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Rigidity theorems for submetries in positive curvature ☆

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ABSTRACT

We derive general structure and rigidity theorems for submetries $f : M \to X$, where M is a complete Riemannian manifold with sectional curvature $\sec M \geq 1$. When applied to a non-trivial Riemannian submersion, it follows that $\operatorname{diam} X \leq \pi/2$. In case of equality, there is a Riemannian submersion $\mathbb{S} \to M$ from a unit sphere, and as a consequence, f is known up to metric congruence. A similar rigidity theorem also holds in the general context of Riemannian foliations.

1. Introduction

The classical so-called Bonnet–Myers theorem implies that a complete Riemannian n-manifold M with sectional curvature, $\sec M \geq 1$ has diameter $\operatorname{diam} M \leq \pi$. Moreover,

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if $\operatorname{diam} M = \pi$, M is isomeric to the unit sphere $\mathbb{S}^n$ by Toponogov's *diameter rigidity theorem*.

The purpose of this note is to analyze and prove analogous rigidity theorems in the general setting of submetries in Riemannian geometry. The example of Riemannian submersions is particularly appealing:

Theorem 1. *Let M be a complete Riemannian manifold with* $\sec M \geq 1$ *and $f : M \to N$ a (non-trivial) Riemannian submersion. Then the base N has* $\operatorname{diam}(N) \leq \frac{\pi}{2}$*, where equality holds if and only if there is a Riemannian submersion (possibly an isometry) $f_1 : \mathbb{S} \to M$, where $\mathbb{S}$ is a unit sphere.*

Here, by non-trivial we of course mean that f is not an isometry. However, our result in particular includes the case of covering maps. For this case we note the interesting fact that there are irreducible space forms $N^n = \mathbb{S}^n/\Gamma_n$ with $\operatorname{diam} N^n$ converging to $\pi/2$ as n goes to infinity [18].

We point out that the conclusion of the theorem yields a *complete metric classification* when the base N has maximal diameter $\pi/2$ (see Corollary 16). This is because Riemannian submersions from the standard unit sphere were classified in [8] and [24], and of course $f \circ f_1 : \mathbb{S} \to N$ is a Riemannian submersion as well.

In particular, if M and N are simply connected, then metrically f is either a Hopf fibration $\mathbb{S}^{2d+1} \to \mathbb{CP}^d$, $\mathbb{S}^{4d+3} \to \mathbb{HP}^d$, $\mathbb{S}^{15} \to \mathbb{S}^8(1/2)$, or the induced fibration $\mathbb{CP}^{2d+1} \to \mathbb{HP}^d$.

It turns out that there is a similar rigidity theorem in the general context of Riemannian foliations.

Theorem 2. *Let $\mathcal{F}$ be a Riemannian foliation on M with leaves of positive dimension, where* $\sec M \geq 1$*. Then any two leaves are at distance at most $\pi/2$ from one another. Moreover, if equality occurs, there is a Riemannian submersion: $\mathbb{S} \to M$.*

This again via the recently completed classification of Riemannian foliations on the unit sphere [8,17,24] yields a complete answer in the case of equality (for details see Corollary 18).

We note that with the exception of the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8(1/2)$ all Riemannian foliations on $\mathbb{S}$ are homogeneous. It is thus natural to wonder about isometric group actions $\mathsf{G} \times M \to M$ with large orbit space, i.e., $\operatorname{diam} M/\mathsf{G} \geq \pi/2$. For this we prove

Theorem 3. *Let G be compact Lie group acting on M isometrically, where* $\sec M \geq 1$*. If* $\operatorname{diam} M/\mathsf{G} \geq \pi/2$*, then one of the following happens:*

(1) *M is a twisted sphere with a suspension action.*
(2) *M is the Cayley plane $\mathbb{OP}^2$ and G has an isolated fixed point.*
(3) *There is a Riemannian submersion $\mathbb{S} \to M$ and the G action on M is induced from a reducible action on $\mathbb{S}$.*

We point out that an isometric G action on the standard sphere $\mathbb{S}$ has orbit space with diameter at least $\pi/2$ if an only if it is a reducible linear action, and that no values are taken between $\pi/2$ and π, and diameter π is equivalent to the action having fixed points, i.e., the action is a suspension. Also, again knowledge of the isometry groups of the standard spaces in question yield complete answers.

For general *singular Riemannian foliations*, we derive a *structure/rigidity result* (see Corollary 22) somewhat similar to Theorem C, an essential difference being that the *homogeneous ones* on the standard sphere $\mathbb{S}$ are known, whereas there is a wealth of inhomogeneous ones (see [22] for recent advances) and a classification is far from known.

