcal{R}$ are the operators defined in (33) and (34), respectively, and $\lambda \in \mathbb{R}$ is a large positive constant to be determined later. Define

$$\tilde{w} = (\boldsymbol{n}, \mathbf{V}) = (\boldsymbol{n}' + \boldsymbol{n}'', \mathbf{V}' + \mathbf{V}'').$$

Our major task is to show that $\tilde{w}$ is exactly the function satisfying all the properties in Lemma 6.

Indeed, it follows from Lemma 8 that

$$\nabla \cdot \boldsymbol{n} = \nabla \cdot (\boldsymbol{n}' + \boldsymbol{n}'') = \nabla \cdot \boldsymbol{n}' - \nabla \cdot \nabla\Delta^{-1}\nabla \cdot \boldsymbol{n}' = 0,$$

and

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}' + \nabla \cdot \mathcal{R}[\mathbf{B}\boldsymbol{n} - \partial_t\boldsymbol{n} - \nabla \cdot \mathbf{V}'] = \mathbf{B}\boldsymbol{n} - \partial_t\boldsymbol{n}.$$

Therefore, $(\boldsymbol{n}, \mathbf{V})$ solves the linear system $\mathscr{L}(\boldsymbol{n}, \mathbf{V}) = 0$.

Direct computations show that

$$\begin{aligned}\boldsymbol{n}' &= \bar{\boldsymbol{n}}h_0(\lambda\tau t + \lambda\xi \cdot x)\phi + \lambda^{-6}\bar{\boldsymbol{n}} \sum_{|\beta|\geq 1, |\alpha+\beta|=6} C_{\alpha,\beta}\partial_x^\alpha \left(h_6(\lambda\tau t + \lambda\xi \cdot x)\right)\partial_x^\beta\phi \\ &= \bar{\boldsymbol{n}}h_0(\lambda\tau t + \lambda\xi \cdot x)\phi + \lambda^{-6}\bar{\boldsymbol{n}} \sum_{|\beta|\geq 1, |\alpha+\beta|=6} \lambda^{|\alpha|}C_{\alpha,\beta}\xi^\alpha h_{6-|\alpha|}(\lambda\tau t + \lambda\xi \cdot x)\partial_x^\beta\phi \end{aligned} \tag{39}$$

and

$$\mathbf{V}' = \bar{\mathbf{V}} h_0(\lambda\tau t + \lambda\xi\cdot x)\phi + \lambda^{-6}\bar{\mathbf{V}} \sum_{|\beta|\geq 1, |\alpha+\beta|=6} \lambda^{|\alpha|} C_{\alpha,\beta}\xi^\alpha h_{6-|\alpha|}(\lambda\tau t + \lambda\xi\cdot x)\partial_x^\beta \phi. \tag{40}$$

It follows from (38) that $\operatorname{supp}(\boldsymbol{n}', \mathbf{V}') \subset \operatorname{supp}\phi$, and $\int_{\mathbb{R}^n}(\boldsymbol{n}', \mathbf{V}')dx = 0$. The expressions (39) and (40) imply that

$$\|(\boldsymbol{n}', \mathbf{V}') - (\bar{\boldsymbol{n}}, \bar{\mathbf{V}})h_0(\lambda\tau t + \lambda\xi\cdot x)\phi\|_{L^\infty(\mathcal{O})} \leq C(|\bar{\boldsymbol{n}}|, |\bar{\mathbf{V}}|, \phi, h_0)\lambda^{-1}. \tag{41}$$

It follows from the property $\bar{\boldsymbol{n}}\cdot\xi = 0$ that

$$\boldsymbol{n}'' = -\lambda^{-6}\nabla\Delta^{-1}\Delta^3\nabla\cdot[\bar{\boldsymbol{n}}h_6(\lambda\tau t + \lambda\xi\cdot x)\phi] = -\lambda^{-6}\nabla\Delta^2[(\bar{\boldsymbol{n}}\cdot\nabla\phi)h_6(\lambda\tau t + \lambda\xi\cdot x)].$$

Hence it holds that $\operatorname{supp}\boldsymbol{n}'' \subset \operatorname{supp}\phi$, $\int_{\mathbb{R}^n}\boldsymbol{n}''dx = 0$ and

$$\|\boldsymbol{n}''\|_{L^\infty(\mathcal{O})} \leq C(|\bar{\boldsymbol{n}}|, \phi, h_0)\lambda^{-1}. \tag{42}$$

Since $\mathbf{B}$ commutes with the Laplacian Δ, direct computations yield that

$$\begin{aligned}
\mathbf{V}'' &= -\mathcal{R}[\partial_t\boldsymbol{n}' + \nabla\cdot\mathbf{V}' + \partial_t\boldsymbol{n}'' - \mathbf{B}\boldsymbol{n}] \\
&= -\lambda^{-6}\mathcal{R}\{\Delta^3[\partial_t\left(\bar{\boldsymbol{n}}h_6\left(\lambda\tau t + \lambda\xi\cdot x\right)\phi\right) + \nabla\cdot\left(\bar{\mathbf{V}}h_6\left(\lambda\tau t + \lambda\xi\cdot x\right)\phi\right)] \\
&\quad - \partial_t\nabla\Delta^2[(\nabla\phi\cdot\bar{\boldsymbol{n}})h_6\left(\lambda\tau t + \lambda\xi\cdot x\right)] \\
&\quad - \mathbf{B}\Delta^3[\bar{\boldsymbol{n}}h_6(\lambda\tau t + \lambda\xi\cdot x)\phi] + \mathbf{B}\nabla\Delta^2[(\nabla\phi\cdot\bar{\boldsymbol{n}})h_6(\lambda\tau t + \lambda\xi\cdot x)]\} \\
&= -\lambda^{-6}\mathcal{R}\{\Delta^3[\left(\bar{\boldsymbol{n}}\partial_t\phi + \bar{\mathbf{V}}\cdot\nabla\phi\right)h_6(\lambda\tau t + \lambda\xi\cdot x)] \\
&\quad - \Delta^2\nabla\partial_t[(\nabla\phi\cdot\bar{\boldsymbol{n}})h_6\left(\lambda\tau t + \lambda\xi\cdot x\right)] \\
&\quad - \Delta^2\mathbf{B}\left[\Delta(\bar{\boldsymbol{n}}h_6(\lambda\tau t + \lambda\xi\cdot x)\phi) - \nabla((\nabla\phi\cdot\bar{\boldsymbol{n}})h_6(\lambda\tau t + \lambda\xi\cdot x))\right]\},
\end{aligned}$$

where $\tau\bar{\boldsymbol{n}} + \bar{\mathbf{V}}\cdot\xi = 0$ has been used in the last equality. Hence $\mathbf{V}''$ can be written as

$$\mathbf{V}'' = -\lambda^{-6}\mathcal{R}[\Delta^2\mathbf{f}],$$

where

$$\begin{aligned}
\mathbf{f} = {}& \Delta[\left(\bar{\boldsymbol{n}}\partial_t\phi + \bar{\mathbf{V}}\cdot\nabla\phi\right)h_6(\lambda\tau t + \lambda\xi\cdot x)] - \nabla\partial_t[(\nabla\phi\cdot\bar{\boldsymbol{n}})h_6\left(\lambda\tau t + \lambda\xi\cdot x\right)] \\
& - \mathbf{B}\{\Delta[\bar{\boldsymbol{n}}h_6(\lambda\tau t + \lambda\xi\cdot x)\phi] - \nabla[(\nabla\phi\cdot\bar{\boldsymbol{n}})h_6(\lambda\tau t + \lambda\xi\cdot x)]\}.
\end{aligned}$$

It follows from Lemma 8 that $\mathcal{R}[\Delta^2 \mathbf{f}]$ is a linear combination of third order derivatives of $\mathbf{f}$. Therefore, $\operatorname{supp} \mathbf{V}'' \subset \operatorname{supp} \mathbf{f} \subset \operatorname{supp} \phi$, and $\int_{\mathbb{R}^n} \mathbf{V}'' dx = 0$. Combining the above computations together yields (30) and that $\operatorname{supp} \tilde{w} \subset \operatorname{supp} \phi$. Furthermore, it holds that

$$\|\mathbf{V}''\|_{L^\infty(\mathcal{O})} = \lambda^{-6} \|\mathcal{R}[\Delta^2 \mathbf{f}]\|_{L^\infty(\mathcal{O})} \leq C(|\bar{\boldsymbol{n}}|, |\bar{\mathbf{V}}|, \phi, h_0)\lambda^{-1},$$

which together with (41) and (42) leads to

$$\|\tilde{w}(x,t) - \bar{w} h_0(\lambda \tau t + \lambda \xi \cdot x)\phi(x,t)\|_{L^\infty(\mathcal{O})} \leq C(\bar{w}, \phi, h_0)\lambda^{-1}. \tag{43}$$

Since for $(x,t) \in \mathcal{O}, \bar{w}(h_0\phi)(x,t) \in [-\mu_2 \bar{w}, \mu_1 \bar{w}]$, it follows from (43) that

$$\begin{aligned} \operatorname{dist}(w + \tilde{w}, [w_1, w_2]) &\leq \operatorname{dist}(w + \bar{w} h_0 \phi, [w_1, w_2]) + |\tilde{w} - \bar{w} h_0(\lambda \tau t + \lambda \xi \cdot x)\phi| \\ &\leq C(\bar{w}, \phi, h_0)\lambda^{-1}. \end{aligned}$$

Define the disjoint open sets $\mathcal{O}_i$ ($i = 1$ and 2) as

$$\mathcal{O}_i = \left\{ (x,t) \in \mathcal{O} : |w + \bar{w} h_0(\lambda \tau t + \lambda \xi \cdot x)\phi(x,t) - w_i| < \min\left(\frac{\varepsilon}{2}, \frac{|w_2 - w_1|}{4} \right) \right\}.$$

It follows from (36) and (37) that $\mathcal{H}^{n+1}(\mathcal{O}_i)$ can be arbitrary close to $\mu_i \mathcal{H}^{n+1}(\mathcal{O})$ for δ sufficiently small and λ sufficiently large. For $(x,t) \in \mathcal{O}_i$, it holds that

$$\begin{aligned} |w + \tilde{w}(x,t) - w_i| &\leq |w + \bar{w} h_0(\lambda \tau t + \lambda \xi \cdot x)\phi - w_i| + |\tilde{w} - \bar{w} h_0(\lambda \tau t + \lambda \xi \cdot x)\phi| \\ &\leq \frac{\varepsilon}{2} + C(\bar{w}, \phi, h_0)\lambda^{-1}. \end{aligned}$$

Therefore, the properties (2) and (3) in Lemma 6 follow by first choosing δ sufficiently small and then λ sufficiently large. The proof of the proposition is finished. □

4. Convex integration

In this section, weak solutions are obtained by iterating strict subsolutions, based on the method of convex integration. The main goal is to show the following proposition.

