NON-UNIQUENESS OF WEAK SOLUTIONS TO HYPERVISCOUS NAVIER-STOKES EQUATIONS - ON SHARPNESS OF J.-L. LIONS EXPONENT

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Abstract. Using the convex integration technique for the three-dimensional Navier-Stokes equations introduced by T. Buckmaster and V. Vicol, it is shown the existence of non-unique weak solutions for the 3D Navier-Stokes equations with fractional hyperviscosity $(-\Delta)^\theta$, whenever the exponent θ is less than J.-L. Lions' exponent 5/4, i.e., when $\theta < 5/4$.

1. Introduction

In this paper we consider the question of non-uniqness of weak solutions to the 3D Navier-Stokes equations with fractional viscosity (FVNSE) on $\mathbb{T}^3$

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^\theta v = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{1}$$

where $\theta \in \mathbb{R}$ is a fixed constant, and for $u \in C^\infty(\mathbb{T}^3)$ the fractional Laplacian is defined via the Fourier transform as

$$\mathcal{F}((-\Delta)^\theta u)(\xi) = |\xi|^{2\theta} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

Definition (weak solutions). *A vector field $v \in C^0_{weak}(\mathbb{R}; L^2(\mathbb{T}^3))$ is called a weak solution to the FVNSE if it solves* (1) *in the sense of distribution.*

When $\theta = 1$, FVNSE (1) is the standard Navier-Stokes equations. J.-L. Lions first considered FVNSE (1) in [17], and showed the existence and uniqueness of weak solutions to the initial value problem, which also satisfied the energy equality, for $\theta \in [5/4, \infty)$ in [18]. Moreover, an analogue of the Caffarelli-Kohn-Nirenberg [6] result was established in [16] for the FVNSE system (1), showing that the Hausdorff dimension of the singular set, in space and time, is bounded by $5 - 4\theta$ for $\theta \in (1, 5/4)$. The existence, uniqueness, regularity and stability of solutions to the FVNSE have been studied in [15, 20, 22, 23] and references therein. Very recently, using the method of convex integration introduced in [11], Colombo, De Lellis and

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De Rosa in [7] showed the non-uniquenss of Leray weak solutions to FVNSE (1) for $\theta \in (0, 1/5)$.

In the recent breakthrough work [5], Buckmaster and Vicol obtained non-uniqueness of weak solutions to the three-dimensional Navier-Stokes equations. They developed a new convex integration scheme in Sobolev spaces using intermittent Beltrami flows which combined concentrations and oscillations. Later, the idea of using intermittent flows was used to study non-uniqueness for transport equations in [19], employing scaled Mikado waves.

The schemes in [5,19] are based on the convex integration framework in Hölder spaces for the Euler equations, introduced by De Lellis and Székelyhidi in [11], subsequently refined in [2,3,9,13], and culminated in the proof of the second half of the Onsager conjecture by Isett in [14]; also see [4] for a shorter proof. For the first half of the Onsager conjecture, see, e.g., [1,8], and the references therein.

The main contribution of this note is to show that the results in Buckmaster-Vicol's paper hold for FVNSE (1) for $\theta < 5/4$:

Theorem 1. *Assume that $\theta \in [1, 5/4)$. Suppose u is a smooth divergence-free vector field, define on $\mathbb{R}_+ \times \mathbb{T}^3$, with compact support in time and satisfies the condition*

$$\int_{\mathbb{T}^3} u(t,x)dx \equiv 0.$$

Then for any given $\varepsilon_0 > 0$, there exists a weak solution v to the FVNSE (1), with compact support in time, satisfying

$$\|v - u\|_{L_t^\infty W_x^{2\theta-1,1}} < \varepsilon_0.$$

As a consequence there are infinitely many weak solutions of the FVNSE (1) which are compactly supported in time; in particular, there are infinitely many weak solutions with initial values zero.

Remark 1. *In the above theorem we assume that $\theta \in [1, 5/4)$. However, using the constructions in [5] with a slightly different choice of parameters, one can actually show that Theorem 1.2 and Theorem 1.3 in [5] hold for the 3D FVNSE, i.e., there exist non-unique weak solutions $v \in C_t^0 W_x^{\beta,2}$, with a different $\beta > 0$, depending on θ. However, in this paper we choose to prove a weaker result, Theorem 1, in order to simplify the presentation while retaining the main idea.*

Remark 2. *For the case $\theta \in (-\infty, 1)$, the same construction also yields weak solutions $v \in C_t^0 L_x^2 \cap C_t^0 W_x^{1,1}$ with a suitable choice of parameters.*

We now make some comments on the analysis in this paper. Using the technique in [5], we adapt a convex integration scheme with intermittent Beltrami flows as the building blocks. The main difficulty in a convex integration scheme for (FVNSE), is the error induced by the frictional viscosity $\nu(-\Delta)^\theta v$, which is greater for a larger exponent θ. This error is controlled by making full use of the concentration effect of intermittent flows introduced in [5]. As it is shown in the crucial estimate (36), the error is controllable only for $\theta < 5/4$. Compared with [5], since our goal is to construct weak solutions $v \in C_t^0 L_{x,weak}^2 \cap L_t^\infty W_x^{2\theta-1,1}$, we adapt a slightly simpler cut-off function and prove only estimates that are sufficient for this purpose.

2. Outline

2.1. Iteration lemma. Following [5], we consider the approximate system

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^\theta v = \nabla \cdot R, \\ \nabla \cdot v = 0, \end{cases} \tag{2}$$

where R is a symmetric 3×3 matrix.

Lemma 1 (Iteration Lemma for L^2 weak solutions)**.** *Let $\theta \in (-\infty, 5/4)$. Assume (v_q, R_q) is a smooth solution to (2) with*

$$\|R_q\|_{L^\infty_t L^1_x} \le \delta_{q+1}, \tag{3}$$

for some $\delta_{q+1} > 0$. Then for any given $\delta_{q+2} > 0$, there exists a smooth solution (v_{q+1}, R_{q+1}) of (2) with

$$\|R_{q+1}\|_{L^\infty_t L^1_x} \le \delta_{q+2}, \tag{4}$$

$$\text{and} \quad \operatorname{supp}_t v_{q+1} \cup \operatorname{supp}_t R_{q+1} \subset N_{\delta_{q+1}}(\operatorname{supp}_t v_q \cup \operatorname{supp}_t R_q). \tag{5}$$

Here for a given set $A \subset \mathbb{R}$, the δ-neighborhood of A is denoted by

$$N_\delta(A) = \{y \in \mathbb{R} : \exists y' \in A, |y - y'| < \delta\}.$$

Furthermore, the increment $w_{q+1} = v_{q+1} - v_q$ satisfies the estimates

$$\|w_{q+1}\|_{L^\infty_t L^2_x} \le C\delta_{q+1}^{1/2}, \tag{6}$$

$$\|w_{q+1}\|_{L^\infty_t W^{2\theta-1,1}_x} \le \delta_{q+2}, \tag{7}$$

where the positive constant C depends only on θ.