The different geometric topics described above are treated simultaneously (cf. our Main Theorem 15) via the study of *submetries* $f : M \to X$, a notion due to Berestovskii [1]. These play an important role in Riemannian as well as in Alexandrov geometry. There is a wealth of examples when $M = \mathbb{S}$ is the standard sphere, but a classification is not in sight. It was proved by Lytchak (Lemma 8.1 in [14]) that the radius rad X of the base X of a submetry $f : Y \to X$ where Y is an Alexandrov space with curv$Y \geq 1$ satisfies rad$X \leq \pi/2$. Here we will consider the diameter rather than the radius and manifold domains in place of Alexandrov spaces.

Our arguments are based on an adaption of the ideas and constructions in [7], where a rigidity theorem for manifolds M with $\sec M \geq 1$ and $\operatorname{diam} M = \pi/2$ was obtained. To keep the exposition tight, familiarity with the methods and results of [7] and [12] is expected. For basic facts and tools from Riemannian geometry, respectively Alexandrov geometry, we refer to [4] and [21], respectively [3].

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2. Submetries

In this section we set up notation and analyze submetries $f : M \to X$, where throughout M is a complete Riemannian manifold with $\sec M \geq 1$ and X is large in the sense that $\operatorname{diam} X \geq \pi/2$.

We will use the notation $|pq|$ to denote the distance between p and q whether in M or in X. For any closed subset L of either M or X, and $r > 0$, we let

$$B(L, r) := \{x \mid |xL| < r\},$$

be the open r-neighborhood of L, and

$$C(L, r) := \{x \mid |xL| \geq r\}$$

its complement.

Recall that by definition, f is a *submetry* if and only if $f(B(p,r)) = B(f(p),r)$ for all $p \in M$ and all $r > 0$. As X is the base of the submetry $f : M \to X$ and $\sec M \geq 1$, we see from the 4-point characterization of a lower curvature bound in [3], that X is an Alexandrov space with $\mathrm{curv} X \geq 1$.

Our investigations will be divided into the cases: (1) $\operatorname{diam} X = \pi$, (2) $\pi/2 < \operatorname{diam} X < \pi$, (3) $\pi/2 = \operatorname{diam} X$. Here the third is naturally further divided into sub cases, depending on the structure of pairs of (dual) convex sets in X and in M at maximal distance $\pi/2$ apart.

The core of our arguments are convexity and critical point theory for non-smooth distance functions. In fact, from distance comparison with the unit 2-sphere, note that

Lemma 1. *For any closed set L, and any $r \geq \pi/2$ the set $C(L,r)$ is locally convex, and (globally) convex except for the case, where $r = \pi/2$ and $C(L,\pi/2)$ contains two points at distance π. Moreover, if $L \subset X$ and $K = f^{-1}(L)$ then $C(K,r) = f^{-1}(C(L,r))$.*

The latter claim follows since f is a submetry. It will be important for us, that if $C(L,r)$ has non-empty boundary, then there is a unique point at maximal distance to the boundary, called its *soul point*. It follows that $C(L,r)$ is contractible and if it is a subset of M, then it and its small metric tubes are topologically discs by Lemma 2.6 in [7].

From Toponogov's *maximal diameter theorem* and its analogue for Alexandrov spaces (cf., e.g., [10] Remark 2.5) we have the following solution to case (1) above:

Proposition 2 *(Metric suspension). Suppose $|xy| = \operatorname{diam} X = \pi$ and $Y = \{z \in X \mid |zx| = |zy| = \pi/2\}$. Then $X = \Sigma Y$ the spherical suspension of Y, $M = \mathbb{S} = \Sigma E$ is a unit sphere with equator E, and f is the suspension of its restriction $f_{|E} : E \to Y$.*

Remark 3. The classification of general submetries from the unit sphere is an open and important problem.

Now suppose $x, y \in X$ are at maximal distance in X, and $\pi/2 < |xy| = \operatorname{diam} X < \pi$. Then by distance comparison y is uniquely determined by x and vice versa. Then for $K = f^{-1}(x)$, $C(K,\pi/2)$ has nonempty boundary consisting of points at distance exactly $\pi/2$ to K. Moreover, $C(K,|xy|) = f^{-1}(y)$ is in the interior of the convex set $C(K,\pi/2)$. Clearly, $f^{-1}(y)$ is at maximal distance to the boundary of $C(K,\pi/2)$, i.e., $f^{-1}(y)$ is a soul point $q \in M$. Reversing the roles of x and y we see that $f^{-1}(x) = p \in M$ is a point as well.