Proposition 1. *If $(\rho, \underline{\mathbf{m}}, \underline{\mathbf{U}}, q)$ is a strict subsolution in $\mathcal{D}$ with constrain sets $K_{(x,t)}$, then there exist infinitely many pairs $(\mathbf{m}, \mathbf{U})$ such that $(\rho, \mathbf{m}, \mathbf{U}, q)$ are subsolutions and*

$$(\mathbf{m}, \mathbf{U})(x,t) \in K_{(x,t)} \text{ for a.e. } (x,t) \in \mathcal{D}, \quad \text{and} \quad \operatorname{supp}(\mathbf{m} - \underline{\mathbf{m}}, \mathbf{U} - \underline{\mathbf{U}}) \subset \overline{\mathcal{D}}. \tag{44}$$

To obtain weak solutions, we use a convex integration scheme to construct strongly convergent sequences of strict subsolutions. This gives a more constructive proof. The scheme is similar to the construction in [9,24], and the convergence relies on the method of controlled weak convergence in [34]. First, we have the following lemma which is a consequence of Lemma 6 and an inductive argument.

Lemma 9. *Suppose that $\tilde{K} \subset K_{\bar{\rho},\bar{q}}$, and $\bar{w} \in$ int conv $\tilde{K}$ is a constant vector. Given a bounded open space–time set $\mathcal{U}$ and $\bar{\varepsilon} > 0$, there exists a $\tilde{w} \in C_c^\infty(\mathcal{U}; \mathbb{R}^n \times \mathbb{S}_0^{n\times n})$ such that*

1. *$\tilde{w}$ satisfies (30);*
2. *there exists a $\gamma = \gamma(\varepsilon, \bar{w}, \tilde{K}) > 0$, which is independent of $\mathcal{U}$, such that*

$$Q_\gamma + \bar{w} + \overline{image(\tilde{w})} \subset \text{int conv } \tilde{K},$$

where $\overline{image(\tilde{w})}$ denotes the closure of the image of $\tilde{w}$ as defined in (11);

3. *the following integral estimate holds*

$$\frac{1}{|\mathcal{U}|} \int_{\mathcal{U}} dist(\bar{w} + \tilde{w}(z), \tilde{K}) dz < \bar{\varepsilon}. \tag{45}$$

Proof. It follows from Lemma 2 that there exists a finite set $\{w_i\}_{i=1}^N \subset \tilde{K}$ such that $\bar{w} \in$ int conv $\{w_i\}_{i=1}^N$. For $\beta \in (0,1)$, define

$$L_\beta = \{w_i^{(\beta)} = \bar{w} + (1-\beta)(w_i - \bar{w})\}_{i=1}^N. \tag{46}$$

It is easy to see that $\bar{w} \in$ int conv L_β for any $\beta \in (0,1)$, and that for any $0 < \beta_2 < \beta_1 < 1$, one has

$$\overline{\text{conv } L_{\beta_1}} \subset \text{int conv } L_{\beta_2} \subset \text{int conv } \tilde{K}.$$

Choose δ such that

$$\delta \max_i |w_i - \bar{w}| < \varepsilon/4. \tag{47}$$

Set $L^{(0)} = L_\delta$ and

$$L^{(j+1)} = L^{(j)} \cup \{\nu_1 w_1' + \nu_2 w_2' : w_i' \in L^{(j)}, w_2' - w_1' \in \Lambda, \nu_i \in (0,1), \nu_1 + \nu_2 = 1\}. \tag{48}$$

Obviously, for any $j \geq 0$, one has

$$L^{(j)} \subset \text{conv } L_\delta. \tag{49}$$

We claim that for $j \geq 0$, one has

$$\left\{ \sum_{i=1}^{j+1} \nu_i w_i' \middle| \, w_i' \in L^{(0)}, \ \nu_i \geq 0, \ \sum_{i=1}^{j+1} \nu_i = 1 \right\} \subset L^{(j)}. \tag{50}$$

Indeed, the inclusion (50) is obviously true for $j = 0$. Suppose the relation (50) holds for $0 \le j \le k$. Given $w = \sum_{i=1}^{k+2} \nu_i w_i'$ for $w_i' \in L^{(0)}$, $\nu_i > 0$, and $\sum_{i=1}^{k+2} \nu_i = 1$, one can represent w as follows

$$w = \frac{\nu_1}{\nu_1 + \nu_2} w' + \frac{\nu_2}{\nu_1 + \nu_2} w'',$$

where

$$w' = (\nu_1 + \nu_2) w_1' + \sum_{i=3}^{k+2} \nu_i w_i' \quad \text{and} \quad w'' = (\nu_1 + \nu_2) w_2' + \sum_{i=3}^{k+2} \nu_i w_i'.$$

Clearly, the induction assumption asserts that w', $w'' \in L^{(k)}$. Furthermore, one has

$$w' - w'' = (\nu_1 + \nu_2)(w_1' - w_2').$$

Since $w_i' \in L^{(0)}$ $(i = 1, 2)$, there exist $w_{j_i} \in \tilde{K}$ $(i = 1, 2)$ such that $w_i' = w_{j_i}^{(\delta)}$. Hence

$$w_1' - w_2' = w_{j_1}^{(\delta)} - w_{j_2}^{(\delta)} = (1 - \delta)(w_{j_1} - w_{j_2}).$$

It follows from Lemma 5 that one has $w_{j_1} - w_{j_2} \in \Lambda$ as long as $w_{j_i} \in \tilde{K}$ $(i = 1, 2)$. Hence we have $w' - w'' \in \Lambda$. This yields $w \in L^{(k+1)}$ and proves the claim.

The relations (49) and (50), together with Carathéodory's theorem, imply that

$$L^{(N_n)} = \text{conv } L_\delta, \quad N_n = \dim(\mathbb{R}^n \times \mathbb{S}_0^{n\times n}).$$

Let $\tau_j = 2^{-(j+1)}$ for $j = 0, 1, 2, \cdots, N_n$. We claim that for any $w \in L^{(j)}$ and any open subset $\mathcal{O} \subset \mathcal{U}$, there exists a $\tilde{w} \in C_c^\infty(\mathcal{O})$ satisfying (30) and

$$w + \overline{\text{image}(\tilde{w})} \subset \text{int conv } L_{\tau_j \delta}, \quad \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \text{dist}(w + \tilde{w}(z), \tilde{K}) dz < (1 - \tau_j)\bar{\varepsilon}. \tag{51}$$

We prove the claim by mathematical induction. First, if $w \in L^{(0)} = L_\delta$, then $w = w_i^{(\delta)}$ for some i. Clearly, $w \in L_\delta \subset \text{int conv } L_{\tau_0 \delta}$ and (30) holds for $\tilde{w} = 0$. Furthermore, it follows from (46) and (47) that

$$\text{dist}(w, \tilde{K}) \le |w_i - w_i^{(\delta)}| \le \max_l |w_l - w_l^{(\delta)}| \le \delta \max_l |w_l - \bar{w}| < \bar{\varepsilon}/4 < (1 - \tau_0)\bar{\varepsilon}.$$

This proves the above claim for the case $j = 0$.

Suppose that the above claim is true for $0 \le j \le k$. If $w \in (L^{(k+1)} \setminus L^{(k)})$, then there exist $w_1', w_2' \in L^{(k)}$ such that

$$w = \nu_1 w_1' + \nu_2 w_2', \quad w_2' - w_1' \in \Lambda, \ \nu_i \in (0, 1), \ \nu_1 + \nu_2 = 1.$$

Given an open subset $\mathcal{O} \subset \mathcal{U}$ and a positive constant ε_0 to be determined later, in view of Lemma 6, there exist a $\tilde{w}_0 \in C_c^\infty(\mathcal{O})$ satisfying (30) and

$$\mathrm{dist}(w + \tilde{w}_0, [w_1', w_2']) < \varepsilon_0 \text{ in } \mathcal{O},$$

and two disjoint open subsets $\mathcal{O}_i \subset \mathcal{O}$ satisfying

$$|w + \tilde{w}_0 - w_i'| < \varepsilon_0 \text{ in } \mathcal{O}_i \text{ and } |\mathcal{H}^{n+1}(\mathcal{O}_i) - \nu_i \mathcal{H}^{n+1}(\mathcal{O})| < \varepsilon_0 \text{ for } i = 1, 2. \tag{52}$$

For $i = 1, 2$, since $w_i' \in L^{(k)}$, it follows from the induction assumption for $j = k$ that there exists a $\tilde{w}_i \in C_c^\infty(\mathcal{O}_i)$ satisfying (30) and

$$w_i' + \overline{\mathrm{image}(\tilde{w}_i)} \subset \mathrm{int\ conv\ } L_{\tau_k \delta}, \quad \frac{1}{|\mathcal{O}_i|} \int\limits_{\mathcal{O}_i} \mathrm{dist}(w_i' + \tilde{w}_i(z), \tilde{K}) dz < (1 - \tau_k)\bar{\varepsilon}. \tag{53}$$

Let $\tilde{w} = \tilde{w}_0 + \tilde{w}_1 + \tilde{w}_2$. Clearly $\tilde{w} \in C_c^\infty(\mathcal{O})$ and satisfies (30). Furthermore, for $z \in \mathcal{O} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$,

$$w + \tilde{w}(z) = w + \tilde{w}_0(z) \in Q_{\varepsilon_0} + [w_1', w_2'] \subset Q_{\varepsilon_0} + \mathrm{conv\ } L_{\tau_k \delta},$$

and for $z \in \mathcal{O}_i$ $(i = 1, 2)$,

$$w + \tilde{w}(z) = (w + \tilde{w}_0(z) - w_i') + (w_i' + \tilde{w}_i(z)) \in Q_{\varepsilon_0} + \mathrm{conv\ } L_{\tau_k \delta}.$$

Since $\overline{\mathrm{conv\ } L_{\tau_k \delta}} \subset \mathrm{int\ conv\ } L_{\tau_{k+1} \delta}$, for ε_0 sufficiently small it holds that

$$Q_{2\varepsilon_0} + \mathrm{conv\ } L_{\tau_k \delta} \subset \mathrm{int\ conv\ } L_{\tau_{k+1} \delta}. \tag{54}$$

Then $w + \overline{\mathrm{image}(\tilde{w})} \subset \mathrm{int\ conv\ } L_{\tau_{k+1} \delta}$.

