GEODESICS IN THE SPACE OF KÄHLER CONE METRICS, I

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Abstract. In this paper, we study the Dirichlet problem of the geodesic equation in the space of Kähler cone metrics $\mathcal{H}_\beta$; that is equivalent to a homogeneous complex Monge-Ampère equation whose boundary values consist of Kähler metrics with cone singularities. Our approach concerns the generalization of the function space defined in Donaldson [25] to the case of Kähler manifolds with boundary; moreover we introduce a subspace $\mathcal{H}_C$ of $\mathcal{H}_\beta$ which we define by prescribing appropriate geometric conditions. Our main result is the existence, uniqueness and regularity of $C^{1,1}_\beta$ geodesics whose boundary values lie in $\mathcal{H}_C$. Moreover, we prove that such geodesic is the limit of a sequence of $C^{2,\alpha}_\beta$ approximate geodesics under the $C^{1,1}_\beta$-norm. As a geometric application, we prove the metric space structure of $\mathcal{H}_C$.

Contents

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1. Introduction

We shall always denote by X a smooth compact Kähler manifold without boundary of complex dimension $n \geq 1$, by $[\omega_0]$ a Kähler class of X, and by $\mathcal{H}$ the space of Kähler metrics in $[\omega_0]$. In their pioneering works, Mabuchi [41], Donaldson [23] and Semmes [49], independently defined the famous Weil-Peterson type metric in $\mathcal{H}$, under which $\mathcal{H}$ becomes a non-positive curved infinite-dimensional symmetric space. (For the study of other such metrics, compare [10, 12, 11].) Semmes [49] pointed out that the geodesic equation in $\mathcal{H}$ is a homogeneous complex Monge-Ampère (HCMA) equation,

$$\begin{cases} (\Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Psi)^{n+1} = 0 & \text{in } X \times R\,, \\ \sum_{1\leq i,j\leq n} (\Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Psi)_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}} > 0 & \text{in } X \times \{z^{n+1}\}\,; \end{cases} \tag{1.1}$$

here R is a cylinder with boundary, and Ω_0 is the pull-back metric of ω_0 under the natural projection.

Geodesics are basic geometric objects in the infinite dimensional manifold $\mathcal{H}$. The relation between the geodesics of $\mathcal{H}$ and the existence and the uniqueness of the cscK metrics was pointed out by Donaldson in [23]. He also conjectured that $\mathcal{H}$ endowed with the Weil-Peterson type metric is geodesically convex and is a metric space. Chen [13] established the existence of $C^{1,1}$ geodesic segments (of bounded mixed derivatives) under smooth Dirichlet conditions and thus verified that the space of Kähler metrics is a metric space. Later, Blocki [7] proved that any $C^{1,1}$ geodesic segment has bounded Hessian when $(X \times R, \Omega_0)$ has nonnegative bisectional curvature. Phong-Sturm [45], Song-Zeltdich [53, 52, 54] approximated the $C^{1,1}$ geodesic by the Bergman geodesics in finite-dimensional symmetric spaces. Later Chen and Tian in [16] improved the partial regularity of the $C^{1,1}$ geodesic, then proved the uniqueness of the extremal metrics. Donaldson [24], Darvas-Lempert [21] and Lempert-Vivas [39] showed that a $C^{1,1}$ geodesic does not need to be smooth in general. On the other hand, the geodesic ray induced by the test configuration is constructed in Arezzo-Tian [1], Chen-Tang [17], Phong-Sturm [46, 47] and Phong-Sturm [46, 47]. The $C^{1,1}$ geodesic ray parallel to a given one is constructed in Chen [14] under the geometric condition "tamed by a bounded ambient geometry". We would like to remark that the existence of $C^{1,1}$ geodesics has been proved by Chen-He [15] in the space of volume forms on a Riemannian manifold, by P.-F. Guan-X. Zhang [33] in Sasakian manifolds and by B. Guan-Q. Li [30] in Hermitian manifolds.

In this paper, our aim is to construct the natural geodesics in the moduli space of all Kähler metrics singular along the divisor D for future study. Let us isolate now the concept, central to our aim, of Kähler cone metric.

Definition 1.1. Let X and $[\omega_0]$ as at the beginning of the paper, and let $D = \sum_{i=1}^{m}(1-\beta_i)V_i$ be a normal crossing, effective smooth divisor of X with $0 < \beta_i \leq 1$ for $1 \leq i \leq m$, where $V_i \subset X$ are irreducible hypersurfaces. Set $\beta := (\beta_1, \dots, \beta_m)$ and call the β_i's the *cone angles.* Given a point p in D, label a local chart (U_p, z^i) centered at p as *local cone chart* where $z^1, \dots z^k$ are the local defining functions of the hypersurfaces where p lies. A *Kähler cone metric* ω of cone angle $2\pi\beta_i$ along V_i, for $1 \leq i \leq m$, is a closed positive $(1,1)$ current and a smooth Kähler metric on the regular part $M := X \setminus D$. In a local cone chart U_p its Kähler form is quasi-isometric to the cone flat metric, which is

$$\omega_{cone} := \frac{\sqrt{-1}}{2}\sum_{i=1}^{k}\beta_i^2|z^i|^{2(\beta_i-1)}dz^i \wedge dz^{\bar{i}} + \sum_{k+1\leq j\leq n} dz^j \wedge dz^{\bar{j}}\,. \tag{1.2}$$

Let $\mathcal{H}_\beta$ be the space of Kähler cone metrics of cone angle $2\pi\beta_i$ along V_i in $[\omega_0]$. It is clear that when $\beta_i = 1$ for all i, $\mathcal{H}_\beta$ consists of all cohomologous smooth Kähler metrics on a compact Kähler manifold. Let s be a global meromorphic section of $[D]$. Let h be an Hermitian metric on $[D]$. It is shown in Donaldson [25] that, for sufficiently small $\delta > 0$,

$$\omega = \omega_0 + \delta\sum_{i=1}^{m}\frac{\sqrt{-1}}{2}\partial\bar{\partial}|s_i|_{h_\Lambda}^{2\beta_i} \tag{1.3}$$

is a Kähler cone metric. Moreover, ω is independent of the choices of ω_0, h_Λ, δ up to quasi-isometry; we call it model metric.

In this paper, we study the geometry of the space of Kähler cone metrics, in particular, the geodesic in $\mathcal{H}_\beta$. Now we clarify the concept of geodesic in $\mathcal{H}_\beta$. A *cone geodesic* is a curve segment $\varphi \in \mathcal{H}_\beta$ for $0 \leq t \leq 1$ which satisfies the natural generalization of the problem (1.1); i.e. we are requiring that $\omega_{\varphi(t)}$ is a Kähler cone metric for any $0 \leq t \leq 1$. In this article, we find the geometric boundary conditions which assure the existence and the uniqueness of the cone geodesic. Those lead to an appropriate choice of a subspace of $\mathcal{H}_\beta$. As we will show in Section 2, the geodesic equation leads to the Dirichlet problem of the HCMA equation with the boundary potentials of cone singularities. The Dirichlet problem of HCMA was studied intensively by many authors under various analytic boundary conditions (see [3, 18, 32, 44, 9]). In our particular environment, the underlying manifold is a product manifold and the curvature conditions on the background metrics play an important role as in the geometric-analysis problems (see the useful tricks we explain at the beginning of Section 3 and Remark 3.1).

The slight difference between our equation and the standard HCMA is that in our case the boundary values allow cone singularities. So the problem is how to choose the appropriate function spaces where the solutions lies. A possible function space could be the edge space. The corresponding elliptic theory is investigated by many authors (see Mazzeo [42], Melrose [43], Schulze [48] and references therein). In our environment, the problem is that the edge space is defined for manifolds without boundary; which is not our case. So, we do not use edge space in this paper. We overcome this problem by generalizing Donaldson's space to the boundary case (see Definition (2.3)), that is more natural for our geometric problem. However, it would be interesting to understand whether the edge space (with some modification) could be defined near the boundary and how to improve the regularity in such space. Finally, it is interesting to see that the cone geodesics are translated as solutions of the HCMA, and then the cone singularities on the boundary travel naturally to the interior of the domain. We hope this phenomenon will be helpful to understand the solution of the complex Monge-Ampère equation.

Now we specify the geometric conditions on the boundary metrics. (The space C^3_β is introduced in Definition 2.2.)

Definition 1.2. Assume D is disjoint union of smooth hypersurfaces and the cone angles β belong to the interval $(0, \frac{1}{2})$. Then, we denote as $\mathcal{H}^3_\beta$ the space of C^3_β ω_0-plurisubharmonic potentials. Moreover, we label as $\mathcal{H}_C \subset \mathcal{H}^3_\beta$ one of the following spaces;

$$\mathfrak{I}_1 = \{\varphi \in \mathcal{H}^3_\beta \text{ such that } \sup Ric(\omega_\varphi) \text{ is bounded}\};$$
$$\mathfrak{I}_2 = \{\varphi \in \mathcal{H}^3_\beta \text{ such that } \inf Ric(\omega_\varphi) \text{ is bounded}\}.$$

In general the Kähler cone metrics do not have bounded geometry. The Riemannian curvature of ω is bounded when the cone angle is less than $\frac{1}{2}$. We will compute that the Levi-Civita connection of the model cone metric defined in (1.3) under the cone coordinates (see (4.3)) is bounded when the cone angle is less than $\frac{2}{3}$. The curvature conditions of the boundary metrics are used to improve the regularities of the weak geodesics. The space $\mathcal{H}_C$ at least contains all Kähler-Einstein cone metrics with the cone angle between 0 and $\frac{1}{2}$ (see Proposition 6.7 in Brendle [8]). The further discussion on the properties of the subspace $\mathcal{H}_C$ will be in the forthcoming paper. In the present work, our main aim is to prove the following result (cf. Theorem 4.5).

Theorem 1.1. *Any two Kähler cone metrics in $\mathcal{H}_C$ are connected by a unique $C^{1,1}_\beta$ cone geodesic. More precisely, it is the limit under the $C^{1,1}_\beta$-norm by a sequence of $C^{2,\alpha}_\beta$ approximate geodesics.*

The notion of approximate geodesic is given in Lemma 6.2. As an application, we prove the following result.

Theorem 1.2. *$\mathcal{H}_C$ is a metric space.*

Concerning geodesics with weak regularity, we should compare the construction in Berndtsson's remarkable paper [4] with our result. It is easy to compute that the volume of the Kähler cone metric belongs to L^p with $p(\beta_i - 1) + 1 > 0$ for any $1 \leq i \leq k$. According to Kolodziej's theorem in [37], there exists a unique Hölder continuous ω_0-plurisubharmonic potential. Berndtsson [4] proved that given two bounded ω_0-plurisubharmonic potentials, there is a bounded geodesic connecting them. Then since the advantage of using the Ding functional (cf. Ding [22]) is that it requires less regularity of the potentials, as observed by Berndtsson, the convexity of the Ding functional along the bounded geodesic is applied to prove the uniqueness of Kähler-Einstein cone metrics (generalizing the Bando-Mabuchi uniqueness theorem [2]). However the cone geodesic we construct here has more regularity across the divisor in a subspace $\mathcal{H}_C$ which still contains the critical metrics. The regularity of the cone geodesic across the divisor is important not only to prove the metric structure as we show in this paper, but also to our further application on existence and uniqueness of cscK cone metrics.

Now we state an application of our main theorem to the smooth Kähler metrics with slightly less geometric conditions than the $C^{1,1}$ geodesic in Chen's theorem [13].

Corollary 1.3. *If the C^3 norm and Ricci curvature upper (or lower) bound of two Kähler potentials are uniformly bounded, then the geodesic connecting them has uniform $C^{1,1}$ bound.*

Now we describe the structure of our paper. In Section 2, we recall the notation and the function spaces introduced by Donaldson [25]. In particular, we define the boundary case. Then, we generalize the Riemannian structure to the space of Kähler cone metrics. The delicate part here is the growth rate near the divisor. In the function space introduced by Donaldson, we derive that the geodesic equation is a HCMA with cone singularities by integration by parts and we explain the construction of the initial metric for the continuity method.

In Section 3, we obtain the *a priori* estimates of the approximate Monge-Ampère equation. It is divided into several steps. The L^∞ estimate is derived from a cone version of the maximum principle and the super-solution of the linear equation obtained in Section 4. In order to find out the proper geometric global conditions, the interior Laplacian estimate is obtained using the techniques of Yau's second order estimate [56] and the Chern-Lu formula (see [19, 40, 55]). In order to prove the boundary Hessian estimate near the divisor, we can not use the the distance function as the barrier function which is introduced in Guan-Spruck [31], since we need a uniform estimate independent of the distance to the divisor. So we choose the auxiliary function by solving the linear equation provided by Section 4. We hope this method could have potentially further applications to the Monge-Ampère equation on manifold with boundary which arises in other geometric problems. In order to obtain the interior gradient estimate near the divisor, we carefully choose an appropriate test function near the divisor.

In Section 4, we solve the linearized equation and prove the $C^{2,\alpha}_\beta$ regularity of the approximate geodesic equation. Both the interior and the boundary Schauder estimates are of the general form. Note that the right hand side of the approximation equation (4.1) contains $\log \Omega^{n+1}$. When applying the Evans-Krylov estimate, we need to bound the first derivative of $\log \Omega^{n+1}$. We will show that it is bounded when the cone angle is less than $\frac{2}{3}$. Thus with these estimates, the existence and the uniqueness of the $C^{1,1}_\beta$ cone geodesic are proved. Moreover, the approximate geodesic is in $C^{2,\alpha}_\beta$. There is an direct application of the interior Schauder estimate to the regularity of the Kähler-Einstein cone metrics. When applying the Evans-Krylov estimate, it is necessary to bound the first derivative of the term on the right hand side of the Kähler-Einstein equation of cone singularities. We show that the gradient of this term is bounded when the cone angle is less than $\frac{2}{3}$. When the cone angle is less than $\frac{1}{2}$, Brendle [8] derived Calabi's third order estimate to prove the existence of Ricci flat Kähler cone metrics.

Section 5 contains the maximum principle and the Hölder continuity of the linearized equation. In particular, the weak Harnack inequality is used to prove the $C^{2,\alpha}_\beta$ regularity of the approximate geodesic equation.

In Section 6, we apply our cone geodesic to prove the metric structure of $\mathcal{H}_C$. Once we establish the $C^{1,1}_\beta$ regularity of cone geodesic, the proof of the metric structure is immediate.

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2. The space of Kähler cone metrics

In this section, we first introduce some pieces of notation and knowledge of Donaldson's program [25], which we will stick to in the remainder of the paper. Let U_p be a local cone chart as in Definition 1.1. Let $W : U_p \setminus D \to U_p \setminus D$ be the change of coordinates given by

(2.1)
$$W(z^1, \cdots, z^n) := (w^1 = |z^1|^{\beta_1 - 1} z^1, \cdots, w^k = |z^k|^{\beta_k - 1} z^k, z^{k+1}, \cdots, z^n) .$$

Now, for any $1 \leq i \leq k$ let $0 \leq \theta_i < 2\pi$, $z^i := \rho_i e^{\sqrt{-1}\theta_i}$ and $r_i := |z^i|^{\beta_i} = |w^i|$; meanwhile, for any $k+1 \leq j \leq n$ let $z^j := x^j + \sqrt{-1}y^j$. Then, let the polar coordinate transformation of $(w^1, \cdots, w^k, z^{k+1}, \cdots, z^n)$ be

$$P : (w^1, \cdots, w^k, z^{k+1}, \cdots, z^n) \to (r_1, \theta_1, \cdots, r_k, \theta_k, x^{k+1}, y^{k+1}, \cdots, x^n, y^n) .$$

Thus we obtain that the expression of the push-forward cone flat metric (1.3) is

(2.2)

$$((P \circ W)^{-1})_* g = \sum_{1\leq i\leq k} [dr_i^2 + \beta_i^2 r_i^2 d\theta_i^2] + \sum_{k+1\leq j\leq n} [(dx^j)^2 + (dy^j)^2] .$$

This flat metric is uniformly equivalent to the standard Euclidean metric. However, letting $\mu_i := \beta_i^{-1} - 1$, we have $w^i = r_i e^{\sqrt{-1}\theta_i} = |w^i|^{-\mu_i} z^i$; moreover, we define

$$\begin{aligned} \varepsilon_i &:= dr_i + \sqrt{-1}\beta_i r_i d\theta_i = \beta_i |w^i|^{1-\mu_i}(w^i)^{-1} dz^i \\ &= \beta_i \left[\left(1 + \frac{\mu_i}{2}\right) |w^i|(w^i)^{-1} dw^i + \frac{\mu_i}{2} |w^i|^{-1} w^i dw^{\bar{i}} \right] , \end{aligned}$$

and we notice that it is not a holomorphic 1-form, since $\partial_{w^{\bar{i}}} \varepsilon_i \neq 0$. Consequently, ε_i and dz^j merely form a local orthonormal basis of the $(1,0)$-forms.

Now we present the function spaces which are introduced by Donaldson in [25]. The Hölder space C^α_β consists in those functions f which are Hölder continuous with respect to a Kähler cone metric. Also, $C^\alpha_{\beta,0}$ denotes the subspace of those functions in C^α_β for which their limit is zero along V_i for any $1 \leq i \leq m$. The Hölder continuous $(1,0)$-forms, in local coordinates U_p, can be expressed as

$$\xi = f_i \varepsilon_i + f_j dz^j ,$$

where the Einstein notation is adopted, $f_i \in C^\alpha_0$ and $f_j \in C^\alpha$. Meanwhile, a Hölder $(1,1)$-form η in local coordinates U_p is of the shape

$$\eta = f_{i_1 \bar{i}_2} \varepsilon_{i_1} \varepsilon_{\bar{i}_2} + f_{i\bar{j}} \varepsilon_i dz^{\bar{j}} + f_{\bar{i}j} \varepsilon_{\bar{i}} dz^j + f_{j_1 \bar{j}_2} dz^{j_1} dz^{\bar{j}_2} ;$$

here the coefficients satisfy $f_{i\bar{j}}, f_{\bar{i}j} \in C^\alpha_0$ and $f_{i_1\bar{i}_2}, f_{j_1\bar{j}_2} \in C^\alpha$. Note that according to this Definition, for any Kähler cone metric $\omega \in C^\alpha_\beta$, around the point $p \in D$, we have a local normal coordinate such that $g_{ij}(p) = \delta_{ij}$.

Definition 2.1. The Hölder space $C^{2,\alpha}_\beta$ is defined by

$$C^{2,\alpha}_\beta = \{ f \mid f, \partial f, \partial\bar{\partial} f \in C^\alpha_\beta \} .$$

Note that the $C^{2,\alpha}_\beta$ space, since it concerns only the mixed derivatives, is different from the usual $C^{2,\alpha}$ space. The definition of higher order space $C^{k,\alpha}_\beta$ depends on the background metrics. In this paper, we use the flat cone metric ω_{cone} (1.2) to define $C^{k,\alpha}_\beta$. It is not hard to see that, by the quasi-isometric mapping W, $\partial\bar{\partial} f \in C^\alpha_\beta$ is equivalent to

$\frac{\partial^2}{\partial w^i \partial w^{\bar{j}}} \in C^\alpha$ for any $1 \le i,j \le n$ under the coordinate $\{w^i\}$. So we say the third derivative of a function belongs to C^α_β if

$$\frac{\partial^3}{\partial w^k \partial w^i \partial w^{\bar{j}}} f \in C^\alpha$$

for any $1 \le i,j,k \le n$. In particular,

Definition 2.2. The Hölder space C^3 is defined by

$$C^3_\beta = \{f | f \in C^{2,\alpha}_\beta \text{ and the third derivative of } f \text{ w.r.t } \omega_{cone} \text{ is bounded}\} .$$

Thus the higher order spaces are defined by induction on the index k. Now we postpone the discussion of the function space for a while, we will continue after introducing the product manifold where the geodesic equation is defined.

We then approach some considerations on the Riemannian geometry of the space of Kähler cone metrics. Recall that $\mathcal{H}^{2,\alpha}_\beta$ is the space of $C^{2,\alpha}_\beta$ ω_0 plurisubharmonic-functions. It is a convex, open set in $C^{2,\alpha}_\beta$. The tangent space of $\mathcal{H}^{2,\alpha}_\beta$ at a point φ is $C^{2,\alpha}_\beta$. We generalize the Donaldson [23], Mabuchi [41], Semmes [49] metric to $\mathcal{H}^{2,\alpha}_\beta$ by associating to $\varphi \in \mathcal{H}^{2,\alpha}_\beta$ and tangent vectors $\psi_1, \psi_2 \in T_\varphi \mathcal{H}^{2,\alpha}_\beta$, the real number

$$\int_M \psi_1 \cdot \psi_2 \omega^n_\varphi . \tag{2.3}$$

The definition makes sense for Kähler cone metrics, since the volume of the Kähler cone metrics is finite. Furthermore, we choose an arbitrary differentiable path $\varphi \in C^1([0,1], \mathcal{H}^{2,\alpha}_\beta)$ and along it, differentiable vector field $\psi \in C^1([0,1], C^{2,\alpha}_\beta)$. We thus define the following derivation of the vector field on $M = X \backslash D$

$$D_t \psi := \frac{\partial \psi}{\partial t} - (\partial \psi, \partial \frac{\partial \varphi}{\partial t})_{g_\varphi} . \tag{2.4}$$

We claim that (2.4) is the Levi-Civita connection of (2.3). The fact that (2.4) is torsion free comes from a point-wise computation on M. Thus, the claim will be accomplished after verifying the metric compatibility, which is done in Proposition 2.2. We first prove an integration by parts formula.

Lemma 2.1. *Assume that φ_1, $|\partial\varphi_1|_\omega$, $|\partial\varphi_2|_\omega$, $|\Delta\varphi_2|_{L^1(\omega)}$ are bounded. Then the following formula holds*

$$\int_M \varphi_1 \Delta \varphi_2 \omega^n = - \int_M (\partial\varphi_1, \partial\varphi_2)_\omega \omega^n .$$

Proof. Choose a cut-off function χ_ϵ which vanishes in a neighborhood of D. Then,

$$\int_M \chi_\epsilon \varphi_1 \Delta \varphi_2 \omega^n = - \int_M \chi_\epsilon (\partial\varphi_1, \partial\varphi_2)_\omega \omega^n - \int_M \varphi_1 (\partial\chi_\epsilon, \partial\varphi_2)_\omega \omega^n .$$

The convergence of the first two terms follows from the Lebesgue dominated convergence theorem. So, it suffices to find a χ_ϵ such that

$$\int_M |\partial\chi_\epsilon|_\omega \omega^n \to 0$$

as $\epsilon \to 0$. Choose

$$\chi_\epsilon := \chi\left(\frac{1}{\epsilon^2}\prod_i |s_i|^2\right),$$

where s_i are the defining functions of D_i and χ is a smooth non-decreasing function such that

$$\begin{cases} \chi = 0 & \text{in } [0,1] \\ 0 \le \chi \le 1 & \text{in } [1,2] \\ \chi = 1 & \text{in } [2,+\infty)\,. \end{cases}$$

Now,

$$|\partial\chi_\epsilon|_\omega \le \chi' \cdot \frac{1}{\epsilon^2}|s_i||s_i|^{1-\beta_i} = \frac{C}{\epsilon^2}|s_i|^{2-\beta_i}\,.$$

So, as $\epsilon \to 0$ we get in the cone chart

$$\begin{aligned}\int_M |\partial\chi_\epsilon|_\omega \omega^n &\le \int_{|s|=r}\int_0^{2\pi}\int_\epsilon^{2\epsilon} \frac{2\pi}{\epsilon^2}|r|^{2-\beta_i}|r|^{2(\beta_i-1)} r dr d\theta dz^2 \wedge \cdots \wedge dz^n \\ &\le \frac{2\pi}{\epsilon^2}\int_\epsilon^{2\epsilon} |r|^{1+\beta_i} dr \le C\epsilon^{\beta_i} \to 0\,.\end{aligned}$$

This completes the proof of the lemma. □

As an application of the above formula, we have

Proposition 2.2. *The connection* (2.4) *is compatible with the metric* (2.3).