From critical point theory for $|x\cdot|$ in X and $|p\cdot|$ in M, it follows that X, respectively M, topologically is the suspension of its space of directions, $\mathbb{S}_x X$ [20], respectively $\mathbb{S}_p M$, the latter being a unit sphere. Moreover, the submetry f induces a submetry $F : \mathbb{S}_p M \to \mathbb{S}_x X$, via its so-called *differential*, $Df : T_p M \to T_x X$, (see [13,14] or [15]), and topologically f is the suspension of F.

All in all, this completes case (2) above and proves

Proposition 4 *(Topological suspension). Suppose $|xy| = \operatorname{diam} X > \pi/2$. Then $f^{-1}(x) = p$ and $f^{-1}(y) = q$ are points in M. Moreover, topologically X is the suspension of an Alexandrov space Y, with $\mathrm{curv} Y \geq 1$, M is a twisted sphere and f is the suspension of a submetry $F : \mathbb{S}^{n-1} \to Y$, where $\dim M = n$.*

We now proceed to investigate the various scenarios when $\operatorname{diam} X = \pi/2$ in case (3). Of course, in this case $r = \pi/2$ for all convex sets $C(L, r)$ we consider. In fact, we now fix $x, y \in X$ at maximal distance in X, i.e., $|xy| = \operatorname{diam} X$, and consider the convex sets

$$B = \{z \in X \mid |zx| \geq \pi/2\} = C(x, \pi/2)\ ,\ B' = \{z \in X \mid |zB| \geq \pi/2\} = C(B, \pi/2)$$

respectively

$$A = \{p \in M \mid |pf^{-1}(x)| \geq \pi/2\} = f^{-1}(B)\ ,\ A' = \{p \in M \mid |pA| \geq \pi/2\} = f^{-1}(B').$$

Note, that in fact, the inequalities $\geq \pi/2$ in the definitions of B, A, etc., can now be replaced by equalities.

It is important to note, that from critical point theory applied to either of the non-smooth distance functions $|A \cdot|$ or $|A' \cdot|$, it follows as in [7] that

$$M = \mathbb{D}(A) \cup \mathbb{D}(A'), \tag{5}$$

where $\mathbb{D}(A)$ and $\mathbb{D}(A')$ are closed distance tubes around A and A', respectively. As mentioned above, if say A' has non-empty boundary, it follows that $\mathbb{D}(A')$ topologically is a disc (Lemma 2.6 in [7]).

There are now two scenarios depending on whether or not A and or A' has non-empty boundary. Whether or not $\operatorname{diam} M > \pi/2$, the above decomposition of M means that the arguments used in Propositions 3.4, 3.5 in [7] carry over verbatim, yielding

Lemma 6. *The dual sets A and A' either both have empty boundary, or they both have non-empty boundary.*

In the (non-rigid) case where both A and A' have non-empty boundary and hence M is a twisted sphere, we are unable to say anything else in complete generality. However, we will see below (Proposition 10) that we can recover essential parts of the structure as in Proposition 4 above, as long as the restricted submetries

$$f : A \to B \quad \text{and} \quad f : A' \to B'$$

satisfy the following natural condition (automatic when $\operatorname{diam} X > \pi/2$).

Definition 7. A submetry $f : A \to B$ *respects the boundary* if whenever $f^{-1}(x) \cap \partial A \neq \emptyset$ then $f^{-1}(x) \subset \partial A$.

There are several natural geometric situations (including the ones discussed in the introduction) where indeed f respects the boundary (see next section). However, here are some examples (not necessarily arising in our context) where the boundary is not respected.

Example 8. 1. Let A be a hemisphere and p a point on the boundary of A. Let $f : A \to [0,\pi] = B$ be the submetry $f := |p \cdot|$, i.e., $f^{-1}(t) = \{x \in A \mid |xp| = t\}$, $0 \le t \le \pi$. Then for any $0 < t < \pi$, $f^{-1}(t) \cap \partial A \neq \emptyset$ but $f^{-1}(t)$ is not contained in ∂A. Note that $f^{-1}(t)$ has non-empty boundary for $0 < t < \pi$.

