REMARKS ON XIAO'S APPROACH OF SLOPE INEQUALITIES

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Abstract. We prove the slope inequality for a relative minimal surface fibration in positive characteristic via Xiao's approach. We also prove a better low bound for the slope of non-hyperelliptic fibrations.

1. Introduction

Let S be a smooth projective surface over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$ and $f : S \to B$ be a fibration with smooth general fiber F of genus g over a smooth projective curve B. Let $\omega_{S/B} := \omega_S \otimes f^*\omega_B^\vee$ be the relative canonical sheaf of f, and $K_{S/B} := K_S - f^*K_B$ be the relative canonical divisor. We say that f is relatively minimal if S contains no (-1)-curve in fibers. The following basic relative invariants are well known:

$$K_{S/B}^2 = (K_S - f^*K_B)^2 = K_S^2 - 8(g-1)(b-1),$$
$$\chi_f = \mathrm{deg} f_*\omega_{S/B} = \chi(\mathcal{O}_S) - (g-1)(b-1).$$

When f is relatively minimal and F is smooth, then $K_{S/B}$ is a nef divisor (see [11]). Under this assumption, the relative invariants satisfy the following remarkable so-called slope inequality.

Theorem 1. *If f is relatively minimal, and the general fiber F is smooth, then*

$$K_{S/B}^2 \geq \frac{4(g-1)}{g}\chi_f. \tag{1.1}$$

When $char(\mathbf{k}) = 0$, this inequality was proved by Xiao (see [12]). For the case of semi-stable fibration, it was proved independently by Cornalba-Harris (see [2]). When $char(\mathbf{k}) = p > 0$, there exist a few

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approaches to prove this inequality (see [9], [13], ect). Some of them require the condition of semi-stable fibration.

In this note, we explain why Xiao's approach still works in the case of $char(\mathbf{k}) = p > 0$. Indeed, Xiao's approach is to study the Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E = f_*\omega_{S/B}$$

and give lower bound of $K^2_{S/B}$ in term of slop $\mu_i = \mu(E_i/E_{i-1})$. Here one of the key points is that semi-stability of E_i/E_{i-1} will imply nefness of $\mathbb{Q}$-divisors $\mathcal{O}_{\mathbb{P}(E_i)}(1) - \mu_i\Gamma_i$ where Γ_i is a fiber of $\mathbb{P}(E_i) \to B$. This is the only place one needs $char(\mathbf{k}) = 0$.

Our observation is that by a result of A. Langer there is an integer k_0 such that, when $k \geq k_0$, the Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E = F^{k*}f_*\omega_{S/B}$$

of $F^{k*}f_*\omega_{S/B}$ has strongly semi-stable E_i/E_{i-1} $(1 \leq i \leq n)$ and that strongly semi-stability of E_i/E_{i-1} implies nefness of $\mathcal{O}_{\mathbb{P}(E_i)}(1) - \mu_i\Gamma_i$. When $f : S \to B$ is a semi-stable fibration, any Frobenius base change $F^k : B \to B$ induces fibration $\tilde{f} : \widetilde{S} \to B$ such that

$$F^{k*}f_*\omega_{S/B} = \tilde{f}_*\omega_{\widetilde{S}/B}, \quad \frac{K^2_{S/B}}{\deg f_*\omega_{S/B}} = \frac{K^2_{\widetilde{S}/B}}{\deg \tilde{f}_*\omega_{\widetilde{S}/B}}.$$

Thus for semi-stable fibration $f : S \to B$ we can assume (without loss of generality) that all E_i/E_{i-1} appearing in Harder-Narasimhan filtration of $E = f_*\omega_{S/B}$ are strongly semi-stable. Then Xiao's approach works for $char(\mathbf{k}) = p > 0$ without any modification. We will show in this note that a slightly modification of Xiao's approach works for any fibration $f : S \to B$. In fact, we will prove the following more general result holds for $char(\mathbf{k}) = p \geq 0$.

Theorem 2. *Let D be a relative nef divisor on $f : S \to B$ such that $D|_F$ is generated by global sections on a general smooth fiber F of $f : S \to B$. Assume that $D|_F$ is a special divisor on F and*

$$A = 2h^0(D|_F) - D \cdot F - 1 > 0.$$

Then

$$D^2 \geq \frac{2D \cdot F}{h^0(D|_F)} \deg(f_*\mathcal{O}_S(D)).$$

Xiao also constructed examples (cf.[12, Example 2]) of hyperelliptic fiberation $f : S \to B$ such that

$$K^2_{S/B} = \frac{4g-4}{g} \deg(f_*\omega_{S/B})$$

and conjectured (cf. [12, Conjecture 1]) that the inequality must be strict for non-hyperelliptic fibrations, i.e., the general fiber F of f is a non-hyperelliptic curve, which was proved by Konno [4, Proposition 2.6]. Lu and Zuo [7] obtained a sharp slope inequality for non-hyperelliptic fibrations, which was generalized to $char(\mathbf{k}) = p > 0$ in [6] for a non-hyperelliptic semi-stable fibration.

