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Propagation of Regularity and Long Time Behavior of 3D Massive Relativistic Transport Equation I: Vlasov–Nordström System

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Abstract: In this paper, we introduce a new set of vector fields for the relativistic transport equation, which is applicable for general Vlasov-wave type coupled systems. By combining the strength of Klainerman vector field method and Fourier method, we prove global regularity and scattering for the 3D massive relativistic Vlasov–Nordström system for small initial data without any compact support assumption, which is widely used in the literature for the study of Vlasov equation.

Contents

1. Introduction

1.1. An overview. In plasma physics and also in general relativity, the dynamics of many physical systems can be described by Vlasov-wave type coupled systems of equations. For example, the motions of electrons and protons, which are particles, in solar wind are subjected to the electromagnetic field created by the particles themselves. The dynamics of particles and electromagnetic field can be described by the Vlasov–Maxwell system in the collisionless case. Moreover, in general relativity, the motion of a collection of collisionless particles in the framework of Einstein's general relativity is described by the Einstein–Vlasov system, which is also a Vlasov-wave type coupled system. For our interest, we restrict ourself to the three dimensions case.

An interesting problem, which is also widely studied in the literature, is to understand the long term regularity for these Vlasov-wave type coupled systems. Most of previous results imposed compact support assumption on the initial data. The long term regularity problem for unrestricted data is less studied.

In this paper and its companion [40], based on two new observations, we prove global regularity for the $3D$ massive relativistic Vlasov–Nordström system (see Sect. 1.2) and the $3D$ massive relativistic Vlasov–Maxwell system (see Sect. 1.3) without imposing compact support assumption on the small initial data.

The first observation is that there exists a new set of vector fields for the massive relativistic Vlasov equation, which depend on a geometric observation about the light cone in (x, v)-space instead of solely in physical space. The second observation, which will be elaborated in [40], is that the spatial derivative in the rotational in v direction, i.e., $v/|v| \times \nabla_x$, plays a role of null structure for the *Vlasov-wave* type nonlinearity of the Vlasov equation.

In this paper, we will elaborate the first observation and the construction of the new set of vector fields, which helps to control the energy of Vlasov part near the light cone. Moreover, we introduce a framework for the Vlasov-wave type coupled system that allows us to combine the strength of Klainerman vector field method and Fourier method. As a result, we prove small data global regularity for the $3D$ relativistic Vlasov–Nordström system without compact support condition. The Fourier method implemented here is motivated by the method developed in the study of relatively simpler Vlasov–Poisson system in [41]. We remark that, thanks to the smallness of coefficient "$m^2/\sqrt{m^2+|v|^2}$" in the relativistic Vlasov–Nodström system (1.1), there is no need to exploit the null structure mentioned in the second observation.

Since the benefit of good coefficients is not available for the relativistic Vlasov–Maxwell system, in [40], we will elaborate the second observation and show how to exploit the hidden null structure, which is related to the structure of the time resonance set, by using a Fourier method.

1.2. The Vlasov–Nordström system. The Vlasov–Nordström system describes the collective motion of collisionless particles interacting by means of their own self-generated gravitational forces under the assumption that the gravitational forces are mediated by a scalar field. This system was proposed by Calogero [4] as a *laboratory* substitution for the more complicated and also more physical Einstein–Vlasov system. We refer readers to [4–6] for more detailed introduction.

Mathematically speaking, the $3D$ relativistic Vlasov–Nordström system reads as follows,

$$
\text{(RVN)}\quad \begin{cases} (\partial_t^2 - \Delta)\phi = \displaystyle\int_{\mathbb{R}^3} \frac{m^2 f}{\sqrt{m^2+|v|^2}} dv, \quad f : \mathbb{R}_t \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \longrightarrow \mathbb{R}_+, \\ \partial_t f + \hat{v}\cdot\nabla_x f - \big((\partial_t + \hat{v}\cdot\nabla_x)\phi(t,x)\big)(4f + v\cdot\nabla_v f) - \dfrac{m^2}{\sqrt{m^2+|v|^2}}\nabla_x\phi\cdot\nabla_v f = 0, \\ f(0,x,v) = f_0(x,v), \quad \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x), \end{cases} \tag{1.1}
$$

where $\hat{v} = v/\sqrt{m^2+|v|^2}$, "$m$" denotes the *mass* of particles, $\phi(t,x)$ denotes the scalar gravitational field generated by the particles and $f(t,x,v)$ denotes the density distribution of the particles. After normalizing the mass of particles, we only need to consider the case $m \in \{0, 1\}$, which corresponds to the massless case and the massive case respectively. Since we are interested in the massive case, from now on, m is normalized to be one.

If the initial particle density $f(0, x, v)$ has compact support, Calogero [5] proved that the $3D$ RVN system (1.1) has a unique global classical solution. However, no asymptotic behavior nor the decay estimate of the scalar field were obtained in [5]. If the initial data are small and moreover the initial scalar field has a compact support, Friedrich [12] proved the global existence of the solution and the decay estimates for the scalar field. Note that the high order derivatives of solution were not studied in [5,12].

Recently, in the spirit of the Klainerman's classic vector field method [25,26] for the nonlinear wave equations, Fajman–Joudioux–Smulevici [9,10] proposed a very interesting modified vector field method to study the propagation of regularity for the solution of the RVN system. For the $3D$ *massless* RVN system, i.e., $m = 0$ in (1.1), and the *massive* RVN system in dimension n, $n \geq 4$, they proved global regularity for suitably small initial data. In particular, the important *compact support assumption* was removed for these scenarios.

For the $3D$ massive RVN system, it was not clear whether the compact support assumption assumed in [5,12] can be removed and the propagation of regularity can be studied at the same time. A recent interesting result by Fajman–Joudioux–Smulevici [10] shows that the $3D$ massive relativistic RVN can be solved forwardly in the sense of the hyperboloid foliation of the space-time for small initial data, which in particular implies the existence of global solution for small initial data with compact support.

For the more physical Einstein–Vlasov system, Taylor [39] proved that global stability holds for the massless case if the initial small particle density has compact support in "x" and also in "v". Recently, Bigorgne–Fajman–Joudioux–Smulevici–Thaller [3] showed global stability of the Minkowski spacetime for the massless Einstein–Vlasov system without compact support assumption. For the massive case, global stability of the Minkowski spacetime also holds if the small initial particle density has compact support in "x" and in "v", see Lindblad–Taylor [29]; or the small perturbation of the Minkowski spacetime is compact in the sense that it coincides with the Schwarzschild data outside a compact set and the small initial particle density has compact support in "x" but not in "v", see Fajman–Joudioux–Smulevici [11]. See also [22,36] for related work on the Vlasov–Poisson system.

1.3. The Vlasov–Maxwell system. First of all, we would like to emphasize the purpose of introducing the Vlasov–Maxwell system in this paper. For the sake of simplicity and conciseness, we don't redo similar computations, which concern the new introduced vector fields, in our second paper [40], which studies the small data global regularity problem for the $3D$ relativistic Vlasov–Maxwell system in details. Therefore, we formulate the related results in a way such that they are applicable for both the RVN system and the RVM system. For readers who are not interested in the Vlasov–Maxwell system, they can skip this subsection and related discussions without having any problems understanding the Vlasov–Nordström system.

In plasma physics, a sufficiently diluted ionized gas or solar wind can be considered as a collisionless plasma, which means that the collision effect between the particles is not as important as the electromagnetic force. The dynamics of the collisionless plasma is described the Vlasov–Maxwell system. We are interested in the physical massive relativistic case, in which the the speed of light is finite, particles are massive and the speed of massive particles is strictly less than the speed of light.

After normalizing the mass of particles and the speed of light to be one, the relativistic Vlasov–Maxwell system with given initial data reads as follows,

$$\text{(RVM)} \quad \begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0, \\ \nabla \cdot E = 4\pi \int f(t, x, v) dv, \qquad \nabla \cdot B = 0, \\ \partial_t E = \nabla \times B - 4\pi \int f(t, x, v) \hat{v} dv, \quad \partial_t B = -\nabla \times E, \\ f(0, x, v) = f_0(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x), \end{cases} \tag{1.2}$$

where $f(t, x, v)$ denotes the density distribution function of particles, (E, B) denote the electromagnetic field, and $\hat{v}$ denotes the relativistic speed. More precisely, $\hat{v} := v/\sqrt{1 + |v|^2}$.

The Cauchy problem of long term regularity for the $3D$ RVM system has been considered by many authors. A remarkable result obtained by the Glassey–Strauss [17] says that the classical solution can be globally extended as long as the particle density has compact support in v for all the time. A new proof of this result based on Fourier analysis was given by Klainerman–Staffilani [27], which adds a new perspective to the study of $3D$ RVM system. See also [15,32,37] for improvements of this result. An interesting line of research is the continuation criterion for the global existence of the Vlasov–Maxwell system instead of assuming the compact support in "v" assumption. In [21], if the initial data decay at rate $|v|^{-7}$ as $|v| \to \infty$ and the assumption that $\|(1 + |v|) f(t, x, v)\|_{L^\infty_x L^1_v}$ remains bounded form all time, then the lifespan of the solution can be continued. Recently, an interesting new continuation criterion was given by Luk–Strain [31], which says that a regular solution can be extended as long as $\|(1 + |v|^2)^{\theta/2} f(t, x, v)\|_{L^q_x L^1_v}$ remains bounded for $\theta > 2/q, 2 < q \leq +\infty$. See also Kunze [28], Pallard [33], and Patel [34] for the recent improvements on the continuation criterion. See also [16,30,31] for results in other physical dimensions.

Although the assumptions in above mentioned results don't depend on the size of data, the assumptions are imposed for all time, which are strong. One can ask whether it is possible to obtain global solution by only imposing assumptions on the initial data. The first positive result was given by Glassey–Strauss [20]. It, roughly speaking, says that if the initial particle density $f(0, x, v)$ has a compact supports in both "x" and "v" and also the electromagnetic field $(E(0), B(0))$ has compact support in "x", and moreover the initial data is suitably small, then there exists a unique classical solution. An improvement by Schaffer [35] shows that a similar result as in [20] also holds without compact support assumption in "v" but with compact support assumption in "x" for both the initial particle density and the electromagnetic field.

It is also interesting to ask for the $3D$ RVM that whether the regularity of solution can be unconditionally propagated for all the time and whether sharp decay estimates hold. This question can be answered in higher physical dimension $n \geq 4$ for small initial data, see Bigorgne [1]. For the two dimensional case, global regularity for large data has been established by Glassey–Strauss [18,19] with compact momentum support assumption and then later extended to the non-compact support case by Luk–Strain [31]. After the completion of [40], Bigorgne [2] also showed sharp decay estimates for the $3D$ massive RVM for small initial data, see also Wei–Yang [42] for global existence of the $3D$ RVM for the partial large initial data, more precisely, the initial particle distribution is small and the initial electromagnetic field is large.

1.4. A rough statement of the main result, the method, and the outline of this paper. Since the precise statement of the main theorem, Theorem 4.1, depends on two sets of vector fields discussed later, we postpone it to Sect. 4.4. In this subsection, to give a sense, we provide a rough theorem for the $3D$ relativistic Vlasov–Nordström system.

Theorem 1.1 (A rough statement). *Given any smooth suitably small localized initial particle density $f(0, x, v)$ and initial data of scalar field $\big(\phi_0(x), \phi_1(x)\big)$, where the particle density $f(0, x, v)$ decays fast but polynomially as (x, v) goes to infinity. Then the $3D$ massive relativistic Vlasov–Nordström (1.1) admits a unique global solution. The regularity of initial data can be globally propagated and the nonlinear solution scatters to a linear solution in a low regularity space. Moreover, the high order energy of solution only grows sub-polynomially and the scalar field and its derivatives have sharp rate $1/(1 + |t|)$ over time.*

In this paper and its companion [40], motivated by the recent progress in the nonlinear dispersive equations, e.g., [7,8,13,14,23,24], we introduce a Vector fields-Fourier method, which combines the strength of both the vector fields method and the Fourier method, to study the small data global regularity problem and the asymptotic behavior of the Vlasov-Wave type coupled system.

From the seminar works of Klainerman [25,26], we know that there exists a set of vector fields which generate the Lorentz group that leaves the Minkowski spactime invariant. One of the main strengths of the Klainerman vector field method is that the decay rate of the nonlinear wave equation is controlled by the L^2-type energy, which involves the vector fields, via the Klainerman–Soblev embedding.

Following this idea, we introduce a new set of vector fields for the Vlasov-type equation, which aims to improve the understanding of the acceleration term $\nabla_v f$, which causes the main difficulty in the small data global regularity problem as ∇_v doesn't commute with the transport operator $\partial_t + \hat{v} \cdot \nabla_x$. This type of estimate is not true

In connection with the space-time resonance analysis, the normal forms, we also appeal to the Fourier method to handle the delicate decay estimates of both the wave part and the density type function of the Vlasov equation. This method has been successfully applied in the small data global regularity problem in water waves and plasmas, see [7,8,23,24]. In this paper and [40], the Fourier method has been used in an essential way. For example, note that the inhomogeneity of the Vlasov–Nordström system (1.1) contains the linear density type function, which is problematic in the energy estimate. To overcome this difficulty, we do the normal form transformation, which involves a Fourier multiplier operator, at the initial stage to utilize the absence of time-resonance set, which essentially comes from the fact that the speed of particles is strictly less than the speed of wave.

The rest of this paper is organized as follows.

- In Sect. 2, we define notations used in this paper, introduce *profiles* for the RVN system (1.1), and prove two basic L^∞_x-type linear estimates.
  Here the profile of a nonlinear solution means the pull back of the nonlinear solution along the linear flow, e.g., the profile of the Vlasov distribution function $f(t, x, v)$ is defined to be $f(t, x + t\hat{v}, v)$.
- In Sect. 3, we introduce the concept of inhomogeneous modulation of light cone in (x, v)-space, construct two sets of vector fields and decompose the bulk derivative "$D_v := \nabla_v - t\nabla_v \hat{v} \cdot \nabla_x$" in terms of the new set of vector fields. In particular, the coefficients of the decompositions of D_v are related to the inhomogeneous modulation.

- In Sect. 4, we set up the high order energy estimates for the Vlasov–Nordström system, give a more precise statement of the main theorem of this paper, Theorem 4.1, and use a bootstrap argument to prove our main theorem.
- In Sect. 5, we control the increment of both the high order energy and the low order energy over time for the profiles of the nonlinear wave equation.
- In Sect. 6, we control the increment of both the high order energy and the low order energy over time for the profiles of the particle distribution function.
- In Appendix A, for the sake of readers, we give detailed computations for the commutation rules between the bulk derivative D_v and the new set of vector fields as well as the case when the vector fields act on the inhomogeneous modulation function.

2. Preliminaries

2.1. Notation. For any two numbers A and B, we use $A \lesssim B$, $A \approx B$, and $A \ll B$ to denote $A \leq CB$, $|A - B| \leq cA$, and $A \leq cB$ respectively, where C is an absolute constant and c is a sufficiently small absolute constant. We use $A \sim B$ to denote the case when $A \lesssim B$ and $B \lesssim A$. For an integer $k \in \mathbb{Z}$, we use "k_+" to denote $\max\{k, 0\}$ and use "k_-" to denote $\min\{k, 0\}$. For any two vectors $\xi, \eta \in \mathbb{R}^3$, we use $\angle(\xi, \eta)$ to denote the angle between "ξ" and "η". Moreover, we use the convention that $\angle(\xi, \eta) \in [0, \pi]$.

For terminology from Fourier theory used in this paper, we refer readers to the seminar work by Stein [38]. For $f \in L^1(\mathbb{R}^3)$, we use both $\widehat{f}(\xi)$ and $\mathcal{F}(f)(\xi)$ to denote the Fourier transform of f, which is defined as follows,

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx.$$

We use $\mathcal{F}^{-1}(g)$ to denote the inverse Fourier transform of $g(\xi)$. Moreover, for a distribution function $f : \mathbb{R}_x^3 \times \mathbb{R}_v^3 \to \mathbb{C}$, we treat "$v$" as a fixed parameter and use the following notation to denote the Fourier transform of $f(x, v)$ in "x",

$$\widehat{f}(\xi, v) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x, v) dx.$$

We fix an even smooth function $\tilde{\psi} : \mathbb{R} \to [0, 1]$, which is supported in $[-3/2, 3/2]$ and equals to "1" in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define $\psi_k, \psi_{\leq k}, \psi_{\geq k} : \mathbb{R}^3 \cup \mathbb{R} \longrightarrow \mathbb{R}$ as follows,

$$\psi_k(x) := \tilde{\psi}(|x|/2^k) - \tilde{\psi}(|x|/2^{k-1}),$$
$$\psi_{\leq k}(x) := \tilde{\psi}(|x|/2^k) = \sum_{l \leq k} \psi_l(x),$$
$$\psi_{\geq k}(x) := 1 - \psi_{\leq k-1}(x).$$

Moreover, we use P_k, $P_{\leq k}$ and $P_{\geq k}$ to denote the projection operators by the Fourier multipliers $\psi_k(\cdot)$, $\psi_{\leq k}(\cdot)$ and $\psi_{\geq k}(\cdot)$ respectively. We use $f_k(x)$ to abbreviate $P_k f(x)$. For $f \in L^1(\mathbb{R}^3)$, we define

$$f^+ := f, \quad P_+[f] := f, \quad f^- := \bar{f}, \quad P_-[f] := \bar{f}. \tag{2.1}$$

Define

$$\chi_k^1 := \{(k_1, k_2) : |k_1 - k_2| \leq 10, k \leq \max\{k_1, k_2\} + 1\}, \tag{2.2}$$

$$\chi_k^2 := \{(k_1, k_2) : k_2 < k_1 - 10, |k - k_1| \leq 1\},$$
$$\chi_k^3 := \{(k_1, k_2) : k_1 < k_2 - 10, |k - k_2| \leq 1\}, \tag{2.3}$$

where χ_k^1 corresponds to the High $\times$ High type interaction and χ_k^2 and χ_k^3 correspond to the High $\times$ Low type interaction and the Low $\times$ High type interaction respectively. We define unit vectors in $\mathbb{R}^3$ as follows,

$$e_1 := (1, 0, 0), \quad e_2 := (0, 1, 0), \quad e_3 := (0, 0, 1).$$

Moreover, we define the following vectors,

$$\begin{aligned} X_i &= e_i \times x, \quad V_i = e_i \times v, \quad \hat{V}_i = e_i \times \hat{v}, \\ \tilde{V}_i &= e_i \times \tilde{v}, \quad \tilde{v} := \frac{v}{|v|}, \\ \tilde{v}_i &:= \tilde{v} \cdot e_i, \quad \hat{v}_i := \hat{v} \cdot e_i. \end{aligned} \tag{2.4}$$

where $i = 1, 2, 3$. As a result of direct computations, the following equalities hold for $\forall u, v \in \mathbb{R}^3$,

$$\begin{aligned} u &= \tilde{v}\tilde{v} \cdot u + \sum_{i=1,2,3} \tilde{V}_i \tilde{V}_i \cdot u, \\ \tilde{v} \cdot \nabla_v \hat{v} &= \frac{\tilde{v}}{(1 + |v|^2)^{3/2}}, \\ \tilde{V}_i \cdot \nabla_v \hat{v} &= \frac{\tilde{V}_i}{(1 + |v|^2)^{1/2}}, \quad i \in \{1, 2, 3\}. \end{aligned} \tag{2.5}$$

For any $k \in \mathbb{Z}$, we define $\mathcal{S}_k^\infty$-norm associated with symbols as follows,

$$\|m(\xi)\|_{\mathcal{S}_k^\infty} := \sum_{|\alpha|=0,1,\ldots,10} 2^{|\alpha|k} \|\mathcal{F}^{-1}[\nabla_\xi^\alpha m(\xi)\psi_k(\xi)]\|_{L^1}. \tag{2.6}$$

Moreover, we define a class of symbol as follows,

$$\mathcal{S}^\infty := \{m(\xi) : \quad \|m(\xi)\|_{\mathcal{S}^\infty} := \sup_{k\in\mathbb{Z}} \|m(\xi)\|_{\mathcal{S}_k^\infty} < \infty\}. \tag{2.7}$$

Definition 2.1. We define a linear operator as follows,

$$Q_i := -R_i|\nabla|^{-1}, Q := (Q_1, Q_2, Q_3), \quad i \in \{1, 2, 3\}, \tag{2.8}$$

where R_i, $i \in \{1, 2, 3\}$, denote the Riesz transforms. Hence, we have

$$Id = \nabla \cdot Q. \tag{2.9}$$

2.2. The profiles of the relativistic Vlasov–Nordström system. The idea of studying the profile of the system instead of the original variables is not new, it has been widely used in the study of nonlinear dispersive equation, e.g., see [7,8,13,14,23,24]. One benefit of studying the profile is that the effect of linear flow has already been taken into account in the transformation, we can focus on the nonlinear effect.

In this subsection, we define the profiles of the Vlasov part and the scalar field part and obtain the evolution equations for the profiles over time. We will also perform similar procedures in Sect. 4.2 when we apply vector fields on the Vlasov–Nordström system.

We first define the profile "$g(t, x, v)$" of the particle distribution function "$f(t, x, v)$ "as follows,

$$g(t, x, v) = f(t, x + \hat{v}t, v), \quad \Longrightarrow f(t, x, v) = g(t, x - \hat{v}t, v). \tag{2.10}$$

As a result of direct computations, the profile $g(t, x, v)$ satisfies the following equation from the system of equations in (1.1),

$$\begin{aligned}\partial_t g(t, x, v) &= (\partial_t + \hat{v} \cdot \nabla_x)\phi(t, x + \hat{v}t)\big(4g(t, x, v) + v \cdot D_v g(t, x, v)\big)\\ &\quad + \frac{\nabla_x \phi(t, x + \hat{v}t)}{\sqrt{1 + |v|^2}} \cdot D_v g(t, x, v).\end{aligned} \tag{2.11}$$

where

$$D_v := \nabla_v - t\nabla_v \hat{v} \cdot \nabla_x, \quad D_{v_i} := \partial_{v_i} - t\partial_{v_i}\hat{v} \cdot \nabla_x, \quad i \in \{1, 2, 3\}. \tag{2.12}$$

Hence, to control the nonlinear effect in (2.11), it's crucial to understand the role of derivative "D_v" acts on the profile g, which will be elaborated in Sect. 3.1.

Next, we define the profile "$h(t)$" and the half wave $u(t)$ of the scalar field part in (1.1) as follows,

$$h(t) := e^{it|\nabla|}u(t), \quad u(t) := (\partial_t - i|\nabla|)\phi(t).$$

Note that, we can recover ϕ and $\partial_t \phi$ from the half wave $u(t)$ and the profile $h(t)$ as follows,

$$\begin{aligned}\partial_t \phi &= \frac{u(t) + \overline{u(t)}}{2},\\ \phi &= \frac{-u(t) + \overline{u(t)}}{2i|\nabla|} := \sum_{\mu \in \{+,-\}} c_\mu |\nabla|^{-1} u^\mu(t),\\ c_\mu &:= i\mu/2, \quad u(t) = e^{-it|\nabla|}h(t).\end{aligned}$$

In terms of the half wave $u(t)$, we can rewrite the equation satisfied by the profile $g(t, x, v)$ in (2.11) as follows,

$$\begin{aligned}\partial_t g(t, x, v) = \sum_{\mu \in \{+,-\}} \big(\frac{1}{2} + c_\mu \hat{v} \cdot R\big)u^\mu(t, x + \hat{v}t)\big(4g(t, x, v)\\ + v \cdot D_v g(t, x, v)\big) + \frac{c_\mu R u^\mu(t, x + \hat{v}t)}{\sqrt{1 + |v|^2}} \cdot D_v g(t, x, v),\end{aligned} \tag{2.13}$$

where $R := \nabla_x/|\nabla|$ denotes the Riesz transforms. Moreover, on the Fourier side, we have

$$\partial_t \widehat{g}(t,\xi,v) = \sum_{\mu\in\{+,-\}} \int_{\mathbb{R}^3} e^{it\hat{v}\cdot(\xi-\eta)-it\mu|\xi-\eta|} \widehat{h^{\mu}}(t,\xi-\eta)\big(a_{\mu}^1(v,\xi-\eta)\widehat{g}(t,\eta,v)$$
$$+ a_{\mu}^2(v,\xi-\eta)\cdot\big(\nabla_v - it\nabla_v\hat{v}\cdot\eta\big)\widehat{g}(t,\eta,v)\big)d\eta, \tag{2.14}$$

where

$$a_{\mu}^1(v,\xi) = \big(2 + i4c_{\mu}\hat{v}\cdot\xi|\xi|^{-1}\big), \quad a_{\mu}^2(v,\xi) = \frac{v}{2}\big(1 + i2c_{\mu}\hat{v}\cdot\xi|\xi|^{-1}\big) + \frac{ic_{\mu}\xi}{\sqrt{1+|v|^2}|\xi|}. \tag{2.15}$$

From the system of equations in (1.1), we can derive the equation satisfied by $u(t)$ as follows,

$$(\partial_t + i|\nabla|)u(t) = \int_{\mathbb{R}^3} \frac{1}{\sqrt{1+|v|^2}} f(t,x,v)dv = \int_{\mathbb{R}^3} \frac{1}{\sqrt{1+|v|^2}} g(t,x-\hat{v}t,v)dv. \tag{2.16}$$

On the Fourier side, we have

$$\partial_t \widehat{h}(t,\xi) = \int_{\mathbb{R}^3} e^{it|\xi|-it\hat{v}\cdot\xi} \frac{1}{\sqrt{1+|v|^2}} \widehat{g}(t,\xi,v)dv. \tag{2.17}$$

Note that, the inhomogeneity of the above equation is linear with respect to the Vlasov part, which, generally speaking, very hard to be controlled directly in the energy estimate.

However, we observe that the nonlinearity in (2.17) is actually oscillating in time. To take the advantage of the oscillation of the phase "$|\xi| - \hat{v}\cdot\xi$" in (2.17) over time, instead of controlling the increment of the profile $h(t)$ over time, we control the following *modified profile*,

$$\widehat{\widetilde{h}}(t,\xi) := \widehat{h}(t,\xi) + \int_{\mathbb{R}^3} e^{it|\xi|-it\hat{v}\cdot\xi} \frac{i}{|\xi|-\hat{v}\cdot\xi} \frac{1}{\sqrt{1+|v|^2}} \widehat{g}(t,\xi,v)dv.$$

Recall the Eqs. (2.14) and (2.17). After doing integration by parts in v once, we can derive the equation satisfied by the modified profile $\widetilde{h}(t)$ as follows,

$$\begin{aligned}\partial_t \widehat{\widetilde{h}}(t,\xi) &= \int_{\mathbb{R}^3} e^{it|\xi|-it\hat{v}\cdot\xi} \frac{1}{\sqrt{1+|v|^2}} \frac{i}{|\xi|-\hat{v}\cdot\xi} \partial_t \widehat{g}(t,\xi,v)dv \\ &= \sum_{\mu\in\{+,-\}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-it\mu|\xi-\eta|-it\hat{v}\cdot\eta} \\ &\quad a_{\mu}^3(v,\xi,\xi-\eta)\widehat{h^{\mu}}(t,\xi-\eta)\widehat{g}(t,\eta,v)d\eta dv\end{aligned} \tag{2.18}$$

where

$$a_{\mu}^3(v,\xi,\xi-\eta) := \frac{ia_{\mu}^1(v,\xi-\eta)}{\sqrt{1+|v|^2}\big(|\xi|-v\cdot\xi\big)} - \nabla_v\cdot\Big(\frac{ia_{\mu}^2(v,\xi-\eta)}{\sqrt{1+|v|^2}\big(|\xi|-v\cdot\xi\big)}\Big). \tag{2.19}$$

where $a_{\mu}^i(v,\xi)$, $i\in\{1,2\}$, are defined in (2.15).

2.3. Linear decay estimates. It is well known that the density of the distribution function decays over time. Now, there are several ways to prove this fact, e.g., performing change of variables, using the vector fields method. We refer readers to a recent result by Wong [43] for more detailed discussion. In [41], we provided another proof for this fact by using a Fourier transform method.

Intuitively speaking, on the Fourier side, we can represent density $\rho(t,x)$ in terms of the profile $g(t,x,v)$ defined in (2.10) as follows,

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv = \int_{\mathbb{R}^3} g(t, x - t\hat{v}, v)dv$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi - it\hat{v}\cdot\xi}\hat{g}(t,\xi,v)d\xi dv.$$

Therefore, from the stationary phase point of view, the decay rate of density function is connected with the regularity. In practice, we can do integration by parts in v in the above integral. As a result, we can gain t^{-1} decay rate by paying the price of losing $|\xi|^{-1}$.

More precisely, the following Lemma for general density type functions holds,

Lemma 2.1. *For any fixed $x \in \mathbb{R}^3$, $a, t \in \mathbb{R}$, s.t, $|t| \geq 1$, $a > -3$, and any given symbol $m(\xi, v) \in L^\infty_v \mathcal{S}^\infty$, the following decay estimate for the density type function holds,*

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi - it\hat{v}\cdot\xi} m(\xi,v)|\xi|^a \widehat{g}(t,\xi,v)dvd\xi\Big|$$
$$\lesssim \sum_{0\leq c_1\leq 5+\lfloor a\rfloor}\Big(\sum_{0\leq c_2\leq 5+\lfloor a\rfloor}\|\nabla_v^{c_1} m(\xi,v)\|_{L^\infty_v\mathcal{S}^\infty}\Big)$$
$$\times\big[|t|^{-3-a}\|(1+|v|)^{5+|a|}\nabla_v^{c_2}\widehat{g}(t,0,v)\|_{L^1_v}$$
$$+|t|^{-4-a}\|(1+|v|)^{5+|a|}(1+|x|)\nabla_v^{c_2} g(t,x,v)\|_{L^1_x L^1_v}\big]. \tag{2.20}$$

Proof. See [41][Lemma 3.1]. □

As in the previous subsection, instead of studying the nonlinear wave equation directly in this paper, we study a nonlinear half wave equation, which is convenient to study on the Fourier side. Hence, we provide a L^∞_x-type decay estimate for the linear half wave equation in the following Lemma.

Lemma 2.2 (The linear decay estimate). *For any $\mu \in \{+,-\}$, the following estimate for the half-wave solution holds,*

$$\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|} m(\xi)\widehat{f}(\xi)\psi_k(\xi)d\xi\Big| \lesssim \min\{2^{k_-}, (1+|t|+|x|)^{-1}\}2^k\|m(\xi)\|_{\mathcal{S}^\infty_k}$$
$$\times\Big(\sum_{0\leq n\leq 1} 2^k\|\widehat{\nabla_x^n f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{2k}\|\nabla_\xi \widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\Big). \tag{2.21}$$

Proof. By using the volume of support of ξ, we have

$$\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|} m(\xi)\widehat{f}(\xi)\psi_k(\xi)d\xi\Big|$$
$$\lesssim 2^{3k}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\widehat{f}(\xi)\psi_k(\xi)\|_{L^\infty_\xi}. \tag{2.22}$$

Hence finishing the proof of the first part of the desired estimate (2.21).

It remains to prove the second part of the desired estimate (2.21).Based on the possible size of t and x, we separate into two cases as follows.

Case 1 If $|x| \geq 3(1+|t|)$ or $|t| \leq 1$.

For this case, we do integration by parts in ξ once. As a result, we have

$$\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|} m(\xi)\widehat{f}(\xi)\psi_k(\xi)d\xi = \int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|} i\nabla_\xi \cdot \Big[\frac{x-\mu t\xi/|\xi|}{\big|x-\mu t\xi/|\xi|\big|^2} m(\xi)\widehat{f}(\xi)\psi_k(\xi)\Big]d\xi.$$

After using the volume of support of ξ for the above equality, we have

$$\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|} m(\xi)\widehat{f}(\xi)\psi_k(\xi)d\xi\Big| \lesssim \frac{2^k\|m(\xi)\|_{\mathcal{S}^\infty_k}}{|x|}\big(2^k\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{2k}\|\nabla_\xi\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\big). \tag{2.23}$$

Now, our desired estimate (2.21) holds from the above estimate and the estimate (2.22) if $|x| \geq 3(1+|t|)$ or $|t| \leq 1$.

Case 2 If $|x| \leq 3(1+|t|)$ and $|t| \geq 1$.

Note that $\nabla_\xi(x\cdot\xi - \mu t|\xi|) = 0$ if and only if $\xi/|\xi| = \mu x/t = \mu x/|x| := \xi_0$. Let $\bar{l}$ be the least integer such that $2^{\bar{l}} \geq 2^{-k/2}(1+|t|)^{-1/2}$. From the volume of support of ξ, we have

$$\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|}\widehat{f}(\xi)m(\xi)\psi_k(\xi)\psi_{\leq\bar{l}}(\angle(\xi,\xi_0))d\xi\Big| \lesssim 2^{3k+2\bar{l}}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\widehat{f}(\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim (1+|t|)^{-1}2^{2k}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}. \tag{2.24}$$

For the case when the angle is localized around 2^l where $l > \bar{l}$, we first do integration by parts in ξ once. As a result, we have

$$\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|}\widehat{f}(\xi)m(\xi)\psi_k(\xi)\psi_l(\angle(\xi,\xi_0))d\xi = I_l^1 + I_l^2,$$

where

$$I_l^1 = \int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|}\frac{i}{t}\frac{x/t+\mu\xi/|\xi|}{\big|x/t+\mu\xi/|\xi|\big|^2}\cdot\nabla_\xi\widehat{f}(\xi)m(\xi)\psi_k(\xi)\psi_l(\angle(\xi,\xi_0))d\xi,$$

$$I_l^2 = \int_{\mathbb{R}^3} e^{ix\cdot\xi - i\mu t|\xi|}\frac{i}{t}\nabla_\xi\cdot\Big[\frac{x/t+\mu\xi/|\xi|}{\big|x/t+\mu\xi/|\xi|\big|^2}m(\xi)\psi_k(\xi)\psi_l(\angle(\xi,\xi_0))\Big]\widehat{f}(\xi)d\xi.$$

From the volume of support of ξ, the following estimate holds for I_l^1,

$$|I_l^1| \lesssim (1+|t|)^{-1}2^{-l}2^{3k+2l}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\nabla_\xi\widehat{f}(\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim (1+|t|)^{-1}2^{3k+l}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\nabla_\xi\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}. \tag{2.25}$$

For I_l^2, we do integration by parts in "ξ" one more time. As a result, the following estimate holds after using the volume of support of ξ,

$$
\begin{aligned}
|I_l^2| &\lesssim (1+|t|)^{-2}2^{-2l}2^{3k+2l}\|m(\xi)\|_{\mathcal{S}_k^\infty}\big(2^{-2k-2l}\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty} \\
&\quad + 2^{-k-l}\|\nabla_\xi \widehat{f}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty}\big) \\
&\lesssim (1+|t|)^{-2}2^{-2l}\|m(\xi)\|_{\mathcal{S}_k^\infty}\big(2^{k}\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty} \\
&\quad + 2^{2k}\|\nabla_\xi \widehat{f}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty}\big). \qquad (2.26)
\end{aligned}
$$

Hence, from (2.25) and (2.26), we have

$$
\begin{aligned}
\sum_{\bar{l}<l\le 2} |I_l^1| + |I_l^2| &\lesssim (1+|t|)^{-1}2^{k}\|m(\xi)\|_{\mathcal{S}_k^\infty}\big(2^{k}\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty} \\
&\quad + 2^{2k}\|\nabla_\xi \widehat{f}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty}\big). \qquad (2.27)
\end{aligned}
$$

Now, our desired estimate (2.21) holds from the estimates (2.22), (2.24) and (2.27) for the case $|x| \le 3(1+|t|)$ and $|t| \ge 1$. $\square$

3. Constructing Vector Fields for the Relativistic Vlasov-Wave Type Coupled System

In this section, based an observation on the light cone $C_t := \{(x,v) : x, v \in \mathbb{R}^3, |t| - |x + t\hat{v}| = 0\}$ in (x,v) space, we construct a new set of vector fields, which will be used to decompose the bulk derivative D_v defined in (2.12). Before that, we first introduce a set of classic vector fields which are applicable for both the Vlasov equation and the nonlinear wave equation. The classic set of vector fields enables us to obtain decay estimate from the energy estimate for the wave equation, which is well known as the Klainerman–Sobolev embedding.

Recall (2.4). For any $i, j = 1, 2, 3$, we define the first set of vector fields for the Vlasov–Nordström system and the Vlasov–Maxwell system as follows,

$$
\begin{aligned}
S &:= t\partial_t + x\cdot\nabla_x, \quad \Omega_{i,j} = x_i\partial_{x_j} - x_j\partial_{x_i}, \\
\Omega_i &= X_i\cdot\nabla_x, \\
\tilde{\Omega}_i &:= V_i\cdot\nabla_v + X_i\cdot\nabla_x, \qquad (3.1) \\
L_i &:= t\partial_{x_i} + x_i\partial_t, \quad \tilde{L}_i := t\partial_{x_i} + x_i\partial_t + \sqrt{1+|v|^2}\partial_{v_i}, \\
L &:= (L_1, L_2, L_3), \ \ \tilde{L} := (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3), \qquad (3.2)
\end{aligned}
$$

where "S", "$\Omega_{i,j}$", and "L_i" are the well-known scaling vector field, rotational vector fields, and the Lorentz vector fields, which all have favorable commutation rules with the linear operator of the nonlinear wave equation, see the classic works of Klainerman [25, 26] for the introduction of the original vector field method. Note that $\Omega_{i,j} \in \{\mu\Omega_i, \mu \in \{+,-\}, i \in \{1,2,3\}\}$ for any $i, j \in \{1,2,3\}$.

As pointed out in Fajman–Joudioux–Smulevici [9] that the commutation rules between the vector fields S, $\tilde{\Omega}_i$, and $\tilde{L}_i$ and with the linear operator of the relativistic Vlasov equation are very favorable. More precisely, we have

$$
[\partial_t + \hat{v}\cdot\nabla_x, S] = \partial_t + \hat{v}\cdot\nabla_x, \quad [\partial_t + \hat{v}\cdot\nabla_x, \tilde{\Omega}_i] = 0,
$$

$$[\partial_t + \hat{v}\cdot\nabla_x, \tilde{L}_i] = \hat{v}_i\big(\partial_t + \hat{v}\cdot\nabla_x\big), \tag{3.3}$$

$$\begin{aligned}[\partial_t + \hat{v}\cdot\nabla_x, \tilde{\Omega}_i] &= [\partial_t + \hat{v}\cdot\nabla_x, V_i\cdot\nabla_v + X_i\cdot\nabla_x]\\ &= (\hat{v}\cdot\nabla_x(X_i))\cdot\nabla_x - \hat{V}_i\cdot\nabla_x = 0,\end{aligned} \tag{3.4}$$

We define the first set of vector fields for the distribution function $f(t, x, v)$ as follows,

$$\mathcal{P}_1 := \{S, \tilde{\Omega}_i, \tilde{L}_i, \partial_{x_i}, i \in \{1, 2, 3\}\}. \tag{3.5}$$

Correspondingly, we define the following set of vector fields for the nonlinear wave part as follows,

$$\mathfrak{P}_1 := \{S, \Omega_i, L_i, \partial_{x_i}, i \in \{1, 2, 3\}\}. \tag{3.6}$$

Lemma 3.1. *For any $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, the following equalities hold,*

$$\begin{aligned}(t^2 - |x|^2)\partial_t &= tS - x\cdot L,\\ (t^2 - |x|^2)\partial_i &= \sum_{j=1,2,3} -x_j\Omega_{ij} + tL_i - x_i S, \quad i \in \{1, 2, 3\}.\end{aligned} \tag{3.7}$$

Proof. Desired identities follow from direct computations. □

Unfortunately, we cannot represent the bulk derivative "D_v" as a "*good*" linear combination of vector fields defined previously, i.e., ∇_x, S, $\tilde{\Omega}_i$, and $\tilde{L}_i$, in the sense that one of coefficients is of size "t", which is too big to control in the long run. This is also why we seek for a new set of vector fields.

3.1. A new set of vector fields. Our new set of vector fields is inspired from the following identity regards the light cone $C_t := \{(x, v) : x, v \in \mathbb{R}^3, |t| - |x + t\hat{v}| = 0\}$ in (x, v) space.

Lemma 3.2. *The following identity holds,*

$$t^2 - |x + \hat{v}t|^2 = \Big(\frac{t}{1 + |v|^2} - \frac{\omega_+(x, v)}{\sqrt{1 + |v|^2}}\Big)\big(t - \sqrt{1 + |v|^2}\omega_-(x, v)\big), \tag{3.8}$$

where

$$\omega_+(x, v) = x\cdot v + \sqrt{(x\cdot v)^2 + |x|^2}, \quad \omega_-(x, v) = x\cdot v - \sqrt{(x\cdot v)^2 + |x|^2}. \tag{3.9}$$

Moreover, the following rough estimates hold,

$$\omega_-(x, v) \lesssim \frac{-|x|}{1 + |v|}, \quad 0 \le \omega_+(x, v) \lesssim |x|(1 + |v|). \tag{3.10}$$

Proof. Note that

$$t^2 - |x + \hat{v}t|^2 = \frac{t^2}{1+|v|^2} - \frac{2tx \cdot v}{\sqrt{1+|v|^2}} - |x|^2$$
$$= \big(\frac{t}{\sqrt{1+|v|^2}} - \omega_+(x,v)\big)\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_-(x,v)\big),$$

where

$$\omega_+(x,v) = x \cdot v + \sqrt{(x\cdot v)^2 + |x|^2}, \quad \omega_-(x,v) = x \cdot v - \sqrt{(x\cdot v)^2 + |x|^2}. \tag{3.11}$$

From the above detailed formulas, the desired estimate (3.10) holds straightforwardly. □

From the equality (3.8) and the estimate (3.10), we know that "$|t| - |x + \hat{v}t| = 0$" if and only if "$t/(1+|v|^2) - \omega_+(x,v)/(\sqrt{1+|v|^2}) = 0$". This fact motivates us to define the *modulation* of the light cone in (x, v)-space as follows, which plays the role of distance to the light cone.