2. In this *tennis ball example*, let $A \subset \mathbb{S}^2$ be a geodesic of length π, and $f : \mathbb{S}^2 \to [0,\pi/2]$ again the submetry $f := |A \cdot|$, i.e., $f^{-1}(t) = \{x \in \mathbb{S}^2 \mid |xA| = t\}$, $0 \le t \le \pi/2$. Then A' is a geodesic arc opposite to A, B is the left end point of $[0,\pi/2]$ and B' the right end point. For the restricted submetry $f : A \to B$, A itself is a fiber having non-empty boundary. Of course $A \cap \partial A \neq \emptyset$ but A is not contained in ∂A.

The importance of "preserving the boundary" is due to the following

Lemma 9. *If $f : A \to B$ respects the boundary, then the distance function $|\cdot \partial A|$ is constant along any "fiber" $f^{-1}(x)$ in the interior of A.*

Proof. Indeed, let $p \in f^{-1}(x)$ and $a \in \partial A$ such that $|pa| = |f^{-1}(x)\ \partial A|$. As $f : A \to B$ respects the boundary and $a \in f^{-1}(f(a)) \cap \partial A$, we see that $f^{-1}(f(a)) \subset \partial A$. Then $|f^{-1}(x)\ f^{-1}(f(a))| \le |pa| = |f^{-1}(x)\ \partial A| \le |f^{-1}(x)\ f^{-1}(f(a))|$. Hence $|pa| = |f^{-1}(x)\ f^{-1}(f(a))|$. Since the fibers $f^{-1}(x)$ and $f^{-1}(f(a))$ are equidistant, for any $p' \in f^{-1}(x)$, there exists $a' \in f^{-1}(f(a))$ such that $|p'a'| = |f^{-1}(x)\ f^{-1}(f(a))| = |pa| = |f^{-1}(x)\,\partial A|$. As $|p'\partial A| \le |p'a'| = |f^{-1}(x)\,\partial A| \le |p'\partial A|$, we see that $|p'\partial A| = |f^{-1}(x)\,\partial A|$ for any $p' \in f^{-1}(x)$. Hence $|\cdot \partial A|$ is constant along any "fiber" $f^{-1}(x)$ in the interior of A. □

Proposition 10 *(Point fibers). Suppose $|xy| = \operatorname{diam} X = \pi/2$ and the convex sets A and A' have non-empty boundary (or are points). If $f_{|A}$ and $f_{|A'}$ respect boundaries, then topologically M is a twisted sphere and f has two point fibers.*

Proof. By Theorem 2.5 in [7], we know that topologically M is a twisted sphere. Since $\sec M \ge 1$, the function $|\cdot \partial A|$ is strictly concave along all geodesics in A not minimizing the distance to ∂A, and hence it attains the maximum value at a unique point $p \in A$. If $f_{|A}$ respects the boundary, by Lemma 9, $|\cdot \partial A|$ is constant along the fiber $f^{-1}(f(p))$. Hence $f^{-1}(f(p))$ is a point fiber. By a similar argument, $f^{-1}(f(p'))$ is also a point fiber, where $p' \in A'$ and $|p'\partial A'| = \max_{q \in A'} |q\partial A'|$. □

Remark 11. Note that under the assumptions of Proposition 10, in fact the distance function $|\cdot \partial A| : A \to \mathbb{R}$ factors through $|\cdot f(\partial A)| : B \to \mathbb{R}$, and so the latter is strictly concave along geodesics in B not minimizing distance to $f(\partial A)$ as well. The associated Sharafutdinov retraction on B lifts to the Sharafutdinov retraction on A.

It seems plausible that a tube around B is homeomorphic to a ball around its soul point (as is the case for A as shown in Lemma 2.6 in [7]), in which case one would recover the full version of Proposition 4, also in this case. One situation, where this indeed is the case, is when X is the orbit space of M by an isometric action (see Corollary 19).

In the remaining case where A and A' are smooth totally geodesic submanifolds we have

Proposition 12 *(Rigidity). Assume* $\operatorname{diam} X = \pi/2$ *and the pair of dual sets* A, A' *have no boundary. Then either*

(1) *There is a Riemannian submersion* $F : \mathbb{S} \to M$ *(possibly an isometry or a covering map) and moreover* $F^{-1}(A)$ *and* $F^{-1}(A')$ *is a pair of totally geodesic dual sub spheres in* $\mathbb{S} = F^{-1}(A) * F^{-1}(A')$.

or

(2) M *is isometric to the Cayley plane* $\mathbb{OP}^2$*, and one of the fibers of* f *is a point.*

Proof. If $\operatorname{diam} M = \pi/2$ this is a direct consequence of the diameter rigidity theorem of [7], and [24]. (Note that case (2) comes from this scenario.)