Here we also remark that our previous observation can be used to prove the following theorem in any characteristic easily.

Theorem 3. *Assume that $f : S \to B$ is a relatively minimal non-hyperelliptic surface fibration over an algebraically closed field of any characteristic, and the general fiber of f is smooth. Then*

$$K^2_{S/B} \geq \min\{\frac{9(g-1)}{2(g+1)}, 4\}\deg f_*\omega_{S/B}. \tag{1.3}$$

Our article is organized as follows. In Section 2, we give a generalization of Xiao's approach, and show that a slightly modification of Xiao's approach works in any characteristic. In Section 3, we prove Theorem 3 via the modification of Xiao's approach and the modified second multiplication map $F^{k*}S^2 f_*\omega_{S/B} \to F^{k*}f_*(\omega_{S/B}^{\otimes 2})$.

2. Xiao's approach and its generalization

We start from an elementary (but important) lemma due to Xiao.

Lemma 1. ([12, Lemma 2]) *Let $f : S \to B$ be a relatively minimal fibration, with a general fiber F. Let D be a divisor on S, and suppose that there are a sequence of effective divisors*

$$Z_1 \geq Z_2 \geq \cdots \geq Z_n \geq Z_{n+1} = 0$$

and a sequence of rational numbers

$$\mu_1 > \mu_2 \cdots > \mu_n, \quad \mu_{n+1} = 0$$

such that for every i, $N_i = D - Z_i - \mu_i F$ is a nef $\mathbb{Q}$-divisor. Then

$$D^2 \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}),$$

where $d_i = N_i \cdot F$.

Proof. Since $N_{i+1} = N_i + (\mu_i - \mu_{i+1})F + (Z_i - Z_{i+1})$, we have

$$\begin{aligned} N_{i+1}^2 &= N_{i+1}N_i + d_{i+1}(\mu_i - \mu_{i+1}) + N_{i+1}(Z_i - Z_{i+1}) \\ &= N_i^2 + (d_i + d_{i+1})(\mu_i - \mu_{i+1}) + (N_i + N_{i+1})(Z_i - Z_{i+1}) \\ &\geq N_i^2 + (d_i + d_{i+1})(\mu_i - \mu_{i+1}). \end{aligned}$$

Thus $N_{i+1}^2 - N_i^2 \geq (d_i + d_{i+1})(\mu_i - \mu_{i+1})$ and

$$D^2 = N_{n+1}^2 = N_1^2 + \sum_{i=1}^{n}(N_{i+1}^2 - N_i^2) \geq \sum_{i=1}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

□

We need some well-known facts about vector bundles on curves. Let B be a smooth projective curve over $\mathbf{k}$, for a vector bundle E on B, the slope of E is defined to be

$$\mu(E) = \frac{\deg E}{\mathrm{rk}(E)}$$

where $\mathrm{rk}(E)$, $\deg E$ denote the rank and degree of E (respectively). Recall that E is said to be semi-stable (resp., stable) if for any nontrivial subbundle $E' \subsetneq E$, we have

$$\mu(E') \leq \mu(E) \quad \text{(resp., <)}.$$

If E is not semi-stable, one has the following well-known theorem

Theorem 4. *(Harder-Narasimhan filtration) For any vector bundle E on B, there is a unique filtration*

$$0 := E_0 \subset E_1 \subset \cdots \subset E_n = E$$

which is the so called Harder-Narasimhan filtration, such that
(1) each quotient E_i/E_{i-1} is semi-stable for $1 \leq i \leq n$,
(2) $\mu_1 > \cdots > \mu_n$, where $\mu_i := \mu(E_i/E_{i-1})$ for $1 \leq i \leq n$.

The rational numbers $\mu_{max}(E) := \mu_1$ and $\mu_{min}(E) := \mu_n$ are important invariants of E. Let $\pi : \mathbb{P}(E) \to B$ be projective bundle and $\pi^*E \to \mathcal{O}_E(1) \to 0$ be the tautological quotient line bundle. Then the following lemma (which was proved by Xiao in another formulation) relating semi-stability of E with nefness of $\mathcal{O}_E(1)$ only holds when $char(\mathbf{k}) = 0$.

Lemma 2. *(*[8, Theorem 3.1]*, See also* [12, Lemma 3]*) Let Γ be a fiber of $\pi : \mathbb{P}(E) \to B$. Then*

$$\mathcal{O}_E(1) - \mu_{min}(E)\Gamma$$

is a nef $\mathbb{Q}$-divisor. In particular, for each sub-bundle E_i in Harder-Narasimhan filtration of E, the divisor

$$\mathcal{O}_{E_i}(1) - \mu_i\Gamma_i$$

is a nef $\mathbb{Q}$-divisor, where Γ_i is a fiber of $\mathbb{P}(E_i) \to B$.