Proof of Theorem 1. Assume Lemma 1 is valid. Let $v_0 = u$. Then

$$\int_{\mathbb{T}^3} \partial_t v_0(t,x)dx = \frac{d}{dt}\int_{\mathbb{T}^3} v_0(t,x)dx \equiv 0.$$

Let

$$R_0 = \mathcal{R}(\partial_t v_0 + \nu(-\Delta)^\theta v_0) + v_0 \otimes v_0 + p_0 I, \quad p_0 = -\frac{1}{3}|v_0|^2,$$

where $\mathcal{R}$ is the symmetric anti-divergence operator established in Lemma 5, below. Clearly (v_0, R_0) solves (2). Set

$$\delta_1 = \|R_0\|_{L^\infty_t L^1_x},$$
$$\delta_{q+1} = 2^{-q}\varepsilon_0, \quad \text{for } q \ge 1.$$

Apply Lemma 1 iteratively to obtain smooth solution (v_q, R_q) to (2). It follows from (6) that

$$\sum \|v_{q+1} - v_q\|_{L^\infty_t L^2_x} = \sum \|w_{q+1}\|_{L^\infty_t L^2_x} \le C \sum \delta_{q+1}^{1/2} < \infty.$$

Thus v_q converge strongly to some $v \in C^0_t L^2_x$. Since $\|R_{q+1}\|_{L^\infty_t L^1_x} \to 0$, as $q \to \infty$, v is a weak solution to the FVNSE (1). Estimate (7) leads to

$$\|v - v_0\|_{L^\infty_t W^{2\theta-1,1}_x} \le \sum_{q=1}^\infty \|w_q\|_{L^\infty_t W^{2\theta-1,1}_x} \le \sum_{q=1}^\infty \delta_{q+1} \le \varepsilon_0.$$

Furthermore, it follows from (5) that

$$\operatorname{supp}_t v \subset \cup_{q\ge0} \operatorname{supp}_t v_q \subset N_{\sum_{q\ge0}\delta_{q+1}}(\operatorname{supp}_t u) \subset N_{\delta_1+\varepsilon_0}(\operatorname{supp}_t u).$$

Now we show the existence of infinitely many weak solutions with initial values zero. Let $u(t,x)=\varphi(t)\sum_{|k|\le N} a_k e^{ik\cdot x}$ with $a_k\neq 0, a_k\cdot k=0, a_{-k}=a_k^*$ for all $|k|\le N$, and $\varphi\in C_c^\infty(\mathbb{R}_+)$. Thus $\nabla\cdot u=0$ satisfies the conditions of the theorem. Hence there exists a weak solution v to (1) close enough to u so that $v\not\equiv 0$. □

3. Iteration scheme

3.1. **Notations and Parameters.** For a complex number $\zeta\in\mathbb{C}$, we denote by ζ^* its complex conjugate. Let us normalize the volume

$$|\mathbb{T}^3|=1.$$

For smooth functions $u\in C^\infty(\mathbb{T}^3)$ and $s\in\mathbb{R}$, we define

$$\mathcal{F}(|\nabla|^s u)(\xi)=|\xi|^s\mathcal{F}(u)(\xi),\quad \xi\in\mathbb{Z}^3.$$

For $M,N\in[-\infty,+\infty]$, denote the Fourier projection of u by

$$\mathcal{F}(\mathbb{P}_{[M,N)}u)=\begin{cases}u(\xi), & M\le\xi<N,\xi\in\mathbb{Z}^3,\\ 0, & \text{otherwise.}\end{cases}$$

We also denote $\mathbb{P}_{\le k}=\mathbb{P}_{[-\infty,k)}$ and $\mathbb{P}_{\ge k}=\mathbb{P}_{[k,+\infty)}$.

Following the notation in [5], we introduce here several parameters σ,r,λ, with

$$0<\sigma<1<r<\lambda<\mu<\lambda^2,\quad \sigma r<1, \tag{8}$$

where $\lambda=\lambda_{q+1}\in 5\mathbb{N}$ is the 'frequency' parameter; σ with $1/\sigma\in\mathbb{N}$ is a small parameter such that $\lambda\sigma\in\mathbb{N}$ parameterizes the spacing between frequencies; $r\in\mathbb{N}$ denotes the number of frequencies along edges of a cube; μ measures the amount of temporal oscillation.

Later σ,r,μ will be chosen to be suitable powers of λ_{q+1}. We also fix a constant $p>1$ which will be chosen later to be close to 1. The constants implicitly in the notation '$\lesssim$' may depend on p but are independent of the parameters σ,r,λ.

3.2. **Intermittent Beltrami flows.** We use intermittent Beltrami flows introduced in [5] as the building blocks. Recall some basic facts of Beltrami waves.

Proposition 1. *([5, Proposition 3.1]) Given $\overline{\xi}\in\mathbb{S}^2\cap\mathbb{Q}^3$, let $A_{\overline{\xi}}\in\mathbb{S}^2\cap\mathbb{Q}^3$ be such that*

$$A_{\overline{\xi}}\cdot\overline{\xi}=0,\quad |A_{\overline{\xi}}|=1,\quad A_{-\overline{\xi}}=A_{\overline{\xi}}.$$

Let Λ be a given finite subset of $\mathbb{S}^2$ such that $-\Lambda=\Lambda$, and $\lambda\in\mathbb{Z}$ be such that $\lambda\Lambda\subset\mathbb{Z}^3$. Then for any choice of coefficients $a_{\overline{\xi}}\in\mathbb{C}$ with $a_{\overline{\xi}}^*=a_{-\overline{\xi}}$ the vector field

$$W(x)=\sum_{\overline{\xi}\in\Lambda}a_{\overline{\xi}}B_{\overline{\xi}}e^{i\lambda\overline{\xi}\cdot x},\qquad \text{with } B_{\overline{\xi}}=\frac{1}{\sqrt{2}}(A_{\overline{\xi}}+i\overline{\xi}\times A_{\overline{\xi}}),$$

is real-valued, divergence-free and satisfies

$$\nabla\times W=\lambda W,\quad \nabla\cdot(W\otimes W)=\nabla\frac{|W|^2}{2}.$$