For $z \in \mathcal{O}_i, w + \tilde{w}(z) = w + \tilde{w}_0(z) + \tilde{w}_i(z)$. It follows from (52) that

$$\begin{aligned} \mathrm{dist}(w + \tilde{w}(z), \tilde{K}) &\le |w + \tilde{w}_0(z) - w_i'| + \mathrm{dist}(w_i' + \tilde{w}_i(z), \tilde{K}) \\ &\le \varepsilon_0 + \mathrm{dist}(w_i' + \tilde{w}_i(z), \tilde{K}), \quad \text{for } z \in \mathcal{O}_i. \end{aligned}$$

Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open subsets of $\mathcal{O}$, it follows from (52) that

$$\begin{aligned} \mathcal{H}^{n+1}(\mathcal{O} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)) &= \mathcal{H}^{n+1}(\mathcal{O}) - \sum_{i=1}^{2} \mathcal{H}^{n+1}(\mathcal{O}_i) \le \sum_{i=1}^{2} |\nu_i \mathcal{H}^{n+1}(\mathcal{O}) - \mathcal{H}^{n+1}(\mathcal{O}_i)| \\ &< 2\varepsilon_0. \end{aligned}$$

It follows from (16) that $\mathrm{dist}(w + \tilde{w}(z), \tilde{K}) \le 2\mathfrak{C}(\bar{\rho}, \bar{q})$. One has

$$\begin{aligned}\int_{\mathcal{O}} \operatorname{dist}(w+\tilde{w}(z), \tilde{K})dz &\le \sum_{i=1}^{2}\int_{\mathcal{O}_i} \operatorname{dist}(w+\tilde{w}(z), \tilde{K})dz + 2\mathfrak{C}(\bar{\rho},\bar{q})|\mathcal{O}\setminus(\cup_i\mathcal{O}_i)| \\ &\le \sum_{i=1}^{2}\int_{\mathcal{O}_i} \varepsilon_0 + \operatorname{dist}(w_i'+\tilde{w}_i(z), \tilde{K})dz + 4\mathfrak{C}(\bar{\rho},\bar{q})\varepsilon_0 \\ &\le (|\mathcal{O}_1|+|\mathcal{O}_2|)(1-\tau_k)\bar{\varepsilon} + C\varepsilon_0,\end{aligned}$$

where (53) is used in the last inequality. Since $\tau_{k+1} < \tau_k$, one has

$$\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \operatorname{dist}(w+\tilde{w}(z), \tilde{K})dz \le (1-\tau_k)\varepsilon + \frac{C\varepsilon_0}{|\mathcal{O}|} < (1-\tau_{k+1})\varepsilon, \tag{55}$$

provided that ε_0 is sufficiently small. One can choose ε_0 depending on $\mathcal{H}^{n+1}(\mathcal{O})$, $\bar{\varepsilon}$, τ_k and τ_{k+1} so that it satisfies (54) and (55). This proves the claim for the existence of $\tilde{w}$ satisfying (30) and (51) for any $w \in L^{(k)}$. Therefore, for $\bar{w} \in \operatorname{conv} L_\delta = L^{(N_n)}$, there exists a $\tilde{w} \in C_c^\infty(\mathcal{U})$ satisfying (30) and

$$\bar{w} + \overline{\operatorname{image}(\tilde{w})} \subset \operatorname{int}\operatorname{conv} L_{\tau_{N_n}\delta}, \quad \frac{1}{|\mathcal{U}|}\int_{\mathcal{U}} \operatorname{dist}(\bar{w}+\tilde{w}(z), \tilde{K})dz < (1-\tau_{N_n})\bar{\varepsilon}. \tag{56}$$

Hence one can choose $\gamma > 0$ sufficiently small such that

$$Q_\gamma + \bar{w} + \overline{\operatorname{image}(\tilde{w})} \subset Q_\gamma + \operatorname{int}\operatorname{conv} L_{\tau_{N_n}\delta} \subset \operatorname{int}\operatorname{conv} \tilde{K}.$$

This finishes the proof of the lemma. □

To deal with variable constrain sets $K_{(x,t)}$, one needs a stability result.

Lemma 10. *(See [11, Lemma 1].) Let K be a compact set. For any compact set $\mathcal{C} \subset$ int conv K there exists a $\delta > 0$ depending on $\mathcal{C}$ and K such that for any compact set L with*

$$d_H(K,L) < \delta \quad \textit{(in Hausdorff distance)}$$

it holds that

$$\mathcal{C} \subset \textit{int conv } L.$$

In view of the above two lemmas, the following key perturbation property holds.

Lemma 11. *Suppose that (ρ, w, q) is a strict subsolution in a bounded open set $\mathcal{D}$ with constrain sets $K_{(x,t)}$. Given any $\varepsilon > 0$, there exists a compact set $\mathcal{C} \subset \mathcal{D}$ and a sequence*

$\{w_k\}$ *such that* (ρ, w_k, q) *are strict subsolutions in* $\mathcal{D}$ *with the same constrain sets* $K_{(x,t)}$ *and satisfy*

$$w_k - w \in C_c^\infty(\mathcal{D}), \quad supp\,(w_k - w) \subset \mathcal{C}, \quad w_k \to w \text{ in } CL^\infty_{w*},$$

and

$$\int_{\mathcal{D}} dist(w_k(x,t), K_{(x,t)})dxdt \le \varepsilon \quad \text{for all } k = 1, 2, \cdots . \tag{57}$$

Proof. Set $C_0 = 2\sup_{z\in\mathcal{D}} \mathfrak{C}(\rho(z), q(z))$ with $\mathfrak{C}(\rho, q)$ defined in (16). Approximate $\mathcal{D}$ by a compact subset $\mathcal{C}'$ such that $\mathcal{H}^{n+1}(\mathcal{D}\setminus\mathcal{C}') < \frac{\varepsilon}{4C_0}$. Given $\zeta \in \mathcal{D}$, for any bounded open set $\mathcal{U}$, applying Lemma 9 to $w(\zeta) \in \text{int conv } K_\zeta$ and $\bar\varepsilon = \frac{\varepsilon}{4|\mathcal{D}|}$ yields that there exists a $v \in C_c^\infty(\mathcal{U})$ satisfying (30) and

$$Q_\gamma + w(\zeta) + \overline{\text{image}(v)} \subset \text{int conv } K_\zeta, \quad \frac{1}{|\mathcal{U}|}\int_{\mathcal{U}} \text{dist}(w(\zeta) + v(z), K_\zeta)dz < \frac{\varepsilon}{4|\mathcal{D}|}. \tag{58}$$

Set $\mathfrak{K}_{\zeta,\gamma} = \{\bar w \in \text{int conv } K_\zeta : \text{dist}(\bar w, \partial \text{ conv } K_\zeta) \ge \gamma/2\}$. The first relation in (58) gives

$$Q_{\gamma/2} + w(\zeta) + \overline{\text{image}(v)} \subset \overline{\mathfrak{K}_{\zeta,\gamma}} \subset \text{int conv } K_\zeta.$$

It follows from Lemma 10 and the continuity of $w(z)$ and K_z that there exists a $r(\zeta) > 0$ such that

$$w(z) + \overline{\text{image}(v)} \subset \overline{\mathfrak{K}_{\zeta,\gamma}} \subset \text{int conv } K_z \quad \text{ for } z \in Q_{r(\zeta)}(\zeta). \tag{59}$$

Furthermore, by shrinking $r(\zeta)$ if necessary, the following estimates hold

$$d_H(K_z, K_\zeta) < \frac{\varepsilon}{8|\mathcal{D}|} \quad \text{ and } \quad |w(z) - w(\zeta)| < \frac{\varepsilon}{8|\mathcal{D}|} \quad \text{ for } z \in Q_{r(\zeta)}(\zeta). \tag{60}$$

Let $\{\mathcal{O}^i = Q_{r(\zeta^i)}(\zeta^i)\}_{i=1}^N$ be a finite covering of the compact set $\mathcal{C}'$ such that $\cup\overline{\mathcal{O}^i} \subset \mathcal{D}$. Set $\mathcal{C} := \cup\overline{\mathcal{O}^i}$ and $r_0 = \frac{1}{2}\min_i r(\zeta^i)$.

It follows from the Whitney covering lemma [32] that, for the open sets $\mathcal{O}^1$ and $\{\mathcal{O}^i \setminus (\bigcup_{j=1}^{i-1}\overline{\mathcal{O}^j})\}_{i=2}^N$, there exist a family of disjoint open cubes $\{\tilde Q^l\}_{l=1}^\infty$ with $\mathcal{H}^{n+1}(\mathcal{C} \setminus (\cup_l \tilde Q^l)) = 0$, each $\tilde Q^l$ lying in some $\mathcal{O}^{i(l)}$. For $k = 1, 2, \cdots$, decomposing the cubes $\tilde Q^l$ further if necessary, yields that there exist finitely many disjoint open cubes $\{Q^{(j,k)}\} = \{Q_{r_{jk}}(z^{(j,k)})\}_{j=1}^{J_k}$ satisfying

$$\max_{1\le j\le J_k} r_{jk} < 2^{-k} r_0, \quad \mathcal{H}^{n+1}\left(\mathcal{C} \setminus \left(\bigcup_{j=1}^{J_k} Q^{(j,k)}\right)\right) < \frac{\varepsilon}{4C_0},$$

and for each $j = 1, \cdots, J_k$, $Q^{(j,k)} \subset \mathcal{O}^{i(j,k)}$ for some $i(j,k)$.

Applying Lemma 9 to $w(\zeta^{i(j,k)})$ with $\mathcal{U} = Q^{(j,k)}$ and $\bar{\varepsilon} = \frac{\varepsilon}{4|\mathcal{D}|}$ yields the existence of $a_{jk}(z) \in C_c^\infty(Q^{(j,k)})$, which satisfies (30),

$$Q_\gamma + w(\zeta^{i(j,k)}) + \overline{\text{image}(a_{jk})} \subset \text{int conv } K_{\zeta^{i(j,k)}}, \tag{61}$$

and

$$\frac{1}{|Q^{(j,k)}|} \int\limits_{Q^{(j,k)}} \text{dist}(w(\zeta^{i(j,k)}) + a_{jk}(z), K_{\zeta^{i(j,k)}})dz < \frac{\varepsilon}{4|\mathcal{D}|}. \tag{62}$$

Let

$$\tilde{w}_k(z) = \sum_{j=1}^{J_k} a_{jk}(z) \quad \text{and} \quad w_k(z) = w(z) + \tilde{w}_k(z).$$

It is clear that $\tilde{w}_k \in C_c^\infty(\mathcal{D})$ and $\text{supp}\tilde{w}_k \subset \cup_j Q^{(j,k)} \subset \mathcal{C}$. Since a_{jk} satisfies (30), it is easy to see that (ρ, w_k, q) solves (17) and (18). It follows from (61) and $Q^{(j,k)} \subset \mathcal{O}^{i(j,k)}$ that for $z \in Q^{(j,k)}$,

$$w_k(z) = w(z) + a_{jk}(z) \in w(z) + \text{image}(a_{jk}) \subset \text{int conv } K_z.$$

For $z \in \mathcal{D} \setminus (\cup_j Q^{(j,k)})$, $w_k(z) = w(z) \in \text{int conv } K_z$. Therefore (ρ, w_k, q) is a strict subsolution in $\mathcal{D}$ with the constrain sets $K_{(x,t)}$.