Theorem 5. *Let D be a relative nef divisor on $f : S \to B$ such that $D|_F$ is generated by global sections on a general smooth fiber F of $f : S \to B$. Assume that $D|_F$ is a special divisor on F and*

$$A = 2h^0(D|_F) - D \cdot F - 1 > 0.$$

Then

$$D^2 \geq \frac{2D \cdot F}{h^0(D|_F)} \deg(f_* \mathcal{O}_S(D)).$$

Proof. For a divisor D on $f : S \to B$, $E = f_*\mathcal{O}_S(D)$ is a vector bundle of rank $h^0(D|_F)$ where F is a general smooth fiber of $f : S \to B$. Let

$$0 := E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be the Harder-Narasimhan filtration of E with $r_i = \mathrm{rk}(E_i)$ and

$$\mu_i = \mu(E_i/E_{i-1}) = \mu_{min}(E_i) \quad (1 \leq i \leq n).$$

Let $\mathcal{L}_i \subset \mathcal{O}_S(D)$ be the image of f^*E_i under sheaf homomorphism

$$f^*E_i \hookrightarrow f^*E = f^*f_*\mathcal{O}_S(D) \to \mathcal{O}_S(D),$$

which is a torsion-free sheaf of rank 1 and is locally free on an open set $U_i \subset S$ of codimension at least 2. Thus there is a morphism (over B)

$$\phi_i : U_i \to \mathbb{P}(E_i)$$

such that $\phi_i^*\mathcal{O}_{E_i}(1) = \mathcal{L}_i|_{U_i}$, which implies that $c_1(\mathcal{L}_i) - \mu_i F$ is nef by Lemma 2. Let $D = c_1(\mathcal{L}_i) + Z_i$ $(1 \leq i \leq n)$. Then we get a sequence of effective divisors $Z_1 \geq Z_2 \geq \cdots \geq Z_n \geq 0$ and a sequence of rational numbers $\mu_1 > \mu_2 \cdots > \mu_n$ such that

$$N_i = D - Z_i - \mu_i F \quad (1 \leq i \leq n)$$

are nef $\mathbb{Q}$-divisors. Note $N_i|_F = c_1(\mathcal{L}_i)|_F \hookrightarrow D|_F$, one has surjection

$$H^1(N_i|_F) \twoheadrightarrow H^1(D|_F).$$

Thus $N_i|_F$ is special since $D|_F$ is special, and

$$d_i = N_i \cdot F \geq 2h^0(\mathcal{L}_i|_F) - 2 = 2r_i - 2, \ (i = 1, ..., n)$$

by Clifford theorem. Since $D|_F$ is generated by global sections, Z_n is supported on fibers of $f : S \to B$ and $d_n = D \cdot F := d_{n+1}$.

When $n = 1$, we have $D^2 = N_1^2 + (D + N_1) \cdot Z_1 + 2\mu_1 D \cdot F$ and

$$D^2 \geq 2\mu_1 D \cdot F = \frac{2D \cdot F}{h^0(D|_F)} \deg(f_*\mathcal{O}_S(D))$$

since Z_1 is supported on fibers of $f : S \to B$ and D is a relative nef divisor. When $n > 1$, by the same reason,

$$D^2 = N_n^2 + 2\mu_n D \cdot F + (N_n + D) \cdot Z_n \geq N_n^2 + 2\mu_n D \cdot F$$

and, by using Lemma 1 to N_n^2, we have

$$\begin{aligned}
D^2 &\geq \sum_{i=1}^{n-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2\mu_n D\cdot F \\
&\geq \sum_{i=1}^{n-1}(2r_i + 2r_{i+1} - 4)(\mu_i - \mu_{i+1}) + 2\mu_n D\cdot F \\
&\geq \sum_{i=1}^{n-1}(4r_i - 2)(\mu_i - \mu_{i+1}) + 2\mu_n D\cdot F \\
&= 4\sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) - 2\mu_1 - (4h^0(D|_F) - 2D\cdot F - 2)\mu_n \\
&= 4\mathrm{deg}(f_*\mathcal{O}_S(D)) - 2\mu_1 - 2A\mu_n
\end{aligned}$$

where we use the equality (which is easy to check) that

$$\mathrm{deg}(f_*\mathcal{O}_S(D)) = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}).$$

Again by $D^2 = N_n^2 + 2\mu_n D\cdot F + (N_n + D)\cdot Z_n$, apply Lemma 1 to $(Z_1 \geq Z_n \geq 0,\ \mu_1 > \mu_n)$, we have

$$D^2 \geq (d_1 + D\cdot F)(\mu_1 - \mu_n) + 2D\cdot F\mu_n \geq D\cdot F(\mu_1 + \mu_n).$$

By using above two inequalities and eliminating μ_1, we have

$$(2 + D\cdot F)D^2 - 4D\cdot F\mathrm{deg}(f_*\mathcal{O}_S(D)) \geq -2(A-1)D\cdot F\mu_n$$

By eliminating μ_n (which is possible since we assume $A > 0$), we have

$$(2A + D\cdot F)D^2 - 4D\cdot F\mathrm{deg}(f_*\mathcal{O}_S(D)) \geq 2(A-1)D\cdot F\mu_1.$$