Proof. We compute

$$\frac{1}{2}\frac{d}{dt}\int_M \psi^2\omega_\varphi^n = \frac{1}{2}\int_M (2\psi\psi' + \psi^2\Delta_\varphi\varphi')\omega_\varphi^n\,.$$

Since ψ^2, $|\partial(\psi^2)|_{g_\varphi}$, $|\partial\varphi'|_{g_\varphi}$, $\Delta_\varphi\varphi'$ are all bounded with respect to g_φ, we are allowed to apply Lemma 2.1 and we get

$$\frac{1}{2}\frac{d}{dt}\int_M \psi^2\omega_\varphi^n = \frac{1}{2}\int_M [2\psi\psi' - 2\psi(\partial\psi, \partial\varphi')_{g_\varphi}]\omega_\varphi^n\,.$$

This completes the proof of the proposition. □

Next, we derive the geodesic equation.

Proposition 2.3. *The geodesic equation satisfies the following equation on M point-wise*

$$\varphi'' - (\partial\varphi', \partial\varphi')_{g_\varphi} = 0\,. \tag{2.5}$$

Proof. Assume that $\varphi(t)|_0^1$ is a path from φ_0 to φ_1, and that $\varphi(s,t) \in C^1([0,1]\times[0,1], C^{2,\alpha}_\beta)$ is a 1-parameter variation of $\varphi(t)|_0^1$ with fixed endpoints. We minimize the length function

$$L(\varphi(s,t)) = \int_0^1 \sqrt{\int_M \left(\frac{\partial\varphi(s,t)}{\partial t}\right)^2 \omega_\varphi^n} \quad dt\ .$$

We are going to compute the variation of $\frac{\partial}{\partial s}L(\varphi(s,t))$; denote $\varphi' = \frac{\partial\varphi}{\partial t}$ and

$$E = \int_M \varphi'^2\omega_\varphi^n\ .$$

Then, using (2.4) and the compatibility property we get

$$\begin{aligned}\frac{\partial}{\partial s}L(\phi(s,t)) &= \int_0^1 \frac{1}{E}\int_M D_s\varphi'\cdot\varphi'\omega_\varphi^n dt = \int_0^1\frac{1}{E}\int_M D_t\frac{\partial}{\partial s}\varphi\cdot\varphi'\omega_\varphi^n dt\\ &= \int_0^1\frac{1}{E}\left[\frac{\partial}{\partial t}\int_M\frac{\partial}{\partial s}\varphi\cdot\varphi'\omega_\varphi^n - \int_M D_t\frac{\partial}{\partial s}\varphi\cdot\varphi'\omega_\varphi^n\right]dt\\ &= -\int_0^1\frac{1}{E}\int_M\frac{\partial}{\partial s}\varphi\cdot D_t\varphi'\omega_\varphi^n dt\ .\end{aligned}$$

The first term in the second line vanishes since the endpoints are fixed. So the geodesic condition reads

$$0 = \frac{\partial}{\partial s}L(\phi(s,t)) = -\int_0^1\frac{1}{E}\int_M\frac{\partial}{\partial s}\varphi\cdot D_t\varphi'\omega_\varphi^n dt$$

which implies that the geodesic equation is

$$D_t\varphi' \equiv 0 \text{ on } M\ .$$

□

Consider the cylinder

$$R = [0,1]\times S^1$$

and introduce the coordinate

$$z^{n+1} = x^{n+1} + \sqrt{-1}y^{n+1}$$

on R. Define

$$\varphi(z', z^{n+1}) = \varphi(z^1,\cdots,z^n,x^{n+1}) = \varphi(z^1,\cdots,z^n,t)$$

on the product manifold $X\times R$ and let π be the natural projection form $X\times R$ to X. We also denote

$$\begin{aligned} z &= (z', z^{n+1}) = (z^1,\cdots,z^n,z^{n+1})\ ,\\ \Omega_0 &= (\pi^{-1})^*\omega_0 + dz^{n+1}\wedge d\bar{z}^{n+1}\ ,\\ \Omega &= (\pi^{-1})^*\omega + dz^{n+1}\wedge d\bar{z}^{n+1}\ ,\\ \Psi &= \varphi - |z^{n+1}|^2\ .\end{aligned}$$

It is a matter of algebra to show that (2.5) could be reduced to a degenerate Monge-Ampère equation. A path $\varphi(t)$ with endpoints φ_0, φ_1 satisfies the geodesic equation (2.5) on X if and only if Ψ satisfies the following Dirichlet problem involving a degenerate complex Monge-Ampère equation

$$\begin{cases} \det(\Omega_{i\bar{j}}+\Psi_{i\bar{j}})=0 & \text{in } M\times R\,, \\ \Psi(z)=\Psi_0 & \text{on } X\times\partial R\,, \\ \sum_{1\le i,j\le n}(\Omega_{i\bar{j}}+\Psi_{i\bar{j}})dz^i dz^{\bar{j}}>0 & \text{in } X\times\{z^{n-1}\}\,. \end{cases} \tag{2.6}$$

Here the following Dirichlet boundary conditions Ψ_0 are satisfied
(2.7)

$$\begin{cases} \Psi_0(z',0)\doteq\Psi(z',\sqrt{-1}y^{n+1})=\varphi_0(z')-(y^{n+1})^2 & \text{on } X\times\{0\}\times S^1, \\ \Psi_0(z',1)\doteq\Psi(z',1+\sqrt{-1}y^{n+1})=\varphi_1(z')-1-(y^{n+1})^2 & \text{on } X\times\{1\}\times S^1. \end{cases}$$

Now we are given a $(n+1)$-dimensional Kähler manifold $\mathfrak{X}=X\times R$ with boundary; the given data of the Dirichlet problem are put on two disjoint copies of X. We also have a divisor $\mathfrak{D}=D\times R$, with D as in Definition (1.1), which intersects transversely the boundary. Let f_i be the local defining function of each irreducible analytic component V_i of $\mathfrak{D}$. Then the transition functions $\frac{f_i}{f_j}$ give a line bundle $[\mathfrak{D}]$ in $\mathfrak{X}$. Let s be a global meromorphic section of $[\mathfrak{D}]$. Let h_Λ be the Hermitian metric on $[\mathfrak{D}]$. There is a small positive δ such that, on $\mathfrak{X}$,

$$\Omega=\Omega_0+\delta\sum_{i=1}^m\frac{\sqrt{-1}}{2}\partial\bar\partial|s_i|^{2\beta_i}_{h_\Lambda} \tag{2.8}$$

is a Kähler cone metric (cf. (1.3)). Moreover, it is also independent of the choices Ω_0, h_Λ, δ up to quasi-isometry.

We can define the Hölder space C^3_β in the interior of $(\mathfrak{X},\mathfrak{D})$ as that one defined on (X,D). On the boundary, near a point p we choose a local holomorphic coordinate chart

$$\{U_p^+;z^i=x^i+iy^i\},1\le i\le 2n+2$$

centered at p. From the discussion above, we see that the boundary of $\mathfrak{X}$ is $x^{n+1}=0$. When U_p^+ does not intersect the divisor $\mathfrak{D}$, the Hölder space is defined in the usual way. So it is sufficient to define a new Hölder space in the coordinates which contain the points of the divisor. We first note that the solution of geodesic equation is independent of the variable y^{n+1}, so the partial derivative on the variable x^{n+1} is the same to the one on the variable z^{n+1}. Next, the quasi-isometric mapping W is still well defined in U_p^+ as follows,

$$W(z^1,\cdots,z^{n+1}):=(w^1=|z^1|^{\beta_1-1}z^1,\cdots,w^k=|z^k|^{\beta_k-1}z^k,z^{k+1},\cdots,z^{n+1})\,.$$

So we can define the Hölder space $C^\alpha_\beta(U_p^+)$ to be the set of functions which are Hölder continuous under $\{z^i\}_{i=1}^{n+1}$ with respect to a Kähler cone metric. Also, $C^\alpha_{\beta,0}(U_p^+)$ denotes the subspace of those functions in $C^\alpha_\beta(U_p^+)$ for which their limit is zero along V_i for any $1\le i\le m$.

The Hölder continuous $(1,0)$-forms, in local coordinates U_p^+, can be expressed as

$$\xi = \sum_i f_i \varepsilon_i + \sum_j f_j dz^j ,$$

where $f_i \in C_0^\alpha(U_p^+)$ and $f_j \in C^\alpha(U_p^+)$. Meanwhile, a Hölder $(1,1)$-form η in local coordinates U_p^+ is of the shape

$$\eta = f_{i_1\bar{i}_2}\varepsilon_{i_1}\varepsilon_{\bar{i}_2} + f_{i\bar{j}}\varepsilon_i dz^{\bar{j}} + f_{\bar{i}j}\varepsilon_{\bar{i}} dz^j + f_{j_1\bar{j}_2} dz^{j_1} dz^{\bar{j}_2} ;$$

here the coefficients satisfy $f_{i\bar{j}}, f_{\bar{i}j} \in C_0^\alpha(U_p^+)$ and $f_{i_1\bar{i}_2}, f_{j_1\bar{j}_2} \in C^\alpha(U_p^+)$. The Hölder space $C_\beta^{2,\alpha}(U_p^+)$ is similarly defined as

$$C_\beta^{2,\alpha}(U_p^+) = \{f \mid f, \partial f, \partial\bar{\partial} f \in C_\beta^\alpha(U_p^+)\} .$$

Then we use the flat cone metric ω_{cone} (1.2) to define the higher order space $C_\beta^{k,\alpha}(U_p^+)$. The boundary C^3 space is defined in the same manner.

Definition 2.3. The Hölder space $C^3(U_p^+)$ is defined by

$$C_\beta^3(U_p^+) = \{f | f \in C_\beta^{2,\alpha}(U_p^+) \text{ and the 3nd derivative of } f \text{ w.r.t } \omega_{cone} \text{ is bounded}\} .$$

Thus the higher order spaces are also defined by induction on the index k in the same way.

In order to apply the maximum principle, we require that the maximum point does not lie on the divisor. On a compact manifold, the technique is used by Jeffres in [36] to overcome this trouble. With the discussion above, we prove this technical auxiliary lemma in our product manifold $\mathfrak{X}$ with boundary. The following lemma will be used several times in this paper. The idea is to choose a appropriate κ, the exponent of the test function S, such that the gradient of S is larger than the gradient of the given function f. Meanwhile, on $\mathfrak{D}$ the value of S is zero, so f and $\tilde{f} = f + S$ have the same value on $\mathfrak{D}$. Then the value of $\tilde{f}(z)$ increases when the point z leaves $\mathfrak{D}$. That implies that $\tilde{f}$ must have maximum points outside $\mathfrak{D}$. We would like to point out an example such as Proposition 3.1 to show how to use it.

Lemma 2.4. *There is a positive constant κ such that $S = ||s||^{2\kappa}$ satisfies the following properties*

(1) $\frac{\sqrt{-1}}{2}\partial\bar{\partial} S \geq \kappa \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log ||s||^2 \geq -C\Omega$,
(2) *for any $a > 0$, when $2\kappa < a\beta$, $|\nabla^a S|^2_\Omega(z)$ goes to infinity as z approaches the divisor $\mathfrak{D}$.*

Proof. Since

$$\begin{aligned}\frac{\sqrt{-1}}{2}\partial\bar{\partial} S &= \frac{\sqrt{-1}}{2} S(\kappa\partial\bar{\partial}\log ||s||^2 + \partial\log S \wedge \bar{\partial}\log S)\\ &\geq \kappa\frac{\sqrt{-1}}{2} S\partial\bar{\partial}\log ||s||^2,\end{aligned}$$

and since $-\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log||s||^2$ is the curvature form of the line bundle under the Hermitian metric h, there is a constant C such that $\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log S \geq -C\Omega$. So we have

$$\frac{\sqrt{-1}}{2}S\partial\bar{\partial}\log S \geq -C\kappa S\Omega \geq -C\kappa\Omega.$$

In order to derive the second conclusion, we compute the first derivative of S along the singular direction. Choosing the basis e, we have $||s||^{2\kappa} = |z|^{2\kappa}||e||^{2\kappa}$, then the main term of $|\nabla^a S|^2_\Omega$ is

$$|z^1|^{4\kappa-2a+2a(1-\beta)}.$$

So it is sufficient to choose $2\kappa < a\beta$ to get that this main term becomes unbounded as z approaches $\mathfrak{D}$. Furthermore, when $\Psi \in C^a_\beta$, $|\nabla^a\Psi|_\Omega$ is bounded, so $\nabla^a S$ grows faster than $\nabla^a\Psi$ near $\mathfrak{D}$. □

In order to apply the continuity method, we first construct the starting metric of the solution path such that it satisfies the boundary conditions. Since Ψ_0 may not be convex along the direction $\frac{\partial}{\partial z^{n+1}}$, we have to extend Ψ_0 to the whole $\mathfrak{X}$ as follows. Let $\tilde{\Psi}_0$ be the line segment between the boundary Kähler cone potentials φ_0 and φ_1; namely, (cf. (2.7))

$$\tilde{\Psi}_0 = t\Psi_0(z',1) + (1-t)\Psi_0(z',0) + t + (y^{n+1})^2 = t\varphi_1 + (1-t)\varphi_0 .$$

Then we choose a function Φ which depends only on z^{n+1} such that

$$\begin{cases} \Phi(z^{n+1}) = 0 & \text{on } \partial\mathfrak{X}, \\ \Phi_{z^{n+1}\bar{z}^{n+1}} > 0 & \text{in } \mathfrak{X} . \end{cases}$$

We denote the new potential by

$$\Psi_1 := \tilde{\Psi}_0 + m\Phi. \tag{2.9}$$

Next we verify that Ψ_1 is a Kähler cone potential on $\mathfrak{X}$.

Proposition 2.5. *Suppose that $\varphi_0, \varphi_1 \in \mathcal{H}_\beta$. Then there exists a large number m such that*

$$\Omega_1 := \Omega + \frac{\sqrt{-1}}{2}\sum_{1\leq i,j\leq n+1}\partial_i\partial_{\bar{j}}\Psi_1 \tag{2.10}$$

is a Kähler cone metric on $(\mathfrak{X}, \mathfrak{D})$.

Proof. The local expression of Ω_{Ψ_1} is

$$\begin{aligned}&\Omega + \frac{\sqrt{-1}}{2}\sum_{1\leq i,j\leq n+1}\partial_i\partial_{\bar{j}}(\tilde{\Psi}_0 + m\Phi)\\ &= t\omega_{\varphi_1} + (1-t)\omega_{\varphi_0} + \frac{\sqrt{-1}}{2}(1 + m\partial_{n+1}\partial_{\overline{n+1}}\Phi)dz^{n+1}\wedge d\bar{z}^{n+1}\\ &+ \frac{1}{\sqrt{2}}(\partial_i\varphi_1 - \partial_i\varphi_0)dz^i dz^{\overline{n+1}} + \frac{1}{\sqrt{2}}(\partial_{\bar{i}}\varphi_1 - \partial_{\bar{i}}\varphi_0)dz^{\bar{i}}dz^{n+1} .\end{aligned}$$

We call $\omega_t := t\omega_{\varphi_1} + (1-t)\omega_{\varphi_0}$ the line segment and $\psi := \varphi_1 - \varphi_0$ the difference of the boundary Kähler cone potentials.

In order to show that Ω_{Ψ_1} is a Kähler cone metric on $\mathfrak{X}$, it suffices to verify two conditions; that it is positive on the regular part M and that Ω_{Ψ_1} is locally quasi-isometric to

$$\Omega_{cone} = \frac{\sqrt{-1}}{2}\sum_{i=1}^{k}(\beta_i^2|z^i|^{2(\beta_i-1)}dz^i\wedge dz^{\bar{i}}) + \sum_{i=k+1}^{n+1}(dz^i\wedge dz^{\bar{i}})\ .$$

Since the determinant of Ω_{Ψ_1} is

$$det(g_t)[1+m\Phi_{n+1,\overline{n+1}} - g_t^{i\bar{j}}\psi_i\psi_{\bar{j}}],$$

the former condition is true once we choose m large enough. The latter condition is verified as $\varphi_0, \varphi_1 \in \mathcal{H}_\beta$. □

3. A priori estimates

In this section, we derive uniform *a priori* estimates for the degenerate equation. With the same background as (2.6), we let $\mathfrak{M} = M\times R$ and recall that Ψ_1 in (2.9) is a Kähler potential in $\mathfrak{M}$, that is

$$\Omega_1 := \Omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Psi_1 > 0,$$

we consider the family of Dirichlet problems for $0\leq\tau\leq 1$,

$$\begin{cases} \det(\Omega_{i\bar{j}} + \Psi_{i\bar{j}}) = \tau e^{\Psi-\Psi_1}\det(\Omega_{i\bar{j}} + \Psi_{1i\bar{j}}) & \text{in } \mathfrak{M}\ , \\ \Psi(z) = \Psi_0 & \text{on } \partial\mathfrak{X}\ , \end{cases} \tag{3.1}$$

in the space $C^{2,\alpha}_\beta$. We will specify the conditions on Ψ_0 in each estimate.

Since the curvature conditions of the background metrics are required when we derive the a priori estimates, we explain an observation on how to choose appropriate background metrics. If we take Ω_1 as the background metric, we obtain an equivalent equation

$$\begin{cases} \det(\Omega_{1i\bar{j}} + \tilde{\Psi}_{i\bar{j}}) = \tau f e^{\tilde{\Psi}}\det(\Omega_{i\bar{j}}) = \tau e^{\tilde{\Psi}}\det(\Omega_{1i\bar{j}}) & \text{in } \mathfrak{M}\ , \\ \tilde{\Psi}(z) = 0 & \text{on } \partial\mathfrak{X}\ , \end{cases} \tag{3.2}$$

where

$$\tilde{\Psi} := \Psi - \Psi_1 \text{ and } f := \frac{\det(\Omega_{i\bar{j}} + \Psi_{1i\bar{j}})}{\det(\Omega_{i\bar{j}})}.$$

In general, given a Kähler cone potential Φ we could take

$$\Omega^A := \Omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Phi, \Psi^A := \Psi - \Phi, \Psi_1^A := \Psi_1 - \Phi, \Psi_0^A := \Psi_0 - \Phi.$$

The new family of Dirichlet problems becomes

$$\begin{cases} \det(\Omega^A_{i\bar{j}} + \Psi^A_{i\bar{j}}) = \tau e^{\Psi^A-\Psi^A_1}\det(\Omega^A_{i\bar{j}} + \Psi^A_{1i\bar{j}}) & \text{in } \mathfrak{M}\ , \\ \Psi(z) = \Psi^A_0 & \text{on } \partial\mathfrak{X}\ . \end{cases} \tag{3.3}$$

The above observation will be particularly useful when we will derive the *a priori* estimates later. Note that the right hand side of the equation is positive as long as τ is positive. When $\tau = 1$, Ψ_1 solves the equation. When τ is zero, (3.1) as well as (3.2) provide a solution of the degenerate equation (2.6).

3.1. L^∞ **estimate.** We will see later that the L^∞ estimate follows from the cone maximum principle (Lemma 5.1) and the global bounded weak solution (Proposition 5.9) provided in Section 5. Applying the logarithm on both sides of (3.2), we have

$$\log \frac{\det(\Omega_{1i\bar{j}} + \tilde{\Psi}_{i\bar{j}})}{\det(\Omega_{1i\bar{j}})} = \log \tau + \tilde{\Psi} \ . \tag{3.4}$$

Proposition 3.1. *(Lower bound of Ψ) For any point $x \in \mathfrak{X}$, the following estimate holds*

$$\Psi(x) \geq \Psi_1(x) \ .$$

Proof. We first apply the second conclusion of Lemma 2.4. Since $\tilde{\Psi} \in C^{2,\alpha}_\beta$, $|\nabla \tilde{\Psi}|^2_\Omega$ is bounded; while, choose $2\kappa < \beta$, then $|\nabla S|^2_\Omega(z)$ goes to infinity as z approaches $\mathfrak{D}$. So ∇S is larger than $\nabla \tilde{\Psi}$. Meanwhile, the value of $\tilde{\Psi}$ and $U = \tilde{\Psi} - \epsilon S$ are the same due to the choice of S. Therefore, $U = \tilde{\Psi} - \epsilon S$ achieves its minimum point p on the regular part $\mathfrak{M}$.

There are two cases, one when p is on the boundary $M \times \partial R$ and the other one when p is in the interior of $\mathfrak{M}$. In the first case, since p is on the regular part of the boundary, then the minimal value is just the boundary value. Thus the inequality holds automatically. Now we explain the second case. The equation (3.4) is rewritten as

$$\log \frac{(\Omega_1 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}(U + \epsilon S))^{n+1}}{\Omega_1^{n+1}} = \log \tau + \tilde{\Psi} \ . \tag{3.5}$$

At the point p the Hessian of U is non-negative $U_{i\bar{i}} \geq 0$; so, after diagonalizing Ω_1 and $\Omega_1 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}(U + \epsilon S)$ simultaneously, (3.5) implies

$$\tau e^{\tilde{\Psi}(p)} \Pi_{i=1}^{n+1} \Omega_{1i\bar{i}} \geq \Pi_{i=1}^{n+1} (\Omega_{1i\bar{i}} + \epsilon S_{i\bar{i}}) \geq (1 - \epsilon C)^{n+1} \Pi_{i=1}^{n+1} \Omega_{1i\bar{i}} \ ,$$

where, at the second inequality, we use the first conclusion of Lemma 2.4. Then we have

$$\tilde{\Psi}(p) \geq \log(1 - \epsilon C)^{n+1} \ .$$

Then for any $x \in \mathfrak{X}$, (1) in Lemma 2.4 implies

$$\begin{aligned} \tilde{\Psi}(x) &= U(x) + \epsilon S(x) \geq U(p) \\ &= \tilde{\Psi}(p) - \epsilon S(p) \geq \log(1 - \epsilon C)^{n+1} - \epsilon \ , \end{aligned}$$

which gives the lower bound of $\tilde{\Psi}$ as ϵ goes to zero. $\square$

Proposition 3.2. *(Upper bound of Ψ) For any point $x \in \mathfrak{X}$, the following estimate holds*

$$\Psi(x) \leq h(x) \ .$$

Proof. From (3.1) the solution is non-negative $\Omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Psi \geq 0$, after taking trace it implies

$$-\triangle\Psi \leq n+1 \ .$$

In order to obtain the lower bound, we then consider the linear equation

$$\begin{cases} \triangle h = -n-1 & \text{in } \mathfrak{M} \ , \\ h = \Psi_0 & \text{on } \partial\mathfrak{X} \ . \end{cases}$$

It is solvable by means of Propositions 5.6 and 5.9. Then the lemma follows from the weak maximum principle of cone metrics (Lemma 5.5). □

Remark 3.1. We could consider the family of equations with parameter $a \in \mathbb{R}$ as

$$\begin{cases} \det(\Omega_{1i\bar{j}} + \tilde{\Psi}_{i\bar{j}}) = \tau e^{a\tilde{\Psi}} \det(\Omega_{1i\bar{j}}) & \text{in } \mathfrak{M} \ , \\ \tilde{\Psi}(z) = 0 & \text{on } \partial\mathfrak{X} \ . \end{cases} \tag{3.6}$$

The approximate equation (3.2) is the former with $a = 1$. That is slightly different from the family considered by Chen [13] with $a = 0$. We would like to indicate that using the estimate in Section 5, the lower bound of the solution of Chen's approximate equation can be proved by applying the maximum principle with respect to the Kähler cone metric Lemma 5.5 to

$$\begin{cases} \det(\Omega_{1i\bar{j}} + \tilde{\Psi}_{i\bar{j}}) \leq \det(\Omega_1) & \text{in } \mathfrak{M} \ , \\ \tilde{\Psi} = 0 & \text{on } \partial\mathfrak{X} \ . \end{cases}$$

The upper and the lower bound of Ψ imply the boundary gradient estimate

$$\sup_{M\times\partial R} |\nabla\Psi|_\Omega \leq \sup_{\mathfrak{X}} |\nabla\Psi_1|_\Omega + \sup_{\mathfrak{X}} |\nabla h|_\Omega \ . \tag{3.7}$$

3.2. **Interior Laplacian estimate.** The content of the present subsection is the statement and proof of three different interior Laplacian estimates (Proposition 3.3).