Furthermore,

$$\langle W\otimes W\rangle:=\fint_{\mathbb{T}^3}W\otimes W\,dx=\sum_{\overline{\xi}\in\Lambda}\frac{1}{2}|a_{(\overline{\xi})}|^2(\mathrm{Id}-\overline{\xi}\otimes\overline{\xi}).$$

Let $\Lambda, \Lambda^+, \Lambda^- \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ be defined by

$$\Lambda^+ = \{\frac{1}{5}(3e_1 \pm 4e_2), \frac{1}{5}(3e_2 \pm 4e_3), \frac{1}{5}(3e_3 \pm 4e_1)\},$$
$$\Lambda^- = -\Lambda^+, \quad \Lambda = \Lambda^+ \cup \Lambda^-.$$

Clearly we have

$$5\Lambda \in \mathbb{Z}^3, \quad \text{and} \quad \min_{\bar{\xi}', \bar{\xi} \in \Lambda, \bar{\xi}' + \bar{\xi} \neq 0} |\bar{\xi}' + \bar{\xi}| \geq \frac{1}{5}. \tag{9}$$

Also it is direct to check that

$$\frac{1}{8} \sum_{\bar{\xi} \in \Lambda} (\mathrm{Id} - \bar{\xi} \otimes \bar{\xi}) = \mathrm{Id}.$$

In fact, representations of this form exist for symmetric matrices close to the identity. We have the following simple variant of [5, Proposition 3.2].

Proposition 2. *Let $B_\varepsilon(\mathrm{Id})$ denote the ball of symmetric matrices, centered at the identity, of radius ε. Then there exist a constant $\varepsilon_\gamma > 0$ and smooth positive functions $\gamma_{(\bar{\xi})} \in C^\infty(B_{\varepsilon_\gamma}(\mathrm{Id}))$, such that*

(1) $\gamma_{(\bar{\xi})} = \gamma_{(-\bar{\xi})}$;
(2) *for each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have the identity*

$$R = \frac{1}{2} \sum_{\bar{\xi} \in \Lambda} \left(\gamma_{(\bar{\xi})}(R)\right)^2 (\mathrm{Id} - \bar{\xi} \otimes \bar{\xi}).$$

Define the Dirichlet kernel

$$D_r(x) = \frac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}, \quad \Omega_r = \{(j,k,l) : j,k,l \in \{-r, \cdots, r\}\}.$$

It has the property that, for $1 < p \leq \infty$,

$$\|D_r\|_{L^p} \lesssim r^{3/2 - 3/p}, \quad \|D_r\|_{L^2} = (2\pi)^3.$$

Following [5], for $\bar{\xi} \in \Lambda^+$, define a directed and rescaled Dirichlet kernel by

$$\eta_{(\bar{\xi})}(t,x) = \eta_{\bar{\xi},\lambda,\sigma,r,\mu}(t,x) = D_r(\lambda\sigma(\bar{\xi} \cdot x + \mu t, A_{\bar{\xi}} \cdot x, (\bar{\xi} \times A_{\bar{\xi}}) \cdot x)), \tag{10}$$

and for $\bar{\xi} \in \Lambda^-$, define

$$\eta_{(\bar{\xi})}(t,x) = \eta_{-(\bar{\xi})}(t,x).$$

Note the important identity

$$\frac{1}{\mu} \partial_t \eta_{(\bar{\xi})}(t,x) = \pm(\bar{\xi} \cdot \nabla) \eta_{(\bar{\xi})}(t,x), \quad \bar{\xi} \in \Lambda^{\pm}. \tag{11}$$

Since the map $x \mapsto \lambda\sigma(\bar{\xi} \cdot x + \mu t, A_{\bar{\xi}} \cdot x, (\bar{\xi} \times A_{\bar{\xi}}) \cdot x)$ is the composition of a rotation by a rational orthogonal matrix mapping $\{e_1, e_2, e_3\}$ to $\{\bar{\xi}, A_{\bar{\xi}}, \bar{\xi} \times A_{\bar{\xi}}\}$, a translation, and a rescaling by integers, for $1 < p \leq \infty$, we have

$$\fint_{\mathbb{T}^3} \eta_{(\bar{\xi})}(t,x)^2(t,x)\,dx = 1, \quad \|\eta_{(\bar{\xi})}\|_{L^\infty_t L^p_x(\mathbb{T}^3)} \lesssim r^{3/2 - 3/p}.$$

Let $W_{(\bar{\xi})}$ be the Beltrami plane wave at frequency λ,

$$W_{(\bar{\xi})} = W_{\bar{\xi},\lambda}(x) = B_{\bar{\xi}} e^{i\lambda \bar{\xi} \cdot x}.$$

Define the intermittent Beltrami wave $\mathbb{W}_{(\overline{\xi})}$ as

$$\mathbb{W}_{(\overline{\xi})}(t,x) := \mathbb{W}_{\overline{\xi},\lambda,\sigma,r,\mu}(t,x) = \eta_{(\overline{\xi})}(t,x)W_{(\overline{\xi})}(x). \tag{12}$$

It follows from the definitions and (9) that

$$\mathbb{P}_{[\frac{\lambda}{2},2\lambda)}\mathbb{W}_{(\overline{\xi})} = \mathbb{W}_{(\overline{\xi})}, \tag{13}$$

$$\mathbb{P}_{[\frac{\lambda}{5},4\lambda)}(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}) = \mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}, \quad \overline{\xi}' \neq -\overline{\xi}. \tag{14}$$

The following properties are immediate from the definitions.