By adding above two inequalities and using definition of A, we have

$$4h^0(D|_F)D^2 - 8D\cdot F\mathrm{deg}(f_*\mathcal{O}_S(D)) \geq 2(A-1)D\cdot F(\mu_1 - \mu_n) \geq 0$$

which is what we want. □

Colloary 1. *(Xiao's inequality) Let $f: S \to B$ be a relatively minimal fibration of genius $g \geq 2$. Then*

$$K_{S/B}^2 \geq \frac{4g-4}{g}\mathrm{deg}(f_*\omega_{S/B}).$$

Proof. Take $D = K_{S/B}$ (the relative canonical divisor), which satisfies all the assumptions in Theorem 5 with $h^0(D|_F) = g$, $D\cdot F = 2g-2$ and $\mathcal{O}_S(D) = \omega_{S/B}$. □

The only obstruction to generalize Xiao's method in positive characteristic is Lemma 2, which is not true in positive characteristic since Frobenius pull-back of a semi-stable bundle may not be semi-stable. However, the following notion of strongly semi-stability has nice property that pull-back under a finite map preserves strongly semi-stability.

Definition 1. *The bundle E is called strongly semi-stable (resp., stable) if its pullback by k-th power F^k is semi-stable (resp., stable) for any integer $k \geq 0$, where F is the Frobenius morphism $B \to B$.*

Lemma 3. *(*[5, Theorem 3.1]*) For any bundle E on B, there exists an integer k_0 such that all of quotients E_i/E_{i-1} $(1 \leq i \leq n)$ appear in the Harder-Narasimhan filtration*

$$0 := E_0 \subset E_1 \subset \cdots \subset E_n = F^{k*}E$$

are strongly semi-stable whenever $k \geq k_0$.

Lemma 4. *For each sub-bundle E_i in the Harder-Narasimhan filtration*

$$0 := E_0 \subset E_1 \subset \cdots \subset E_n = F^{k*}E$$

of $F^{k*}E$ (when $k \geq k_0$), the divisor $\mathcal{O}_{E_i}(1) - \mu_i \Gamma_i$ is a nef $\mathbb{Q}$-divisor, where Γ_i is a fiber of $\mathbb{P}(E_i) \to B$ and $\mu_i = \mu(E_i/E_{i-1})$.

Proof. The proof is just a modification of [8, Theorem 3.1] since pullback of strongly semi-stable bundles under a finite morphism are still strongly semi-stable. One can see [8, Theorem 3.1, Page 464] for more details. ☐

We now can prove, by the same arguments, that Theorem 5 still holds in positive characteristic.

Theorem 6. *Let D be a relative nef divisor on $f : S \to B$ such that $D|_F$ is generated by global sections on a general smooth fiber F of $f : S \to B$. Assume that $D|_\Gamma$ is a special divisor on F and*

$$A = 2h^0(D|_F) - D \cdot F - 1 > 0.$$

Then

$$D^2 \geq \frac{2D \cdot F}{h^0(D|_F)} \deg(f_*\mathcal{O}_S(D)).$$

Proof. It is enough to prove the theorem when $f : S \to B$ is defined over a base field $\mathbf{k}$ of characteristic $p > 0$. Let $F_S : S \to S$ denote the Frobenius morphism over $\mathbf{k}$. Then we have the following commutative diagram (for any integer $k \geq k_0$):

$$\begin{array}{ccc} S & \xrightarrow{F_S^k} & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{F^k} & B \end{array} .$$

For a divisor D on $f: S \to B$, $E = f_*\mathcal{O}_S(D)$ is a vector bundle of rank $h^0(D|_F)$ where F is a general smooth fiber of $f: S \to B$. Let

$$0 := E_0 \subset E_1 \subset \cdots \subset E_n = F^{k*}E$$

be the Harder-Narasimhan filtration of $F^{k*}E$ with $r_i = \mathrm{rk}(E_i)$ and

$$\mu_i = \mu(E_i/E_{i-1}) = \mu_{min}(E_i) \quad (1 \leq i \leq n)$$

where we choose $k \geq k_0$ such that all quotients E_i/E_{i-1} appears in above filtration are strongly semi-stable.

Let $\mathcal{L}_i \subset F_S^{k*}\mathcal{O}_S(D)$ be the image of f^*E_i under sheaf homomorphism

$$f^*E_i \hookrightarrow f^*F^{k*}E = F_S^{k*}f^*f_*\mathcal{O}_S(D) \to F_S^{k*}\mathcal{O}_S(D) = \mathcal{O}_S(p^kD),$$

which is a torsion-free sheaf of rank 1 and is locally free on an open set $U_i \subset S$ of codimension at least 2. Thus there is a morphism (over B)

$$\phi_i : U_i \to \mathbb{P}(E_i)$$

such that $\phi_i^*\mathcal{O}_{E_i}(1) = \mathcal{L}_i|_{U_i}$, which implies that $c_1(\mathcal{L}_i) - \mu_i F$ is nef by Lemma 4. Let $p^kD = c_1(\mathcal{L}_i) + Z_i$ $(1 \leq i \leq n)$. Then we get a sequence of effective divisors $Z_1 \geq Z_2 \geq \cdots \geq Z_n \geq 0$ and a sequence of rational numbers $\mu_1 > \mu_2 \cdots > \mu_n$ such that