We remark that in Lemma 3.4 below, we could choose different background metrics. As a result, constants would have different dependence on geometric quantities.

Proposition 3.3. *There are three constants C_i, for $i = 1, 2, 3$ such that*

$$\sup_{\mathfrak{X}}(n+1+\triangle\Psi) \leq C_i \sup_{\partial\mathfrak{X}}(n+1+\triangle\Psi) \ . \tag{3.8}$$

The constants respectively depend on

$C_1 = C_1(\inf Riem(\Omega), \sup \mathrm{Ric}(\Omega_1), \sup \mathrm{tr}_\Omega \Omega_1, \mathrm{Osc}\,\Psi, \mathrm{Osc}\,\Psi_1)$;

$C_2 = C_2(|Riem(\Omega_1)|_{L^\infty}, \sup \mathrm{tr}_\Omega \Omega_1, \sup \mathrm{tr}_{\Omega_1} \Omega, \mathrm{Osc}\,\Psi)$;

$C_3 = C_3(\sup Riem(\Omega), \inf \mathrm{Ric}(\Omega_1), \sup \mathrm{tr}_\Omega \Omega_1, \mathrm{Osc}\,\Psi, \mathrm{Osc}\,\Psi_1)$.

Remark 3.2. The estimates work for any given Kähler cone metric Ω.

We first consider the equation (3.1). We denote

$$F := \log \tau + \log f + \Psi - \Psi_1 = \log \frac{\det(\Omega_{i\bar{j}} + \Psi_{i\bar{j}})}{\det(\Omega_{i\bar{j}})} \ . \tag{3.9}$$

We calculate $\Delta'(n+1+\Delta\Psi)$ of our equation and explain later how to change the background metric.

Lemma 3.4. *The following formula holds*

$$\Delta'(n+1+\Delta\Psi) = g^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}} - \mathrm{tr}_\Omega \mathrm{Ric}(\Omega_1)$$
$$+ \Delta\Psi - \Delta\Psi_1 + g'^{k\bar{l}} R^{i\bar{j}}{}_{k\bar{l}} g'_{i\bar{j}} \ .$$

Proof. Since $g'_{i\bar{j}} = g_{i\bar{j}} + \Psi_{i\bar{j}}$, when we take $-\partial_k \partial_{\bar{l}}$ on both sides we get

$$-\partial_k \partial_{\bar{l}} g'_{i\bar{j}} = R_{i\bar{j}k\bar{l}} - \Psi_{i\bar{j}k\bar{l}} \ . \tag{3.10}$$

Since the Riemannian curvature is defined by

$$R'_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g'_{i\bar{j}} + g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}},$$

inserting the latter in (3.10) and taking the trace with respect to $g'^{k\bar{l}}$ and $g^{i\bar{j}}$ we have

$$g^{i\bar{j}} R'_{i\bar{j}} = g^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}} + g'^{k\bar{l}} R_{k\bar{l}} - g^{i\bar{j}} g'^{k\bar{l}} \Psi_{i\bar{j}k\bar{l}} \ . \tag{3.11}$$

Since

$$\Delta'(n+1+\Delta\Psi) = g'^{k\bar{l}} g^{i\bar{l}} \Psi_{i\bar{j}k\bar{l}} + g'^{k\bar{l}} R^{i\bar{j}}{}_{k\bar{l}} \Psi_{i\bar{j}} \ ,$$

inserting the latter in (3.11) we get

$$\Delta'(n+1+\Delta\Psi) = g^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}} - g^{i\bar{j}} R'_{i\bar{j}} + g'^{k\bar{l}} R_{k\bar{l}} + g'^{k\bar{l}} R^{i\bar{j}}{}_{k\bar{l}} \Psi_{i\bar{j}} \ .$$

Since (3.9) implies $R'_{i\bar{j}} = R_{i\bar{j}} - F_{i\bar{j}}$ we therefore have

$$\Delta'(n+1+\Delta\Psi) = g^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}} - S(\Omega) + \Delta F + g'^{k\bar{l}} R^{i\bar{j}}{}_{k\bar{l}} g'_{i\bar{j}} \ .$$

Then the lemma follows from the formula

$$\Delta F = \Delta(\log f + \Psi - \Psi_1) = -\mathrm{tr}_\Omega \mathrm{Ric}(\Omega_1) + \mathrm{S}(\Omega) + \Delta\Psi - \Delta\Psi_1 \ .$$

This completes the proof of the lemma. □

The following formula follows from the Schwarz inequality. See Yau [56], and Siu [51, page 73].

$$g^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}} \geq \frac{|\partial(n+1+\Delta\Psi)|^2}{n+1+\Delta\Psi} \ . \tag{3.12}$$

Lemma 3.5. *There is a constant C depending on* $\sup \mathrm{Ric}(\Omega_1)$, $\sup \mathrm{tr}_\Omega \Omega_1$, $\inf_{\mathrm{i}\neq\mathrm{k}} \mathrm{R}_{\mathrm{i}\bar{\mathrm{i}}\mathrm{k}\bar{\mathrm{k}}}$ *such that*

$$\Delta'(\log(n+1+\Delta\Psi)) \geq -C(1+\mathrm{tr}_{\Omega'}\Omega).$$

Proof. We compute

$$\Delta'(\log(n+1+\Delta\Psi)) = \frac{\Delta'(n+1+\Delta\Psi)}{n+1+\Delta\Psi} - \frac{|\partial(n+1+\Delta\Psi)|^2}{n+1+\Delta\Psi} .$$

Thus, by combining Lemma 3.4 with (3.12), we have

$$\begin{aligned}\Delta'(\log(n+1+\Delta\Psi)) &\geq \frac{-\mathrm{tr}_\Omega \mathrm{Ric}(\Omega_1) + \Delta\Psi - \Delta\Psi_1 + \mathrm{g}'^{\mathrm{k}\bar{\mathrm{l}}}\mathrm{R}^{\mathrm{i}\bar{\mathrm{j}}}{}_{\mathrm{k}\bar{\mathrm{l}}}\mathrm{g}'_{\mathrm{i}\bar{\mathrm{j}}}}{n+1+\Delta\Psi} \\ &\geq -C(1+\frac{1}{n+1+\Delta\Psi} + \mathrm{tr}_{\Omega'}\Omega) .\end{aligned}$$

Thus the lemma follows from $\frac{1}{n+1+\Delta\Psi} \leq \frac{1}{1+\Psi_{i\bar{i}}} \leq \mathrm{tr}_{\Omega'}\Omega$. □

Proof. (proof of constant C_1) Denote

$$Z := \log(n+1+\Delta\Psi) - K\Psi + \epsilon S ,$$

with K to be chosen. According to Lemma 2.4, with appropriate κ, the maximum point p of Z stays in the interior of $\mathfrak{M}$. Since $\Delta'\Psi = n+1-\mathrm{tr}_{\Omega'}\Omega$, and $\Delta' S \geq -C\mathrm{tr}_{\Omega'}\Omega$ (Lemma 2.4), then at p there holds

$$0 \geq \Delta' Z \geq -C(1+\mathrm{tr}_{\Omega'}\Omega) - \mathrm{K}(\mathrm{n}+1-\mathrm{tr}_{\Omega'}\Omega) - \epsilon \mathrm{C}\mathrm{tr}_{\Omega'}\Omega .$$

Now we choose K such that $-C+K-\epsilon C > 0$ to obtain the upper bound of $\mathrm{tr}_{\Omega'}\Omega(\mathrm{p})$. From the arithmetic-geometric-mean inequality we have

$$\begin{aligned}(n+1+\Delta\Psi)^{\frac{1}{n}} \cdot e^{-\frac{F}{n}} &= \left(\sum_{i=1}^{n+1} \frac{1}{\prod_{k=1,k\neq i}^{n+1}(1+\Psi_{k\bar{k}})}\right)^{\frac{1}{n}} \\ &\leq \sum_{k=1}^{n+1} \frac{1}{1+\Psi_{k\bar{k}}} = \mathrm{tr}_{\Omega'}\Omega .\end{aligned}$$

Since $F = \log\tau + \log f + \Psi - \Psi_1$, so, $n+1+\Delta\Psi$ is bounded from above at p depending on $\sup \mathrm{Ric}(\Omega_1)$, $\sup \mathrm{tr}_\Omega \Omega_1$, $\inf_{i\neq k} R_{i\bar{i}k\bar{k}}$, $\sup \Psi$, and $\inf \Psi_1$. For any $x \in \mathfrak{X}$, there holds $Z(x) \leq \sup_{\partial\mathfrak{X}} Z + Z(p)$. Hence,

$$\begin{aligned} n+1+\Delta\Psi &= e^{Z+K\Psi-\epsilon S} \\ &\leq e^{\sup_{\partial\mathfrak{X}} Z + Z(p) + K\sup\Psi} \\ &\leq \sup_{\partial\mathfrak{X}}(n+1+\Delta\Psi) e^{-K\inf_{\mathfrak{X}}\Psi_0 + 1 + Z(p) + K\sup\Psi} . \end{aligned} \tag{3.13}$$

This formula gives precisely the claimed inequality (3.8) for the first constant C_1. □

Proof. (proof of constant C_2) Now the same argument as in Lemma 3.4, applied to equation (3.2), gives the following formula

$$\Delta'(n+1+\Delta_1\tilde{\Psi}) = {g_1}^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} \partial_{\bar{l}} g'_{p\bar{j}} \partial_k g'_{i\bar{q}} - S(\Omega_1) + \Delta_1\tilde{\Psi} + g'^{k\bar{l}} {R_1}^{i\bar{j}}{}_{k\bar{l}} g'_{i\bar{j}} \ .$$

Then, still following an argument similar to that used in the first part of this subsection, we get a constant C which depends on $\sup S(\Omega_1)$, $\inf_{i\neq k} R_{i\bar{i}k\bar{k}}(\Omega_1)$, $\operatorname{Osc}\Psi$, such that

$$n+1+\Delta_1\tilde{\Psi} \leq C \sup_{\partial\mathfrak{X}}(n+1+\Delta_1\tilde{\Psi}) \ .$$

Since Ω and Ω_1 are L^∞ equivalent, we have

$$n+1+\Delta\Psi \leq C(\sup \mathrm{tr}_{\Omega_1}\Omega)(\sup \mathrm{tr}_{\Omega}\Omega_1)\cdot \sup_{\partial\mathfrak{X}}(\mathrm{n}+1+\Delta\Psi) \ . \tag{3.14}$$

This formula gives precisely the second constant C_2 for claimed inequality (3.8). Here the conditions $\inf Riem(\Omega_1)$ and $\sup S(\Omega_1)$ are bounded are equivalent to the L^∞ bound of the Riemannian curvature of Ω_1. □

Proof. (proof of constant C_3) Now we use the Chern-Lu formula (see [19, 40, 55]) to derive the second order estimate. We get the formula of

$$\mathrm{tr}_{\Omega'}\Omega = \mathrm{n}+1-\Delta'\Psi.$$

This following identity is interpreted as the energy identity of the harmonic map id between (M,g') to (M,g).

$$\Delta'(\mathrm{tr}_{\Omega'}\Omega) = \mathrm{R}'^{\mathrm{i}\bar{\mathrm{j}}}\mathrm{g}_{\mathrm{i}\bar{\mathrm{j}}} - \mathrm{g}'^{\mathrm{i}\bar{\mathrm{j}}}\mathrm{g}'^{\mathrm{k}\bar{\mathrm{l}}}\mathrm{R}_{\mathrm{i}\bar{\mathrm{j}}\mathrm{k}\bar{\mathrm{l}}} - \mathrm{g}^{\mathrm{i}\bar{\mathrm{j}}}\mathrm{g}'^{\mathrm{k}\bar{\mathrm{l}}}\mathrm{g}'^{\mathrm{p}\bar{\mathrm{q}}}\partial_{\bar{\mathrm{l}}}\mathrm{g}'_{\mathrm{p}\bar{\mathrm{j}}}\partial_{\mathrm{k}}\mathrm{g}'_{\mathrm{i}\bar{\mathrm{q}}} \ . \tag{3.15}$$

The Schwarz inequality implies

$$g'^{k\bar{l}}\partial_k g'^{i\bar{j}} g_{i\bar{j}} \partial_{\bar{l}} g'^{p\bar{q}} g_{p\bar{q}} \leq -(g'^{k\bar{l}} g'^{p\bar{j}} g_{i\bar{j}} \partial_{\bar{l}} g'_{p\bar{q}} \partial_k g'^{i\bar{q}})(g'^{i\bar{j}} g_{i\bar{j}}) \ . \tag{3.16}$$

Now we use the equation (3.4).

Lemma 3.6. *The following formula holds*

$$\triangle'(\log tr_{\Omega'}\Omega) \geq -(n+1) - C(\mathrm{tr}_{\Omega'}\Omega),$$

with C that depends on $\inf \mathrm{Ric}(\Omega_1)$*,* $\sup_{i\neq k} R_{i\bar{i}k\bar{k}}(\Omega)$*,* $\sup \mathrm{tr}_\Omega\Omega_1$*.*

Proof. We apply (3.15) and (3.16) to obtain

$$\begin{aligned} \triangle'[\log tr_{\Omega'}\Omega] &= \frac{\triangle'(tr_{\Omega'}\Omega)}{tr_{\Omega'}\Omega} - \frac{g'^{k\bar{l}}\partial_k g'^{i\bar{j}} g_{i\bar{j}} \partial_{\bar{l}} g'^{p\bar{q}} g_{p\bar{q}}}{(tr_{\Omega'}\Omega)^2} \\ &\geq \frac{R'^{i\bar{j}} g_{i\bar{j}} - g'^{i\bar{j}} g'^{k\bar{l}} R_{i\bar{j}k\bar{l}}}{tr_{\Omega'}\Omega} \ . \end{aligned}$$

From (3.4) we have

$$\begin{aligned} \mathrm{Ric}' &= \mathrm{Ric}(\Omega) - \frac{\sqrt{-1}}{2}\partial\bar{\partial}F \\ &= \mathrm{Ric}(\Omega) + \mathrm{Ric}(\Omega_1) - \mathrm{Ric}(\Omega) - \Omega' + \Omega_1 \ , \end{aligned}$$

then $\mathrm{Ric}' \geq (\inf \mathrm{Ric}(\Omega_1)+1)\Omega_1 - \Omega'$ and

$$\begin{aligned} R'^{i\bar{j}} g_{i\bar{j}} &\geq (\inf \mathrm{Ric}(\Omega_1)+1) g'^{i\bar{l}} g'^{k\bar{j}} g_{1,k\bar{l}} g_{1,i\bar{j}} - g'^{i\bar{l}} g'^{k\bar{l}} g'_{k\bar{l}} g_{i\bar{j}} \\ &\geq -C(\mathrm{tr}_{\Omega'}\Omega)^2 \cdot (\mathrm{tr}_{\Omega}\Omega_1)^2 - (\mathrm{n}+1)\mathrm{tr}_{\Omega}\Omega_1 \ , \end{aligned}$$

where C is a positive constant depending on $\inf \mathrm{Ric}(\Omega_1)$. Then we have

$$\Delta'(\log \mathrm{tr}_{\Omega'}\Omega) \geq -(\mathrm{n}+1) - \mathrm{C}(\mathrm{tr}_{\Omega'}\Omega) \ ,$$

where C depends on $\inf \mathrm{Ric}(\Omega_1)$, $\sup_{i\neq k} R_{i\bar{i}k\bar{k}}$, $\sup \mathrm{tr}_{\Omega}\Omega_1$. This completes the proof of the lemma. □

Consider $Z_1 := \log \mathrm{tr}_{\Omega'}\Omega - \mathrm{C}'\Psi + \epsilon \mathrm{S}$, such it has a maximum point p which stays away from $\mathfrak{D}$, and with C' to be chosen. Then

$$\Delta' Z_1 \geq -(n+1) - C\mathrm{tr}_{\Omega'}\Omega - \mathrm{C}'((\mathrm{n}+1) - \mathrm{tr}_{\Omega'}\Omega) - \mathrm{C}\epsilon \mathrm{tr}_{\Omega'}\Omega \ .$$

Now we choose C' such that $C' - C - C\epsilon > 0$ and we have at p, $\mathrm{tr}_{\Omega'}\Omega \leq \mathrm{C}$. In the same vein as the first part of the subsection we compute that for any $x \in \mathfrak{X}$ there holds

$$\begin{aligned} \log \mathrm{tr}_{\Omega'}\Omega(\mathrm{x}) &= Z_1(x) + C'\Psi - \epsilon S + \sup_{\partial \mathfrak{X}} \log \mathrm{tr}_{\Omega'}\Omega \\ &\leq Z_1(p) + C' \sup \Psi + \sup_{\partial \mathfrak{X}} \log \mathrm{tr}_{\Omega'}\Omega \end{aligned}$$

Using the arithmetic-geometric-mean inequality we have

$$(\mathrm{tr}_{\Omega}\Omega')^{\frac{1}{\mathrm{n}}} \leq \mathrm{tr}_{\Omega'}\Omega \mathrm{e}^{\frac{\mathrm{F}}{\mathrm{n}}} \leq \mathrm{C} \sup_{\partial \mathfrak{X}} \mathrm{tr}_{\Omega'}\Omega \ , \tag{3.17}$$

where C depends on $\inf \mathrm{Ric}(\Omega_1)$, $\sup_{i\neq k} R_{i\bar{i}k\bar{k}}(\Omega)$, $\sup \mathrm{tr}_{\Omega}\Omega_1$, $\mathrm{Osc}\,\Psi$, $\inf \Psi_1$. This formula gives precisely the third constant of formula (3.8). □

We could choose Ω_1 as the background metric and repeat the estimate, but it would not provide more information. The three constants C_i are determined by the formulas (3.13), (3.14) and (3.17), respectively. This concludes the proof of Proposition 3.3.

3.3. **Boundary Hessian estimate.** The boundary Hessian estimate for real and complex Monge-Ampère equation is developed in [9, 35, 31, 29, 13]. The difficulty that arises in our problem is the estimate near the singular varieties V_i. The distance function can not be used in our problem, since we need the uniform estimate which is independent of the distance to the divisor $\mathfrak{D}$. We overcome this difficulty by multiplying singular terms with proper weight and using the linear theory developed in Section 5 to construct an appropriate barrier function which is independent of the distance function.

Proposition 3.7. *The following boundary estimate holds*

$$\sup_{X\times\partial R} |\sqrt{-1}\partial\bar{\partial}\Psi|_{\Omega} \leq C(\sup_{\mathfrak{X}} |\partial\Psi|_{\Omega} + 1) \ .$$

The constant C depends on $|\partial \tilde{g}_{1\alpha\bar{\beta}}|$, $|\Psi|$, $|\partial\Psi_1|_{\Omega}$, $|\partial\Psi_0|_{\Omega}$, $|\partial\bar{\partial}\Psi_0|$.

Proof. Fix a point $p \in M \times \partial R$, and consider $U_p \subset M \times R$ an open neighborhood of p. Recall that we denote by Ψ an *a priori* solution of the equation

$$\det(\Omega_{i\bar{j}} + \Psi_{i\bar{j}}) = \tau f e^{\Psi - \Psi_1} \det(\Omega_{i\bar{j}}) ,$$

whose boundary values are given by the datum Ψ_0. The tangent-tangent term of the boundary Hessian estimate follows from the boundary value directly. Since the boundary is flat, the normal-normal term follows from the construction of the approximate geodesic equation

$$[\varphi'' - (\partial\varphi', \partial\varphi')_{g_\varphi}] \det \omega_\varphi = \Omega_\Psi^{n+1} = \tau e^{\Psi - \Psi_1} \det(\Omega_{1i\bar{j}}) ,$$

i.e.

$$\left| \frac{\partial}{\partial z^{n+1}} \frac{\partial}{\partial \overline{z^{n+1}}} \Psi \right|_{\Omega; X \times \partial R} \leq \sum_i \left| \frac{\partial}{\partial z^{n+1}} \frac{\partial}{\partial z^{\bar{i}}} \Psi \right|_{\Omega; X \times \partial R} + C . \tag{3.18}$$

The constant C depends on $|\Psi_1|$, $|\Psi_0|$, $|\partial\bar{\partial}\Psi_0|_\Omega$, and $\det(\Omega_{1i\bar{j}})$. The quantity $\det(\Omega_{1i\bar{j}})$ depends on the boundary value and the chosen function Φ in Proposition 2.5. Then the aim of the present subsection is to derive the mixed tangent-normal estimate on the boundary.

We put

$$\triangle' := \sum_{\alpha,\beta=1}^{n+1} g'^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} .$$

The elliptic operator $\triangle'$ allows the use of the maximum principle in Section 5.1.

Our idea is to construct a barrier function and apply the maximum principle locally in a small neighborhood of the point $p \in M \times \partial R$. Since the second order derivatives of Ψ blow up near the singular points where $\mathfrak{D}$ intersects $X \times \partial R$, we need to prove that the estimates do not depend on the choice of the diameter of the small neighborhood U_p.

Let us suppose that the open neighborhood $U_p \subset X \times R$ is a coordinate chart near p (cf. Definition 1.1) for the first n variables; moreover, the coordinate $z^{n+1} := x + \sqrt{-1}y$ in U_p locally parametrizes the Riemann surface R. Next, let us define the function $v : U_p \to \mathbb{R}$ as

$$v := (\Psi - \Psi_1) + sx - Nx^2 , \tag{3.19}$$

where N, s are constants which depend only on $M \times R$, the background metric g and the datum Ψ_0, and they will be determined later in (3.21) and (3.22) respectively. Also, let us fix the small neighborhood of the origin $\Omega_\delta := (M \times R) \cap B_\delta(0) \subset U_p$ with small radius $\delta < 1$. We require that Ω_δ does not intersect $\mathfrak{D}$. We will show that the estimate does not depend on the choice of δ.

We first prove the following lemma.

Lemma 3.8. *The following inequalities hold*

$$\begin{cases} \triangle' v \leq -\frac{\epsilon}{4}\left(1+\sum_{\alpha,\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\alpha\bar\beta}\right) & \text{in } \Omega_\delta \\ v \geq 0 & \text{on } \partial\Omega_\delta\ , \end{cases} \tag{3.20}$$

where $\epsilon > 0$ is a constant depending on the lower bound of Ω_{Ψ_0}.