Proposition 3. *([5, Proposition 3.4]) Let $a_{\overline{\xi}} \in \mathbb{C}$ be constants with $a_{\overline{\xi}}^* = a_{-\overline{\xi}}$. Let*

$$W(x) = \sum_{\overline{\xi}\in\Lambda} a_{\overline{\xi}}\mathbb{W}_{(\overline{\xi})}(x).$$

Then $W(x)$ is real valued. Moreover, for each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have

$$\sum_{\overline{\xi}\in\Lambda}\Big(\gamma_{(\overline{\xi})}(R)\Big)^2 \fint_{\mathbb{T}^3}\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(-\overline{\xi})} = \sum_{\overline{\xi}\in\Lambda}\Big(\gamma_{(\overline{\xi})}(R)\Big)^2 B_{\overline{\xi}}\otimes B_{-\overline{\xi}} = R.$$

Proposition 4. *([5, Proposition 3.5]) For any $1 < p \leq \infty, N \geq 0, K \geq 0$:*

$$\|\nabla^N\partial_t^K\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \lesssim \lambda^N(\lambda\sigma r\mu)^K r^{3/2-3/p}, \tag{15}$$

$$\|\nabla^N\partial_t^K\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \lesssim (\lambda\sigma r)^N(\lambda\sigma r\mu)^K r^{3/2-3/p}. \tag{16}$$

3.3. **Perturbations.** Let $\psi(t)$ be a smooth cut-off function such that

$$\psi(t) = 1 \text{ on } \operatorname{supp}_t R_q, \quad \operatorname{supp}\psi(t) \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q), \quad |\psi'(t)| \leq 2\delta_{q+1}^{-1}. \tag{17}$$

Take a smooth increasing function χ such that

$$\chi(s) = \begin{cases} 1, & 0 \leq s < 1 \\ s, & s \geq 2 \end{cases},$$

and set

$$\rho(t,x) = \varepsilon_\gamma^{-1}\delta_{q+1}\chi(\delta_{q+1}^{-1}|R_q(t,x)|)\psi^2(t).$$

where ε_γ is the constant in Proposition 2. Then clearly

$$\operatorname{supp}_t\rho \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q). \tag{18}$$

It follows from the above definition that

$$|R_q|/\rho = \varepsilon_\gamma\frac{|R_q|}{\delta_{q+1}\chi(\delta_{q+1}^{-1}|R_q(t,x)|)\psi^2} \leq \varepsilon_\gamma \implies \mathrm{Id} - R_q/\rho \in B_{\varepsilon_\gamma}(\mathrm{Id}) \text{ on } \operatorname{supp} R_q.$$

Therefore, the amplitude functions

$$a_{(\overline{\xi})}(t,x) := \rho^{1/2}(t,x)\gamma_{(\overline{\xi})}(\mathrm{Id} - \rho(t,x)^{-1}R_q(t,x))$$

are well-defined and smooth. Define the velocity perturbation to be $w = w_{q+1}$:

$$
\begin{aligned}
w &= w^{(p)} + w^{(c)} + w^{(t)},\\
w^{(p)} &= \sum_{\overline{\xi}\in\Lambda} a_{(\overline{\xi})}\mathbb{W}_{(\overline{\xi})} = \sum_{\overline{\xi}\in\Lambda} a_{(\overline{\xi})}(t,x)\eta_{(\overline{\xi})}(t,x)B_{\overline{\xi}}e^{i\lambda\overline{\xi}\cdot x},\\
w^{(c)} &= \frac{1}{\lambda_{q+1}}\sum_{\overline{\xi}\in\Lambda}\nabla\left(a_{(\overline{\xi})}\eta_{(\overline{\xi})}\right)\times W_{(\overline{\xi})},\\
w^{(t)} &= \frac{1}{\mu}\sum_{\overline{\xi}\in\Lambda^+}\mathbb{P}_{LH}\mathbb{P}_{\neq 0}\left(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\right),
\end{aligned}
$$

where $\mathbb{P}_{LH} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$ is the Leray-Helmholtz projection into divergence-free vector field, and $\mathbb{P}_{\neq 0}f = f - \fint_{\mathbb{T}^3} f dx$. It is well-known that $\mathbb{P}_{LH}$ is bounded on $L^p, 1 < p < \infty$ (see , e.g., [12]). It follows from Proposition 3 that

$$
\sum_{\overline{\xi}\in\Lambda} a^2_{(\overline{\xi})}\fint_{\mathbb{T}^3}\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(-\overline{\xi})}dx = \rho\mathrm{Id} - R_q. \tag{19}
$$

3.4. **Estimates for perturbations.**

Lemma 2. *The following bounds hold:*

$$
\begin{aligned}
\|\rho\|_{L^\infty_t L^1_x} &\le C\delta_{q+1}, &(20)\\
\|\rho^{-1}\|_{C^0(\operatorname{supp} R_q)} &\lesssim \delta^{-1}_{q+1}, &(21)\\
\|\rho\|_{C^N_{t,x}} &\le C(\delta_{q+1}, \|R_q\|_{C^N}), &(22)\\
\|a_{(\overline{\xi})}\|_{L^\infty_t L^2_x} &\lesssim \|\rho\|^{1/2}_{L^\infty_t L^1_x} \lesssim \delta^{1/2}_{q+1}, &(23)\\
\|a_{(\overline{\xi})}\|_{C^N_{t,x}} &\le C(\delta_{q+1}, \|R_q\|_{C^N}). &(24)
\end{aligned}
$$

Proof. It follows from (3) that

$$
\begin{aligned}
\|\rho(t,\cdot)\|_{L^1_x} &= \int_{|R_q|\le\delta_{q+1}}\rho + \int_{|R_q|>\delta_{q+1}}\rho \lesssim \delta_{q+1} + \int_{|R_q|>\delta_{q+1}}|R_q|\\
&\le C\delta_{q+1}.
\end{aligned}
$$

It is direct to verify (21) and (23), while (22) and (24) follow from (17) and (21). □

Now we can estimate the time support of w_{q+1}:

$$
\operatorname{supp}_t w_{q+1} \subset \operatorname{supp}_t \rho \subset \operatorname{supp}\psi \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q). \tag{25}
$$

We need the following Lemma, which is a variant of [5, Lemma 3.6].

Lemma 3. *([19, Lemma 2.1]) Let $f, g \in C^\infty(\mathbb{T}^3)$, and g is $(\mathbb{T}/N)^3$ periodic, $N \in \mathbb{N}$. Then for $1 \le p \le \infty$,*

$$
\|fg\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^p} + C_pN^{-1/p}\|f\|_{C^1}\|g\|_{L^p}.
$$

Let us denote

$$
\mathcal{C}_N = C(\sup_{\overline{\xi}\in\Lambda}\|a_{(\overline{\xi})}\|_{C^N_{t,x}}) \tag{26}
$$

to be some polynomials depending on $\sup_{\overline{\xi}\in\Lambda}\|a_{(\overline{\xi})}\|_{C^N_{t,x}}$.

Lemma 4. *Suppose the parameters satisfy* (8) *and*

$$r^{3/2} \leq \mu. \tag{27}$$

Then the following estimates for the perturbations hold:

$$\|w_{q+1}^{(p)}\|_{L_t^\infty L_x^2} \lesssim \delta_{q+1}^{1/2} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1, \tag{28}$$

$$\|w_{q+1}\|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\mathcal{C}_1, \tag{29}$$

$$\|w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} + \|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} \lesssim (\sigma r + \mu^{-1} r^{3/2}) r^{3/2-3/p}\mathcal{C}_1, \tag{30}$$

$$\|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} + \|\partial_t w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2, \tag{31}$$

$$\||\nabla|^N w_{q+1}\|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\lambda_{q+1}^N \mathcal{C}_{N+1}, \tag{32}$$

for $1 < p < \infty, N \geq 1$.