It follows from (60) and (62) that

$$\begin{aligned} \int\limits_{Q^{(j,k)}} \text{dist}(w_k(z), K_z)dz &= \int\limits_{Q^{(j,k)}} \text{dist}(w(z) + a_{jk}(z), K_z)dz \\ &\le \int\limits_{Q^{(j,k)}} \text{dist}(w(\zeta^{i(j,k)}) + a_{jk}(z), K_{\zeta^{i(j,k)}}) + \frac{\varepsilon}{4|\mathcal{D}|}dz \\ &\le \frac{|Q^{(j,k)}|}{2|\mathcal{D}|}\varepsilon. \end{aligned}$$

Thus one has

$$\begin{aligned} \int\limits_{\mathcal{D}} \text{dist}(w_k(z), K_z)dz &= \sum_{j=1}^{J_k} \int\limits_{Q^{(j,k)}} \text{dist}(w_k(z), K_z)dz + \int\limits_{\mathcal{D}\setminus(\cup_j Q^{(j,k)})} \text{dist}(w_k(z), K_z)dz \\ &\le |\mathcal{D}|\frac{\varepsilon}{2|\mathcal{D}|} + C_0|\mathcal{D} \setminus (\cup_j Q^{(j,k)})| \le \frac{\varepsilon}{2} + C_0|\mathcal{D} \setminus \mathcal{C}'| \\ &\le \varepsilon. \end{aligned}$$

Let $\mathscr{Q}^{(j,k)}$ be the projection of $Q^{(j,k)}$ on $\mathbb{R}^n$ and denote by $z^{(j,k)} = (x^{(j,k)}, t^{(j,k)})$. Notice that

$$a_{jk} \in C_c^\infty(Q^{(j,k)}), \qquad \int a_{jk}(x,t)dx = 0, \qquad \|a_{jk}\|_{L^\infty} \leq 2C_0. \tag{63}$$

For $\phi \in C_c^\infty(\mathbb{R}^n)$, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \tilde{w}_k \phi dx\right| &= \left|\sum_{j=1}^{J_k} \int_{Q^{(j,k)}} a_{jk}\phi dx\right| = \left|\sum_{j=1}^{J_k} \int_{Q^{(j,k)}} a_{jk}(x,t)(\phi(x) - \phi(x^{(j,k)}))dx\right| \\ &\leq 2C_0|\operatorname{supp}\phi| \max_{x\in\mathscr{Q}^{(j,k)}} |\phi(x) - \phi(x^{(j,k)})|.\end{aligned}$$

Using $\operatorname{diam}(\mathscr{Q}^{(j,k)}) \leq \sqrt{n}2^{-k}r_0$ yields

$$\left|\int \tilde{w}_k \phi\, dx\right| \to 0 \quad \text{uniformly in } t \text{ as } k \to \infty.$$

It follows from the density argument that $\tilde{w}_k \to 0$ in CL^∞_{w*}. This completes the proof of the lemma. □

Now we are in position to prove Proposition 1.

Proof of Proposition 1. Denote $\underline{w} = (\underline{\mathbf{m}}, \underline{\mathbf{U}})$. Define

$$\begin{aligned}X^0 = \{w \in C_{loc}(\bar{\mathbb{R}}_+; L^\infty_{w*}) : (\rho, w, q) &\text{ are strict subsolutions in } \mathcal{D} \\ &\text{with constrain sets } K_{(x,t)} \text{ and } w - \underline{w} \in C_c^\infty(\mathcal{D})\}.\end{aligned} \tag{64}$$

Obviously, $\underline{w} \in X^0$. This implies that X^0 is non-empty. It follows from (16) that X^0 is a bounded set in $L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$. Let X be the closure of X^0 in $C_{loc}(\bar{\mathbb{R}}_+; L^\infty_{w*}(\mathbb{R}^n))$ topology. Observe that X is metrizable due to the Banach–Alaoglu theorem and hence there is a metric d_X such that (X, d_X) forms a complete metric space (see [14]). Our aim is to show that the set of $w = (\mathbf{m}, \mathbf{U})$ satisfying (44) is an infinite dense set in X.

Given any $w' \in X^0$ and $\varepsilon_0 > 0$, we shall construct a sequence $\{w_k\} \subset X^0$ which converge to a $w \in X$, such that

$$d_X(w_k, w') \leq \varepsilon_0, \tag{65}$$

and for any compact subset $\mathcal{C} \subset \mathcal{D}$, we have

$$w_k \to w \text{ strongly in } L^2(\mathcal{C}) \text{ and } \int_{\mathcal{C}} \operatorname{dist}(w_k(z), K_z)dz \to 0. \tag{66}$$

Assuming the existence of such a sequence w_k, it is easy to see that the limit $w = (\mathbf{m}, \mathbf{U})$ satisfies (44) and $d_X(w', w) \le \varepsilon_0$. Therefore, the set for $w = (\mathbf{m}, \mathbf{U})$ satisfying (44) is a dense set in X^0 and hence also a dense set in X. It follows from (66) that $w \notin X^0$. Thus $\{w_k\}$ must be an infinite sequence in X^0, which yields that X must be an infinite set. Therefore, a dense set in X which satisfies (44) is also infinite.

Let $\{\mathcal{D}_j\}_{j=1}^{\infty}$ be a family of bounded open subsets of $\mathcal{D}$ such that $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots$, and $\mathcal{D} = \cup \mathcal{D}_i$. In the case that $\mathcal{D}$ is bounded, one can take $\mathcal{D}_j = \mathcal{D}$ for $j = 1, 2, \cdots$. Set $w_1 = w'$. If, for some $k \ge 1$, $w_k \in X^0$ has been obtained, then we construct w_{k+1} as follows. Since (ρ, w_k, q) is a strict subsolution in $\mathcal{D}$ with constrain sets $K_{(x,t)}$, one has $w_k(z) \in \text{int conv } K_z$ for $z \in \mathcal{D}$, which yields

$$\int_{\mathcal{D}_1} \text{dist}(w_k(z), K_z) dz > 0.$$

Note that (ρ, w_k, q) is also a strict subsolution in $\mathcal{D}_{k+1} \subset \mathcal{D}$. Applying Lemma 11 to w_k with $\mathcal{D}_{k+1}$, there exists a sequence $\{v^{(i,k)}\}_{i=1}^{\infty}$ such that $(\rho, v^{(i,k)}, q)$ are strict subsolutions in $\mathcal{D}_{k+1}$ with constrain sets $K_{(x,t)}$, and satisfy

$$v^{(i,k)} - w_k \in C_c^{\infty}(\mathcal{D}_{k+1}), \quad v^{(i,k)} \to w_k \text{ in } CL^{\infty}_{w*} \quad \text{as } i \to \infty,$$

and

$$\int_{\mathcal{D}_{k+1}} \text{dist}(v^{(i,k)}(z), K_z) dz \le \min \left\{ 2^{-k}, \frac{1}{2} \int_{\mathcal{D}_1} \text{dist}(w_k(z), K_z) dz \right\}. \tag{67}$$

Since $v^{(i,k)}(z) \in \text{int conv } K_z$ for $z \in \mathcal{D}_{k+1}$ and $\text{supp}\,(v^{(i,k)} - w_k) \subset \mathcal{D}_{k+1}$, it follows that $v^{(i,k)}(z) \in \text{int conv } K_z$ for $z \in \mathcal{D}$. It is easy to see that $(\rho, v^{(i,k)}, q)$ are strict subsolutions in $\mathcal{D}$. Since $v^{(i,k)} - \underline{w} = (v^{(i,k)} - w_k) + (w_k - \underline{w}) \in C_c^{\infty}(\mathcal{D})$, it holds that $v^{(i,k)} \in X^0$. Furthermore, since $v^{(i,k)} \to w_k$ in CL^{∞}_{w*} as $i \to \infty$ and d_X is the metric induced by the topology of CL^{∞}_{w*}, there exists an i_k sufficiently large such that

$$\max_{1 \le j \le k} \left| \int_{\mathcal{D}_j} (v^{(i_k,k)} - w_k) w_k dz \right| < \min \left\{ 2^{-k}, \frac{1}{100|\mathcal{D}_k|} \left(\int_{\mathcal{D}_1} \text{dist}(w_k(z), K_z) dz \right)^2 \right\} \tag{68}$$

and

$$d_X(v^{(i_k,k)}, w_k) < 2^{-(k+1)} \varepsilon_0. \tag{69}$$

Set $w_{k+1} = v^{(i_k,k)}$. Then w_{k+1} satisfies

$$\int_{\mathcal{D}_{k+1}} \operatorname{dist}(w_{k+1}, K_z)dz \le \min\left\{2^{-k}, \frac{1}{2}\int_{\mathcal{D}_1} \operatorname{dist}(w_k(z), K_z)dz\right\}, \tag{70}$$

$$\max_{1\le j\le k}\left|\int_{\mathcal{D}_j}(w_{k+1}-w_k)w_k dz\right| < \min\left\{2^{-k}, \frac{1}{100|\mathcal{D}_k|}\left(\int_{\mathcal{D}_1}\operatorname{dist}(w_k(z), K_z)dz\right)^2\right\}, \tag{71}$$

and

$$d_X(w_{k+1}, w_k) < 2^{-(k+1)}\varepsilon_0. \tag{72}$$

It follows from (72) that $\{w_k\}$ is a Cauchy sequence in X and thus w_k converges to some $w \in X$ which satisfies (65). We now show that for any fixed j, $\{w_k\}$ converges strongly in $L^2(\mathcal{D}_j)$.

Fix any $j \ge 1$. For $k \ge j$, since $\mathcal{D}_1 \subset \mathcal{D}_j \subset \mathcal{D}_{k+1}$, it follows from (67) that

$$\begin{aligned}\int_{\mathcal{D}_j}|w_{k+1}-w_k|dz &\ge \int_{\mathcal{D}_j}\operatorname{dist}(w_k, K_z)dz - \int_{\mathcal{D}_j}\operatorname{dist}(w_{k+1}, K_z)dz\\ &\ge \int_{\mathcal{D}_j}\operatorname{dist}(w_k, K_z)dz - \int_{\mathcal{D}_{k+1}}\operatorname{dist}(w_{k+1}, K_z)dz\\ &\ge \frac{1}{2}\int_{\mathcal{D}_j}\operatorname{dist}(w_k, K_z)dz.\end{aligned}$$

Thus applying Hölder's inequality yields

$$\|w_{k+1}-w_k\|^2_{L^2(\mathcal{D}_j)} \ge \frac{1}{|\mathcal{D}_j|}\left(\int_{\mathcal{D}_j}|w_{k+1}-w_k|dz\right)^2 \ge \frac{1}{4|\mathcal{D}_j|}\left(\int_{\mathcal{D}_j}|w_{k+1}-w_k|dz\right)^2.$$

Combining this with (71) yields that

$$\|w_{k+1}\|^2_{L^2(\mathcal{D}_j)} - \|w_k\|^2_{L^2(\mathcal{D}_j)} = \|w_{k+1}-w_k\|^2_{L^2(\mathcal{D}_j)} - 2\int_{\mathcal{D}_j}(w_k - w_{k+1})w_k dz \ge 0.$$

Therefore, $\{\|w_k\|^2_{L^2(\mathcal{D}_j)}\}_{k=j}^\infty$ is a non-decreasing sequence. This, together with

$$\|w_k\|_{L^2(\mathcal{D}_j)} \le \mathfrak{C}(\rho, q)|\mathcal{D}_j|^{1/2},$$

which implies the uniform boundedness of the sequence $\{\|w_k\|^2_{L^2(\mathcal{D}_j)}\}_{k=j}^\infty$, yields that $\{\|w_k\|^2_{L^2(\mathcal{D}_j)}\}_{k=j}^\infty$ is a convergent sequence. It follows from (71) that for $k > m \ge j$, the following estimates hold

$$\begin{aligned}\|w_k - w_m\|^2_{L^2(\mathcal{D}_j)} &\le 2(\|w_k - w_{k-1}\|^2_{L^2(\mathcal{D}_j)} + \cdots + \|w_{m+1} - w_m\|^2_{L^2(\mathcal{D}_j)}) \\ &= 2\sum_{l=m}^{k-1}\left(\|w_{l+1}\|^2_{L^2(\mathcal{D}_j)} - \|w_l\|^2_{L^2(\mathcal{D}_j)} - 2\int_{\mathcal{D}_j}(w_{l+1} - w_l)w_l dz\right) \\ &\le 2(\|w_k\|^2_{L^2(\mathcal{D}_j)} - \|w_m\|^2_{L^2(\mathcal{D}_j)}) + 2^{-m+3}.\end{aligned}$$

Hence $\{w_k\}$ converges strongly in $L^2(\mathcal{D}_j)$. The estimates (65) and (66) for $\{w_k\}$ follow from (72) and (70). The proof of the proposition is finished. □

As in [5,14,18], it is possible to construct weak solutions with initial data satisfying (76), which are useful for constructing admissible solutions. It is straightforward to extend Definition 3 to the cases where the time domain $\bar{\mathbb{R}}_+$ is replaced by $[-\tau, \infty)$ for some $\tau \ge 0$.