Proof. By means of the equation (3.19) and the linearity of $\triangle'$, let us first consider the term $\triangle'(\Psi - \Psi_1)$. Here the remark to do is that, as the metric $g_{\alpha\bar\beta} + \Psi_{1\alpha\bar\beta}$ is L^∞ equivalent to $g_{\alpha\bar\beta}$ in $X \times R$, then we can find a uniform constant ϵ such that $g_{\alpha\bar\beta} + \Psi_{1\alpha\bar\beta} > \epsilon g_{\alpha\bar\beta}$ holds point-wise in Ω_δ (could be in the whole $\mathfrak{X}$). Notice that the lower bound of Ω_{Ψ_1} depends on the lower bound of Ω_{Ψ_0}. We conclude, using the remark, that just by definition there holds

$$\begin{aligned}\triangle'(\Psi-\Psi_1) &= \sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta}\left[\left(g_{\alpha\bar\beta}+\Psi_{\alpha\bar\beta}\right)-\left(g_{\alpha\bar\beta}+\Psi_{1\alpha\bar\beta}\right)\right] \\ &= n+1-\sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\Psi_1\alpha\bar\beta} \leq n+1-\epsilon\sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\alpha\bar\beta}\ .\end{aligned}$$

It is clear that $\triangle' x = 0$ and $\triangle' x^2 = 2g'^{(n+1)\overline{n+1}}$. Thus, we have

$$\begin{aligned}\triangle' v &= \triangle'(\Psi-\Psi_1) + s\triangle' x - 2Ng'^{(n+1)\overline{n+1}} \\ &\leq n+1-\epsilon\sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\alpha\bar\beta} - 2Ng'^{(n+1)\overline{n+1}} \\ &= n+1+\left(-\frac{\epsilon}{2}\right)\sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\alpha\bar\beta} - 2Ng'^{(n+1)\overline{n+1}} - \frac{\epsilon}{2}\sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\alpha\bar\beta}\ .\end{aligned}$$

Without loss of generality we can prove the inequality in the local normal coordinate such that, at the origin, there holds $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$. We have, at the origin,

$$\begin{aligned}& Ng'^{(n+1)\overline{n+1}} + \frac{\epsilon}{4}\sum_{\alpha,\bar\beta=1}^{n+1} g'^{\alpha\bar\beta} g_{\alpha\bar\beta} \\ &= \left(N+\frac{\epsilon}{4}\right)\frac{1}{1+\Psi_{(n+1)\overline{n+1}}} + \frac{\epsilon}{4}\sum_{j=1}^{n}\frac{1}{1+\Psi_{j\bar j}} \\ &\geq (n+1)\left[\left(N+\frac{\epsilon}{4}\right)\cdot\left(\frac{\epsilon}{4}\right)^n \prod_{\alpha=1}^{n+1}\frac{1}{1+\Psi_{\alpha\bar\alpha}}\right]^{\frac{1}{n+1}}\ .\end{aligned}$$

Since, still at the origin, there holds

$$\prod_{\alpha=1}^{n+1}\frac{1}{1+\Psi_{\alpha\bar\alpha}} = \frac{\Omega^{n+1}}{\Omega_\Psi^{n+1}} = \frac{1}{\tau f}e^{\Psi_1-\Psi} \geq e^{\inf(\Psi_1-\Psi)}\frac{1}{\sup f}\ ,$$

then we choose the constant N large enough so that

(3.21)
$$-2(n+1)\left[\left(N+\frac{\epsilon}{4}\right)\left(\frac{\epsilon}{4}\right)^n e^{\inf(\Psi_1-\Psi)}\frac{1}{\sup f}\right]^{\frac{1}{n+1}}+n+1<-\frac{\epsilon}{4}\ .$$

Here N depends on $\inf(\Psi_1-\Psi)$, $\sup f=\sup\frac{\Omega_1^{n+1}}{\Omega^{n+1}}$ and ϵ.

To fully achieve the claim (3.20), we have to verify the condition on $\partial\Omega_\delta$. On $\partial\Omega_\delta\cap\partial(M\times R)$, there holds $v=0$. On $\partial\Omega_\delta\cap\mathrm{Int}\,(M\times R)$, there holds, since $\Psi\geq\Psi_1$,

$$v\geq(s-Nx)x\geq(s-N\delta)x.$$

So we choose $s=2N$ such that

$$(s-N\delta)x\geq 0. \tag{3.22}$$

This completes the proof of the lemma. □

Now, we come to construct the auxiliary function u. We construct a nonnegative boundary value ϕ such that ϕ only vanishes on the point p. For example, $\phi=\Psi_0-\Psi_0(p)+e^{|\Psi_0-\Psi_0(p)|}-1$. Then we solve the equation $\triangle_g u_\parallel=-n-1$ with the boundary value ϕ. According to the maximum principle for the cone metrics (cf. Proposition 5.5), we have $u_\parallel\geq 0$. Meanwhile, we choose a smooth nonnegative function $u_\perp$ of z^{n+1} monotonic along $\frac{\partial}{\partial z^{n+1}}$ such that it vanishes on the boundary and strictly larger than $u_\parallel+1$ in the interior of $\mathfrak{X}$, since $u_\parallel$ is bounded. Now, we define the function u by adding up $u_\parallel$ and $u_\perp$.

We need to change the variables via the map W defined at (4.3), extended as the identity on the variable z^{n+1}; we mark functions and operators transformed under W with ˜ on the top. Finally, under W coordinate functions become, for $1\leq i\leq n$, $w^i=x^i+\sqrt{-1}y^i$; then, we define $D_i:=\frac{\partial}{\partial x^i}$, for $1\leq i\leq 2n$. With the above notations, we define the function $h:U_p\to\mathbb{R}$ as

$$h:=\lambda_1\tilde{v}+\lambda_2\tilde{u}+\lambda_3\cdot D_i(\tilde{\Psi}-\tilde{\Psi}_1)\ ,$$

for one fixed $1\leq i\leq n$ and three constants λ_1, λ_2 and λ_3 determined below.

We emphasize that till the end of the subsection, the index $1\leq i\leq n$ is fixed; we recall that the cone angle β_i is equal to one for the directions corresponding to $k+1\leq i\leq n$.

We notice that at the origin (or point p), the value of h is zero. We define ρ_i as the distance from p to the divisor only along the coordinate w^i. We shrink Ω_δ to be the set containing such points whose distance to p less than half the distance from p to D. So, on $\partial\Omega_\delta\cap\partial(M\times R)$ there holds $\frac{\rho_i}{2}\leq|w^i|\leq 2\rho_i$ and $\tilde{u}\geq 1$; then, letting λ_3 be the smallest eigenvalue of the inverse matrix of $W_*\Omega$, there holds for $q\in\partial\Omega_\delta\cap$

Int $(M \times R)$

$$\begin{aligned} h(q) &\geq \lambda_2 \tilde{u}(q) - \lambda_3 |D_i(\tilde{\Psi} - \tilde{\Psi}_1)(q)| \\ &\geq \lambda_2 - C|\partial(\Psi - \Psi_1)(q)|_\Omega \\ &\geq 0\,, \end{aligned}$$

where the last inequality is true provided $\lambda_2 = 1 + C|\partial(\Psi - \Psi_1)(q)|_\Omega$ with C that depends on background metric Ω. Let us come to analyze $\tilde{\triangle}' h$.

Lemma 3.9. *There exist λ_1 depending on λ_2, $\lambda_4 = |D_i \log \tilde{\Omega}_1^{n+1}| + |\partial\Psi|_\Omega + |\partial\Psi_1|_\Omega$, $\lambda_5 = |D_i \tilde{g}_{1\alpha\bar{\beta}}|_\Omega$, such that*

$$\tilde{\triangle}' h \leq 0.$$

Proof. By our preliminary work, we read off (3.20) an estimate for $\triangle' v = \tilde{\triangle}' \tilde{v}$. About $\tilde{\triangle}' \tilde{u}$, we compute

$$\tilde{\triangle}' \tilde{u} = \triangle' u = \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} u_{\alpha\bar{\beta}} \leq C \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\,, \tag{3.23}$$

where C is a constant depending on Ψ_0 and $u_\perp$. Finally, as Ψ is a solution to $\Omega_\Psi^{n+1} = e^F \Omega^{n+1}$ with $F = \log\tau + \log f + \Psi - \Psi_1$, we differentiate this equation under coordinate w^i, and we get

$$\begin{aligned} &\tilde{\triangle}' D_i(\tilde{\Psi} - \tilde{\Psi}_1) \\ &= \sum_{\alpha,\beta=1}^{n+1} D_i \log \tilde{\Omega}_1^{n+1} + D_i \tilde{\Psi} - D_i \tilde{\Psi}_1 - \sum_{\alpha,\beta=1}^{n+1} (\tilde{g})'^{\alpha\bar{\beta}} D_i \tilde{g}_{1\alpha\bar{\beta}}\,. \end{aligned}$$

We end up with the estimate for $\tilde{\triangle}' D_i(\tilde{\Psi} - \tilde{\Psi}_0)$,

$$\tilde{\triangle}' D_i(\tilde{\Psi} - \tilde{\Psi}_1) \leq \lambda_4 + \lambda_5 \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\,. \tag{3.24}$$

There, $\lambda_4 := |\partial \log \tilde{\Omega}_1^{n+1}| + |\partial\Psi|_\Omega + |\partial\Psi_1|_\Omega$, $\lambda_5 := |\partial \tilde{g}_{1\alpha\bar{\beta}}|_\Omega$. We conclude the following estimate for $\tilde{\triangle}' h$ by means of (3.23) and (3.24);

$$\begin{aligned} &\tilde{\triangle}' h = \lambda_1 \tilde{\triangle}' \tilde{v} + \lambda_2 \tilde{\triangle}' \tilde{u} + \lambda_3 \tilde{\triangle}' D_i(\tilde{\Psi} - \tilde{\Psi}_1) \\ &\leq -\lambda_1 \frac{\epsilon}{4} \left(1 + \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right) + \lambda_2 \cdot C \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} + \left[\lambda_4 + \lambda_5 \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right] \\ &\leq \left[-\frac{\epsilon}{4}\lambda_1 + \lambda_2 \cdot C + \lambda_4 + \lambda_5\right] \cdot \left(1 + \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right) < 0\,, \end{aligned}$$

after choosing λ_1 properly. □

(Completion of the proof of Proposition 3.7.) To summarize, we get $h \geq 0$ on $\partial\Omega_\delta$ and $\tilde{\triangle}' h < 0$ in Ω_δ in the weak sense. So, by the weak maximum principle, we get that $h \geq 0$ in Ω_δ. Since $h(0) = 0$, then we have (recall $z^{n+1} = x + \sqrt{-1}\, y$)

$$\frac{\partial h}{\partial x}(0) \geq 0, \quad \frac{\partial h}{\partial y}(0) \geq 0 \ .$$

In particular, we compute

$$\frac{\partial h}{\partial x} = \lambda_1 \frac{\partial(\tilde{\Psi} - \tilde{\Psi}_1)}{\partial x} + s - 2Nx + \lambda_2 \frac{\partial \tilde{u}}{\partial x} + \lambda_3 \frac{\partial}{\partial x} D_i(\tilde{\Psi} - \tilde{\Psi}_1) \ ,$$

which leads to

$$\lambda_3 \frac{\partial}{\partial x} D_i(\tilde{\Psi} - \tilde{\Psi}_1)(0) \geq -s - \lambda_1 \frac{\partial(\tilde{\Psi} - \tilde{\Psi}_1)}{\partial x}(0) - \lambda_2 \frac{\partial \tilde{u}}{\partial x}(0) \ .$$

Combining the above inequality with $\frac{\partial}{\partial y} D_i(\tilde{\Psi} - \tilde{\Psi}_1) = 0$, and adding the inequalities, we get that for any $1 \leq i \leq n$

$$\frac{\partial}{\partial z^{n+1}} \frac{\partial}{\partial w^{\bar{i}}} (\tilde{\Psi} - \tilde{\Psi}_1)(0) \geq -C \ ,$$

where C depends on λ_1, λ_2, $|\partial\Psi|_\Omega$, $|\partial\Psi_1|_\Omega$ and $|\partial u|_\Omega$. We repeat the same argument for $D_i = -\frac{\partial}{\partial x^i}$ and for $D_i = -\frac{\partial}{\partial y^i}$ and we conclude that the tangent-normal derivative is bounded, for $1 \leq i \leq n$, by

(3.25)
$$\left| \frac{\partial}{\partial z^{n+1}} \frac{\partial}{\partial z^{\bar{i}}} \Psi \right|_\Omega (0) = \left| \frac{\partial}{\partial z^{n+1}} \frac{\partial}{\partial w^{\bar{i}}} \Psi \right| (0) \geq \left| \frac{\partial}{\partial z^{n+1}} \frac{\partial}{\partial w^{\bar{i}}} \Psi_0 \right| (0) + C \ ,$$

where again C depends on λ_1, λ_2, $|\partial\Psi|_\Omega$, $|\partial\Psi_1|_\Omega$ and $|\partial u|_\Omega$. Note from the construction of Ψ_1 that the derivatives of Ψ_1 are controlled by the corresponding derivatives of Ψ_0. As (3.25) clearly coincides with (3.18), this completes the proof of the proposition. □

3.4. **Interior gradient estimate.** We directly calculate the norm of the gradient to obtain the differential inequality in Proposition 3.14. Gradient estimates were obtained by Cherrier and Hanani [20, 34] for Hermitian manifolds and later by Blocki [6] for the Kähler case. Since (3.2) has singularity along the divisor, in order to apply the maximum principle, we need to choose an appropriate test function near the divisor.

We define the following functions, where $\epsilon > 0$ and $\gamma : \mathbb{R} \to \mathbb{R}$ are not yet specified

$$B := |\partial\Psi|^2 = g^{i\bar{j}} \Psi_i \Psi_{\bar{j}} \ , \quad D := |\partial\Psi|^2_{g'} = g'^{k\bar{l}} \Psi_k \Psi_{\bar{l}},$$
$$Z := \log B - \gamma(\Psi), \quad K := Z - \sup_{\mathfrak{D}} Z + \epsilon S \ .$$

Consider κ and $S = ||s||^{2\kappa}$ as in Lemma 2.4. Recall that $0 < \alpha < \mu = \beta^{-1} - 1$.

Lemma 3.10. *Suppose that $\Psi \in C^{2,\alpha}_\beta$ with $\alpha > 0$ and $\beta_i < \frac{1}{2}$, for all $1 \le i \le k$. Then for any κ which satisfies $\beta_i \le 2\kappa < (1+\alpha)\beta_i$, for all $1 \le i \le k$, the function $K = Z - \sup_{\mathfrak{D}} Z + \epsilon S$ achieves its maximum away from $\mathfrak{D}$ and $|\partial S|_\Omega \le C$.*

Proof. The second claim follows directly from the formula (cf. Lemma 2.4)

$$g^{1\bar{1}}_\Omega \frac{\partial}{\partial z^1} S \frac{\partial}{\partial z^{\bar{1}}} S = O(|z^1|^{2(1-\beta)+4\kappa-2})$$

and the fact that the exponent is non-negative.

Now we verify the first statement. We only concern one direction $\frac{\partial}{\partial z^1}$ perpendicular to one component of the divisor defined by $z^1 = 0$, as other directions are verified similarly. We have

$$\partial_1 Z = B^{-1} g^{i\bar{j}} (\nabla_1 \nabla_i \Psi \cdot \Psi_{\bar{j}} + \Psi_i \cdot \partial_1 \partial_{\bar{j}} \Psi) - \gamma' \Psi_1 \ .$$

In order to prove $Z \in C^{1,\alpha}_\beta$, it suffices to prove that $\nabla_1 \nabla_i \Psi \in C^\alpha_\beta$, which follows from [8, Proposition A.1]. On the other hand, $|\partial_1^{1+\alpha} S| = O(|z^1|^{2\kappa-(1+\alpha)\beta})$ with negative power. Thus we see that S grows extremely faster than $Z - \sup_{\mathfrak{D}} Z$ near the divisor. Since $Z - \sup_{\mathfrak{D}} Z$ is non-positive on $\mathfrak{D}$ while S vanishes along $\mathfrak{D}$, we obtain that the maximum point of $Z - \sup_{\mathfrak{D}} Z + \epsilon S$ must be achieved on $\mathfrak{M}$. □

With the lemmas above, we could assume that p in the interior of $\mathfrak{M}$ is the maximum point of K and choose normal coordinates around p. We get at p,

$$g_{i\bar{j}} = \delta_{ij} \text{ and } \frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{i\bar{j}}}{\partial z^{\bar{j}}} = 0 \ ;$$

so

$$\Psi_{i\bar{j}} = \Psi_{i\bar{i}} \delta_{ij} \text{ and } g'^{i\bar{j}} = \frac{\delta_{ij}}{1+\Psi_{i\bar{i}}} \ .$$

We have

$$\triangle' K = B^{-1} \sum_{k,i,j} \frac{1}{1+\Psi_{k\bar{k}}} [R_{i\bar{j}k\bar{k}} \Psi_j \Psi_{\bar{i}} + \Psi_{k\bar{k}i} \Psi_{\bar{i}} + \Psi_i \Psi_{k\bar{k}\bar{i}} + \Psi_{ik} \Psi_{\bar{i}\bar{k}} + \Psi_{i\bar{k}} \Psi_{\bar{i}k}]$$
$$- \gamma' \triangle' \Psi - \gamma'' D - B^{-2} g'^{k\bar{l}} B_k B_{\bar{l}} + \epsilon \triangle' S \ .$$

We deal with these terms by means of the next lemmas.

Lemma 3.11. *The following inequality holds*

$$B^{-1} \sum_{k,i,j} \frac{1}{1+\Psi_{k\bar{k}}} R_{i\bar{j}k\bar{k}} \Psi_i \Psi_{\bar{j}} - \gamma' \triangle' \Psi \ge \sum_{k,i,j} \frac{1}{1+\Psi_{k\bar{k}}} (\inf R_{i\bar{j}k\bar{k}} + \gamma') - (n+1)\gamma' \ .$$

Proof. From

$$\triangle' \Psi = \sum_k \frac{\Psi_{k\bar{k}}}{1+\Psi_{k\bar{k}}} = n+1 - \sum_k \frac{1}{1+\Psi_{k\bar{k}}} \ ,$$

we have the lemma. □

Lemma 3.12. *The following formula holds*

$$B^{-1}\sum_{k,i}\frac{1}{1+\Psi_{k\bar k}}\Psi_i\Psi_{k\bar k\bar i}=B^{-1}\sum_{k,i}\frac{1}{1+\Psi_{k\bar k}}\Psi_{\bar i}\Psi_{k\bar k i}\le 1+|\partial\log f|_\Omega+|\partial\Psi_1|_\Omega\ .$$

Proof. Differentiating the equation (3.9), we have

$$g'^{i\bar j}(\partial_k g_{i\bar j}+\Psi_{i\bar j k})-g^{i\bar j}\partial_k g_{i\bar j}=\partial_k F\ ,$$

or

$$\sum_i\frac{\Psi_{i\bar i k}}{1+\Psi_{i\bar i}}=F_k=\partial_k\Psi+\partial_k(\log f-\Psi_1)\ . \tag{3.26}$$

Then (3.26) implies

$$\begin{aligned}B^{-1}\sum_{k,i}\frac{1}{1+\Psi_{k\bar k}}\Psi_i\Psi_{k\bar k\bar i}&=B^{-1}\sum_{k,i}\frac{1}{1+\Psi_{k\bar k}}\Psi_{\bar i}\Psi_{k\bar k i}\\&=B^{-1}\Psi_iF_{\bar i}\\&=1+B^{-1}\Psi_i\tilde F_{\bar i}\\&\le 1+|\partial\tilde F|_\Omega\ .\end{aligned}$$

Here $\tilde F=\log f-\Psi_1$, and this completes the proof of the lemma. □

Lemma 3.13. *The following formula holds*

$$-B^{-2}g'^{k\bar l}B_kB_{\bar l}+B^{-1}\sum_{k,i,j}\frac{1}{1+\Psi_{k\bar k}}[\Psi_{ik}\Psi_{\bar i\bar k}+\Psi_{k\bar k}\Psi_{\bar k k}]\ge-(\gamma'+\epsilon)-\epsilon B^{-1}|\partial S|_\Omega\ .$$

Proof. At p we have

$$0=(Z_k+\epsilon S_k)(p)=B^{-1}B_k-\gamma'\Psi_k+\epsilon S_k\ '$$

i.e.

$$B^{-1}B_k=\gamma'\Psi_k-\epsilon S_k\ .$$

Also,

$$0=(Z_{\bar l}+\epsilon S_{\bar l})(p)=B^{-1}B_{\bar l}-\gamma'\Psi_{\bar l}+\epsilon S_{\bar l}\ ,$$

i.e.

$$B^{-1}B_{\bar l}=\gamma'\Psi_{\bar l}-\epsilon S_{\bar l}\ .$$

Since at p we have

$$B_k=\Psi_{ik}\Psi_{\bar i}+\Psi_i\Psi_{\bar i k}\ ,$$

and

$$B_{\bar l}=\Psi_{i\bar l}\Psi_{\bar i}+\Psi_i\Psi_{\bar i\bar l}\ ,$$

we obtain

$$\Psi_{ik}\Psi_{\bar i}=B(\gamma'\Psi_k-\epsilon S_k)-\Psi_i\Psi_{\bar i k}$$

and

$$\Psi_i\Psi_{\bar i\bar l}=B(\gamma'\Psi_{\bar l}-\epsilon S_{\bar l})-\Psi_{i\bar l}\Psi_{\bar i}\ .$$

So

$$\begin{aligned} g'^{k\bar{l}}B_kB_{\bar{l}} &= g'^{k\bar{l}}[\Psi_{ik}\Psi_{\bar{i}}\Psi_{j\bar{l}}\Psi_{\bar{j}} + \Psi_i\Psi_{\bar{i}k}\Psi_j\Psi_{\bar{j}\bar{l}} + \Psi_{ik}\Psi_{\bar{i}}\Psi_j\Psi_{\bar{j}\bar{l}} + \Psi_{i\bar{l}}\Psi_{\bar{i}}\Psi_j\Psi_{\bar{j}k}] \\ &= g'^{k\bar{l}}\{B(\gamma'\Psi_k - \epsilon S_k)\Psi_{j\bar{l}}\Psi_{\bar{j}} - \Psi_{i\bar{l}}\Psi_{\bar{i}}\Psi_j\Psi_{\bar{j}k} + \Psi_{ik}\Psi_{\bar{i}}\Psi_j\Psi_{\bar{j}\bar{l}}\}\ ' \end{aligned}$$

using the normal coordinates at p and assuming $\gamma' > 0$ we have

$$\begin{aligned} g'^{k\bar{l}}B_kB_{\bar{l}} &= \sum_{k,i,j}\frac{1}{1+\Psi_{k\bar{k}}}\{B(\gamma'\Psi_k - \epsilon S_k)\Psi_{k\bar{k}}\Psi_{\bar{k}} - (\Psi_{k\bar{k}}\Psi_{\bar{k}})^2 + \Psi_{ik}\Psi_{\bar{i}}\Psi_j\Psi_{\bar{j}\bar{k}}\} \\ &\le B\sum_{k,i,j}\frac{1}{1+\Psi_{k\bar{k}}}\{(\gamma'\Psi_k - \epsilon S_k)(\Psi_{k\bar{k}}+1)\Psi_{\bar{k}} - (\Psi_{k\bar{k}})^2 + \Psi_{ik}^2\} \\ &\le (\gamma'+\epsilon)B^2 + \epsilon B|\partial S|_\Omega + B\sum_{k,i}\frac{1}{1+\Psi_{k\bar{k}}}\{-(\Psi_{k\bar{k}})^2 + \Psi_{ik}^2\}\ . \end{aligned}$$

So

$$\begin{aligned} &-B^{-2}g'^{k\bar{l}}B_kB_{\bar{l}} + B^{-1}\sum_{k,i,j}\frac{1}{1+\Psi_{k\bar{k}}}[\Psi_{ik}\Psi_{\bar{i}\bar{k}} + \Psi_{k\bar{k}}\Psi_{\bar{k}k}] \\ &\ge -(\gamma'+\epsilon) - \epsilon B^{-1}|\partial S|_\Omega + 2B^{-1}\sum_k\frac{(\Psi_{k\bar{k}})^2}{1+\Psi_{k\bar{k}}}\ . \end{aligned}$$

□

Proposition 3.14. *We have the gradient estimate*

$$\sup_{\mathfrak{X}}|\partial\Psi|^2_\Omega \le C_i \text{ for } i = 1,2\ .$$

The constants C_i depend on, respectively,

$$C_1 = C_1(\inf_{i\neq k}R_{i\bar{i}k\bar{k}}(\Omega), \sup|\partial\log f|_\Omega, \sup|\partial\Psi_1|_\Omega, \operatorname{Osc}_{\mathfrak{X}}\Psi, \operatorname{Osc}_{\partial\mathfrak{X}}\Psi_1)\ ,$$

$$C_2 = C_2(\inf_{i\neq k}R_{i\bar{i}k\bar{k}}(\Omega_1), \sup|\partial\Psi_1|_\Omega, \sup\operatorname{tr}_\Omega\Omega_1, \sup\operatorname{tr}_{\Omega_1}\Omega, \operatorname{Osc}_{\mathfrak{X}}\Psi, \operatorname{Osc}_{\mathfrak{X}}\Psi_1)\ .$$