Proof. Since $\mathbb{W}_{(\overline{\xi})}$ is $(\mathbb{T}/\lambda\sigma)^3$ periodic, it follows from (15), (23), and Lemma 3 that

$$\begin{aligned}
\|w_{q+1}^{(p)}\|_{L_t^\infty L_x^2} &\lesssim \sum_{\overline{\xi}\in\Lambda} (\|a_{(\overline{\xi})}\|_{L_t^\infty L_x^2} + (\lambda_{q+1}\sigma)^{-1/2}\|a_{(\overline{\xi})}\|_{C^1})\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^2} \\
&\lesssim \delta_{q+1}^{1/2} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1.
\end{aligned}$$

In view of (8), (15) and (16) yield that

$$\begin{aligned}
\|w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda} \|a_{(\overline{\xi})}\|_{C^0}\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \lesssim r^{3/2-3/p}\mathcal{C}_0, \\
\|w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\overline{\xi}\in\Lambda} \left(\|\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} + \|\nabla\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p}\right) \|a_{(\overline{\xi})}\|_{C^1}\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim (\sigma r) r^{3/2-3/p}\mathcal{C}_1, \\
\|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1} \sum_{\overline{\xi}\in\Lambda^+} \|a_{(\overline{\xi})}^2 \eta_{(\overline{\xi})}^2 \overline{\xi}\|_{L_t^\infty L_x^p} \lesssim \mu^{-1} \sum_{\overline{\xi}\in\Lambda^+} \|a_{(\overline{\xi})}^2\|_{C^0}\|\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^{2p}}^2 \\
&\lesssim \mu^{-1} r^{3-3/p}\mathcal{C}_0,
\end{aligned}$$

where the boundedness of $\mathbb{P}_{LH}$ and $\mathbb{P}_{\neq 0}$ on L^p, for $1 < p < \infty$, is used in the first inequality of the estimate for $\|w_{q+1}^{(t)}\|_{L_t^\infty L_x^p}$. In the same way, we can estimate

$$\begin{aligned}
\|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda} \|\partial_t a_{(\overline{\xi})}\|_{C^0}\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} + \|a_{(\overline{\xi})}\|_{C^0}\|\partial_t \mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_1, \\
\|\partial_t w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1} \sum_{\overline{\xi}\in\Lambda} \|a_{(\overline{\xi})}\|_{C_{t,x}^2} \Big(\|\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} + \|\nabla\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} + \|\partial_t\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\qquad + \|\partial_t\nabla\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p}\Big) \lesssim \sigma r \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2 \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2.
\end{aligned}$$

For $N \geq 1$, Using (15) and (16), we obtain that

$$\begin{aligned}
\|\nabla^N w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\bar{\xi}\in\Lambda}\sum_{k=0}^{N} \|\nabla^k a_{(\bar{\xi})}\|_{C^0}\|\nabla^{N-k}\mathbb{W}_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_N, \\
\|\nabla^N w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1}\sum_{\bar{\xi}\in\Lambda}\sum_{m=0}^{N}\sum_{k=0}^{m} \lambda_{q+1}^{N-m}\|\nabla^{k+1} a_{(\bar{\xi})}\|_{C^0}\|\nabla^{m-k}\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\
&\quad + \lambda_{q+1}^{-1}\sum_{\bar{\xi}\in\Lambda}\sum_{m=0}^{N}\sum_{k=0}^{m} \lambda_{q+1}^{N-m}\|\nabla^{k} a_{(\bar{\xi})}\|_{C^0}\|\nabla^{m-k+1}\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_{N+1}, \\
\|\nabla^N w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1}\sum_{\bar{\xi}\in\Lambda}\sum_{m=0}^{N} \|\nabla^{N-m}(a_{(\bar{\xi})}^2)\|_{C^0}\sum_{k=0}^{m}\|\nabla^k\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^{2p}}\|\nabla^{m-k}\eta_{(\bar{\xi})}\|_{L_t^\infty L_x^{2p}} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\frac{(\sigma r)^N r^{3/2}}{\mu}\mathcal{C}_N \lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_N,
\end{aligned}$$

where we use (8) and (27). □

3.5. **Estimates for the stress.** Let us recall the following operator in [11].

Lemma 5 (symmetric anti-divergence)**.** *There exists a linear operator $\mathcal{R}$, of order -1, mapping vector fields to symmetric matrices such that*

$$\nabla\cdot\mathcal{R}(u) = u - \fint_{\mathbb{T}^3} u, \tag{33}$$

with standard Calderon-Zygmund estimates, for $1 < p < \infty$,

$$\|\mathcal{R}\|_{L^p\to W^{1,p}} \lesssim 1, \quad \|\mathcal{R}\|_{C^0\to C^0} \lesssim 1, \quad \|\mathcal{R}\mathbb{P}_{\neq 0}u\|_{L^p} \lesssim \||\nabla|^{-1}\mathbb{P}_{\neq 0}u\|_{L^p}. \tag{34}$$

Proof. Suppose $u \in C^\infty(\mathbb{T}^3,\mathbb{R}^3)$ is a smooth vector field. Define

$$\mathcal{R}(u) = \frac{1}{4}\left(\nabla\mathbb{P}_{LH}v + (\nabla\mathbb{P}_{LH}v)^T\right) + \frac{3}{4}\left(\nabla v + (\nabla v)^T\right) - \frac{1}{2}(\nabla\cdot v)\mathrm{Id}$$

where $v \in C^\infty(\mathbb{T}^3,\mathbb{R}^3)$ is the unique solution to $\Delta v = u - \fint_{\mathbb{T}^3} u$ with $\fint_{\mathbb{T}^3} v = 0$.

It is direct to verify that $\mathcal{R}(u)$ is a symmetric matrix field depending linearly on u and satisfies (33). Note that $\mathcal{R}$ is a constant coefficient ellitpic operator of order -1. We refer to [12] for the Calderon-Zygmund estimates $\|\mathcal{R}\|_{L^p\to W^{1,p}} \lesssim 1$ and $\|\mathcal{R}\mathbb{P}_{\neq 0}u\|_{L^p} \lesssim \||\nabla|^{-1}\mathbb{P}_{\neq 0}u\|_{L^p}$. Combining these with Sobolev embeddings, we have $\|\mathcal{R}u\|_{C^\alpha} \lesssim \|\mathcal{R}u\|_{W^{1,4}} \lesssim \|u\|_{L^4} \lesssim \|u\|_{C^0}$, with $\alpha = 1/4$. □

We have the following variant of [5, Lemma B.1] in [5].