Corollary 2. *Suppose that $(\rho, \underline{\mathbf{m}}, \underline{\mathbf{U}}, q)$ is a strict subsolution in $\mathcal{D} \subset \mathbb{R}^n \times [-\tau, \infty)$ for some $\tau > 0$, such that $\mathcal{D}_{t=0} = \{x|(x, 0) \in \mathcal{D}\} \subset \mathbb{R}^n$ is a non-empty open set and $\rho(\cdot, 0) = \rho_0$ with*

$$|\underline{\mathbf{m}}|^2(x, 0) = n\rho_0(x)q(x, 0) \quad \text{for a.e. } x \in (\mathbb{R}^n \setminus \mathcal{D}_{t=0}), \tag{73}$$

and

$$(\underline{\mathbf{m}}, \underline{\mathbf{U}})(x, t) \in K_{\rho(x,t),q(x,t)} \quad \text{for a.e. } (x, t) \in (\mathbb{R}^n \times \mathbb{R}_+ \setminus \mathcal{D}). \tag{74}$$

Then there exists an $\mathbf{m}^\diamond \in L^\infty(\mathbb{R}^n)$ with

$$\mathbf{m}^\diamond(x) \in (K_{(x,0)}|_{\mathbb{R}^n}) \quad \text{for a.e. } x \in \mathcal{D}_{t=0}, \tag{75}$$

and

$$|\mathbf{m}^\diamond|^2(x) = n\rho_0(x)q(x, 0) \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{76}$$

such that the Cauchy problem (1) and (3) has infinitely many weak solutions $(\rho, \mathbf{m}^\flat)$ satisfying

$$\mathbf{m}^\flat(x, t) \in (K_{(x,t)}|_{\mathbb{R}^n}) \quad \text{for a.e. } (x, t) \in \mathcal{D}, \tag{77}$$

and

$$|\mathbf{m}^\flat|^2(x, t) = n\rho(x, t)q(x, t) \quad \text{for a.e. } (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{78}$$

where $K_{(x,t)}|_{\mathbb{R}^n} = \{\boldsymbol{n} \in \mathbb{R}^n : (\boldsymbol{n}, \mathbf{V}) \in K_{(x,t)}$ for some $\mathbf{V} \in \mathbb{S}_0^{n\times n}\}$.

Furthermore, there exists a sequence of divergence-free vector fields $\{\tilde{\mathbf{m}}_j\}$ such that

$$\tilde{\mathbf{m}}_j \in C_c^\infty(\mathcal{D}) \quad \text{and} \quad \underline{\mathbf{m}} + \tilde{\mathbf{m}}_j \to \mathbf{m}^\flat \text{ in } C_{loc}(\bar{\mathbb{R}}_+; L^\infty_{w*}(\mathbb{R}^n)). \tag{79}$$

Proof. Denote $\underline{w} = (\underline{\mathbf{m}}, \underline{\mathbf{U}})$. Define

$$X_\tau^0 = \{w \in C_{loc}([-\tau,\infty); L^\infty_{w*}) : (\rho, w, q) \text{ are strict subsolutions in } \mathcal{D} \\ \text{with constrain sets } K_{(x,t)} \text{ and } w - \underline{w} \in C_c^\infty(\mathcal{D})\}.$$

Let X_τ be the closure of X_τ^0 in $C_{loc}([-\tau,\infty); L^\infty_{w*})$ and d_{X_τ} be the associated metric induced by $C_{loc}([-\tau,\infty); L^\infty_{w*})$. Given $w \in X_\tau^0$, $\varepsilon > 0$, and a bounded open subset $\Omega_0 \subset \mathcal{D}_{t=0}$, if one modifies slightly the proof of Lemma 11, then there exists a sequence of functions $\{w_k\} \subset X_\tau^0$, which satisfies $d_{X_\tau}(w_k, w) \to 0$,

$$\operatorname{supp}(w_k - w) \subset \Omega_0 \times (-\varepsilon, \varepsilon), \text{ and } \int_{\Omega_0} \operatorname{dist}(w_k(x,0), K_{(x,0)}) dx \le \varepsilon. \tag{80}$$

For $\xi \in \Omega_0$ and any bounded open set $\mathcal{U}$, it follows from Lemma 9, Lemma 10, and the continuity of $w(z)$ and K_z that there exist $v \in C_c^\infty(\mathcal{U})$ and $r(\xi) > 0$ such that v satisfies (30) and

$$\frac{1}{|\mathcal{U}|} \int_{\mathcal{U}} \operatorname{dist}(w(\xi,0) + v(x,t), K_{(\xi,0)})\, dxdt < \frac{\varepsilon}{4|\Omega_0|}. \tag{81}$$

Furthermore, for $z \in Q_{r(\xi)}(\xi, 0)$, we have

$$w(z) + \overline{\operatorname{image}(v)} \subset \operatorname{int\,conv} K_z, \;\; d_H(K_z, K_{(\xi,0)}) < \frac{\varepsilon}{8|\Omega_0|},$$

and

$$|w(z) - w(\xi,0)| < \frac{\varepsilon}{8|\Omega_0|}.$$

Let $\mathscr{C}'$ be a compact subset of Ω_0 with $\mathcal{H}^n(\mathcal{D}_{t=0} \setminus \mathscr{C}') < \frac{\varepsilon}{4C_0}$, where $C_0 = 2\sup_{\mathcal{D}} \mathfrak{C}(\rho, q)$ with $\mathfrak{C}(\rho, q)$ defined in (16). Denote by $\mathscr{Q}_{r(\xi)}(\xi)$ the space cube of length $r(\xi)$ centered at ξ. Let $\{O^i = \mathscr{Q}_{r(\xi^i)}(\xi^i)\}_{i=1}^N$ be a finite covering of $\mathscr{C}'$ and set $r_0 = \min_i r(\xi^i)$. For $k \in \mathbb{N}$, there exist $\{\mathscr{Q}^{(j,k)}\} = \{\mathscr{Q}_{r_{jk}}(\xi^{(j,k)})\}_{j=1}^{J_k}$ such that $\mathcal{H}^n(\mathscr{C}' \setminus (\bigcup_{j=1}^{J_k} \mathscr{Q}^{(j,k)})) < \frac{\varepsilon}{4C_0}$ and

$$r_{jk} \le \min(2^{-k} r_0, 2^{-1}\varepsilon), \quad \mathscr{Q}^{(j,k)} \subset O^{i(j,k)} \text{ for some } i(j,k), \text{ for } j = 1, \cdots, J_k.$$

Denote $Q^{(j,k)} = \mathscr{Q}^{(j,k)} \times (-\frac{1}{2} r_{jk}, \frac{1}{2} r_{jk})$. Applying Lemma 9 to $w(\xi^{i(j,k)})$ with $\mathcal{U} = Q^{(j,k)}$ and $\bar{\varepsilon} = \frac{\varepsilon}{4|\Omega_0|}$ yields the existence of $\tilde{a}_{jk}(z) \in C_c^\infty(Q^{(j,k)})$ satisfying (30) and

$$\frac{1}{|Q^{(j,k)}|} \int_{Q^{(j,k)}} \operatorname{dist}(w(\xi^{i(j,k)}, 0) + \tilde{a}_{jk}(x,t), K_{(\xi^{i(j,k)},0)})\, dxdt < \frac{\varepsilon}{4|\Omega_0|}. \tag{82}$$

Since $Q^{(j,k)} = \mathscr{Q}^{(j,k)} \times (-\frac{1}{2}r_{jk}, \frac{1}{2}r_{jk})$, there must be an $s^{(j,k)} \in (-\frac{1}{2}r_{jk}, \frac{1}{2}r_{jk})$ such that

$$\frac{1}{|\mathscr{Q}^{(j,k)}|} \int\limits_{\mathscr{Q}^{(j,k)}} \operatorname{dist}(w(\xi^{i(j,k)}, 0) + \tilde{a}_{jk}(x, s^{(j,k)}), K_{(\xi^{i(j,k)},0)})dx < \frac{\varepsilon}{4|\Omega_0|}.$$

Set

$$a_{jk}(x,t) = \tilde{a}_{jk}(x, t + s^{(j)}) \text{ and } w_k = w + \sum_{j=1}^{J_k} a_{jk}.$$

It is easy to see that $a_{jk} \in C_c^\infty(\mathscr{Q}_{r_{jk}}(\xi^{i(j,k)}) \times (-\varepsilon, \varepsilon))$, which implies $w_k \in X_\tau^0$. Now the same as the proof of Lemma 11, we can prove (80).

Using (80) instead of Lemma 11, the proof of Proposition 1 can be adapted to obtain $\{w^{(k)}\} \subset X_\tau^0$ with $w^{(1)} = (\underline{\mathbf{m}}, \underline{\mathbf{U}})$, satisfying the estimates

$$w^{(k+1)} = w^{(k)} \text{ for } t \notin (-2^{-k}, 2^{-k}) \text{ and } d_X(w^{(k+1)}, w^{(k)}) < 2^{-k}.$$

Furthermore, for any compact subset $\mathscr{C} \subset \mathcal{D}_{t=0}$, $\{w^{(k)}(\cdot, 0)\}$ is a strongly convergent sequence in $L^2(\mathscr{C})$ and

$$\int\limits_{\mathscr{C}} \operatorname{dist}(w^{(k)}(x,0), K_{(x,0)})dx \to 0 \text{ as } k \to \infty.$$

Let $w' = (\mathbf{m}', \mathbf{U}')$ be the limit of $w^{(k)}$ in X. Set $\mathbf{m}^\diamond(x) = \mathbf{m}'(x,0)$. It is easy to see that

$$(\mathbf{m}', \mathbf{U}')(x,0) \in K_{(x,0)} \quad \text{a.e. in } \mathcal{D}_{t=0} \quad \text{and} \quad \operatorname{supp}(\mathbf{m}' - \underline{\mathbf{m}}, \mathbf{U}' - \underline{\mathbf{U}}) \subset \overline{\mathcal{D}}.$$

Since $w' = w^{(k)}$ for $t \notin (-2^{-k}, 2^{-k})$ for any $k \geq 1$, (ρ, w', q) is a strict subsolution in $\mathcal{D}^{t=0} = \mathcal{D} \setminus \overline{\{(x,0)|(x,0) \in \mathcal{D}\}}$. Applying Proposition 1 to (ρ, w', q) in $\mathcal{D}^{t=0}$ yields the existence of infinitely many pairs $(\mathbf{m}^\flat, \mathbf{U}^\flat)$ satisfying

$$(\mathbf{m}^\flat, \mathbf{U}^\flat) \in X_\tau,\ (\mathbf{m}^\flat, \mathbf{U}^\flat)(\cdot, 0) = (\mathbf{m}', \mathbf{U}')(\cdot, 0),\ (\mathbf{m}^\flat, \mathbf{U}^\flat)(x,t) \in K_{(x,t)} \ a.e. \text{ in } \mathcal{D}. \quad (83)$$

It is clear that (75) and (77) are satisfied. Since $K_{(x,t)} \subset K_{\rho(x,t),q(x,t)}$, it is easy to see that (76) and (78) follow from (73) and (74).