Proof. We assume $B(p) \ge 1$, otherwise we are done. We compute

$$\begin{aligned} \Delta'(Z - \sup_{\mathfrak{D}}Z + \epsilon S) &\ge \sum_{i,j,k}\frac{1}{1+\Psi_{k\bar{k}}}\left(\inf_{i\neq k}R_{i\bar{i}k\bar{k}} + \gamma'\right) - (n+1)\gamma' - \gamma''D - 1 \\ &\quad - |\partial\log f|_\Omega - |\partial\Psi_1|_\Omega - |\gamma'+\epsilon| - \epsilon B^{-1}|\partial S|_\Omega - C\epsilon\operatorname{tr}_{\Omega'}\Omega \\ &= \left(\inf_{i\neq k}R_{i\bar{i}k\bar{k}} + \gamma' - C\epsilon\right)\operatorname{tr}_{\Omega'}\Omega - \gamma''\mathrm{D} - 1 - |\partial\log \mathrm{f}|_\Omega - |\partial\Psi_1|_\Omega \\ &\quad - (n+1)\gamma' - (n+2)\gamma' - \epsilon - \epsilon|\partial S|_\Omega\ . \end{aligned}$$

We choose an appropriate γ, for example

$$\gamma(t) := -C'e^{\sup\Psi - t},\ \text{ where } C' := \inf_{i\neq k}R_{i\bar{i}k\bar{k}} - C\epsilon + 1\ .$$

We notice that $|\nabla S|$ is bounded by means of Lemma 3.10. Then,

$$D + \operatorname{tr}_{\Omega'}\Omega(\mathrm{p}) \le \mathrm{C} = \mathrm{C}(\inf_{\mathrm{i}\neq\mathrm{k}}\mathrm{R}_{\mathrm{i}\bar{\mathrm{i}}\mathrm{k}\bar{\mathrm{k}}},\ \sup|\partial\log\mathrm{f}|_\Omega,\ \sup|\partial\Psi_1|_\Omega)\ .$$

Since

$$(\mathrm{tr}_{\Omega}\Omega')^{\frac{1}{n}} = (n+1+\Delta\Psi)^{\frac{1}{n}} \leq \mathrm{tr}_{\Omega'}\Omega \cdot \mathrm{e}^{\frac{\mathrm{F}}{\mathrm{n}}} ,$$

so $B \leq \mathrm{tr}_{\Omega}\Omega' \cdot \mathrm{D} \leq \mathrm{C}$. Moreover, for any $x \in \mathrm{Int}\,(\mathfrak{X})$, there holds

$$\begin{aligned}\log B(x) &= K(x) + \gamma(\Psi)(x) - \epsilon S(x) + \sup_{\mathfrak{D}} Z \\ &\leq K(p) + \sup_{\partial\mathfrak{X}} K + \gamma(\Psi)(x) - \epsilon S(x) + \sup_{\mathfrak{D}} Z \\ &= \log B(p) - \gamma(\Psi)(p) - \sup_{\mathfrak{D}} Z + \epsilon S(p) + \sup_{\partial\mathfrak{X}} K + \gamma(\Psi)(x) - \epsilon S(x) + \sup_{\mathfrak{D}} Z \\ &\leq \log B(p) - \gamma(\Psi)(p) + \gamma(\Psi)(x) + \sup_{\partial\mathfrak{X}}(\log B - \sup_{\mathfrak{D}} \log B - \gamma(\Psi)) + C \,.\end{aligned}$$

Here we use the assumption that $B \geq 1$, so $\log B \geq 0$. Similarly to former arguments, we change the background metric and we consider

$$\log \frac{\det(\Omega_{1,i\bar{j}} + \tilde{\Psi}_{i\bar{j}})}{\det(\Omega_{1,i\bar{j}})} = F_1 = \log\tau + \tilde{\Psi} \,.$$

We arrive at

$$\sup_{\mathfrak{X}} |\partial\tilde{\Psi}|^2_{\Omega_1} \leq C \,.$$

As a result, the proof of the proposition follows from

$$\sup_{\mathfrak{X}} |\partial\Psi|_{\Omega} \leq \sup_{\mathfrak{X}} \mathrm{tr}_{\Omega}\Omega_1 \cdot (\sup_{\mathfrak{X}} |\partial\tilde{\Psi}|_{\Omega_1} + \sup_{\mathfrak{X}} |\partial\Psi_1|_{\Omega_1}) \,.$$

□

4. Solving the geodesic equation

In this section, we assume that the components of D are smooth and disjoint.

4.1. Existence of the $C^{1,1}_\beta$ cone geodesic.

In the present subsection we are dealing with the Dirichlet problem for the family of approximate geodesic equation (3.2). In order to apply the a priori estimates in Section 3, we require that the pair (Ω, Ω_1) satisfies that $|\partial \log \tilde{\Omega}_1^{n+1}|$, $|\partial \log \frac{\Omega_1^{n+1}}{\Omega^{n+1}}|$ are bounded and one of the following conditions hold

- $|Riem(\Omega_1)|$ is bounded;
- $\inf Riem(\Omega_1)$ and $\sup Riem(\Omega)$ are bounded;
- $\sup \mathrm{Ric}(\Omega_1)$ and $\inf Riem(\Omega)$ are bounded;
- $\inf \mathrm{Ric}(\Omega_1)$ and $|Riem(\Omega)|$ are bounded.

Then we reduce these conditions to geometric conditions on the boundary potentials φ_0 and φ_1 as follows.

The boundedness of the connection of the background cone metric ω in (1.3) is computed in the following lemma for $0 < \beta_1 < \frac{2}{3}$. It was also computed for $0 < \beta_1 < \frac{1}{2}$ in Brendle [8].

Lemma 4.1. *The connection of ω is bounded for $0 < \beta_1 < \frac{2}{3}$ under the coordinate chart $\{w^i\}$.*

Proof. Since there exists a smooth function ρ such that $\delta|s|^{2\beta_1}_{h_\Lambda} = \rho|z^1|^{2\beta_1}$, we can rewrite (1.3) as

$$\begin{aligned}\omega = \omega_0 &+ \frac{\sqrt{-1}}{2}|z^1|^{2\beta_1}\rho_{k\bar{l}}dz^k \wedge dz^{\bar{l}} \\ &+ \frac{\sqrt{-1}}{2}\beta_1|z^1|^{2(\beta_1-1)}(z^1\rho_k dz^k \wedge dz^{\bar{1}} + z^{\bar{1}}\rho_{\bar{l}}dz^1 \wedge dz^{\bar{l}}) \\ &+ \frac{\sqrt{-1}}{2}\beta_1^2\rho|z^1|^{2(\beta_1-1)}dz^1 \wedge dz^{\bar{1}}\end{aligned}$$

for k, l from 2 to n. By means of the change of coordinates (4.3), as $w^i = |z^i|^{\beta_1-1}z^i$, we have, for $i \in \{1, \cdots, n\}$

$$\frac{\partial w^i}{\partial z^i} = \frac{\beta_i+1}{2}|z^i|^{\beta_i-1}; \frac{\partial w^i}{\partial z^{\bar{i}}} = \frac{\beta_i-1}{2}|z^i|^{\beta_i-3}z^iz^i.$$

Meanwhile,

$$\frac{\partial z^i}{\partial w^i} = \frac{1+\beta_i}{2\beta_i}|w^i|^{\frac{1-\beta_i}{\beta_i}}; \frac{\partial z^i}{\partial w^{\bar{i}}} = \frac{1-\beta_i}{2\beta_i}|w^i|^{\frac{1-3\beta_i}{\beta_i}}w^iw^i.$$

The components of the model cone metrics under the variables w^i become

$$\begin{aligned}\tilde{g}_{1\bar{1}} &= [(\frac{1+\beta_1}{2\beta_1})^2|w^1|^{\frac{2}{\beta_1}-2} + (\frac{1-\beta_1}{2\beta_1})^2|w^1|^{\frac{2}{\beta_1}-2}]g_{1\bar{1}} \circ W^{-1} \\ &= \frac{1+\beta_1^2}{2\beta_1^2}|w^1|^{\frac{2}{\beta_1}-2}[g_{01\bar{1}} \circ W^{-1} + |w^1|^2\rho_{1\bar{1}} \\ &+ \beta_1|w^1|^{\frac{2}{\beta_1}-2}(|w^1|^{\frac{1}{\beta_1}-1}w^1\rho_1 + |w^1|^{\frac{1}{\beta_1}-1}w^{\bar{1}}\rho_{\bar{1}}) + \beta_1^2\rho|w^1|^{2-\frac{2}{\beta_1}}] \\ &= \frac{1+\beta_1^2}{2\beta_1^2}[g_{01\bar{1}} \circ W^{-1}|w^1|^{\frac{2}{\beta_1}-2} + |w^1|^{\frac{2}{\beta_1}}\rho_{1\bar{1}} + \beta_1(w^1\rho_1 + w^{\bar{1}}\rho_{\bar{1}} + \beta_1^2\rho)], \\ \tilde{g}_{1\bar{l}} &= \frac{1+\beta_1}{2\beta_1}[|w^1|^{\frac{1}{\beta_1}-1}g_{01\bar{l}} \circ W^{-1} + |w^1|^{\frac{1}{\beta_1}+1}\rho_{1\bar{l}} \circ W^{-1} + \beta_1 w^1\rho_{\bar{l}} \circ W^{-1}], \\ \tilde{g}_{k\bar{l}} &= g_{0k\bar{l}} \circ W^{-1} + |w^1|^2\rho_{k\bar{l}} \circ W^{-1}.\end{aligned}$$

Now, the connection of ω depends on the first derivative with respect to w^i. We check them one by one. Note that ρ is smooth on w^k for $1 \leq k \leq n$.

$$\begin{aligned}&\frac{\partial}{\partial w^1}\tilde{g}_{1\bar{1}} = O(|w^1|^{\frac{2}{\beta_1}-3} + |w^1|^{\frac{2}{\beta_1}-1}); \\ &\frac{\partial}{\partial w^i}\tilde{g}_{1\bar{1}} = O(1); \\ &\frac{\partial}{\partial w^i}\tilde{g}_{1\bar{l}} = O(1); \\ &\frac{\partial}{\partial w^1}\tilde{g}_{k\bar{l}} = \frac{\partial}{\partial w^i}\tilde{g}_{k\bar{l}} = O(1).\end{aligned}$$

Now let us check $\frac{\partial}{\partial w^1}\tilde{g}_{1\bar{l}}$. It contains three terms. The first term is

$$\begin{aligned}&\frac{\partial}{\partial w^1}(|w^1|^{\frac{1}{\beta_1}-1}g_{01\bar{l}}\circ W^{-1})\\&=\frac{\partial}{\partial w^1}[(|w^1|^{\frac{1}{\beta_1}-1}|w^1|)(|w^1|^{-1}g_{01\bar{l}}\circ W^{-1})].\end{aligned}$$

Since $g_{01\bar{l}}\circ W^{-1}$ is also smooth and converges to zero as w^1 goes to zero, then this first term is $O(|w^1|^{\frac{1}{\beta_1}-1})$. The second and third term are both $O(1)$. Thus we conclude that when $0<\beta_1<\frac{2}{3}$, the connection is bounded. □

As a corollary, we arrive at the boundedness of the connection of Ω_1.

Corollary 4.2. *When $0<\beta_1<\frac{2}{3}$ and $\varphi_0,\varphi_1\in C^3_\beta$, the connection of Ω_1 is bounded.*

Proof. From Lemma 4.1 and the expression of Ω in (2.8), we know that the connection of Ω is bounded for $0<\beta_1<\frac{2}{3}$. Recall the formula (2.10) of Ω_1; we have

$$\begin{aligned}\Omega_1&=t\omega_\phi+(1-t)\omega_\varphi+\frac{\sqrt{-1}}{2}(1+m\partial_{n+1}\partial_{\overline{n+1}}\Phi)dz^{n+1}\wedge d\bar{z}^{n+1}\\&+\frac{1}{\sqrt{2}}\partial_i(\phi-\varphi)dz^i dz^{\overline{n+1}}+\frac{1}{\sqrt{2}}\partial_{\bar{i}}(\phi-\varphi)dz^{\bar{i}}dz^{n+1}.\end{aligned}$$

We have that the components of Ω_1 are, for $2\le i,j\le n$,

$$\begin{aligned}&(g_1)_{1\bar{1}}=t(g_{\varphi_0})_{1\bar{1}}+(1-t)(g_{\varphi_1})_{1\bar{1}};\\&(g_1)_{1\bar{i}}=t(g_{\varphi_0})_{1\bar{i}}+(1-t)(g_{\varphi_1})_{1\bar{i}};\\&(g_1)_{1\overline{n+1}}=\partial_1(\varphi_0-\varphi_1);\\&(g_1)_{i\bar{j}}=t(g_{\varphi_0})_{i\bar{j}}+(1-t)(g_{\varphi_1})_{i\bar{j}};\\&(g_1)_{i\overline{n+1}}=\partial_i(\varphi_0-\varphi_1);\\&(g_1)_{n+1\overline{n+1}}=1+m\partial_{n+1}\partial_{\overline{n+1}}\Phi.\end{aligned}$$

Thus the corollary follows from $\varphi_0,\varphi_1\in C^3_\beta$. □

Lemma 4.3. *Suppose that $\varphi_0,\varphi_1\in C^3_\beta$ have curvature lower (resp. upper) bound. Then Ω_1 has also curvature lower (resp. upper) bound.*

Proof. Since the formula of the bisectional curvature is

$$R_{i\bar{j}k\bar{l}}=-\frac{\partial^2 g_{i\bar{j}}}{\partial z^k\partial z^{\bar{l}}}+g^{p\bar{q}}\frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}\frac{\partial g_{i\bar{q}}}{\partial z^k},$$

we have for $1 \leq i, j, k, l \leq n$ and $\phi = t\varphi_1 + (1-t)\varphi_0$,

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}(g_1) &= tR_{i\bar{j}k\bar{l}}(g(\varphi_1)) + (1-t)R_{i\bar{j}k\bar{l}}(g(\varphi_0)) \\
&\quad - tg(\varphi_1)^{p\bar{q}}\partial_{\bar{l}}g(\varphi_1)_{p\bar{j}}\partial_k g(\varphi_1)_{i\bar{q}} \\
&\quad - (1-t)g(\varphi_0)^{p\bar{q}}\partial_{\bar{l}}g(\varphi_0)_{p\bar{j}}\partial_k g(\varphi_0)_{i\bar{q}} \\
&\quad + \sum_{1\leq p,q\leq n} g_1^{p\bar{q}}\partial_{\bar{l}}g(\phi)_{p\bar{j}}\partial_k g(\phi)_{i\bar{q}} \\
&\quad + \sum_{1\leq p\leq n} g_1^{p\overline{n+1}}\partial_{\bar{l}}g(\phi)_{p\bar{j}}\frac{1}{\sqrt{2}}\partial_k\partial_i[\varphi_1-\varphi_0] \\
&\quad + \sum_{1\leq q\leq n} g_1^{n+1\bar{q}}\partial_k g(\phi)_{i\bar{q}}\frac{1}{\sqrt{2}}\partial_{\bar{l}}\partial_{\bar{j}}[\varphi_1-\varphi_0] \\
&\quad + \frac{1}{2}g_1^{n+1\overline{n+1}}\partial_k\partial_i[\varphi_1-\varphi_0]\partial_{\bar{l}}\partial_{\bar{j}}[\varphi_1-\varphi_0].
\end{aligned}$$

Also,

$$\begin{aligned}
R_{i\bar{j}k\overline{n+1}}(g_1) &= -\partial_k(\varphi_1-\varphi_0)_{i\bar{j}} \\
&\quad + \sum_{1\leq p,q\leq n} g_1^{p\bar{q}}(\varphi_1-\varphi_0)_{p\bar{j}}\partial_k g(\phi)_{i\bar{q}} \\
&\quad + g_1^{p\overline{n+1}}(\varphi_1-\varphi_0)_{p\bar{j}}\frac{1}{\sqrt{2}}\partial_k\partial_i[\varphi_1-\varphi_0]; \\
R_{i\bar{j}(n+1)\overline{n+1}}(g_1) &= \sum_{1\leq p,q\leq n} mg_1^{p\bar{q}}(\varphi_1-\varphi_0)_{p\bar{j}}(\varphi_1-\varphi_0)_{i\bar{q}}; \\
R_{i\overline{n+1}(n+1)\overline{n+1}}(g_1) &= \sum_{q=1}^{n} mg_1^{n+1\bar{q}}\bar{\partial}_{n+1}\partial_{n+1}\bar{\partial}_{n+1}\Psi(\varphi_1-\varphi_0)_{i\bar{q}}; \\
R_{n+1\overline{n+1}(n+1)\overline{n+1}}(g_1) &= -m\partial_{n+1}\bar{\partial}_{n+1}\partial_{n+1}\bar{\partial}_{n+1}\Psi \\
&\quad + m^2 g_1^{n+1\overline{n+1}}\bar{\partial}_{n+1}\bar{\partial}_{n+1}\partial_{n+1}\Psi\partial_{n+1}\bar{\partial}_{n+1}\partial_{n+1}\Psi.
\end{aligned}$$

The connection and the lower bound of the curvature of φ_0 and φ_1 are bounded. So the curvature of Ω_1 is also bounded below. The upper bound follows in the same way. □

Corollary 4.4. *Suppose that $0 < \beta_1 < \frac{2}{3}$, $\varphi_0, \varphi_1 \in C^3_\beta$ and their Ricci curvature have lower (upper) bound. Then the Ricci curvature of Ω_1 also has lower (resp. upper) bound.*

Proof. We use the formulas of the Riemannian curvature in Lemma 4.3, and we take the trace to obtain the Ricci curvature. Then the lemma follows directly. □

Since inf $Riem(\Omega_1)$ is bounded for $0 < \beta_1 < \frac{1}{2}$ (see [8]), we introduce the following subspaces of Kähler cone metrics. When $0 < \beta_1 < \frac{1}{2}$, we define

$$\mathfrak{I}_1 := \{\varphi \in \mathcal{H}^3_\beta | \sup \mathrm{Ric}(\omega_\varphi) \text{ is bounded}\};$$

$$\mathfrak{I}_2 := \{\varphi \in \mathcal{H}^3_\beta | \inf \mathrm{Ric}(\omega_\varphi) \text{ is bounded}\}.$$

Theorem 4.5. *Assume that two Kähler cone potentials φ_0, φ_1 are both in $\mathfrak{I}_i$ $i = 1, 2$. Then they are connected by a $C^{1,1}_\beta$ cone geodesic.*

Proof. Note that the right hand side of the equation is positive as long as τ is positive. When τ is zero, (3.1) provides a solution of the geodesic equation (2.6).

We denote the set of solvable times of (3.1) by

$$I = \{\tau \in (0, 1] | (3.1)_\tau \text{ is solvable in } C^{2,\alpha}_\beta\} .$$

Automatically, $\Psi = \Psi_1$ satisfies the equation at $\tau = 1$, so the set I is not empty.

For any $0 < \tau \leq 1$, assuming that $\omega(\tau_0)$ solves the equation (3.1), Proposition 5.21 provides a unique solution in $C^{2,\alpha}_\beta$ to the following linearized equation

$$\begin{cases} \triangle_{\tau_0} v - v = f & \text{in } \mathfrak{M} , \\ v = u & \text{on } \partial\mathfrak{X} , \end{cases}$$

for any $f \in C^\alpha_\beta$ and $u \in C^{2,\alpha}_\beta$. So the linearized operator at τ_0 is invertible, and thus I is open. So the solvable time can be extended beyond τ_0.

The a priori estimates in Section 3, with one of the geometry conditions in $\mathfrak{I}_1$ or $\mathfrak{I}_2$ assures the uniform $C^{1,1}_\beta$ bound of $\varphi(t)$ which is independent of τ. Two estimates in the next subsections improve $C^{2,\alpha}_\beta$ regularity of the solution of (3.1) before $\tau = 0$. Thus, we can solve th approximate equation till $\tau = 0$. With the uniform $C^{1,1}_\beta$ bound, after taking a subsequence t_i we have a weak limit $\varphi = \lim_{t_i \to 0} \varphi(t_i)$ under a $C^{1,\alpha}_\beta$ norm. In Section 4.4, we prove the uniqueness of a weak solution. Hence the theorem is proved completely. □

4.2. **Interior Schauder estimate:** $\tau > 0$. We first prove the $C^{2,\alpha}_\beta$ estimate for a general equation.

$$\log \Omega^{n+1}_\Psi = \log \Omega^{n+1} + F. \tag{4.1}$$

Proposition 4.6. *Assume that we have the second order estimate of Ψ. Then the following estimate holds for the solution of* (4.1) *on any small ball $B \subset \mathfrak{X}$*

$$|\sqrt{-1}\partial\bar{\partial}\Psi|_{C^\alpha_\beta(B)} \leq C , \tag{4.2}$$

where C depends on $|\partial \log \tilde{\Omega}^{n+1}|_{L^q}$, $|\log \Omega^{n+1}|_{C^\alpha_\beta}$, $|\partial\Psi|_\infty$,$|\triangle\Psi|_\infty$, $|\partial\tilde{F}|_{L^q}$, $|F|_{C^\alpha_\beta}$, where $q > 2n + 2$.

Proof. Choose a small ball $B_d(p)$ around p in the interior of $\mathfrak{X}$. When $B_d(p)$ does not intersect $\mathfrak{D}$, this proposition follows directly from the standard Evans-Krylov estimate. So it's sufficient to fix a point $p \in \mathfrak{D}$. We consider (4.1) in $B_d(p)$. The distance d is measured with respect to the flat cone metric g.