Lemma 6. *Let $a \in C^2(\mathbb{T}^3)$. For $1 < p < \infty$, and any $f \in L^p(\mathbb{T}^3)$, we have*

$$\||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k}f)\|_{L^p} \lesssim k^{-1}(\|a\|_{L^\infty} + \|\nabla^2 a\|_{L^\infty})\|f\|_{L^p}. \tag{35}$$

Proof of Lemma 6. We follow the proof in [5]. Note that

$$|\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k}f) = |\nabla|^{-1}\mathbb{P}_{\geq k/2}(\mathbb{P}_{\leq k/2}a\mathbb{P}_{\geq k}f) + |\nabla|^{-1}\mathbb{P}_{\neq 0}(\mathbb{P}_{\geq k/2}a\mathbb{P}_{\geq k}f).$$

As direct consequences of the Littlewood-Paley decomposition and Schauder estimates we have the bounds (see, for example, [12])

$$\|\mathbb{P}_{\le k/2}\|_{L^\infty\to L^\infty}\lesssim 1,\quad \||\nabla|^{-1}\mathbb{P}_{\ge k/2}\|_{L^p\to L^p}\lesssim k^{-1},\quad \||\nabla|^{-1}\mathbb{P}_{\neq 0}\|_{L^p\to L^p}\lesssim 1.$$

Combining these bounds with Hölder's inequality and the embedding $W^{1,4}(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$, we obtain

$$\begin{aligned}
&\||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\ge k}f)\|_{L^p}\lesssim k^{-1}\|\mathbb{P}_{\le k/2}a\mathbb{P}_{\ge k}f\|_{L^p}+\|\mathbb{P}_{\ge k/2}a\mathbb{P}_{\ge k}f\|_{L^p}\\
&\lesssim k^{-1}(\|\mathbb{P}_{\le k/2}a\|_{L^\infty}+k\|\mathbb{P}_{\ge k/2}a\|_{L^\infty})\|f\|_{L^p}\lesssim k^{-1}(\|a\|_{L^\infty}+k\|\nabla\mathbb{P}_{\ge k/2}a\|_{L^4})\|f\|_{L^p}\\
&\lesssim k^{-1}(\|a\|_{L^\infty}+k\||\nabla|^{-1}\mathbb{P}_{\ge k/2}|\nabla|\nabla\mathbb{P}_{\ge k/2}a\|_{L^4})\|f\|_{L^p}\lesssim k^{-1}(\|a\|_{L^\infty}+\|\nabla^2\mathbb{P}_{\ge k/2}a\|_{L^4})\|f\|_{L^p}\\
&\lesssim k^{-1}(\|a\|_{L^\infty}+\|\nabla^2 a\|_{L^\infty})\|f\|_{L^p}.
\end{aligned}$$

□

It follows from the definition of w_{q+1} that

$$\int_{\mathbb{T}^3} w_{q+1}dx=\int_{\mathbb{T}^3}\frac{1}{\lambda_{q+1}}\sum_{\overline{\xi}\in\Lambda}\nabla\left(a_{(\overline{\xi})}\eta_{(\overline{\xi})}W_{(\overline{\xi})}\right)dx+\int_{\mathbb{T}^3}\frac{1}{\mu}\sum_{\overline{\xi}\in\Lambda^+}\mathbb{P}_{LH}\mathbb{P}_{\neq 0}\left(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\right)dx=0.$$

Hence $\int_{\mathbb{T}^3}\nu(-\Delta)^\theta w_{q+1}dx=0$ and $\frac{d}{dt}\int_{\mathbb{T}^3}w_{q+1}dx=0$. We obtain R_{q+1} by plugging $v_{q+1}=v_q+w_{q+1}$ in (2), using (33) and the assumption that (v_q,R_q) solves (2):

$$\begin{aligned}
\nabla\cdot R_{q+1}&=\nabla\cdot\left[\mathcal{R}(\nu(-\Delta)^\theta w_{q+1}+\partial_t w^{(p)}_{q+1}+\partial_t w^{(c)}_{q+1})+v_q\otimes w_{q+1}+w_{q+1}\otimes v_q\right]\\
&\quad+\nabla\cdot\left[(w^{(c)}_{q+1}+w^{(t)}_{q+1})\otimes w_{q+1}+w^{(p)}_{q+1}\otimes(w^{(c)}_{q+1}+w^{(t)}_{q+1})\right]\\
&\quad\left[\nabla\cdot(w^{(p)}_{q+1}\otimes w^{(p)}_{q+1}-R_q)+\partial_t w^{(t)}_{q+1}\right]+\nabla(p_{q+1}-p_q)\\
&:=\nabla\cdot(\widetilde{R}_{linear}+\widetilde{R}_{corrector}+\widetilde{R}_{oscillation})+\nabla(p_{q+1}-p_q).
\end{aligned}$$

It follows from Lemma 4 that

$$\begin{aligned}
\|\widetilde{R}_{corrector}\|_{L^\infty_t L^p_x}&\lesssim\left(\|w^{(c)}_{q+1}\|_{L^\infty_t L^{2p}_x}+\|w^{(t)}_{q+1}\|_{L^\infty_t L^{2p}_x}\right)\left(\|w_{q+1}\|_{L^\infty_t L^{2p}_x}+\|w^{(p)}_{q+1}\|_{L^\infty_t L^{2p}_x}\right)\\
&\lesssim(\sigma r+\mu^{-1}r^{3/2})r^{3-3/p}\mathcal{C}_1.
\end{aligned}$$

Noting that $\nabla\times\dfrac{w^{(p)}_{q+1}}{\lambda_{q+1}}=w^{(p)}_{q+1}+w^{(c)}_{q+1}$, Lemma 4 and (34) yield that

$$\begin{aligned}
&\|\widetilde{R}_{linear}\|_{L^\infty_t L^p_x}\\
&\lesssim\lambda_{q+1}^{-1}\|\partial_t\mathcal{R}\nabla\times(w^{(p)}_{q+1})\|_{L^\infty_t L^p_x}+\|\mathcal{R}(\nu(-\Delta)^\theta w_{q+1})\|_{L^\infty_t L^p_x}\\
&\quad+\|v_q\otimes w_{q+1}+w_{q+1}\otimes v_q\|_{L^\infty_t L^p_x}\\
&\lesssim\lambda_{q+1}^{-1}\|\partial_t w^{(p)}_{q+1}\|_{L^\infty_t L^p_x}+\||\nabla|^{2\theta-1}w_{q+1}\|_{L^\infty_t L^p_x}+\|v_q\|_{C^0}\|w_{q+1}\|_{L^\infty_t L^p_x}\\
&\lesssim\sigma\mu r^{5/2-3/p}\mathcal{C}_2+r^{3/2-3/p}(\lambda^{2\theta-1}_{q+1}+\|v_q\|_{C^0})\mathcal{C}_3.
\end{aligned}\tag{36}$$

This is the crucial estimate to control the fractional viscosity. If we assume that $p\sim 1, r\sim\lambda_{q+1}^{-1}$, we must have $\theta<5/4$ in order that the second term in (36) is small for λ_{q+1} sufficiently large.