Now assume $\operatorname{diam} M > \pi/2$, and thus M by [12] is a sphere topologically. In this case we need to show that A and A' are dual spheres with diameter π. This is immediately obvious in the exceptional case of Lemma 1, where A contains two points at distance π, i.e., M is the unit sphere.

Now suppose, that both A and A' are totally geodesic submanifolds of dimensions $a \geq 1$, and $a' \geq 1$. Arguing exactly as Propositions 3.4, 3.5 and Theorem 3.6 in [7], one shows that any unit normal vector to either of A and A' defines a minimal geodesic to the other, and that the corresponding maps $\mathbb{S}_p^\perp \to A'$, $p \in A$ and $\mathbb{S}_{p'}^\perp \to A$, $p' \in A'$ are Riemannian submersions, where $\mathbb{S}_p^\perp$ is the unit normal sphere of A at p. We are going to show that A is a sphere with diameter π and so is A' by similar arguments. By transversality and (5), we see that

$$(i_A)_* : \pi_k(A) \to \pi_k(M)$$

is injective for $k \leq \dim(M) - a' - 2$ and surjective for $k \leq \dim(M) - a' - 1$, where $i_A : A \to M$ is the inclusion map. It follows that $a > \dim(M) - a' - 2$ since A is a closed submanifold of M and M topologically is a sphere. On the other hand, since A and A' are disjoint totally geodesic submanifolds of M, we have $a + a' \leq \dim(M) - 1$ by Frankel's Theorem [6]. Thus $a + a' = \dim(M) - 1$ and hence $\dim(S_{p'}^\perp) = \dim(A)$. It follows that the Riemannian submersion $\mathbb{S}_{p'}^\perp \to A$ is a covering map.

By the arguments in the proof of Lemma 4.1 and Proposition 4.2 in [7], we see that A is either simply connected or a closed geodesic of length 2π. In the simply connected case, A is a constant curvature 1 sub-sphere since $\mathbb{S}^{\perp}_{p'} \to A$ is a covering map. Because A is also convex, the diameter of M is π and we are done. Otherwise A is a closed geodesic of length 2π and again $\mathrm{diam}(M) = \pi$ since A is convex. $\square$

Remark 13 *(Fat joins).* Since $f \circ F : \mathbb{S} \to X$ is a submetry, to understand the possible submetries occurring in (1) above, it suffices to describe submetries $g : \mathbb{S} = F^{-1}(A) * F^{-1}(A') \to X$, where of course $g_{|F^{-1}(A)} : F^{-1}(A) \to B$ and $g_{|F^{-1}(A')} : F^{-1}(A') \to B'$ are submetries from standard dual sub spheres of $\mathbb{S}$. For this we note that any point of $x \in B$ is joined to any point $x' \in B'$ by a minimal geodesic of length $\pi/2$. Moreover, for any direction $\xi \in S^{\perp}_x$ orthogonal to B, there is a minimal geodesic from x to B' with direction ξ and the maps

$$P_x : S^{\perp}_x \to B', x \in B, \quad \text{and similarly} \quad P_{x'} : S^{\perp}_{x'} \to B, x' \in B'$$

are submetries, where $S^{\perp}_x$ is the directions of X at x orthogonal to B. Now pick points $p \in g^{-1}(x)$ and $p' \in g^{-1}(x')$ and note that the submetry $\mathbb{S}^{\perp}_p \to B'$ which is the composition of the isometry $\mathbb{S}^{\perp}_p \to F^{-1}(A')$ with $g_{|F^{-1}(A')} : \mathbb{S}^{\perp}_p \equiv F^{-1}(A') \to B'$ also can be written as $P_x \circ Dg_p : \mathbb{S}^{\perp}_p \to B'$ and similarly for p'. These data completely describe g.

From the above remark it follows that the rigid submetries from part (1) of the above proposition, can be thought of as "fat joins" of the restricted submetries $f_{|A}$ and $f_{|A'}$

3. Manifold submetries

A common feature of submetries $f : M \to X$ arising from Riemannian submersions, isometric group actions, or more generally from (singular) Riemannian foliations is that all "fibers", $f^{-1}(x)$, $x \in X$ are (smooth) closed submanifolds of M (allowing a discrete set of points) and every geodesic on M that is perpendicular at one point to a fiber remains perpendicular to every fiber it meets. We will refer to such a submetry as a *manifold submetry* (this is actually a special case of the notion of *splitting submetries* introduced by Lytchak in his thesis [14]).