$$N_i = p^kD - Z_i - \mu_i F \quad (1 \leq i \leq n)$$

are nef $\mathbb{Q}$-divisors. Let $d_i = N_i \cdot F = \deg(\mathcal{L}_i|_F)$, then

$$d_n = p^kD \cdot F := d_{n+1}$$

since $D|_F$ is generated by global sections and Z_n is supported on fibers of $f: S \to B$. For $1 \leq i < n$, there are $r_i = \mathrm{rk}(E_i)$ sections

$$\{s_1,\ \dots,\ s_{r_i}\} \in H^0(\mathcal{O}_S(D)|_F)$$

such that $\mathcal{L}_i|_F \subset \mathcal{O}_S(p^kD)|_F$ is generated by the global sections $s_1^{p^k},\ \dots,\ s_{r_i}^{p^k}$.

Since $\mathcal{O}_S(D)|_F$ is special, the sub-sheaf $L_i \subset \mathcal{O}_S(D)|_F$ generated by

$$\{s_1,\ \dots,\ s_{r_i}\} \in H^0(\mathcal{O}_S(D)|_F)$$

is special. Thus $\deg(L_i) \geq 2r_i - 2$ by Clifford theorem. Then we have

$$d_i = N_i \cdot F = \deg(\mathcal{L}_i|_F) = p^k\deg(L_i) \geq p^k(2r_i - 2) \ (1 \leq i \leq n).$$

When $n = 1$, which means that $E = f_*\mathcal{O}_S(D)$ is strongly semi-stable, the same proof of Theorem 5 implies

$$D^2 \geq 2\mu_1 D \cdot F = \frac{2D \cdot F}{h^0(D|_F)} \deg(f_*\mathcal{O}_S(D)).$$

When $n > 1$, since Z_n is supported on fibers of $f : S \to B$, we have

$$p^{2k}D^2 = N_n^2 + 2\mu_n p^k D \cdot F + (N_n + p^k D) \cdot Z_n \geq N_n^2 + 2\mu_n p^k D \cdot F.$$

By $d_i \geq p^k(2r_i - 2)$ and using Lemma 1 to N_n^2, we have

$$\begin{aligned}
p^{2k}D^2 &\geq \sum_{i=1}^{n-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2\mu_n p^k D \cdot F \\
&\geq \sum_{i=1}^{n-1} p^k(2r_i + 2r_{i+1} - 4)(\mu_i - \mu_{i+1}) + 2\mu_n p^k D \cdot F \\
&\geq p^k \sum_{i=1}^{n-1}(4r_i - 2)(\mu_i - \mu_{i+1}) + 2\mu_n p^k D \cdot F \\
&= 4p^k \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) - 2p^k\mu_1 - p^k(4h^0(D|_F) - 2D \cdot F - 2)\mu_n \\
&= 4p^k \deg(F^{k*}f_*\mathcal{O}_S(D)) - 2p^k\mu_1 - 2p^k A\mu_n
\end{aligned}$$

where we set $\mu_{n+1} = 0$ and use the equality (which is easy to check)

$$\deg(F^{k*}f_*\mathcal{O}_S(D)) = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}).$$

By $p^{2k}D^2 = N_n^2 + 2\mu_n p^k D \cdot F + (N_n + p^k D) \cdot Z_n$, apply Lemma 1 to $(Z_1 \geq Z_n \geq 0,\ \mu_1 > \mu_n)$, we have

$$p^{2k}D^2 \geq (d_1 + p^k D \cdot F)(\mu_1 - \mu_n) + 2p^k D \cdot F\mu_n \geq p^k D \cdot F(\mu_1 + \mu_n).$$

Altogether, we have the following inequalities

$$p^k D^2 \geq 4\deg(F^{k*}f_*\mathcal{O}_S(D)) - 2\mu_1 - 2A\mu_n \tag{2.1}$$

$$p^k D^2 \geq D \cdot F(\mu_1 + \mu_n) \tag{2.2}$$

By using (2.1) and (2.2), eliminating μ_1, we have

$$(2 + D \cdot F)p^k D^2 - 4D \cdot F\deg(F^{k*}f_*\mathcal{O}_S(D)) \geq -2(A - 1)D \cdot F\mu_n$$

By eliminating μ_n (which is possible since we assume $A > 0$), we have

$$(2A + D \cdot F)p^k D^2 - 4D \cdot F\deg(F^{k*}f_*\mathcal{O}_S(D)) \geq 2(A - 1)D \cdot F\mu_1.$$