It remains to estimate $\widetilde{R}_{oscillation}$, which can be handled in the same way as in [5]. It follows from (19) that

$$\begin{aligned}
&\nabla\cdot(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}-R_q)=\nabla\cdot(\sum_{\overline{\xi},\overline{\xi}'\in\Lambda}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{W}_{\overline{\xi}}\otimes\mathbb{W}_{(\overline{\xi}')}-R_q)\\
&=\nabla\cdot(\sum_{\overline{\xi},\overline{\xi}'\in\Lambda}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')})+\nabla\rho\\
&:=\sum_{\overline{\xi},\overline{\xi}'\in\Lambda}E_{(\overline{\xi},\overline{\xi}')}+\nabla\rho.
\end{aligned}$$

Since $E_{(\overline{\xi},\overline{\xi}')}$ has zero mean, we can split it as

$$\begin{aligned}
E_{(\overline{\xi},\overline{\xi}')}+E_{(\overline{\xi}',\overline{\xi})}&=\mathbb{P}_{\neq 0}\left(\nabla(a_{(\overline{\xi})}a_{(\overline{\xi}')})\cdot(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi}'}\otimes\mathbb{W}_{(\overline{\xi})}))\right)\\
&\quad+\mathbb{P}_{\neq 0}\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi}'}\otimes\mathbb{W}_{(\overline{\xi})})\right)\\
&:=E_{(\overline{\xi},\overline{\xi}',1)}+E_{(\overline{\xi},\overline{\xi}',2)}.
\end{aligned}$$

Using (15), (34) and (35), we obtain

$$\begin{aligned}
\|\mathcal{R}E_{(\overline{\xi},\overline{\xi}',1)}\|_{L_t^\infty L_x^p}&\lesssim\||\nabla|^{-1}E_{(\overline{\xi},\overline{\xi}',1)}\|_{L_t^\infty L_x^p}\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}(\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\|_{C^1}+\|\nabla^2(a_{(\overline{\xi})}a_{(\overline{\xi}')})\|_{C^1})\|\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}\|_{L_t^\infty L_x^p}\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\|_{C^3}\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^{2p}}\|\mathbb{W}_{(\overline{\xi}')}\|_{L_t^\infty L_x^{2p}}\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}r^{3-3/p}\mathcal{C}_3.
\end{aligned}$$

Recall the vector identity $A\cdot\nabla B+B\cdot\nabla A=\nabla(A\cdot B)-A\times(\nabla\times B)-B\times(\nabla\times A)$. For $\overline{\xi},\overline{\xi}'\in\Lambda$, using the anti-symmetry of the cross product, we can write

$$\begin{aligned}
&\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{(\overline{\xi}')}\otimes\mathbb{W}_{(\overline{\xi})})\\
&=\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)+\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\left(W_{(\overline{\xi})}\cdot\nabla W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\cdot\nabla W_{(\overline{\xi})}\right)\\
&=\left(W_{(\overline{\xi}')}\cdot\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right)W_{(\overline{\xi})}+\left(W_{(\overline{\xi})}\cdot\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right)W_{\overline{\xi}'}+\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\nabla\left(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\right).
\end{aligned}$$

For the term $E_{(\overline{\xi},\overline{\xi}',2)}$, first consider the case $\overline{\xi}+\overline{\xi'}\neq 0$. It follows from the above identity and (14) that

$$\begin{aligned}
&a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{(\overline{\xi}')}\otimes\mathbb{W}_{(\overline{\xi})})\\
&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)\\
&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\cdot\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)\\
&\quad+a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\nabla\left(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\right)\right)\\
&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\cdot\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)\\
&\quad+\nabla\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\right)-\nabla\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\right)\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\right)\\
&\quad-a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\left(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\right)\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right),
\end{aligned}$$

where the second term is a pressure, the third can be estimated analogously to $E_{(\bar\xi,\bar\xi',1)}$. Also note that the first and fourth term can estimated analogously. Using (16), (34) and (35), we obtain

$$\begin{aligned}
&\|\mathcal{R}\left(a_{(\bar\xi)}a_{(\bar\xi')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\bar\xi)}\eta_{(\bar\xi')}\right)\cdot\left(W_{(\bar\xi)}\otimes W_{(\bar\xi')}+W_{(\bar\xi')}\otimes W_{(\bar\xi)}\right)\right)\right)\|_{L_t^\infty L_x^p}\\
&\lesssim \lambda_{q+1}^{-1}(\|a_{(\bar\xi)}a_{(\bar\xi')}\|_{C^1}+\|\nabla^2(a_{(\bar\xi)}a_{(\bar\xi')})\|_{C^1})\|\nabla\left(\eta_{(\bar\xi)}\eta_{(\bar\xi')}\right)\|_{L_t^\infty L_x^p}\\
&\lesssim \sigma r^{4-3/p}\mathcal{C}_3.
\end{aligned}$$