For manifold submetries we have the following important result, a special case of which is due to Boltner [2], Lemma 2.4 (cf. also [5] Lemma 4.1):

Lemma 14. *Let $f : M \to X$ be a manifold submetry between a complete Riemannian manifold M and an Alexandrov space X. Then f respects the boundary ∂C of any closed convex subset $C \subset M$ saturated by fibers of f, i.e., with $C = f^{-1}(f(C))$.*

Proof. Suppose there is a fiber $f^{-1}(x)$, $x \in X$ such that $f^{-1}(x) \cap \partial C \neq \emptyset$, we want to show $f^{-1}(x) \subset \partial C$. If not, there are two points $p, q \in f^{-1}(x)$ such that $p \in C^{\circ}$, the interior of C, and $q \in \partial C$. Since $f^{-1}(x)$ is a smooth closed submanifold and C is saturated by the fibers of f, the tangent space of $f^{-1}(x)$ at q is tangent to the boundary ∂C. Then by the convexity of C, there is a geodesic γ in M passing through q, perpendicular to $f^{-1}(x)$ such that $\gamma(0) = q$ and $q_1 := \gamma(\epsilon) \in C^\circ$, $q_2 := \gamma(-\epsilon) \notin C$ for $\epsilon > 0$ sufficiently small. By Proposition 1.17 in [2], $f \circ \gamma$ can be lifted to a geodesic $\tilde{\gamma}$ at p. Let $p_1 := \tilde{\gamma}(\epsilon)$, $p_2 := \tilde{\gamma}(-\epsilon)$. Since $\gamma_{|[0,\epsilon]} \subset C$, so is $\tilde{\gamma}_{|[0,\epsilon]}$ as C is saturated by the fibers over $f(C)$. Then as $p \in C^\circ$, by taking ϵ small enough, we see that $p_2 = \tilde{\gamma}(-\epsilon) \in C^\circ \subset C$. This, however, is impossible since q_2 and p_2 are in the same fiber and C is saturated by the fibers over $f(C)$. $\square$

Summarizing and combining the results Propositions 2, 4, 10 and 12 from the previous section we have in particular,

Theorem 15 *(Submetry rigidity). Let $f : M \to X$ be manifold submetry with* $\sec M \geq 1$ *and* $\operatorname{diam} X \geq \pi/2$. *Then one of the following happens:*

(1) *M is a twisted sphere and f has two point fibers at maximal distance* $\operatorname{diam} X$.
(2) *M is isometric to the Cayley plane and f has one point fiber.*
(3) *There is a Riemannian submersion $F : \mathbb{S} = \mathbb{S}_- * \mathbb{S}_+ \to M$, so that $F(\mathbb{S}_\pm)$ are totally geodesic submanifolds of M saturated by fibers of f at maximal distance $\pi/2$.*

We now restrict attention to geometric objects of common interest where more precise information can be derived.

The simplest is of course that of *Riemannian submersions* $f : M \to N$, which in particular are locally trivial bundles with manifold fiber. This obviously rules out the cases (1), (2) in Theorem 15, since f does not have any point fibers. In particular, we have $\operatorname{diam} N \leq \pi/2$ and case (3) of Theorem 15 applies. This proves Theorem 1 of the introduction.

From the classification of Riemannian submersions from unit spheres in [8,24] and the fact that composition of Riemannian submersions is a Riemannian submersion, this yields the following analogue of Toponogov's maximal diameter theorem:

Corollary 16 *(Submersion rigidity). Let M be a Riemannian n-manifold with* $\sec M \geq 1$. *Then for any non-trivial Riemannian submersion $f : M \to N$, we have* $\operatorname{diam} N \leq \pi/2$. *Moreover, in case of equality, the following:*

(1) *Hopf fibrations $\mathbb{S}^{2d+1} \to \mathbb{CP}^d$, $\mathbb{S}^{4d+3} \to \mathbb{HP}^d$, $\mathbb{S}^{15} \to \mathbb{S}^8(1/2)$;*
(2) *The $\mathbb{Z}_2$ cover, $\mathbb{CP}^{2d+1} \to \mathbb{CP}^{2d+1}/\mathbb{Z}_2$;*
(3) *The covering maps $\mathbb{S}^n \to \mathbb{S}^n/\Gamma$, where Γ acts reducibly on $\mathbb{R}^{n+1}$,*

provide an exhaustive list up to metric congruence modulo composition and factorization.