By adding above two inequalities and using definition of A, we have

$$4h^0(D|_F)p^kD^2 - 8\deg(F^{k*}f_*\mathcal{O}_S(D)) \geq 2(A-1)D\cdot F(\mu_1-\mu_n) \geq 0$$

which and $\deg(F^{k*}f_*\mathcal{O}_S(D)) = p^k\deg(f_*\mathcal{O}_S(D))$ imply

$$D^2 \geq \frac{2D\cdot F}{h^0(D|_F)}\deg(f_*\mathcal{O}_S(D)).$$

□

Colloary 2. *Let $f: S\to B$ be a relatively minimal fibration of genius $g\geq 2$ over an algebraically closed field of characteristic $p\geq 0$. Then*

$$K^2_{S/B} \geq \frac{4g-4}{g}\deg(f_*\omega_{S/B}).$$

Proof. Take $D = K_{S/B}$ (the relative canonical divisor), which satisfies all the assumptions in Theorem 6 with $h^0(D|_F) = g$, $D\cdot F = 2g-2$ and $\mathcal{O}_S(D) = \omega_{S/B}$. □

3. Slopes of non-hyperelliptic fibrations

Xiao has constructed examples (cf.[12, Example 2]) of hyperelliptic fiberation $f: S\to B$ such that

$$K^2_{S/B} = \frac{4g-4}{g}\deg(f_*\omega_{S/B})$$

and has conjectured (cf. [12, Conjecture 1]) that the inequality must be strict for non-hyperelliptic fibrations.

Proposition 1. *Let $f: S\to B$ be a non-hyperelliptic fibration of genus $g\geq 3$, if $f_*\omega_{S/B}$ is strongly semi-stable, then*

$$K^2_{S/B} \geq \frac{5g-6}{g}\deg(f_*\omega_{S/B}). \tag{3.1}$$

Proof. By Max Noether's theorem, the second multiplication map

$$\varrho: S^2f_*\omega_{S/B} \to f_*(\omega^{\otimes 2}_{S/B})$$

is generically surjective for non-hyperelliptic fibrations $f: S\to B$. Let

$$S^2f_*\omega_{S/B} \twoheadrightarrow \mathcal{F} := \varrho(S^2f_*\omega_{S/B}) \subset f_*(\omega^{\otimes 2}_{S/B}).$$

Then $\mathcal{F}$ is a vector bundle of rank $\mathrm{rk}(f_*(\omega^{\otimes 2}_{S/B})) = 3g-3$, and

$$\deg(\mathcal{F}) \leq \deg(f_*(\omega^{\otimes 2}_{S/B})) = K^2_{S/B} + \deg(f_*\omega_{S/B}). \tag{3.2}$$

On the other hand, semi-stability of $S^2f_*\omega_{S/B}$ implies

$$\deg(\mathcal{F}) \geq (3g-3)\mu(S^2f_*\omega_{S/B}) = \frac{6g-6}{g}\deg(f_*\omega_{S/B}). \tag{3.3}$$

Then (3.2) and (3.3) imply the required inequality (3.1). □

If $E = f_*\omega_{S/B}$ is not strongly semi-stable, let

$$0 := E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset \widetilde{E} = F^{k*}E \tag{3.4}$$

be the Harder-Narasimhan filtration of $F^{k*}E$ with $r_i = \mathrm{rk}(E_i)$ and

$$\mu_i = \mu(E_i/E_{i-1}) = \mu_{min}(E_i) \quad (1 \le i \le n)$$

where we choose $k \ge k_0$ such that all quotients E_i/E_{i-1} appears in above filtration are strongly semi-stable. The second multiplication map induces a multiplication map, which is still denoted by ϱ,

$$\varrho : S^2\widetilde{E} = F^{k*}S^2 f_*\omega_{S/B} \to F^{k*}f_*(\omega_{S/B}^{\otimes 2}).$$

Let $\widetilde{\mathcal{F}} = F^{k*}\mathcal{F} = \varrho(S^2\widetilde{E}) \subset F^{k*}f_*(\omega_{S/B}^{\otimes 2})$ be the image of ϱ, then

$$K_{S/B}^2 \ge \frac{1}{p^k}\deg(\widetilde{\mathcal{F}}) - \deg(f_*\omega_{S/B}).$$

Thus the question is to find a good lower bound of $\deg(\widetilde{\mathcal{F}})$, where

$$0 \to \widetilde{\mathcal{K}} := \ker(\varrho) \to S^2\widetilde{E} \xrightarrow{\varrho} \widetilde{\mathcal{F}} \to 0.$$

Note that for any filtration

$$0 := \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n := \widetilde{\mathcal{F}} \tag{3.5}$$

of $\widetilde{\mathcal{F}}$, $\deg(\mathcal{F}_i/\mathcal{F}_{i-1}) \ge (\mathrm{rk}(\mathcal{F}_i) - \mathrm{rk}(\mathcal{F}_{i-1}))\mu_{min}(\mathcal{F}_i)$. If $\mu_{min}(\mathcal{F}_i) \ge a_i$,

$$\deg(\widetilde{\mathcal{F}}) \ge \sum_{i=1}^{n} \mathrm{rk}(\mathcal{F}_i)(a_i - a_{i+1}). \tag{3.6}$$

One of choices of the filtration (3.5) is induced by the Harder-Narasimhan filtration (3.4) of $\widetilde{E} = F^{k*}f_*\omega_{S/B}$ (similar with [7]):

$$\mathcal{F}_i = \varrho(E_i \otimes E_i) \subset \widetilde{\mathcal{F}}.$$

The following lemma implies that $\mu_{min}(\mathcal{F}_i) \ge 2\mu_i$ for all $1 \le i \le n$.