In $B_d(p) \setminus \mathfrak{D}$, we use the local holomorphic coordinate chart

$$f^i(z^i) = (z^i)^{\beta_i}$$

and

(4.3)

$$W(z^1, \cdots, z^n) := (w^1 = f^1(z^1), \cdots, w^s = f^s(z^s), z^{i+1}, \cdots, z^{n+1}) .$$

Here above, for $1 \leq i \leq s$ we have that z^i is the singular direction with cone angle β_i. Then on $\mathbb{C} \setminus 0$, f^i is holomorphic when the right hand is restricted to the principal branch and the coordinate transformation satisfies

$$\begin{cases} \dfrac{\partial w^i}{\partial z^i} = \beta_i (z^i)^{\beta_i - 1}, 1 \leq i \leq s; \\ \dfrac{\partial w^i}{\partial z^i} = 1, s \leq i \leq n+1; \\ \dfrac{\partial w^{\bar{i}}}{\partial z^i} = \dfrac{\partial w^i}{\partial z^j} = \dfrac{\partial w^{\bar{i}}}{\partial z^j} = 0, j \neq i, 1 \leq i, j \leq n+1. \end{cases}$$

We rewrite (4.1) in the coordinate system $\{w^i\}$, we use the notation $\widetilde{*}$ over the quantities to denote the corresponding ones after pulling back or pushing forward. Under (4.3), (4.1) becomes

$$\log \tilde{\Omega}^n_{\tilde{\varphi}} := \tilde{h}. \tag{4.4}$$

So, we fix a $1 \leq k \leq n$ and, by taking $\frac{\partial}{\partial w^k}$ on both sides of (4.4) we get

$$\tilde{g'}^{i\bar{j}} (\tilde{g}_{i\bar{j}k} + \tilde{\varphi}_{i\bar{j}k}) = \tilde{h}_k .$$

Taking $\frac{\partial}{\partial w^l}$ on both sides of the above equation, we have

$$- \tilde{g'}^{p\bar{j}} \tilde{g'}^{i\bar{q}} (\tilde{g}_{p\bar{q}\bar{l}} + \tilde{\varphi}_{p\bar{q}\bar{l}})(\tilde{g}_{i\bar{j}k} + \tilde{\varphi}_{i\bar{j}k}) + \tilde{g'}^{i\bar{j}} (\tilde{g}_{i\bar{j}k\bar{l}} + \tilde{\varphi}_{i\bar{j}k\bar{l}}) = \tilde{h}_{k\bar{l}} .$$

Let g be the local potential of $g_{i\bar{j}}$ in $B_d(p)$, the existence of such g is guaranteed by the linear theory in Section 5. We introduce the notation

$$V := g + \varphi \text{ and } \tilde{V} := \tilde{g} + \tilde{\varphi}.$$

Then

$$\tilde{g'}^{i\bar{j}} \tilde{V}_{k\bar{l}i\bar{j}} = \tilde{g'}^{p\bar{j}} \tilde{g'}^{i\bar{q}} \tilde{V}_{p\bar{q}k} \tilde{V}_{i\bar{j}\bar{l}} + \tilde{h}_{k\bar{l}}.$$

In conclusion, we have

$$\tilde{\triangle}' \tilde{V}_{k\bar{l}} \geq \tilde{h}_{k\bar{l}}. \tag{4.5}$$

Then we pull back the differential inequality above by the map W,

$$\tilde{V}_{k\bar{l}} = \frac{\partial z^k}{\partial w^k}\frac{\partial z^{\bar{l}}}{\partial w^{\bar{l}}} V_{k\bar{l}} \text{ and } \tilde{h}_{k\bar{l}} = \frac{\partial z^k}{\partial w^k}\frac{\partial z^{\bar{l}}}{\partial w^{\bar{l}}} h_{k\bar{l}}$$

and denote the weight

$$\sigma^{k\bar{l}} = \frac{\partial z^k}{\partial w^k}\frac{\partial z^{\bar{l}}}{\partial w^{\bar{l}}}.$$

Under the coordinate transformation, we see that for $1 \leq i \leq s$ and $s+1 \leq j \leq n$

$$\begin{cases} \sigma^{i\bar{i}} = \beta_i^{-2}(z^i)^{2-2\beta_i} = O(|z^i|^{2-2\beta_i}) \\ \sigma^{i\bar{j}} = \beta_i^{-1}(z^i)^{1-\beta_i} = O(|z^i|^{1-\beta_i}), \\ \sigma^{j\bar{j}} = 1. \end{cases} \tag{4.6}$$

Thus the weight σ is equivalent to g_{cone} as well as to g', according to the second order estimate of φ. Thus, (4.5) becomes in $B_d(p)\backslash\mathfrak{D}$ under the coordinate system $\{z^i\}$,

$$\triangle'[\sigma^{k\bar{l}}V_{k\bar{l}}] \geq \sigma^{k\bar{l}}h_{k\bar{l}}. \tag{4.7}$$

At last given any direction $\eta \in \mathbb{C}^n$, with $|\eta| = 1$, we denote

$$\partial_\eta := \sum_k \eta^k \frac{\partial}{\partial z^k}.$$

Also, we set

$$V_{\eta\bar{\eta}} := \partial^2_{\eta\bar{\eta}} V = \sum_{k,l} \eta^k \eta^{\bar{l}} \frac{\partial}{\partial z^k}\frac{\partial}{\partial z^{\bar{l}}} V.$$

We then define

$$u_\eta := \sum_{k,l} \eta^k \eta^{\bar{l}} \sigma^{k\bar{l}} V_{k\bar{l}}.$$

By using (4.7) , we have the following differential inequality in $B_d(p)\backslash\mathfrak{D}$ in coordinate system $\{z^i\}$,

$$\triangle'(u_\eta) = \sum_{k,l} \eta^k \eta^{\bar{l}} \triangle'(\sigma^{k\bar{l}}V_{k\bar{l}}) \geq \sum_{k,l} \eta^k \eta^{\bar{l}} \sigma^{k\bar{l}} h_{k\bar{l}}\ . \tag{4.8}$$

We emphasize again that the differential inequality (4.8) is defined outside the divisor and its coefficient is a Kähler cone metric.

Let us now introduce the following symbols. We denote, for $\lambda \in \mathbb{N}$

$$M_{\lambda\cdot\eta} := \sup_{B_{\lambda\cdot d}(p)} u_\eta, \quad m_{\lambda\cdot\eta} := \inf_{B_{\lambda\cdot d}(p)} u_\eta\ .$$

Applying Proposition 5.11 (weak Harnack inequality) to $M_{2\eta} - u_\eta$, we have that there exists a $q > 2n+2$ such that

$$\left\{ d^{-2n-2} \int_{B_d(p)} (M_{2\eta} - u_\eta)^p \Omega^{n+1} \right\}^{\frac{1}{p}} \leq C\left\{M_{2\eta} - M_\eta + K\right\}\ . \tag{4.9}$$

Here

$$K := d^{1-\frac{2n+2}{q}} \|\partial_z h\|_q$$

We denote $\tilde{h}_{\bar{l}} := g^{k\bar{l}}\tilde{h}_k$ on the coordinate system $\{w^i\}$. Using the coordinate change from $\{z^i\}$ to $\{w^i\}$, we need to cut the cone point. While, along the cutting line in the one branch w-disk D_w, the value of $\tilde{h}$ might not be equal. We let

$$\|\partial_w \tilde{h}\|_q = \|\partial_w \log \tilde{\Omega}^{n+1} + \partial_w \tilde{F}\|_q.$$

Now let us check the term $\|\partial_z \log \Omega^{n+1}\|_q$. We could choose a ϵ-tubular neighbourhood of the cutting line as D_ϵ. Thus from Corollary 4.2, we have that there is a uniform constant C which is independent of ϵ such that

$$\begin{aligned}&\|\partial_z \log \Omega^{n+1}\|_{q;D_w\setminus D_\epsilon}\\ &= \|\partial_w \log \tilde{\Omega}^{n+1}\|_{q;D_w\setminus D_\epsilon}\\ &\leq CVol(D_w \setminus D_\epsilon)\\ &\leq CVol(D_w).\end{aligned}$$

So we could take $\epsilon \to 0$, and the integral above is still finite. Meanwhile, $\|\partial_z F\|_{q;D_w\setminus D_\epsilon} = \|\partial_w \tilde{F}\|_{q;D_w\setminus D_\epsilon}$ is bounded by means of Lebesgue's dominated convergence theorem, when $\|\partial_w \tilde{F}\|_{q;D_w}$ is finite.

In order to obtain the inverse inequality for

$$u_\eta - m_{2\eta}$$

we use the concavity of the Monge-Ampère operator. Fix any two points

$$Q_2 \in B_{2d}(p) \text{ and } Q_1 \in B_d(p),$$

without loss of generality, we assume that the distance from Q_2 to $\mathfrak{D}$ is longer than the distance from Q_1 to $\mathfrak{D}$. From the formula of the flat metric (1.2), we see that

$$\sigma^{i\bar{j}}(Q_2) > \sigma^{i\bar{j}}(Q_1).$$

We set

$$g'(t) := (1-t)g'(Q_2) + tg'(Q_1),$$

and

$$a^{i\bar{j}} = \int_0^1 g'^{i\bar{j}}(t)dt.$$

When Q_1, Q_2 are on D, we choose and sequence of points Q_1^k, Q_2^l which are outside D such that $Q_1^k \to Q_1$ and $Q_2^l \to Q_2$. From the equation (4.1), we have

$$\begin{aligned}&h(Q_1^k) - h(Q_2^l) = \log\det(g'_{i\bar{j}}(Q_1^k)) - \log\det(g'_{i\bar{j}}(Q_2^l))\\ &= \int_0^1 g'(t)^{i\bar{j}}dt(V(Q_1^k) - V(Q_2^l))_{i\bar{j}} = a^{i\bar{j}}(V(Q_1^k) - V(Q_2^l))_{i\bar{j}}\ . \qquad (4.10)\end{aligned}$$

Now, we define

$$\begin{aligned}
(\tilde{g'})_{i\bar{j}}(t) &= (1-t)(\sigma(Q_2^l))^{i\bar{j}}(g'(Q_2^l))_{i\bar{j}} + t(\sigma(Q_1^k))^{i\bar{j}}(g'(Q_1^k))_{i\bar{j}} \\
\tilde{a}^{i\bar{j}} &:= \int_0^1 (\tilde{g'})^{-1}(t)dt, \text{ if one of } i,j \text{ is in } 1,\cdots,s; \\
\tilde{a}^{i\bar{j}} &:= a^{i\bar{j}}, s+1 \le i,j \le n+1.
\end{aligned}$$

Since $g'(t)$ is L^∞-equivalent to g, for $1 \le i,j \le n+1$, the matrix $\tilde{a}^{i\bar{j}}$ is positive definite and its eigenvalues range between the positive constants λ and Λ. Thus, we can apply [28, Lemma 17.13] (see also [50, Section (4.3)]); we get that there exists a finite set of unit vectors $\gamma_1, \cdots, \gamma_N \in \mathbb{C}^{n+1}$ and positive numbers λ^*, Λ^* depending only on n, λ, Λ such that the matrix $\tilde{a}^{i\bar{j}}$ can be written as

$$\tilde{a}^{i\bar{j}} = \sum_{\nu=1}^N b_\nu \gamma_{\nu i}\gamma_{\nu\bar{j}} \ .$$

Here $\lambda^* \le b_\nu \le \Lambda^*$ for any $1 \le \nu \le N$. As a result, we can express the matrix $a^{i\bar{j}}$ in terms of b_ν and the vectors γ_ν. Thus, we continue from (4.10) and we write

$$\begin{aligned}
h(Q_1^k) - h(Q_2^l) &= \sum_{i,j} \tilde{a}^{i\bar{j}}[(\sigma(Q_1^k))^{i\bar{j}}(V(Q_1^k))_{i\bar{j}} - (\sigma(Q_2^l))^{i\bar{j}}(V(Q_2^l))_{i\bar{j}}] \\
&= \sum_{\nu=1}^N b_\nu \sum_{i,j} \gamma_{\nu i}\gamma_{\nu\bar{j}}[(\sigma(Q_1^k))^{i\bar{j}}(V(Q_1^k))_{i\bar{j}} - (\sigma(Q_2^l))^{i\bar{j}}(V(Q_2^l))_{i\bar{j}}] \\
&= C\sum_{\nu=1}^N b_\nu(u_{\gamma_\nu}(Q_1^k) - u_{\gamma_\nu}(Q_2^l)) \ .
\end{aligned}$$

Since both end sides are C^α_β, letting $Q_1^k \to Q_1$ and $Q_2^l \to Q_2$, we have

$$h(Q_1) - h(Q_2) = C\sum_{\nu=1}^N b_\nu(u_{\gamma_\nu}(Q_1) - u_{\gamma_\nu}(Q_2)) \ .$$

From the final decomposition of the cone matrix above, we conclude that for a fixed $1 \le l \le N$,

(4.11)
$$Cb_l(u_{\gamma_l}(Q_1) - u_{\gamma_l}(Q_2)) \le h(Q_1) - h(Q_2) + C\sum_{\nu\neq l} b_\nu(u_{\gamma_\nu}(Q_2) - u_{\gamma_\nu}(Q_1)) \ .$$

We now fix $1 \le \nu \le N$, $\lambda = 1,2$ and we denote

$$w(\lambda\cdot d) := \sum_{\nu=1}^N \mathrm{Osc}_{B_{\lambda\cdot d}(p)}\, u_{\gamma_\nu}.$$

From (4.11), since $Q_1 \in B_d(p)$ and $Q_2 \in B_{2d}(p)$ we get

$$u_{\gamma_l}(Q_1) - m_{2l} \le C\{d^\alpha |h|_{C^\alpha_\beta} + \sum_{\nu\neq l}(M_{2\gamma_\nu} - u_{\gamma_\nu}(Q_1))\}.$$

Applying the inequality (4.9), we have

$$\begin{aligned}
&\left\{ d^{-2n-2}\int_{B_d(p)} (\sum_{\nu\neq l} M_{2\gamma_\nu} - u_{\gamma_\nu})^p \Omega^{n+1}\right\}^{\frac{1}{p}} \\
&\le N^{\frac{1}{p}} \sum_{\nu\neq l}\left\{ d^{-2n-2}\int_{B_d(p)} (M_{2\gamma_\nu} - u_{\gamma_\nu})^p \Omega^{n+1}\right\}^{\frac{1}{p}} \\
&\le C\left\{ \sum_{\nu\neq l}(M_{2\gamma_\nu} - M_{\gamma_\nu}) + K\right\} \\
&\le C\left\{ w(2d) - w(d) + K\right\} \qquad (4.12)
\end{aligned}$$

which entails, by integrating of $(u_{\gamma_l}(Q_1) - m_{2l})^p$ on $B_d(p)$ with respect to Ω and using (4.12)

$$\begin{aligned}
&\left\{ d^{-2n-2}\int_{B_d(p)} (u_{\gamma_l}(Q_1) - m_{2l})^p \Omega^{n+1}\right\}^{\frac{1}{p}} \\
&\le C\left\{ d^\alpha |h|_{C^\alpha_\beta} + \{d^{-2n-2}\int_{B_d(p)} (\sum_{\nu\neq l} M_{2\gamma_\nu} - u_{\gamma_\nu})^p \Omega^{n+1}\}^{\frac{1}{p}}\right\} \\
&\le C\left\{ d^\alpha |h|_{C^\alpha_\beta} + w(2d) - w(d) + K\right\} . \qquad (4.13)
\end{aligned}$$

At the last inequality we used (4.12). Now, we combine (4.9) and (4.13) to obtain

$$w(2d) \le C\left\{ d^\alpha |h|_{C^\alpha_\beta} + w(2d) - w(d) + K\right\} .$$

Then, using the Iteration Lemma 8.23 in [28], we have $u_\eta \in C^\alpha_\beta$, for all $\eta \in \mathbb{C}^{n+1}$. So $\Delta V \in C^\alpha_\beta$ and $V \in C^{2,\alpha}_\beta$ follows from Proposition 5.20. This gives (4.2) and completes the proof of the proposition. □

In conclusion, we obtain the conical Evans-Krylov estimate of the geodesic equation (3.1).

Proposition 4.7. *Assume* $0 < \beta < \frac{2}{3}$ *and that* φ_0, φ_1 *are in* $\mathfrak{I}_{\mathrm{i}}$, $i = 1, 2$. *Then the* $C^{1,1}_\beta$ *solution* Ψ *of the approximate geodesic equation* (3.1) *belongs to* $C^{2,\alpha}_\beta$ *in the interior of* $\mathfrak{X}$.

Proof. Considering the geodesic equation (3.1), then $F = \log\tau + \log\frac{\Omega_1^{n+1}}{\Omega^{n+1}} + \Psi - \Psi_1$. Since $\Omega \in C^\alpha_\beta$, we have $\log\Omega^{n+1} \in C^\alpha_\beta$. Moreover, $\varphi_0, \varphi_1 \in C^{2,\alpha}_\beta$, so $\log\Omega_1^{n+1} \in C^\alpha_\beta$. Thus we have $F \in C^\alpha_\beta$. When $0 < \beta_1 < \frac{2}{3}$, Lemma 4.1, Lemma 4.2 and $\varphi_0, \varphi_1 \in C^3_\beta$ imply that $\partial\tilde{F}$ is bounded. □

Our argument presented above follows Evans-Krylov's estimate [26, 27, 38]. We also used Blocki's observation in [5] that $F \in W^{1,q}$ is sufficient to get the estimate. In our problem, since $V_{k\bar{l}}$ is singular along the direction which is perpendicular to $\mathfrak{D}$, we multiply by the weight. In the next Section, we will develop the linear theory including the weak Harnack inequality for the linear equation and with cone coefficient which is used in the proof above.

4.3. **Boundary Schauder estimate: $\tau > 0$.** We adapt Krylov's method [38] (also cf. [28]) for the boundary estimate to our cone case. We notice that the linear equation is of divergence form, so the Harnack inequality and maximum principle proved in the next section can be applied here. The boundary of $\mathfrak{X}$ is $X \times \partial R$, which is a manifold with $2n + 1$ real dimension. Under the local coordinate $z^i = x^{n+1} + iy^{n+1}$, the boundary is defined by $x^{n+1} = 0$. Denote $x' = \{x^1, y^1, \cdots, x^n, y^n, y^{n+1}\}$.

Proposition 4.8. *Assume $0 < \beta_1 < \frac{2}{3}$ and that φ_0, φ_1 are in $\mathfrak{I}_i$, $i = 1, 2$. Then the $C^{1,1}_\beta$ solution Ψ of the approximate geodesic equation* (3.1) *belongs to $C^{2,\alpha}_\beta$ on the boundary of $\mathfrak{X}$.*

Proof. Recall that the approximate geodesic equation is
(4.14)
$$\begin{cases} \log\det(\Omega_{\Psi_1 i\bar{j}} + \tilde{\Psi}_{i\bar{j}}) = h = \log\tau + \tilde{\Psi} + \log\det(\Omega_{\Psi_1 i\bar{j}}) & \text{in } \mathfrak{M}\,, \\ \tilde{\Psi}(z) = 0 & \text{on } \partial\mathfrak{X}\,. \end{cases}$$
We first see that the tangent-tangent direction of the boundary estimate equals to the same estimate of the boundary values. Then the normal-normal estimate follows from the approximate geodesic equation
$$[\varphi'' - (\partial\varphi', \partial\varphi')_{g_\varphi}]\det\omega_\varphi = \Omega^{n+1}_\Psi = \tau e^{\Psi-\Psi_1}\det(\Omega_{1i\bar{j}})$$
with the estimates of the tangent-normal direction and the tangent-tangent direction. We differentiate (4.1) with respect to ∂_k for a fixed $k \in 1, \cdots, n$, and we get
$$\triangle'\tilde{\Psi}_k = h_k - g^{i\bar{j}}_\Psi g(\Psi_1)_{i\bar{j}k}\,.$$
We use the flat metric as the weighted metric to derive the differential equation of $u = \sqrt{g^{k\bar{k}}}\tilde{\Psi}_k$. Then we obtain that u satisfies
$$\triangle' u = \sqrt{g^{k\bar{k}}}(h_k - g^{i\bar{j}}_\Psi g(\Psi_1)_{i\bar{j}k})\,.$$
We denote the right hand side as f. According to Lemma 4.2, f is bounded when $0 < \beta < \frac{2}{3}$ and $\varphi_0, \varphi_1 \in C^3_\beta$. Note that u vanishes on the boundary $\partial\mathfrak{X}$. We fix a point p on the boundary, and we take

coordinates z^i centered at p. We introduce the following domains for a small radius d.

$$
\begin{aligned}
&|x'|^2_{\beta_1} = |z^1|^{2\beta_1} + \sum_{i=2}^{n} |z^i|^2 + |y^{n+1}|^2,\ B_d(p) = \{z \in M||z-p|_{\beta_1} \le d\},\\
&B_1 = B_d(p) \times \{x^{n+1}|0 \le |x^{n+1}| \le \delta d, x^{n+1} \ge 0\},\\
&B_1^3 = B_d(p) \times \{x^{n+1}|\delta d \le |x^{n+1}| \le 3\delta d, x^{n+1} \ge 0\},\\
&S_2 = B_d(p) \times \{|x^{n+1}| = 2\delta d, x^{n+1} \ge 0\},\\
&B_2 = B_{2d}(p) \times \{x^{n+1}|0 \le |x^{n+1}| \le 2\delta d, x^{n+1} \ge 0\},\\
&B_4 = B_{4d}(p) \times \{x^{n+1}|0 \le |x^{n+1}| \le 4\delta d, x^{n+1} \ge 0\}.
\end{aligned}
$$

Here, $\delta \ll 1$ is a small positive constant such that $v := \frac{u}{x^{n+1}}$ is strictly positive on S_2. We assume that v is nonnegative on B_4; then $u \ge 0$.

We use the barrier function

$$
w = \left[(4 - \frac{|x'|^2_{\beta_1}}{d^2}) \inf_{S_2} v + (1 + d \sup |f|) \frac{\frac{x^{n+1}}{2d} - \delta}{\sqrt{\delta}} \right] x^{n+1}.
$$

We first prove that on the boundary of B_2, $w \le u$. On $|x^{n+1}| = 2\delta d$, we have $w \le 4x^{n+1} \inf_{S_2} v \le u$; on $|x^{n+1}| = 0$, we have $w = 0 \le u$; on $|x'|^2_{\beta_1} = 2d$, $w \le 0 \le u$. Then, in B_2 we compute $\triangle' w = -\frac{\inf_{S_2} v}{d^2} x^{n+1} + (1 + d \sup |f|) \frac{1}{2d\sqrt{\delta}} \ge f$. According to the maximum principle Lemma 5.1, we have $w \le u$ on B_2. As a result, we obtain in B_1,

$$
\begin{aligned}
v &\ge (4 - \frac{|x'|^2_{\beta_1}}{d^2}) \inf_{S_2} v + (1 + d \sup |f|) \frac{\frac{x^{n+1}}{2d} - \delta}{\sqrt{\delta}}\\
&\ge 2 \inf_{S_2} v - d \sup |f|. \qquad (4.15)
\end{aligned}
$$

Note that δ only needs to be an arbitrarily small constant.

Now, notice that $\triangle' u$ is of the divergence form, we could apply the interior Harnack inequality (Proposition 5.12) to $\triangle' u = f$ on B_1^3; since now $\frac{u}{3\delta d} \le v \le \frac{u}{\delta d}$ we obtain

$$
\sup_{B_1^3} v \le C(\inf_{B_1^3} v + \sup |f|).
$$

Here C depends on ω. Since $\inf_{B_1^3} v \le \inf_{S_2} v$, using (4.15), we have

$$
\sup_{B_1^3} v \le C(\inf_{B_1} v + d \sup |f|). \qquad (4.16)
$$

Replacing in the former arguments, v by $v - \inf_{B_4} v$ and then by $\sup_{B_4} v - v$, noticing that they are both positive, and finally adding the resulting inequalities (4.16), we arrive at the following inequality,

$$
\mathrm{Osc}_{B_1} v \le \frac{C-1}{C} \mathrm{Osc}_{B_4} v + 2d \sup |f|.
$$

Then by the iteration Lemma 8.23 in [28], we have the Hölder estimate of v for any $d \leq d_0$,

$$\mathrm{Osc}_{B_d}\, v \leq C\frac{d^\alpha}{d_0^\alpha}(\mathrm{Osc}_{B_{d_0}}\, v + d_0 \sup |f|).$$

For any q in $\mathfrak{X}$, choose $d = |p-q|_{\beta_1}$ and $d_0 = diam(\mathfrak{X})$, we obtain the Hölder continuity of v as

$$\frac{|v(p)-v(q)|}{|p-q|^\alpha_{\beta_1}} \leq C(d_0^{-\alpha}\sup_{B_{d_0}}|v| + d_0 \sup |f|).$$

Since u vanishes on the boundary and depends trivially on the variable y^{n+1}, we have $\partial_{z^{n+1}}u$ is C^α_β. Thus the proposition is proved. □

4.4. **Uniqueness of the $C^{1,1}_\beta$ cone geodesic.** In Theorem 4.5, we have obtained the existence of a $C^{1,1}_\beta$ cone geodesic. Our present goal is to prove its uniqueness. Suppose that Φ_i for $i = 1, 2$ are two cone geodesic segments, which correspond to the solutions $\Psi_{\tau_i} \in C^{2,\alpha}_\beta$ of

$$\begin{cases} \frac{\det(\Omega_{\Psi_{\tau_i}})}{\det(\Omega_1)} = \tau_i e^{a(\Psi_{\tau_i}-\Psi_1)} & \text{in } \mathfrak{M}\ , \\ \Psi_{\tau_i} = \Psi_{0i} & \text{on } \partial\mathfrak{X}\ , \end{cases}$$

for $i = 1, 2$ and $\tau_i \in [0,1]$. Since $\Psi_{\tau_i} \to \Psi_i$ in $C^{1,\alpha}_\beta$ as $\tau_i \to 0$, then for any $\epsilon > 0$ we can find two values τ_1, τ_2 such that

$$\sup_{\mathfrak{X}} |\Psi_i - \Psi_{\tau_i}| \leq \epsilon\ .$$

So, we compute

$$\log\det(\Omega_{\Psi_{\tau_1}}) - \log\det(\Omega_{\Psi_{\tau_2}}) = \int_0^1 g_t^{i\bar{j}} dt (\Psi_{\tau_1} - \Psi_{\tau_2})_{i\bar{j}} > a(\Psi_{\tau_1} - \Psi_{\tau_2})\ ,$$

where $g_t = tg_{\Psi_{\tau_1}} + (1-t)g_{\Psi_{\tau_2}}$ and $a \geq 0$. Now, applying Lemma 5.5 we have,

$$\sup_{\mathfrak{X}}(\Psi_{\tau_1} - \Psi_{\tau_2}) \leq \sup_{\partial\mathfrak{X}}(\Psi_{01} - \Psi_{02})\ .$$

So we have

$$\begin{aligned}\sup_{\mathfrak{X}}(\Psi_1 - \Psi_2) &\leq \sup_{\mathfrak{X}}(\Psi_{\tau_1} - \Psi_1) + \sup_{\mathfrak{X}}(-\Psi_{\tau_2} + \Psi_{\tau_1}) + \sup_{\mathfrak{X}}(\Psi_{\tau_2} - \Psi_2) \\ &\leq 2\epsilon + \sup_{\partial\mathfrak{X}}(\Psi_{01} - \Psi_{02})\ .\end{aligned}$$

Then, switching Ψ_1 and Ψ_2 and letting $\epsilon \to 0$, we end up with

$$\sup_{\mathfrak{X}}|\Psi_1 - \Psi_2| \leq \sup_{\partial\mathfrak{X}}|\Psi_{01} - \Psi_{02}|\ .$$

The above inequality proves the uniqueness of a cone geodesic segment with prescribed boundary values.