Now consider $E_{(\bar\xi,-\bar\xi,2)}$. We can write

$$\begin{aligned}
&\nabla\cdot(\mathbb{W}_{(\bar\xi)}\otimes\mathbb{W}_{(-\bar\xi)}+\mathbb{W}_{(-\bar\xi)}\otimes\mathbb{W}_{(\bar\xi)})=\left(W_{(-\bar\xi)}\cdot\nabla\eta^2_{(\bar\xi)}\right)W_{(\bar\xi)}+\left(W_{(\bar\xi)}\cdot\nabla\eta^2_{(\bar\xi)}\right)W_{(-\bar\xi)}\\
&=(A_{\bar\xi}\cdot\nabla\eta^2_{(\bar\xi)})A_{\bar\xi}+((\bar\xi\times A_{\bar\xi})\cdot\nabla\eta^2_{(\bar\xi)})(\bar\xi\times A_{\bar\xi})=\nabla\xi^2_{(\bar\xi)}-(\bar\xi\cdot\nabla\eta^2_{(\bar\xi)})\bar\xi=\nabla\eta^2_{(\bar\xi)}-\frac{\bar\xi}{\mu}\partial_t\eta^2_{(\bar\xi)},
\end{aligned}$$

where we use (11) and the fact that $\{\bar\xi, A_{\bar\xi}, \bar\xi\times A_{\bar\xi}\}$ forms an orthonormal basis of $\mathbb{R}^3$. Therefore, we can write

$$\begin{aligned}
E_{(\bar\xi,-\bar\xi,2)}&=\mathbb{P}_{\neq0}\left(a^2_{(\bar\xi)}\nabla\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\bar\xi)}-a^2_{(\bar\xi)}\frac{\bar\xi}{\mu}\partial_t\eta^2_{(\bar\xi)}\right)\\
&=\nabla\left(a^2_{(\bar\xi)}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\bar\xi)}\right)-\mathbb{P}_{\neq0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\bar\xi)})\nabla a^2_{(\bar\xi)}\right)\\
&\quad-\mu^{-1}\partial_t\mathbb{P}_{\neq0}\left(a^2_{(\bar\xi)}\eta^2_{(\bar\xi)}\bar\xi\right)+\mu^{-1}\mathbb{P}_{\neq0}\left(\partial_t\left(a^2_{(\bar\xi)}\right)\eta^2_{(\bar\xi)}\bar\xi\right).
\end{aligned}$$

Using the identity $\mathrm{Id}-\mathbb{P}_{LH}=\nabla\Delta^{-1}\operatorname{div}$, we obtain

$$\begin{aligned}
&\sum_{\bar\xi}E_{(\bar\xi,-\bar\xi,2)}+\partial_t w^{(t)}_{q+1}=\nabla\sum_{\bar\xi}\left(a^2_{(\bar\xi)}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\bar\xi)}\right)-\nabla\sum_{\bar\xi}\mu^{-1}\Delta^{-1}\nabla\cdot\partial_t\left(a^2_{(\bar\xi)}\eta^2_{(\bar\xi)}\bar\xi\right)\\
&\quad-\sum_{\bar\xi}\mathbb{P}_{\neq0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\bar\xi)})\nabla a^2_{(\bar\xi)}\right)+\mu^{-1}\sum_{\bar\xi}\mathbb{P}_{\neq0}\left(\partial_t\left(a^2_{(\bar\xi)}\right)\eta^2_{(\bar\xi)}\bar\xi\right),
\end{aligned}$$

where the first and second terms are pressure terms. Using (16), (34) and (35), we obtain

$$\begin{aligned}
\|\mathcal{R}\mathbb{P}_{\neq0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\bar\xi)})\nabla a^2_{(\bar\xi)}\right)\|_{L_t^\infty L_x^p}&\lesssim(\lambda_{q+1}\sigma)^{-1}\|\eta_{(\bar\xi)}\|^2_{L_t^\infty L_x^{2p}}\mathcal{C}_3\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}r^{3-3/p}\mathcal{C}_3.
\end{aligned}$$

It follows from (16) and (34) that

$$\begin{aligned}
\mu^{-1}\|\mathcal{R}\mathbb{P}_{\neq0}\left(\partial_t\left(a^2_{(\bar\xi)}\right)\eta^2_{(\bar\xi)}\bar\xi\right)\|_{L_t^\infty L_x^p}&\lesssim\mu^{-1}\|\partial_t\left(a^2_{(\bar\xi)}\right)\eta^2_{(\bar\xi)}\bar\xi\|_{L_t^\infty L_x^p}\\
&\lesssim\mu^{-1}r^{3-3/p}\mathcal{C}_1.
\end{aligned}$$

Let us now give the explicit definition of $\widetilde{R}_{oscillation}$:

$$\widetilde{R}_{oscillation} = \sum_{\overline{\xi},\overline{\xi}'\in\Lambda} \mathbb{P}_{\neq 0}\left(\nabla(a_{(\overline{\xi})}a_{(\overline{\xi}')})\cdot(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi}'}\otimes\mathbb{W}_{(\overline{\xi})}))\right)$$

$$+\sum_{\overline{\xi},\overline{\xi}'\in\Lambda,\overline{\xi}\neq\overline{\xi}'} a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\cdot\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)$$

$$-\sum_{\overline{\xi},\overline{\xi}'\in\Lambda,\overline{\xi}\neq\overline{\xi}'} \nabla\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\right)\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\right)$$

$$-\sum_{\overline{\xi},\overline{\xi}'\in\Lambda,\overline{\xi}\neq\overline{\xi}'} a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\left(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\right)\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right)$$

$$-\sum_{\overline{\xi}\in\Lambda}\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)+\mu^{-1}\sum_{\overline{\xi}\in\Lambda}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right).$$

Finally, we estimate the time support of R_{q+1}. Using (25) we obtain

$$\operatorname{supp}_t R_{q+1}\subset\operatorname{supp}_t w_{q+1}\cup\operatorname{supp}_t R_q\subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q).$$

Now we choose the parameters r,σ,μ. Fix α so that

$$\max\{0,\frac{2}{3}(2\theta-1)\}<\alpha<1,$$

which is possible since $\theta\in(-\infty,5/4)$. Fix

$$r=\lambda_{q+1}^{\alpha},\quad \sigma=\lambda_{q+1}^{-(\alpha+1)/2},\quad \mu=\lambda_{q+1}^{(5\alpha+1)/4}. \tag{37}$$

Clearly (27) is satisfied. Choose $p>1$ sufficiently close to 1 so that

$$-\frac{\alpha+1}{2}+\frac{5\alpha+1}{4}+\left(\frac{5}{2}-\frac{3}{p}\right)\alpha<0,\quad \left(\frac{3}{2}-\frac{3}{p}\right)\alpha+\max(0,2\theta-1)<0,$$

$$-\frac{5\alpha+1}{4}+\left(\frac{9}{2}-\frac{3}{p}\right)\alpha<0,\quad -\frac{1-\alpha}{2}+\left(3-\frac{3}{p}\right)\alpha<0.$$

Note that $\mathcal{C}_N$ is independent of λ_{q+1}, due to (24). Combining the above estimates with Lemma 4, it is easy to check that, by taking λ_{q+1} sufficiently large, we arrive at (4), (6) and (7). This completes the proof of Lemma 1.

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