Since $(\mathbf{m}^\flat, \mathbf{U}^\flat) \in X_\tau$, there exist $\{(\mathbf{m}_j, \mathbf{U}_j)\} \subset X_\tau^0$ converging to $(\mathbf{m}^\flat, \mathbf{U}^\flat)$ in X_τ. Let $\tilde{\mathbf{m}}_j = \mathbf{m}_j - \underline{\mathbf{m}}$. It follows from the definition of X_τ^0 that one has $\tilde{\mathbf{m}}_j \in C_c^\infty(\mathcal{D})$. Since $\mathbf{m}_j$ and $\underline{\mathbf{m}}$ both solve (17), $\tilde{\mathbf{m}}_j$ is a divergence-free vector field. This finishes the proof of the corollary. □

5. Proof of the main results

5.1. Constructions of finite-states admissible solutions

The starting points of the constructions are the piecewise constant stationary strict subsolutions inspired by [14].

Proof of Theorem 1. Let χ be a constant satisfying $\chi \geq \max_i p(\bar{\rho}_i)$. Define

$$\bar{q}_i = -p(\bar{\rho}_i) + \chi \quad \text{and} \quad I = \{i : \bar{q}_i > 0\}.$$

For each $i \in I$, Lemma 3 yields the existence of a subset of N_n^* states $\mathcal{K}_i \subset K_{\bar{\rho}_i,\bar{q}_i}$ such that $(0,0) \in \text{int conv } \mathcal{K}_i$. For $i \notin I$, define $\mathcal{K}_i = \{(0,0)\}$. Set

$$q(x,t) = \bar{q}_i \text{ and } K_{(x,t)} = \mathcal{K}_i \quad \text{for } x \in \Omega_i. \tag{84}$$

Let $\mathcal{D} = (\bigcup_{i\in I} \Omega_i) \times (-1,\infty)$. It is easy to see that $(\rho, 0, 0, q)$ is a strict subsolution in $\mathcal{D}$ with the constrain set $K_{(x,t)}$. Corollary 2 yields the existence of infinitely many weak solutions $(\rho, \mathbf{m}^\flat)$ to the system (1) for some initial data $(\rho_0, \mathbf{m}^\diamond)$.

For $i \notin I$, $q = 0$ in Ω_i. Hence it follows from (76) and (78) that

$$(\rho, \mathbf{m}^\flat)(x,t) = (\bar{\rho}_i, 0) \quad \text{for a.e. } (x,t) \in \Omega_i \times \bar{\mathbb{R}}_+.$$

For $i \in I$, denote $\mathcal{K}_i = \{\bar{w}^l\}_{l=1}^{N_n^*}$ and $w = (\mathbf{m}^\flat, \dfrac{\mathbf{m}^\flat \otimes \mathbf{m}^\flat}{\rho} - q\mathbf{I})$. The properties of $(\mathbf{m}^\flat, \mathbf{U}^\flat)$ in (83) show that $w \in X_\tau$ and $w(x,t) \in \{\bar{w}^l\}_{l=1}^{N_n^*}$ a.e. in $\Omega_i \times \bar{\mathbb{R}}_+$. Hence w has at most N_n^* states in Ω_i. Since $0 \in \text{int conv } \{\bar{w}^l\}_{l=1}^{N_n^*}$, one can uniquely represent 0 as a convex combination of $\{\bar{w}^l\}_{l=1}^{N_n^*}$, i.e.,

$$0 = \sum_{l=1}^{N_n^*} \mu_l \bar{w}^l, \quad \text{where } 0 < \mu_l < 1 \text{ and } \sum_{l=1}^{N_n^*} \mu_l = 1. \tag{85}$$

It follows from the proof of Corollary 2 that there exist $\{w_k\} \subset X_\tau^0$ with

$$w_k \to w \text{ in } CL^\infty_{w*},\ w_1 = 0,\ w_{k+1} - w_k = \sum_{j=1}^{J_k} a_{jk} \text{ for } k \geq 1,$$

where a_{jk} satisfies $\operatorname{supp} a_{jk} \subset Q^{(j,k)} \subset \Omega_{l(j)} \times \bar{\mathbb{R}}_+$ for some $l(j)$ and $\int a_{jk}(x,t)dx = 0$ due to (63). Hence $\int_{\Omega_i} w_k(x,t)dx = 0$. For any compact time interval $J \subset \bar{\mathbb{R}}_+$, one has

$$\int\limits_{\Omega_i \times J} w(x,t)dxdt = \lim_k \int\limits_{\Omega_i \times J} w_k(x,t)dxdt = 0.$$

Since $w(x,t) \in \{\bar{w}^l\}_{l=1}^{N_n^*}$ a.e. in $\Omega_i \times \bar{\mathbb{R}}_+$, it holds that

$$0 = \int\limits_{\Omega_i \times J} w(x,t)dxdt = \sum \nu_l \bar{w}^l, \qquad \text{where } \nu_l = \mathcal{H}^{n+1}\left(\{(x,t) \in \Omega_i \times J : w(x,t) = \bar{w}^l\}\right).$$

It follows from (85) that $\nu_l = \mu_l|\Omega_i \times J| > 0$. Hence $(\rho, \mathbf{m}^\flat)$ has exactly N_n^* states in Ω_i.

It remains to consider the energy inequality (5). It follows from (76) and (78) that

$$\int\limits_0^\infty \int\limits_\Omega \left(\mathcal{I}(\rho) + \frac{|\mathbf{m}^\flat|^2}{2\rho}\right)\partial_t\varphi + \int\limits_\Omega \left(\mathcal{I}(\rho_0(x)) + \frac{|\mathbf{m}_0(x)|^2}{2\rho_0}\right)\varphi(x,0)dx = 0.$$

It follows from (79) that there exist divergence-free vector fields $\tilde{\mathbf{m}}_j \in C_c^\infty(\mathcal{D})$ with $\tilde{\mathbf{m}}_j \to \mathbf{m}^\flat$ in CL^∞_{w*}. From the definition of $\mathcal{D}$, it holds that $\tilde{\mathbf{m}}_j = 0$ on $\partial\Omega_i$ for $i \in I$ and $\tilde{\mathbf{m}}_j = 0$ in Ω_l for $l \notin I$. Hence one has

$$\int\limits_0^\infty \int\limits_\Omega P(\rho,q)\tilde{\mathbf{m}}_j \cdot \nabla\varphi dxdt = \sum_{i\in I} P(\bar{\rho}_i, \bar{q}_i) \int\limits_0^\infty \int\limits_{\Omega_i} \tilde{\mathbf{m}}_j \cdot \nabla\varphi dxdt = 0,$$

where

$$P(\rho,q) = \rho^{-1}(\mathcal{I}(\rho) + \frac{n}{2}q + p(\rho)).$$

Therefore, one has

$$\begin{aligned}
&\int\limits_0^\infty \int\limits_\Omega \left(\mathcal{I}(\rho) + \frac{|\mathbf{m}^\flat|^2}{2\rho} + p(\rho)\right)\frac{\mathbf{m}^\flat}{\rho} \cdot \nabla\varphi dxdt \\
&= \int\limits_0^\infty \int\limits_\Omega P(\rho,q)\mathbf{m}^\flat \cdot \nabla\varphi dxdt = \lim_j \int\limits_0^\infty \int\limits_\Omega P(\rho,q)\tilde{\mathbf{m}}_j \cdot \nabla\varphi dxdt = 0.
\end{aligned}$$

Since $\mathbf{B}$ satisfies the condition (6), it follows that $(\rho, \mathbf{m}^\flat)$ satisfies the entropy condition (5). □

5.2. Constructions of admissible solutions with a general source

Given ρ_0 satisfying (8), we use the acoustic potential Ψ introduced in [18] to construct a strict subsolution such that ρ becomes constant in finite time.

Proof of Theorem 3. We first consider a simple case where $\rho_0 = \rho^\sharp$ with $\rho^\sharp$ a fixed constant. Let $q = e^{-2\beta t}\delta_0$ for some $\delta_0 > 0$, where β is defined in (7). It follows from (23) that $(\rho^\sharp, 0, 0, q)$ is a strict subsolution in $\mathcal{D} = \mathbb{T}^n \times (-1, \infty)$ with constrain sets $K_{\rho^\sharp, q(x,t)}$.

Then Corollary 2 yields the existence of infinitely many weak solutions $(\rho^\sharp, \mathbf{m}^\flat)$ to the system (1) with initial data $(\rho^\sharp, \mathbf{m}^\diamond)$. Since $\nabla\cdot\mathbf{m}^\flat = -\partial_t\rho^\sharp = 0$, it follows from (7), (76), and (78) that

$$\partial_t\left(\mathcal{I}(\rho^\sharp) + \frac{1}{2}\frac{|\mathbf{m}^\flat|^2}{\rho^\sharp}\right) + \nabla\cdot\left[\left(\mathcal{I}(\rho^\sharp) + \frac{1}{2}\frac{|\mathbf{m}^\flat|^2}{\rho^\sharp}\right)\frac{\mathbf{m}^\flat}{\rho^\sharp}\right] - \mathbf{B}\frac{\mathbf{m}^\flat}{\rho^\sharp}\cdot\mathbf{m}^\flat \le \frac{n}{2}(\partial_t q + 2\beta q) = 0.$$

Therefore, the energy inequality (5) holds for the weak solution $(\rho^\sharp, \mathbf{m}^\flat)$.