Definition 3.1. We define the following function as the *modulation* with respect to the "light cone" in (x, v)-space

$$d(t,x,v) := \frac{t}{1+|v|^2} - \frac{\omega_+(x,v)}{\sqrt{1+|v|^2}}. \tag{3.12}$$

Since we will only care about the distance with respect to the "light cone" when "$|x| + |x \cdot v| \gtrsim 1$", we define an *inhomogeneous modulation* $\tilde{d}(t,x,v)$ as follows,

$$\tilde{d}(t,x,v) := \frac{t}{1+|v|^2} - \frac{\omega(x,v)}{\sqrt{1+|v|^2}}, \tag{3.13}$$

where $\omega(x, v)$ is defined as follows,

$$\omega(x,v) = \psi_{\geq 0}(|x|^2 + (x\cdot v)^2)\omega_+(x,v)$$
$$= \psi_{\geq 0}(|x|^2 + (x\cdot v)^2)\big(x \cdot v + \sqrt{(x\cdot v)^2 + |x|^2}\big). \tag{3.14}$$

From (3.8) and (3.10), we have

$$|\tilde{d}(t,x,v)| \lesssim 1 + |t - |x + \hat{v}t||. \tag{3.15}$$

With the above definition of the inhomogeneous modulation function $\tilde{d}(t,x,v)$, the construction of our new set of vector fields is motivated from a good decomposition of the derivative D_v. Recall (2.12). Instead of "naturally" decompose D_v into two parts, which are "∇_v" and "$t\nabla_v \hat{v} \cdot \nabla_x$", we decompose D_v into two parts asfollows,

$$D_v = \nabla_v - t\nabla_v \hat{v}\cdot\nabla_x$$
$$= \underbrace{\nabla_v - \sqrt{1+|v|^2}\omega(x,v)\nabla_v\hat{v}\cdot\nabla_x}_{\text{Part I}}$$
$$- \underbrace{(t-\sqrt{1+|v|^2}\omega(x,v))\nabla_v\hat{v}\cdot\nabla_x}_{\text{Part II}}. \tag{3.16}$$

The main intuition of the above "*good*" decomposition is that *it is more promising to control the burden of extra degree of modulation than the burden of extra degree of "t" over time for the nonlinear wave solution.*

With the above motivation, we define,

$$K_v := \nabla_v - \sqrt{1+|v|^2}\omega(x,v)\nabla_v\hat{v}\cdot\nabla_x, \quad S^v := \tilde{v}\cdot\nabla_v,$$
$$S^x := \tilde{v}\cdot\nabla_x, \quad \Omega_i^v = \tilde{V}_i\cdot\nabla_v, \quad \Omega_i^x = \tilde{V}_i\cdot\nabla_x, \tag{3.17}$$

where $i\in\{1,2,3\}$, $\tilde{v}$ and $\tilde{V}_i$ are defined in (2.4).

Moreover, we define a set of vector fields as follows,

$$\widehat{S}^v := \tilde{v}\cdot K_v = S^v - \frac{\omega(x,v)}{1+|v|^2}S^x,$$
$$\widehat{\Omega}_i^v := \tilde{V}_i\cdot K_v = \Omega_i^v - \omega(x,v)\Omega_i^x, \quad K_{v_i} := K_v\cdot e_i, \quad i\in\{1,2,3\}. \tag{3.18}$$

Note that we used the equalities in (2.5) in the above equation.

We remark that the vector fields defined in (3.18) will be applied on the profile "$g(t,x,v)$" instead of the original distribution "$f(t,x,v)$". Also, it's not difficult to find the corresponding vector fields act on the original distribution function $f(t,x,v)$. For example, from (2.10), we have

$$\begin{aligned} K_v g(t,x,v) &= \big(\nabla_v - \sqrt{1+|v|^2}\omega(x,v)\nabla_v\hat{v}\cdot\nabla_x\big)\big(f(t,x+\hat{v}t,v)\big)\\ &= (\nabla_v f)(t,x+\hat{v}t,v) + (t-\sqrt{1+|v|^2}\omega(x,v))\nabla_v\hat{v}\cdot\nabla_x f(t,x+\hat{v}t,v)\\ &=: (\widetilde{K}_v f)(t,x+\hat{v}t,v), \end{aligned}$$

where

$$\widetilde{K}_v := \nabla_v + (t-\sqrt{1+|v|^2}\omega(x-\hat{v}t,v))\nabla_v\hat{v}\cdot\nabla_x. \tag{3.19}$$

As a result of direct computation, we know that $\widetilde{K}_v$ commutates with the linear transport operator "$\partial_t + \hat{v}\cdot\nabla_x$" of the relativistic Vlasov equation.

With the above defined new set of vector fields, in the following Lemma, we decompose the bulk derivative "D_v" in terms of the new set of vector fields.

Lemma 3.3. *The following two decompositions holds for "D_v",*

$$D_v = (\tilde{v}\widehat{S}^v + \tilde{V}_i\widehat{\Omega}_i^v) - \tilde{d}(t,x,v)\Big(\frac{\tilde{v}S^x}{\sqrt{1+|v|^2}} + \sum_{i=1,2,3}\sqrt{1+|v|^2}\tilde{V}_i\Omega_i^x\Big), \tag{3.20}$$

$$D_v = \tilde{v}\widehat{S}^v - \tilde{d}(t,x,v)\frac{\tilde{v}S^x}{\sqrt{1+|v|^2}} + \sum_{i=1,2,3}\frac{\tilde{V}_i}{|v|}\tilde{\Omega}_i - \frac{\tilde{V}_i}{|v|}\Big(X_i\cdot\tilde{v}S^x$$
$$+ \sum_{j=1,2,3}(X_i+\hat{V}_i t)\cdot\tilde{V}_j\Omega_j^x\Big). \tag{3.21}$$

Proof. Recall (2.12), (3.17), and (3.18). From (2.5), we have

$$D_v = \tilde{v} S^v - \frac{t\tilde{v} S^x}{(1+|v|^2)^{3/2}} + \sum_{i=1,2,3} \tilde{V}_i \Omega_i^v - \frac{t\tilde{V}_i\Omega_i^x}{\sqrt{1+|v|^2}} \tag{3.22}$$

$$= \sum_{i=1,2,3} (\tilde{v}\widehat{S}^v + \tilde{V}_i \widehat{\Omega}_i^v) - \tilde{d}(t,x,v)\big(\frac{\tilde{v} S^x}{\sqrt{1+|v|^2}} + \sqrt{1+|v|^2}\tilde{V}_i\Omega_i^x\big). \tag{3.23}$$

Hence, finishing the proof of (3.20).

Now, let's proceed to prove the desired equality (3.21). Recall (3.1) and (3.17), we have

$$\Omega_i^v = \frac{1}{|v|}\tilde{\Omega}_i - \frac{X_i}{|v|}\cdot\nabla_x. \tag{3.24}$$

From (2.5), (3.22) and (3.24), we have

$$\begin{aligned}
D_v &= \tilde{v}\widehat{S}^v - \tilde{d}(t,x,v)\frac{\tilde{v} S^x}{\sqrt{1+|v|^2}} + \sum_{i=1,2,3} \frac{\tilde{V}_i}{|v|}\tilde{\Omega}_i - \frac{\tilde{V}_i}{|v|}(X_i + \hat{V}_i t)\cdot\nabla_x \\
&= \tilde{v}\widehat{S}^v - \tilde{d}(t,x,v)\frac{\tilde{v} S^x}{\sqrt{1+|v|^2}} + \sum_{i=1,2,3} \frac{\tilde{V}_i}{|v|}\tilde{\Omega}_i - \frac{\tilde{V}_i}{|v|}(X_i + \hat{V}_i t) \\
&\quad \cdot\big(\tilde{v} S^x + \sum_{j=1,2,3} \tilde{V}_j\Omega_j^x\big) \\
&= \tilde{v}\widehat{S}^v - \tilde{d}(t,x,v)\frac{\tilde{v} S^x}{\sqrt{1+|v|^2}} + \sum_{i=1,2,3} \frac{\tilde{V}_i}{|v|}\tilde{\Omega}_i - \frac{\tilde{V}_i}{|v|}\big[X_i\cdot\tilde{v} S^x \\
&\quad + \sum_{j=1,2,3} (X_i + \hat{V}_i t)\cdot\tilde{V}_j\Omega_j^x\big].
\end{aligned}$$

Hence, finishing the proof of (3.21). □

Remark 3.1. Because the coefficients of Ω_i^x in the first decomposition (3.20) are of size "$(1+|v|)|\tilde{d}(t,x,v)|$", we use it when "$|v|$" is relatively small and use the second decomposition (3.21) when "$|v|$" is relatively large.

Thanks to the coefficient $1/\sqrt{1+|v|^2}$ in the Vlasov–Nordström system, see (1.1), the first decomposition of "D_v" in (3.20) is sufficient for the Vlasov–Nordström system.

Motivated from the two decompositions in the above Lemma, we define the following set of vector fields, which act on *the profile* $g(t,x,v)$,

$$\mathfrak{P}_2 := \{\Gamma_i, \quad i\in\{1,\ldots,17\}\}, \tag{3.25}$$

where

$$\begin{aligned}
\Gamma_1 &= \psi_{\geq 1}(v)\widehat{S}^v, \\
\Gamma_2 &:= \psi_{\geq 1}(v) S^x, \\
\Gamma_{i+2} &:= \psi_{\geq 1}(v)\widehat{\Omega}_i^v, \\
\Gamma_{i+5} &:= \psi_{\geq 1}(v)\Omega_i^x,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}\Gamma_{i+8} &:= \psi_{\leq 0}(v) K_{v_i},\\ \Gamma_{i+11} &:= \psi_{\leq 0}(v)\partial_{x_i},\\ \Gamma_{i+14} &:= \tilde{\Omega}_i, \quad i = 1, 2, 3. \end{aligned} \tag{3.27}$$

Correspondingly, we define the following associated set of vector fields which act on the original distribution function $f(t, x, v)$,

$$\mathcal{P}_2 := \{\widehat{\Gamma}_i, \quad i \in \{1, \ldots, 17\}\}, \tag{3.28}$$

where

$$\begin{aligned}\widehat{\Gamma}_1 &= \psi_{\geq 1}(v)\tilde{v}\cdot \widetilde{K}_v,\\ \widehat{\Gamma}_2 &:= \psi_{\geq 1}(v) S^x,\\ \Gamma_{i+2} &:= \psi_{\geq 1}(v)\tilde{V}_i \cdot \widetilde{K}_v,\\ \widehat{\Gamma}_{i+5} &:= \psi_{\geq 1}(v)\Omega_i^x, \qquad (3.29)\\ \widehat{\Gamma}_{i+8} &:= \psi_{\leq 0}(v)\widetilde{K}_{v_i},\\ \widehat{\Gamma}_{i+11} &:= \psi_{\leq 0}(v)\partial_{x_i},\\ \widehat{\Gamma}_{i+14} &:= \tilde{\Omega}_i, \quad i = 1, 2, 3. \end{aligned} \tag{3.30}$$

For the convenience of notation, we don't distinguish these two sets of vector fields ($\mathfrak{P}_2$ and $\mathcal{P}_2$) if there is no confusion. For simplicity, we define a set of notations to represent the above defined vector fields uniformly.

Definition 3.2. For any vectors $e = (e_1, \ldots, e_n) \in \mathbb{R}^n$ and $f = (f_1, \ldots, f_m) \in \mathbb{R}^m$, where $e_1, \ldots, e_n, f_1, \ldots, f_m \in \mathbb{R}$, we define

$$e \circ f := (e_1, \ldots, e_n, f_1, \ldots, f_m), \quad |e| := \sum_{i=1,\ldots,n} |e_i|, \quad \Longrightarrow |e\circ f| = |e| + |f|.$$

Let

$$\begin{aligned}&\mathcal{A} := \{\vec{a} : \vec{a} \in \{0,1\}^{10}, |\vec{a}| = 0, 1\}, \quad \vec{0} := (0, \ldots, 0),\\ &\vec{a}_i := (0, \ldots, \underbrace{1}_{i\text{-th}}, \ldots, 0), \quad if\, \vec{0}, \vec{a}_i \in \mathcal{A}, \quad \mathcal{B} := \cup_{k\in\mathbb{N}_+}\mathcal{A}^k.\\ &\Gamma^{\vec{0}} := Id, \quad \Gamma^{\vec{a}_1} := S, \quad \Gamma^{\vec{a}_{i+1}} := \partial_{x_i},\\ &\Gamma^{\vec{a}_{i+4}} := \Omega_i, \quad \Gamma^{\vec{a}_{i+7}} := L_i, \quad i = 1, 2, 3, \end{aligned} \tag{3.31}$$

$$\begin{aligned}&\tilde{\Gamma}^{\vec{0}} := Id, \quad \tilde{\Gamma}^{\vec{a}_1} := S,\\ &\tilde{\Gamma}^{\vec{a}_{i+1}} := \partial_{x_i}, \quad \tilde{\Gamma}^{\vec{a}_{i+4}} := \tilde{\Omega}_i,\\ &\tilde{\Gamma}^{\vec{a}_{i+7}} := \tilde{L}_i, \quad i = 1, 2, 3. \end{aligned} \tag{3.32}$$

We represent the high order derivatives of the first set of vector field $\mathfrak{P}_1$ and $\mathcal{P}_1$ (see (3.6) and (3.5)) through composition as follows,

$$\Gamma^{\alpha_1\circ\alpha_2} := \Gamma^{\alpha_1}\Gamma^{\alpha_2}, \quad \tilde{\Gamma}^{\alpha_1\circ\alpha_2} := \tilde{\Gamma}^{\alpha_1}\tilde{\Gamma}^{\alpha_2} \quad \alpha_1, \alpha_2 \in \mathcal{B}. \tag{3.33}$$

Definition 3.3. We define

$$
\begin{aligned}
&\mathcal{K} := \{\vec{e}:\ \vec{e}\in\{0,1\}^{17}, |\vec{e}|=0,1\}, \quad \vec{0} := (0,\ldots,0), \vec{e}_i \\
&\quad := (0,\ldots,\underbrace{1}_{i\text{-th}},\ldots,0), \quad if\ \vec{0},\vec{e}_i\in\mathcal{K}, \\
&\mathcal{S} := \cup_{k\in\mathbb{N}_+}\mathcal{K}^k, \quad \Lambda^{\vec{0}} := Id, \\
&\widehat{\Gamma}^{\vec{0}} := Id, \quad \Lambda^{\vec{e}_i} := \Gamma_i, \\
&\widehat{\Gamma}^{\vec{e}_i} := \widehat{\Gamma}_i, \quad \Gamma_i\in\mathfrak{P}_2, \widehat{\Gamma}_i\in\mathcal{P}_2, \quad \vec{e}_i\in\mathcal{K},
\end{aligned}
$$

where "$\mathcal{P}_2$" is defined in (3.28) and "$\mathfrak{P}_2$" is defined in (3.25). Hence, we can represent the high order derivatives of the second set of vector fields for the profile "$g(t,x,v)$" through the composition defined as follows,

$$
\Lambda^{e\circ f} := \Lambda^e\Lambda^f, \quad \widehat{\Gamma}^{e\circ f} := \widehat{\Gamma}^e\widehat{\Gamma}^f, \quad e,f\in\mathcal{S}.
$$

Definition 3.4. For any $\kappa,\gamma\in\mathcal{S}$, we define the equivalence relation between "κ" and "γ" as follows,

$$
\begin{aligned}
&\kappa\sim\gamma \text{ and } \Lambda^\kappa\sim\Lambda^\gamma, \\
&\quad \text{if and only if } \Lambda^\kappa h(x,v) = \Lambda^\gamma h(x,v) \text{ for any differentiable function } h(x,v),
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
&\kappa\nsim\gamma \text{ and } \Lambda^\kappa\nsim\Lambda^\gamma, \text{ if and only if} \\
&\quad \Lambda^\kappa h(x,v)\neq\Lambda^\gamma h(x,v) \text{ for some differentiable function } h(x,v).
\end{aligned} \tag{3.35}
$$

Very similarly, we can define the corresponding equivalence relation for $\alpha_1,\alpha_2\in\mathcal{B}$.

Note that, for any $\beta\in\mathcal{S}$ and $\alpha\in\mathcal{B}$, there exists a unique expansion such that

$$
\beta\sim\iota_1\circ\cdots\iota_{|\beta|}, \quad \iota_i\in\mathcal{K}, |\iota_i|=1, \quad i\in\{1,\ldots,|\beta|\}, \tag{3.36}
$$

$$
\alpha\sim\gamma_1\circ\cdots\gamma_{|\alpha|}, \quad \gamma_i\in\mathcal{A}, |\gamma_i|=1, \quad i\in\{1,\ldots,|\alpha|\}. \tag{3.37}
$$

With the above notation, without the complexity caused the constant coefficients, we can represent the Leibniz rule by the equality as follows,

$$
\Lambda^\beta(fg) = \sum_{\beta_1,\beta_2\in\mathcal{S},\beta_1+\beta_2=\beta}\Lambda^{\beta_1}f\Lambda^{\beta_2}g, \quad \beta\in\mathcal{S}, \tag{3.38}
$$

where f and g are two smooth functions.

To capture the effect of different sizes of coefficients in the decompositions of "D_v" in (3.20) and (3.21) and effect of good efficients in the Vlasov–Nordström system, we define an index "$c_{\mathrm{vn}}(\iota)$" for the Vlasov–Nordström system, which classifies derivatives in $\mathfrak{P}_2$ into good derivatives, bad derivatives, and ordinary derivatives by setting $c_{\mathrm{vn}}(\iota)$ to 1, -1, and 0 respectively.

As a comparison, we also define the corresponding index "$c_{\mathrm{vm}}(\iota)$" for the Vlasov–Maxwell system. For simplicity, we will not redo similar computations and will use part of results obtained in this paper directly in [40]. Hence, related results will be formulated in both indexes.

Definition 3.5. For any $\iota \in \mathcal{K}$, we define an index "$c_{\mathrm{vn}}(\iota)$" for the Vlasov–Nordström system (1.1) and indexes "$c_{\mathrm{vm}}(\iota)$" for the Vlasov–Maxwell system (1.2) as follow,

$$c_{\mathrm{vn}}(\iota) = \begin{cases} 1 & \text{if } \Lambda^\iota \sim \widehat{S}^v \\ -1 & \text{if } \Lambda^\iota \sim \widehat{\Omega}_i, i \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases},$$

$$c_{\mathrm{vm}}(\iota) = \begin{cases} 1 & \text{if } \Lambda^\iota \sim \widehat{S}^v \text{or } \Omega_i^x, i \in \{1,2,3\} \\ \\ 0 & \text{otherwise} \end{cases}. \tag{3.39}$$

Moreover, for any $\beta \in \mathcal{S}$, we have $\beta \sim \iota_1 \circ \cdots \iota_{|\beta|}$, $\iota_i \in \mathcal{K}/\{\vec{0}\}$. We define

$$c_{\mathrm{vn}}(\beta) = \sum_{i=1,\ldots,|\beta|} c_{\mathrm{vn}}(\iota_i), \quad c_{\mathrm{vm}}(\beta) = \sum_{i=1,\ldots,|\beta|} c_{\mathrm{vm}}(\iota_i). \tag{3.40}$$

With the above defined notation, we can reformulate the results in Lemma 3.3 systematically as follows.

Lemma 3.4. *The following two decompositions for "D_v" holds,*

$$D_v = \sum_{\rho \in \mathcal{K}, |\rho|=1} d_\rho(t,x,v)\Lambda^\rho = \sum_{\rho \in \mathcal{K}, |\rho|=1} e_\rho(t,x,v)\Lambda^\rho, \tag{3.41}$$

where the detailed formulas of coefficients $d_\rho(t,x,v)$ and $e_\rho(t,x,v)$ are given as follow,

$$d_\rho(t,x,v) = \begin{cases} \tilde{v}\psi_{\geq -1}(v) & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v)\widehat{S}^v \\ \tilde{v}\tilde{d}(t,x,v)(1+|v|^2)^{-1/2}\psi_{\geq -1}(v) & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v) S^x \\ \tilde{V}_i\psi_{\geq -1}(v) & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v)\widehat{\Omega}_i^v, i=1,2,3 \\ \tilde{V}_i\tilde{d}(t,x,v)(1+|v|^2)^{1/2}\psi_{\geq -1}(v) & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v)\Omega_i^x, i=1,2,3 \\ \psi_{\leq 2}(v) & \textit{if } \Lambda^\rho \sim \psi_{\leq 0}(v) K_{v_i}, i=1,2,3 \\ -\tilde{d}(t,x,v)(1+|v|^2)\nabla_v \hat{v}_i \psi_{\leq 2}(v) & \textit{if } \Lambda^\rho \sim \psi_{\leq 0}(v)\partial_{x_i}, i=1,2,3 \\ 0 & \textit{if } \Lambda^\rho \sim \tilde{\Omega}_i, i=1,2,3 \end{cases}, \tag{3.42}$$

$$e_\rho(t,x,v) = \begin{cases} \tilde{v}\psi_{\geq -1}(v) & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v)\widehat{S}^v \\ -\psi_{\geq -1}(v)\Big(\dfrac{\tilde{d}(t,x,v)\tilde{v}}{(1+|v|^2)^{1/2}} + \dfrac{\tilde{V}_i(X_i\cdot\tilde{v})}{|v|}\Big) & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v) S^x \\ 0 & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v)\widehat{\Omega}_i^v, i=1,2,3 \\ -\psi_{\geq -1}(v)|v|^{-1}\tilde{V}_j(X_j+\hat{V}_j t)\cdot\tilde{V}_i & \textit{if } \Lambda^\rho \sim \psi_{\geq 1}(v)\Omega_i^x, i=1,2,3 \\ \psi_{\leq 2}(v) & \textit{if } \Lambda^\rho \sim \psi_{\leq 0}(v) K_{v_i}, i=1,2,3 \\ -\psi_{\leq 2}(v)\tilde{d}(t,x,v)(1+|v|^2)\nabla_v \hat{v}_i & \textit{if } \Lambda^\rho \sim \psi_{\leq 0}(v)\partial_{x_i}, i=1,2,3 \\ \psi_{\geq -1}(v)|v|^{-1}\tilde{V}_i & \textit{if } \Lambda^\rho \sim \tilde{\Omega}_i, i=1,2,3 \end{cases}. \tag{3.43}$$

Moreover, the following estimates hold,

$$\sum_{\rho\in\mathcal{K},|\rho|=1} \big|(1+|v|)^{-1-c_{vn}(\rho)} d_\rho(t,x,v)\big| + \big|(1+|v|)^{1-c_{vn}(\rho)}\tilde{v}\cdot d_\rho(t,x,v)\big| + \big|(1+|v|)^{-c_{vm}(\rho)} d_\rho(t,x,v)\big| \lesssim 1+|\tilde{d}(t,x,v)|. \tag{3.44}$$

Proof. The above results follow directly from the results in Lemma 3.3, which will be used for the case when $|v|\gtrsim 1$ and the validity of the following decomposition, which will be used for the case when $|v|\lesssim 1$,

$$D_v = K_v - (1+|v|^2)\tilde{d}(t,x,v)\nabla_v(\hat{v}\cdot\nabla_x).$$

Recall the detailed formulas of $d_\rho(t,x,v)$ in (3.42) and the definition of indexes $c_{\mathrm{vn}}(\iota)$ and $c_{\mathrm{vm}}(\iota)$ in (3.39). Our desired estimate (3.44) holds from straightforward computations. □

3.2. Commutation rules. In this subsection, we mainly obtain two types of commutation rules.

The first type of commutation rules apply to the commutation between the classic vector fields associated with the wave equation in $\mathfrak{P}_1$ (see (3.6)), and general Fourier multiplier operators. This type of commutation rules appears when we try to prove that the scalar field ϕ not only has sharp decay rate over time but also has extra $1/(1+||t|-|x||)$ decay rate with respect to the light cone.

The second type of commutation rules apply to the commutation between the vector fields defined in previous subsection and the derivative "D_v". This type of commutation rules appears when we do high order energy estimate for the profile $g(t,x,v)$ of the Vlasov part.

Definition 3.6. For any linear operator T and any $\alpha\in\mathcal{B}$, $|\alpha|=1$, we use the following notation to denote the commutator between T and $\Gamma^\alpha\in\mathfrak{P}_1$,

$$T_\alpha := (T)_\alpha := [\Gamma^\alpha, T]. \tag{3.45}$$

We use "T_k " to denote $T\circ P_k$, where $k\in\mathbb{Z}$.

We explicitly compute the commutator "T_α" if "T" is a Fourier multiplier operator as follows.

Lemma 3.5. *For any Fourier multiplier operator T, which has a Fourier multiplier $m(\xi)$, we have*

$$T_\alpha = \widetilde{T}_\alpha\circ\hat{T}_\alpha, \tag{3.46}$$

where

$$\hat{T}_\alpha = \begin{cases} \partial_t & \text{if } \Gamma^\alpha\in\{L_1,L_2,L_3\} \\ Id & \text{otherwise}, \end{cases},$$

$$\widetilde{T}_\alpha = \begin{cases} -\mathcal{F}^{-1}\circ\big(\xi\cdot\nabla_\xi m(\xi)\big)\circ\mathcal{F} & \text{if } \Gamma^\alpha = S, \\ 0 & \text{if } \Gamma^\alpha=\partial_{x_i}, i=1,2,3, \\ \mathcal{F}^{-1}\circ\big((e_i\times\xi)\cdot\nabla_\xi m(\xi)\big)\circ\mathcal{F} & \text{if } \Gamma^\alpha=\Omega_i, i=1,2,3, \\ \mathcal{F}^{-1}\circ\big(i\partial_{\xi_i} m(\xi)\big)\circ\mathcal{F} & \text{if } \Gamma^\alpha = L_i, i=1,2,3. \end{cases} \tag{3.47}$$

Proof. Since ∂_{x_i} commutes with T, it is easy to see that the commutator is zero for this case. Now, we consider the case when $\Gamma^\alpha = S$. Note that the following equality holds on the Fourier side,

$$\begin{aligned}\mathcal{F}[[S,T]f](t,\xi) &= -\big(3+\xi\cdot\nabla_\xi\big)\big(m(\xi)\widehat{f}(t,\xi)\big)\\ &\quad + m(\xi)\big(3+\xi\cdot\nabla_\xi\big)\widehat{f}(t,\xi)\\ &= -\big(\xi\cdot\nabla_\xi m(\xi)\big)\widehat{f}(t,\xi).\end{aligned}$$

If $\Gamma^\alpha = \Omega_i, i \in \{1,2,3\}$, we know that the following equality holds on the Fourier side,

$$\begin{aligned}\mathcal{F}[[\Omega_i,T]f](t,\xi) &= (e_i\times\xi)\cdot\nabla_\xi\big(m(\xi)\widehat{f}(t,\xi)\big)\\ &\quad - m(\xi)(e_i\times\xi)\cdot\nabla_\xi\widehat{f}(t,\xi)\\ &= \big((e_i\times\xi)\cdot\nabla_\xi m(\xi)\big)\widehat{f}(t,\xi).\end{aligned}$$

Lastly, we consider the case when $\Gamma^\alpha = L_i, i \in \{1,2,3\}$. For this case, we have

$$\begin{aligned}\mathcal{F}[[L_i,T]f](t,\xi) &= \mathcal{F}[x_i T(\partial_t f) - T(x_i\partial_t f)](t,\xi)\\ &= i\partial_{\xi_i}\big(m(\xi)\widehat{\partial_t f}(t,\xi)\big) - im(\xi)\partial_{\xi_i}\widehat{\partial_t f}(t,\xi) = i\partial_{\xi_i}m(\xi)\widehat{\partial_t f}(t,\xi).\end{aligned}$$

Hence finishing the proof. □

Recall the equality (3.7) in Lemma 3.1. We use the following notation to represent it systematically,

$$\begin{aligned}(|t|^2-|x|^2)\partial_i &= \sum_{\alpha\in\mathcal{B},|\alpha|=1} c_{\alpha,i}(t,x)\Gamma^\alpha,\ i\in\{1,2,3\},\\ c_\alpha(t,x) &= (c_{\alpha,1}(t,x), c_{\alpha,2}(t,x), c_{\alpha,3}(t,x)),\end{aligned} \tag{3.48}$$

where "$c_\alpha(t,x)$" denotes the unique determined vector coefficient associated with Γ^α in (3.7). Moreover, we have

$$\sum_{\alpha\in\mathcal{B},|\alpha|=1} |c_\alpha(t,x)| + |t\partial_t c_\alpha(t,x)| \lesssim (|t|+|x|), \quad \sum_{\alpha\in\mathcal{B},|\alpha|=1} |\nabla_x c_\alpha(t,x)| \lesssim 1. \tag{3.49}$$

With the above notation and the commutation rules in Lemmas 3.5, in 3.6, we prove a Fourier version of the equality (3.7) in Lemma 3.1. It enables us to prove that the scalar field decays at rate $1/((1+|t|+|x|)(1+||t|-|x||))$, see Lemma 6.3 for more details.

Lemma 3.6. *For any given Fourier multiplier operator T with Fourier symbol $m(\xi)$, the following equalities hold for any $k \in \mathbb{Z}$,*

$$\begin{aligned}&(|t|^2-|x|^2)^3 T_k[f](t,x)\\ &\quad = \sum_{i=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 3} \tilde{c}^i_\alpha(t,x)\tilde{T}^i_{k,\alpha}(\partial_t^i f^\alpha) + (|t|^2-|x|^2)e_\alpha(t,x)\tilde{T}^3_{k,\alpha}((\partial_t^2-\Delta)f),\\ &\quad (|t|^2-|x|^2)^3 T_k[\partial_t f](t,x) = \sum_{i=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 3}\end{aligned} \tag{3.50}$$

$$\begin{aligned}&\mathfrak{c}^i_\alpha(t,x)\widehat{T}^i_{k,\alpha}(\partial_t^i f^\alpha)(t,x) + (|t|^2-|x|^2)\mathfrak{e}^1_\alpha(t,x)\\ &\quad\times \widehat{T}^3_{k,\alpha}((\partial_t^2-\Delta)f^\alpha)(t,x) + (|t|^2-|x|^2)^2\mathfrak{e}^2_\alpha(t,x)\widehat{T}^4_{k,\alpha}((\partial_t^2-\Delta)f^\alpha)(t,x),\end{aligned} \tag{3.51}$$

where the coefficients $\tilde{c}^i_\alpha(t,x)$, $\mathfrak{c}^i_\alpha(t,x)$, $i=0,1,2$, $e_\alpha(t,x)$, $\mathfrak{e}^1_\alpha(t,x)$, *and* $\mathfrak{e}^2_\alpha(t,x)$ *satisfy the estimates in* (3.59), (3.60), (3.63), (3.64), *and* (3.65).

Moreover, the symbols $\tilde{m}^i_{k,\alpha}(\xi)$ *of the Fourier multiplier operators "*$\tilde{T}^i_{k,\alpha}(\cdot)$*",* $i\in\{0,1,2,3\}$, *and the symbols* $\widehat{m}^i_{k,\alpha}(\xi)$ *of the Fourier multiplier operators* $\widehat{T}^i_{k,\alpha}(\cdot)$ *satisfy the estimates in* (3.58) *and* (3.62) *respectively.*

Proof. Recall the definition of commutator in (3.45) and the definition of operator "Q" in (2.8). From the equality (3.48), we have

$$\begin{aligned}(|t|^2-|x|^2)T_k[f](t,x) &= (|t|^2-|x|^2)\nabla_x\cdot Q\circ T_k(f)(t,x)\\ &= \sum_{\alpha\in\mathcal{B},|\alpha|=1} c_\alpha(t,x)\cdot\Gamma^\alpha Q\circ T_k(f)(t,x)\\ &= \sum_{\alpha\in\mathcal{B},|\alpha|=1} c_\alpha(t,x)\\ &\quad\cdot\big[Q\circ T_k(\Gamma^\alpha f)(t,x)+\big(Q\circ T_k\big)_\alpha(f)(t,x)\big]. \end{aligned}\tag{3.52}$$

Very similarly, after doing this process one more time, we have

$$\begin{aligned}&(|t|^2-|x|^2)^2T_k[f](t,x)\\ &= \sum_{\alpha\in\mathcal{B},|\alpha|=1} c_\alpha(t,x)\cdot\Big[(|t|^2-|x|^2)\nabla\cdot Q\circ\big[Q\circ T_k(\Gamma^\alpha f)\\ &\quad+\big(Q\circ T_k\big)_\alpha(f)\big](t,x)\Big]\\ &= \sum_{\alpha_1,\alpha_2\in\mathcal{B},|\alpha_1|=|\alpha_2|=1} c_{\alpha_1}(t,x)\cdot\Big[c_{\alpha_2}(t,x)\cdot\big[Q\circ Q\circ T_k(\Gamma^{\alpha_2\circ\alpha_1}f)\\ &\quad+Q\circ(Q\circ T_k)_{\alpha_1}(\Gamma^{\alpha_2}f)\\ &\quad+(Q\circ Q\circ T_k)_{\alpha_2}(\Gamma^{\alpha_1}f)\\ &\quad+(Q\circ(Q\circ T_k)_{\alpha_1})_{\alpha_2}(f)\big](t,x). \end{aligned}\tag{3.53}$$

Note that the following commutation rule holds for any linear Fourier multiplier operator K,

$$\begin{aligned}&[\Gamma^\alpha, K\circ\partial_t]=K_\alpha\circ\partial_t+K[\Gamma^\alpha,\partial_t],\\ &[\Gamma^\alpha,\partial_t]=\begin{cases}-\partial_t & \text{if } \Gamma^\alpha=S\\ -\partial_{x_i} & \text{if } \Gamma^\alpha=L_i,\ i\in\{1,2,3\}\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{3.54}$$

Therefore, from the equality in (3.53), we have

$$(|t|^2-|x|^2)^2T_k[f](t,x)=\sum_{\alpha\in\mathcal{B},|\alpha|\le 2}\sum_{i=0,1,2}\hat{c}^i_\alpha(t,x)T^i_{k,\alpha}(\partial_t^i f^\alpha),\tag{3.55}$$

where the symbol $m^i_{k,\alpha}(\xi)$ of the Fourier multiplier $T^i_{k,\alpha}$ and the coefficient $\hat{c}^i_\alpha(t,x)$ satisfy the following estimate,

$$\sum_{\alpha\in\mathcal{B},|\alpha|\le 2}\sum_{i=0,1,2}2^{ik}\|m^i_{k,\alpha}(\xi)\|_{\mathcal{S}^\infty_k}\lesssim 2^{-2k},$$

$$\sum_{\alpha\in\mathcal{B},|\alpha|\leq 2} |\hat{c}_\alpha^i(t,x)| + (|t|+|x|)|\nabla_x \hat{c}_\alpha^i(t,x)| \lesssim (|t|+|x|)^2. \tag{3.56}$$

Lastly, we do this process one more time. Note that the commutator contains the time derivative "∂_t" when $\Gamma^\alpha = L_i$, $i \in \{1,2,3\}$. For simplicity, we don't want to introduce an operator with a third order time derivative "∂_t^3". Hence, we appeal to the nonlinear wave equation itself for the second order time derivative. More precisely, from (3.55), we have

$$\begin{aligned}
(|t|^2-|x|^2)^3 T_k[f](t,x) = &\sum_{\alpha\in\mathcal{B},|\alpha|\leq 2}\sum_{i=0,1} \hat{c}_\alpha^i(t,x)(|t|^2-|x|^2)\nabla_x \\
&\cdot Q\circ T_{k,\alpha}^i(\partial_t^i f^\alpha) \\
&+\hat{c}_\alpha^2(t,x)(|t|^2-|x|^2)\nabla_x\cdot\nabla_x T_{k,\alpha}^2(f^\alpha) \\
&+(|t|^2-|x|^2)\hat{c}_\alpha^2(t,x)T_{k,\alpha}^2((\partial_t^2-\Delta)f^\alpha), \\
= &\sum_{\alpha\in\mathcal{B},|\alpha|\leq 2}\sum_{\rho\in\mathcal{B},|\rho|=1}\sum_{i=0,1} \hat{c}_\alpha^i(t,x)c_\rho(t,x) \\
&\cdot\big(Q\circ T_{k,\alpha}^i(\Gamma^\rho\partial_t^i f^\alpha) + (Q\circ T_{k,\alpha}^i)_\rho(\partial_t^i f^\alpha)\big) \\
&+\hat{c}_\alpha^2(t,x)c_\rho(t,x)\cdot\big[(\nabla_x T_{k,\alpha}^2)(\Gamma^\rho f^\alpha) + (\nabla_x T_{k,\alpha}^2)_\rho(f^\alpha)\big] \\
&+(|t|^2-|x|^2)\hat{c}_\alpha^2(t,x)T_{k,\alpha}^2((\partial_t^2-\Delta)f^\alpha).
\end{aligned} \tag{3.57}$$

From the results in Lemma 3.5 and the equalities in (3.54) and (3.57), the following equality holds for some uniquely determined coefficients $\tilde{c}_\alpha^i(t,x)$ and $e_\alpha(t,x)$,

$$\begin{aligned}
(|t|^2-|x|^2)^3 T_k[f](t,x) = &\sum_{i=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 3} \\
&\tilde{c}_\alpha^i(t,x)\tilde{T}_{k,\alpha}^i(\partial_t^i f^\alpha)(t,x) \\
&+(|t|^2-|x|^2)e_\alpha(t,x)\tilde{T}_{k,\alpha}^3((\partial_t^2-\Delta)f)(t,x),
\end{aligned}$$

where the following estimate holds for the symbol $\tilde{m}_{k,\alpha}^i(\xi)$ of the Fourier multiplier operator "$\tilde{T}_{k,\alpha}^i(\cdot)$",

$$\sum_{i=0,1,2} 2^{ik}\|\tilde{m}_{k,\alpha}^i(\xi)\|_{\mathcal{S}_k^\infty} \lesssim 2^{-3k}, \quad \|\tilde{m}_{k,\alpha}^3(\xi)\|_{\mathcal{S}_k^\infty} \lesssim 2^{-4k}. \tag{3.58}$$

Moreover, for $i\in\{0,1,2\}$, $\alpha\in\mathcal{B}$, $|\alpha|\leq 3$, from the estimate (3.49), the following estimates hold for the coefficients $\tilde{c}_\alpha^i(t,x)$ and $e_\alpha(t,x)$,

$$|\tilde{c}_\alpha^i(t,x)| + |t\partial_t\tilde{c}_\alpha^i(t,x)| \lesssim (|t|+|x|)^3, \quad |e_\alpha(t,x)| + |t\partial_t e_\alpha(t,x)| \lesssim (|t|+|x|)^2, \tag{3.59}$$

$$|\nabla_x\tilde{c}_\alpha^i(t,x)| \lesssim (|t|+|x|)^2, \quad |\nabla_x e_\alpha(t,x)| \lesssim (|t|+|x|). \tag{3.60}$$

With minor modifications, we can derive the following equality

$$(|t|^2-|x|^2)^3 T_k[\partial_t f](t,x) = \sum_{i=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 3} \mathfrak{c}_\alpha^i(t,x)\widehat{T}_{k,\alpha}^i(\partial_t^i f^\alpha)(t,x)$$

$$+(|t|^2-|x|^2)\mathfrak{e}^1_\alpha(t,x)\widehat{T}^3_{k,\alpha}((\partial_t^2-\Delta)f^\alpha)(t,x)$$
$$+(|t|^2-|x|^2)^2\mathfrak{e}^2_\alpha(t,x)\widehat{T}^4_{k,\alpha}((\partial_t^2-\Delta)f^\alpha)(t,x), \tag{3.61}$$

where the symbols $\widehat{m}^i_{k,\alpha}(\xi)$ of the Fourier multiplier operators $\widehat{T}^i_{k,\alpha}(\cdot)$, $i\in\{0,1,2,3,4\}$, satisfy the following estimates,

$$\sum_{i=0,1,2} 2^{ik}\|\widehat{m}^i_{k,\alpha}(\xi)\|_{\mathcal{S}^\infty_k}\lesssim 2^{-2k},$$
$$\|\widehat{m}^3_{k,\alpha}(\xi)\|_{\mathcal{S}^\infty_k}\lesssim 2^{-3k},$$
$$\|\widehat{m}^4_{k,\alpha}(\xi)\|_{\mathcal{S}^\infty_k}\lesssim 2^{-2k}. \tag{3.62}$$

Moreover, the uniquely determined coefficients $\mathfrak{c}^i_\alpha(t,x)$, $i\in\{0,1,2\}$, $\mathfrak{e}^1_\alpha(t,x)$, and $\mathfrak{e}^2_\alpha(t,x)$, satisfy the following estimates,

$$\sum_{i=0,1,2}|\mathfrak{c}^i_\alpha(t,x)|+|t\partial_t\mathfrak{c}^i_\alpha(t,x)|\lesssim(|t|+|x|)^3, \tag{3.63}$$

$$|\mathfrak{e}^1_\alpha(t,x)|+|t\partial_t\mathfrak{e}^1_\alpha(t,x)|\lesssim(|t|+|x|)^2,$$
$$|\mathfrak{e}^2_\alpha(t,x)|+|t\partial_t\mathfrak{e}^2_\alpha(t,x)|\lesssim(|t|+|x|), \tag{3.64}$$

$$\sum_{i=0,1,2}|\nabla_x\mathfrak{c}^i_\alpha(t,x)|\lesssim(|t|+|x|)^2,\quad |\nabla_x\mathfrak{e}^1_\alpha(t,x)|\lesssim|t|+|x|,$$
$$|\nabla_x\mathfrak{e}^2_\alpha(t,x)|\lesssim 1. \tag{3.65}$$

□

Now, we proceed to the second type of commutation rules. For simplicity, we define a set of vector fields as follows,

$$X_1 := \psi_{\geq 1}(v)\tilde{v}\cdot D_v,$$
$$X_{i+1} = \psi_{\geq 1}(v)\tilde{V}_i\cdot D_v,$$
$$X_{i+4} = \psi_{\leq 0}(v)D_{v_i},\quad i=1,2,3, \tag{3.66}$$

From (3.66), we have

$$D_v=\tilde{v}X_1+\tilde{V}_iX_{i+1}+e_iX_{i+4}:=\sum_{i=1,\dots 7}\alpha_i(v)X_i, \tag{3.67}$$

where

$$\alpha_1(v):=\psi_{\geq -1}(v)\tilde{v},\quad \alpha_{i+1}(v):=\psi_{\geq -1}(v)\tilde{V}_i,$$
$$\alpha_{i+4}(v):=\psi_{\leq 2}(v)e_i,\quad i=1,2,3. \tag{3.68}$$

As a basic step, we first consider the first order commutation rule, i.e., the case when $\rho\in\mathcal{K}$, $|\rho|=1$.