5. Linearized equation

In this section we consider the general linear elliptic equation

$$\begin{cases} Lv &= g^{i\bar{j}} v_{i\bar{j}} + b^i v_i + cv = f + \partial_i h^i \\ v &= v_0 \end{cases} \tag{5.1}$$

in the space $(\mathfrak{X}, \mathfrak{D})$ defined in Section 2. Here $g^{i\bar{j}}$ is the inverse matrix of a Kähler cone metric Ω in $H^{2,\alpha}_{\beta}$. Moreover, we are given the following data.

$$b^i, h^i \in C^{1,\alpha}_{\beta}; c, f \in C^{\alpha}_{\beta} \text{ and } v_0 \in C^{2,\alpha}_{\beta} \ . \tag{5.2}$$

We also denote the vector field $h^i \partial_i$ to be $\mathbf{h}$.

This type of equation has been studied via the general edge calculus theory (cf. Mazzeo [42] and references therein). However, we consider in this paper a Kähler manifold with boundary. The edge space is not defined near the boundary. Recently, Donaldson introduced a function space on a closed Kähler manifold which fits well with our geometric problem. In Section 2, Definition 2.3, we generalized Donaldsons space to the boundary case and thus introduced a Hölder space. Now we study (5.1) in this Hölder space. We collect here the analytic results on the linear equation (5.1) which are not only used in previous arguments above but also for our further applications.

5.1. The maximum principle and the weak solution.

We say that v is the solution of (5.1) if it satisfies this equation on $\mathfrak{X} \setminus \mathfrak{D}$ and belongs to $C^{2,\alpha}_{\beta}$. From the theory of the elliptic equation, we know that V is smooth outside $\mathfrak{D}$. The delicate part here is always the estimate near the divisor. We first prove a maximum principle for the Kähler cone metric.

Lemma 5.1. *Assume that v satisfies $Lv \geq 0$ (resp. $Lv \leq 0$) with $c < 0$, then the maximum (minimum) is achieved on the boundary i.e.*

$$\sup_{\mathfrak{X}} v = \sup_{\partial\mathfrak{X} \setminus \partial\mathfrak{D}} v \qquad \left(\inf_{\mathfrak{X}} v = \inf_{\partial\mathfrak{X} \setminus \partial\mathfrak{D}} v \right) .$$

Proof. Set $u = v + \epsilon S$ and $S = \|s\|^{2\kappa}$ with $(1+\alpha)\beta > 2\kappa \geq \beta$. Then $|\partial S|_g$ is bounded. Suppose that p is the maximum point of u. According to Lemma 2.4, p cannot be on $\mathfrak{D}$. So either p stays on the boundary $\partial\mathfrak{X} \setminus \partial\mathfrak{D}$ or in the interior of $\mathfrak{X} \setminus \mathfrak{D}$. Then in the latter case, at the maximum point p we have

$$0 \leq Lv = Lu - \epsilon LS \leq cu - \epsilon(\triangle_g S + b^i S_i + cS) \leq cu + \epsilon C \ .$$

Here we use $b^i S_i \geq -|b^i|^2_g - |\partial S|^2_g$ and the first conclusion in Lemma 2.4, $\triangle_g S \geq -C$. Combining these inequalities we obtain

$$u(p) \leq \epsilon C \ .$$

Then at any point $x \in \mathfrak{X}$, we have the following relation

$$v(x) = u(x) - \epsilon S \leq u(p) \leq \sup_{\partial\mathfrak{X}\setminus\partial\mathfrak{D}} v + \epsilon C \ ,$$

since S is nonnegative. Similarly, we shall use $u = v - \epsilon F$ instead of $Lv \leq 0$. As a result, the proposition follows as $\epsilon \to 0$. □

Now we use the maximum principle to deduce the uniqueness of solutions of the elliptic equation (5.1).

Corollary 5.2. *If v_1, v_2 are two solutions of the linearized equation* (5.1) *with $c < 0$, then $v_1 = v_2$.*

The singular volume form ω^{n+1} with respect to the cone metric gives a measure on the manifold $\mathfrak{X}$. As a consequence, the $L^p(\mathfrak{X}, g)$ space is defined in the usual way. The $W^{1,p}(\mathfrak{X}, g)$ space furthermore requires that the derivatives satisfy $\int_{\mathfrak{X}} |\nabla f|^p_\Omega \omega^{n+1} < \infty$.

Definition 5.1. The weak solution in $W^{1,2}$ of (5.1) is defined, for any $\eta \in W^{1,2}_0$, in the sense of distributions;

$$\mathcal{L}(v,\eta) = \int_{\mathfrak{X}} [g^{i\bar{j}} v_i \eta_{\bar{j}} - b^i v_i \eta - cv\eta]\omega^{n+1} = \int_{\mathfrak{X}} [-\eta f - h^i \eta_i]\omega^{n+1}. \tag{5.3}$$

Note that our weak solution is defined globally.

The following lemmas follow directly from the local lifting $P \circ W$ (cf. (2.2)).

Lemma 5.3. *(Sobolev imbedding) Assume that $f \in W^{1,2}_0$. Then there is a constant C depending on n, β such that*

$$||f||_{\frac{2n+2}{n}} \leq C||f||_{W^{1,2}} \ .$$

Lemma 5.4. *(Kondrakov compact imbedding) The imbedding $W^{1,2}_0 \to L^p$ for $1 \leq p < \frac{2n+2}{n}$ is compact.*

Lemma 5.5. *(Weak maximum principle) Let $v \in W^{1,2}$ satisfy $Lv \geq 0 (\leq 0)$ in $\mathfrak{X}$ with $c \leq 0$. Then*

$$\sup_{\mathfrak{X}} v \leq \sup_{\partial\mathfrak{X}} v^+ \qquad \left(\inf_{\mathfrak{X}} v \geq \sup_{\partial\mathfrak{X}} v^- \right) \ .$$

Proof. From the definition of weak solution we have that $Lv \geq 0$ implies $\mathcal{L}(v,\eta) \leq 0$. Then for $\eta \geq 0$, we have

$$\int_{\mathfrak{X}^+} [g^{i\bar{j}} v_i \eta_{\bar{j}} - b^i v_i \eta]\omega^{n+1} \leq 0 \ ,$$

where $\mathfrak{X}^+ = \{x \in \mathfrak{X} | v(x) \geq 0\}$. Let $v^+ = \max\{0, v\}$. If $b^i = 0$, letting $\eta = \sup\{0, v - \sup_{\partial\mathfrak{X}} v^+\}$, we have

$$\int_{\mathfrak{X}^+} |\nabla \eta|^2 \omega^{n+1} \leq 0 \ .$$

So $|\nabla\eta|^2 = 0$ on $\mathfrak{X}^+ \setminus \mathfrak{D}$. Since $\eta = 0$ at the maximum point on the boundary of $\mathfrak{X}^+$, we obtain $\eta = 0$ on $\mathfrak{X}^+ \setminus \mathfrak{D}$. Since the measure of $\mathfrak{D}$ is zero, we could modify the value of η such that $\eta = 0$ on the whole $\mathfrak{X}$. Then the lemma follows for $b^i = 0$. When $b^i \neq 0$, using the Sobolev inequality (5.3), the proof is the same as that of Theorem 8.1 in [28]. □

Then this lemma and a standard argument by means of the Fredholm alternative theorem implies the uniqueness and the existence of the weak solution.

Proposition 5.6. *The linear equation* (5.1) *with* $c \leq 0$ *has a unique weak solution in* $W^{1,2}$.

5.2. **Hölder estimates.** We remark that in this subsection, all results hold for normal-crossing divisors D with more than one component. The normal crossing condition means that at each point, the divisor locally looks like the intersection of coordinate hyperplanes. So at each intersection point, we can have a coordinate system $\{z^i; 1 \leq i \leq n+1\}$ such that $\{z^i; 1 \leq i \leq k\}$, for some k, denote the singular directions and the reminders are the smooth directions. In the following proofs, we mainly check one singular direction, since for the case of multiple singular directions, the proof still holds by checking multiple integrals. We shall emphasize this point in each proof.

The Hölder estimates derived in this subsection are used in the proof of both the interior and boundary Schauder estimates of the approximate geodesic equation. Before stating the proposition on the global and local boundedness, we require some technical lemmas which will be useful later. Denote

$$\omega_0 = dz^1 \wedge d\overline{z}^1 + \cdots + dz^{n+1} \wedge d\overline{z}^{n+1}.$$

Then locally in a neighborhood U_p near $p \in \mathfrak{D}$,

$$\omega_0^{n+1} = n! \cdot dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^{n+1} \wedge d\overline{z}^{n+1},$$

and then we have that there is a bounded function h such that

$$\omega^{n+1} = \beta^2 |z^1|^{2(\beta-1)} \omega_0^{n+1} e^h \ .$$

n the case of k singular directions,

$$\omega^{n+1} = \Pi_{1\leq i\leq k} \beta_i^2 |z^i|^{2(\beta_i-1)} \omega_0^{n+1} e^h \ .$$

Finally, let $m = 2n+2$.

Lemma 5.7. *There is a constant C depending on $|\mathbf{h}|_\infty$ such that, for any $s > \frac{1}{\beta}$, the following inequality holds*

$$\left(\int_{U_p} f^p \omega^{n+1} \right)^{\frac{1}{p}} \leq C \left(\int_{U_p} f^{sp} \omega_0^{n+1} \right)^{\frac{1}{sp}} \ .$$

Proof. Let $z^1 = \rho e^{i\theta}$ and compute

$$\left(\int_{U_p} f^p \omega^{n+1}\right)^{\frac{1}{p}} = \left(\int_{U_p(z')}\int_0^{r_2}\int_0^{2\pi} f^p\beta^2\rho^{2(\beta-1)}e^h\omega_0^{n+1}\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{U_p(z')}\int_0^{r_2}\int_0^{2\pi} f^{sp}e^h\omega_0^{n+1}\right)^{\frac{1}{sp}} \cdot \left(\int_{U_p(z')}\int_0^{r_2}\int_0^{2\pi} (\rho^{2\beta-2})^t e^h\omega_0^{n+1}\right)^{\frac{1}{tp}} .$$

Here $\frac{1}{s}+\frac{1}{t}=1$. Since $t<\frac{1}{1-\beta}$, the second term is bounded, we have $s>\frac{1}{\beta}$, which concludes the proof. In the case of multiple singular directions, the proof is accomplished by means of continuing the above argument inductively along the direction z^i ranging from 2 to k in $U_p(z')$. □

Lemma 5.8. *There is a constant C depending on β and $|\mathbf{h}|_\infty$ such that, for any $s>1$, the following formula holds*

$$\left(\int_{U_l} f^p\omega_0^{n+1}\right)^{\frac{1}{p}} \leq C\left(\int_{U_l} f^{sp}\omega^{n+1}\right)^{\frac{1}{sp}} .$$

Proof. Again we compute in polar coordinates

$$\left(\int_{U_p} f^p\omega_0^{n+1}\right)^{\frac{1}{p}} = \left(\int_{U_p(z')}\int_0^{r_2}\int_0^{2\pi} f^p\rho^{\frac{2(\beta-1)}{s}}\rho^{-\frac{2(\beta-1)}{s}}\omega_0^{n+1}\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{U_p} f^{sp}\beta^{-2}e^{-h}\omega^{n+1}\right)^{\frac{1}{sp}} \left(\int_{U_p(z')}\int_0^{r_2}\int_0^{2\pi} \rho^{-\frac{2(\beta-1)t}{s}}\omega_0^{n+1}\right)^{\frac{1}{tp}} .$$

Here s,t are two positive constants such that $\frac{1}{s}+\frac{1}{t}=1$. The second term is bounded as $\frac{t}{s}>\frac{1}{\beta-1}$ which is trivially satisfied. In the case of multiple singular directions, the proof is accomplished again by the iteration of the argument. □

The proofs of the following propositions is in the same vein as the proofs in Chapter 8 in [28]. However, by the lemmas stated above, we need a careful analysis in the charts which intersect the divisor.

Proposition 5.9. *(Global boundedness) If v is a $W^{1,2}$ sub-solution (respectively super-solution) of* (5.1) *in $\mathfrak{X}$ satisfying $v\leq 0$ (resp.$v\geq 0$) on $\partial\mathfrak{X}$; moreover, if $f\in L^{\frac{q}{2}}$ and $h^i\in L^q$, $i=1,\cdots,n+1$ with $q>m$ then there is a constant C depending on $|b^i|_g$, $|c|_\infty$, q, β such that*

$$\sup_{\mathfrak{X}} v(-v) \leq C(\|v^+(v^-)\|_2 + \|f\|_{\frac{q}{2}} + \|h^i\|_q).$$

Proof. Assume that v is a $W^{1,2}$ sub-solution of (5.1). We are going to use the De Giorgi-Nash-Moser iteration as in [28, Theorem 8.15].

Denote $k = \|f\|_{\frac{q}{2}} + \|h^i\|_q$. Choose $w = v^+ + k$ and $\eta = \int_k^w a^2 s^{2(a-1)} ds$ for $a \geq 1$ in $\mathcal{L}(v, \eta)$. With the Sobolev inequality Lemma 5.3, we have

$$\|w\|_{\frac{(2n+2)a}{n};\omega} \leq (C(a+1))^{\frac{1}{a}} \|w\|_{2a;\omega} .$$

We use Lemma 5.7 and Lemma 5.8 on the coordinates which intersect the divisor $\mathfrak{D}$ and the Hölder inequality in the remainder coordinates. After patching them together via a partition of the unity we have, for $s > \frac{1}{\beta} \geq 1$,

$$\|w\|_{\frac{(2n+2)as}{n};\omega_0} \leq (C(a+1))^{\frac{1}{a}} \|w\|_{2as;\omega_0} .$$

Now we follow a standard iteration argument; using the interpolation inequality we have with $\chi = \frac{n+1}{n}$

$$\|w\|_{\frac{\chi^{N2s}}{n};\omega_0} \leq C\|w\|_{\frac{2}{s};\omega_0} .$$

Finally, letting $N \to \infty$ and using Lemma 5.8 again, we get the proposition. □

Denote as d the distance measured via the Kähler cone metric ω.

Proposition 5.10. *(Local boundedness) Suppose that v is a $W^{1,2}$ sub-solution of* (5.1) *and suppose that $f \in L^q$, and $h^i \in L^q$, $i = 1, \cdots, n+1$ with $q > m$. Then for any ball $B_{2d}(y) \subset \mathfrak{X}$ and any $p > 1$ there is a constant C depending on $(|b^i|_g + |c|_\infty)d$, q, β, p such that*

$$\sup_{B_d(y)} v(-v) \leq C(d^{-\frac{m}{p}} \|v^+(v^-)\|_{L^p(B_{2d}(y))} + d^{2(1-\frac{m}{2q})} \|f\|_{\frac{q}{2}} + d^{1-\frac{m}{q}} \|h^i\|_q) .$$

Proof. We will prove the local boundedness of the homogeneous equation. The general case follows by means of using $v + d^{2(1-\frac{m}{2q})}\|f\|_{\frac{q}{2}} + d^{1-\frac{m}{q}} \|h^i\|_q$ instead of v. Then v would be a weak sub-solution of (5.1) with $f = 0$ and $h^i = 0$; namely $\mathcal{L}(v, \eta) \leq 0$. Assume $d = 1$ and take the test function to be $\eta^2 v^\alpha$ for $\eta \in C_0^1(B_4)$ and $\alpha > 0$. Then we have for $w := v^{\frac{\alpha+1}{2}}$

$$\|\eta w\|_{\frac{2n+2}{n};\omega} \leq C \cdot (\|w\partial\eta\|_{2;\omega} + \|w\eta\|_{2;\omega}) .$$

Using Lemma 5.7 and Lemma 5.8 we obtain, on any open set U_p which intersects the divisor D for $s > \frac{1}{\beta}$

$$\|\eta w\|_{\frac{2n+2}{s(n)};\omega_0} \leq C[\|w\partial\eta\|_{2;\omega} + \|w\eta\|_{2s;\omega_0}] .$$

We claim that for the first addendum on the right hand side it holds

$$\|w\partial\eta\|_{2;\omega} \leq C\|w\partial\eta\|_{2s;\omega_0}$$

with $s > \frac{1}{\beta}$. Again, by means of Lemma 5.7 and Lemma 5.8 we compute

$$\|w\partial\eta\|_{2;\omega} = \left[\int_{U_p} w^2 (\sum_{i=1}^{k} \partial_{z_i}\eta\partial_{z_{\bar{i}}}\eta |z^i|^{2(1-\beta_i)} \frac{1}{\beta^2} + \sum_{i=k+1}^{n+1} \partial_{z_i}\eta\partial_{z_{\bar{i}}}\eta)\omega^{n+1}\right]^{\frac{1}{2}}$$

$$\leq \left[\int_{U_p} w^2 (\sum_{i=1}^{k} \partial_{z_i}\eta\partial_{z_{\bar{i}}}\eta) e^h \omega_0^{n+1} + C\left(\int_{U_p} \sum_{i=k+1}^{n+1} w^{2s} (\partial_{z_i}\eta\partial_{z_{\bar{i}}}\eta)^s \omega^{n+1}\right)^{\frac{1}{s}}\right]^{\frac{1}{2}}$$

$$\leq C\left(\int_{U_p} w^{2s} |\partial\eta|^{2s}_{\omega_0} \omega_0^{n+1}\right)^{\frac{1}{2s}} ,$$

where to get the last step we used the Hölder inequality on the first term.

So standard argument with Lemma 5.8 implies

$$\|v\|_{\infty;B_1,\omega_0} \leq C\|v\|_{ps;B_2,\omega_0} \leq C\|v\|_{ps^2;B_2,\omega} .$$

The local boundedness follows from the next observation; $B_1(0,\omega) \subset B_1(0,\omega_0)$ which follows from the distance inequality,

$$\sqrt{\sum_{i=1}^{k} |z^1|^2 + \sum_{i=k+1}^{n} |z^i|^2} \leq \sqrt{\sum_{i=1}^{k} |z^1|^{2\beta_i} + \sum_{i=k+1}^{n} |z^i|^2} \leq 1.$$

□

Proposition 5.11. *(Weak Harnack inequality) Suppose that v is a $W^{1,2}$ super-solution of (5.1), non-negative in a ball $B_{4d}(y) \subset \mathfrak{X}$ and suppose that $f \in L^{\frac{q}{2}}$ and $h^i \in L^q$, $i = 1, \cdots, n+1$ with $q > m$. Then, for any $\frac{n+1}{n} > p > 1$ there is a constant C depending on $(|b^i|_g + |c|_\infty)d$, q, β, p such that*

$$d^{-\frac{m}{p}}\|v\|_{L^p(B_{2d}(y))} \leq C\left\{\inf_{B_{d(y)}} v + d^{2(1-\frac{m}{q})}||f||_{\frac{q}{2}} + d^{1-\frac{m}{q}}\|h^i\|_q\right\} . \tag{5.4}$$

Proof. We assume $d = a$ and argue as in the proof of the local boundedness with different test function. Thus it suffices to prove, for the weak super-solution of (5.1) with vanishing right hand side, that there is a $p > 0$ and constant C such that

$$\int_{B_2} v^{-p}\omega^{n+1} \int_{B_2} v^p \omega^{n+1} \leq C . \tag{5.5}$$

Choose a test function of the form $\eta^2 v^\alpha$ and let $w := \log v$ and $\alpha = -1$. Here η is the cut-off function defined in Lemma 2.1. We have by the Cauchy Schwarz's inequality for small ϵ_1 and ϵ_2,

$$\int_{B_r} |\partial w|^2 \omega^{n+1} \leq \frac{2}{\epsilon_1}\int_{\mathfrak{X}} |\partial\eta|^2\omega^{n+1} + 2\left(\frac{|b^i|_0}{4\epsilon_2} + |c|_0\right)\int_{\mathfrak{X}} \eta^2\omega^{n+1} .$$

Since $(\mathfrak{X},\omega)$ has finite volume, the second term is bounded. Concerning the first term, we compute,

$$\int_{U_p} |\partial\eta|^2\omega^{n+1} \leq C\int_0^{2\pi}\int_0^r t^{4-2\beta+2(\beta-1)}dtd\theta \leq Cr^3.$$

We conclude that $\int_{B_r}|\partial w|\omega^{n+1}$ is bounded.

Next we claim that $\int_{B_r}|\partial w|_0\omega_0^{n+1}$ is also bounded. To prove the claim, let's compute

$$\int_{B_r}|\partial w|_0\omega_0^{n+1} = \int_{B_r}(|\sum_{i=1}^k \partial_{z^i}w|_0^2 + \sum_{i=k+1}^{n+1}|\partial_{z^i}w|_0^2)^{\frac{1}{2}}\beta^{-2}\Pi_{1\leq i\leq k}|z^i|^{2(1-\beta_i)}e^{-h}\omega^{n+1}.$$

The second term is bounded, since h and $|z^i|$ are bounded. For the first term, when $i = 1,\cdots,k$, we first consider the case when $|\partial_{z^i}w|_0 \leq 1$. Then its boundedness follows from the finiteness of the volume. The second case is when $|\partial_{z^i}w|_0 > 1$. In this second case $|\partial_{z^i}w|_0 < |\partial_{z^i}w|_0^2$ and so its integral is bounded by $\int_{B_r}|\partial w|^2\omega^{n+1}$. The claim thus holds.

Now we apply the Moser-Trudinger inequality (see [28, Theorem 7.21]) with respect to ω_0. Thus there exists a constant p_0 such that

$$\int_{B_3} e^{p_0|w-w_0|}\omega_0^{n+1}$$

is bounded and so is

$$\int_{B_3} v^{p_0}\omega_0^{n+1}\int_{B_3} v^{-p_0}\omega_0^{n+1}\ .$$

From Lemma 5.7 we have, for some $s_0 > \beta^{-1}$,

$$\int_{B_3} v^{\frac{p_0}{s_0}}\omega^{n+1}\int_{B_3} v^{\frac{-p_0}{s_0}}\omega^{n+1} \leq C\left(\int_{B_3} v^{p_0}\omega_0^{n+1}\int_{B_3} v^{-p_0}\omega_0^{n+1}\right)^{\frac{1}{s_0}} \leq C\ .$$

The above inequality gives the wanted inequality (5.5) with $p = \frac{p_0}{s_0}$. The proof of the proposition is therefore achieved. □

As a result we have the following estimates.