Remark 17. As examples of factorization and composition in Corollary 16, we mention that $\mathbb{S}^{4d+3} \to \mathbb{HP}^d$ factors through $\mathbb{S}^{4d+3}/\mathsf{K} \to \mathbb{HP}^d$, where $\mathsf{K} \subset \mathbb{S}^3$ acts as a sub action of the Hopf action, and $\mathbb{S}^{4d+3} \to \mathbb{CP}^{2d+1}$ composed with $\mathbb{CP}^{2d+1} \to \mathbb{CP}^{2d+1}/\mathbb{Z}_2$ yield $\mathbb{S}^{4d+3} \to \mathbb{CP}^{2d+1}/\mathbb{Z}_2$.

We next move to *Riemannian foliations* $\mathcal{F}$ on M. Here not all leaves need to be closed, so we consider the decomposition $\bar{\mathcal{F}}$ of M by the closures of the leaves in $\mathcal{F}$. It is well known [19] that this yields a manifold submetry $f : M \to M/\bar{\mathcal{F}} =: X$ and $\bar{\mathcal{F}}$ is a singular Riemannian foliation. Again, when all leaves have non-zero dimension, the cases (1), (2) in Theorem 15 are excluded, and in particular $\operatorname{diam} X \leq \pi/2$ for this case as well. In particular Theorem 2 follows and specifically we have,

Corollary 18 *(Foliation rigidity). Let M be a Riemannian n-manifold with* $\sec M \geq 1$ *and $\mathcal{F}$ a Riemannian foliation on M with leaves of positive dimensions. Then any two leaves in M are at distance at most $\pi/2$ from one another. Moreover, assuming that M is simply connected, in case of equality we have the following up to metric congruence:*

(1) *M is $\mathbb{S}^n$, and $\mathcal{F}$ is given by an almost free isometric action by R, $\mathbb{S}^1$ or $\mathbb{S}^3$, or $\mathcal{F}$ is the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8(1/2)$,*
(2) *M is $\mathbb{CP}^{2d+1}$ and $\mathcal{F}$ is induced from a* 3*-dimensional foliation from (1).*

The proof is a direct consequence of Theorem 15 and the classification of Riemannian foliations on the unit sphere [8,17,24]. The only additional input needed, is the obvious fact that any Riemannian foliation $\mathcal{F}$ on the base N of a Riemannian submersion $M \to N$ lifts to M with leaf dimension increasing by the dimension of the fiber.

We now consider the case of a *homogeneous singular foliation*, i.e., the case where our manifold submetry $f : M \to M/\mathsf{G} =: X$ is the orbit map for an isometric action by a compact Lie group G on M. Recall that in the case of G actions on the standard sphere $\mathbb{S}$, simple convexity arguments yield the following: The orbit space $\mathbb{S}/\mathsf{G}$ has diameter $\geq \pi/2$ if and only if it is a reducible linear action. No values are taken between $\pi/2$ and π, and diameter π is equivalent to the action having fixed points, i.e., the action is a suspension.

In general we have:

Corollary 19 *(Group action rigidity). Let G be compact Lie group acting on M^n isometrically, where* $\sec M \geq 1$. *If* $\operatorname{diam} M/\mathsf{G} \geq \pi/2$, *then one of the following happens:*

(1) *M is a twisted sphere with action topologically the suspension of a linear action on $\mathbb{S}^{n-1}$.*
(2) *M is the Cayley plane $\mathbb{OP}^2$ and G has an isolated fixed point.*
(3) *M is isometric to either $\mathbb{S}^n$ or to the base of a Riemannian submersion $\mathbb{S} \to M$, and the action on M is induced from a reducible linear $\hat{\mathsf{G}}$-action on $\mathbb{S}$.*

Proof. By our discussion in previous section, the only case not covered by Theorem 15 is when $\operatorname{diam}(M^n/\mathsf{G}) = \pi/2$ and the convex sets A and A' have non-empty boundary. In this case it remains to show that the G action on M^n is topologically the suspension of a linear action on $\mathbb{S}^{n-1}$.

The proof consist of two parts: (1) Find $\epsilon > 0$ so that the closed ϵ neighborhoods, $D(A, \epsilon)$ and $D(A', \epsilon)$ of A and A' are G equivariantly homeomorphic to the closed ϵ balls of their soul points. (2) Show that $M - (B(A, \epsilon) \cup B(A', \epsilon))$ is G equivariantly homeomorphic to the product $\partial D(A, \epsilon) \times [0, 1]$.

The proof of part (1) is given in Lemma 1.5 and Lemma 2.6 in [7] when no group action is present. However, all constructions in the proof there are invariant under an isometric group action. As A and A' are G invariant by construction, we see that part (1) holds.