Lemma 3.7. *For any $\rho \in \mathcal{K}$, $|\rho| = 1$, and $i \in \{1, \ldots, 7\}$, the following commutation rules hold,*

$$[X_i, \Lambda^\rho] = \sum_{\kappa \in \mathcal{K}, |\kappa|=1} \tilde{d}^{\kappa}_{\rho,i}(t,x,v)\Lambda^\kappa, \quad \tilde{d}^{\kappa}_{\rho,i}(t,x,v) := \tilde{c}_i^{\rho,\kappa}(x,v)\tilde{d}(t,x,v) + \hat{c}_i^{\rho,\kappa}(x,v), \tag{3.69}$$

where the coefficients $\tilde{c}_i^{\rho,\kappa}(t,x,v)$ and $\hat{c}_i^{\rho,\kappa}(t,x,v)$ satisfy the following rough estimates,

$$|\tilde{c}_i^{\rho,\kappa}(x,v)| + |\hat{c}_i^{\rho,\kappa}(x,v)| \lesssim \min\{(1+|v|)^{1+c_{vn}(\kappa)-c_{vn}(\rho)}, (1+|v|)^{c_{vm}(\kappa)-c_{vm}(\rho)}\}, \tag{3.70}$$

$$|\Lambda^\beta\big(\tilde{c}_i^{\rho,\kappa}(x,v)\big)| + |\Lambda^\beta\big(\hat{c}_i^{\rho,\kappa}(x,v)\big)| \lesssim (1+|v|)^{|\beta|+2}(1+|x|)^{|\beta|+2}, \quad \beta \in \mathcal{S}. \tag{3.71}$$

In particular, for the case when $i = 1$, the following improved estimate holds,

$$|\tilde{c}_1^{\rho,\kappa}(x,v)| + |\hat{c}_1^{\rho,\kappa}(x,v)| \lesssim (1+|v|)^{-1+c_{vn}(\kappa)-c_{vn}(\rho)}. \tag{3.72}$$

Moreover, if $i(\kappa) - i(\rho) > 0$, where $i(\kappa)$ denotes the total number of vector fields Ω_i^x in Λ^κ, then the following improved estimate holds for the coefficients of the commutation rule in (3.69),

$$|\hat{c}_i^{\rho,\kappa}(x,v)| \lesssim (1+|v|)^{-1+c_{vm}(\kappa)-c_{vm}(\rho)}. \tag{3.73}$$

Proof. Postponed to Appendix A. See the proof of Lemma A.2 in Appendix A. □

In the process of commutation between high order vector fields and X_i, it is unavoidable that the vector field $\Lambda^\kappa, \kappa \in \mathcal{K}$, might act on the coefficients. From the equality (3.69) and the estimate (3.71) in Lemma 3.7, it would be sufficient to consider the case when Λ^κ hits the inhomogeneous modulation $\tilde{d}(t,x,v)$. For this case, we have the following Lemma.

Lemma 3.8. *For any $\rho \in \mathcal{K}$, $|\rho| = 1$, the following equality holds,*

$$\Lambda^\rho(\tilde{d}(t,x,v)) := \tilde{d}_\rho(x,v) = \tilde{d}(t,x,v)e_1^\rho(x,v) + e_2^\rho(x,v), \tag{3.74}$$

where the coefficients $b_\rho^1(t,x,v)$ and $b_\rho^2(t,x,v)$ are some explicit coefficients and satisfy the following estimate,

$$\sum_{i=1,2} |e_i^\rho(x,v)| \lesssim 1, \quad \sum_{i=1,2} |\Lambda^\beta(e_i^\rho(x,v))| \lesssim (1+|x|)^{|\beta|}(1+|v|)^{|\beta|}, \quad \beta \in \mathcal{S}. \tag{3.75}$$

Proof. Postponed to Appendix A. See the proof of Lemma A.1 in Appendix A. □

Now we are ready to introduce the high order commutation rules, which are basic tools to compute the equation satisfied by the high order derivatives of the profile "$g(t,x,v)$".

We will be very precise about the estimate of the top order coefficients, which matter very much in the energy estimate. However, the estimate of the lower order coefficients in (3.78), i.e., the case when $|\kappa| \le |\beta| - 1$, are rough because it's not necessary to be precise once we set a hierarchy for weight functions associated with the derivatives of the profile $g(t,x,v)$.

Lemma 3.9. *For any $i \in \{1, \ldots, 7\}$, and $\beta \in \mathcal{S}$, we have*

$$[X_i, \Lambda^\beta] = Y_i^\beta + \sum_{\kappa\in\mathcal{S}, |\kappa|\leq|\beta|-1} \big[\tilde{d}(t,x,v)\tilde{e}_{\beta,i}^{\kappa,1}(x,v) + \tilde{e}_{\beta,i}^{\kappa,2}(x,v)\big]\Lambda^\kappa, \tag{3.76}$$

where Y_i^β denotes the top order commutators and it is given as follows,

$$Y_i^\beta = \sum_{\kappa\in\mathcal{S}, |\kappa|=|\beta|, |i(\kappa)-i(\beta)|\leq 1} \big[\tilde{d}(t,x,v)\tilde{e}_{\beta,i}^{\kappa,1}(x,v) + \tilde{e}_{\beta,i}^{\kappa,2}(x,v)\big]\Lambda^\kappa, \tag{3.77}$$

where $i(\kappa)$ denotes the total number of vector fields Ω_i^x in Λ^κ.

For any $i \in \{1, \ldots, 7\}$, and $\kappa \in \mathcal{S}$, the following estimates hold for the coefficients $\tilde{e}_{\beta,i}^{\kappa,1}(x,v)$ and $\tilde{e}_{\beta,i}^{\kappa,2}(x,v)$,

$$|\tilde{e}_{\beta,i}^{\kappa,1}(x,v)| + |\tilde{e}_{\beta,i}^{\kappa,2}(x,v)| \lesssim (1+|x|)^{|\beta|-|\kappa|+2}(1+|v|)^{|\beta|-|\kappa|+4}, \quad \text{when } |\kappa| \leq |\beta|-1, \tag{3.78}$$

$$|\tilde{e}_{\beta,i}^{\kappa,1}(x,v)| + |\tilde{e}_{\beta,i}^{\kappa,2}(x,v)| \lesssim \min\{(1+|v|)^{1+c_{vn}(\kappa)-c_{vn}(\beta)}, (1+|v|)^{c_{vm}(\kappa)-c_{vm}(\beta)}\}, \text{ when } |\kappa| = |\beta|, \tag{3.79}$$

$$|\Lambda^\rho \tilde{e}_{\beta,i}^{\kappa,1}(x,v)| + |\Lambda^\rho \tilde{e}_{\beta,i}^{\kappa,2}(x,v)| \lesssim (1+|x|)^{|\rho|+|\beta|-|\kappa|+2}(1+|v|)^{|\rho|+|\beta|-|\kappa|+4}. \tag{3.80}$$

In particular, the following improved estimate holds if $i = 1$,

$$|\tilde{e}_{\beta,i}^{\kappa,1}(x,v)| + |\tilde{e}_{\beta,i}^{\kappa,2}(x,v)| \lesssim (1+|v|)^{-1+c_{vn}(\kappa)-c_{vn}(\beta)}, \quad \text{when } |\kappa| = |\beta|. \tag{3.81}$$

Moreover, if $i(\kappa) - i(\beta) > 0$ and $|\kappa| = |\beta|$, then the following improved estimate holds for the coefficients $\tilde{e}_{\beta,i}^{\kappa,2}(x,v)$ in (3.76),

$$|\tilde{e}_{\beta,i}^{\kappa,2}(x,v)| \lesssim (1+|v|)^{-1+c_{vm}(\kappa)-c_{vm}(\beta)}. \tag{3.82}$$

Proof. From the equality (3.69) in Lemma 3.7, we have

$$\begin{aligned}
[X_i, \Lambda^\beta] &= [X_i, \Lambda^{\iota_1}\cdots\Lambda^{\iota_{|\beta|-1}}]\Lambda^{\iota_{|\beta|}} + \Lambda^{\iota_1}\cdots\Lambda^{\iota_{|\beta|-1}}[X_i, \Lambda^{\iota_{|\beta|}}] \\
&= [X_i, \Lambda^{\iota_1}\cdots\Lambda^{\iota_{|\beta|-1}}]\Lambda^{\iota_{|\beta|}} + \Lambda^{\iota_1}\cdots\Lambda^{\iota_{|\beta|-1}}\Big(\sum_{\gamma_1\in\mathcal{K}, |\gamma_1|=1} \tilde{d}_{\iota_{|\beta|},i}^{\gamma_1}(t,x,v)\Lambda^{\gamma_1}\Big) \\
&= [X_i, \Lambda^{\iota_1}\cdots\Lambda^{\iota_{|\beta|-1}}]\Lambda^{\iota_{|\beta|}} \\
&\quad + \sum_{\kappa_1,\rho_1\in\mathcal{S}, \kappa_1+\rho_1=\iota_1\circ\cdots\circ\iota_{|\beta|-1}} \sum_{\gamma_1\in\mathcal{K}, |\gamma_1|=1} \big(\Lambda^{\kappa_1}(\tilde{d}_{\iota_{|\beta|},i}^{\gamma_1}(t,x,v))\Lambda^{\rho_1}\Lambda^{\gamma_1}\big).
\end{aligned}$$

By induction, from the above equality, we have

$$[X_i, \Lambda^\beta] = \sum_{j=2\cdots,|\beta|-1} \sum_{\kappa_j,\rho_j\in\mathcal{S}, |\kappa_j|\geq 1, \kappa_j+\rho_j=\iota_1\circ\cdots\circ\iota_{|\beta|-j}} \sum_{\gamma_j\in\mathcal{K}, |\gamma_j|=1} \Lambda^{\kappa_j}(\tilde{d}_{\iota_{|\beta|-j+1},i}^{\gamma_j}(t,x,v))\Lambda^{\rho_j\circ\gamma_j\circ\iota_{|\beta|-j+2}\circ\cdots\iota_{|\beta|}}$$

$$+ \sum_{\kappa_1,\rho_1\in\mathcal{S},|\kappa_1|\geq 1,\kappa_1+\rho_1=\iota_1\circ\cdots\circ\iota_{|\beta|-1}} \sum_{\gamma_1\in\mathcal{K},|\gamma_1|=1} \Lambda^{\kappa_1}(\tilde{d}^{\gamma_1}_{\iota_{|\beta|},i}(t,x,v))\Lambda^{\rho_1\circ\gamma_1} + Y_i^{\beta}, \tag{3.83}$$

where $\tilde{d}^{\gamma}_{\iota,i}(t,x,v)$ is defined in (3.69) and Y_i^{β} is defined as follows,

$$Y_i^{\beta} := \sum_{\gamma\in\mathcal{K},|\gamma|=1} \tilde{d}^{\gamma}_{\iota_1,i}(t,x,v)\Lambda^{\gamma\circ\iota_2\circ\cdots\circ\iota_{|\beta|}} + \tilde{d}^{\gamma}_{\iota_{|\beta|},i}(t,x,v)\Lambda^{\iota_1\circ\cdots\iota_{|\beta|-1}\circ\gamma}$$
$$+ \sum_{i=2\cdots,|\beta|-1} \tilde{d}^{\gamma}_{\iota_{|\beta|-i+1},i}(t,x,v)\Lambda^{\iota_1\circ\cdots\iota_{|\beta|-i}\circ\gamma\circ\iota_{|\beta|-i+2}\circ\cdots\iota_{|\beta|}}. \tag{3.84}$$

From (3.83) and (3.74), we have

$$[X_i,\Lambda^{\beta}] = \sum_{\substack{j=2\cdots,|\beta|-1\\ \kappa_j,\rho_j\in\mathcal{S},|\kappa_i|\geq 1\\ \kappa_j+\rho_j=\iota_1\circ\cdots\circ\iota_{|\beta|-i}}} \sum_{\gamma_j\in\mathcal{K},|\gamma_i|=1} \big(\tilde{d}(t,x,v)e^{\gamma_j,\kappa_j;1}_{\iota_{|\beta|-j+1},i}(x,v) + e^{\gamma_j,\kappa_j;2}_{\iota_{|\beta|-j+1},i}(x,v)\big)\Lambda^{\rho_j\circ\gamma_j\circ\iota_{|\beta|-j+2}\circ\cdots\iota_{|\beta|}}$$
$$+ \sum_{\kappa_1,\rho_1\in\mathcal{S},|\kappa_1|\geq 1,\kappa_1+\rho_1=\iota_1\circ\cdots\circ\iota_{|\beta|-1}} \sum_{\gamma_1\in\mathcal{K},|\gamma_1|=1} \big(\tilde{d}(t,x,v)e^{\gamma_1,\kappa_1;1}_{\iota_{|\beta|},i}(x,v)$$
$$+ e^{\gamma_1,\kappa_1;2}_{\iota_{|\beta|},i}(x,v)\big)\Lambda^{\rho_1\circ\gamma_1} + Y_i^{\beta}.$$

Hence, our desired equality (3.76) holds for some determined coefficients $\tilde{e}^{\kappa,1}_{\beta,k}(x,v)$ and $\tilde{e}^{\kappa,2}_{\beta,k}(x,v)$, whose explicit formulas are not pursued here.

Recall (3.84). Our desired equality (3.77) and desired estimates (3.79), (3.81), and (3.82) hold from the decomposition (3.69) and the estimates (3.70), (3.72), and (3.73) in Lemma 3.7. The desired estimate (3.78) follows from (3.69), (3.70), and (3.71) in Lemma 3.7 and (3.74) and (3.75) in Lemma 3.8. □

4. Set-Up of the Energy Estimate

4.1. The equation satisfied by the high order derivatives of the profile $g(t,x,v)$. In this subsection, our main goal is to compute the equation satisfied by the high order derivatives of the Vlasov–Nordström system.

Recall (1.1). For the sake of readers, we restate the equation satisfied by "$f(t,x,v)$" as follows,

$$\partial_t f + \hat{v}\cdot\nabla_x f = \big((\partial_t + \hat{v}\cdot\nabla_x)\phi(t,x)\big)(4f + v\cdot\nabla_v f) + \frac{1}{\sqrt{1+|v|^2}}\nabla_x\phi\cdot\nabla_v f$$

For any $\alpha\in\mathcal{B}$, we define

$$f^{\alpha}(t,x,v) := \tilde{\Gamma}^{\alpha} f(t,x,v), \quad \phi^{\beta}(t,x) := \Gamma^{\beta}\phi(t,x). \tag{4.1}$$

Note that the following equality holds,

$$\tilde{\Gamma}^{\alpha}\big((\partial_t + \hat{v}\cdot\nabla_v)f\big) = \sum_{\beta,\gamma\in\mathcal{B},\beta+\gamma=\alpha} \tilde{\Gamma}^{\beta}\big((\partial_t + \hat{v}\cdot\nabla_x)\phi(t,x)\big)\tilde{\Gamma}^{\gamma}(4f + v\cdot\nabla_v f)$$
$$+ \frac{1}{\sqrt{1+|v|^2}}\Gamma^{\beta}\big(\nabla_x\phi\big)\cdot\tilde{\Gamma}^{\gamma}\nabla_v f.$$

As a result of direct computations, the following commutation rules hold for any i, $j \in \{1,2,3\}$,

$$[\partial_{v_i}, S] = 0, \quad [\partial_{v_i}, \tilde{\Omega}_j] = \partial_{v_i}V_j\cdot\nabla_v, \quad [\partial_{v_i}, \tilde{L}_j] = \frac{v_i}{\sqrt{1+|v|^2}}\partial_{v_j}, \tag{4.2}$$

$$[\partial_{x_i}, S] = \partial_{x_i}, \quad [\partial_{x_i}, \tilde{\Omega}_j] = \partial_{x_i}X_j\cdot\nabla_x, \quad [\partial_{x_i}, \tilde{L}_j] = \delta_{ij}\partial_t, \tag{4.3}$$

$$[v\cdot\nabla_v, \tilde{\Omega}_j] = 0, \quad [v\cdot\nabla_v, S] = 0, \quad [v\cdot\nabla_v, \tilde{L}_j] = \frac{-1}{\sqrt{1+|v|^2}}\partial_{v_j}. \tag{4.4}$$

From the above commutation rules and the commutation rules in (3.3) and (3.4), we have

$$\begin{aligned}(\partial_t + \hat{v}\cdot\nabla_v)f^{\alpha} = &\sum_{\beta,\gamma\in\mathcal{B},|\beta|+|\gamma|\le|\alpha|} \big((\partial_t + \hat{v}\cdot\nabla_x)\phi^{\beta}(t,x)\big)\\ &\big(a^1_{\alpha;\beta,\gamma}(v)f^{\gamma} + a^2_{\alpha;\beta,\gamma}(v)v\cdot\nabla_v f^{\gamma}\big)\\ &+ \big(a^3_{\alpha;\beta,\gamma}(v)\nabla_x\phi^{\beta}(t,x) + a^4_{\alpha;\beta,\gamma}(v)\partial_t\phi^{\beta}(t,x)\big)\cdot\nabla_v f^{\gamma},\end{aligned} \tag{4.5}$$

where $a^i_{\alpha;\beta,\gamma}(v)$, $i\in\{1,2,3,4\}$, are some determined coefficients, whose explicit formulas are not pursued here. The following rough estimate holds for any $\beta,\gamma\in\mathcal{B}$, s.t., $|\beta|+|\gamma|\le|\alpha|$,

$$|a^1_{\alpha;\beta,\gamma}(v)| + |a^2_{\alpha;\beta,\gamma}(v)| + (1+|v|)\big(|a^3_{\alpha;\beta,\gamma}(v)| + |a^4_{\alpha;\beta,\gamma}(v)|\big) \lesssim 1. \tag{4.6}$$

Similar to the profiles defined in Sect. 2.2, we define the profile of $f^{\alpha}(t,x,v)$ as follows,

$$g^{\alpha}(t,x,v) := f^{\alpha}(t, x+\hat{v}t, v), \quad \Longrightarrow f^{\alpha}(t,x,v) = g^{\alpha}(t, x-\hat{v}t, v).$$

From (4.5), we can compute the equation satisfied by the profile $g^{\alpha}(t,x,v)$ as follows,

$$\begin{aligned}\partial_t g^{\alpha}(t,x,v) = &\sum_{\beta,\gamma\in\mathcal{B},|\beta|+|\gamma|\le|\alpha|} \big((\partial_t\phi^{\beta}\\ &+ \hat{v}\cdot\nabla_x\phi^{\beta})(t, x+\hat{v}t)\big)\\ &\big(a^1_{\alpha;\beta,\gamma}(v)g^{\gamma} + a^2_{\alpha;\beta,\gamma}(v)v\cdot D_v g^{\gamma}\big)(t,x,v)\\ &+ \big(a^3_{\alpha;\beta,\gamma}(v)\nabla_x\phi^{\beta}\\ &+ a^4_{\alpha;\beta,\gamma}(v)\partial_t\phi^{\beta}\big)(t, x+\hat{v}t)\cdot D_v g^{\gamma}(t,x,v),\end{aligned} \tag{4.7}$$

Now, we apply the second set of vector fields on $g^{\alpha}(t,x,v)$. For any $\beta\in\mathcal{S}$ and any $\alpha\in\mathcal{B}$, we define

$$g^{\alpha}_{\beta}(t,x,v) := \Lambda^{\beta}g^{\alpha}(t,x,v), \quad \beta \sim \iota_1\circ\iota_2\circ\cdots\circ\iota_{|\beta|},$$

$$\iota_i \in \mathcal{K}, |\iota_i| = 1, \quad i = 1, \ldots, |\beta|. \tag{4.8}$$

Note that $[\partial_t, \Lambda^\beta] = 0$. From (4.7) and (4.8), based on the order of derivatives, we classify the nonlinearities of $\partial_t g^\alpha_\beta(t, x, v)$ as follows,

$$\partial_t g^\alpha_\beta(t, x, v) = K(t, x + \hat{v}t, v) \cdot D_v g^\alpha_\beta(t, x, v) + h.o.t^\alpha_\beta(t, x, v) + l.o.t^\alpha_\beta(t, x, v), \tag{4.9}$$

where

$$K(t, x + \hat{v}t, v) := v\big(\partial_t\phi(t, x + \hat{v}t) + \hat{v} \cdot \nabla_x\phi(t, x + \hat{v}t)\big) + \frac{1}{\sqrt{1 + |v|^2}}\nabla_x\phi(t, x + \hat{v}t), \tag{4.10}$$

and "$h.o.t^\alpha_\beta(t, x, v)$" denotes all the terms in which the total number of derivatives act on $g(t, x, v)$ is "$|\alpha| + |\beta|$" and "$l.o.t^\alpha_\beta(t, x, v)$" denotes all the terms in which the total number of derivatives act on $g(t, x, v)$ is strictly less than "$|\alpha|+|\beta|$". We remind readers that the case of all vector fields act on the scalar field is included in "$l.o.t^\alpha_\beta(t, x, v)$".

For any $\alpha \in \mathcal{B}, \beta \in \mathcal{S}$, from the decomposition of D_v in (3.67), we decompose "$K(t, x + \hat{v}t, v) \cdot D_v g^\alpha_\beta(t, x, v)$" as follows,

$$K(t, x + \hat{v}t, v) \cdot D_v g^\alpha_\beta(t, x, v) = \sum_{i=1,\ldots,7} K_i(t, x + \hat{v}t, v) \cdot X_i g^\alpha_\beta(t, x, v), \tag{4.11}$$

where

$$K_1(t, x + \hat{v}t, v) = \psi_{\geq -1}(v)|v|\big(\partial_t\phi(t, x + \hat{v}t) + \hat{v} \cdot \nabla_x\phi(t, x + \hat{v}t)\big) + \frac{\alpha_1(v)}{\sqrt{1 + |v|^2}}\nabla_x\phi(t, x + \hat{v}t), \tag{4.12}$$

$$K_{i+1}(t, x + \hat{v}t, v) = \frac{\alpha_{i+1}(v)}{\sqrt{1 + |v|^2}}\nabla_x\phi(t, x + \hat{v}t), \quad i \in \{1, 2, 3\}, \tag{4.13}$$

$$K_{i+4}(t, x + \hat{v}t, v) = v_i\psi_{\leq 2}(v)\big(\partial_t\phi(t, x + \hat{v}t) + \hat{v} \cdot \nabla_x\phi(t, x + \hat{v}t)\big) + \frac{\alpha_{i+4}(v)}{\sqrt{1 + |v|^2}}\nabla_x\phi(t, x + \hat{v}t), \quad i \in \{1, 2, 3\}. \tag{4.14}$$

Based on the source of the high order terms, recall (3.67), we classify the high order terms $h.o.t^\alpha_\beta(t, x, v)$ as follows,

$$h.o.t^\alpha_\beta(t, x, v) = \sum_{i=1,2,3} h.o.t^\alpha_{\beta;i}(t, x, v) \tag{4.15}$$

where

$$h.o.t^\alpha_{\beta;1}(t, x, v) = \sum_{\iota+\kappa=\beta, |\iota|=1, \iota,\kappa\in\mathcal{S}, i=1,\ldots,7} \Lambda^\iota(K_i(t, x + \hat{v}t, v))X_i g^\alpha_\kappa(t, x, v), \tag{4.16}$$

$$h.o.t^\alpha_{\beta;2}(t, x, v) = \sum_{|\rho|\leq 1, |\gamma|=|\alpha|-1} a^2_{\alpha;\rho,\gamma}(v)\big(\partial_t\phi^\rho(t, x + \hat{v}t)$$

$$
\begin{aligned}
&+\hat{v}\cdot\nabla_x\phi^\rho(t,x+\hat{v}t)\big)v\cdot D_v g^\gamma_\beta(t,x,v)\big)+\big(a^3_{\alpha;\rho,\gamma}(v)\\
&\times\nabla_x\phi^\rho(t,x+\hat{v}t)+a^4_{\alpha;\rho,\gamma}(v)\partial_t\phi^\rho(t,x+\hat{v}t)\big)\cdot D_v g^\gamma_\beta(t,x,v)\\
&+4\big(\partial_t\phi(t,x+\hat{v}t)+\hat{v}\cdot\nabla_x\phi(t,x+\hat{v}t)\big)g^\alpha_\beta(t,x,v), \qquad (4.17)
\end{aligned}
$$

$$
h.o.t^\alpha_{\beta;3}(t,x,v)=\sum_{i=1,\dots,7}K_i(t,x+\hat{v}t,v)Y_i^\beta g^\alpha(t,x,v). \qquad (4.18)
$$

Similarly, we classify the low order terms "$l.o.t^\alpha_\beta(t,x,v)$" as follows,

$$
l.o.t^\alpha_\beta(t,x,v)=\sum_{i=1,\dots,4}l.o.t^\alpha_{\beta;i}(t,x,v), \qquad (4.19)
$$

where

$$
l.o.t^\alpha_{\beta;1}(t,x,v)=\sum_{i=1,\dots,7}K_i(t,x+\hat{v}t,v)\big([\Lambda^\beta,X_i]-Y_i^\beta\big)g^\alpha(t,x,v), \qquad (4.20)
$$

$$
\begin{aligned}
l.o.t^\alpha_{\beta;2}(t,x,v)=&\sum_{|\rho|\le 1,|\gamma|=|\alpha|-1}a^2_{\alpha;\rho,\gamma}(v)\big(\partial_t\phi^\rho(t,x+\hat{v}t)\\
&+\hat{v}\cdot\nabla_x\phi^\rho(t,x+\hat{v}t)\big)[\Lambda^\beta,v\cdot D_v]g^\gamma(t,x,v)\big)\\
&+\big(a^3_{\alpha;\rho,\gamma}(v)\nabla_x\phi^\rho+a^4_{\alpha;\rho,\gamma}(v)\partial_t\phi^\rho\big)(t,x+\hat{v}t)\\
&\cdot\alpha_i(v)[\Lambda^\beta,X_i]g^\gamma(t,x,v)\\
&+\sum_{\iota+\kappa=\beta,|\iota|=1,\iota,\kappa\in\mathcal{S},i=1,\dots,7}\\
&\Lambda^\iota(K_i(t,x+\hat{v}t,v))[\Lambda^\kappa,X_i]g^\alpha(t,x,v), \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
l.o.t^\alpha_{\beta;3}(t,x,v)=&\sum_{\substack{\iota,\kappa\in\mathcal{S},\beta_1,\gamma_1,\beta_2,\gamma_2\in\mathcal{B},\\|\rho|+|\beta_1|>11,|\beta_1|+|\gamma_1|\le|\alpha|\\|\rho|+|\beta_2|>11,|\beta_2|+|\gamma_2|\le|\alpha|}}\\
&\Lambda^\rho\Big((\partial_t\phi^{\beta_1}(t,x+\hat{v}t)+\hat{v}\cdot\nabla_x\phi^{\beta_1}(t,x+\hat{v}t))\Big)\Lambda^\kappa\Big(a^1_{\alpha;\beta_1,\gamma_1}(v)\\
&\times g^{\gamma_1}(t,x,v)+a^2_{\alpha;\beta_1,\gamma_1}(v)v\cdot D_v g^{\gamma_1}(t,x,v)\Big)\\
&+\Lambda^\rho\Big(a^3_{\alpha;\beta_2,\gamma_2}(v)\nabla_x\phi^{\beta_2}(t,x+\hat{v}t)\cdot\alpha_i(v)\Big)\Lambda^\kappa\Big(X_i g^{\gamma_2}(t,x,v)\Big), \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
l.o.t^\alpha_{\beta;4}(t,x,v)=&\sum_{\substack{\iota,\kappa\in\mathcal{S},\beta_1,\gamma_1,\beta_2,\gamma_2\in\mathcal{B},\\|\rho|+|\beta_1|\le 11,|\beta_1|+|\gamma_1|\le|\alpha|\\|\rho|+|\beta_2|\le 11,|\beta_2|+|\gamma_2|\le|\alpha|\\|\gamma_1|\le|\alpha|+|\rho|-1,|\gamma_2|\le|\alpha|+|\rho|-2}}\\
&\Lambda^\rho\Big((\partial_t\phi^{\beta_1}(t,x+\hat{v}t)\\
&+\hat{v}\cdot\nabla_x\phi^{\beta_1}(t,x+\hat{v}t))\Big)\Lambda^\kappa\Big(a^1_{\alpha;\beta_1,\gamma_1}(v)g^{\gamma_1}(t,x,v)\\
&+a^2_{\alpha;\beta_2,\gamma_2}(v)v\cdot D_v g^{\gamma_2}(t,x,v)\Big)\\
&+\Lambda^\rho\Big[\Big(a^3_{\alpha;\beta_2,\gamma_2}(v)\nabla_x\phi^{\beta_2}(t,x+\hat{v}t)\\
&+a^4_{\alpha;\beta_2,\gamma_2}(v)\partial_t\phi^{\beta_2}(t,x+\hat{v}t)\Big)\cdot\alpha_i(v)\Big]\Lambda^\kappa\Big(X_i g^{\gamma_2}(t,x,v)\Big), \qquad (4.23)
\end{aligned}
$$

where $l.o.t^{\alpha}_{\beta;1}(t,x,v)$ arises from the low order commutator between X_i and Λ^{β}, see (3.76) in Lemma 3.9, $l.o.t^{\alpha}_{\beta;2}(t,x,v)$ arises from the commutator between X_i and Λ^{κ}, $\kappa \in \mathcal{S}$, $|\kappa| = |\beta| - 1$ or between X_i and Λ^{β} when there is only one derivative hits on $K^i(t,x,v)$, $l.o.t^{\alpha}_{\beta;3}(t,x,v)$ arises from the case when there are at least twelve derivatives hit on the nonlinear wave part, and $l.o.t^{\alpha}_{\beta;4}(t,x,v)$ denotes all the other low order terms, in which there are at most eleven derivatives hit on the nonlinear wave part and the total number of derivatives hit on $g(t,x,v)$ is strictly less than $|\alpha| + |\beta|$.

To study the term of type $\Lambda^{\rho}(f(t,x+\hat{v}t))$ in $l.o.t^{\alpha}_{\beta;3}(t,x,v)$, see (4.22), the following Lemma is helpful.

Lemma 4.1. *The following identity holds for any* $\rho \in \mathcal{S}$,

$$\Lambda^{\rho}\big(f(t,x+\hat{v}t)\big) = \sum_{\iota\in\mathcal{B}, |\iota|\le|\rho|} c^{\iota}_{\rho}(t,x,v) f^{\iota}(t,x+\hat{v}t) \tag{4.24}$$

where the coefficients $c^{\iota}_{\rho}(x,v)$, $\iota \in \mathcal{B}$, $|\iota| \le |\rho|$, *satisfy the following estimate,*

$$|c^{\iota}_{\rho}(t,x,v)| \lesssim (1+|x|)^{|\rho|-|\iota|}(1+|v|)^{|\rho|-|\iota|} \min\{(1+|v|)^{-c_{vn}(\rho)}, (1+|v|)^{|\rho|-c_{vm}(\rho)}\}. \tag{4.25}$$

For any $\kappa \in \mathcal{S}$, *the following rough estimate holds*

$$|\Lambda^{\kappa}\big(c^{\iota}_{\rho}(t,x,v)\big)| \lesssim (1+|x|)^{|\kappa|+|\rho|-|\iota|}(1+|v|)^{|\kappa|+|\rho|-|\iota|} \min\{(1+|v|)^{-c_{vn}(\rho)}, (1+|v|)^{|\rho|-c_{vm}(\rho)}\}. \tag{4.26}$$

Moreover, the following improved estimate holds if $\Lambda^{\rho} \sim \Omega^x_i$ *or* $\widehat{\Omega}^v_i$, $i \in \{1,2,3\}$, $|\rho| = 1$,

$$|c^{\iota}_{\rho}(t,x,v)| \lesssim (1+|v|)^{-c_{vm}(\rho)}. \tag{4.27}$$

Proof. To calculate $\Lambda^{\rho}\big(f(t,x+\hat{v}t)\big)$, $\rho \in \mathcal{S}$, we induct on the size of $|\rho|$. Since the case when $\rho = \vec{0}$ is trivial, we first consider the case $|\rho| = 1$, i.e., $\rho \in \mathcal{K}/\vec{0}$. Recall (3.17). It is easy to see that the following equalities hold from direct computations,

$$\widehat{S}^v f(t,x+\hat{v}t) = \tilde{d}(t,x,v)(1+|v|^2)^{-1/2} S^x f(t,x+\hat{v}t), \tag{4.28}$$

$$\widehat{\Omega}^v_i f(t,x+\hat{v}t) = \sqrt{1+|v|^2}\tilde{d}(t,x,v)\Omega^x_i f(t,x+\hat{v}t),$$

$$\tilde{\Omega}_i\big(f(t,x+\hat{v}t)\big) = (\Omega_i f)(t,x+\hat{v}t), \tag{4.29}$$

$$K_{v_i} f(t,x+\hat{v}t) = (1+|v|^2)\tilde{d}(t,x,v)\partial_{v_i}\hat{v}\cdot\nabla_x f(t,x+\hat{v}t), \quad i \in \{1,2,3\}. \tag{4.30}$$

Recall the equality (3.8) in Lemma 3.2 and the equality (3.7) in Lemma 3.1. We know that the following equality holds for some uniquely determined coefficient $b_{\alpha}(t,x,v)$,

$$\tilde{d}(t,x,v)\nabla_x f(t,x+\hat{v}t) = \sum_{\alpha\in\mathcal{B}, |\alpha|=1} b_{\alpha}(t,x,v)\Gamma^{\alpha} f(t,x+\hat{v}t),$$

where the coefficient $b_{\alpha}(t,x,v)$ satisfies the following rough estimate,

$$|b_{\alpha}(t,x,v)| \lesssim 1, \quad |\Lambda^{\beta} b_{\alpha}(t,x,v)| \lesssim (1+|x|)^{|\beta|}(1+|v|)^{|\beta|}, \quad \text{where } \beta \in \mathcal{S}. \tag{4.31}$$

From the above rough estimate (4.31) and the equalities (4.28), (4.29), and (4.30), we know that our desired equality (4.24) and the desired estimates (4.25), (4.26) and (4.27) hold for the case $|\rho| = 1$.

Now, we proceed to consider the case when $\rho \in \mathcal{S}$, $|\rho| > 1$. Recall (3.36), we have

$$\rho \sim \iota_1 \circ \cdots \iota_{|\rho|}, \quad \iota_i \in \mathcal{K}/\vec{0}, \quad i = 1, \ldots, |\rho|.$$

From the equality (4.24) for the case $|\rho| = 1$ and the above equivalence relation, we have

$$\Lambda^{\rho}\big(f(t, x + \hat{v}t, v)\big) = \sum_{\iota \in \mathcal{B}, |\iota| = 1} \Lambda^{\iota_1 \circ \cdots \iota_{|\rho|-1}}\Big(c^{\iota}_{\iota_{|\rho|}, \iota}(t, x, v) f^{\iota}(t, x + \hat{v}t)\Big).$$

After keeping iterating the above process, our desired estimate (4.25) holds from the Leibniz rule. □

4.2. The modified profiles of the scalar field. Recall (3.31), (3.32), and (3.33). We know that the following equality holds,

$$\Gamma^{\alpha} - \tilde{\Gamma}^{\alpha} = \sum_{\beta, \gamma \in \mathcal{B}, |\beta| + |\gamma| \le |\alpha|, |\beta| \ge 1} a_{\alpha;\beta,\gamma}(v) \cdot \nabla_v^{\beta} \tilde{\Gamma}^{\gamma}, \tag{4.32}$$

where "$a_{\alpha;\beta,\gamma}(v)$", $\beta, \gamma \in \mathcal{B}$, are some determined coefficients, whose explicit formulas are not pursued here and the vector field "∇_v^{β}", $\beta \in \mathcal{B}$, is defined as follows,

$$\nabla_v^{\beta} := \nabla_v^{\gamma_1} \circ \cdots \nabla_v^{\gamma_{|\beta|}}, \quad \beta \sim \gamma_1 \circ \cdots \gamma_{|\beta|}, \quad \gamma_i \in \mathcal{A}, |\gamma_i| = 1, i \in \{1, \ldots, |\beta|\},$$

$$\nabla_v^{\gamma} = \begin{cases} V_i \cdot \nabla_v & \text{if } \gamma = \vec{a}_{i+4}, i = 1, 2, 3 \\ \sqrt{1 + |v|^2}\partial_{v_i} & \text{if } \gamma = \vec{a}_{i+7}, i = 1, 2, 3 \\ Id & \text{otherwise,} \end{cases} \quad \text{where } \gamma \in \mathcal{A}. \tag{4.33}$$

Due to the fact that ∇_v may hit the coefficients during the expansion, we have $|\beta| + |\gamma| \le |\alpha|$ instead of $|\beta| + |\gamma| = |\alpha|$ in (4.32). Let

$$a_{\alpha:\vec{0},\alpha}(v) := 1, \Longrightarrow \Gamma^{\alpha} = \sum_{\beta, \gamma \in \mathcal{B}, |\beta| + |\gamma| \le |\alpha|} a_{\alpha;\beta,\gamma}(v) \cdot \nabla_v^{\beta} \tilde{\Gamma}^{\gamma}. \tag{4.34}$$

With the above equality, we are ready to compute the equation satisfied by the high order derivatives of the scalar field, i.e., $\Gamma^{\alpha}\phi$. Recall (1.1). From (4.34), we have,

$$\Gamma^{\alpha}\big((\partial_t^2 - \Delta)\phi\big) = \sum_{\beta, \gamma \in \mathcal{B}, |\beta| + |\gamma| \le |\alpha|} \int_{\mathbb{R}^3} a_{\alpha;\beta,\gamma}(v) \cdot \nabla_v^{\beta} \tilde{\Gamma}^{\gamma}\Big(\frac{f}{\sqrt{1 + |v|^2}}\Big) dv. \tag{4.35}$$

After doing integration by parts in "v", we derive the equation satisfied by ϕ^{α} as follows,

$$(\partial_t^2 - \Delta)\phi^{\alpha} = \sum_{\gamma \in \mathcal{B}, |\gamma| \le |\alpha|} \int_{\mathbb{R}^3} \tilde{a}_{\alpha;\gamma}(v) f^{\gamma} dv, \tag{4.36}$$

where $a_{\alpha;\gamma}(v)$, $\gamma \in \mathcal{B}$, $|\gamma| \le |\alpha|$, are some coefficients, whose explicit formulas are not so important and will not be pursued here. From (4.33), we know that the following equality and rough estimate holds,

$$\tilde{a}_{\alpha;\alpha}(v) = (1 + |v|^2)^{-1/2}, \quad |\tilde{a}_{\alpha;\gamma}(v)| \lesssim (1 + |v|)^{|\alpha| - |\gamma|}. \tag{4.37}$$

Similar to the definitions of the half wave $u(t)$ and the profile $h(t)$ in Sect. 2.2, we define

$$u^\alpha(t) := (\partial_t - i|\nabla|)\phi^\alpha(t), \quad h^\alpha(t) := e^{it|\nabla|}u^\alpha(t). \tag{4.38}$$

Hence, we can recover $\partial_t\phi^\alpha$ and ϕ^α from the half wave $u^\alpha(t)$ and the profile $h^\alpha(t)$ as follow,

$$\partial_t\phi^\alpha(t) = \frac{u^\alpha(t) + \overline{u^\alpha(t)}}{2}, \quad \phi^\alpha(t) \\ = \frac{-u^\alpha(t) + \overline{u^\alpha(t)}}{2i|\nabla|} = \sum_{\mu\in\{+,-\}} c_\mu |\nabla|^{-1}(u^\alpha)^\mu(t), \quad u^\alpha(t) := e^{-it|\nabla|}h^\alpha(t). \tag{4.39}$$

From (4.35), we can derive the equation satisfied by $u^\alpha(t)$ as follows,

$$(\partial_t + i|\nabla|)u^\alpha(t) = \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} \int_{\mathbb{R}^3} \tilde{a}_{\alpha;\gamma}(v) f^\gamma dv \\ = \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} \int_{\mathbb{R}^3} \tilde{a}_{\alpha;\gamma}(v) g^\gamma(t, x - \hat{v}t, v) dv. \tag{4.40}$$

On the Fourier side, we have

$$\partial_t\widehat{h^\alpha}(t,\xi) = \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} \int_{\mathbb{R}^3} \tilde{a}_{\alpha;\gamma}(v) e^{it|\xi| - it\hat{v}\cdot\xi} \widehat{g^\gamma}(t,\xi,v) dv. \tag{4.41}$$

Correspondingly, we can write the equation satisfied by $g^\alpha(t, x, v)$ in (4.7) in terms of profiles on Fourier side as follows,

$$\partial_t\widehat{g^\alpha}(t,\xi,v) = \sum_{\beta,\gamma\in\mathcal{B},|\beta|+|\gamma|\le|\alpha|} \sum_{\mu\in\{+,-\}} \int_{\mathbb{R}^3} e^{it(\xi-\eta)\cdot\hat{v} - it\mu|\xi-\eta|} \widehat{(h^\beta)^\mu}(t,\xi-\eta) \\ \Big[\Big(\frac{1}{2} + ic_\mu \hat{v}\cdot\frac{\xi-\eta}{|\xi-\eta|}\Big) a^1_{\alpha;\beta,\gamma}(v) \\ \times \widehat{g^\gamma}(t,\eta,v) + \Big(\Big(\frac{1}{2} + ic_\mu\hat{v}\cdot\frac{\xi-\eta}{|\xi-\eta|}\Big) a^2_{\alpha;\beta,\gamma}(v) v + \frac{1}{2} a^4_{\alpha;\beta,\gamma}(v) \\ + ic_\mu a^3_{\alpha;\beta,\gamma}(v) \frac{\xi-\eta}{|\xi-\eta|}\Big) \cdot \big(\nabla_v - it\eta\big) \widehat{g^\gamma}(t,\eta,v)\Big] d\eta. \tag{4.42}$$

Similar to the modified profile defined in Sect. 2.2, to take the advantage of the oscillation of the phase over time, instead of controlling the increment of the profile, we control the following *modified profile*,

$$\widehat{\widehat{h^\alpha}}(t,\xi) := \widehat{h^\alpha}(t,\xi) + \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} \int_{\mathbb{R}^3} e^{it|\xi| - it\hat{v}\cdot\xi} \frac{i\tilde{a}_{\alpha;\gamma}(v)}{|\xi| - \hat{v}\cdot\xi} \widehat{g^\gamma}(t,\xi,v) dv. \tag{4.43}$$

Define

$$\widetilde{\phi^{\alpha}}(t) := \frac{-e^{-it|\nabla|}\widetilde{h^{\alpha}}(t) + \overline{e^{-it|\nabla|}\widetilde{h^{\alpha}}(t)}}{2i|\nabla|},$$
$$\widetilde{\partial_t\phi^{\alpha}}(t) := \frac{e^{-it|\nabla|}\widetilde{h^{\alpha}}(t) + \overline{e^{-it|\nabla|}\widetilde{h^{\alpha}}(t)}}{2}, \tag{4.44}$$
$$E_{\alpha;\gamma}(f)(t,x) := \mathcal{F}^{-1}\Big[\int_{\mathbb{R}^3} e^{-it\hat{v}\cdot\xi} \frac{-i\tilde{a}_{\alpha;\gamma}(v)}{(|\xi| - \hat{v}\cdot\xi)}\widehat{f}(t,\xi,v)dv\Big](x). \tag{4.45}$$

Hence, from (4.39) and (4.43), we have

$$\phi^{\alpha}(t) = \widetilde{\phi^{\alpha}}(t) - \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} |\nabla|^{-1}\big(\mathrm{Im}\big[E_{\alpha;\gamma}(g^{\gamma})(t)\big]\big),$$
$$\partial_t\phi^{\alpha}(t) = \widetilde{\partial_t\phi^{\alpha}}(t) + \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} \big(\mathrm{Re}\big[E_{\alpha;\gamma}(g^{\gamma})(t)\big]\big). \tag{4.46}$$

Recall the equations (4.41), (4.43), and (4.42). After doing integration by parts in "v" once, which moves the "∇_v" derivative in front of $\nabla_v\widehat{g^{\gamma}}(t,\eta,v)$ around, we obtain the equation for the modified profile $\widehat{\widetilde{h^{\alpha}}}(t,\xi)$ as follows,

$$\begin{aligned}
\partial_t\widehat{\widetilde{h^{\alpha}}}(t,\xi) &= \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|}\int_{\mathbb{R}^3} e^{it|\xi|-it\hat{v}\cdot\xi}\frac{i\tilde{a}_{\alpha;\gamma}(v)}{|\xi|-\hat{v}\cdot\xi}\partial_t\widehat{g^{\gamma}}(t,\xi,v)dv \\
&= \sum_{\substack{\gamma\in\mathcal{B},|\gamma|\le|\alpha|,\mu\in\{+,-\}\\ \beta,\kappa\in\mathcal{B},|\beta|+|\kappa|\le|\gamma|}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-it\mu|\xi-\eta|-it\hat{v}\cdot\eta}\widehat{g^{\kappa}}(t,\eta,v)\widehat{(h^{\beta})^{\mu}}(t,\xi-\eta) \\
&\quad \Big[\frac{i\tilde{a}_{\alpha;\gamma}(v)}{|\xi|-\hat{v}\cdot\xi}\Big(\frac{1}{2}+ic_{\mu}\hat{v}\cdot\frac{\xi-\eta}{|\xi-\eta|}\Big) \\
&\quad \times a^{1}_{\gamma;\beta,\kappa}(v) - \nabla_v\cdot\Big(\frac{i\tilde{a}_{\alpha;\gamma}(v)}{|\xi|-\hat{v}\cdot\xi}\Big(\Big(\frac{1}{2}+ic_{\mu}\hat{v}\cdot\frac{\xi-\eta}{|\xi-\eta|}\Big)a^{2}_{\gamma;\beta,\kappa}(v)v \\
&\quad +\frac{1}{2}a^{4}_{\gamma;\beta,\kappa}(v) \\
&\quad +ic_{\mu}a^{3}_{\gamma;\beta,\kappa}(v)\frac{\xi-\eta}{|\xi-\eta|}\Big)\Big)\Big]d\eta dv \\
&= \sum_{\beta,\gamma\in\mathcal{B},|\beta|+|\gamma|\le|\alpha|}\sum_{\mu\in\{+,-\}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-it\mu|\xi-\eta|-it\hat{v}\cdot\eta}\big(m_1(\xi,v)\widehat{a}^{\mu,1}_{\alpha;\beta,\gamma}(v) \\
&\quad +m_2(\xi,v)\widehat{a}^{\mu,2}_{\alpha;\beta,\gamma}(v)\cdot\frac{\xi-\eta}{|\xi-\eta|}\big) \\
&\quad \times\widehat{g^{\gamma}}(t,\eta,v)\widehat{(h^{\beta})^{\mu}}(t,\xi-\eta)d\eta dv,
\end{aligned} \tag{4.47}$$

where $\widehat{a}^{\mu,i}_{\alpha;\beta,\gamma}(v)$, $\mu\in\{+,-\}$, $i\in\{1,2\}$, $\beta,\gamma\in\mathcal{B}$, s.t., $|\beta|+|\gamma|\le|\alpha|$, are some determined coefficients and $m_i(\xi,v)$, $i\in\{1,2\}$, are some determined symbol, whose explicit formulas are not pursued here.