Consider now the case that ρ_0 is a small perturbation of $\rho^\sharp$ under the assumption (8). Let $h(t)$ be a smooth cut-off function satisfying that

$$h \in C_c^\infty((-1,1)), \quad h(0) = 1, \quad 0 \le h(t) \le 1, \quad \text{and} \quad |h'(t)| \le 2. \tag{86}$$

Define

$$\rho(x,t) = [1-h(t)]\rho^\sharp + h(t)\rho_0(x), \quad q(x,t) = p(\rho^\sharp) - p(\rho(x,t)) + \chi(t), \tag{87}$$

and

$$\mathbf{m}(x,t) = h'(t)\nabla\Psi(x), \quad \mathbf{U} = \mathcal{R}_{\mathbb{T}^n}[-h''(t)\nabla\Psi(x) + h'(t)\mathbf{B}\nabla\Psi(x)], \tag{88}$$

where $\chi(t)$ is a function to be determined and Ψ is the unique solution in $\mathcal{L}(\mathbb{T}^n)$ of the problem

$$\Delta\Psi = \rho^\sharp - \rho_0 \quad \text{in } \mathbb{T}^n.$$

It follows from standard estimates for the Poisson equation [20] and Lemma 7 that one has

$$\|\Psi\|_{C^{2,\alpha}(\mathbb{T}^n)} \le C\varepsilon \quad \text{and} \quad \|\mathbf{U}\|_{C^{1,\alpha}(\mathbb{T}^n)} \le C(|h'(t)| + |h''(t)|)\varepsilon. \tag{89}$$

Furthermore, the density ρ defined in (87) can be estimated as

$$0 < \check{\rho} \le \rho(x,t) \le \hat{\rho}, \quad \|\rho^\sharp - \rho\|_{L^\infty(\mathbb{T}^n)} \le h(t)\sup|\nabla\rho_0| \le h(t)\varepsilon. \tag{90}$$

The direct computation shows that $(\rho, \mathbf{m}, \mathbf{U}, q)$ solves (17) and (18). One has

$$\frac{\mathbf{m}\otimes\mathbf{m}}{\rho} - \mathbf{U} - q\mathbf{I} = \frac{(h')^2}{\rho}\nabla\Psi\otimes\nabla\Psi - \mathbf{U} - \left(p(\rho^\sharp) - p(\rho) + \chi\right)\mathbf{I}.$$

It follows from (89) and (90) that $\dfrac{\mathbf{m}\otimes\mathbf{m}}{\rho} - \mathbf{U} < q\mathbf{I}$ holds, provided one can show

$$\chi(t) > C(\hat{\rho}, \check{\rho})[h'(t)^2\varepsilon + |h'(t)| + |h''(t)| + h(t)]\varepsilon \tag{91}$$

for $t \in (-\tau_0, \infty)$ with some small $\tau_0 > 0$. Then $(\rho, \mathbf{m}, \mathbf{U}, q)$ is a strict subsolution in $\mathcal{D} = \mathbb{T}^n \times (-\tau_0, \infty)$ with constrain sets $K_{\rho(x,t),q(x,t)}$. Therefore Corollary 2 yields the existence of infinitely many weak solutions $(\rho, \mathbf{m}^\flat)$ to the system (1) with initial data $(\rho_0, \mathbf{m}^\diamond)$. It remains to check that these weak solutions are admissible.

Recalling that β is defined in (7), it follows from (76) and (78) that

$$\begin{aligned}
&\partial_t \left(\mathcal{I}(\rho) + \frac{|\mathbf{m}^\flat|^2}{2\rho}\right) + \nabla \cdot \left[\left(\mathcal{I}(\rho) + \frac{|\mathbf{m}^\flat|^2}{2\rho} + p(\rho)\right)\frac{\mathbf{m}^\flat}{\rho}\right] - \frac{\mathbf{m}^\flat \cdot (\mathbf{B}\mathbf{m}^\flat)}{\rho} \\
&\le \frac{n}{2}[\partial_t \chi(t) - \partial_t p(\rho)] + \partial_t \mathcal{I}(\rho) + n\beta[p(\rho^\sharp) - p(\rho(x,t)) + \chi(t)] \\
&\quad + \nabla \cdot \left[\frac{\mathcal{I}(\rho) + p(\rho) + (n/2)(p(\rho^\sharp) - p(\rho))}{\rho}\mathbf{m}^\flat\right] + \frac{n}{2}\chi(t)\nabla \cdot \left(\frac{\mathbf{m}^\flat}{\rho}\right).
\end{aligned} \tag{92}$$

For $t \in [1, \infty)$, since $\operatorname{supp} h \subset (-1, 1)$, it follows that $(\rho, \mathbf{m}, \mathbf{U}, q) = (\rho^\sharp, 0, 0, \chi)$ for $t \ge 1$, thus reducing to the case of constant density. Hence if $\chi(1) > 0$, then $\chi(t)$ can be defined on $[1, \infty)$ satisfying (91) and the entropy condition (5).

For $t \in [0, 1)$, in view of (89), (90) and the estimates

$$\begin{aligned}
&|\nabla \cdot \mathbf{m}^\flat| = |\partial_t \rho| \le |h'(t)|\varepsilon, \quad |p(\rho^\sharp) - p(\rho)| \le C(\hat{\rho})(\rho^\sharp - \rho) \le C(\hat{\rho})\varepsilon, \\
&|\mathbf{m}^\flat| = \sqrt{n\rho[p(\rho^\sharp) - p(\rho) + \chi]} \le C(\hat{\rho})(\sqrt{\varepsilon} + \sqrt{\chi}),
\end{aligned}$$

it is not hard to see that the weak solution $(\rho, \mathbf{m}^\flat)$ satisfies

$$\partial_t \left(\mathcal{I}(\rho) + \frac{|\mathbf{m}^\flat|^2}{2\rho}\right) + \nabla \cdot \left[\left(\mathcal{I}(\rho) + \frac{|\mathbf{m}^\flat|^2}{2\rho} + p(\rho)\right)\frac{\mathbf{m}^\flat}{\rho}\right] - \frac{\mathbf{m}^\flat \cdot (\mathbf{B}\mathbf{m}^\flat)}{\rho} \le 0, \tag{93}$$

as long as χ satisfies

$$\partial_t \chi(t) \le -2\beta\chi - C(\check{\rho}, \hat{\rho}, \beta)(\chi^{3/2} + \chi + \chi^{1/2} + 1)\varepsilon \quad \text{for } t \in [0, 1). \tag{94}$$

Therefore there exists a $\bar{\varepsilon}$ depending on $\check{\rho}, \hat{\rho}$, and β such that if $\varepsilon < \bar{\varepsilon}$, it is possible to find a smooth function $\chi(t)$ satisfying (91) and (94) on $[0, 1]$. Hence the solution $(\rho, \mathbf{m}^\flat)$ satisfies the entropy condition (5). Observe that for sufficiently small ε, one may take $\delta = \chi(0)$ to be small as well. Finally, the estimate (9) follows from the constructions. □

5.3. Constructions of admissible solutions with general density

The aim is to remove the small-oscillation assumption (8) on ρ_0 in the case that $\mathbf{B}$ satisfies the condition (6). However, since (94) is a Riccati-type ordinary differential inequality, the assumption seems essential for the smooth subsolution ansatz in [18]. The key idea is the construction of a non-smooth subsolution ansatz based on the ansatz in [18], which transits to piecewise constant states before the blow-up time.

Proof of Theorem 2. It is assumed that ρ_0 is piecewise Lipschitz. Therefore there exists a family of mutually disjoint open sets $\{\Omega_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ with $\rho_0 \in W^{1,\infty}(\Omega_i)$,

$$\mathbb{R}^n = \bigcup_{i=1}^{\infty} \overline{\Omega_i} \quad \text{and} \quad \mathcal{H}^n\left(\bigcup_{i=1}^{\infty} \partial\Omega_i\right) = 0, \tag{95}$$

and there are positive constants $\check{\rho}$, $\hat{\rho}$, $\bar{\varrho}$ such that

$$0 < \check{\rho} \le \inf \rho_0(x) \le \sup \rho_0(x) \le \hat{\rho} < \infty \quad \text{and} \quad \sup_i \|\nabla\rho_0\|_{L^\infty(\Omega_i)} \le \bar{\varrho} < \infty. \tag{96}$$

Let $T \in (0,1)$ and $\vartheta \in (0,T)$ be two positive constants to be determined. It follows from the Whitney covering lemma [32, Theorem 3, p. 16] that there exits a countable family of mutually disjoint open cubes $\{\mathcal{Q}^{(j)} = \mathcal{Q}_{r_j}(x^{(j)})\}_{j=1}^{\infty}$ such that

$$\bigcup_{i=1}^{\infty} \Omega_i = \bigcup_{j=1}^{\infty} \overline{\mathcal{Q}^{(j)}}, \quad \overline{\mathcal{Q}^{(j)}} \subset \Omega_{i(j)} \text{ for some } i(j),$$

and

$$r_j \le \min\left[\operatorname{dist}(\mathcal{Q}^{(j)}, \partial\Omega_{i(j)}), (1+|x^{(j)}|^2)^{-\frac{n+1}{2}}(1+\bar{\varrho})^{-1}\vartheta\right] \text{ for } j \in \mathbb{N}.$$

Since every $\mathcal{Q}^{(j)}$ lies in some $\Omega_{i(j)}$, one has $\rho_0 \in W^{1,\infty}(\mathcal{Q}^{(j)})$.

Let Ψ_j be the unique solution in $\mathcal{L}(\mathcal{Q}^{(j)})$ to the problem

$$\begin{cases} \Delta\Psi_j = \rho_j^\sharp - \rho_0 & \text{in } \mathcal{Q}^{(j)}, \\ \dfrac{\partial\Psi_j}{\partial\nu} = 0 & \text{on } \partial\mathcal{Q}^{(j)}, \end{cases}$$

where

$$\rho_j^\sharp = \frac{1}{|\mathcal{Q}^{(j)}|}\int_{\mathcal{Q}^{(j)}} \rho_0(x)dx. \tag{97}$$

Let h be a smooth function satisfying (86) and define

$$h_T(t) = h(t/T) \in C_c^\infty((-T,T)).$$

For $(x,t) \in \mathcal{Q}^{(j)} \times (-1,\infty)$, set

$$\rho_j(x,t) = [1-h_T(t)]\rho_j^\sharp + h_T(t)\rho_0(x), \quad \mathbf{m}_j(x,t) = h_T'(t)\nabla\Psi_j(x),$$

and

$$q_j(x,t) = -p(\rho_j(x,t)) + \chi(t),$$

where $\chi(t)$ is a positive function to be determined satisfying

$$\chi(t) = \chi(T) > p(\hat{\rho}) \quad \text{for } t \in [T, \infty). \tag{98}$$

Recall that $\mathcal{H}^n(\mathbb{R}^n \setminus (\cup_j \mathscr{Q}^{(j)})) = 0$, it suffices to define $\rho, \mathbf{m}, q$ on each cube as

$$\rho = \rho_j, \ \mathbf{m} = \mathbf{m}_j, \ q = q_j, \ \text{ in } \mathscr{Q}^{(j)} \times (-1, \infty). \tag{99}$$

Let $\Phi(y) = r_j^{-2} \Psi_j(x^{(j)} + r_j y)$ and $f(y) = \rho_j^\sharp - \rho_0(x^{(j)} + r_j y)$. Then Φ solves

$$\begin{cases} \Delta \Phi = f \text{ in } \mathscr{Q}_1, \\ \dfrac{\partial \Phi}{\partial \nu} = 0 \text{ on } \partial \mathscr{Q}_1. \end{cases} \tag{100}$$

Since

$$\|f\|_{C^1(\mathscr{Q}_1)} \le r_j \|\nabla \rho_0\|_{L^\infty(\mathscr{Q}^{(j)})} \le (1 + |x^{(j)}|^2)^{-\frac{n+1}{2}} \vartheta,$$

it follows from the standard estimates for the Poisson equation [20] that

$$\|\nabla \Phi\|_{C^{1,\alpha}(\mathscr{Q}_1)} \le C(1 + |x^{(j)}|^2)^{-\frac{n+1}{2}} \vartheta.$$

Hence it holds that

$$\|\nabla \Psi_j\|_{L^\infty(\mathscr{Q}^{(j)})} \le r_j \|\nabla \Phi\|_{L^\infty(\mathscr{Q}_1)} \le C(1 + |x^{(j)}|^2)^{-(n+1)} \vartheta^2. \tag{101}$$