Lemma 5. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two bundles over a smooth projective curve with all quotients in the Harder-Narasimhan of $\mathcal{E}_1$ and $\mathcal{E}_2$ are strongly semi-stable. Then we have*

$$\mu_{min}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mu_{min}(\mathcal{E}_1) + \mu_{min}(\mathcal{E}_2).$$

Proof. It is clear that $\mu_{min}(\mathcal{E}_1 \otimes \mathcal{E}_2) \le \mu_{min}(\mathcal{E}_1) + \mu_{min}(\mathcal{E}_2)$ by [10, Proposition 3.5 (3)]. Thus is enough to show

$$\mu_{min}(\mathcal{E}_1 \otimes \mathcal{E}_2) \ge \mu_{min}(\mathcal{E}_1) + \mu_{min}(\mathcal{E}_2).$$

By Lemma 3, there is a k_0 such that for all $k \geq k_0$, all quotients in the Harder-Narasimhan filtration of $F^{k*}(\mathcal{E}_1 \otimes \mathcal{E}_2)$ are strongly semi-stable.

Let $F^{k*}(\mathcal{E}_1 \otimes \mathcal{E}_2) \twoheadrightarrow \mathcal{Q}$ be the strongly semi-stable quotient with

$$\mu(\mathcal{Q}) = \mu_{min}(F^{k*}(\mathcal{E}_1 \otimes \mathcal{E}_2)) \leq p^k \mu_{min}(\mathcal{E}_1 \otimes \mathcal{E}_2).$$

Applying [10, Proposition 3.5(4)] on the nontrivial morphism

$$F^{k*}\mathcal{E}_1 \to (F^{k*}\mathcal{E}_2)^\vee \otimes \mathcal{Q},$$

we have $\mu_{max}((F^{k*}\mathcal{E}_2)^\vee \otimes \mathcal{Q}) \geq \mu_{min}(F^{k*}\mathcal{E}_1)$ and

$$\mu_{max}((F^{k*}\mathcal{E}_2)^\vee \otimes \mathcal{Q}) = \mu(\mathcal{Q}) - \mu_{min}(F^{k*}\mathcal{E}_2)$$

since all quotients $gr_i^{\mathrm{HN}}(\mathcal{E}_2)$ and $\mathcal{Q}$ are strongly semi-stable. Then

$$\mu(\mathcal{Q}) \geq \mu_{min}(F^{k*}\mathcal{E}_1) + \mu_{min}(F^{k*}\mathcal{E}_2) = p^k(\mu_{min}(\mathcal{E}_1) + \mu_{min}(\mathcal{E}_2)),$$

where the last equality holds since all $gr_i^{\mathrm{HN}}(\mathcal{E}_1)$ and $gr_i^{\mathrm{HN}}(\mathcal{E}_2)$ are strongly semi-stable, which implies that

$$\mu_{min}(\mathcal{E}_1 \otimes \mathcal{E}_2) \geq \mu_{min}(\mathcal{E}_1) + \mu_{min}(\mathcal{E}_2).$$

□

A lemma of [7] provides the lower bound of $\mathrm{rk}(\mathcal{F}_i)$. To state it, recall that in the proof of Theorem 6, each E_i defines a morphism

$$\phi_{L_i} : F \to \mathbb{P}^{r_i - 1}$$

on the general fiber F of $f : S \to B$, where $L_i \subset \omega_F$ is generated by global sections $\{s_1, \ \dots, \ s_{r_i}\} \subset H^0(\mathcal{O}_S(K_{S/B})|_F) = H^0(\omega_F)$.

Definition 2. *Let $\tau_i : C_i \to \phi_{L_i}(F)$ be the normalization of $\phi_{L_i}(F)$,*

$$g_i = g(C_i)$$

be the genius of C_i and $\psi_i : F \to C_i$ be the morphism such that

$$\phi_{L_i} = \tau_i \cdot \psi_i.$$

Let $c_i = \deg(\phi_{L_i}) = \deg(\psi_i)$. Then $c_i | c_{i-1}$ for all $1 \leq i \leq n$ and

$$r_1 < r_2 < \cdots < r_{n-1} < r_n = g, \quad g_1 \leq g_2 \leq \cdots \leq g_{n-1} \leq g_n = g.$$

Lemma 6. *(*[7, Lemma 2.6]*) For each $1 \leq i \leq n$, we have*

$$\mathrm{rk}(\mathcal{F}_i) \geq \begin{cases} 3r_i - 3, & \text{if } r_i \leq g_i + 1; \\ 2r_i + g_i - 1, & \text{if } r_i \geq g_i + 2. \end{cases}$$