From the rough estimates in (4.6) and (4.37), we have

$$\sum_{i=1,2} |\widehat{a}^{i}_{\alpha;\beta,\gamma}(v)| \lesssim (1+|v|)^{|\alpha|-|\gamma|},$$
$$\sum_{i=1,2,0\leq a\leq 5} \|(1+|v|)^{-10}\nabla_v^a m_i(\xi, v)\|_{\mathcal{S}^\infty_k} \lesssim 2^{-k}, \quad k\in\mathbb{Z}. \tag{4.48}$$

4.3. Constructing the energy for the Vlasov–Nordström system.

4.3.1. Control of the profile $g(t,x,v)$ We define the high order energy for the profile $g(t,x,v)$ of the distribution function as follows,

$$E^{f}_{\text{high}}(t) := E^{f;1}_{\text{high}}(t) + E^{f;2}_{\text{high}}(t),$$
$$E^{f:1}_{\text{high}}(t) := \sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|=N_0} \|\omega^{\alpha}_{\beta}(x,v) g^{\alpha}_{\beta}(t,x,v)\|_{L^2_{x,v}}, \tag{4.49}$$
$$E^{f:2}_{\text{high}}(t) := \sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|<N_0} \|\omega^{\alpha}_{\beta}(x,v) g^{\alpha}_{\beta}(t,x,v)\|_{L^2_{x,v}}, \tag{4.50}$$

where $g^{\alpha}_{\beta}(t,x,v)$ is defined in (4.8) and the weight function $\omega^{\alpha}_{\beta}(x,v)$ is defined as follows,

$$\omega^{\alpha}_{\beta}(t,x,v) = (1+|x|^2+(x\cdot v)^2+|v|^{20})^{20N_0-10(|\alpha|+|\beta|)} (1+|v|)^{c_{\text{vn}}(\beta)}, \tag{4.51}$$

where the index $c_{\text{vn}}(\beta)$ is defined in (3.40). We separate out $E^{f:1}_{\text{high}}(t)$ as the strictly top order energy.

It's worth to explain why we make such a choice of the weight function in (4.51). (i) We set up a hierarchy for the order of weight function $\omega^{\alpha}_{\beta}(t,x,v)$. Note that the total number of derivatives act on $g^{\alpha}_{\beta}(t,x,v)$ is $|\alpha|+|\beta|$. The more derivatives act the profile, the lower order weight function we use for $\omega^{\alpha}_{\beta}(t,x,v)$. (ii) Comparing with the ordinary derivatives of the profile, we expect that the good derivatives of the profile can propagate more weight in "$|v|$"; (iii) We choose an anisotropic weight in "x" in the definition of the weight function $\omega^{\alpha}_{\beta}(x,v)$ in (4.51) to guarantee that the following Lemma holds, which plays an essential role in the energy estimate.

Lemma 4.2. *For any $\alpha\in\mathcal{B}$, $\beta\in\mathcal{S}$, s.t., $|\alpha|+|\beta|\leq N_0$, the following estimate holds for any $x, v\in\mathbb{R}^3$,*

$$\Big[\Big|\frac{v\cdot D_v\omega^{\alpha}_{\beta}(x,v)}{\omega^{\alpha}_{\beta}(x,v)}\Big| + \Big|\frac{D_v\omega^{\alpha}_{\beta}(x,v)}{\omega^{\alpha}_{\beta}(x,v)}\Big|\Big]\frac{1}{1+||t|-|x+\hat{v}t||} \lesssim 1. \tag{4.52}$$

Remark 4.1. Essentially speaking, the estimate (4.52) says that, through a good choice of the weight function $\omega^{\alpha}_{\beta}(x,v)$, the loss of size t, which comes from the coefficient of D_v, can be controlled by the distance with respect to the light cone when D_v hits the weight function. The desired estimate (4.52) is important in the energy estimate of the distribution function for the case when all derivatives hits on $\nabla_v f$, or equivalently $D_v g(t,x,v)$, see Proposition 6.1 in Sect. 6.1.

Remark 4.2. Thanks to the presence of coefficient "$1/\sqrt{1+|v|^2}$" in the Vlasov–Nordström system (see (1.1)), we don't need an estimate as strong as in (4.52) to close the energy estimate for the Vlasov–Nordström system. However, for the purpose of being more applicable in the study of the Vlasov-wave type system, e.g., the Vlasov–Maxwell system, we prove a stronger estimate here.

Proof. Recall the decomposition of D_v in (3.22). We have

$$\frac{D_v \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)} = \tilde{v}\Big(\frac{\tilde{v}\cdot\nabla_v \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)} - \frac{t}{(1+|v|^2)^{3/2}}\frac{\tilde{v}\cdot\nabla_x \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big) + \sum_{i=1,2,3}\tilde{V}_i\Big(\frac{\tilde{V}_i\cdot\nabla_v \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)} - \frac{t}{(1+|v|^2)^{1/2}}\frac{\tilde{V}_i\cdot\nabla_x \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big). \tag{4.53}$$

From the explicit formula of $\omega_\beta^\alpha(x,v)$ in (4.51), we have

$$(1+|v|)\Big|\frac{\tilde{v}\cdot\nabla_v \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| + \sum_{i=1,2,3}\Big|\frac{\tilde{V}_i\cdot\nabla_v \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| \lesssim 1, \tag{4.54}$$

$$\Big|\frac{\tilde{v}\cdot\nabla_x \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| \lesssim \frac{x\cdot\tilde{v} + (x\cdot v)|v|}{1+|x|^2+(x\cdot v)^2+|v|^{10}} \lesssim \frac{1+|v|}{1+|x|+|x\cdot v|+|v|^5}, \tag{4.55}$$

$$\sum_{i=1,2,3}\Big|\frac{\tilde{V}_i\cdot\nabla_x \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| \lesssim \frac{|x|}{1+|x|^2+(x\cdot v)^2+|v|^{10}} \lesssim \frac{1}{1+|x|+|x\cdot v|+|v|^5}. \tag{4.56}$$

Recall the decomposition (4.53). From the estimates (4.54), (4.55), and (4.56), we know that the desired estimate estimate (4.52) holds easily if $|x| \geq 3|t|$.

It remains to consider the case when $|x| \leq 3|t|$. For this case, we have

$$\frac{1}{1+||t|-|x+\hat{v}t||} \sim \frac{1+|t|}{1+|t|+|t^2-|x+\hat{v}t|^2|} = \frac{1+|t|}{1+|t|+\Big|\frac{t^2}{1+|v|^2} - \frac{2tx\cdot v}{\sqrt{1+|v|^2}} - |x|^2\Big|}. \tag{4.57}$$

Based on the size of $|x|$ and $x\cdot v$, we separate into two cases as follows.

- If $|x| \geq 2^{-10}|t|/(1+|v|)$ or $|x\cdot v| \geq 2^{-10}|t|/(1+|v|)$, then from the estimates (4.55) and (4.56), we have

$$\frac{|t|}{1+|v|^2}\Big|\frac{\tilde{v}\cdot\nabla_x \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| + \sum_{i=1,2,3}\frac{|t|}{1+|v|}\Big|\frac{\tilde{V}_i\cdot\nabla_x \omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| \lesssim 1. \tag{4.58}$$

- If $|x| \leq 2^{-10}|t|/(1+|v|)$ and $|x\cdot v| \leq 2^{-10}|t|/(1+|v|)$, then from the estimate (4.57), we have

$$\frac{1}{1+||t|-|x+\hat{v}t||} \lesssim \frac{1+|v|^2}{1+|t|}.$$

Therefore, from the above estimate and the estimates (4.55) and (4.56), we have,

$$\Big[\frac{|t|}{1+|v|^2}\Big|\frac{\tilde{v}\cdot\nabla_x\omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big| + \sum_{i=1,2,3}\frac{|t|}{1+|v|}\Big|\frac{\tilde{V}_i\cdot\nabla_x\omega_\beta^\alpha(x,v)}{\omega_\beta^\alpha(x,v)}\Big|\Big]\frac{1}{1+||t|-|x+\hat{v}t||}\lesssim 1. \tag{4.59}$$

To sum up, for any $x, v \in \mathbb{R}^3$, our desired estimate (4.52) holds from the decomposition (4.53) and the estimates (4.54), (4.58), and (4.59). □

Similar to the study of the Vlasov–Poisson system in [41], from the decay estimate (2.20) of the average of the distribution function in Lemma 2.1, we know that the zero frequency of the distribution function plays the leading role in the decay estimate. With this intuition, to ensure $E_{\alpha;\gamma}(g^\gamma)(t)$ defined in (4.45) has sharp decay rate, we define a lower order energy for the profile $g(t,x,v)$ as follows,

$$E_{\text{low}}^f(t) := \sum_{\gamma\in\mathcal{B},a\in\mathbb{Z}_+,a+|\gamma|\le N_0}\|\widetilde{\omega_\gamma^a}(v)\big(\nabla_v^a\widehat{g^\gamma}(t,0,v)-\nabla_v\cdot\widetilde{g}_{a,\gamma}(t,v)\big)\|_{L_v^2},$$
$$\widetilde{\omega_\gamma^a}(v) := (1+|v|)^{20N_0-10(a+|\gamma|)} \tag{4.60}$$

where the correction term $\widetilde{g}_{a,\gamma}(t,v)$, which is introduced for the purpose of avoiding losing derivatives in the study of $\nabla_v^a\widehat{g^\gamma}(t,0,v)$, is defined as follows,

$$\widetilde{g}_{a,\gamma}(t,v) := \begin{cases}\int_0^t\int_{\mathbb{R}^3}K(s,x+\hat{v}s,v)\nabla_v^a g^\gamma(s,x,v)dxds & \text{if } a+|\gamma|=N_0\\ 0 & \text{if } a+|\gamma|<N_0,\end{cases} \tag{4.61}$$

where $K(t,x+\hat{v}t,v)$ is defined in (4.10).

4.3.2. Control of the profiles and the modified profiles of the scalar field For the nonlinear wave part, we define a high order energy as follows,

$$\begin{aligned}E_{\text{high}}^\phi(t) := \sup_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{B},|\alpha|\le N_0} & 2^k\|\widehat{h^\alpha}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty}\\ & +2^k\|\widehat{\widetilde{h^\alpha}}(t,\xi)\psi_k(\xi)\|_{L_\xi^\infty}+2^{k/2}\|\nabla_\xi\widehat{\widetilde{h^\alpha}}(t,\xi)\psi_k(\xi)\|_{L_\xi^2}\\ & +\|\widehat{h^\alpha}(t,\xi)\|_{L_\xi^2}+\|\widehat{\widetilde{h^\alpha}}(t,\xi)\|_{L_\xi^2}.\end{aligned} \tag{4.62}$$

The first part of energy $E_{\text{high}}^\phi(t)$, which is stronger than L^2 at low frequencies, controls the low frequency part of the profiles $h^\alpha(t)$; the second part of energy $E_{\text{high}}^\phi(t)$, which has the same scaling level as the first part of energy $E_{\text{high}}^\phi(t)$, aims to control the first order weighted norm of the modified profiles $\widetilde{h^\alpha}(t)$; the third part of energy $E_{\text{high}}^\phi(t)$, controls the high frequency part of the profiles $h^\alpha(t)$ and the modified profiles $\widetilde{h^\alpha}(t)$.

Motivated from the linear decay estimate of half wave equation in Lemma 2.2, to prove sharp decay estimate for the nonlinear wave part, we define a low order energy for the profiles $h^{\alpha}(t)$, $\alpha \in \mathcal{B}$ as follows,

$$E_{\text{low}}^{\phi}(t) := \sum_{\substack{n=0,1,2,3\\ \alpha\in\mathcal{B},|\alpha|\leq 20-3n}} \|h^{\alpha}(t)\|_{X_n} + (1+|t|)\|\partial_t h^{\alpha}(t)\|_{X_n} + (1+|t|)^2\|\partial_t \nabla_x (1+|\nabla_x|)^{-1} h^{\alpha}(t)\|_{X_n}, \tag{4.63}$$

where the X_n-normed space is defined as follows,

$$\|h\|_{X_n} := \sup_{k\in\mathbb{Z}} 2^{(n+1)k}\|\nabla_\xi^n \widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}, \quad n \in \{0,1,2,3\}. \tag{4.64}$$

4.4. A precise statement of main theorem. With previous preparations, we are ready to state the main theorem.

Theorem 4.1 (A precise statement). *Let $N_0 = 200$, $\delta \in (0, 10^{-9}]$. Suppose that the given initial data $f_0(x,v)$, $\phi_0(x)$, $\phi_1(x)$ of the 3D relativistic Vlasov–Nordström system (1.1) satisfy the following smallness assumption,*

$$\sum_{|\alpha_1|+|\alpha_2|\leq N_0} \|(1+|x|^2+(x\cdot v)^2+|v|^{20})^{30N_0}\nabla_v^{\alpha_1}\nabla_x^{\alpha_2} f_0(x,v)\|_{L^2_x L^2_v} + \sum_{\alpha\in\mathcal{B},|\alpha|\leq N_0} \sum_{n\in\{0,1,2,3\}} \|\Gamma^{\alpha}\big(|\nabla|\phi_0(x)\big)\|_{L^2} + \|\Gamma^{\alpha}\big(|\nabla|\phi_0(x)\big)\|_{X_n} + \|\Gamma^{\alpha}\big(\phi_1(x)\big)\|_{L^2} + \|\Gamma^{\alpha}\big(\phi_1(x)\big)\|_{X_n} \leq \epsilon_0, \tag{4.65}$$

where the X_n-normed space is defined in (4.64) and ϵ_0 is some sufficiently small constant. Then the relativistic Vlasov–Nordström system (1.1) admits a global solution and scatters to a linear solution. Moreover, the following estimate holds over time,

$$\sup_{t\in[0,\infty)} (1+t)^{-\delta}\big[E_{high}^{f}(t) + E_{high}^{\phi}(t)\big] + E_{low}^{f}(t) + E_{low}^{\phi}(t) \lesssim \epsilon_0, \tag{4.66}$$

As byproducts of the above estimate, we have the following decay estimates for the derivatives of the average of the distribution function and the derivatives of the scalar field,

$$\sup_{t\in[0,\infty)} \sum_{|\alpha|\leq N_0-20} (1+|t|)^{(3+|\alpha|)/p}\Big|\int_{\mathbb{R}^3} \nabla_x^{\alpha}\big(f(t,x,v)\big)^p dv\Big|^{1/p} \lesssim \epsilon_0, \quad \textit{where } p \in [1,\infty)\cap\mathbb{Z}, \tag{4.67}$$

$$\sup_{t\in[0,\infty)} \sum_{|\alpha|\leq 10} (1+|t|)(1+||t|-|x||)^{|\alpha|+1}\big(\big|\nabla_x^{\alpha}\partial_t\phi(t,x)\big| + \big|\nabla_x^{\alpha}\nabla_x\phi(t,x)\big|\big) \lesssim \epsilon_0. \tag{4.68}$$

4.5. Proof of the main theorem: A bootstrap argument. In next two sections, we will estimate the increment of both the low order energy and the high order energy for the nonlinear wave part and the Vlasov part with respect to time. Those estimates allow us to prove the desire theorem, Theorem 4.1, by using a bootstrap argument.

We state our bootstrap assumption as follows,

$$\sup_{t\in[0,T]} (1+t)^{-\delta}\big[E^{f;1}_{\text{high}}(t) + E^{\phi}_{\text{high}}(t)\big] + (1+t)^{-\delta/2} E^{f;2}_{\text{high}}(t) + E^{f}_{\text{low}}(t) + E^{\phi}_{\text{low}}(t) \leq \epsilon_1 := \epsilon_0^{5/6}, \tag{4.69}$$

where T is the supremum of times such that the above bootstrap assumption holds.

Recall the definition of correction term in (4.61), From the linear decay estimate (6.55) of the scalar field in Lemma 6.3, we know that the following estimate holds for any $t \in [0, T]$, under the bootstrap assumption (4.69),

$$\sum_{\gamma\in\mathcal{B}, a\in\mathbb{Z}_+, a+|\gamma|\leq N_0} \|\widetilde{\omega^{a}_{\gamma}}(v)\widetilde{g}_{a,\gamma}(t,v)\|_{L^2_v} \lesssim \int_0^t (1+|s|)^{-2+2\delta}\epsilon_1^2 ds \lesssim \epsilon_0. \tag{4.70}$$

From the estimate (6.38) in Proposition 6.2, the estimate (6.5) in Proposition 6.1, the estimate (5.1) in Proposition 5.1, and the estimate (5.6) in Proposition 5.2, the following improved estimate holds under the bootstrap assumption (4.69),

$$\sup_{t\in[0,T]} (1+t)^{-\delta}\big[E^{f;1}_{\text{high}}(t) + E^{\phi}_{\text{high}}(t)\big] + (1+t)^{-\delta/2} E^{f;2}_{\text{high}}(t) + E^{f}_{\text{low}}(t) + E^{\phi}_{\text{low}}(t) \lesssim \epsilon_0. \tag{4.71}$$

Therefore, we can close the bootstrap argument and extend "T" to infinity. As a result, we have

$$\sup_{t\in[0,\infty)} (1+t)^{-\delta}\big[E^{f;1}_{\text{high}}(t) + E^{\phi}_{\text{high}}(t)\big] + (1+t)^{-\delta/2} E^{f;2}_{\text{high}}(t) + E^{f}_{\text{low}}(t) + E^{\phi}_{\text{low}}(t) \lesssim \epsilon_0, \tag{4.72}$$

which also implies our desired estimate (4.66).

The desired decay estimate (4.67) follows from the estimate (4.72) and the decay estimate (2.20) in Lemma 2.1. The desired decay estimate (4.68) follows directly from the estimate (4.72), the equality (3.7) in Lemma 3.1, and the decay estimate (6.55) in Lemma 6.3.

Lastly, we explain the scattering property of the nonlinear solution. From the estimate (6.38) in Proposition 6.2 and the definition of "$E^{\phi}_{\text{low}}(t)$" in (4.63), we can construct a limit for the profiles of the Vlasov–Nordström system by integrating the profile with respect to time from zero up to infinity. After pulling back the limit along the linear flow, we have our desired scattering linear solution.

Hence finishing the proof of Theorem 4.1.

5. Energy Estimates for the Nonlinear Wave Part

This section is devoted to control both the low order energy $E^{\phi}_{\text{low}}(t)$ defined in (4.63) and the high order energy $E^{\phi}_{\text{high}}(t)$ defined in (4.62) of the profiles of the scalar field over time. For the low order energy estimate, our main result is summarized in Proposition 5.1. For the high order energy estimate, our main result is summarized in Proposition 5.2.

The main tools used to prove Propositions 5.1 and 5.2 are some linear estimates and some bilinear estimates, which are postponed to Sects. 5.1 and 5.2 for the sake of clarity of presentation.

Proposition 5.1. *Under the bootstrap assumption* (4.69)*, the following estimate holds for any* $t \in [0, T]$*,*

$$E^{\phi}_{low}(t) \lesssim E^{f}_{low}(t) + (1+|t|)^{-1} E^{f}_{high}(t) + \epsilon_0. \tag{5.1}$$

Proof. Recall (4.63). We first estimate the X_n-norm of $\partial_t h^{\alpha}(t)$. Recall (4.41). From the estimate of coefficients in (4.37), the estimate (5.16) in Lemma 5.1, we have

$$\begin{aligned}\sum_{0\le n\le 3}\ \sum_{\alpha\in\mathcal{B},|\alpha|\le 20-3n} &(1+|t|)\\ &\|\partial_t h^{\alpha}(t)\|_{X_n} + (1+|t|)^2 \Big\|\frac{\nabla_x}{1+|\nabla_x|}\partial_t h^{\alpha}(t)\Big\|_{X_n}\\ &\lesssim E^{f}_{\text{low}}(t) + (1+|t|)^{-1} E^{f}_{\text{high}}(t).\end{aligned} \tag{5.2}$$

Now, it remains to estimate the X_n-norm of $h^{\alpha}(t)$. Recall (4.43). As a result of direct computation, we know that the symbol $1/(|\xi| - \hat{v}\cdot\xi)$ verifies the estimate (5.15). From the estimate of coefficients in (4.37) and the estimate (5.16) in Lemma 5.1, we have

$$\begin{aligned}\sum_{n=0,1,2,3}\ \sum_{|\alpha|\le 20-3n} &\|\widetilde{h^{\alpha}}(t) - h^{\alpha}(t)\|_{X_n}\\ &\lesssim E^{f}_{\text{low}}(t) + (1+|t|)^{-1} E^{f}_{\text{high}}(t).\end{aligned} \tag{5.3}$$

Hence, it would be sufficient to estimate the X_n-norm of the modified profiles $\widetilde{h^{\alpha}}(t)$. Recall (4.47) and (5.24). We know that $\partial_t \widetilde{h^{\alpha}}(t,\xi)$ is a linear combination of bilinear forms defined in (5.24). Therefore, from the estimate (5.31) in Lemma 5.2, we have

$$\begin{aligned}\sum_{n=0,1,2,3}\ \sum_{|\alpha|\le 20-3n} &\|\partial_t \widetilde{h^{\alpha}}(t)\|_{X_n}\\ &\lesssim (1+|t|)^{-2} E^{\phi}_{\text{low}}(t) E^{f}_{\text{high}}(t) \lesssim (1+|t|)^{-2+\delta}\epsilon_1^2.\end{aligned} \tag{5.4}$$

Hence, from the above estimate (5.4) and the estimate (5.3), we have

$$\begin{aligned}\sum_{n=0,1,2,3}\ \sum_{|\alpha|\le 20-3n} &\|\widetilde{h^{\alpha}}(t)\|_{X_n} + \|h^{\alpha}(t)\|_{X_n}\\ &\lesssim \epsilon_0 + \int_0^t (1+|s|)^{-2+\delta}\epsilon_1^2 ds \lesssim \epsilon_0.\end{aligned} \tag{5.5}$$

To sum up, our desired estimate (5.1) holds from the estimates (5.2) , (5.3), and (5.5). □

Proposition 5.2. *Under the bootstrap assumption* (4.69)*, the following estimate holds for any* $t \in [0, T]$,

$$E^{\phi}_{high}(t) \lesssim E^{f}_{high}(t) + (1+|t|)^{\delta}\epsilon_0. \tag{5.6}$$

Proof. Recall (4.62). Based on the components of $E^{\phi}_{\mathrm{high}}(t)$, we divide the high order energy estimate into three parts as follow.

• **Case 1** L^∞_ξ-estimate of $\widehat{h^\alpha}(t,\xi)$ and $\widehat{\widehat{h^\alpha}}(t,\xi)$.
Recall (4.43). We know that the following estimate holds for any $\alpha \in \mathcal{B}$, $|\alpha| \le N_0$,

$$\sup_{k\in\mathbb{Z}} 2^k \|\big(\widehat{\widehat{h^\alpha}}(t,\xi) - \widehat{h^\alpha}(t,\xi)\big)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim \sum_{\gamma\in\mathcal{B},|\gamma|\le|\alpha|} \|(1+|v|)^{5+4(|\alpha|-|\gamma|)}\widehat{g^\gamma}(t,\xi,v)\|_{L^\infty_\xi L^1_v} \lesssim E^{f}_{\mathrm{high}}(t). \tag{5.7}$$

Hence, it would be sufficient to estimate the L^∞_ξ-norm of the frequency localized $\widehat{\widehat{h^\alpha}}(t,\xi)$. Recall the equation satisfied by $\partial_t\widehat{\widehat{h^\alpha}}(t,\xi)$ in (4.47) . From the estimate (5.31) in Lemma 5.2, which is used when the profile $h(t)$ has relatively more derivatives, and the estimate (5.42) in Lemma 5.3, which is used when $g(t,x,v)$ has relatively more derivatives, we have

$$\sup_{k\in\mathbb{Z}} 2^k \|\partial_t\widehat{\widehat{h^\alpha}}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim (1+|t|)^{-2+\delta}\big(E^{\phi}_{\mathrm{high}}(t) + E^{\phi}_{\mathrm{low}}(t)\big)E^{f}_{\mathrm{high}}(t) \lesssim (1+|t|)^{-2+2\delta}\epsilon_1^2. \tag{5.8}$$

Therefore, from the estimates (5.7) and (5.8), we have

$$\sum_{|\alpha|\le N_0}\sup_{k\in\mathbb{Z}} 2^k \|\widehat{\widehat{h^\alpha}}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim \epsilon_0,$$
$$\sum_{|\alpha|\le N_0}\sup_{k\in\mathbb{Z}} 2^k \|\widehat{h^\alpha}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim E^{f}_{\mathrm{high}}(t) + \epsilon_0. \tag{5.9}$$

• **Case 2** L^2-estimate of $\widehat{h^\alpha}(t,\xi)$ and $\widehat{\widehat{h^\alpha}}(t,\xi)$.
From the estimate (5.7), after dyadically decomposing frequency "ξ", we have

$$\begin{aligned}\|\big(\widehat{\widehat{h^\alpha}}(t,\xi) - \widehat{h^\alpha}(t,\xi)\big)\|_{L^2_\xi} &\lesssim \sum_{k\ge 0}\sum_{|\gamma|\le|\alpha|} 2^{-k}\|(1+|v|)^{5+4(|\alpha|-|\gamma|)}\widehat{g^\gamma}(t,\xi,v)\|_{L^1_v L^2_\xi} \\ &\quad + \sum_{k\le 0} 2^{3k/2}\|\big(\widehat{\widehat{h^\alpha}}(t,\xi) - \widehat{h^\alpha}(t,\xi)\big)\psi_k(\xi)\|_{L^\infty_\xi} \\ &\lesssim E^{f}_{\mathrm{high}}(t) + \sum_{|\gamma|\le|\alpha|} \|\omega^{\vec{0}}_\gamma(x,v)g^\gamma(t,x,v)\|_{L^2_x L^2_v} \lesssim E^{f}_{\mathrm{high}}(t).\end{aligned} \tag{5.10}$$

Hence, it would be sufficient to estimate the L^2-norm of $\widehat{\widehat{h^\alpha}}(t,\xi)$. After using the first estimate of (5.49) in Lemma 5.4 for the case when there are more derivatives act on the profile $g(t,x,v)$ and using the second estimate of (5.49) for the case when there are more derivatives act on the profile $h(t)$, we have

$$\|\partial_t \widehat{\widetilde{h^\alpha}}(t,\xi)\|_{L^2_\xi} \lesssim (1+t)^{-1} E^f_{\mathrm{high}}(t) E^\phi_{\mathrm{low}}(t) + (1+t)^{-2} E^f_{\mathrm{high}}(t) E^\phi_{\mathrm{high}}(t) \lesssim (1+|t|)^{-1+\delta}\epsilon_1^2.$$

From the above estimate and the estimate (5.10), we have

$$\sum_{\alpha\in\mathcal{B},|\alpha|\le N_0} \|\widehat{h^\alpha}(t,\xi)\|_{L^2_\xi} + \|\widehat{\widetilde{h^\alpha}}(t,\xi)\|_{L^2_\xi} \lesssim E^f_{\mathrm{high}}(t) + \epsilon_0 + \int_0^t (1+|s|)^{-1+\delta}\epsilon_1^2 ds \lesssim E^f_{\mathrm{high}}(t) + (1+|t|)^\delta \epsilon_0. \tag{5.11}$$

• **Case 3** L^2_ξ-estimate of $\nabla_\xi \widehat{\widetilde{h^\alpha}}(t,\xi)$.

Recall the equation satisfied by $\widetilde{h^\alpha}(t,\xi)$ in (4.47). Based on the size of $|\gamma|$ in (4.47), we separate into two subcases.

If $|\gamma| \le N_0 - 10$, we use the decomposition (4.43) for $\widehat{h^\beta}(t,\xi)$. As a result, we have

$$\begin{aligned}
\partial_t \widehat{\widetilde{h^\alpha}}(t,\xi) = & \sum_{\beta,\gamma\in\mathcal{B},|\beta|+|\gamma|\le|\alpha|} \\
& \sum_{\mu\in\{+,-\}} \Big[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-it\mu|\xi-\eta|-it\hat{v}\cdot\eta} \\
& \big(m_2(\xi,v)\widehat{a}^{\mu,2}_{\alpha;\beta,\gamma}(v)\cdot\frac{\xi-\eta}{|\xi-\eta|} + m_1(\xi,v) \\
& \times \widehat{a}^{\mu,1}_{\alpha;\beta,\gamma}(v)\big)\widehat{g^\gamma}(t,\eta,v)\widehat{(\widetilde{h^\beta})^\mu}(t,\xi-\eta)d\eta dv \\
& + \sum_{\kappa\in\mathcal{B},|\kappa|\le|\beta|} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-it\hat{u}\cdot(\xi-\eta)-it\hat{v}\cdot\eta} \\
& \big(\widehat{a}^{\mu,2}_{\alpha;\beta,\gamma}(v)\cdot\frac{\xi-\eta}{|\xi-\eta|} \\
& \times m_2(\xi,v) + m_1(\xi,v)\widehat{a}^{\mu,1}_{\alpha;\beta,\gamma}(v)\big)\widehat{g^\gamma}(t,\eta,v)\frac{i\mu\tilde{a}_{\beta;\kappa}(u)}{|\xi-\eta|-\mu\hat{u}\cdot(\xi-\eta)} \\
& \widehat{(g^\kappa)^\mu}(t,\xi-\eta,u)d\eta dv du\Big].
\end{aligned} \tag{5.12}$$

For the first integral in (5.12), we apply the estimates (5.66) and (5.67) in Proposition 5.3. For the second integral in (5.12), we apply the estimate (5.98) in Lemma 5.9. If $|\gamma| \ge N_0 - 10$, we apply the estimate (5.66) in Proposition 5.3.

To summarize, the following estimate holds for any fixed $k\in\mathbb{Z}$,

$$\begin{aligned}
2^{k/2}\|\partial_t\nabla_\xi \widehat{\widetilde{h^\alpha}}(t,\xi)\psi_k(\xi)\|_{L^2_\xi} \lesssim & \big((1+|t|)^{-1}2^{k_-} + (1+|t|)^{-2+\delta}\big)E^\phi_{\mathrm{low}}(t)E^f_{\mathrm{high}}(t) \\
& + (1+|t|)^{-2}E^f_{\mathrm{high}}(t)\big(E^f_{\mathrm{high}}(t) + E^\phi_{\mathrm{high}}(t)\big) \\
\lesssim & (1+|t|)^{-1+\delta}2^{k_-}\epsilon_1^2 + (1+|t|)^{-2+2\delta}\epsilon_1^2.
\end{aligned} \tag{5.13}$$

Hence, from the above estimate, we have

$$\sum_{|\alpha|\leq N_0} 2^{k/2}\|\nabla_\xi \widehat{h^\alpha}(t,\xi)\psi_k(\xi)\|_{L^2_\xi}$$
$$\lesssim \epsilon_0 + \int_0^t \big[(1+s)^{-1+\delta}2^{k_-} + (1+s)^{-2+2\delta}\big]\epsilon_1^2 ds \lesssim \epsilon_0 + (1+t)^\delta 2^{k_-}\epsilon_0. \quad (5.14)$$

To sum up, recall again (4.62), our desired estimate (5.6) holds from the estimates (5.9), (5.11), and (5.14). □

5.1. Linear estimates and a bilinear estimate in the L^∞_ξ-type space. In this subsection, we mainly prove several L^∞_ξ-type linear estimates for the density type functions, which are summarized in Lemma 5.1, and a L^∞_ξ-type bilinear estimate, which is summarized in Lemma 5.2.

Lemma 5.1. *Given any $n\in\mathbb{N}_+$, s.t., $n\leq 10$, and any symbol $m(\xi,v)$ such that the following estimate holds,*

$$\sup_{k\in\mathbb{Z}} \sum_{i=0,1,\ldots,10,0\leq a\leq 15} 2^{ik-(n-1)k}\|(1+|v|)^{-20-4i}$$
$$\nabla_\xi^i \nabla_v^a m(\xi,v)\psi_k(\xi)\|_{L^\infty_\xi L^\infty_v} \lesssim 1, \quad (5.15)$$

the following estimate holds for any $i\in\{0,1,2,3\}$,

$$\|\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\widehat{g}(t,\xi,v)dv\|_{X_i}$$
$$\lesssim \sum_{0\leq b\leq i+n} (1+|t|)^{-n}\|(1+|v|)^{30}\nabla_v^b \widehat{g}(t,0,v)\|_{L^1_v}$$
$$+ \sum_{\beta\in\mathcal{S},|\beta|\leq i+n}$$
$$(1+|t|)^{-n-1}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta g(t,x,v)\|_{L^2_x L^2_v}. \quad (5.16)$$

Moreover, for any fixed $k\in\mathbb{Z}$ and any differentiable function $\tilde{g}(t,v):\mathbb{R}_t\times\mathbb{R}^3_v\to\mathbb{R}^3$, the following L^∞_ξ-type estimate holds,

$$2^k\|\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\widehat{g}(t,\xi,v)\psi_k(\xi)dv\|_{L^\infty_\xi}$$
$$\lesssim 2^{nk}\Big(\|(1+|v|)^{20}\big(\widehat{g}(t,0,v)-\nabla_v\cdot\tilde{g}(t,v)\big)\|_{L^1_v}$$
$$+(1+|t|2^k)\|(1+|v|)^{20}\tilde{g}(t,v)\|_{L^1_v}$$
$$+2^k\|(1+|x|+|v|)^{30}g(t,x,v)\|_{L^2_x L^2_v}\Big). \quad (5.17)$$

Proof. Note that for any $i\in\{0,1,2,3\}$, the following equalityholds,

$$\nabla_\xi^i\Big(\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\widehat{g}(t,\xi,v)dv\Big) = \sum_{0\le l_1+l_2\le i} H^{\mu}_{i;l_1,l_2}(t,\xi), \tag{5.18}$$

where

$$H^{\mu}_{i;l_1,l_2}(t,\xi) = \sum_{0\le k\le l_1}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} c^{l_2,k}_{i,l_1}\big(it\big)^{l_1} \big(\frac{\xi}{|\xi|}-\mu\hat{v}\big)^{l_1-k}\nabla_\xi^{i-l_1-l_2}\big(\big(\frac{\xi}{|\xi|}-\mu\hat{v}\big)^{k} m(\xi,v)\big)\nabla_\xi^{l_2}\widehat{g}(t,\xi,v)dv, \tag{5.19}$$

where the $c^{l_2,k}_{i,l_1}$ are some uniquely determined coefficients. From the above detailed formula of $H^{\mu}_{i;l_1,l_2}(t,\xi)$, we know that our desired estimate (5.16) is trivial if $|t|\le 1$. From now on, we restrict ourself to the case $|t|\ge 1$.

If $l_2=0$, then we separate out the zero frequency of "$\widehat{g}(t,\xi,v)$" as follows,

$$\begin{aligned} H^{\mu}_{i;l_1,0}(t,\xi) = \sum_{0\le k\le l_1}&\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} c^{l_2,k}_{i,l_1}\big(it\big)^{l_1}\big(\frac{\xi}{|\xi|}-\mu\hat{v}\big)^{l_1-k}\nabla_\xi^{i-l_1} \\ &\times\big(\big(\frac{\xi}{|\xi|}-\mu\hat{v}\big)^{k} m(\xi,v)\big)\widehat{g}(t,0,v)dv \\ &+\int_0^1\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} c^{l_2,k}_{i,l_1}\big(it\big)^{l_1} \\ &\big(\frac{\xi}{|\xi|}-\mu\hat{v}\big)^{l_1-k}\nabla_\xi^{i-l_1}\big(\big(\frac{\xi}{|\xi|}-\mu\hat{v}\big)^{k} m(\xi,v)\xi\cdot\nabla_\xi\widehat{g}(t,s\xi,v)dvds. \end{aligned} \tag{5.20}$$

For the first integral of $H^{\mu}_{i;l_1,0}(t,\xi)$ in (5.20), we do integration by parts in "v" $n+l_1$ times. For the second integral of $H^{\mu}_{i;l_1,0}(t,\xi)$ in (5.20) and $H^{\mu}_{i;l_1,l_2}(t,\xi)$, $|l_2|>0$, in (5.19), we do integration by parts in "v" $n+l_1+1$ times. As a result, from the estimate of symbol in (5.15), the following estimate holds for any fixed $k\in\mathbb{Z}$, i, l_1, and l_2,

$$\begin{aligned} 2^{(i+1)k}\|H^{\mu}_{i;l_1,l_2}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} &\lesssim \sum_{0\le b\le n+l_1}(1+|t|)^{-n}\|(1+|v|)^{30}\nabla_v^b\widehat{g}(t,0,v)\|_{L^1_v} \\ &+\sum_{1\le a\le i-l_1}(1+|t|)^{-n-1} \\ &\times\big(2^{(a-1)k}+2^{ak}\big)\|(1+|v|)^{30}\nabla_\xi^a\nabla_v^b\widehat{g}(t,\xi,v)\psi_k(\xi)\|_{L^\infty_\xi L^1_v} \\ &\lesssim \sum_{0\le b\le n+i}(1+|t|)^{-n}\|(1+|v|)^{30}\nabla_v^b\widehat{g}(t,0,v)\|_{L^1_v} \\ &+\sum_{\beta\in\mathcal{S},|\beta|\le i+n}(1+|t|)^{-n-1} \\ &\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta g(t,x,v)\|_{L^2_xL^2_v}. \end{aligned} \tag{5.21}$$

Hence our desired estimate (5.16) holds from (5.18) and the above estimate (5.21).