To see part (2), start with the G invariant function $|\cdot A|$. Smoothing $|\cdot A|$ by the technique of [12], we get a smooth G invariant function h as close to $|\cdot A|$ as we wish. Since $|\cdot A|$ has no critical points in $M - (A \cup A')$ it follows that h also has no critical points in $M - (A \cup A')$ (cf. [7,12]). In particular, its G invariant gradient vector field ∇h is nonzero everywhere in $M - (B(A, \epsilon/2) \cup B(A', \epsilon/2))$ and moreover transverse to its boundary. Note that the radial vector fields on $(B(A, \epsilon) - A)$ as well as its negative on $(B(A', \epsilon) - A')$ are G-invariant as well and of course transverse to the boundaries. Using partition of unity, we construct a G invariant vector field which is nonzero everywhere in $M - (A \cup A')$ and transversal to $\partial D(A, \epsilon/2)$ and $\partial D(A', \epsilon/2)$. This proves part (2). $\square$

Remark 20. Since the full isometry group is known for all spaces in (2) and (3), it is possible to get exhaustive and complete statements in those cases.

In case (1), G has two fixed points at maximal distance $\operatorname{diam} M/\mathsf{G} \geq \pi/2$ and $M = \mathbb{D}^n \cup_f \mathbb{D}^n$, where $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is a G invariant diffeomorphism and the action of G on $\mathbb{S}^{n-1}$ is the isotropy action at a fixed point.

Of course, the positively curved twisted sphere M above can potentially be exotic, namely if f does not extend to the disc $\mathbb{D}^n$. We note that if G acts transitively on $\mathbb{S}^{n-1}$, then f is actually linear (cf. Lemma 2.6 of [11]) and hence it extends. Also in [9] it was proved that if G acts by cohomogeneity one on $\mathbb{S}^{n-1}$, then f is isotopic to a linear map and hence again extends. For much smaller actions, we of course expect there to be exotic G gluing diffeomorphism. However, at the moment we do not know of any theory that addresses the following problem:

Problem 21. Given a linear G action on $\mathbb{S}^{n-1}$. Which properties of the action prevents the existence of exotic G equivariant diffeomorphisms.

We conclude our note by a description of singular Riemannian foliations $\mathcal{F}$ on M with $\sec M \geq 1$ and $\operatorname{diam} M/\mathcal{F} \geq \pi/2$. As we will indicate below, a structure as in Corollary 19 is expected, and this in turn to a large extent will reduce their study to those on unit spheres. Here considerable progress has been made recently. On the one hand numerous new examples have been found [22], and it has been shown that they are all algebraic [16].

From Theorem 15, we have

Corollary 22. *Let $\mathcal{F}$ be a singular Riemannian foliation (all leaves are closed) on M^n with* $\sec M \geq 1$. *If there are leaves at distance at least $\pi/2$, then one of the following happens:*

(1) *M is a twisted sphere and $\mathcal{F}$ has two point leaves at maximal distance.*
(2) *M is the Cayley plane $\mathbb{OP}^2$ and $\mathcal{F}$ has a point leaf.*
(3) *M is isometric to either $\mathbb{S}^n$ or to the base of a Riemannian submersion $\mathbb{S} \to M$, and the lifted foliation $\hat{\mathcal{F}}$ is reducible in the sense that there are two dual spheres $\mathbb{S}_\pm \subset \mathbb{S}^n = \mathbb{S}_- * \mathbb{S}_+$ that are saturated by leaves from $\hat{\mathcal{F}}$.*

Remark 23. In case (1) it is likely that topologically the foliation is the suspension of a singular Riemannian foliation on $\mathbb{S}^{n-1}$. As in the homogeneous case what remains is only the case where $\operatorname{diam} X = \pi/2$ and the convex dual sets A and A' in M have non-empty boundary. Again there are two steps as in the proof of Corollary 19 above. The local structure of $X = M/\mathcal{F}$ allows one to carry out part (1) as in the homogeneous case. As for part (2) the newly constructed *average operator* associated to singular Riemannian foliations on unit spheres $\mathbb{S}$ [16] is likely to yield a gradient like vector field for the distance function $|\cdot, A|$ tangent to the strata of the leaf through any point $x \in M - (B(A,\epsilon) \cup B(A',\epsilon))$ and transversal to $\partial D(A,\epsilon)$ and $\partial D(A',\epsilon)$ [23]. This will suffice to prove the desired statement.

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