Thus

$$\|\mathbf{m}\|_{L^\infty(\mathbb{R}^n)} = |h_T'(t)| \sup_j \|\nabla \Psi_j\|_{L^\infty(\mathscr{Q}^{(j)})} \le C T^{-1} \vartheta^2, \quad \|\partial_t \mathbf{m}\|_{L^\infty(\mathbb{R}^n)} \le C T^{-2} \vartheta^2.$$

Since the cubes $\{\mathscr{Q}^{(j)}\}$ have disjoint interiors, so

$$\begin{aligned} \sum_j \|\nabla \Psi_j\|_{L^1(\mathscr{Q}^{(j)})} &\le \sum_j \|\nabla \Psi_j\|_{L^\infty(\mathscr{Q}^{(j)})} |\mathscr{Q}^{(j)}| \\ &\le \sum_{l=0}^\infty \sum_{2^l \le |x^{(j)}| \le 2^{l+1}} \|\nabla \Psi_j\|_{L^\infty} |\mathscr{Q}^{(j)}| + \sum_{|x^{(j)}| \le 1} \|\nabla \Psi_j\|_{L^\infty} |\mathscr{Q}^{(j)}| \\ &\le \sum_{l=0}^\infty (1 + |2^l|^2)^{-(n+1)} \vartheta^2 |\mathscr{Q}_{2^{l+1}+1}| + \vartheta^2 |\mathscr{Q}_2| \le C \vartheta^2 \left(\sum_{l=0}^\infty 2^{-l} + 1 \right) \le C \vartheta^2. \end{aligned}$$

Thus

$$\|\mathbf{m}\|_{L^1(\mathbb{R}^n)} = |h_T'(t)| \sum_j \|\nabla \Psi_j\|_{L^1(\mathscr{Q}^{(j)})} = C T^{-1} \vartheta^2, \quad \|\partial_t \mathbf{m}\|_{L^1(\mathbb{R}^n)} \le C T^{-2} \vartheta^2.$$

Hence we can define

$$\mathbf{U} = \mathcal{R}[-\partial_t \mathbf{m} + \mathbf{B}\mathbf{m}].$$

It follows from Property (4) of Lemma 8 that

$$\|\mathbf{U}\|_{C^\alpha(\mathbb{R}^n)} \le CT^{-2}\vartheta^2, \tag{102}$$

where the constant C depends only on α, n and the constant matrix $\mathbf{B}$. Since

$$\nu \cdot \mathbf{m}_j = h_T'(t)\nu \cdot \nabla\Psi_j = 0 \quad \text{ on } \partial\mathcal{Q}^{(j)}, \tag{103}$$

and that $\partial_t \rho_j + \nabla \cdot \mathbf{m}_j = 0$ inside $\mathcal{Q}^{(j)}$, it holds that for $\varphi \in C_c^\infty(\mathbb{R}^n \times (-1, \infty))$

$$\int_{-1}^{\infty}\int_{\mathbb{R}^n} \rho\partial_t\varphi + \mathbf{m}\cdot\nabla\varphi\, dxdt = \sum_j \int_{-1}^{\infty}\int_{\mathcal{Q}^{(j)}} \rho_j\partial_t\varphi + \mathbf{m}_j\cdot\nabla\varphi\, dxdt = 0.$$

Furthermore, direct computations yield

$$\partial_t\mathbf{m} + \nabla\cdot\mathbf{U} + \nabla(p+q) - \mathbf{B}\mathbf{m} = \partial_t\mathbf{m} + \nabla\cdot\mathcal{R}[-\partial_t\mathbf{m} + \mathbf{B}\mathbf{m}] + \nabla\chi(t) - \mathbf{B}\mathbf{m} = 0,$$

where $p + q = \chi$ is used. It follows that $(\rho, \mathbf{m}, \mathbf{U}, q)$ solves (17) and (18) in the sense of distribution.

Noting $\operatorname{supp} h_T \subset (-T, T)$ and denoting $q_j^\sharp = \chi(T) - p(\rho_j^\sharp)$, yield

$$(\mathbf{m}, \mathbf{U}) = (0, 0) \quad \text{ for } t \in [T, \infty), \quad (\rho, q) = (\rho_j^\sharp, q_j^\sharp) \quad \text{ for } (x,t) \in \mathcal{Q}^{(j)} \times [T, \infty). \tag{104}$$

Since

$$\frac{\mathbf{m}\otimes\mathbf{m}}{\rho} - \mathbf{U} - q\mathbf{I} \le \left(\frac{|\mathbf{m}|^2}{\rho} + |\mathbf{U}| + p(\rho) - \chi(t)\right)\mathbf{I},$$

it follows from (101) and (102) that $\dfrac{\mathbf{m}\otimes\mathbf{m}}{\rho} - \mathbf{U} < q\mathbf{I}$ holds, provided that

$$\chi(t) > p(\hat\rho) + C(\check\rho, \hat\rho)T^{-2}\vartheta^2 1_{(-T,T)}(t), \tag{105}$$

where $1_{(-T,T)}(t)$ is the characteristic function of $(-T, T)$. Suppose that (105) holds in $(-\tau_0, T)$ for some small $\tau_0 > 0$. Let

$$\tilde{\mathcal{D}} = \bigcup_j \mathcal{Q}^{(j)} \times (-\tau_0, T), \quad \mathcal{D}^{(j)} = \mathcal{Q}^{(j)} \times (T, \infty), \quad \mathcal{D} = \tilde{\mathcal{D}} \cup \left(\bigcup_j \mathcal{D}^{(j)}\right).$$

For each j, Lemma 3 yields a set of N_n^* states $\mathcal{K}_j \subset K_{\rho_j^\sharp, q_j^\sharp}$ such that $(0,0) \in \text{int conv } \mathcal{K}_j$. Set

$$K_{(x,t)} = \begin{cases} K_{\rho(x,t), q(x,t)} & \text{in } \tilde{\mathcal{D}}, \\ \mathcal{K}_j & \text{in } \mathcal{D}^{(j)}. \end{cases}$$

It follows from (104) and (105) that $(\rho, \mathbf{m}, \mathbf{U}, q)$ is a strict subsolution in $\mathcal{D}$ with constrain sets $K_{(x,t)}$. Therefore Corollary 2 yields infinitely many weak solutions $(\rho, \mathbf{m}^\flat)$ to the system (1) with initial data $(\rho_0, \mathbf{m}^\diamond)$.

For $t \in [T, \infty)$, in view of (104), as in the proof of Theorem 1, we can show that $(\rho, \mathbf{m})$ has exactly N_n^* states in $\mathscr{Q}^{(j)} \times [T, \infty)$, and that the energy inequality (5) holds in $[T, \infty)$.

For $t \in [0, T)$, observe that in view of (79) and (103), it suffices to show that the energy inequality (5) holds in each $\mathscr{Q}^{(j)}$. It follows from (76) and (78) that

$$\frac{|\mathbf{m}^\flat|^2}{2\rho} = \frac{nq_j}{2} \quad \text{in} \quad \mathscr{Q}^{(j)}.$$

Recall that $\rho = \rho_j$ and $q = q_j$ in $\mathscr{Q}^{(j)}$. Since $\mathbf{B}$ satisfies the condition (6), the entropy condition (5) holds provided that

$$\begin{aligned} &\partial_t \left(\mathcal{I}(\rho_j) + \frac{|\mathbf{m}^\flat|^2}{2\rho} \right) + \nabla \cdot \left\{ \left(\mathcal{I}(\rho_j) + \frac{|\mathbf{m}^\flat|^2}{2\rho} + p(\rho_j) \right) \frac{\mathbf{m}^\flat}{\rho} \right\} \\ &= \partial_t \left(\mathcal{I}(\rho_j) + \frac{nq_j}{2} \right) + \nabla \cdot \left\{ \left(\mathcal{I}(\rho_j) + \frac{nq_j}{2} + p(\rho_j) \right) \frac{\mathbf{m}^\flat}{\rho_j} \right\} \le 0 \quad \text{in } \mathscr{Q}^{(j)} \times [0, T). \quad (106) \end{aligned}$$

Direct computations yield

$$\begin{aligned} &\partial_t \left(\mathcal{I}(\rho_j) + \frac{nq_j}{2} \right) + \nabla \cdot \left\{ \left(\mathcal{I}(\rho_j) + \frac{nq_j}{2} + p(\rho_j) \right) \frac{\mathbf{m}^\flat}{\rho_j} \right\} \\ &= \left(\frac{n}{2} \partial_t \chi(t) - \frac{n}{2} \partial_t p(\rho_j) + \partial_t \mathcal{I}(\rho_j) \right) + \nabla \cdot \left[\left(\mathcal{I}(\rho_j) + (1 - \frac{n}{2}) p(\rho_j) \right) \frac{\mathbf{m}^\flat}{\rho_j} \right] \\ &\quad + \frac{n}{2} \chi(t) \nabla \cdot \left(\frac{\mathbf{m}^\flat}{\rho_j} \right) \\ &= \frac{n}{2} \left\{ \partial_t \chi(t) - \chi(t) \frac{\rho_j \partial_t \rho_j + \mathbf{m}^\flat \cdot \nabla \rho_j}{\rho_j^2} + \mathbf{m}^\flat \cdot \nabla \left[\frac{2\mathcal{I}(\rho_j) + (2-n) p(\rho_j)}{n\rho_j} \right] \right. \\ &\qquad \left. - \left[\frac{2\mathcal{I}(\rho_j) + (2-n) p(\rho_j)}{n\rho_j} + p'(\rho_j) - \frac{2}{n} \mathcal{I}'(\rho_j) \right] \partial_t \rho_j \right\}, \end{aligned}$$

where $\nabla \cdot \mathbf{m}^\flat = -\partial_t \rho_j$ in $\mathscr{Q}^{(j)}$ is used in the last equality. Furthermore, the construction gives

$$|\partial_t \rho_j| = |h_T'(t)(\rho_0 - \rho_j^\sharp)| \le |h_T'(t)| r_j \|\nabla \rho_0\|_{L^\infty(\mathscr{Q}^{(j)})} \le C T^{-1} \vartheta,$$
$$|\nabla \rho_j| \le \|\nabla \rho_0\|_{L^\infty(\mathscr{Q}^{(j)})} \le \overline{\varrho}, \quad \check{\rho} \le \rho_j \le \hat{\rho}, \quad |\mathbf{m}^\flat| = \sqrt{n \rho_j q_j} \le (n\hat{\rho})^{1/2} \sqrt{\chi},$$

therefore, the energy inequality (106) follows from

$$\chi'(t) \le -C_3 \overline{\varrho} \chi^{3/2} - C_2 \chi - C_1 \overline{\varrho} \chi^{1/2} - C_0, \quad 0 \le t < T, \tag{107}$$

where the positive constants $C_l = C_l(\check{\rho}, \hat{\rho})$ ($l = 0, 1, 2, 3$) are independent of $\mathscr{Q}^{(j)}$ and the choices of the constants T and ϑ. Therefore, choosing the constants T and ϑ to be sufficiently small, we can find a smooth function χ satisfying (98), (105), and (107) on $[0, \infty)$. It is not hard to see that T can be chosen to be of the order of $C(\check{\rho}, \hat{\rho})\overline{\varrho}^{-1}$. This finishes the proof of the theorem. □

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