Now we proceed to prove the desired estimate (5.17). Note that the following equality holds after subtracting the correction term $\nabla_v \cdot \tilde{g}(t,v)$ and doing integration by parts in v,

$$\begin{aligned}
&\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\widehat{g}(t,\xi,v)\psi_k(\xi)dv \\
&= \int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\big(\widehat{g}(t,0,v) \\
&\quad - \nabla_v \cdot \tilde{g}(t,v)\big)\psi_k(\xi)dv \\
&\quad - \int_{\mathbb{R}^3} \nabla_v\big(e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\big)\cdot \tilde{g}(t,v)\psi_k(\xi)dv \\
&\quad + \int_0^1 \int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{v}\cdot\xi} m(\xi,v)\xi \\
&\quad \cdot\nabla_\xi \widehat{g}(t,s\xi,v)\psi_k(\xi)dvds. \qquad (5.22)
\end{aligned}$$

Therefore, our desired estimate (5.17) holds from the above equality and the estimate of symbol in (5.15). □

For any fixed $l \in \{0,1\}$ and any given symbol $m(\xi,v)$ such that the following estimate holds,

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \sum_{n=0,1,2,3} \sum_{0\le a\le 5} 2^{lk+nk} \\
&\quad \|(1+|v|)^{-20}\nabla_\xi^n \nabla_v^a m(\xi,v)\psi_k(\xi)\|_{L_v^\infty \mathcal{S}_k^\infty} \lesssim 1, \qquad (5.23)
\end{aligned}$$

we define a bilinear operator $T_\mu(\cdot,\cdot)(t,\xi)$, which represents the Vlasov-wave type interaction, as follows,

$$\begin{aligned}
T_\mu(h,f)(t,\xi) &:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} \\
&\quad m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)\widehat{f}(t,\eta,v)d\eta dv. \qquad (5.24)
\end{aligned}$$

Based on the relative size of $|\xi-\eta|/|\xi|$, we separate $T_\mu(h,f)(t,\xi)$ into two parts as follows,

$$T_\mu(h,f)(t,\xi) = T_\mu^1(h,f)(t,\xi) + T_\mu^2(h,f)(t,\xi), \qquad (5.25)$$

$$\begin{aligned}
T_\mu^1(h,f)(t,\xi) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} \\
&\quad m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)\widehat{f}(t,\eta,v)\psi_{\ge -10}(|\xi-\eta|/|\xi|)d\eta dv, \qquad (5.26)\\
T_\mu^2(h,f)(t,\xi) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)} \\
&\quad m(\xi,v)\widehat{h^\mu}(t,\eta)\widehat{f}(t,\xi-\eta,v)\psi_{< -10}(|\eta|/|\xi|)d\eta dv, \qquad (5.27)
\end{aligned}$$

where we changed the coordinates $\eta \longrightarrow \xi-\eta$ in $T_\mu^2(h,f)(t,\xi)$. Note that the following equality holds for any$n \in \{0,1,2,3\}$,

$$\nabla_\xi^n\big(T_\mu^i(h,f)(t,\xi)\big) = \sum_{0\le j\le n} T_{\mu,j}^{n,i}(t,\xi), \quad i=1,2, \tag{5.28}$$

where

$$T_{\mu,j}^{n,1}(t,\xi) = \sum_{0\le a\le j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} c_{a,j}^{n,1}(it)^j\big(\frac{\xi}{|\xi|} - \mu\frac{\xi-\eta}{|\xi-\eta|}\big)^a \nabla_\xi^{n-j}\big[\big(\frac{\xi}{|\xi|}-\mu\frac{\xi-\eta}{|\xi-\eta|}\big)^{j-a} \times m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)\psi_{\ge -10}(|\xi - \eta|/|\xi|)\big)\big]\widehat{f}(t,\eta,v)d\eta dv, \tag{5.29}$$

$$T_{\mu,j}^{n,2}(t,\xi) = \sum_{0\le a\le j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)} c_{a,j}^{n,2}(it)^j\big(\frac{\xi}{|\xi|}-\hat{v}\big)^a\nabla_\xi^{n-j} \big[\big(\frac{\xi}{|\xi|}-\hat{v}\big)^{j-a} m(\xi,v) \times \widehat{f}(t,\xi-\eta,v)\psi_{<-10}(|\eta|/|\xi|)\big)\big] \widehat{h^\mu}(t,\eta)d\eta dv, \tag{5.30}$$

where $c_{a,j}^{n,1}$ and $c_{a,j}^{n,2}$ are some determined constants, whose explicit formulas are not pursued here.

Our desired bilinear estimate is summarized in the following Lemma.

Lemma 5.2. *For any $n\in\{0,1,2,3\}$, any $l\in\{0,1\}$, and any given symbol "$m(\xi,v)$" that satisfies the estimate (5.23), the following estimate holds for the bilinear form $T_\mu(h,f)(t,\xi)$ defined in (5.24),*

$$\sup_{k\in\mathbb{Z}} 2^{(n+1)k}\|\nabla_\xi^n\big(T_\mu(h,f)(t,\xi)\big)\psi_k(\xi)\|_{L_\xi^\infty} \lesssim \sum_{\beta\in\mathcal{S},|\beta|\le n+4}(1+|t|)^{-3+l}\big(\sum_{0\le c\le n}\sum_{0\le b\le n-c}\sum_{|\alpha|\le c}\|h^\alpha\|_{X_b}\big) \times\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L_x^2L_v^2}. \tag{5.31}$$

Proof. Recall the decompositions in (5.25) and (5.28). To control X_n-norm of $T_\mu(h,f)(t,\xi)$, it would be sufficient to control L^∞-norm of $T_{\mu,j}^{n,1}(t,\xi)$ and $T_{\mu,j}^{n,2}(t,\xi)$.

Recall (5.29). We first do integration by parts in" η" j-times for $T_{\mu,j}^{n,1}(t,\xi)$. As a result, we have

$$T_{\mu,j}^{n,1}(t,\xi) = \sum_{0\le a\le j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} c_{a,j}^{n,1}\nabla_\eta\cdot \Big[\frac{i\mu\frac{\xi-\eta}{|\xi-\eta|}-i\hat{v}}{\big|\frac{(\xi-\eta)}{|\xi-\eta|}-\mu\hat{v}\big|^2}\circ\nabla_\eta\cdot\Big[\frac{i\mu\frac{\xi-\eta}{|\xi-\eta|}-i\hat{v}}{\big|\frac{(\xi-\eta)}{|\xi-\eta|}-\mu\hat{v}\big|^2}\circ\cdots\circ\nabla_\eta\cdot\Big[\frac{i\mu\frac{\xi-\eta}{|\xi-\eta|}-i\hat{v}}{\big|\frac{(\xi-\eta)}{|\xi-\eta|}-\mu\hat{v}\big|^2}$$

$$\times \big[\big(\frac{\xi}{|\xi|} - \mu\frac{\xi-\eta}{|\xi-\eta|}\big)^a \nabla_\xi^{n-j}\big[\big(\frac{\xi}{|\xi|}$$
$$-\mu\frac{\xi-\eta}{|\xi-\eta|}\big)^{j-a} m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)$$
$$\psi_{\geq -10}(|\xi-\eta|/|\xi|)\big)\big]\widehat{f}(t,\eta,v)\big]\cdots\big]\Big]d\eta dv,$$

Now, we apply the dyadic decomposition for "$\xi-\eta$" and "η". As a result, we have

$$T^{n,1}_{\mu,j}(t,\xi)\psi_k(\xi) = \sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2} H^{\mu,n,j}_{k,k_1,k_2}(t,\xi), \tag{5.32}$$

where

$$H^{\mu,n,j}_{k,k_1,k_2}(t,\xi) = \sum_{0\leq a\leq j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} c^{n,1}_{a,j}\nabla_\eta$$
$$\cdot\Big[\frac{i\mu\frac{\xi-\eta}{|\xi-\eta|}-i\hat{v}}{\big|\frac{(\xi-\eta)}{|\xi-\eta|}-\mu\hat{v}\big|^2}\circ\nabla_\eta\cdot\Big[\frac{i\mu\frac{\xi-\eta}{|\xi-\eta|}-i\hat{v}}{\big|\frac{(\xi-\eta)}{|\xi-\eta|}-\mu\hat{v}\big|^2}$$
$$\circ\cdots\big[\big(\frac{\xi}{|\xi|}-\mu\frac{\xi-\eta}{|\xi-\eta|}\big)^a\nabla_\xi^{n-j}\big[\big(\frac{\xi}{|\xi|}$$
$$-\mu\frac{\xi-\eta}{|\xi-\eta|}\big)^{j-a} m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)$$
$$\times\psi_{\geq -10}(|\xi-\eta|/|\xi|)\big)\big]\widehat{f}(t,\eta,v)\big]$$
$$\circ\cdots\big]\Big]\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta dv, \tag{5.33}$$

Recall (5.30), after doing dyadic decomposition for $\xi-\eta$ and η, we have

$$T^{n,2}_{\mu,j}(t,\xi) = \sum_{(k_1,k_2)\in\chi_k^3} K^{\mu,n,j}_{k,k_1,k_2}(t,\xi) \tag{5.34}$$

where

$$K^{\mu,n,j}_{k,k_1,k_2}(t,\xi) = \sum_{0\leq a\leq j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)} c^{n,2}_{a,j}(it)^j$$
$$\big(\frac{\xi}{|\xi|}-\hat{v}\big)^a\nabla_\xi^{n-j}\big[\big(\frac{\xi}{|\xi|}-\hat{v}\big)^{j-a} m(\xi,v)$$
$$\times\widehat{f}(t,\xi-\eta,v)\psi_{<-10}(|\eta|/|\xi|)\big)\big]\widehat{h^\mu}(t,\eta)\psi_{k_1}(\eta)\psi_k(\xi)\psi_{k_2}$$
$$\times(\xi-\eta)d\eta dv. \tag{5.35}$$

Note that, from the estimate of symbol "$m(\xi,v)$" in (5.23), the desired estimate (5.31) is trivial if $|t|\leq 1$. Therefore, from now on, we restrict ourself to the case $|t|\geq 1$.

- The estimate of $H^{\mu,n,j}_{k,k_1,k_2}$.

Recall (5.32) and (5.33). Note that $(k_1,k_2)\in\chi_k^1\cup\chi_k^2$. After using the volume of support of η and the estimate of symbol $m(\xi,v)$ in (5.23), we have

$$2^{(n+1)k}\|H^{\mu,n,j}_{k,k_1,k_2}\psi_k(\xi)\|_{L^\infty_\xi}$$

$$\lesssim \sum_{0\le a\le 4}\sum_{0\le |c|\le n} 2^{(n+1-l)k+3k_2-4k_{2,+}}\|(1+|v|)^{4n+10}\nabla_\xi^c \widehat{\nabla_x^a f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}$$
$$\times\Big(\sum_{0\le b\le n-c} 2^{-bk}\|\nabla_\xi^{n-c-b}\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\Big)$$
$$\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 4} 2^{-k_1+(1-l)k+3k_{2,-}}\Big(\sum_{0\le c\le n}\sum_{0\le b\le n-c}\sum_{\alpha\in\mathcal{B},|\alpha|\le c}\|h^\alpha(t)\|_{X_b}\Big)\|(1$$
$$+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}. \tag{5.36}$$

After doing integration by parts in v four times for "$H^{\mu,n,j}_{k,k_1,k_2}$", the following estimate holds,

$$2^{(n+1)k}\|H^{\mu,n,j}_{k,k_1,k_2}\psi_k(\xi)\|_{L^\infty_\xi}$$
$$\lesssim |t|^{-4}2^{(n+1-l)k-k_2}\Big[\sum_{0\le c\le n}\sum_{0\le d\le 4}\Big(\sum_{0\le b\le n-c}2^{-bk}\|\nabla_\xi^{n-c-b}\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\Big)$$
$$\times\|(1+|v|)^{4n+10}\nabla_v^d\nabla_\xi^c\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\Big]$$
$$\lesssim |t|^{-4}2^{-k_1+(1-l)k-k_2}\Big(\sum_{0\le c\le n}\sum_{0\le b\le n-c}\sum_{|\alpha|\le c}\|h^\alpha\|_{X_b}\Big)\Big(\sum_{\beta\in\mathcal{S},|\beta|\le 4}\|(1+|x|^2$$
$$+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}\Big). \tag{5.37}$$

Therefore, from (5.36) and (5.37), the following estimate holds for any fixed $k\in\mathbb{Z}$,

$$\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2} 2^{(n+1)k}\|H^{\mu,n,j}_{k,k_1,k_2}\psi_k(\xi)\|_{L^\infty_\xi}$$
$$\lesssim\Big(\sum_{k_2,2^{k_2}\le|t|^{-1}}2^{(3-l)k_2}+\sum_{k_2,2^{k_2}\ge|t|^{-1}}|t|^{-4}2^{-(1+l)k_2}\Big)$$
$$\times\Big(\sum_{0\le c\le n}\sum_{0\le b\le n-c}\sum_{|\alpha|\le c}\|h^\alpha\|_{X_b}\Big)\Big(\sum_{\beta\in\mathcal{S},|\beta|\le 4}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}\Big)$$
$$\lesssim |t|^{-3+l}\Big(\sum_{0\le c\le n}\sum_{0\le b\le n-c}\sum_{|\alpha|\le c}\|h^\alpha\|_{X_b}\Big)\Big(\sum_{\beta\in\mathcal{S},|\beta|\le 4}\|(1+|x|^2$$
$$+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}\Big). \tag{5.38}$$

- The estimate of $K^{\mu,n,j}_{k,k_1,k_2}$.
  Recall (5.34) and (5.35). From the estimate of symbol "$m(\xi,v)$" in (5.23), the following estimate holds after using the volume of support of "η",

$$2^{(n+1)k}\|K^{\mu,n,j}_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi}$$
$$\lesssim |t|^j2^{(n+1-l)k+3k_1}\Big[\sum_{0\le b\le n-j}2^{-bk_2}\|(1+|v|)^{4n+10}\nabla_\xi^{n-j-b}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}$$
$$\times\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\Big]$$
$$\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le n-j}|t|^j2^{(j+1-l)k+2k_1}\|h\|_{X_0}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}. \tag{5.39}$$

Moreover, after doing integration by parts in "v" "$(j+3)$" times for "$K^{\mu,n,j}_{k,k_1,k_2}(t,\xi)$" in (5.35) and using the volume of support of "η", we have

$$\begin{aligned}
&2^{(n+1)k}\|K^{\mu,n,j}_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi}\\
&\quad\lesssim (1+|t|)^{-3}2^{(n-j-l-2)k+3k_1}\\
&\quad\quad\times\Big[\sum_{0\le b\le n-j}\sum_{0\le a\le j+3}2^{-bk_2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|(1\\
&\quad\quad+|v|)^{4n+10}\nabla_v^a\nabla_\xi^{n-j-b}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\Big]\\
&\quad\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le n+3}(1+|t|)^{-3}2^{2k_1-(2+l)k}\|h\|_{X_0}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.40}$$

Combining the estimates (5.39) and (5.40), we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\chi^3_k}2^{(n+1)k}\|K^{\mu,n,j}_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi}\\
&\quad\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le n+3}\|h\|_{X_0}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^1_v}\min\big\{(1\\
&\quad\quad+|t|)^j2^{(j+3-l)k},(1+|t|)^{-3}2^{-lk}\big\}\\
&\quad\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le n+3}(1+|t|)^{-3+l}\|h\|_{X_0}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.41}$$

To sum up, our desired estimate (5.31) holds from the estimates (5.38) and (5.41). □

5.2. Bilinear estimates in the high order energy space: Vlasov-Wave type interaction. In this subsection, we mainly prove several bilinear estimates for the Vlasov-Wave type interaction in different function spaces. As we have seen in the proof of Propositions 5.1 and 5.2, these bilinear estimates play important roles in the estimate of the high order energy $E^{\phi}_{\text{high}}(t)$ (see (4.62)) of the nonlinear wave part.

We first prove a L^∞_ξ-type bilinear estimate.

Lemma 5.3. *Given any symbol "$m(\xi,v)$" that satisfies the estimate (5.23) with $l=1$, the following estimate holds for the bilinear form $T_\mu(h,f)(t,\xi)$ defined in (5.24),*

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}2^k\|T_\mu(h,f)(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\quad\lesssim\sum_{n=0,1,2,\alpha\in\mathcal{B},|\alpha|\le 4}(1+|t|)^{-2+\delta}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.42}$$

Proof. Recall (5.24). After doing dyadic decompositions for two inputs, we have

$$T_\mu(h,f)(t,\xi)\psi_k(\xi)=\sum_{i=1,2,3}\sum_{(k_1,k_2)\in\chi^i_k}T^\mu_{k,k_1,k_2}(t,\xi),\tag{5.43}$$

where $\chi_k^i, i \in \{1,2,3\}$, are defined in (2.2) and (2.3) and the detailed formula of $T^{\mu}_{k,k_1,k_2}(t,\xi)$ is given as follows,

$$T^{\mu}_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} m(\xi,v)\widehat{h^{\mu}}(t,\xi-\eta)\widehat{f}(t,\eta,v)\psi_{k_1}(\xi - \eta)\psi_{k_2}(\eta)\psi_k(\xi)d\eta dv. \tag{5.44}$$

From the above detailed formula, our desired estimates (5.42) and (5.49) hold easily if $|t| \leq 1$. From now on, we restrict ourself to the case $|t| \geq 1$. Recall (5.44). After doing integration by parts in η once for $T^{\mu}_{k,k_1,k_2}(t,\xi)$, we have

$$T^{\mu}_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} m(\xi,v)\psi_k(\xi)\frac{i}{t}\nabla_\eta \cdot\Big(\frac{\mu(\xi-\eta)/|\xi-\eta| - \hat{v}}{|\mu(\xi-\eta)/|\xi-\eta| - \hat{v}|^2} \times \widehat{h^{\mu}}(t,\xi-\eta)\widehat{f}(t,\eta,v)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\Big)d\eta dv, \tag{5.45}$$

From the estimate of symbol "$m(\xi,v)$" in (5.23) and the volume of support of η, we have

$$\begin{aligned} 2^k\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi} &\lesssim |t|^{-1}2^{3\min\{k_1,k_2\}}\big(2^{-\min\{k_1,k_2\}}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi} + \|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big) \\ &\quad\times\big(\|(1+|v|)^{10}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v} + 2^{\min\{k_1,k_2\}}\|(1 + |v|)^{10}\nabla_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\big) \\ &\lesssim \sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\leq 4} |t|^{-1}2^{\min\{k_1,k_2\}-2k_{1,+}}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2 + |v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \end{aligned} \tag{5.46}$$

Similarly, the following estimate holds after doing integration by parts in "η" twice,

$$2^k\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi} \lesssim \sum_{n=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 4} |t|^{-2}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \tag{5.47}$$

Combining the estimates (5.46) and (5.47), the following estimate holds for any fixed $k \in \mathbb{Z}$,

$$\begin{aligned} \sum_{i=1,2,3,(k_1,k_2)\in\chi_k^i} 2^k\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi} &\lesssim \sum_{n=0,1,2,k_1\in\mathbb{Z},|\alpha|\leq 4} |t|^{-2+\delta}2^{\delta k_1 - 2\delta k_{1,+}}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v} \\ &\quad\times\|h^\alpha(t)\|_{X_n} \lesssim \sum_{n=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 4} |t|^{-2+\delta}\big\|h^\alpha(t)\|_{X_n}\|(1+|x|^2 + |v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \end{aligned} \tag{5.48}$$

Hence finishing the proof of the desired estimate (5.42). □

Now, we prove a L^2_ξ-type bilinear estimate.

Lemma 5.4. *Given any symbol "$m(\xi, v)$" that satisfies the estimate (5.23) with $l = 1$, the following estimate holds for the bilinear form $T_\mu(h, f)(t, \xi)$ defined in (5.24),*

$$\begin{aligned}
&\|T_\mu(h, f)(t,\xi)\|_{L^2_\xi}\\
&\lesssim \min\Big\{ \sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\le 4} (1+|t|)^{-1}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20} f(t,x,v)\|_{L^2_x L^2_v},\\
&\sum_{\beta\in\mathcal{S},|\beta|\le 3} (1+|t|)^{-2}\|h(t)\|_{L^2}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_{x,v}}\Big\}. \qquad (5.49)
\end{aligned}$$

Proof. Note that the decomposition (5.43) and the equality (5.44) still hold and our desired estimate (5.49) hold straightforwardly if $|t| \le 1$. Hence, we restrict ourself to the case $|t| \ge 1$.

- Proof of the first estimate in (5.49). Based on the possible sizes of k, k_1, k_2, we separate into three sub-cases as follow.

Subcase 1 If $k \le 0$. From the estimate (5.46) and the volume of support of ξ, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z},k\le 0}\ \sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2\cup\chi_k^3} \|T^\mu_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim \sum_{k\in\mathbb{Z},k\le 0}\ \sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\le 4} \|(1+|x|^2+|v|^2)^{20} f(t,x,v)\|_{L^2_x L^2_v}\\
&\quad\times|t|^{-1}2^{k/2}\big\|h^\alpha(t)\|_{X_n} \lesssim \sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\le 4} |t|^{-1}\big\|h^\alpha(t)\|_{X_n}\|(1+|x|^2\\
&\quad+|v|^2)^{20} f(t,x,v)\|_{L^2_x L^2_v}. \qquad (5.50)
\end{aligned}$$

Subcase 2 If $k \ge 0$ and $(k_1, k_2) \in \chi_k^1 \cup \chi_k^2$.

Note that we have $k_1 \ge k - 10$ and $k_2 \le k + 10$ for the subcase we are considering. Recall (5.45). From the estimate of symbol "$m(\xi, v)$" in (5.23) and the volume of support of "ξ" and "η", we have

$$\begin{aligned}
&\|T^\mu_{k,k_1,k_2}(t,\xi)\|_{L^2}\\
&\lesssim \sum_{\alpha\in\mathcal{B},|\alpha|\le 4} |t|^{-1}2^{k/2+3k_2-4k_{1,+}}\big(2^{-k_2}\|\widehat{h^\alpha}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi} + \|\nabla_\xi\widehat{h^\alpha}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big)\\
&\quad\times\big(\|(1+|v|)^{10}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v} + 2^{k_2}\|(1+|v|)^{10}\nabla_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\big)\\
&\lesssim \sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\le 4} |t|^{-1}2^{k_2-3k_{1,+}/2}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2\\
&\quad+|v|^2)^{20} f(t,x,v)\|_{L^2_x L^2_v}. \qquad (5.51)
\end{aligned}$$

From the above estimate, wehave

$$\sum_{k\in\mathbb{Z},k\geq 0}\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}$$
$$\lesssim\sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\leq 4}|t|^{-1}\|h^{\alpha}(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.52)$$

Subcase 3 If $k\geq 0$ and $(k_1,k_2)\in\chi_k^3$.

For this case, we first switch the role of "$\xi-\eta$" and "η" in (5.45). Instead of using the volume of support of "ξ", we use the Minkowski inequality. As a result, from the estimate of symbol "$m(\xi,v)$" in (5.23), we have

$$\sum_{k\in\mathbb{Z},k\geq 0,(k_1,k_2)\in\chi_k^3}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}$$
$$\lesssim\sum_{k\in\mathbb{Z},k\geq 0}\sum_{k_1\leq k-5,|k-k_2|\leq 5}\sum_{\alpha\in\mathcal{B},|\alpha|\leq 4}|t|^{-1}2^{3k_1-k-4k_{1,+}}$$
$$\times\big(\|(1+|v|)^{10}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_vL^2_\xi}+2^{k_1}\|(1$$
$$+|v|)^{10}\nabla_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_vL^2_\xi}\big)\big(2^{-k_1}\|\widehat{h^{\alpha}}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}$$
$$+\|\nabla_\xi\widehat{h^{\alpha}}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big)\lesssim\sum_{n=0,1}\sum_{\alpha\in\mathcal{B},|\alpha|\leq 4}|t|^{-1}\|h^{\alpha}(t)\|_{X_n}\|(1+|x|^2$$
$$+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.53)$$

To sum up, recall the decomposition (5.43), our desired first estimate in (5.49) holds from the estimates (5.50), (5.52) and (5.53).

- Proof of the second estimate in (5.49). Based on the size of k_1 and k_2, we separate into three subcases as follows.

Subcase 1 If $(k_1,k_2)\in\chi_k^1$.

For this case, we have $|k_1-k_2|\leq 10$ and $k\leq k_1+10$. Recall (5.44). After using the volume of "ξ", the estimate of symbol "$m(\xi,v)$" in (5.23), the L^2-L^2 type bilinear estimate, and the volume of support of "η", we have

$$\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\lesssim 2^{k/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(t,\xi)\|_{L^2_\xi L^1_v}$$
$$\lesssim 2^{k/2+3k_2/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(t,\xi)\|_{L^\infty_\xi L^1_v}. \quad (5.54)$$

Moreover, after first doing integration by parts in "v" three times and then using the volume of "ξ" and "η", the following estimate holds from the estimate of symbol "$m(\xi,v)$" in (5.23),

$$\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\lesssim\sum_{0\leq a\leq 3}|t|^{-3}2^{k/2-3k_2/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1$$
$$+|v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\psi_{k_2}(t,\xi)\|_{L^\infty_\xi L^1_v}. \quad (5.55)$$

From (5.54) and (5.55), wehave

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\sum_{(k_1,k_2)\in\chi_k^1}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\\&\lesssim\sum_{\beta\in\mathcal{S},|\beta|\leq 3}\big[\big(\sum_{k_1\in\mathbb{Z},2^{k_1}\leq|t|^{-1}}\sum_{k\leq k_1+10}2^{k/2+3k_1/2}\big)\\&\quad+\big(\sum_{k_1\in\mathbb{Z},2^{k_1}\geq|t|^{-1}}\sum_{k\leq k_1+10}|t|^{-3}2^{k/2-3k_1/2}\big)\big]\|h(t)\|_{L^2}\\&\quad\times\|(1+|x|^2+|v|^2)^{20}\Lambda^{\beta}f(t,x,v)\|_{L^2_{x,v}}\lesssim\sum_{\beta\in\mathcal{S},|\beta|\leq 3}|t|^{-2}\|h(t)\|_{L^2}\|(1+|x|^2\\&\quad+|v|^2)^{20}\Lambda^{\beta}f(t,x,v)\|_{L^2_{x,v}}.\end{aligned}\tag{5.56}$$

Subcase 2 If $(k_1,k_2)\in\chi_k^2$.

For this case we have $|k-k_1|\leq 10, k_2\leq k_1-5$. Recall (5.44). After using the volume of "η" and the estimate of symbol "$m(\xi,v)$" in (5.23), the following estimate holds,

$$\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\lesssim 2^{-k+3k_2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\|_{L^\infty_\xi L^1_v}.\tag{5.57}$$

Moreover, after first doing integration by parts in "v" three times and then using the volume of support of "η ", the following estimate holds from the estimate of symbol $m(\xi,v)$ in (5.23),

$$\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\lesssim\sum_{0\leq a\leq 3}2^{-k}|t|^{-3}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1+|v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\|_{L^\infty_\xi L^1_v}.\tag{5.58}$$

Therefore, from the orthogonality in L^2_ξ and the estimates (5.57) and (5.58) derived above, we have

$$\begin{aligned}&\|\sum_{k\in\mathbb{Z}}\sum_{(k_1,k_2)\in\chi_k^2}T^{\mu}_{k,k_1,k_2}(t,\xi)\|^2_{L^2}\\&\lesssim\sum_{|k-k_1|\leq 10}\big(\sum_{k_2\leq k_1-5}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\big)^2\lesssim\sum_{\beta\in\mathcal{S},|\beta|\leq 3}\sum_{k_1\in\mathbb{Z}}\big[\\&\quad\big(\sum_{2^{k_2}\leq|t|^{-1}}2^{2k_2}+\sum_{2^{k_2}\geq|t|^{-1}}|t|^{-3}2^{-k_2}\big)\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1+|x|^2\\&\quad+|v|^2)^{20}\Lambda^{\beta}f(t,x,v)\|_{L^2_{x,v}}\big]^2\\&\lesssim\sum_{\beta\in\mathcal{S},|\beta|\leq 3}\sum_{k_1\in\mathbb{Z}}|t|^{-4}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|^2_{L^2_\xi}\|(1+|x|^2+|v|^2)^{20}\Lambda^{\beta}f(t,x,v)\|^2_{L^2_{x,v}}\\&\lesssim\sum_{\beta\in\mathcal{S},|\beta|\leq 3}|t|^{-4}\|h(t)\|^2_{L^2}\|(1+|x|^2+|v|^2)^{20}\Lambda^{\beta}f(t,x,v)\|^2_{L^2_{x,v}}.\end{aligned}\tag{5.59}$$

Subcase 3 If $(k_1,k_2)\in\chi_k^3$.

For this case we have $|k-k_2|\leq 10$ and $k_1\leq k-5$. Recall (5.44). On one hand, after using the L^2-L^∞ type bilinear estimate, the $L^\infty\to L^2$ type Sobolev embedding, the estimate of symbol "$m(\xi, v)$" in (5.23) and the volume of support of the frequency variable, we have

$$\begin{aligned}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2} &\lesssim 2^{-k+3k_1/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1\\ &\quad+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^2_\xi L^1_v}\\ &\lesssim 2^{-k+3k_1/2+3k_2/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1\\ &\quad+|v|)^{25}\widehat{f}(t,\xi,v)\|_{L^\infty_\xi L^1_v}.\end{aligned} \tag{5.60}$$

On the other hand, we first do integration by parts in "v" three times and then use the L^2-L^∞ type bilinear estimate, the $L^\infty \to L^2$ type Sobolev embedding, the estimate of symbol "$m(\xi, v)$" in (5.23) and the volume of support of the frequency. As a result, we have

$$\begin{aligned}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2} &\lesssim \sum_{0\le a\le 3}|t|^{-3}2^{-k-3k_2+3k_1/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1\\ &\quad+|v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^2_\xi L^1_v}\\ &\lesssim \sum_{0\le a\le 3}|t|^{-3}2^{-k+3k_1/2-3k_2/2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\|(1\\ &\quad+|v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\|_{L^\infty_\xi L^1_v}.\end{aligned} \tag{5.61}$$

Therefore, from the estimates (5.60) and (5.61), we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\sum_{(k_1,k_2)\in\chi_k^3}\|T^{\mu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\\ &\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 3}\Big(\sum_{k_2\in\mathbb{Z},2^{k_2}\le|t|^{-1}}\sum_{k_1\le k_2-5}2^{3k_1/2+k_2/2}\\ &\quad+\sum_{k_2\in\mathbb{Z},2^{k_2}\ge|t|^{-1}}\sum_{k_1\le k_2-5}|t|^{-3}2^{3k_1/2-5k_2/2}\Big)\|h(t)\|_{L^2}\\ &\quad\times\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_{x,v}}\\ &\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 3}|t|^{-2}\|h(t)\|_{L^2}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_{x,v}}.\end{aligned} \tag{5.62}$$

To sum up, recall the decomposition (5.43), our desired second estimate in (5.49) holds from the estimates (5.56), (5.59) and (5.62). Hence finishing the proof. □

Our last bilinear estimates concern the weighted L^2-type estimate, which corresponds to the last part of the high order energy defined in (4.62). Recall (5.25), (5.26), and (5.27). We have

$$\nabla_\xi\big(T_\mu(h,f)(t,\xi)\big) = O^1(t,\xi)+O^2(t,\xi), \tag{5.63}$$

where

$$O^1(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta}\nabla_\xi\big(m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)\psi_{\ge -10}(|\xi$$

$$- \eta|/|\xi|)\big)\widehat{f}(t,\eta,v)$$
$$+ e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)}\nabla_\xi$$
$$\times\big(m(\xi,v)\psi_{<-10}(|\eta|/|\xi|)\widehat{f}(t,\xi-\eta,v)\big)\widehat{h^\mu}(t,\eta)d\eta dv, \tag{5.64}$$

$$O^2(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} it\big(\frac{\xi}{|\xi|} - \mu\frac{\xi-\eta}{|\xi-\eta|}\big)m(\xi,v)\widehat{h^\mu}(t,\xi$$
$$-\eta)\widehat{f}(t,\eta,v)\psi_{\geq -10}(|\xi-\eta|/|\xi|)$$
$$+ e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)} it\big(\frac{\xi}{|\xi|} - \hat{v}\big)m(\xi,v)\widehat{h^\mu}(t,\eta)\widehat{f}(t,\xi$$
$$-\eta,v)\psi_{<-10}(|\eta|/|\xi|)d\eta dv. \tag{5.65}$$

For the sake of clarity, we first summarize our desired two weighted L^2-type bilinear estimates in the following Proposition. The proof of this proposition consists of four parts, which will be elaborated in the next four Lemmas.

Proposition 5.3. *Given any symbol "$m(\xi,v)$" that satisfies the estimate* (5.23) *with $l = 1$ and any fixed $k \in \mathbb{Z}$, the following estimate holds for the bilinear form $T_\mu(h,f)(t,\xi)$ defined in* (5.24)*,*

$$2^{k/2}\|\nabla_\xi\big(T_\mu(h,f)(t,\xi)\big)\psi_k(\xi)\|_{L^2}$$
$$\lesssim \sum_{0\leq n\leq 3}\sum_{\alpha\in\mathcal{B},|\alpha|\leq 4}\big((1+|t|)^{-1}2^{k_-} + (1+|t|)^{-2+\delta}\big)\|h^\alpha(t)\|_{X_n}$$
$$\times\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \tag{5.66}$$

Moreover, the following bilinear estimate also holds,

$$2^{k/2}\|\nabla_\xi\big(T_\mu(h,f)(t,\xi)\big)\psi_k(\xi)\|_{L^2}$$
$$\lesssim \sum_{\beta\in\mathcal{S},|\beta|\leq 4}(1+|t|)^{-2}\big(\sup_{k\in\mathbb{Z}} 2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)$$
$$\times\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}. \tag{5.67}$$

Proof. Recall (5.63), (5.64), and (5.65). The desired estimates (5.66) and (5.67) follows directly from estimate (5.68) in Lemma 5.5, estimate (5.77) in Lemma 5.6, estimate (5.84) in Lemma 5.7, and estimate (5.93) in Lemma 5.8. □

Lemma 5.5. *For "$O^1(t,\xi)$" defined in* (5.64)*, the following estimate holds for any fixed $k \in \mathbb{Z}$,*

$$2^{k/2}\|O^1(t,\xi)\psi_k(\xi)\|_{L^2}$$
$$\lesssim \sum_{0\leq n\leq 3}\sum_{\alpha\in\mathcal{B},|\alpha|\leq 4}\big((1+|t|)^{-1}2^{k_-} + (1+|t|)^{-2+\delta}\big)\|h^\alpha(t)\|_{X_n}\|(1+|x|^2$$
$$+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \tag{5.68}$$

Proof. Firstly, we do dyadic decomposition for two frequencies "$\xi-\eta$" and "η". As a result, we have

$$O^1(t,\xi)\psi_k(\xi) = \sum_{i=1,2,3}\sum_{(k_1,k_2)\in\chi_k^i} O^1_{k,k_1,k_2}(t,\xi), \tag{5.69}$$

where $\chi_k^i, i \in \{1, 2, 3\}$ are defined in (2.2) and (2.3),

$$O^1_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta}\nabla_\xi\big(m(\xi,v)\widehat{h^\mu}(t,\xi-\eta)\psi_{\geq -10}(|\xi-\eta|/|\xi|)\big)$$
$$\times \widehat{f}(t,\eta,v)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\psi_k(\xi)d\eta dv, \quad \text{when } (k_1,k_2)\in\chi_k^1\cup\chi_k^2, \tag{5.70}$$

$$O^1_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)}\nabla_\xi\big(m(\xi,v)\widehat{f}(t,\xi-\eta,v)\psi_{<-10}(|\eta|/|\xi|)\big)$$
$$\times \widehat{h^\mu}(t,\eta)\psi_{k_1}(\eta)\psi_{k_2}(\xi-\eta)\psi_k(\xi)d\eta dv, \quad \text{when } (k_1,k_2)\in\chi_k^3. \tag{5.71}$$

From the above detailed formulas of $O^1_{k,k_1,k_2}(t,\xi)$, the desired estimate (5.68) holds straightforwardly if $|t|\leq 1$. It would be sufficient to consider the case $|t|\geq 1$. For any fixed $k\in\mathbb{Z}$, we separate into two cases as follows.

• **Case** 1: If $(k_1,k_2)\in\chi_k^1\cup\chi_k^2$. For this case we have $k_1\geq k-10$, $k_2\leq k_1+10$. Recall (5.70) and the estimate of symbol $m(\xi,v)$ in (5.23). After doing integration by parts in "η" once, we have

$$\begin{aligned}
2^{k/2}&\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2} \lesssim 2^{2k}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^\infty_\xi} \lesssim (1+|t|)^{-1}2^{k+3k_2}\big[\|(1\\
&+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\\
&\times\big(2^{-k-k_2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}+2^{-\min\{k,k_2\}}\|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\\
&+\|\nabla^2_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big)\\
&+\big(2^{-\min\{k,k_2\}}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}+\|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big)\|(1\\
&+|v|)^{25}\nabla_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\big]\\
q\lesssim&\sum_{n=0,1,2}\sum_{\alpha\in\mathcal{B},|\alpha|\leq 4}(1+|t|)^{-1}2^{2k_2-k_1-2k_{1,+}}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2\\
&+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.72}$$

Similarly, after doing integration by parts in "η" twice, the following estimate holds,

$$\begin{aligned}
&2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2}\\
&\lesssim\sum_{n=0,1,2,3}\sum_{\alpha\in\mathcal{B},|\alpha|\leq 4}(1+|t|)^{-2}2^{k_2-k_1}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.73}$$

To sum up, after interpolating the estimates (5.72) and (5.73), we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2}2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2}\\
&\quad\lesssim\sum_{\substack{n=0,1,2,3\\ \alpha\in\mathcal{B},|\alpha|\leq 4}}(1+|t|)^{-2+\delta}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.74}$$

• **Case** 2: If $(k_1, k_2) \in \chi_k^3$. Recall (5.71). After doing integration by parts in "η" once, we have

$$O^1_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)}\frac{1}{t}\nabla_\eta\cdot\Big[\frac{-i\mu\eta/|\eta|+i\hat{v}}{|\mu\eta/|\eta|-\hat{v}|^2}\nabla_\xi\big(m(\xi,v)\widehat{f}(t,\xi - \eta, v)\psi_{<-10}(|\eta|/|\xi|)\big) \times \widehat{h^\mu}(t,\eta)\psi_{k_1}(\eta)\psi_{k_2}(\xi-\eta)\Big]\psi_k(\xi)d\eta dv.$$

If ∇_ξ doesn't hit on $\widehat{f}(t,\xi-\eta,v)$ in the above integral, we use the volume of support of ξ and η. If ∇_ξ does hit on $\widehat{f}(t,\xi-\eta,v)$, we first use the L^2-L^∞ type bilinear estimate by putting $\widehat{f}(t,\xi-\eta,v)$ in L^2 and the other input in L^∞ and then use the volume of support of η. As a result, the following estimate holds,

$$\begin{aligned}
&2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2}\\
&\lesssim (1+|t|)^{-1}2^{-k/2+3k_1}\Big[\big(2^{-k_1}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi} + \|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\eta)\|_{L^\infty_\xi}\big)\\
&\quad\times\big(2^{k/2}\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v} + \|(1+|v|)^{25}\nabla_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_v L^2_\xi}\\
&\quad+2^{k_1}\|(1+|v|)^{25}\nabla^2_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_vL^2_\xi}\big)\Big].
\end{aligned}\tag{5.75}$$

From the above estimate (5.75), we have

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\chi_k^3} 2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2}\\
&\lesssim \sum_{k_1\le k}\sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\le 4}(1+|t|)^{-1}2^{k_1-2k_{1,+}}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}\\
&\quad\times\|h^\alpha(t)\|_{X_n}\lesssim\sum_{n=0,1,\alpha\in\mathcal{B},|\alpha|\le 4}(1+|t|)^{-1}2^{k_-}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2\\
&\quad+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.76}$$

Combining estimates (5.74) and (5.76), our desired estimate (5.68) holds. □

Lemma 5.6. *For "$O^1(t,\xi)$" defined in* (5.64)*, the following estimate holds for any fixed $k\in\mathbb{Z}$,*

$$\begin{aligned}
&2^{k/2}\|O^1(t,\xi)\psi_k(\xi)\|_{L^2}\\
&\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le 3}(1+|t|)^{-2}\big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2_\xi}\big)\\
&\quad\times\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.77}$$

Proof. Recall (5.69), (5.70), and (5.71). Since the desired estimate (5.77) is trivial when $|t|\le 1$, we restrict ourself to the case when $|t|\ge 1$. We separate into two cases as follows.