Proposition 5.12. *(The Harnack inequality) For any $B_{4d}(y) \subset \mathfrak{X}$, suppose that v is a non-negative $W^{1,2}$ solution of* (5.1) *with homogeneous right hand side in a ball $B_{4d}(y) \subset \mathfrak{X}$. Then, there is a constant C depending on $(|b^i|_g + |c|_\infty)d$, β such that*

$$\sup_{B_d} v \leq C\inf_{B_d} v. \tag{5.6}$$

Proposition 5.13. *(Interior Hölder estimate) Suppose that v is a $W^{1,2}$ solution of* (5.1) *in $\mathfrak{X}$ and suppose that $f \in L^{\frac{q}{2}}$ and $h^i \in L^q$ with*

$q > m$. Then, for any $B_{d_0}(y) \subset \mathrm{Int}\,\mathfrak{X}$ and $d \leq d_0$, there is a constant $C(|b^i|_g, |c|_\infty, d_0, q)$ and $\alpha((|b^i|_g + |c|_\infty)d_0, q)$ such that

$$osc_{B_d(y)} v \leq C d^\alpha (d_0^{-\alpha} \sup_{B_{d_0}(y)} |v| + d^{2(1-\frac{m}{q})} ||f||_{\frac{q}{2}} + d^{1-\frac{m}{q}} \|h^i\|_q).$$

5.2.1. *Local estimates at the boundary.* Consider a point $y \in \partial\mathfrak{X}$ and using the local holomorphic coordinate in the half space $\bar{\mathbb{R}}^{2n+2}_+ = \{x | x^{n+1} \geq 0\}$, here x^{n+1} is the real part of the variable z^{n+1}. Then the coordinate chart near y becomes a domain T in $\bar{\mathbb{R}}^{2n+2}_+$. Recall that we assumed v_0 in $C^\alpha_\beta(\partial\mathfrak{X})$ in (5.2). We let

$$M := \sup_{\partial\mathfrak{X}} \cap B_{2d} v\,, \qquad m := \inf_{\partial\mathfrak{X}} \cap B_{2d} v\ .$$

Moreover we extend v from the half space to the whole space $\mathbb{R}^{2n+2}$.

$$v^+_M := \begin{cases} \sup\{v(x), M\}, & x \in T \\ M & x \notin T\ . \end{cases}$$

$$v^-_m := \begin{cases} \inf\{v(x), m\}, & x \in T \\ m & x \notin T\ . \end{cases}$$

Just by following the proof of interior estimates, we obtain the following results.

Proposition 5.14. *(Local boundedness at the boundary) Suppose that v is a $W^{1,2}$ sub-solution of (5.1) and suppose that $f \in L^q$, and $h^i \in L^q$, $i = 1, \cdots, n+1$ with $q > m$. Then for any ball $B_{2d}(y)$ and any $p > 1$ there is a constant C depending on $(|b^i|_g + |c|_\infty)d$, q, β, p such that*

$$\sup_{B_d(y)} v^+_M \leq C(d^{-\frac{m}{p}} \|v^+_M\|_{L^p(B_{2d}(y))} + d^{2(1-\frac{m}{q})} ||f||_{\frac{q}{2}} + d^{1-\frac{m}{q}} \|h^i\|_q).$$

Proposition 5.15. *(Weak Harnack inequality at the boundary) Suppose that v is a $W^{1,2}$ super-solution of (5.1), non-negative in a ball $B_{4d}(y) \cap T$ and suppose that $f \in L^{\frac{q}{2}}$ and $h^i \in L^q$, $i = 1, \cdots, n+1$ with $q > m$. Then, for any $\frac{n+1}{n} > p > 1$ there is a constant C depending on $(|b^i|_g + |c|_\infty)d$, q, β, p such that*

$$d^{-\frac{m}{p}} \|v^-_m\|_{L^p(B_{2d}(y))} \leq C(\inf_{B_{d(y)}} v^-_m + d^{2(1-\frac{m}{q})} ||f||_{\frac{q}{2}} + d^{1-\frac{m}{q}} \|h^i\|_q).$$

Proposition 5.16. *(Höder estimate at the boundary) Suppose that v is a $W^{1,2}$ solution of (5.1) in $\mathfrak{X}$ and $f \in L^{\frac{q}{2}}$ and $h^i \in L^q$, $i = 1, \cdots, n+1$ with $q > m$. Suppose that y is on the boundary of $\mathfrak{X}$. Then, for any $B_{d_0}(y)$ and $d \leq d_0$, there is a constant $C(|b^i|_g, |c|_\infty, d_0, q)$ and $\alpha((|b^i|_g + |c|_\infty)d_0, q)$ such that*

$$\mathrm{Osc}_{B_d(y) \cap \mathfrak{X}}\, v \leq C\{d^\alpha (d_0^{-\alpha} \sup_{B_{d_0}(y) \cap T} |v| + d^{2(1-\frac{m}{q})} ||f||_{\frac{q}{2}} + d^{1-\frac{m}{q}} \|h^i\|_q) + \mathrm{Osc}_{B_{\sqrt{d_0 d}}(y) \cap \partial\mathfrak{X}}\, v\}.$$

5.3. **The Dirichlet problem of the linearized problem.** Fix $0 < \beta < 1$, and write

$$\mu := \beta^{-1} - 1.$$

Denote by G the Green function of the standard cone metric

$$dr^2 + \beta^2 r^2 d\theta^2 + \sum_{3\leq i\leq m} ds_i^2$$

and by T one of the second order operators

$$\frac{\partial^2}{\partial s^i \partial s^j}, r^{-1}\frac{\partial^2}{\partial\theta\partial s^i}, \frac{\partial^2}{\partial r\partial s^i} .$$

It is shown by Donaldson in [25, Proposition 4] that the polyhomogeneous expansion of the Green function around the singular set D is

$$G = \sum_{j,k} a_{j,k}(s) r^{\nu+2j} \cos k(\theta - \theta') . \tag{5.7}$$

Donaldson proved the following Schauder estimate.

Proposition 5.17. *(Donaldson [25]) Suppose that $\alpha \in (0,\mu)$, then there exists a constant C which depends only on α, m, β such that for all functions $\rho \in C_c^\infty(\mathbb{R}^m)$, we have*

$$[i\partial\bar\partial(G\rho)]_{C^\alpha_\beta} \leq C[\rho]_{C^\alpha_\beta} .$$

In our problem, the interior Schauder estimate follows by applying Proposition 5.17.

Lemma 5.18. *Supposing that $\alpha \in (0,\mu)$, then there exists a constant C which depends only on α, n, β such that for all functions $\rho \in C_c^\infty(\mathbb{R}^{2n+2})$, we have*

$$[v]_{C^\alpha_\beta} \leq C[[\rho]_{C^\alpha_\beta} + [h^i]_{C^{1,\alpha}_\beta}] .$$

Proof. The weak solution of $\triangle_g v = f + \partial_i h^i$ has the form

$$v = \int_{R^{2n+2}} [G(x,y)f(y) + < \nabla G(x,y), \mathbf{h}(y) >_g] dg(y).$$

We need to deal with the divergence term on the left hand side of (5.1). We estimate the second derivatives of this term, we put one derivative on G and the other one on $\mathbf{h}$. Thus the lemma holds. □

When we consider the Schauder estimate near the boundary, we notice that our manifold is a product manifold $X \times R$, and $R = [0,1] \times S^1$. Since the solution

$$\Psi(z^1, \cdots, z^n, x^{n+1} + \sqrt{-1}y^{n+1}) = \Psi(z^1, \cdots, z^n, x^{n+1})$$

along $y^{n+1} \in S^1$, we work on $\bar{\mathbb{R}}^{m+1}_+$, which is the closure of the upper half space of $\mathbb{R}^{m+1}$ when we consider the local model near the boundary. Letting $B_2(x_0), B_1(x_0)$ be the balls with center x_0 on $\bar{\mathbb{R}}^{m+1}_+$, we want to consider the Schauder estimate for the Dirichlet problem of the Laplacian operator $\triangle_g$ with respect to the standard cone metric

$$g = dr^2 + \beta^2 r^2 d\theta^2 + \sum_{3\leq i\leq m} ds_i^2 + ds^2$$

on $\mathbb{R}^2 \times \mathbb{R}^{m-2} \times \bar{\mathbb{R}}_+$. Letting $f \in C_c^{\alpha}(\bar{\mathbb{R}}_+^{m+1})$, $\mathbf{h} \in C_c^{1,\alpha}(\bar{\mathbb{R}}_+^{m+1})$, we would look for the solution v which satisfies the Laplacian equation

$$\triangle_g v = f + \partial_i h^i$$

and the boundary condition $v = 0$ on $\mathbb{R}^m \times \{x^{n+1} = 0\}$. Note that the standard cone metric is equivalent to the Euclidean metric. We declare v a weak solution if it satisfies the following identity, for any $\phi \in C_c^{\infty}(\bar{\mathbb{R}}_+^{m+1})$,

$$\int_{\bar{\mathbb{R}}_+^{m+1}} < \nabla\phi, \nabla v >_g dg = \int_{\bar{\mathbb{R}}_+^{m+1}} f v dg + \int_{\bar{\mathbb{R}}_+^{m+1}} < \mathbf{h}, \nabla v >_g dg.$$

The left hand side is a bounded coercive bilinear form and the right hand side is bounded by $||\phi||_{L^2}||v||_{W^{1,2}}$. So the Lax-Milgram theorem shows that there exists a weak solution of the Laplacian equation. The Green function is defined to be the kernel function $G(x, y)$ such that

$$v = \int_{\bar{\mathbb{R}}_+^{m+1}} [G(x, y)f + < \nabla G(x, y), \mathbf{h}(y) >_g] dg(y). \tag{5.8}$$

Since ρ has compact support within $\bar{\mathbb{R}}_+^{m+1}$, the asymptotic behavior of this Green function is the same as (5.7). Also, the second derivative of the second term is obtained by putting one derivative on $\mathbf{h}$.

Now we consider the Schauder estimate of v.

Lemma 5.19. *Supposing that $\alpha \in (0, \mu)$, then there exists a constant C which depends only on α, n, β such that for all functions $\rho \in C_c^{\infty}(\bar{\mathbb{R}}_+^{m+1})$, we have*

$$[v]_{C^{\alpha}_{\beta}} \leq C[[\rho]_{C^{\alpha}_{\beta}} + [h^i]_{C^{1,\alpha}_{\beta}}] .$$

Proof. On the whole space $\mathbb{R}^2 \times \mathbb{R}^{m-1}$, we use G_0 to denote the Green function defined by (5.7). So on the half space $\mathbb{R}^2 \times \mathbb{R}^{m-2} \times \bar{\mathbb{R}}_+$, our Green function is written down by the reflexion method for any $x, x' \in \mathbb{R}^2 \times \mathbb{R}^{m-2}$, $s, s' \in \bar{\mathbb{R}}_+$,

$$G(x, x^{n+1}; x', x^{n+1\prime}) = G_0(x, s; x', x^{n+1\prime}) - G_0(x, x^{n+1}; x', -x^{n+1\prime}). \tag{5.9}$$

So G has the same asymptotic behaviour of G_0 around the divisor. Furthermore, when i or j is not equal to the singular direction 1, the $\partial_i\bar{\partial}_j$ estimate follows exactly the same line of [25, Theorem 1]. The $\partial_1\bar{\partial}_1$ estimate follows from the equation (see [28, Section 4.4]). □

Now we patch the local estimates together to the whole manifold by the partition of unity in the standard way.

Proposition 5.20. *Fix α with $0 < \alpha < \mu = \beta^{-1} - 1$. Then there is a constant C depending on β, n, α, $|b^i|_g$, $|c|_\infty$ such that for all the functions $f \in C^\alpha_\beta$ and $h^i \in C^{1,\alpha}_\beta$ we have the Schauder estimate of the weak solution of the equation* (5.1)

$$|v|_{C^{2,\alpha}_\beta} \le C(|v|_{L^\infty} + |f|_{C^\alpha_\beta} + \sum_i |h^i|_{C^{1,\alpha}_\beta}) .$$

Combining the existence and uniqueness of the weak solution Proposition 5.6, we obtain

Proposition 5.21. *There exists a unique solution of* (5.1) *with data as* (5.2) *in $C^{2,\alpha}_\beta$.*

The linear theory in this section immediately implies the $\partial\bar{\partial}$-lemma with cone singularities.

6. The metric space structure

In this section we apply our geodesic to study the geometry of the space of Kähler cone metrics. We equip the space of Kähler cone metrics with the following normalization condition; we ask any Kähler cone potential φ with respect to the background model metric ω to satisfies $I(\varphi) = 0$, where

$$I_\omega(\varphi) = \frac{1}{V}\int_M \varphi\omega^n - \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\varphi\wedge\bar{\partial}\varphi\wedge\omega^i\wedge\omega_\varphi^{n-i-1}.$$

In particular, the functional $I(\varphi)$ is well defined along any $C^{1,1}_\beta$ geodesic. We show that the space of cone metrics has a structure of metric space following the approach in [13]. We said that $\varphi(t)$ is an ϵ-approximate geodesic if it solves

$$(\varphi'' - |\partial\varphi'|^2_{g_\varphi})\det g_\varphi = \epsilon f \det g, \tag{6.1}$$

where $f = \frac{\det\Omega_1}{\det\Omega} = |m\Phi_{n+1,\overline{n+1}} - \partial(\varphi(1) - \varphi(0))|_\Omega$. Recall that the energy is defined as $E := \int_0^1\int_M \varphi'(t)\omega^n_{\varphi(t)}dt$. Along any $C^{1,1}_\beta$ geodesic, there holds

$$\frac{1}{2}\left|\frac{d}{dt}E\right| = \left|\int_M \varphi'(\varphi'' - |\partial\varphi|^2_{g_\varphi})\omega^n_\varphi\right| \le \epsilon\sup_{\mathfrak{X}}|\phi'|\cdot\sup_{\mathfrak{X}}|f|\cdot\mathrm{Vol}\ . \tag{6.2}$$

We show the positivity of the length of any non-trivial geodesic segment and the geodesic approximation lemma. We omit the proof here, since along any $C^{1,1}_\beta$ geodesic, all the inequalities are well defined.

Proposition 6.1. *Let $\varphi(t)$ be a $C^{1,1}_\beta$ geodesic from 0 to φ, and $I(\varphi) = 0$. Then the following inequality holds*

$$\int_0^1\sqrt{\int_M(\varphi')^2\frac{\omega^n_\varphi}{n!}dt} \ge \mathrm{Vol}^{-\frac{1}{2}}\left(\sup\left(\int_{\varphi>0}\varphi\frac{\omega^n_\varphi}{n!}, \int_{\varphi<0}\varphi\frac{\omega^n_0}{n!}\right)\right) .$$

In particular, the length of any non-constant $C^{1,1}_\beta$ geodesic is positive.

Lemma 6.2. *Let $\mathcal{H}_C \subset \mathcal{H}_\beta$ be as in Definition 1.2. Also, let $C_i := \varphi_i(s) : [0,1] \to \mathcal{H}_C$, for $i = 1, 2$, be two smooth curves. Then, for a small enough ϵ_0, there is a two-parameter family of curves*

$$C(s,\epsilon) : \varphi(t,s,\epsilon) : [0,1] \times [0,1] \times (0,\epsilon_0] \to \mathcal{H}$$

such that the following properties hold:

1. *Fixed s, ϵ, then $C(s,\epsilon) \in C^{2,\alpha}_\beta$ is an ϵ-approximate geodesic from $\varphi_1(s)$ to $\varphi_2(s)$.*
2. *There exists a uniform constant C such that*
$$|\varphi| + \left|\frac{\partial \varphi}{\partial t}\right| + \left|\frac{\partial \varphi}{\partial s}\right| < C; \quad 0 \le \frac{\partial^2 \varphi}{\partial t^2} < C; \quad \frac{\partial^2 \varphi}{\partial s^2} < C\ .$$
3. *Fixed any s, the limit in $C^{1,1}_\beta$ of $C(s,\epsilon)$ as $\epsilon \to 0$ is the unique geodesic arc from $\varphi_1(s)$ to $\varphi_2(s)$.*
4. *There exists a uniform constant C such that, about the energy $E(t,s,\epsilon)$ along the curve $C(s,\epsilon)$, there holds*
$$\sup_{t,s} \left|\frac{\partial E}{\partial t}\right| \le \epsilon \cdot C \cdot \mathrm{Vol}\ .$$

With the geodesic approximation lemma above, the triangular inequality and the differentiability property of the distance function follow immediately.

Theorem 6.3. *Suppose that $\phi = \varphi(s) : [0,1] \to \mathcal{H}_\beta$ is a smooth curve, and let p be a base point of $\mathcal{H}$. Then, the length of the geodesic arc between p and φ is less than the sum of the length of the geodesic arc between from p to $\phi(0)$ and the length of the curve from $\phi(0)$ to $\phi(s)$.*

Theorem 6.4. *The distance function given by the length of the geodesic arc is a differentiable function.*

References

[1] C. Arezzo and G. Tian. Infinite geodesic rays in the space of Kähler potentials. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2(4):617–630, 2003.

[2] S. Bando and T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 11–40. North-Holland, Amsterdam, 1987.

[3] E. Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37(1):1–44, 1976.

[4] B. Berndtsson. Curvature of vector bundles associated to holomorphic fibrations. *Ann. of Math. (2)*, 169(2):531–560, 2009.

[5] Z. Blocki. Interior regularity of the complex Monge-Ampère equation in convex domains. *Duke Math. J.*, 105(1):167–181, 2000.

[6] Z. Blocki. A gradient estimate in the Calabi-Yau theorem. *Math. Ann.*, 344(2):317–327, 2009.

[7] Z. Blocki. On geodesics in the space of Kähler metrics. *Advanced Lectures in Mathematics*, 21(1):3–20, 2012.

[8] S. Brendle. Ricci flat Kähler metrics with edge singularities. *Int. Math. Res. Not. IMRN*, (24):5727–5766, 2013.
[9] L. Caffarelli, J. J. Kohn, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations. *Comm. Pure Appl. Math.*, 38(2):209–252, 1985.
[10] S. Calamai. The Calabi metric for the space of Kähler metrics. *Math. Ann.*, 353(2):373–402, 2012.
[11] S. Calamai, D. Petrecca, and K. Zheng. The geodesic problem for the Dirichlet metric and the Ebin metric on the space of Sasakian metrics arXiv:1405.1211.
[12] S. Calamai and K. Zheng. The Dirichlet and the weighted metrics for the space of Kähler metrics. *Math. Ann.*, DOI: 10.1007/s00208-015-1188-x.
[13] X. Chen. The space of Kähler metrics. *J. Differential Geom.*, 56(2):189–234, 2000.
[14] X. Chen. Space of Kähler metrics. III. On the lower bound of the Calabi energy and geodesic distance. *Invent. Math.*, 175(3):453–503, 2009.
[15] X. Chen and W. He. The space of volume forms. *Int. Math. Res. Not. IMRN*, (5):967–1009, 2011.
[16] X. Chen and G. Tian. Geometry of Kähler metrics and foliations by holomorphic discs. *Publ. Math. Inst. Hautes Études Sci.*, (107):1–107, 2008.
[17] X. Chen and Y. Tang. Test configuration and geodesic rays. *Astérisque*, (321):139–167, 2008. Géométrie différentielle, physique mathématique, mathématiques et société. I.
[18] S. S. Chern, H. I. Levine, and L. Nirenberg. Intrinsic norms on a complex manifold. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 119–139. Univ. Tokyo Press, Tokyo, 1969.
[19] S. S. Chern. On holomorphic mappings of hermitian manifolds of the same dimension. In *Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966)*, pages 157–170. Amer. Math. Soc., Providence, R.I., 1968.
[20] P. Cherrier. Équations de Monge-Ampère sur les variétés hermitiennes compactes. *Bull. Sci. Math. (2)*, 111(4):343–385, 1987.
[21] T. Darvas and L. Lempert. Weak geodesics in the space of Kähler metrics. *Math. Res. Lett.*, 19(5):1127–1135, 2012.
[22] W. Y. Ding. Remarks on the existence problem of positive Kähler-Einstein metrics. *Math. Ann.*, 282(3):463–471, 1988.
[23] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.
[24] S. K. Donaldson. Holomorphic discs and the complex Monge-Ampère equation. *J. Symplectic Geom.*, 1(2):171–196, 2002.
[25] S. K. Donaldson. Kähler metrics with cone singularities along a divisor. In *Essays in mathematics and its applications*, pages 49–79. Springer, Heidelberg, 2012.
[26] L. C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.*, 35(3):333–363, 1982.
[27] L. C. Evans. Classical solutions of the Hamilton-Jacobi-Bellman equation for uniformly elliptic operators. *Trans. Amer. Math. Soc.*, 275(1):245–255, 1983.
[28] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[29] B. Guan. The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function. *Comm. Anal. Geom.*, 6(4):687–703, 1998.
[30] B. Guan and Q. Li. Complex Monge-Ampère equations and totally real submanifolds. *Adv. Math.*, 225(3):1185–1223, 2010.
[31] B. Guan and J. Spruck. Boundary-value problems on S^n for surfaces of constant Gauss curvature. *Ann. of Math. (2)*, 138(3):601–624, 1993.
[32] P. Guan. The extremal function associated to intrinsic norms. *Ann. of Math. (2)*, 156(1):197–211, 2002.
[33] P. Guan and X. Zhang. Regularity of the geodesic equation in the space of Sasakian metrics. *Adv. Math.*, 230(1):321–371, 2012.
[34] A. Hanani. Équations du type de Monge-Ampère sur les variétés hermitiennes compactes. *J. Funct. Anal.*, 137(1):49–75, 1996.
[35] D. Hoffman, H. Rosenberg, and J. Spruck. Boundary value problems for surfaces of constant Gauss curvature. *Comm. Pure Appl. Math.*, 45(8):1051–1062, 1992.
[36] T. D. Jeffres. Uniqueness of Kähler-Einstein cone metrics. *Publ. Mat.*, 44(2):437–448, 2000.
[37] S. Kolodziej. The complex Monge-Ampère equation. *Acta Math.*, 180(1):69–117, 1998.
[38] N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(1):75–108, 1983.
[39] L. Lempert and L. Vivas. Geodesics in the space of Kähler metrics. *Duke Math. J.*, 162(7):1369–1381, 2013.
[40] Y. Lu. Holomorphic mappings of complex manifolds. *J. Differential Geometry*, 2:299–312, 1968.
[41] T. Mabuchi. Some symplectic geometry on compact Kähler manifolds. I. *Osaka J. Math.*, 24(2):227–252, 1987.
[42] R. Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.
[43] R. B. Melrose. *Pseudodifferential operators, corners and singular limits*. ICM-90. Mathematical Society of Japan, Tokyo, 1990. A plenary address presented at the International Congress of Mathematicians held in Kyoto, August 1990.
[44] R. Moriyón. Regularity of the Dirichlet problem for the degenerate complex Monge-Ampère equation. *Comm. Pure Appl. Math.*, 35(1):1–27, 1982.
[45] D. H. Phong and J. Sturm. The Monge-Ampère operator and geodesics in the space of Kähler potentials. *Invent. Math.*, 166(1):125–149, 2006.
[46] D. H. Phong and J. Sturm. Test configurations for K-stability and geodesic rays. *J. Symplectic Geom.*, 5(2):221–247, 2007.
[47] D. H. Phong and J. Sturm. Regularity of geodesic rays and Monge-Ampère equations. *Proc. Amer. Math. Soc.*, 138(10):3637–3650, 2010.
[48] B. W. Schulze. *Pseudo-differential operators on manifolds with singularities*, volume 24 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1991.
[49] S. Semmes. Complex Monge-Ampère and symplectic manifolds. *Amer. J. Math.*, 114(3):495–550, 1992.
[50] Y. T. Siu. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, volume 8 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1987.
[51] Y. T. Siu. The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. *Ann. of Math. (2)*, 127(3):585–627, 1988.

[52] J. Song and S. Zelditch. Convergence of Bergman geodesics on $\mathbf{CP}^1$. *Ann. Inst. Fourier (Grenoble)*, 57(7):2209–2237, 2007. Festival Yves Colin de Verdière.

[53] J. Song and S. Zelditch. Bergman metrics and geodesics in the space of Kähler metrics on toric varieties. *Anal. PDE*, 3(3):295–358, 2010.

[54] J. Song and S. Zelditch. Test configurations, large deviations and geodesic rays on toric varieties. *Adv. Math.*, 229(4):2338–2378, 2012.

[55] S. T. Yau. A general Schwarz lemma for Kähler manifolds. *Amer. J. Math.*, 100(1):197–203, 1978.

[56] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

[57] H. Yin. Ricci flow on surfaces with conical singularities. *J. Geom. Anal.*, 20(4), 970–995, 2010.

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