In particular, if ϕ_{L_i} is a birational morphism, then

$$\mathrm{rk}(\mathcal{F}_i) \geq 3r_i - 3.$$

Lemma 7. *Let d_i' be the degree of $\phi_{L_i}(F) \subset \mathbb{P}^{r_i-1}$, $\ell = \min\{\, i \mid c_i = 1\}$,*

$$I = \{1 \le i \le \ell - 1 \mid r_i \ge g_i + 2\,\}.$$

Then we have

$$p^k K_{S/B}^2 \ge \sum_{i\in I}(3r_i + 2g_i - 2)(\mu_i - \mu_{i+1}) + \sum_{i\notin I}(5r_i - 6)(\mu_i - \mu_{i+1}),$$

$$p^k K_{S/B}^2 \ge 2\sum_{i\in I} c_i d_i'(\mu_i - \mu_{i+1}) + \sum_{i\notin I}(4r_i - 2)(\mu_i - \mu_{i+1}) - 2\mu_n.$$

Proof. The first inequality is from (3.6) by taking $a_i = 2\mu_i$ and using estimate of $\mathrm{rk}(\mathcal{F}_i)$ in Lemma 6. The second inequality is from

$$p^{2k} K_{S/B}^2 \ge \sum_{i=1}^{n-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + p^k(4g - 4)\mu_n$$

by using $d_i \le d_{i+1}$, $d_i = p^k c_i d_i'$ and $c_i d_i' \ge 2r_i - 2$. □

Proposition 2. *If* $\min\{\, c_i \,|\, i \in I\} \ge 3$, *then*

$$K_{S/B}^2 \ge \frac{9(g-1)}{2(g+1)}\deg f_*\omega_{S/B}.$$

Proof. When $\min\{\, c_i \,|\, i \in I\} \ge 3$, use $d_i' \ge r_i - 1$ and Lemma 7,

$$p^k K_{S/B}^2 \ge \sum_{i\in I}(3r_i - 2)(\mu_i - \mu_{i+1}) + \sum_{i\notin I}(5r_i - 6)(\mu_i - \mu_{i+1})$$

$$p^k K_{S/B}^2 \ge \sum_{i\in I}(6r_i - 6)(\mu_i - \mu_{i+1}) + \sum_{i\notin I}(4r_i - 2)(\mu_i - \mu_{i+1}) - 2\mu_n.$$

Take the average of above two inequalities, we have

$$K_{S/B}^2 \ge \frac{9}{2}\deg f_*\omega_{S/B} - \frac{4\mu_1 + \mu_n}{p^k}. \tag{3.7}$$

On the other hand, by Lemma 1, we have $K_{S/B}^2 \ge \frac{(2g-2)(\mu_1+\mu_n)}{p^k}$ and $K_{S/B}^2 \ge \frac{2g-2}{p^k}\mu_1$, which and (3.7) implies the required inequality. □

Proposition 3. *If* $\min\{\, c_i \,|\, i \in I\} = 2$ *and* $g_i \ge \frac{g-1}{4}$ *for* $i \in I$ *with* $c_i = 2$. *Then we have*

$$K_{S/B}^2 > \frac{9(g-1)}{2(g+1)}\deg f_*\omega_{S/B}.$$

Proof. It is a matter to estimate d_i'. Since $\phi_{L_i}(F) \subset \mathbb{P}^{r_i-1}$ is an irreducible non-degenerate curve of degree d_i', we have in general $d_i' \ge r_i - 1$ and more precisely the so called Castelnuovo's bound

$$d_i' - 1 \ge \frac{g_i}{m_i} + \frac{m_i + 1}{2}(r_i - 2)$$

where $m_i = [\frac{d'_i - 1}{r_i - 2}]$ is the positive integer defined by $d'_i - 1 = m_i(r_i - 2) + \varepsilon_i$ with $0 \leq \varepsilon_i < 1$ (see [1, Chapter III, 2]).

Let $I_1 = \{ i \in I \,|\, c_i = 2\}$. Then for any $i \in I_1$, $d'_i \geq r_i - 1 + g_i$ by Castelnuovo's bound (since $r_i \geq g_i + 2 \geq 2$). On the other hand,

$$8g_i \geq 2g - 2 \geq 2d'_i \geq 2r_i - 2 + 2g_i$$

implies that $3g_i \geq r_i - 1$, which implies that

$$(3r_i + 2g_i - 2) + 2c_i d'_i \geq 9r_i - 8, \quad \forall\, i \in I_1,$$

thus $(3r_i + 2g_i - 2) + 2c_i d'_i \geq 9r_i - 8$ for all $i \in I$. Then the required inequality follows the same arguments in Proposition 2. □

Proposition 4. *(*[3, Theorem 3.1, 3.2]*) If there is an $i \in I$ such that $c_i = 2$ and $g_i < \frac{g-1}{4}$. Then*

$$K^2_{S/B} \geq \frac{4(g-1)}{g - g_i} \mathrm{deg} f_* \omega_{S/B}.$$

Proof of Theorem 3. When $n = 1$ (i.e. $f_* \omega_{S/B}$ strongly semistable), Theorem 3 is true by Proposition 1. When $n > 1$, Theorem 3 is a consequence of Proposition 2, Proposition 3 and Proposition 4 since we have $g_i \geq 1$ if $c_i = 2$. □

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