• **Case 1** If $(k_1,k_2)\in\chi_k^1\cup\chi_k^3$. For this case, we have $k_1\le k_2+10$. Recall the estimate of symbol "$m(\xi,v)$" in (5.23). On the one hand, after using the volume of support of ξ and the Cauchy-Schwarz inequality for the integration with respect to" η", the following estimate holds,

$$\begin{aligned}
&2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim 2^{k+3k_1/2}\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\big(2^{-k}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\\
&\quad+\|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\big)\\
&\lesssim 2^{k_1+\max\{k,k_1\}}\big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2\\
&\quad+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.78}$$

On the other hand, after doing integration by parts in "v" three times, using the volume of support of "ξ" and the L^2-L^2 type bilinear estimate, we have,

$$\begin{aligned}
&2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim\sum_{0\le a\le 3}|t|^{-3}2^{k-3k_2+3k_1/2}\big(2^{-k}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}+\|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\big)\\
&\quad\times\|(1+|v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\\
&\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le 3}|t|^{-3}2^{-3k_2+k_1+\max\{k,k_1\}}\big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\quad+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.79}$$

From (5.78) and (5.79), we have

$$\begin{aligned}
&\sup_{(k_1,k_2)\in\chi_k^1\cup\chi_k^3}2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le 3}\Big(\sum_{k,2^k\le|t|^{-1}}2^{2k}+\sum_{k,2^k\ge|t|^{-1}}2^{-k}|t|^{-3}\Big)\big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\quad+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\\
&\quad\times\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}\lesssim\sum_{\beta\in\mathcal{S},|\beta|\le 3}|t|^{-2}\big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\quad+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.80}$$

• **Case 2** If $(k_1,k_2)\in\chi_k^2$. For this case, we have $k_2\le k_1-10$ and $|k-k_1|\le 10$. After using the L^2-L^∞ type bilinear estimate and the volume of support of η, we have

$$\begin{aligned}
&2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim 2^{-k/2+3k_2}\big(2^{-k}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}+\|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\|(1\\
&\quad+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\\
&\lesssim 2^{3k_2-k}\big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2\\
&\quad+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned}\tag{5.81}$$

After doing integration by parts in "v" three times and using the $L^2 - L^\infty$ type bilinear estimate and the volume of support of the frequency of the input putted in L^∞_x, we have

$$\begin{aligned}
&2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim \sum_{|\beta|\le 3} |t|^{-3}2^{-k/2}\big(2^{-k}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi} + \|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\big)\\
&\quad\times\|(1+|v|)^{25}\nabla_v^\beta \widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\\
&\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 3} (1+|t|)^{-3}2^{-k}\big(\sup_{k\in\mathbb{Z}} 2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\quad+2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.82)
\end{aligned}$$

From (5.81) and (5.82), we have

$$\begin{aligned}
&\sup_{(k_1,k_2)\in\chi_k^2} 2^{k/2}\|O^1_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi} \lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 3}\Big(\sum_{k,2^k\le|t|^{-1}} 2^{2k} + \sum_{k,2^k\ge|t|^{-1}} 2^{-k}|t|^{-3}\Big)\\
&\quad\times\big(\sup_{k\in\mathbb{Z}} 2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2\\
&\quad+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}\\
&\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 3} |t|^{-2}\big(\sup_{k\in\mathbb{Z}} 2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{k/2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^2}\big)\|(1+|x|^2\\
&\quad+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.83)
\end{aligned}$$

To sum up, our desired estimate (5.77) holds from (5.80) and (5.83). □

Lemma 5.7. *For "$O^2(t,\xi)$" defined in (5.65), the following estimate holds for any fixed $k\in\mathbb{Z}$,*

$$\begin{aligned}
2^{k/2}\|O^2(t,\xi)\psi_k(\xi)\|_{L^2} &\lesssim \sum_{0\le n\le 3}\sum_{\alpha\in\mathcal{B},|\alpha|\le 3}\big((1+|t|)^{-1}2^{k_-} + (1+|t|)^{-2+\delta}\big)\\
&\quad\times\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20} f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.84)
\end{aligned}$$

Proof. Recall (5.65). After doing dyadic decomposition for "$\xi-\eta$" and "η", we have

$$O^2(t,\xi)\psi_k(\xi) = \sum_{i=1,2,3}\sum_{(k_1,k_2)\in\chi_k^i} O^2_{k,k_1,k_2}(t,\xi),$$

where

$$\begin{aligned}
O^2_{k,k_1,k_2}(t,\xi) = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta} it\Big(\frac{\xi}{|\xi|} - \mu\frac{\xi-\eta}{|\xi-\eta|}\Big)m(\xi,v)\psi_{\ge -10}(|\xi\\
&-\eta|/|\xi|)\widehat{h^\mu}(t,\xi-\eta)\\
&\times\widehat{f}(t,\eta,v)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta dv, \quad \textit{if} \quad (k_1,k_2)\in\chi_k^1\cup\chi_k^2,
\end{aligned} \quad (5.85)$$

$$O^2_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\eta|-it\hat{v}\cdot(\xi-\eta)} it\big(\frac{\xi}{|\xi|} - \hat{v}\big) m(\xi,v)\psi_{<-10}(|\eta|/|\xi|)\widehat{h^{\mu}}(t,\eta) \times \widehat{f}(t,\xi-\eta,v)\psi_k(\xi)\psi_{k_1}(\eta)\psi_{k_2}(\xi-\eta)d\eta dv, \quad \textit{if} \quad (k_1,k_2)\in\chi^3_k. \tag{5.86}$$

From the above detailed formulas of "O^2_{k,k_1,k_2}", our desired estimate (5.84) holds straightforwardly if $|t|\le 1$. Hence, from now on, we restrict ourself to the case $|t|\ge 1$.

• **Case 1** If $(k_1,k_2)\in\chi^1_k\cup\chi^2_k$. For this case, we have $k_1\ge k-10$ and $k_2\le k_1+10$. Recall (5.85). After doing integration by parts in "η" twice for O^2_{k,k_1,k_2}, we have

$$O^2_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t|\xi-\eta|-it\hat{v}\cdot\eta}\frac{i}{t}\nabla_\eta\cdot\Big[\frac{i\mu(\xi-\eta)/|\xi-\eta|-i\hat{v}}{\big|\mu(\xi-\eta)/|\xi-\eta|-\hat{v}\big|^2}\nabla_\eta \cdot\Big[\frac{i\mu(\xi-\eta)/|\xi-\eta|-i\hat{v}}{\big|\mu(\xi-\eta)/|\xi-\eta|-\hat{v}\big|^2} \times\big(\frac{\xi}{|\xi|}-\mu\frac{\xi-\eta}{|\xi-\eta|}\big)m(\xi,v)\psi_{\ge-10}(|\xi-\eta|/|\xi|)\widehat{h^{\mu}}(t,\xi-\eta)\widehat{f}(t,\eta,v)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\Big]\Big]d\eta dv.$$

After using the volume of support of "ξ" and "η", the following estimate holds,

$$\begin{aligned}
&2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim |t|^{-1}2^{k+3k_2}\big[\big(2^{-2k_2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi} + 2^{-k_2}\|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\\
&\quad + \|\nabla^2_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big)\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\\
&\quad + \big(2^{-k_2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi} + \|\nabla_\xi\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\big)\\
&\quad \times\|(1+|v|)^{25}\nabla_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v} + \|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|(1\\
&\quad +|v|)^{25}\nabla^2_\xi\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\big]\\
&\lesssim |t|^{-1}2^{k_2-2k_{2,+}}\big(\sum_{n=0,1,2,\alpha\in\mathcal{B},|\alpha|\le 4}\|h^\alpha(t)\|_{X_n}\big)\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned} \tag{5.87}$$

Moreover, after doing integration by parts in "η" three times first and then using the volume of support of ξ and η, the following estimate holds,

$$\begin{aligned}
&2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\
&\lesssim \sum_{n=0,1,2,3,\alpha\in\mathcal{B},|\alpha|\le 4}|t|^{-2}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}.
\end{aligned} \tag{5.88}$$

After interpolating the estimates (5.87) and(5.88), we have

$$\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2} 2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}$$
$$\lesssim \sum_{0\leq n\leq 3,\alpha\in\mathcal{B},|\alpha|\leq 4} |t|^{-2+\delta}\|h^\alpha(t)\|_{X_n}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.89)$$

• **Case** 2: If $(k_1,k_2)\in\chi_k^3$. Recall (5.86). For this case we have $k_1\leq k_2-10$, $|k-k_2|\leq 10$. On one hand, the following estimate holds after using the volume of support of η and the Minkowski inequality,

$$2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\lesssim 2^{-k/2+3k_1}(1+|t|)\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|(1$$
$$+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_vL^2_\xi}$$
$$\lesssim (1+|t|)2^{2k_1}\min\{2^{-k/2},2^k\}\|h(t)\|_{X_0}\|(1+|x|^2$$
$$+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.90)$$

On the other hand, after doing integration by parts in "η" three times first and then using the volume of support of η and the Minkowski inequality, we have

$$2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}$$
$$\lesssim \sum_{0\leq b\leq 3}\sum_{0\leq a\leq 3-b}|t|^{-2}2^{-k/2+3k_1-(3-b-a)k_1}\|(1+|v|)^{25}\nabla_\xi^b\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_vL^2_\xi}$$
$$\times\|\nabla_\xi^a\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\lesssim \sum_{0\leq n\leq 3,\alpha\in\mathcal{B},|\alpha|,0\leq b\leq 3}|t|^{-2}2^{-k/2-k_1}\|(1$$
$$+|v|)^{25}\nabla_\xi^b\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^1_vL^2_\xi}\|h^\alpha(t)\|_{X_n}$$
$$\lesssim \sum_{0\leq n\leq 3,\alpha\in\mathcal{B},|\alpha|\leq 3}|t|^{-2}2^{-k_1}\min\{2^{-k/2},2^k\}\|h^\alpha(t)\|_{X_i}\|(1+|x|^2$$
$$+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.91)$$

Therefore, from (5.90) and (5.91), we have

$$\sum_{(k_1,k_2)\in\chi_k^3} 2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}$$
$$\lesssim \sum_{0\leq n\leq 3}\sum_{|\alpha|\leq 3}\Big(\sum_{2^{k_1}\leq|t|^{-1}}|t|2^{2k_1}+\sum_{2^{k_1}\geq|t|^{-1}}|t|^{-2}2^{-k_1}\Big)\min\{2^{-k/2},2^k\}\|h^\alpha(t)\|_{X_n}$$
$$\times\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}\lesssim \sum_{0\leq n\leq 3,|\alpha|\leq 3}|t|^{-1}2^{k_-}\|h^\alpha(t)\|_{X_n}\|(1$$
$$+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \quad (5.92)$$

To sum up, our desired estimate (5.84) holds after combining the estimates (5.89) and (5.92). □

Lemma 5.8. *For "$O^2(t,\xi)$" defined in (5.65), the following estimate holds for any $k \in \mathbb{Z}$,*

$$2^{k/2}\|O^2(t,\xi)\psi_k(\xi)\|_{L^2} \lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 4} \sup_{k\in\mathbb{Z}}(1+|t|)^{-2}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\|(1+|x|^2 + |v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_x L^2_v}. \tag{5.93}$$

Proof. Recall (5.85) and (5.86). From the estimate of symbol in (5.23), the following estimate holds after using the volume of support of "ξ" and "η",

$$2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi} \lesssim 2^{k+3\min\{k_1,k_2\}}(1+|t|)\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|(1 + |v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}. \tag{5.94}$$

Moreover, after doing integration by parts in "v" four times first and then using the volume of support of "ξ" and "η", we have

$$\begin{aligned}&2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\ &\lesssim \sum_{0\le a\le 4}(1+|t|)^{-3}2^{k+3\min\{k_1,k_2\}-4k_2}\|\widehat{h}(t,\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|(1\\ &\quad + |v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}.\end{aligned} \tag{5.95}$$

Therefore, from the estimates (5.94) and (5.95), the following estimate holds for any $k \in \mathbb{Z}$,

$$\begin{aligned}&\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2\cup\chi_k^3} 2^{k/2}\|O^2_{k,k_1,k_2}(t,\xi)\|_{L^2_\xi}\\ &\lesssim \sum_{0\le a\le 4}\Big(\sum_{k,2^k\le(1+|t|)^{-1}}(1+|t|)2^{3k} + \sum_{k,2^k\ge(1+|t|)^{-1}}(1+|t|)^{-3}2^{-k}\Big)\\ &\quad\times\Big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\Big)\|(1+|v|)^{25}\nabla_v^a\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v}\\ &\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 4}(1+|t|)^{-2}\Big(\sup_{k\in\mathbb{Z}}2^k\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\Big)\|(1+|x|^2\\ &\quad + |v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_x L^2_v}.\end{aligned} \tag{5.96}$$

Hence finishing the proof of the desired estimate (5.93). □

5.3. Bilinear estimates in the high order energy space: Vlasov-Vlasov type interaction. In this subsection, we prove a bilinear estimate in the weighted L^2-type space for the Vlasov-Vlasov type interaction, which, more precisely, is the interaction between two density type functions. This bilinear estimate has been used in the high order energy estimate of the scalar field in the proof of Proposition 5.2. More precisely, the estimate of the second integral in (5.12), which can be viewed as a linear combination of bilinear forms defined in (5.97).

Lemma 5.9. *For any symbols $m_1(\xi, v), m_2(\xi, v)$ that satisfy (5.23) with "$l = 1$" and any two localized distribution functions $f, g : \mathbb{R}_t \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \longrightarrow \mathbb{R}$, we define a bilinear operator as follows,*

$$K^{\mu}(g, f)(t, \xi) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi| - i\mu t\hat{u}\cdot(\xi-\eta) - it\hat{v}\cdot\eta} m_1(\xi, v) m_2(\xi-\eta, u)\widehat{g}(t, \xi - \eta, u)\widehat{f}(t, \eta, v) d\eta du dv. \tag{5.97}$$

Then the following bilinear estimate holds for any fixed $k \in \mathbb{Z}$,

$$2^{k/2}\|\nabla_\xi\big(K^{\mu}(g, f)(t, \xi)\big)\psi_k(\xi)\|_{L^2_\xi} \lesssim \sum_{\beta\in\mathcal{S}, |\beta|\le 5} (1+|t|)^{-2}\|(1+|x|^2+|v|^2)^{20} g(t, x, v)\|_{L^2_x L^2_v}\|(1+|x|^2 + |v|^2)^{20}\Lambda^{\beta} f(t, x, v)\|_{L^2_x L^2_v}. \tag{5.98}$$

Proof. Similar to the decomposition we did in (5.63), we first separate "$\nabla_\xi(K^{\mu}(g, f)(t, \xi))$" into two parts and then do dyadic decompositions for "$\xi - \eta$" and "η". As a result, the following decompositions hold,

$$\nabla_\xi(K^{\mu}(g, f)(t, \xi))\psi_k(\xi) = \sum_{i=1,2} K^i_k(g, f)(t, \xi), \quad K^i_k(g, f)(t, \xi) = \sum_{(k_1,k_2)\in\chi^l_k, l=1,2,3} K^{k;i}_{k_1,k_2}(g, f)(t, \xi), \tag{5.99}$$

where

$$K^{k;1}_{k_1,k_2}(g, f)(t, \xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi| - i\mu t\hat{u}\cdot(\xi-\eta) - it\hat{v}\cdot\eta}\nabla_\xi\big(m_1(\xi, v)m_2(\xi-\eta, u)\widehat{g}(t, \xi - \eta, u)\psi_{\ge -10}(|\xi-\eta|/|\xi|)\big) \times \widehat{f}(t, \eta, v)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta) d\eta du dv, \quad \textit{if } (k_1, k_2) \in \chi^1_k \cup \chi^2_k, \tag{5.100}$$

$$K^{k;1}_{k_1,k_2}(g, f)(t, \xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi| - i\mu t\hat{u}\cdot\eta - it\hat{v}\cdot(\xi-\eta)}\nabla_\xi\big(m_1(\xi, v)\widehat{f}(t, \xi-\eta, v)\psi_{<-10}(|\eta|/|\xi|)\big) \times\widehat{g}(t, \eta, u)m_2(\eta, u)\psi_k(\xi)\psi_{k_1}(\eta)\psi_{k_2}(\xi-\eta) d\eta du dv, \quad \textit{if } (k_1, k_2) \in \chi^3_k, \tag{5.101}$$

$$K^{k;2}_{k_1,k_2}(g, f)(t, \xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi| - i\mu t\hat{u}\cdot(\xi-\eta) - it\hat{v}\cdot\eta} it\Big(\frac{\xi}{|\xi|} - \mu\hat{u}\Big) m_1(\xi, v)m_2(\xi - \eta, u)\widehat{g}(t, \xi - \eta, u) \times\widehat{f}(t, \eta, v)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta) d\eta du dv, \quad \textit{if } (k_1, k_2) \in \chi^1_k \cup \chi^2_k, \tag{5.102}$$

$$\begin{aligned} &K^{k;2}_{k_1,k_2}(g,f)(t,\xi) \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\mu t\hat{u}\cdot\eta - it\hat{v}\cdot(\xi-\eta)} it\big(\frac{\xi}{|\xi|}-\hat{v}\big) m_1(\xi,v)m_2(\eta,u)\widehat{g}(t,\eta,u) \\ &\quad \times \widehat{f}(t,\xi-\eta,v)\psi_k(\xi)\psi_{k_1}(\eta)\psi_{k_2}(\xi-\eta)d\eta du dv, \quad \textit{if } (k_1,k_2)\in\chi_k^3. \end{aligned} \tag{5.103}$$

From the above detailed formulas, our desired estimate (5.98) holds straightforwardly if $|t|\le 1$. Hence, we restrict ourself to the case $|t|\ge 1$.

- **Case** 1: If $(k_1,k_2)\in\chi_k^1\cup\chi_k^2$. For this case we have $k_1\ge k-10$ and $k_2\le k_1+10$. On one hand, from the size of support of "ξ" and "η" and the estimate of symbols "$m_i(\xi,v)$", $i\in\{1,2\}$, in (5.23), we have

$$\begin{aligned} &2^{k/2}\|K^{k;1}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi} \\ &\lesssim 2^{2k-k-k_1-\min\{k,k_1\}+3k_2}\big(\|(1+|u|)^{25}\widehat{g}(t,\xi,u)\psi_{k_1}(\xi)\|_{L^\infty_\xi L^1_u} \\ &\quad + 2^{k_1}\|(1+|u|)^{25}\nabla_\xi\widehat{g}(t,\xi,u)\psi_{k_1}(\xi)\|_{L^\infty_\xi L^1_u}\big)\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_v} \\ &\lesssim 2^{2k_2}\|(1+|x|^2+|v|^2)^{20}g(t,x,v)\|_{L^2_xL^2_v}\|(1+|x|^2+|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \end{aligned} \tag{5.104}$$

Following the similar strategy, recall (5.102), the following estimate holds for $K^{k;2}_{k_1,k_2}(g,f)(t,\xi)$,

$$\begin{aligned} 2^{k/2}\|K^{k;2}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi} &\lesssim |t|2^{3k_2}\|(1+|x|^2+|v|^2)^{20}g(t,x,v)\|_{L^2_xL^2_v}\|(1+|x|^2 \\ &\quad +|v|^2)^{20}f(t,x,v)\|_{L^2_xL^2_v}. \end{aligned} \tag{5.105}$$

On the other hand, we first do integration by parts in v three times for $K^{k;1}_{k_1,k_2}$ and do integration by parts in v four times for $K^{k;2}_{k_1,k_2}$ and then use the volume of support of ξ and η. As a result, we have

$$\begin{aligned} &2^{k/2}\|K^{k;1}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi}+2^{k/2}\|K^{k;2}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi} \\ &\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 5}(1+|t|)^{-3}2^{-k_2}\|(1+|x|^2+|v|^2)^{20}g(t,x,v)\|_{L^2_xL^2_v}\|(1+|x|^2 \\ &\quad +|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v}. \end{aligned} \tag{5.106}$$

To sum up, from the estimates (5.104), (5.105), and (5.106), the following estimate holds,

$$\begin{aligned} &\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2}\sum_{i=1,2}2^{k/2}\|K^{k;i}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi} \\ &\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 5}\Big(\sum_{2^{k_2}\le|t|^{-1}}(2^{2k_2}+|t|2^{3k_2})+\sum_{2^{k_2}\ge|t|^{-1}}|t|^{-3}2^{-k_2}\Big) \\ &\quad \times\|(1+|x|^2+|v|^2)^{20}g(t,x,v)\|_{L^2_xL^2_v}\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_xL^2_v} \end{aligned}$$

$$\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 5} |t|^{-2}\|(1+|x|^2+|v|^2)^{20} g(t,x,v)\|_{L^2_x L^2_v}\|(1+|x|^2 + |v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_x L^2_v}. \tag{5.107}$$

- **Case 2** If $(k_1,k_2)\in\chi_k^3$. For this case, we have $k_1\le k-10$ and $|k_2-k|\le 10$.

On one hand, similar to the strategy used in obtaining the estimates (5.104) and (5.105), the following estimate holds from the size of support of ξ and η and the estimate of symbols $m_i(\xi,v)$ in (5.23),

$$\begin{aligned}
&\sum_{i=1,2} 2^{k/2}\|K^{k;i}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi}\\
&\lesssim 2^{2k-k-k_1+3k_1}\big(1+|t|2^{k_1}\big)\big(2^{-k_1}\|(1+|v|)^{25}\widehat{f}(t,\xi,v)\psi_{k_1}(\xi)\|_{L^\infty_\xi L^1_v}\\
&\quad + 2^k\|(1+|v|)^{25}\nabla_\xi \widehat{f}(t,\xi,v)\psi_{k_1}(\xi)\|_{L^\infty_\xi L^1_v}\big)\|(1+|u|)^{25}\widehat{g}(t,\xi,u)\psi_{k_2}(\xi)\|_{L^\infty_\xi L^1_u}\\
&\lesssim \big(2^{k_1+k_2}+|t|2^{2k_1+k_2}\big)\|(1+|x|^2+|v|^2)^{20} g(t,x,v)\|_{L^2_x L^2_v}\|(1+|x|^2\\
&\quad +|v|^2)^{20} f(t,x,v)\|_{L^2_x L^2_v}.
\end{aligned} \tag{5.108}$$

On the other hand, similar to the strategy used in the estimate (5.106), we do integration by parts in v three times for $K^{k;1}_{k_1,k_2}$ in (5.101) and do integration by parts in v four times for $K^{k;2}_{k_1,k_2}$ in (5.103). As a result, the following estimate holds after using the volume of support of ξ and η,

$$\begin{aligned}
&\sum_{i=1,2} 2^{k/2}\|K^{k;i}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi}\\
&\quad\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 5} |t|^{-3}2^{2k-k-2k_1+3k_1-3k_2}\|(1+|x|^2+|v|^2)^{20} g(t,x,v)\|_{L^2_x L^2_v}\\
&\qquad\times\|(1+|x|^2+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_x L^2_v}.
\end{aligned} \tag{5.109}$$

From the estimates (5.108) and (5.109), the following estimate holds,

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\chi_k^3}\sum_{i=1,2} 2^{k/2}\|K^{k;i}_{k_1,k_2}(g,f)(t,\xi)\|_{L^2_\xi}\\
&\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 5}\Big(\sum_{2^{k_2}\le |t|^{-1}}\sum_{k_1\le k_2+10}\big(2^{k_1+k_2}+|t|2^{2k_1+k_2}\big)\\
&\quad+\sum_{2^{k_2}\ge |t|^{-1}}\sum_{k_1\le k_2+10}|t|^{-3}2^{k_1-2k_2}\Big)\|(1+|x|^2+|v|^2)^{20} g(t,x,v)\|_{L^2_x L^2_v}\|(1+|x|^2\\
&\quad+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_x L^2_v}\\
&\lesssim \sum_{\beta\in\mathcal{S},|\beta|\le 5}|t|^{-2}\|(1+|x|^2+|v|^2)^{20} g(t,x,v)\|_{L^2_x L^2_v}\|(1+|x|^2\\
&\quad+|v|^2)^{20}\Lambda^\beta f(t,x,v)\|_{L^2_x L^2_v}.
\end{aligned} \tag{5.110}$$

To sum up, our desired estimate (5.98) holds from the decomposition (5.99) and the estimates (5.107) and (5.110). □

6. Energy Estimate for the Vlasov Part

In this section, we estimate both the low order energy $E^{f}_{\mathrm{low}}(t)$ defined in (4.60) and the high order energy $E^{f}_{\mathrm{high}}(t)$ defined in (4.49) of the profile $g(t, x, v)$ of the Vlasov part.

The main ingredients are some general linear estimates and bilinear estimates, which will be used as black boxes first in Sects. 6.1 and 6.2. We will prove these estimates in Sect. 6.3.

6.1. The high order energy estimate for the Vlasov part. Recall (4.49) and (4.9). As a result of direct computations, the following equality holds for any fixed $t \in [0, T]$, $\alpha \in \mathcal{B}$, $\beta \in \mathcal{S}$, s.t., $|\alpha| + |\beta| \leq N_0$,

$$\frac{1}{2}\|\omega^{\alpha}_{\beta}(x, v)g^{\alpha}_{\beta}(t, x, v)\|^2_{L^2_{x,v}} - \frac{1}{2}\|\omega^{\alpha}_{\beta}(x, v)g^{\alpha}_{\beta}(0, x, v)\|^2_{L^2_{x,v}}$$
$$= \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(\omega^{\alpha}_{\beta}(x, v)\big)^2 g^{\alpha}_{\beta}(t, x, v)\partial_t g^{\alpha}_{\beta}(t, x, v)dxdv = \sum_{i=1,2,3} I^{\alpha}_{\beta;i}, \quad (6.1)$$

where

$$I^{\alpha}_{\beta;1} = \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(\omega^{\alpha}_{\beta}(x, v)\big)^2 g^{\alpha}_{\beta}(s, x, v)K(s, x + \hat{v}s, v) \cdot D_v g^{\alpha}_{\beta}(s, x, v)dxdvds, \quad (6.2)$$

$$I^{\alpha}_{\beta;2} = \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(\omega^{\alpha}_{\beta}(x, v)\big)^2 g^{\alpha}_{\beta}(s, x, v)l.o.t^{\alpha}_{\beta}(s, x, v)dxdvds, \quad (6.3)$$

$$I^{\alpha}_{\beta;3} = \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(\omega^{\alpha}_{\beta}(x, v)\big)^2 g^{\alpha}_{\beta}(s, x, v)h.o.t^{\alpha}_{\beta}(s, x, v)dxdvds. \quad (6.4)$$

The main result of this subsection is summarized in the following Proposition. For the sake of readers and for clarity, we give a concise proof of Proposition 6.1 by separating out two independent Lemmas first and then proving the validities of these two Lemmas.

Proposition 6.1. *Under the bootstrap assumption* (4.69)*, the following estimate holds for any* $t \in [0, T]$,

$$E^{f;1}_{high}(t) \lesssim (1 + t)^{\delta}\epsilon_0, \quad E^{f;2}_{high}(t) \lesssim (1 + t)^{\delta/2}\epsilon_0 \quad (6.5)$$

Proof. Recall the definition of the high order energy in (4.49) and the decomposition in (6.1).

We first estimate $I^{\alpha}_{\beta;2}$ and $I^{\alpha}_{\beta;3}$. Recall (6.3) and (6.4). From the estimates (6.8) and (6.9) in Lemma 6.1 and the estimates (6.13) and (6.14) in Lemma 6.2, the following estimate holds from the $L^2_{x,v} - L^2_{x,v}$ type bilinear estimate,

$$\sum_{|\alpha|+|\beta|=N_0} |I^{\alpha}_{\beta;2}|+|I^{\alpha}_{\beta;3}| \lesssim \int_0^t (1 + s)^{-1+2\delta}\epsilon_0 ds \lesssim (1 + t)^{2\delta}\epsilon_0, \quad \sum_{|\alpha|+|\beta|<N_0} |I^{\alpha}_{\beta;2}| + |I^{\alpha}_{\beta;3}|$$
$$\lesssim (1 + t)^{\delta}\epsilon_0. \quad (6.6)$$

Now, it remains to estimate $I^{\alpha}_{\beta;1}$. Recall (6.2). Note that

$$D_v = \nabla_v - t\nabla_v \hat{v}\cdot\nabla_x, \quad \Longrightarrow g^{\alpha}_{\beta}(t,x,v)D_v g^{\alpha}_{\beta}(t,x,v) = \frac{1}{2}D_v\big(g^{\alpha}_{\beta}(t,x,v)\big)^2.$$

Therefore, after doing integration by parts in x and v, the following equality holds,

$$I^{\alpha}_{\beta;1} = \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{2}\big(\omega^{\alpha}_{\beta}(x,v)g^{\alpha}_{\beta}(t,x,v)\big)^2\Big[\frac{2K(t,x+\hat{v}t,v)\cdot D_v\omega^{\alpha}_{\beta}(x,v)}{\omega^{\alpha}_{\beta}(x,v)}$$
$$+D_v\cdot K(t,x+\hat{v}t,v)\Big]dxdv.$$

Recall (4.10). From the estimate (4.52) in Lemma 4.2, and the decay estimate (6.55) in Lemma 6.3, we have

$$\big|D_v\cdot K(t,x+\hat{v}t,v)\big| + \Big|\frac{K(t,x+\hat{v}t,v)\cdot D_v\omega^{\alpha}_{\beta}(x,v)}{\omega^{\alpha}_{\beta}(x,v)}\Big|$$
$$\lesssim (1+|t|)^{-1}E^{\phi}_{\text{low}}(t) \lesssim (1+|t|)^{-1}\epsilon_1. \tag{6.7}$$

To sum up, our desired estimate (6.5) holds from the above estimate (6.7), the $L^2_{x,v} - L^2_{x,v} - L^{\infty}_{x,v}$ type multi-linear estimate, and the estimate (6.6). □

Lemma 6.1. *Under the bootstrap assumption* (4.69)*, the following estimate holds for any* $t\in[0,T]$*,*

$$\sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|=N_0}\|\omega^{\alpha}_{\beta}(x,v)h.o.t^{\alpha}_{\beta}(t,x,v)\|_{L^2_xL^2_v} \lesssim (1+|t|)^{-1+\delta}\epsilon_0. \tag{6.8}$$

$$\sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|<N_0}\|\omega^{\alpha}_{\beta}(x,v)h.o.t^{\alpha}_{\beta}(t,x,v)\|_{L^2_xL^2_v} \lesssim (1+|t|)^{-1+\delta/2}\epsilon_0. \tag{6.9}$$

Proof. Recall (4.15). Motivated from the the decomposition of $h.o.t^{\alpha}_{\beta}(t,x,v)$, we separate into three cases as follows.

- **Case** 1: The estimate of $h.o.t^{\alpha}_{\beta;1}(t,x,v)$.

Recall the equations (4.11), (4.16), and the first decomposition of D_v in (3.41) in Lemma 3.4. From the estimate of coefficients in the estimate (3.44) in Lemma 3.4 and the estimate (4.25) in Lemma 4.1 and the decay estimate (6.55) in Lemma 6.3, we have

$$\sum_{|\alpha|+|\beta|=N_0}\|\omega^{\alpha}_{\beta}(x,v)h.o.t^{\alpha}_{\beta;1}(t,x,v)\|_{L^2_{x,v}}$$
$$\lesssim \sum_{\rho,\iota\in\mathcal{S},\gamma\in\mathcal{B},|\rho|+|\gamma|\leq N_0,|\iota|\leq 1}\big(\|(1+|\tilde{d}(t,x,v)|)\partial_t\phi^{\iota}(t,x+\hat{v}t)\|_{L^{\infty}_{x,v}}$$
$$+\|(1+|\tilde{d}(t,x,v)|)\nabla_x\phi^{\iota}(t,x+\hat{v}t)\|_{L^{\infty}_{x,v}}\big)\|\omega^{\gamma}_{\rho}(x,v)g^{\gamma}_{\rho}(t,x,v)\|_{L^2_{x,v}}$$
$$\lesssim (1+|t|)^{-1}E^{f}_{\text{high}}(t)E^{\phi}_{\text{low}}(t). \tag{6.10}$$

- **Case** 2: The estimate of $h.o.t^{\alpha}_{\beta;2}(t,x,v)$.

Recall the equation (4.17) and the first decomposition of D_v in (3.41) in Lemma 3.4. From the estimate of coefficients in the estimate (3.44) in Lemma 3.4, the estimate of coefficients in (4.6) and the decay estimate (6.55) in Lemma 6.3, we have

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|=N_0} \|\omega^\alpha_\beta(x,v) h.o.t^\alpha_{\beta;2}(t,x,v)\|_{L^2_{x,v}} \\
&\lesssim \sum_{\rho,\iota\in\mathcal{S},\gamma\in\mathcal{B},|\rho|+|\gamma|\le N_0,|\iota|\le 1} \big(\|(1+|\tilde{d}(t,x,v)|)\partial_t\phi^\iota(t,x+\hat{v}t)\|_{L^\infty_{x,v}} \\
&\quad + \|(1+|\tilde{d}(t,x,v)|)\nabla_x\phi^\iota(t,x+\hat{v}t)\|_{L^\infty_{x,v}}\big)\|\omega^\gamma_\rho(x,v) g^\gamma_\rho(t,x,v)\|_{L^2_{x,v}} \\
&\lesssim (1+|t|)^{-1} E^f_{\text{high}}(t) E^\phi_{\text{low}}(t). \qquad (6.11)
\end{aligned}$$

• **Case** 3: The estimate of $h.o.t^\alpha_{\beta;3}(t,x,v)$.

Recall (4.18), the detailed formula of Y^β_i in (3.77), and the detailed formula of $K_i(t,x,v)$, $i\in\{1,\dots,7\}$, in (4.12), (4.13), and (4.14). From the estimates of coefficients in (3.79) and (3.81) in Lemma 3.9, the $L^2_{x,v}-L^\infty_{x,v}$ type bilinear estimate and the decay estimate (6.55) in Lemma 6.3, we have

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|=N_0} \|\omega^\alpha_\beta(x,v) h.o.t^\alpha_{\beta;3}(t,x,v)\|_{L^2_{x,v}} \\
&\lesssim \sum_{\rho,\iota\in\mathcal{S},\gamma\in\mathcal{B},|\rho|+|\gamma|\le N_0,|\iota|\le 1} \big(\|(1+|\tilde{d}(t,x,v)|)\partial_t\phi^\iota(t,x+\hat{v}t)\|_{L^\infty_{x,v}} \\
&\quad + \|(1+|\tilde{d}(t,x,v)|)\nabla_x\phi^\iota(t,x+\hat{v}t)\|_{L^\infty_{x,v}}\big)\|\omega^\gamma_\rho(x,v) g^\gamma_\rho(t,x,v)\|_{L^2_{x,v}} \\
&\lesssim (1+|t|)^{-1} E^f_{\text{high}}(t) E^\phi_{\text{low}}(t). \qquad (6.12)
\end{aligned}$$

To sum up, our desired estimate (6.8) holds from the estimates (6.10), (6.11), and (6.12). With minor modifications, the desired estimate (6.9) holds very similarly. □

Lemma 6.2. *Under the bootstrap assumption* (4.69)*, the following estimate holds for any fixed time* $t\in[0,T]$*,*

$$\sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|=N_0} \|\omega^\alpha_\beta(x,v) l.o.t^\alpha_\beta(t,x,v)\|_{L^2_x L^2_v} \lesssim (1+|t|)^{-1+\delta}\epsilon_0. \qquad (6.13)$$

$$\sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|<N_0} \|\omega^\alpha_\beta(x,v) l.o.t^\alpha_\beta(t,x,v)\|_{L^2_x L^2_v} \lesssim (1+|t|)^{-1+\delta/2}\epsilon_0. \qquad (6.14)$$

Proof. Since the case $|t|\le 1$ is trivial, it would be sufficient to consider the case $|t|\ge 1$. Recall the decomposition of $l.o.t^\alpha_\beta(t,x,v)$ in (4.19). Based on the size of the total number of derivatives acting on the scalar field, we separate into two cases as follows.

• **Case** 1: The estimate of $l.o.t^\alpha_{\beta;i}(t,x,v)$, $i\in\{1,2,4\}$.

Recall (4.20), (4.21), and (4.23). Note that there are at most ten derivatives hit on the nonlinear wave part. Recall the commutation rule between Λ^β and X_i in (3.76) and the equality (4.24). From the estimate of coefficients in (3.78), (3.79), (3.80), (4.25) and (4.26), the following estimate holds from the linear decay estimate (6.55) in Lemma 6.3 and the $L^2_{x,v}-L^\infty_{x,v}$ type bilinearestimate,

$$\sum_{i=1,2,4}\sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|=N_0}\|\omega^{\alpha}_{\beta}(x,v)l.o.t^{\alpha}_{\beta;i}(t,x,v)\|_{L^2_xL^2_v}$$
$$\lesssim\sum_{\rho,\gamma\in\mathcal{B},\kappa\in\mathcal{S},|\gamma|+|\kappa|\le N_0,|\rho|\le 13}\sum_{u\in\{\partial_t\phi^{\rho},\nabla\phi^{\rho}\}}\|\omega^{\gamma}_{\kappa}(x,v)g^{\gamma}_{\kappa}(t,x,v)\|_{L^2_{x,v}}\|(1$$
$$+|\tilde{d}(t,x,v)|)u(t,x+\hat{v}t)\|_{L^\infty_{x,v}}$$
$$\lesssim(1+|t|)^{-1}E^{f}_{\mathrm{high}}(t)E^{\phi}_{\mathrm{low}}(t)\lesssim(1+|t|)^{-1+\delta}\epsilon_1^2. \tag{6.15}$$

• **Case** 2: The estimate of $l.o.t^{\alpha}_{\beta;3}(t,x,v)$.

The main difficulty is caused by the case in which the scalar field has the maximal number of derivatives. For this case, we are forced to put the scalar field in the energy space. The main ingredients in the estimate of $l.o.t^{\alpha}_{\beta;3}(t,x,v)$ are the decay estimate of the density type function and the fact that the space-resonance set is relatively small, which is nontrivial.

Recall (4.22). From the equality (4.24) in Lemma 4.1, we have

$$l.o.t^{\alpha}_{\beta;3}(t,x,v)$$
$$=\sum_{\substack{\iota,\kappa\in\mathcal{S},\beta_1,\gamma_1,\beta_2,\gamma_2\in\mathcal{B},\\|\rho|+|\beta_1|>11,|\beta_1|+|\gamma_1|\le|\alpha|\\|\rho|+|\beta_2|>11,|\beta_2|+|\gamma_2|\le|\alpha|}}\Lambda^{\rho}\big((\partial_t\phi^{\beta_1}(t,x+\hat{v}t)$$
$$+\hat{v}\cdot\nabla_x\phi^{\beta_1}(t,x+\hat{v}t))\big)\Lambda^{\kappa}\big(a^1_{\alpha;\beta_1,\gamma_1}(v)g^{\gamma_1}(t,x,v)+a^2_{\alpha;\beta_1,\gamma_1}(v)v\cdot D_vg^{\gamma_1}(t,x,v)\big)$$
$$+\Lambda^{\rho}\big(\big(a^3_{\alpha;\beta_2,\gamma_2}(v)\nabla_x\phi^{\beta_2}+a^4_{\alpha;\beta_2,\gamma_2}(v)\partial_t\phi^{\beta_2}\big)(t,x+\hat{v}t)\cdot\alpha_i(v)\big)\Lambda^{\kappa}\big(X_ig^{\gamma_2}(t,x,v)\big), \tag{6.16}$$

From the equalities (3.76) and (3.77) in Lemma 3.9 and the first decomposition of D_v in (3.41) in Lemma 3.4, we have

$$\Lambda^{\kappa}\big(X_ig^{\gamma_2}(t,x,v)\big)$$
$$=\big[\alpha_i(v)\cdot D_v\circ\Lambda^{\kappa}+[\Lambda^{\kappa},X_i]\big]g^{\gamma_2}(t,x,v)$$
$$=\sum_{\rho\in\mathcal{K},|\rho|=1}\alpha_i(v)\cdot d_{\rho}(t,x,v)\Lambda^{\rho\circ\kappa}g^{\gamma_2}(t,x,v)$$
$$+Y^{\kappa}_ig^{\gamma_2}(t,x,v)+\sum_{\kappa'\in\mathcal{S},|\kappa'|\le|\kappa|-1}\big[\tilde{d}(t,x,v)\tilde{e}^{\kappa',1}_{\kappa,i}(x,v)$$
$$+\tilde{e}^{\kappa',2}_{\kappa,i}(x,v)\big]\Lambda^{\kappa'}g^{\gamma_2}(t,x,v). \tag{6.17}$$

From (6.16) and (6.17), and the detailed formula of $d_{\rho}(t,x,v)$ in (3.42), we can rewrite "$l.o.t^{\alpha}_{\beta;3}(t,x,v)$" as follows

$$l.o.t^{\alpha}_{\beta;3}(t,x,v)$$
$$=\sum_{\substack{\rho\in\mathcal{S},\kappa_1,\kappa_2\in\mathcal{B},|\kappa_2|\le|\alpha|,|\rho|\le|\beta|,\\|\rho|+|\kappa_1|+|\kappa_2|\le|\alpha|+|\beta|\\|\rho|+|\kappa_2|\le|\alpha|+|\beta|-12}}\big[\big(\tilde{d}(t,x,v)\widehat{e}^{1}_{\kappa_1,\kappa_2,\rho}(t,x,v)$$
$$+\widehat{e}^{2}_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)\partial_t\phi^{\kappa_1}(t,x+\hat{v}t)$$

$$+\left(\tilde{d}(t,x,v)\widehat{e}^3_{\kappa_1,\kappa_2,\rho}(t,x,v)+\widehat{e}^4_{\kappa_1,\kappa_2,\rho}(t,x,v)\right)\cdot\nabla_x\phi^{\kappa_1}(t,x+\hat{v}t)\big]g^{\kappa_2}_\rho(t,x,v), \tag{6.18}$$

where the coefficients $\widehat{e}^i_{\kappa_1,\kappa_2,\rho}(t,x,v)$, $i\in\{1,2,3,4\}$, satisfy the following rough estimate,

$$\sum_{i=1,\dots,4}|\widehat{e}^i_{\kappa_1,\kappa_2,\rho}(t,x,v)|\lesssim(1+|x|^2+|v|^2)^{|\alpha|+2|\beta|-2|\rho|-|\kappa_2|+5}, \tag{6.19}$$

which can be derived from the estimate of coefficients in (4.6), the estimate (4.25) in Lemma 4.1 and the estimates (3.78), (3.79), and (3.80) in Lemma 3.9.

Recall (6.18). After doing dyadic decomposition for the wave part, the following decomposition holds,

$$l.o.t^\alpha_{\beta;3}(t,x,v)=\sum_{k\in\mathbb{Z}}H_k(t,x,v), \tag{6.20}$$

where

$$\begin{aligned}&H_k(t,x,v)\\&:=\sum_{\substack{\rho\in\mathcal{S},\kappa_1,\kappa_2\in\mathcal{B},|\kappa_2|\le|\alpha|,|\rho|\le|\beta|,\\|\rho|+|\kappa_1|+|\kappa_2|\le|\alpha|+|\beta|\\|\rho|+|\kappa_2|\le|\alpha|+|\beta|-12}}\big[\big(\tilde{d}(t,x,v)\widehat{e}^1_{\kappa_1,\kappa_2,\rho}(t,x,v)\\&\quad+\widehat{e}^2_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)(\partial_t\phi^{\kappa_1})_k(t,x+\hat{v}t)\\&\quad+\big(\tilde{d}(t,x,v)\widehat{e}^3_{\kappa_1,\kappa_2,\rho}(t,x,v)+\widehat{e}^4_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)\big(\nabla\phi^{\kappa_1}\big)_k(t,x+\hat{v}t)\big]g^{\kappa_2}_\rho(t,x,v).\end{aligned}$$

Based on the possible size of k, we separate into the low frequency case and the high frequency case as follows.

- If $k\le 0$.

Recall the decomposition in (4.43). From the estimate (5.17) in Lemma 5.1, the estimate of correction term in (4.70) and the L^∞_ξ-type estimate of the modified profile in (5.9), we have

$$\sum_{\kappa\in\mathcal{B},|\kappa|\le N_0}\|P_k[\partial_t\phi^\kappa]\|_{L^2}+\|P_k[\nabla_x\phi^\kappa]\|_{L^2}\lesssim 2^{k/2}\epsilon_1+|t|2^{3k/2}\epsilon_1. \tag{6.21}$$

After using the $L^2_x-L^\infty_x L^2_v$ type estimate, the estimate of coefficients in (6.19), the estimate (6.21), and the decay estimate (2.20) in Lemma 2.1, the following estimate holds if $2^k\le|t|^{-1}$,

$$\begin{aligned}\|\omega^\alpha_\beta(x,v)H_k(t,x,v)\|_{L^2_xL^2_v}&\lesssim\big(2^{k/2}+|t|2^{3k/2}\big)\epsilon_1(1+|t|)^{-1/2}E^{f;2}_{\text{high}}(t)\\&\lesssim|t|^{-1/2+\delta/2}2^{k/2}\big(1+|t|2^k\big)\epsilon_0.\end{aligned} \tag{6.22}$$

Note that, we estimate the inhomogeneous modulation $\tilde{d}(t,x,v)$ roughly from the above by $1+|t|$ in the above estimate.

If $|t|^{-1}\le 2^k\le 1$, from the decompositions in (4.46), the following decomposition holds for H_k,

$$H_k(t,x,v)=H^1_k(t,x,v)+H^2_k(t,x,v), \tag{6.23}$$

where

$$
\begin{aligned}
H_k^1(t,x,v) := \sum_{\substack{\rho\in\mathcal{S},\kappa_1,\kappa_2\in\mathcal{B},|\kappa_2|\le|\alpha|,|\rho|\le|\beta|,\\ |\rho|+|\kappa_1|+|\kappa_2|\le|\alpha|+|\beta|\\ |\rho|+|\kappa_2|\le|\alpha|+|\beta|-12}} & \Big[\big(\tilde{d}(t,x,v)\widehat{e}^1_{\kappa_1,\kappa_2,\rho}(t,x,v)\\
&+\widehat{e}^2_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)(\widetilde{\partial_t\phi^{\kappa_1}})_k(t,x+\hat{v}t)\\
&+\big(\tilde{d}(t,x,v)\widehat{e}^3_{\kappa_1,\kappa_2,\rho}(t,x,v)+\widehat{e}^4_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)\\
&\cdot\big(\nabla_x\widetilde{\phi^{\kappa_1}}\big)_k(t,x+\hat{v}t)\Big]g_\rho^{\kappa_2}(t,x,v), \qquad (6.24)\\
H_k^2(t,x,v) := \sum_{\substack{\rho\in\mathcal{S},\kappa_1,\kappa_2,\eta\in\mathcal{B},|\kappa_2|\le|\alpha|,|\rho|\le|\beta|\\ |\rho|+|\kappa_1|+|\kappa_2|\le|\alpha|+|\beta|,|\eta|\le|\kappa_1|\\ |\rho|+|\kappa_2|\le|\alpha|+|\beta|-12}} & \Big[\big(\tilde{d}(t,x,v)\widehat{e}^1_{\kappa_1,\kappa_2,\rho}(t,x,v)\\
&+\widehat{e}^2_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)\mathrm{Re}[E_{\kappa_1;\eta}(g_k^\eta)(t,x+\hat{v}t)]\\
&-\big(\tilde{d}(t,x,v)\widehat{e}^3_{\kappa_1,\kappa_2,\rho}(t,x,v)+\widehat{e}^4_{\kappa_1,\kappa_2,\rho}(t,x,v)\big)\\
&\cdot R\big(\mathrm{Im}\big(E_{\kappa_1;\eta}(g_k^\eta)(t,x+\hat{v}t)\big)\big)\Big]g_\rho^{\kappa_2}(t,x,v). \qquad (6.25)
\end{aligned}
$$

From the estimate of coefficients in (6.19), we have

$$
\begin{aligned}
&\|\omega_\beta^\alpha(x,v)H_k^1(t,x,v)\|^2_{L_x^2L_v^2}\\
&\lesssim \sum_{\substack{\kappa,\gamma\in\mathcal{B},|\kappa|\le N_0,|\gamma|\le|\alpha|\\ |\rho|+|\gamma|\le N_0-12,}}\ \sum_{u\in\{\widetilde{\partial_t\phi^\kappa},\nabla_x\widetilde{\phi^\kappa}\}}(1+t)^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}G_\rho^\gamma(t,x,v)|u_k(t,x+\hat{v}t)|^2dxdv,
\end{aligned}
\tag{6.26}
$$

where $G_\rho^\gamma(t,x,v)$ is some determined function that satisfies the following estimate,

$$
|G_\rho^\gamma(t,x,v)|\lesssim|\omega_\beta^\alpha(x,v)(1+|x|^2+|v|^2)^{|\alpha|+2|\beta|-2|\rho|-|\gamma|+5}g_\rho^\gamma(t,x,v)|^2.
$$

Recall (4.44). From the estimate (6.26), the multilinear estimate (6.60) in Lemma 6.4, the estimates of modified profiles $\widehat{h^\alpha}(t,\xi)$ in (5.9) and (5.14), and the hierarchy between the different orders of weight functions, we have

$$
\begin{aligned}
&\|\omega_\beta^\alpha(x,v)H_k^1(t,x,v)\|_{L_x^2L_v^2}\\
&\lesssim\sum_{|\rho|+|\kappa|\le N_0-7}\|(1+|x|^2+|v|^2)^{|\alpha|+2|\beta|-2|\rho|-|\gamma|+22}\omega_\beta^\alpha(x,v)g_\rho^\kappa(t,x,v)\|_{L_x^2L_v^2}\\
&\times t^{-3/2}2^{-k/2}\big(\epsilon_0+|t|^{2\delta}2^{k_-}\epsilon_0\big)\lesssim|t|^{-3/2+\delta/2}2^{-k/2}\epsilon_1^2+|t|^{-3/2+3\delta}2^{-k/2+k_-}\epsilon_1^2.
\end{aligned}
\tag{6.27}
$$

It remains to estimate $H_k^2(t,x,v)$. Recall (6.25), (4.45), and (4.46). Note that $H_k^2(t,x,v)$ is a linear combination of bilinear forms that will be defined in (6.71). From the estimate of coefficients in (6.19) and the bilinear estimate (6.72) in Lemma 6.5 and the estimate of correction term in (4.70), we have

$$
\|\omega_\beta^\alpha(x,v)H_k^2(t,x,v)\|_{L_x^2L_v^2}\lesssim\big(|t|^{-1+\delta/2}+|t|^{-2+\delta/2}2^{-k}+|t|^{-2+2\delta}2^k\big)\epsilon_0. \tag{6.28}
$$

To sum up, from the decomposition (6.23) and the estimates (6.22), (6.27), and (6.28), we have

$$\sum_{k\in\mathbb{Z},k\leq 0}\|\omega_\beta^\alpha(x,v)H_k(t,x,v)\|_{L_x^2L_v^2}$$
$$\lesssim \Big[\Big(\sum_{2^k\leq|t|^{-1}}|t|^{-1/2+\delta/2}2^{k/2}\big(1+|t|2^k\big)\epsilon_0\Big)+\Big(\sum_{|t|^{-1}\leq 2^k\leq 2}\big(|t|^{-1+\delta/2}$$
$$+|t|^{-3/2+\delta/2}2^{-k/2}+|t|^{-2+\delta/2}2^{-k}+|t|^{-3/2+2\delta}2^{k/2}\big)\epsilon_0\Big]$$
$$\lesssim (1+t)^{-1+\delta/2}\log(1+|t|)\epsilon_0. \tag{6.29}$$

- If $k\geq 0$.
  From the estimate of coefficients in (6.19) and the bilinear estimate (6.73) in Lemma 6.5, we have

$$\sum_{k\geq 0,k\in\mathbb{Z}}\|\omega_\beta^\alpha(x,v)H_k^2(t,x,v)\|_{L_x^2L_v^2}\lesssim\sum_{k\geq 0}(1+t)^{-2+2\delta}2^{-k}\epsilon_1^2\lesssim(1+|t|)^{-2+2\delta}\epsilon_0. \tag{6.30}$$

Now, it would be sufficient to estimate "$H_k^1(t,x,v)$". Similar to the strategy used in obtaining the estimate (6.26), we have

$$\|\sum_{k\in\mathbb{Z},k\geq 0}\omega_\beta^\alpha(x,v)H_k^1(t,x,v)\|_{L_x^2L_v^2}^2\lesssim\sum_{k_1,k_2\in\mathbb{Z},k_1,k_2\geq 0}(1+t)^2K_{k_1,k_2}, \tag{6.31}$$

where

$$K_{k_1,k_2}:=\sum_{\substack{\kappa,\gamma\in\mathcal{B},|\kappa|\leq N_0\\|\rho|+|\gamma|\leq N_0-12,|\gamma|\leq|\alpha|}}\ \sum_{u,v\in\{\widetilde{\partial_t\phi^\kappa},\nabla_x\widetilde{\phi^\kappa}\}}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}G_{\rho;u}^{\gamma;v}(t,x,v)u_{k_1}(t,x$$
$$+\hat{v}t)v_{k_2}(t,x+\hat{v}t)dxdv\Big|, \tag{6.32}$$

where $G_{\rho;u}^{\gamma;v}(t,x,v)$, $u,v\in\{\widetilde{\partial_t\phi^\kappa},\nabla_x\widetilde{\phi^\kappa}\}$, are some determined function that satisfies the following estimate,

$$\sum_{\substack{a,b\in\mathbb{Z}_+,a+b\leq 5\\u,v\in\{\widetilde{\partial_t\phi^\kappa},\nabla_x\widetilde{\phi^\kappa}\}}}|\nabla_x^a\nabla_v^bG_{\rho;u}^{\gamma;v}(t,x,v)|$$
$$\lesssim\sum_{|\iota|+|\kappa|\leq|\rho|+|\gamma|+5}|\omega_\beta^\alpha(x,v)(1+|x|^2+|v|^2)^{|\alpha|+2|\beta|-2|\iota|-|\kappa|+10}g_\iota^\kappa(t,x,v)|^2. \tag{6.33}$$

Firstly, we consider the case $|k_1-k_2|\geq 5$. Recall (6.32). From the orthogonality in L^2 on the Fourier side, we know that the frequency of "$G_{\rho;i}^{\gamma}(t,x,v)$" is localized around $2^{\max\{k_1,k_2\}}$. Hence, from the above estimate (6.33), and the trilinear estimate (6.61) in Lemma 6.4, we have

$$\sum_{\substack{k,k_1,k_2\in\mathbb{Z},k_1,k_2\geq 0\\|k_1-k_2|\geq 5}}|K_{k_1,k_2}|$$

$$\lesssim \sum_{\substack{i=1,2,3,0\le a\le 4\\ \rho\in\mathcal{S},\gamma\in\mathcal{B},|\gamma|\le|\alpha|,|\rho|+|\gamma|\le N_0-12}} \sum_{\substack{k,k_1,k_2\in\mathbb{Z},k_1,k_2\ge 0\\ |k_1-k_2|\ge 5}} 2^{-\max\{k_1k_2\}}|t|^{-5+2\delta}\epsilon_1^2$$
$$\times\|(1+|x|^2)(1+|v|^{25})\nabla_x\nabla_v^a G^\gamma_{\rho;i}(t,x,v)\|_{L^1_{x,v}} \lesssim |t|^{-5+4\delta}\epsilon_0^2. \quad (6.34)$$

Lastly, we consider the case $|k_1-k_2|\le 5$. Recall (6.32). Again, from the estimate (6.33), the trilinear estimate (6.60) in Lemma 6.4, and the Cauchy-Schwarz inequality, we have

$$\sum_{\substack{k,k_1,k_2\in\mathbb{Z}\\ k_1,k_2\ge 0\\ |k_1-k_2|\le 5}} |K_{k_1,k_2}|$$
$$\lesssim \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1,k_2\ge 0\\ |k_1-k_2|\le 5}} \sum_{\substack{i=1,2,3,|a|\le 4\\ \rho\in\mathcal{S},\gamma\in\mathcal{B},|\gamma|\le|\alpha|\\ |\rho|+|\gamma|\le N_0-12}} (1+|t|)^{-5}\|(1+|x|^2)(1+|v|^{25})\nabla_v^\alpha G^\gamma_{\rho;i}(t,x,v)\|_{L^1_{x,v}}$$
$$\times\big(2^{-k_1/2}E^\phi_{\text{high}}(t) + \sum_{\iota\in\mathcal{B},|\iota|\le N_0}\|\widehat{h^\iota}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\big(2^{-k_2/2}E^\phi_{\text{high}}(t)$$
$$+\sum_{\iota\in\mathcal{B},|\iota|\le N_0}\|\widehat{h^\iota}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\big)$$
$$\lesssim |t|^{-5}\big(E^\phi_{\text{high}}(t)E^f_{\text{high}}(t)\big)^2 \lesssim |t|^{-5+4\delta}\epsilon_0^2. \quad (6.35)$$

From the estimates (6.31), (6.34), and (6.35), we have

$$\|\sum_{k\in\mathbb{Z},k\ge 0}\omega^\alpha_\beta(x,v)H^1_k(t,x,v)\|_{L^2_xL^2_v} \lesssim |t|^{-3/2+2\delta}\epsilon_0. \quad (6.36)$$

To sum up, recall the decompositions (6.20) and (6.23), the following estimate holds from the estimates (6.29), (6.30), and (6.36),

$$\sum_{\alpha\in\mathcal{B},\beta\in\mathcal{S},|\alpha|+|\beta|=N_0}\|\omega^\alpha_\beta(x,v)l.o.t^\alpha_{\beta;3}(t,x,v)\|_{L^2_{x,v}} \lesssim (1+|t|)^{-1+\delta}\epsilon_0. \quad (6.37)$$

Therefore, our desired estimate (6.13) holds from the estimates (6.15) and the above estimate (6.37).

Since the correction term $\widetilde{g}_{\alpha,\gamma}(t,v)$ defined in (4.61), which contributes the logarithmic growth in the estimate (6.29), equals zero if $|\alpha|<N_0$, with minor modifications in the above argument, our desired estimate (6.14) holds similarly. □

6.2. The low order energy estimate for the Vlasov part. In this subsection, as summarized in Proposition 6.2, we show that the low order energy $E^f_{\text{low}}(t)$ are uniformly bounded over time.

The main ideas of proving this fact are same as the ones we used in a relatively simpler system, Vlasov–Poisson system, in [41]. For the sake of readers, we explain concisely the main ideas in [41] as follows: (i) The main obstruction preventing the nonlinearities of the Vlasov–Nordström system decay faster comes from the low frequency part of the scalar field, which, intuitively speaking, is of size $1/t$. (ii) In the worst case, the decay rate over time is compensated by the spatial derivative. The fact depends on two observations. Firstly, since the output frequency is zero in the definition of low order energy, see (4.60), the frequency of the scalar field is exactly same as the frequency of the Vlasov part. Lastly, there is a spatial derivative in the bulk term $t\nabla_v \hat{v}\cdot\nabla_x g(t,x,v)$.

Proposition 6.2. *Under the bootstrap assumption* (4.69)*, the following estimate holds for any* $t\in[0,T]$,

$$\big|E^{f}_{low}(t)\big| \lesssim \epsilon_0 + \int_0^t (1+s)^{-3/2+3\delta}\epsilon_1^2 ds \lesssim \epsilon_0. \tag{6.38}$$

Proof. Recall the definition of the low order energy $E^{f}_{\mathrm{low}}(t)$ in (4.60), the definition of correction term in (4.61), the equation satisfied by $g(t,x,v)$ in (2.11) and the equation satisfied by $g^{\gamma}(t,x,v)$ in (4.7). For the sake of simplicity in notation, we focus on the case $|\gamma|=0$, i.e., $\gamma = Id$. With minor modification, we can show general case $\gamma\in\mathcal{B}, |\gamma|\le N_0$, similarly.

Let $\widetilde{\omega^a}(v) := \widetilde{\omega^a_{Id}}(v)$. Note that, the following equality holds for any a s.t., $0\le a\le N_0$,

$$\begin{aligned}
&\partial_t\big(\nabla^a\widehat{g}(t,0,v) - \nabla_v\cdot g_{a,Id}(t,v)\big)\\
&= \sum_{b_1+c_1=a}\int_{\mathbb{R}^3} 4\nabla_v^{b_1}\big((\partial_t+\hat{v}\cdot\nabla_x)\phi(t,x+\hat{v}t)\big)\nabla_v^{c_1} g(t,x,v)\\
&\quad + \sum_{b_2+c_2=a,|c_2|\le N_0-1} \nabla_v^{b_2}(\partial_t\phi(t,x+\hat{v}t))\big(c^{a,1}_{b_2,c_2}(v)\nabla_v^{c_2+1} g(t,x,v)\\
&\quad - t c^{a,2}_{b_2,c_2}\nabla_x\nabla_v^{c_2} g(t,x,v)\big)\\
&\quad + \nabla_v^{b_2}(\nabla_x\phi(t,x+\hat{v}t))\big(c^{a,3}_{b_2,c_2}(v)\nabla_v^{c_2+1} g(t,x,v) - t c^{a,4}_{b_2,c_2}(v)\nabla_x\nabla_v^{c_2} g(t,x,v)\big)dx,
\end{aligned} \tag{6.39}$$

where $c^{a,i}_{b_2,c_2}(v)$, $i\in\{1,2,3,4\}$, are some uniquely determined coefficients that satisfy the following estimate,

$$|c^{a,i}_{b_2,c_2}(v)| \lesssim (1+|v|), \quad \text{for any } i\in\{1,2,3,4\}. \tag{6.40}$$

Since our desired estimate (6.38) holds straightforwardly for the case $|t|\le 1$, we focus on the case when $|t|\ge 1$. Note that, after using the equality (3.7) repeatedly, the following equality holds for any $h\in L^\infty_t H^{N_0}(\mathbb{R}^3)$,

$$\begin{aligned}
\nabla_v^b h(t,x+\hat{v}t) &= \sum_{0\le c\le b} t^c c_{b,c}(v)\nabla_x^c h(t,x+\hat{v}t)\\
&= \sum_{0\le c\le b,\gamma\in\mathcal{B},|\gamma|=c} t^c\big||t|-|x+\hat{v}t|\big|^{-c} C_{b,\gamma}(x,v)\Gamma^\gamma h(t,x+\hat{v}t),
\end{aligned} \tag{6.41}$$

where the uniquely determined coefficients $C_{b,\gamma}(x,v)$ satisfy the following estimate,

$$|C_{b,\gamma}(x,v)| \lesssim 1, \quad |\nabla_x C_{b,\gamma}(x,v)| \lesssim 1/(|t|+|x+\hat{v}t|). \tag{6.42}$$

Note that $||t|-|x+\hat{v}t|| \gtrsim |t|(1+|v|^2)^{-1}$ if $|x|(1+|v|^2) \le 2^{-10}|t|$. This fact motivates us to decompose (6.39) into two parts as follows,

$$\begin{aligned}&\partial_t\big(\nabla^a\widehat{g}(t,0,v) - \nabla_v\cdot g_{a,Id}(t,v)\big)\\&= \sum_{\kappa\in\mathcal{B},c_1,c_2\in\mathbb{Z}_+,|\kappa|=c_1,c_1+c_2\le N_0,c_2\le N_0-1} H^a_{c_1,c_2}(t,v) + K^{a,\kappa}_{c_1,c_2}(t,v),\end{aligned} \tag{6.43}$$

where

$$\begin{aligned}H^a_{c_1,c_2}(t,v) = &\int_{\mathbb{R}^3}\psi_{>-10}(|x|(1+|v|^2)/(1+|t|))|t|^{c_1}\big[\nabla_x^{c_1}\partial_t\phi(t,x\\&+\hat{v}t)\big(\alpha^{a;1}_{c_1,c_2}(v)\nabla_v^{c_2+1}g(t,x,v) - t\alpha^{a;2}_{c_1,c_2}(v)\\&\times\nabla_x\nabla_v^{c_2}g(t,x,v)\big) + \nabla_x^{c_1+1}\phi(t,x+\hat{v}t)\big(\alpha^{a;3}_{c_1,c_2}(v)\nabla_v^{c_2+1}g(t,x,v)\\&-t\alpha^{a;4}_{c_1,c_2}(v)\nabla_x\nabla_v^{c_2}g(t,x,v)\big)\big]dx\\K^{a,\kappa}_{c_1,c_2}(t,v) = &\int_{\mathbb{R}^3}c_{c_1}(t,x,v)\big[\partial_t\Gamma^\kappa\phi(t,x+\hat{v}t)\big(b^{a,\kappa;1}_{c_1,c_2}(x,v)\nabla_v^{c_2+1}g(t,x,v)\\&-tb^{a,\kappa;2}_{c_1,c_2}(x,v)\nabla_x\nabla_v^{c_2}g(t,x,v)\big)\\&+\nabla_x\Gamma^\kappa\phi(t,x+\hat{v}t)\big(b^{a,\kappa;3}_{c_1,c_2}(x,v)\nabla_v^{c_2+1}g(t,x,v)\\&-tb^{a,\kappa;4}_{c_1,c_2}(x,v)\nabla_x\nabla_v^{c_2}g(t,x,v)\big)\big]dx,\end{aligned}$$

where

$$\begin{aligned}&c_{c_1}(t,x,v) := \psi_{\le-10}(|x|(1+|v|^2)/(1+|t|))|t|^{c_1}||t|-|x+\hat{v}t||^{-c_1},\\&\Longrightarrow |c_{c_1}(t,x,v)| \lesssim (1+|v|)^{2c_1}.\end{aligned} \tag{6.44}$$

From the estimates (6.40) and (6.42), the coefficients $\alpha^{a;i}_{c_1,c_2}(v)$ and $b^{a,\kappa;i}_{c_1,c_2}(x,v)$, $i\in\{1,2,3,4\}$, satisfy the following estimate,

$$|\alpha^{a;i}_{c_1,c_2}(v)| + |b^{a,\kappa;i}_{c_1,c_2}(x,v)| + |t||\nabla_x b^{a,\kappa;i}_{c_1,c_2}(x,v)| \lesssim (1+|v|). \tag{6.45}$$

Due to the high order weight function $\omega^{\vec{0}}_\beta(t,x,v)$ (see (4.51)) we associated with $\Lambda^\beta g(t,x,v)$, the inverse of weight function provides fast decay rate if $|x|(1+|v|^2) \gtrsim (1+|t|)$. As a result, the following estimate holds after using the $L^2_x - L^2_x L^2_v$ type bilinear estimate,

$$\|\widetilde{\omega^a}(v)H^a_{c_1,c_2}(t,v)\|_{L^2_v} \lesssim (1+|t|)^{-2}E^\phi_{\text{high}}(t)E^f_{\text{high}}(t) \lesssim (1+|t|)^{-2+2\delta}\epsilon_1^2. \tag{6.46}$$

It remains to estimate $K^{a,\kappa}_{c_1,c_2}(t,v)$. We emphasize the fact that, due to the cutoff function $\psi_{\le-10}(|x|(1+|v|^2)/(1+|t|))$ in $K^{a,\kappa}_{c_1,c_2}(t,v)$, the following estimate holds,

$$||t|-|x+\hat{v}t|| \gtrsim |t|(1+|v|^2)^{-1}.$$

After doing dyadic decomposition for the scalar field, we have

$$K^{a,\kappa}_{c_1,c_2}(t,v) = \sum_{k\in\mathbb{Z}} J^{a,\kappa;k}_{c_1,c_2}(t,v), \tag{6.47}$$

$$
\begin{aligned}
J_{c_1,c_2}^{a,\kappa;k}(t,v) := \int_{\mathbb{R}^3} & c_{c_1}(t,x,v)\big[P_k\big(\partial_t\Gamma^\kappa\phi\big)(t,x+\hat{v}t)\big(b_{c_1,c_2}^{a,\kappa;1}(x,v)\nabla_v^{c_2+1}g(t,x,v) \\
& - t b_{c_1,c_2}^{a,\kappa;2}(x,v)\nabla_x\nabla_v^{c_2}g(t,x,v)\big) \\
& + P_k\big(\nabla_x\Gamma^\kappa\phi\big)(t,x+\hat{v}t)\big(b_{c_1,c_2}^{a,\kappa;3}(x,v)\nabla_v^{c_2+1}g(t,x,v) \\
& - t b_{c_1,c_2}^{a,\kappa;4}(x,v)\nabla_x\nabla_v^{c_2}g(t,x,v)\big)\big]dx. \qquad (6.48)
\end{aligned}
$$

Based on the size of k, we separate into two case as follows.

- If $k \geq 0$.

If $c_1 \leq 10$, we use the $L_x^\infty - L_x^1 L_v^2$-type bilinear estimate by putting $P_k[\partial_t\Gamma^\kappa\phi](t,x+\hat{v}t)$ and $P_k[\nabla_x\Gamma^\kappa\phi](t,x+\hat{v}t)$ in L_x^∞ and the linear decay estimate (6.55) in Lemma 6.3. If $c_1 \geq 10$, we redo the argument used in the estimate H_k (see (6.20)) in the proof of Lemma 6.2. Recall the estimates (6.30) and (6.36). As a result, from the estimate of coefficients in (6.45), we have

$$
\begin{aligned}
\sum_{k\geq 0}\|\widetilde{\omega^a}(v)J_{c_1,c_2}^{a,\kappa;k}\|_{L_v^2} &\lesssim \sum_{k\geq 0}(1+|t|)^{-2}2^{-k}\big(E_{\mathrm{high}}^{\phi}(t)+E_{\mathrm{low}}^{\phi}(t)\big)E_{\mathrm{high}}^{f}(t)+|t|^{-3/2+2\delta}\epsilon_0 \\
&\lesssim |t|^{-3/2+2\delta}\epsilon_0. \qquad (6.49)
\end{aligned}
$$

- If $k \leq 0$.

For this case, we first do integration by parts in "x" in (6.48) to move the spatial derivative "∇_x" in front of $\nabla_x\nabla_v^{\gamma_2}g(t,x,v)$.

If $c_1 \leq 10$, similar to the strategies used in the case $k \geq 0$, we use the $L_x^\infty - L_x^1 L_v^2$-type bilinear estimate. As a result, from the linear decay estimates (2.21) in Lemma 2.2 and the estimate (6.55) in Lemma 6.3 for the scalar field and the estimate of coefficients in (6.45), we have

$$
\begin{aligned}
\sum_{k\in\mathbb{Z},k\leq 0}\|\widetilde{\omega^a}(v)J_{c_1,c_2}^{a,\kappa;k}\|_{L_v^2} &\lesssim \sum_{k\in\mathbb{Z},k\leq 0}\min\{(1+|t|)^{-2},(1+|t|)^{-1}2^k\}E_{\mathrm{low}}^{\phi}(t)E_{\mathrm{high}}^{f}(t) \\
&\lesssim (1+|t|)^{-2+4\delta}\epsilon_1^2. \qquad (6.50)
\end{aligned}
$$

If $c_1 \geq 10$, then we rerun the argument used in the estimate of H_k, $k \leq 0$, in the proof of Lemma 6.2. More precisely, if $2^k \leq |t|^{-1}$, then with minor modifications in the estimate (6.22), we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z},2^k\leq|t|^{-1}}\|\widetilde{\omega^a}(v)J_{c_1,c_2}^{a,\kappa;k}\|_{L_v^2} \\
&\lesssim \sum_{k\in\mathbb{Z},2^k\leq|t|^{-1}}\big(2^{k/2}+|t|2^{3k/2}\big)\epsilon_0(1+2^k|t|)(1+|t|)^{-3/2}E_{\mathrm{high}}^{f;2}(t) \\
&\lesssim (1+|t|)^{-2+\delta}\epsilon_0. \qquad (6.51)
\end{aligned}
$$

If $|t|^{-1} \leq 2^k \leq 1$, then similar to what we did in (6.23) and (6.48), we use the decomposition of the profile in (4.46) for ϕ^{γ_1} and $\partial_t\phi^{\gamma_1}$. Thanks to the extra spatial derivative that comes from the integration by parts in "x", the following estimate holds from the estimate (6.60) in Lemma 6.4 and the estimate (6.73) in Lemma 6.5,

$$
\sum_{k\in\mathbb{Z},|t|^{-1}\leq 2^k\leq 1}\|\widetilde{\omega^a}(v)J_{c_1,c_2}^{a,\kappa;k}\|_{L_v^2}
$$

$$\lesssim \sum_{k\in\mathbb{Z},|t|^{-1}\leq 2^k\leq 1} |t|^{-3/2+3\delta}2^{k/2}\epsilon_1^2 + |t|^{-2}\epsilon_1^2 + |t|^{-3}2^{-k}\epsilon_1^2$$
$$\lesssim |t|^{-3/2+3\delta}\epsilon_0. \tag{6.52}$$

To sum up, from the estimates (6.50), (6.51), and (6.52), the following estimate holds,

$$\sum_{\kappa\in\mathcal{B},c_1,c_2\in\mathbb{Z}_+,|\kappa|=c_1,c_1+c_2\leq N_0,c_2\leq N_0-1}\ \sum_{k\in\mathbb{Z},k\leq 0} \|\widetilde{\omega^a}(v)J^{a,\kappa;k}_{c_1,c_2}\|_{L^2_v} \lesssim |t|^{-3/2+3\delta}\epsilon_0. \tag{6.53}$$

From the estimates (6.49) and (6.53) and the decomposition (6.47), we have

$$\sum_{\kappa\in\mathcal{B},c_1,c_2\in\mathbb{Z}_+,|\kappa|=c_1,c_1+c_2\leq N_0,c_2\leq N_0-1} \|\widetilde{\omega^a}(v)K^{a,\kappa}_{c_1,c_2}(t,v)\|_{L^2_v} \lesssim |t|^{-3/2+3\delta}\epsilon_0. \tag{6.54}$$

Recall the decomposition (6.43). Our desired estimate (6.38) follows directly from the estimates (6.46) and (6.54). □

6.3. Toolkit. In this subsection, we prove some basic estimates used in the previous two subsections.

The first basic estimate is a linear decay estimate for the scalar field part, which shows that the nonlinear solutions have sharp decay ate $1/\big((1+|t|)(1+||t|-|x||)\big)$ as long as the low order energy of scalar field doesn't grow over time. It is a natural application of the linear decay estimate of the half wave in Lemma 2.2.

Lemma 6.3. *Given a Fourier multiplier operator T with Fourier multiplier symbol $m(\xi)\in\mathcal{S}^\infty$, the following estimate holds,*

$$\sum_{u\in\{\partial_t\phi^\rho,\nabla_x\phi^\rho\},\rho\in\mathcal{B},|\rho|\leq 13} |T(u)(t,x)|$$
$$\lesssim (1+|t|)^{-1}(1+||t|-|x||)^{-1}\|m(\xi)\|_{\mathcal{S}^\infty}E^{\phi}_{low}(t). \tag{6.55}$$

Proof. We first do dyadic decomposition for the frequency of u. As a result, we have

$$T(u)(t,x)=\sum_{k\in\mathbb{Z}} T_k(u)(t,x),\quad T_k=T\circ P_k.$$

Recall (4.38) and (4.39). From the linear decay estimate (2.21) in Lemma 2.2, we have

$$\sum_{u\in\{\nabla\phi^\rho,\partial_t\phi^\rho\},\rho\in\mathcal{B},|\rho|\leq 13} |T_k(u)(t,x)|$$
$$\lesssim \sum_{i=0,1,2,\alpha\in\mathcal{B},|\alpha|\leq 15} (1+|t|+|x|)^{-1}2^{k-2k_+}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|P_k h^\alpha(t)\|_{X_i}. \tag{6.56}$$

From the above estimate, our desired estimate (6.55) holds straightforwardly if $||t|-|x||\leq 1$. Hence, it would be sufficient to consider the case $||t|-|x||\geq 1$.

From the equalities (3.50) and (3.51) in Lemma 3.6, we can trade regularities for the decay rates of the distance with respect to the light cone. More precisely, from the estimates of coefficients in (3.59), (3.63), and (3.64), the estimates of symbols in (3.62), and the linear decay estimate (2.21) in Lemma 2.2, we have

$$\begin{aligned}&\sum_{u\in\{\nabla\phi^{\rho},\partial_t\phi^{\rho}\},\rho\in\mathcal{B},|\rho|\le 13}\big|\big(|t|-|x|\big)^3 T_k(u)(t,x)\big|\\&\lesssim\sum_{\alpha\in\mathcal{B},|\alpha|\le 16}2^{-2k}(1+|t|+|x|)^{-1}\|m(\xi)\|_{\mathcal{S}^\infty_k}\big(\|P_k h^{\alpha}(t)\|_{X_0}\\&\quad+\|P_k h^{\alpha}(t)\|_{X_1}\big)+\big(2^{-5k}2^{k+3k}+||t|-|x||2^{-4k}2^{k+3k}\\&\quad+||t|-|x||^2 2^{-3k}2^{k+3k}\big)\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\widehat{\partial_t h^{\alpha}}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}.\end{aligned}\tag{6.57}$$

Recall the low order energy of the nonlinear wave part $E^{\phi}_{\mathrm{low}}(t)$ in (4.63). From the above estimate, we have

$$\begin{aligned}&\sum_{u\in\{\nabla\phi^{\rho},\partial_t\phi^{\rho}\},\rho\in\mathcal{B},|\rho|\le 13}\big|T_k(u)(t,x)\big|\lesssim(||t|-|x||)^{-1}\|m(\xi)\|_{\mathcal{S}^\infty_k}E^{\phi}_{\mathrm{low}}(t)\\&\times\big[(1+|t|)^{-1}2^{-2k}(||t|-|x||)^{-2}+(1+|t|)^{-1}2^{-k}(||t|-|x||)^{-1}+(1+|t|)^{-2}2^{-k}\big].\end{aligned}\tag{6.58}$$

Therefore, after optimizing the estimates (6.56) and (6.58), we have

$$\sum_{u\in\{\nabla\phi^{\rho},\partial_t\phi^{\rho}\},\rho\in\mathcal{B},|\rho|\le 13}|T(u)(t,x)|\lesssim(1+|t|)^{-1}(||t|-|x||)^{-1}\|m(\xi)\|_{\mathcal{S}^\infty_k}E^{\phi}_{\mathrm{low}}(t).$$

Hence finishing the proof of the desired estimate (6.55). □

Next, we prove two trilinear estimates, which are used in the estimate of $H^1_k(t,x,v)$ in (6.24). The key feature of the desired trilinear estimates is that we exploit the smallness of the space-resonance set to get a improved decay estimate, which is better than applying the decay estimate of the density type function in Lemma 2.20 directly.

Lemma 6.4. *Given any fixed signs $\mu,\nu\in\{+,-\}$, fixed time $t\in\mathbb{R}_+$, fixed $k_1,k_2\in\mathbb{Z}$. Moreover, given any functions $f_1,f_2:\mathbb{R}_t\times\mathbb{R}^3_x\to\mathbb{C}$, and any distribution function $g:\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v\to\mathbb{R}$, we define a trilinear form as follows,*

$$\begin{aligned}T(f_1,f_2,g):=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}&e^{-i\mu t|\nabla|}P_{k_1}[f_1](t,x+\hat{v}t)\\&e^{-i\nu t|\nabla|}P_{k_2}[f_2](t,x+\hat{v}t)g(t,x,v)\,dx\,dv,\end{aligned}\tag{6.59}$$

then the following estimate holds,

$$\begin{aligned}&|T(f_1,f_2,g)|\\&\lesssim\sum_{0\le a\le 4}(1+|t|)^{-5}\|(1+|x|)^2(1+|v|)^{25}\nabla_v^a g(t,x,v)\|_{L^1_{x,v}}\big(2^{-k_{1,-}}\|\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\\&\quad+\|\nabla_\xi\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\big(2^{-k_{2,-}}\|\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}+\|\nabla_\xi\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\big).\end{aligned}\tag{6.60}$$

Moreover, if $|k_1-k_2|\ge 5$, then the following estimate holds,

$$|T(f_1,f_2,g)|$$

$$\lesssim \sum_{0\leq a\leq 4} (1+|t|)^{-5} 2^{-\max\{k_1,k_2\}} \|(1+|x|)^2(1+|v|)^{25}\nabla_x\nabla_v^a g(t,x,v)\|_{L^1_{x,v}}$$
$$\big(2^{-k_{1,-}}\|\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}$$
$$+\|\nabla_\xi\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\big(2^{-k_{2,-}}\|\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}+\|\nabla_\xi\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\big). \tag{6.61}$$

Proof. Recall (6.59). Note that the following equality holds on the Fourier side,

$$T(f_1,f_2,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi - it\mu|\xi-\eta| - it\nu|\eta|}\overline{\widehat{g}(t,\xi,v)}\widehat{f_1}(t,\xi - \eta)\widehat{f_2}(t,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta d\xi dv. \tag{6.62}$$

From the above formula of $T(f_1,f_2,g)$, our desired estimate (6.60) holds straightforwardly if $|t|\leq 1$. Hence, we restrict ourself to the case $|t|\geq 1$.

Firstly, we do integration by parts in "ξ" once. As a result, we have

$$T(f_1,f_2,g) = T_1(f_1,f_2,g) + T_2(f_1,f_2,g), \tag{6.63}$$

where

$$T_1(f_1,f_2,g) := \frac{i}{t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi - it\mu|\xi-\eta| - it\nu|\eta|}\nabla_\xi\overline{\widehat{g}(t,\xi,v)} \cdot\Big(\frac{\hat{v}-\mu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\mu(\xi-\eta)/|\xi-\eta|\big|^2}\widehat{f_1}(t,\xi-\eta) \times\psi_{k_1}(\xi-\eta)\Big)\widehat{f_2}(t,\eta)\psi_{k_2}(\eta)d\eta d\xi dv,$$

$$T_2(f_1,f_2,g) := \frac{i}{t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi - it\mu|\xi-\eta| - it\nu|\eta|}\overline{\widehat{g}(t,\xi,v)}\nabla_\xi \cdot\Big(\frac{\hat{v}-\mu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\mu(\xi-\eta)/|\xi-\eta|\big|^2}\widehat{f_1}(t,\xi-\eta) \times\psi_{k_1}(\xi-\eta)\Big)\widehat{f_2}(t,\eta)\psi_{k_2}(\eta)d\eta d\xi dv.$$

For $T_1(f_1,f_2,g)$, we do integration by parts in "ξ" one more time. As a result, we have

$$T_1(f_1,f_2,g) := \frac{-1}{t^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi - it\mu|\xi-\eta| - it\nu|\eta|}\widehat{f_2}(t,\eta)\psi_{k_2}(\eta)\nabla_\xi \cdot\Big[\frac{\hat{v}-\mu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\mu(\xi-\eta)/|\xi-\eta|\big|^2} \times\nabla_\xi\overline{\widehat{g}(t,\xi,v)}\cdot\Big(\frac{\hat{v}-\mu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\mu(\xi-\eta)/|\xi-\eta|\big|^2}\psi_{k_1}(\xi-\eta)\widehat{f_1}(t,\xi-\eta)\Big)\Big]d\eta d\xi dv. \tag{6.64}$$

For $T_2(f_1,f_2,g)$, we first switch the role of "$\xi-\eta$" and "η" and then do integration by parts in "ξ" once. As a result, we have

$$T_2(f_1,f_2,g) := \frac{i}{t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi - it\mu|\eta| - it\nu|\xi-\eta|}\overline{\widehat{g}(t,\xi,v)}\nabla_\eta\cdot\Big(\frac{\hat{v}-\mu\eta/|\eta|}{\big|\hat{v}-\mu\eta/|\eta|\big|^2}\widehat{f_1}(t,\eta)$$

$$
\begin{aligned}
&\times\psi_{k_1}(\eta)\big)\widehat{f_2}(t,\xi-\eta)\psi_{k_2}(\xi-\eta)d\eta d\xi dv\\
=\ &\frac{-1}{t^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi-it\mu|\eta|-it\nu|\xi-\eta|}\nabla_\xi\\
&\cdot\Big[\frac{\hat{v}-\nu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\nu(\xi-\eta)/|\xi-\eta|\big|^2}\overline{\widehat{g}(t,\xi,v)}\widehat{f_2}(t,\xi-\eta)\psi_{k_2}(\xi-\eta)\Big]\\
&\times\nabla_\eta\cdot\Big(\frac{\hat{v}-\mu\eta/|\eta|}{\big|\hat{v}-\mu\eta/|\eta|\big|^2}\widehat{f_1}(t,\eta)\psi_{k_1}(\eta)\Big)d\eta d\xi dv.
\end{aligned}
$$

After doing dyadic decomposition for "ξ", we have the following decomposition,

$$
T_1(f_1,f_2,g)=\sum_{k\in\mathbb{Z}}T_1^k(f_1,f_2,g),\quad T_2(f_1,f_2,g)=\sum_{k\in\mathbb{Z}}T_2^k(f_1,f_2,g),\tag{6.65}
$$

where

$$
\begin{aligned}
T_1^k(f_1,f_2,g):=\ &\frac{-1}{t^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi-it\mu|\xi-\eta|-it\nu|\eta|}\widehat{f_2}(t,\eta)\psi_{k_2}(\eta)\nabla_\xi\\
&\cdot\Big[\frac{\hat{v}-\mu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\mu(\xi-\eta)/|\xi-\eta|\big|^2}\\
&\times\nabla_\xi\overline{\widehat{g}(t,\xi,v)}\cdot\Big(\frac{\hat{v}-\mu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\mu(\xi-\eta)/|\xi-\eta|\big|^2}\psi_{k_1}(\xi-\eta)\widehat{f_1}(t,\xi\\
&-\eta)\Big)\Big]\psi_k(\xi)d\eta d\xi dv,
\end{aligned}\tag{6.66}
$$

$$
\begin{aligned}
T_2^k(f_1,f_2,g):=\ &\frac{-1}{t^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it\hat{v}\cdot\xi-it\mu|\eta|-it\nu|\xi-\eta|}\nabla_\xi\\
&\cdot\Big[\frac{\hat{v}-\nu(\xi-\eta)/|\xi-\eta|}{\big|\hat{v}-\nu(\xi-\eta)/|\xi-\eta|\big|^2}\overline{\widehat{g}(t,\xi,v)}\widehat{f_2}(t,\xi-\eta)\\
&\times\psi_{k_2}(\xi-\eta)\Big]\nabla_\eta\cdot\Big(\frac{\hat{v}-\mu\eta/|\eta|}{\big|\hat{v}-\mu\eta/|\eta|\big|^2}\widehat{f_1}(t,\eta)\psi_{k_1}(\eta)\Big)\psi_k(\xi)d\eta d\xi dv.
\end{aligned}\tag{6.67}
$$

On one hand, if we use the volume of support of "ξ" and the Cauchy-Schwarz inequality for the integration with respect to "η", then the following estimate holds,

$$
\begin{aligned}
&|T_1^k(f_1,f_2,g)|+|T_2^k(f_1,f_2,g)|\\
&\lesssim\sum_{0\le a\le 2}(1+|t|)^{-2}2^{3k}\big(\|(1+|v|)^{10}\nabla_\xi^a\widehat{g}(t,\xi,v)\|_{L^\infty_\xi L^1_v}\big(2^{-k_{1,-}}\|\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\\
&+\|\nabla_\xi\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\big(2^{-k_{2,-}}\|\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\\
&+\|\nabla_\xi\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\big).
\end{aligned}\tag{6.68}
$$

On the other hand, if we do integration by parts in "v"four times for $T_1^k(f_1,f_2,g)$ and $T_2^k(f_1,f_2,g)$, then the following estimate holds after using the volume of support of "ξ" and the Cauchy-Schwarz inequality for the integration with respect to "η",

$$
|T_1^k(f_1,f_2,g)|+|T_2^k(f_1,f_2,g)|
$$

$$\lesssim \sum_{0\le a\le 4, 0\le b\le 2} (1+|t|)^{-6} 2^{-k} \|(1+|v|)^{25}\nabla_v^a \nabla_\xi^b \widehat{g}(t,\xi,v)\|_{L^\infty_\xi L^1_v}$$
$$\times \big(2^{-k_{1,-}}\|\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}$$
$$+\|\nabla_\xi \widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\big(2^{-k_{2,-}}\|\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}+\|\nabla_\xi \widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\big). \tag{6.69}$$

Recall the decompositions in (6.63) and (6.65). From the estimates (6.68) and (6.69), we have

$$|T(f_1, f_2, g)|$$
$$\lesssim \sum_{0\le a\le 4} (1+|t|)^{-5}\|(1+|x|)^2(1+|v|)^{25}\nabla_v^a g(t,x,v)\|_{L^1_{x,v}}\big(2^{-k_{1,-}}\|\widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}$$
$$+\|\nabla_\xi \widehat{f_1}(t,\xi)\psi_{k_1}(\xi)\|_{L^2}\big)\big(2^{-k_{2,-}}\|\widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}+\|\nabla_\xi \widehat{f_2}(t,\xi)\psi_{k_2}(\xi)\|_{L^2}\big). \tag{6.70}$$

Hence finishing the proof of the desired estimate (6.60).

Note that $|k-\max\{k_1,k_2\}|\le 10$ if $|k_1-k_2|\ge 5$. Hence the desired estimate (6.61) follows directly from (6.69). □

Lastly, we prove two bilinear estimates for the Vlasov-Vlasov type interaction, which have been used in the estimate of $H_k^2(t,x,v)$ in (6.25).

For any fixed sign $\mu\in\{+,-\}$, any two distribution functions $f_1(t,x,v)$ and $f_2(t,x,v)$, any fixed $k\in\mathbb{Z}$, any symbol $m(\xi,v)\in L^\infty_v\mathcal{S}^\infty_k$, and any differentiable coefficient $c(v)$, we define a bilinear operator as follows,

$$B_k(f_1,f_2)(t,x,v) := f_1(t,x,v)E(P_k[f_2(t)])(x+t\hat{v}), \tag{6.71}$$

where

$$E(P_k[f])(t,x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{-i\mu t\hat{u}\cdot\xi}c(u)m(\xi,u)\psi_k(\xi)\widehat{f}(t,\xi,u)d\xi du.$$

As summarized in the following Lemma, we have two bilinear estimates for the above defined bilinear operator.

Lemma 6.5. *For any fixed $t\in\mathbb{R}$, $|t|\ge 1$, and any localized differentiable function $f_3(t,v):\mathbb{R}_t\times\mathbb{R}^3_v\longrightarrow\mathbb{C}$, the following bilinear estimate holds for the bilinear operators defined in (6.71),*

$$\|B_k(f_1,f_2)(t,x,v)\|_{L^2_xL^2_v}$$
$$\lesssim \sum_{|\alpha|\le 5}\big(\|m(\xi,v)\|_{L^\infty_v\mathcal{S}^\infty_k}+\|m(\xi,v)\|_{L^\infty_v\mathcal{S}^\infty_k}\big)\big[|t|^{-2}2^k\|\big(|c(v)|+|\nabla_v c(v)|\big)f_3(t,v)\|_{L^2_v}$$
$$+|t|^{-3}2^k\|(1+|v|+|x|)^{20}c(v)f_2(t,x,v)\|_{L^2_xL^2_v}$$
$$+|t|^{-3}\|c(v)\big(\widehat{f_2}(t,0,v)-\nabla_v\cdot f_3(t,v)\big)\|_{L^2_v}\big]$$
$$\times\|(1+|v|+|x|)^{20}\nabla_v^\alpha f_1(t,x,v)\|_{L^2_xL^2_v},\quad \text{if } k\in\mathbb{Z}, |t|^{-1}\lesssim 2^k\le 1. \tag{6.72}$$

Alternatively, the following bilinear estimate holds for any $k\in\mathbb{Z}$,

$$\|B_k(f_1,f_2)(t,x,v)\|_{L^2_xL^2_v}$$

$$\lesssim \sum_{|\alpha|\leq 5} \min\{|t|^{-3}, 2^{3k}\} \|m(\xi, v)\|_{L^\infty_v \mathcal{S}^\infty_k} \|(1+|v|+|x|)^{20} c(v) f_2(t,x,v)\|_{L^2_x L^2_v}$$
$$\times \|(1+|v|+|x|)^{20} \nabla_v^\alpha f_1(t,x,v)\|_{L^2_x L^2_v}. \tag{6.73}$$

Proof. See [41][Lemma 3.2& Lemma 3.3]. □

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Appendix A. Commutation Rules

Although it is tedious to compute the commutation rules and check all the first order derivatives in $\mathcal{S}$ acting on the inhomogeneous modulation $\tilde{d}(t,x,v)$, for the sake of readers, we do detailed computations in this appendix. Hence proving Lemmas 3.7 and 3.8

In the following lemma, we compute several basic quantities to be used later, which also directly imply our desired results in Lemma 3.8.

Lemma A.1. *For any $\rho \in \mathcal{S}$, $|\rho| = 1$, the following equality holds,*

$$\Lambda^\rho(\tilde{d}(t,x,v)) = e_1^\rho(x,v)\tilde{d}(t,x,v) + e_2^\rho(x,v), \quad D_v(\tilde{d}(t,x,v)) = \hat{e}_1(x,v)\tilde{d}(t,x,v) + \hat{e}_2(x,v), \tag{A.1}$$

where the coefficients satisfy the following estimate,

$$|e_1^\rho(x,v)| + |e_2^\rho(x,v)| + |\hat{e}_1(x,v)| + |\hat{e}_2(x,v)| \lesssim 1, \quad |\hat{e}_2(x,v)|\psi_{\geq 2}(|x|) = 0. \tag{A.2}$$

Moreover, the following rough estimate holds for any $\beta \in \mathcal{S}$,

$$\sum_{i=1,2} |\Lambda^\beta e_i^\rho(x,v)| + |\Lambda^\beta \hat{e}_i(x,v)| \lesssim (1+|x|)^{|\beta|}(1+|v|)^{|\beta|}. \tag{A.3}$$

Proof. First of all, we compute two basic quantities. Recall (3.9). For any $\mu \in \{+,-\}$, we have

$$\begin{aligned}
\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big)\omega_\mu(x,v) &= \big[\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big)\big((x\cdot v) + \mu\sqrt{(x\cdot v)^2 + |x|^2}\big)\big] \\
&= \big[(\tilde{V}_i\cdot x) + \mu\frac{(x\cdot v)(\tilde{V}_i\cdot x)}{\sqrt{(x\cdot v)^2+|x|^2}} - \mu\frac{t}{\sqrt{1+|v|^2}}\frac{\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big] \\
&= \frac{-\mu\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}} - (x\cdot v + \mu\sqrt{(x\cdot v)^2+|x|^2})\big) \\
&= \frac{-\mu\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_\mu(x,v)\big),
\end{aligned} \tag{A.4}$$

$$\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\omega_\mu(x,v) = \big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\big((x\cdot v) + \mu\sqrt{(x\cdot v)^2+|x|^2}\big)$$
$$= (\tilde{v}\cdot x + \mu\frac{(x\cdot v)\tilde{v}\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}) - \frac{t}{(1+|v|^2)^{3/2}}(\tilde{v}\cdot v + \mu\frac{(x\cdot v)(\tilde{v}\cdot v) + x\cdot\tilde{v}}{\sqrt{(x\cdot v)^2+|x|^2}})$$
$$= \frac{-\mu x\cdot\tilde{v}}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_\mu(x,v)\big) - \frac{t|v|}{(1+|v|^2)^{3/2}}. \tag{A.5}$$

Therefore,

$$\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_\mu(x,v)\big)$$
$$= \frac{\mu\tilde{v}\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_\mu(x,v)\big), \tag{A.6}$$
$$\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big)\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_\mu(x,v)\big)$$
$$= \frac{\mu\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_\mu(x,v)\big). \tag{A.7}$$

Now, we consider the case with the cutoff function defined in (3.14). From the equalities (A.4), we have,

$$\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big)\omega(x,v)$$
$$= \frac{-\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}} - \omega_+(x,v)\big)\psi_{\geq 0}((x\cdot v)^2+|x|^2)$$
$$+2\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\big((x\cdot v)(\tilde{V}_i\cdot x)$$
$$-\frac{t}{\sqrt{1+|v|^2}}\tilde{V}_i\cdot x\big) = c_i(x,v)\tilde{d}(t,x,v) + e_i(x,v), \tag{A.8}$$

where $c_i(x,v)$ and $e_i(x,v)$, $i\in\{1,2,3\}$, are defined as follows,

$$c_i(x,v) = -\sqrt{1+|v|^2}\big[\psi_{\geq 0}((x\cdot v)^2+|x|^2)\frac{\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}$$
$$+2\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\tilde{V}_i\cdot x\big], \tag{A.9}$$
$$e_i(x,v) = \frac{\tilde{V}_i\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\omega_+(x,v)\psi_{<0}((x\cdot v)^2+|x|^2)\psi_{\geq 0}((x\cdot v)^2+|x|^2)$$
$$+2\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\big((x\cdot v)(\tilde{V}_i\cdot x) - \omega(x,v)\tilde{V}_i\cdot x\big). \tag{A.10}$$

From the detailed formulas of $c_i(x,v)$ and $e_i(x,v)$ in (A.9) and (A.10), we have

$$|c_i(x,v)| \lesssim 1+|v|, \quad |e_i(x,v)| \lesssim 1, \quad |e_i(x,v)|\psi_{\geq 2}(|x|) = 0. \tag{A.11}$$

From the equality (A.5), we have

$$\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\omega(x,v)$$

$$= \Big[\frac{-\tilde{v}\cdot x}{\sqrt{(x\cdot v)^2+|x|^2}}\big(\frac{t}{\sqrt{1+|v|^2}}-\omega_+(x,v)\big)-\frac{t|v|}{(1+|v|^2)^{3/2}}\Big]$$
$$\times\psi_{\geq 0}(|x|^2+(x\cdot v)^2)+2\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\big((x\cdot v)\tilde{v}\cdot x$$
$$-\frac{t}{(1+|v|^2)^{1/2}}\tilde{v}\cdot x\big)$$
$$= \tilde{c}(x,v)\tilde{d}(t,x,v)-\frac{t|v|}{(1+|v|^2)^{3/2}}\psi_{\geq 0}((x\cdot v)^2+|x|^2)+\tilde{e}(x,v), \tag{A.12}$$

where

$$\tilde{c}(x,v) = -x\cdot\tilde{v}\sqrt{1+|v|^2}\Big[\frac{\psi_{\geq 0}((x\cdot v)^2+|x|^2)}{\sqrt{(x\cdot v)^2+|x|^2}}+2\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\omega_+(x,v)\Big], \tag{A.13}$$

$$\tilde{e}(x,v) = 2\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)(\tilde{v}\cdot x)\big(x\cdot v-\omega(x,v)\big)$$
$$+\omega_+(x,v)\frac{\tilde{v}\cdot x}{\sqrt{|x|^2+(x\cdot v)^2}}\psi_{\geq 0}((x\cdot v)^2+|x|^2)\psi_{<0}((x\cdot v)^2+|x|^2). \tag{A.14}$$

From the detailed formulas of $\tilde{c}(x,v)$ and $\tilde{e}(x,v)$ in (A.13) and (A.14), we have

$$|\tilde{c}(x,v)|\lesssim 1,\quad |\tilde{e}(x,v)|\lesssim 1,\quad |\tilde{e}(x,v)|\psi_{\geq 2}(|x|)=0. \tag{A.15}$$

Now, we are ready to compute the quantity $\Lambda^{\rho}(\tilde{d}(t,x,v))$, where $\rho\in\mathcal{S}$, $|\rho|=1$. Recall that

$$\tilde{d}(t,x,v) = \frac{1}{\sqrt{1+|v|^2}}\big(\frac{t}{\sqrt{1+|v|^2}}-\omega(x,v)\big) \tag{A.16}$$
$$= \frac{1}{\sqrt{1+|v|^2}}\big(\frac{t}{\sqrt{1+|v|^2}}-(x\cdot v+\sqrt{(x\cdot v)^2+|x|^2})\psi_{\geq 0}((x\cdot v)^2+|x|^2). \tag{A.17}$$

A direct computation gives us the following equality,

$$\nabla_x\tilde{d}(t,x,v) = \frac{-1}{\sqrt{1+|v|^2}}\big(v+\frac{(x\cdot v)v+x}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\psi_{\geq 0}\big((x\cdot v)^2+|x|^2\big)$$
$$-\frac{\omega_+(x,v)}{\sqrt{1+|v|^2}}\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\big((x\cdot v)v+x\big). \tag{A.18}$$

Therefore, from the above equality (A.18), we have

$$|\tilde{v}\cdot\nabla_x\tilde{d}(t,x,v)|\lesssim 1,\quad \sum_{i=1,2,3}|\tilde{V}_i\cdot\nabla_x\tilde{d}(t,x,v)|\lesssim(1+|v|)^{-1}. \tag{A.19}$$

Moreover, from the equalities (A.8) and (A.12), we know that the following two equalities hold,

$$\big(\Omega_i^v-\frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big)\big(\frac{t}{\sqrt{1+|v|^2}}-\omega(x,v)\big)$$

$$= -\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big)\omega(x,v) = -\big(c_i(x,v)\tilde{d}(t,x,v)+e_i(x,v)\big), \quad (A.20)$$

$$\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\big(\frac{t}{\sqrt{1+|v|^2}} - \omega(x,v)\big)$$
$$= -\frac{t|v|}{(1+|v|^2)^{3/2}} - \big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\omega(x,v)$$
$$= -\tilde{c}(x,v)\tilde{d}(t,x,v) - \frac{t|v|}{(1+|v|^2)^{3/2}}\psi_{<0}((x\cdot v)^2+|x|^2) - \tilde{e}(x,v)$$
$$= \tilde{\tilde{c}}(x,v)\tilde{d}(t,x,v) + \tilde{\tilde{e}}(x,v), \quad (A.21)$$

where

$$\tilde{\tilde{c}}(x,v) := -\tilde{c}(x,v) - \frac{|v|}{\sqrt{1+|v|^2}}\psi_{<0}((x\cdot v)^2+|x|^2), \quad (A.22)$$

$$\tilde{\tilde{e}}(x,v) := -\tilde{e}(x,v) - \frac{|v|}{1+|v|^2}\omega_+(x,v)\psi_{\geq 0}((x\cdot v)^2+|x|^2)\psi_{<0}((x\cdot v)^2+|x|^2). \quad (A.23)$$

From the detailed formulas in above equalities (A.22) and (A.23) and the estimate of coefficients in (A.15), we have

$$|\tilde{\tilde{c}}(x,v)| + |\tilde{\tilde{e}}(x,v)| \lesssim 1, \quad |\tilde{\tilde{e}}(x,v)|\psi_{\geq 2}(|x|) = 0. \quad (A.24)$$

Recall (A.16). To sum up, from the equalities (A.18), (A.20), and (A.21) and the decomposition of "D_v" in (3.22), we know that the desired equalities in (A.1) hold for some uniquely determined coefficients.

Moreover, our desired estimates (A.2) and (A.3) hold from the estimate (A.19) and the estimates of coefficients in (A.11) and (A.24).

□

Our desired results in Lemma 3.7 follow directly from the following Lemma.

Lemma A.2. *For any $\rho \in \mathcal{K}$, $|\rho| = 1$, and $i \in \{1,\dots,7\}$, the following commutation rule holds,*

$$[X_i, \Lambda^\rho] = \sum_{\kappa\in\mathcal{K},|\kappa|=1}\big(\tilde{c}_i^{\rho,\kappa}(x,v)\tilde{d}(t,x,v) + \hat{c}_i^{\rho,\kappa}(x,v)\big)\Lambda^\kappa, \quad (A.25)$$

where the coefficients $\tilde{c}_i^{\rho,\kappa}(t,x,v)$ and $\hat{c}_i^{\rho,\kappa}(t,x,v)$ satisfy the following rough estimates,

$$|\tilde{c}_i^{\rho,\kappa}(x,v)| + |\hat{c}_i^{\rho,\kappa}(x,v)| \lesssim \min\{(1+|v|)^{1+c_{vn}(\kappa)-c_{vn}(\rho)}, (1+|v|)^{c_{vm}(\kappa)-c_{vm}(\rho)}\}, \quad (A.26)$$

$$|\Lambda^\beta\big(\tilde{c}_i^{\rho,\kappa}(x,v)\big)| + |\Lambda^\beta\big(\hat{c}_i^{\rho,\kappa}(x,v)\big)| \lesssim (1+|v|)^{|\beta|+2}(1+|x|)^{|\beta|+2}, \quad \beta\in\mathcal{S}. \quad (A.27)$$

If $i(\kappa) - i(\rho) > 0$, where $i(\kappa)$ denotes the total number of vector fields Ω_i^x in Λ^κ, then the following improved estimate holds for the coefficients $\hat{c}_i^{\rho,\kappa}(x,v)$,

$$|\hat{c}_i^{\rho,\kappa}(x,v)| \lesssim (1+|v|)^{-1+c_{vm}(\kappa)-c_{vm}(\rho)}. \quad (A.28)$$

Moreover, for the case when $i = 1$, the following improved estimate holds,

$$|\tilde{c}_1^{\rho,\kappa}(x,v)| + |\hat{c}_1^{\rho,\kappa}(x,v)| \lesssim (1+|v|)^{-1+c_{vn}(\kappa)-c_{vn}(\rho)}. \quad (A.29)$$

Proof. Note that the following commutation rules hold for any differentiable coefficients "$c_1(v)$" and "$c_2(v)$" and any two vector fields "X" and "Y"

$$\begin{aligned}&[c_1(v)X, c_2(v)Y] = c_1(v)(Xc_2(v))Y + c_1(v)c_2(v)[X, Y] - c_2(v)(Yc_1(v))X,\\ &[c_1(v)\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big), c_2(v)\cdot\nabla_x] = c_1(v)\big(\tilde{v}\cdot\nabla_v c_2(v)\big)\cdot\nabla_x,\\ &[c_1(v)\big(\Omega_i^v - \frac{t}{(1+|v|^2)^{1/2}}\Omega_i^x\big), c_2(v)\cdot\nabla_x] = c_1(v)\big(\tilde{V}_i\cdot\nabla_v c_2(v)\big)\\ &\cdot\nabla_x, \quad i\in\{1,2,3\}. \end{aligned} \tag{A.30}$$

Moreover, the following commutation rules hold for the vector field $\tilde{\Omega}_i$,$i \in \{1, 2, 3\}$,

$$\begin{aligned}&[S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x, X_i\cdot\nabla_x + V_i\cdot\nabla_v] = \frac{1}{|v|}\big(\Omega_i^v - \frac{t}{(1+|v|^2)^{3/2}}\Omega_i^x\big)\\ &\quad - \frac{1}{|v|}\Omega_i^v + \frac{t}{(1+|v|^2)^{3/2}}\Omega_i^x = 0,\\ &[\Omega_j^v - \frac{t}{(1+|v|^2)^{1/2}}\Omega_j^x, X_i\cdot\nabla_x + V_i\cdot\nabla_v] = \frac{1}{|v|}\big(V_j\cdot\nabla_v V_i - V_i\cdot\nabla_v V_j\big)\cdot\nabla_v\\ &\quad - \frac{t}{(1+|v|^2)^{1/2}}\frac{1}{|v|}\big(V_j\cdot\nabla_x(X_i) - V_i\cdot\nabla_v V_j\big)\cdot\nabla_x\\ &\quad = \sum_{k=1,2,3}\epsilon_{ij}^k\big(\Omega_k^v - \frac{t}{(1+|v|^2)^{1/2}}\Omega_k^x\big), \end{aligned} \tag{A.31}$$

where $\epsilon_{ij}^k \in \{0, 1\}$ are some uniquely determined coefficients.

Hence, to prove the desired equality (A.25) and the desired estimates (A.26), (A.27), (A.28), and (A.29), it would be sufficient to consider the case when $\Lambda^\rho \in \{\psi_{\geq 1}(v)\widehat{S}^v, \psi_{\geq 1}(v)\widehat{\Omega}_i^v, \psi_{<1}(v)K_{v_i}, i \in \{1, 2, 3\}\}$. Moreover, from the equality (A.30), it would be sufficient to compute the commutation rules without the cutoff functions $\psi_{\geq 1}(v)$ and $\psi_{\leq 0}(v)$ during computations.

We first consider the case when $\Lambda^\rho = \psi_{\geq 1}(v)\widehat{S}^v$ and $X_i = \psi_{\geq 1}(v)\tilde{v}\cdot D_v$. From the equality (A.12), we have

$$\begin{aligned}&[\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big), \big(S^v - \frac{\omega(x,v)}{1+|v|^2}S^x\big)]\\ &= -[\frac{t}{(1+|v|^2)^{3/2}}S^x, S^v] - [S^v, \frac{\omega(x,v)}{1+|v|^2}S^x] + [\frac{t}{(1+|v|^2)^{3/2}}S^x, \frac{\omega(x,v)}{1+|v|^2}S^x]\\ &= \tilde{v}\cdot\nabla_v\big(\frac{t}{(1+|v|^2)^{3/2}}\big)S^x - \big(S^v\big(\frac{\omega(x,v)}{1+|v|^2}\big)\big)S^x + \frac{t}{(1+|v|^2)^{5/2}}\big(S^x(\omega(x,v))\big)S^x\\ &= -\frac{3t|v|}{(1+|v|^2)^{5/2}}S^x + \frac{2|v|\omega(x,v)}{(1+|v|^2)^2}S^x - \frac{1}{1+|v|^2}\big[\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big)\omega(x,v)\big]S^x\\ &= -\frac{3t|v|}{(1+|v|^2)^{5/2}}S^x + \frac{2|v|\omega(x,v)}{(1+|v|^2)^2}S^x\\ &\quad - \frac{1}{1+|v|^2}\big[\tilde{c}(x,v)\tilde{d}(t,x,v) + \tilde{e}(x,v) - \frac{t|v|}{(1+|v|^2)^{3/2}}\\ &\quad \times\psi_{\geq 0}((x\cdot v)^2 + |x|^2)\big]S^x = \big(\widehat{c}(x,v)\tilde{d}(t,x,v) + \widehat{e}(x,v)\big)S^x, \end{aligned} \tag{A.32}$$

where

$$\widehat{c}(x, v) = -\frac{1}{1+|v|^2}\Big(\tilde{c}(x, v) + \frac{2|v|}{(1+|v|^2)^{1/2}} + \frac{|v|}{(1+|v|^2)^{1/2}}\psi_{<0}((x\cdot v)^2 + |x|^2)\Big), \tag{A.33}$$

$$\widehat{e}(x, v) = -\frac{1}{1+|v|^2}\Big(\tilde{e}(x, v) + \frac{|v|}{1+|v|^2}\omega(x, v)\psi_{<0}((x\cdot v)^2 + |x|^2)\Big), \tag{A.34}$$

where $\tilde{c}(x, v)$ and $\tilde{e}(x, v)$ are defined in (A.13) and (A.14) respectively.
From the detailed formulas of $\widehat{c}(x, v)$ and $\widehat{e}(x, v)$ in (A.33) and (A.34) and the estimate of coefficients in (A.15), we have

$$|\widehat{c}(x, v)| + |\widehat{e}(x, v)| \lesssim (1+|v|)^{-2}. \tag{A.35}$$

Next, we consider the case $\Lambda^\rho = \psi_{\geq 1}(v)\widehat{\Omega}_i^v$, $X_i = \psi_{\geq 1}(v)\tilde{v}\cdot D_v$. Recall (A.12), we have

$$\begin{aligned}
&[\big(S^v - \frac{t}{(1+|v|^2)^{3/2}}S^x\big), \big(\Omega_i^v - \omega(x, v)\Omega_i^x\big)] \\
&= [S^v, \Omega_i^v] - [S^v, \omega(x, v)\Omega_i^x] - [\frac{t}{(1+|v|^2)^{3/2}}S^x, \Omega_i^v] \\
&\quad + [\frac{t}{(1+|v|^2)^{3/2}}S^x, \omega(x, v)\Omega_i^x] = \\
&\quad - \frac{1}{|v|}\Omega_i^v - (S^v\omega(x, v))\Omega_i^x - \omega(x, v)[S^v, \Omega_i^x] \\
&\quad + \frac{t}{(1+|v|^2)^{3/2}}[\Omega_i^v, S^x] \\
&\quad + \frac{t}{(1+|v|^2)^{3/2}}\big(S^x\omega(x, v)\big)\Omega_i^x = -\frac{1}{|v|}\Omega_i^v + \frac{1}{|v|}\frac{t}{(1+|v|^2)^{3/2}}\Omega_i^x \\
&\quad - \big(\tilde{c}(x, v)\tilde{d}(t, x, v) + \tilde{e}(x, v) - \frac{t|v|}{(1+|v|^2)^{3/2}} \\
&\quad \times \psi_{\geq 0}((x\cdot v)^2 + |x|^2)\big)\Omega_i^x = \\
&\quad - \frac{1}{|v|}\widehat{\Omega}_i + \big(\hat{c}_i(x, v)\tilde{d}(t, x, v) + \hat{e}_i(x, v)\big)\Omega_i^x,
\end{aligned} \tag{A.36}$$

where

$$\hat{c}_i(x, v) = -\tilde{c}(x, v) + \frac{\sqrt{1+|v|^2}}{|v|} - \frac{|v|}{(1+|v|^2)^{1/2}}\psi_{<0}((x\cdot v)^2 + |x|^2), \tag{A.37}$$

$$\hat{e}_i(x, v) = -\tilde{e}(x, v) - \frac{|v|}{1+|v|^2}\omega(x, v)\psi_{<0}((x\cdot v)^2 + |x|^2). \tag{A.38}$$

From the above detailed formulas and the estimate of coefficients in (A.15), we have

$$\big(|\hat{c}_i(x,v)| + |\hat{e}_i(x,v)|\big)\psi_{\geq -10}(v) \lesssim 1. \tag{A.39}$$

Next, we consider the case $\Lambda^\rho = \psi_{\geq 1}(v)\widehat{S^v}$, $X_i = \psi_{\geq 1}(v)\tilde{V}_i \cdot D_v$. Recall (A.8), we have

$$\begin{aligned}
&[\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big), \big(S^v - \frac{\omega(x,v)}{(1+|v|^2)}S^x\big)] \\
&= [\Omega_i^v, S^v] - [\frac{t}{\sqrt{1+|v|^2}}\Omega_i^x, S^v] - [\Omega_i^v, \frac{\omega(x,v)}{(1+|v|^2)}S^x] \\
&\quad + [\frac{t}{\sqrt{1+|v|^2}}\Omega_i^x, \frac{\omega(x,v)}{(1+|v|^2)}S^x] \\
&= \frac{1}{|v|}\Omega_i^v + \tilde{v}\cdot\nabla_v\big(\frac{t}{\sqrt{1+|v|^2}}\big)\Omega_i^x + \frac{t}{\sqrt{1+|v|^2}}[S^v, \Omega_i^x] \\
&\quad - \big(\Omega_i^v(\frac{\omega(x,v)}{(1+|v|^2)})\big)S^x - \frac{\omega(x,v)}{1+|v|^2}[\Omega_i^v, S^x] \\
&\quad + \frac{t}{\sqrt{1+|v|^2}}\big(\Omega_i^x(\frac{\omega(x,v)}{1+|v|^2})\big)S^x \\
&= \frac{1}{|v|}\big(\Omega_i^v - \frac{\omega(x,v)}{1+|v|^2}\Omega_i^x \\
&\quad - \frac{t|v|^2}{(1+|v|^2)^{3/2}}\Omega_i^x\big) - \frac{1}{1+|v|^2}\big(c_i(x,v)\tilde{d}(t,x,v) + e_i(x,v)\big)S^x \\
&= \frac{1}{|v|}\widehat{\Omega}_i^v + \big[\widehat{\tilde{c}}_i(x,v))\tilde{d}(t,x,v) \\
&\quad + \widehat{\tilde{e}}_i(x,v))\big]\cdot(S^x, \Omega_1^x, \Omega_2^x, \Omega_3^x),
\end{aligned} \tag{A.40}$$

where

$$\begin{aligned}
\widehat{\tilde{c}}_i(x,v)) &= -\big(\frac{c_i(x,v)}{1+|v|^2}, \frac{|v|\delta_{1i}}{(1+|v|^2)^{1/2}}, \\
&\quad \frac{|v|\delta_{2i}}{(1+|v|^2)^{1/2}}, \frac{|v|\delta_{3i}}{(1+|v|^2)^{1/2}}\big), \\
\widehat{\tilde{e}}_i(x,v) &= -\big(\frac{e_i(x,v)}{1+|v|^2}, 0, 0, 0\big).
\end{aligned} \tag{A.41}$$

From the above detailed formulas and the estimate (A.11), we have

$$\begin{aligned}
&|\widehat{\tilde{c}}_{i;1}(x,v))| \lesssim (1+|v|)^{-1}, \\
&\sum_{l=2,3,4} |\widehat{\tilde{c}}_{i;l}(x,v))| \lesssim 1, \\
&|\widehat{\tilde{e}}_i(x,v))| \lesssim (1+|v|)^{-2}.
\end{aligned} \tag{A.42}$$

Next, we consider the case $\Lambda^\rho = \psi_{\geq 1}(v)\widehat{\Omega}_j^v$, $X_i = \psi_{\geq 1}(v)\tilde{V}_i \cdot D_v$, $i, j \in \{1,2,3\}$. Recall (A.8), we have

$$[\big(\Omega_i^v - \frac{t}{\sqrt{1+|v|^2}}\Omega_i^x\big), \big(\Omega_j^v - \omega(x,v)\Omega_j^x\big)] = [\Omega_i^v, \Omega_j^v]$$

$$
\begin{aligned}
&-[\frac{t}{\sqrt{1+|v|^2}}\Omega_i^x, \Omega_j^v] + \frac{t}{\sqrt{1+|v|^2}}\big(\Omega_i^x\omega(x,v)\big)\Omega_j^x \\
&-[\Omega_i^v, \omega(x,v)\Omega_j^x] = \big(\tilde{V}_i\cdot\nabla_v\tilde{V}_j - \tilde{V}_j\cdot\nabla_v\tilde{V}_i\big)\cdot\nabla_v \\
&+\frac{t}{\sqrt{1+|v|^2}}\big(\tilde{V}_j\cdot\nabla_v\tilde{V}_i\big)\cdot\nabla_x - \omega(x,v)\big(\tilde{V}_i\cdot\nabla_v\tilde{V}_j\big)\cdot\nabla_x \\
&-\big(\Omega_i^v\omega(x,v)\big)\Omega_j^x + \frac{t}{\sqrt{1+|v|^2}}\big(\Omega_i^x\omega(x,v)\big)\Omega_j^x = \frac{(e_j\times e_i)\times v}{|v|^2}\cdot\big(\nabla_v - \omega(x,v)\nabla_x\big) \\
&+\sqrt{1+|v|^2}\tilde{d}(t,x,v)\big((\tilde{V}_j\cdot\nabla_v)\tilde{V}_i\big)\cdot\nabla_x - \big[c_i(x,v)\tilde{d}(t,x,v) + e_i(x,v)\big]\Omega_j^x. \quad (A.43)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\nabla_x &= \tilde{v}S^x + \sum_{k=1,2,3}\tilde{V}_k\Omega_k^x, \Longrightarrow (\tilde{V}_j\cdot\nabla_v)\tilde{V}_i\cdot\nabla_x = \big((\tilde{V}_j\cdot\nabla_v)\tilde{V}_i\big)\cdot\tilde{v}S^x \\
&+ \sum_{k=1,2,3}\big((\tilde{V}_j\cdot\nabla_v)\tilde{V}_i\big)\cdot\tilde{V}_k\Omega_k^x, \\
&(e_j\times e_i)\times v = \epsilon_{ij}^k\tilde{V}_k,
\end{aligned}
$$

where $\epsilon_{ij}^k \in \{0,1\}$, are some uniquely determined constants. Hence,

$$
(A.43) = \frac{\epsilon_{ij}^k}{|v|}\widehat{\Omega}_k^v + \big[a_{i,j}(x,v)\tilde{d}(t,x,v) + e_{i,j}(x,v)\big]\cdot(S^x, \Omega_1^x, \Omega_2^x, \Omega_3^x), \quad (A.44)
$$

where

$$
\begin{aligned}
a_{i,j}(x,v) &= (a_{i,j}^0(x,v), a_{i,j}^1(x,v), a_{i,j}^2(x,v), a_{i,j}^3(x,v)), \\
a_{i,j}^0(x,v) &= \sqrt{1+|v|^2}\big((\tilde{V}_j\cdot\nabla_v)\tilde{V}_i\big)\cdot\tilde{v}, \quad (A.45)\\
a_{i,j}^k(x,v) &= \sqrt{1+|v|^2}\big((\tilde{V}_j\cdot\nabla_v)\tilde{V}_i\big)\cdot\tilde{V}_k - c_i(x,v)\delta_{jk}, \quad k=1,2,3, \\
e_{i,j}(x,v) &= -(0, e_i(x,v)\delta_{j1}, e_i(x,v)\delta_{j2}, e_i(x,v)\delta_{j3}). \quad (A.46)
\end{aligned}
$$

From the above detailed formulas and the estimate of coefficients in (A.11), we have

$$
|a_{i,j}(x,v)|\psi_{\geq -10}(v) \lesssim 1, \quad |e_{i,j}(x,v)| \lesssim 1. \quad (A.47)
$$

Lastly, we consider the case when $X_i = \psi_{\leq 0}(v)D_{v_i}$, $i \in \{1,2,3\}$. For this case we have $|v| \lesssim 1$, which means that our desired estimates in (A.26), (A.27), (A.28), and (A.29) are trivial. Hence, it would be sufficient to verify that the desired equality (A.25) holds. Note that the following commutation rules hold for any $i, j \in \{1,2,3\}$,

$$
\begin{aligned}
&[\partial_{v_i} - t\partial_{v_i}\hat{v}\cdot\nabla_x, \partial_{v_j} - \omega(x,v)\sqrt{1+|v|^2}\partial_{v_j}\hat{v}\cdot\nabla_x] = -[t\partial_{v_i}\hat{v}\cdot\nabla_x, \partial_{v_j}] \\
&\quad -[\partial_{v_i}, \omega(x,v)\sqrt{1+|v|^2}\partial_{v_j}\hat{v}\cdot\nabla_x] \\
&\quad +[t\partial_{v_i}\hat{v}\cdot\nabla_x, \omega(x,v)\sqrt{1+|v|^2}\partial_{v_j}\hat{v}\cdot\nabla_x] = t\partial_{v_i}\partial_{v_j}\hat{v}\cdot\nabla_x \\
&\quad -\partial_{v_i}\big(\omega(x,v)\sqrt{1+|v|^2}\partial_{v_j}\hat{v}\big)\cdot\nabla_x \\
&\quad +t\partial_{v_i}\hat{v}\cdot\nabla_x\omega(x,v)\sqrt{1+|v|^2}\partial_{v_j}\hat{v}\cdot\nabla_x = (t-\omega(x,v)\sqrt{1+|v|^2})\partial_{v_i}\partial_{v_j}\hat{v}\cdot\nabla_x
\end{aligned}
$$

$$+\left[t\partial_{v_i}\hat{v}\cdot\nabla_x\omega(x,v)\sqrt{1+|v|^2}-\partial_{v_i}\left(\omega(x,v)\sqrt{1+|v|^2}\right)\right]\partial_{v_j}\hat{v}\cdot\nabla_x. \tag{A.48}$$

Note that

$$\begin{aligned}
&t\partial_{v_i}\hat{v}\cdot\nabla_x\omega(x,v)\sqrt{1+|v|^2}-\partial_{v_i}\left(\omega(x,v)\sqrt{1+|v|^2}\right)\\
&=t\Big[\big(v_i+\frac{(x\cdot v)v_i+x_i}{\sqrt{(x\cdot v)^2+|x|^2}}\big)-\frac{v_i}{(1+|v|^2)}\big(|v|^2+\frac{(x\cdot v)(1+|v|^2)}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\Big]\psi_{\geq 0}((x\cdot v)^2+|x|^2)\\
&\quad+2t\omega_+(x,v)\Big(\psi'_{\geq 0}((x\cdot v)^2+|x|^2)((x\cdot v)v_i+x_i)\\
&\quad-\frac{v_i}{1+|v|^2}\psi'_{\geq 0}((x\cdot v)^2+|x|^2)(x\cdot v)(1+|v|^2)\Big)\\
&\quad-\frac{v_i}{\sqrt{1+|v|^2}}\omega(x,v)-\sqrt{1+|v|^2}\big(x_i+\frac{(x\cdot v)x_i}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\psi_{\geq 0}((x\cdot v)^2+|x|^2)\\
&\quad-2\omega_+(x,v)\sqrt{1+|v|^2}\psi'_{\geq 0}((x\cdot v)^2+|x|^2)(x\cdot v)x_i\\
&=t\big(\frac{v_i}{1+|v|^2}+\frac{x_i}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\psi_{\geq 0}((x\cdot v)^2+|x|^2)\\
&\quad+2t\omega_+(x,v)x_i\psi'_{\geq 0}((x\cdot v)^2+|x|^2)-\frac{v_i}{\sqrt{1+|v|^2}}\omega(x,v)\\
&\quad-\sqrt{1+|v|^2}\omega(x,v)\frac{x_i}{\sqrt{(x\cdot v)^2+|x|^2}}-2\sqrt{1+|v|^2}\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)(x\cdot v)x_i\\
&=\psi_{\geq 0}((x\cdot v)^2+|x|^2)\big(t-\sqrt{1+|v|^2}\omega_+(x,v)\big)\big(\frac{v_i}{1+|v|^2}+\frac{x_i}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\\
&\quad+2x_i\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\big(t-\sqrt{1+|v|^2}(x\cdot v)\big)=\mathfrak{c}_i(x,v)\tilde{d}(t,x,v)+\mathfrak{e}_i(x,v),
\end{aligned} \tag{A.49}$$

where

$$\begin{aligned}
\mathfrak{c}_i(x,v)&=\psi_{\geq 0}((x\cdot v)^2+|x|^2)\big(v_i+\frac{x_i(1+|v|^2)}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\\
&\quad+2x_i\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)(1+|v|^2),
\end{aligned} \tag{A.50}$$

$$\begin{aligned}
\mathfrak{e}_i(x,v)&=-\omega_+(x,v)\psi_{\geq 0}((x\cdot v)^2+|x|^2)\psi_{<0}((x\cdot v)^2\\
&\quad+|x|^2)\big(\frac{v_i}{\sqrt{1+|v|^2}}+\frac{x_i\sqrt{1+|v|^2}}{\sqrt{(x\cdot v)^2+|x|^2}}\big)\\
&\quad+2x_i\omega_+(x,v)\psi'_{\geq 0}((x\cdot v)^2+|x|^2)\sqrt{1+|v|^2}\big(\omega(x,v)-x\cdot v\big).
\end{aligned} \tag{A.51}$$

Combining equalities in (A.48) and (A.49), we have

$$\begin{aligned}
&[\partial_{v_i}-t\partial_{v_i}\hat{v}\cdot\nabla_x,\partial_{v_j}-\omega(x,v)\sqrt{1+|v|^2}\partial_{v_j}\hat{v}\cdot\nabla_x]\\
&\quad=\left[\tilde{\mathfrak{c}}_{i,j}(x,v)\tilde{d}(t,x,v)+\tilde{\mathfrak{e}}_{i,j}(x,v)\right]\cdot\nabla_x,
\end{aligned} \tag{A.52}$$

where

$$\tilde{\mathfrak{c}}_{i,j}(x,v)=(1+|v|^2)\partial_{v_i}\partial_{v_j}\hat{v}+\mathfrak{c}_i(x,v)\partial_{v_j}\hat{v},\quad \tilde{\mathfrak{e}}_{i,j}(x,v)=\mathfrak{e}_i(x,v)\partial_{v_j}\hat{v}. \tag{A.53}$$

To sum up, our desired equality (A.25) holds from the equalities (A.31), (A.32), (A.36), (A.40), (A.44), and (A.52). Moreover, recall the definition of indexes in (3.39), our desired estimates (A.26), (A.27), (A.28), and (A.29) hold from the estimates of coefficients in (A.35), (A.39), (A.42), and (A.47). □

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