MULTIPLICITY ONE THEOREM FOR THE GINZBURG-RALLIS MODEL: THE TEMPERED CASE

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Abstract. Following the method developed by Waldspurger and Beuzart-Plessis in their proof of the local Gan-Gross-Prasad conjecture, we prove a local trace formula for the Ginzburg-Rallis model. By applying this trace formula, we prove the multiplicity one theorem for the Ginzburg-Rallis model over the tempered Vogan L-packets. In some cases, we also prove the epsilon dichotomy conjecture which gives a relation between the multiplicity and the exterior cube epsilon factor. This is a sequel work of [Wan15] in which we proved the geometric side of the trace formula.

1. Introduction and Main Result

1.1. **Main results.** This paper is a continuation of [Wan15]. For an overview of the Ginzburg-Rallis model, see Section 1 of [Wan15]. We recall from there the definition of the Ginzburg-Rallis models and the relevant conjectures.

Let F be a p-adic field or the real field, and let D be the unique quaternion algebra over F. Take $P = P_{2,2,2} = MU$ be the standard parabolic subgroup of $G(F) = \mathrm{GL}_6(F)$ whose Levi part M is isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, and whose unipotent radical U consists of elements of the form

$$u = u(X,Y,Z) := \begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix}. \tag{1.1}$$

We define a character ξ on $U(F)$ by

$$\xi(u(X,Y,Z)) := \psi(\mathrm{tr}(X) + \mathrm{tr}(Y)) \tag{1.2}$$

where ψ is a non-trivial additive character on F. It's clear that the stabilizer of ξ is the diagonal embedding of $\mathrm{GL}_2(F)$ into $M(F)$, which is denoted by $H_0(F)$. For a given character χ of $F^\times$, we define a

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character ω of $H_0(F)$ to be $\omega(h) := \chi(\det(h))$. Combining ω and ξ, we get a character $\omega \otimes \xi$ on $H(F) := H_0(F) \ltimes U(F)$. The pair (G, H) is the Ginzburg-Rallis model introduced by D. Ginzburg and S. Rallis in their paper [GR00]. Let π be an irreducible admissible representation of $G(F)$ with central character χ^2. We define the multiplicity $m(\pi)$ to be the dimension of the Hom space $Hom_{H(F)}(\pi, \omega \otimes \xi)$.

On the other hand, define $G_D(F) = \mathrm{GL}_3(D)$, similarly we can define U_D, $H_{0,D}$ and H_D. We can also define the character $\omega_D \otimes \xi_D$ on $H_D(F)$ in a similar way. Then for an irreducible admissible representation π_D of $G_D(F)$ with central character χ^2, we can also talk about the Hom space $Hom_{H_D(F)}(\pi_D, \omega_D \otimes \xi_D)$, whose dimension is denoted by $m(\pi_D)$.

The purpose of this paper is to study the multiplicity $m(\pi)$ and $m(\pi_D)$. It was proved by Nien in [N06] over a p-adic local field, and by Jiang-Sun-Zhu in [JSZ11] for an archimedean local field that both multiplicities are less or equal to 1: $m(\pi)$, $m(\pi_D) \leq 1$. In other words, the pairs (G, H) and (G_D, H_D) are Gelfand pairs. In this paper, we are interested in the relations between $m(\pi)$ and $m(\pi_D)$ under the local Jacquet-Langlands correspondence established in [DKV84]. The following local conjecture has been expected since the work of [GR00], and was first discussed in details by Jiang in his paper [J08].

Conjecture 1.1 (Jiang,[J08])**.** *For any irreducible admissible generic representation π of $GL_6(F)$, let π_D be the local Jacquet-Langlands correspondence of π to $GL_3(D)$ if it exists, and zero otherwise. We still assume that the central character of π is χ^2. Then we have*

$$m(\pi) + m(\pi_D) = 1. \tag{1.3}$$

The assertion in Conjecture 1.1 can be formulated in terms of Vogan L-packets. For details, see Section 1 of the previous paper [Wan15].

In the author's previous paper [Wan15], by proving the geometric side of a local trace formula for the Ginzburg-Rallis model, we proved Conjecture 1.1 when F is a p-adic field and π is an irreducible supercuspidal representation of $GL_6(F)$. In this paper, by proving the spectral side of the trace formula, together with the geometric side in the previous paper [Wan15], we will prove Conjecture 1.1 when F is a p-adic field and π is an irreducible tempered representation of $GL_6(F)$. We will also prove Conjecture 1.1 when $F = \mathbb{R}$ and π is an irreducible tempered representation of $GL_6(F)$.

Theorem 1.2. *Let F be a p-adic field or $F = \mathbb{R}$. For any irreducible tempered representation π of $\mathrm{GL}_6(F)$, Conjecture 1.1 holds.*

Our proof of Theorem 1.2 uses the method developed by Waldspurger and Beuzart-Plessis in their proof of the local Gan-Gross-Prasad conjecture ([W10], [W12], [B12], [B15]). In the p-adic case, the key ingredient of the proof is a local trace formula for the Ginzburg-Rallis model, which will be called the trace formula in this paper for simplicity, unless otherwise specified. To be specific, let $f \in C_c^\infty(Z_G(F)\backslash G(F), \chi^{-2})$ be a strongly cuspidal function (see Section 3 for the definition of strongly cuspidal functions). We define the function $I(f, \cdot)$ on $H(F)\backslash G(F)$ to be

$$I(f,x) = \int_{Z_H(F)\backslash H(F)} f(x^{-1}hx)\, \omega \otimes \xi(h) dh.$$

Then we define

$$I(f) = \int_{H(F)\backslash G(F)} I(f,g) dg. \tag{1.4}$$

$I(f)$ is the distribution in the trace formula.

Now we define the spectral and the geometric sides of the trace formula. The geometric side $I_{geom}(f)$ has already been defined in the previous paper [Wan15]. The main ingredient in its definition is the germ of θ_f. Here θ_f is a distribution associated to f via the weighted orbital integral. We refer the readers to Section 7.2 for detailed description of $I_{geom}(f)$.

For the spectral side, we define

$$I_{spec}(f) = \int_{\Pi_{temp}(G,\chi^2)} \theta_f(\pi) m(\bar{\pi}) d\pi$$

where $\Pi_{temp}(G, \chi^2)$ is the set of all irreducible tempered representations of $G(F) = \mathrm{GL}_6(F)$ with central character χ^2, $d\pi$ is some measure on $\Pi_{temp}(G, \chi^2)$ as defined in Section 2.8, and $\theta_f(\pi)$ is defined in Section 3.3 via the weighted character. Then the trace formula we will prove in this paper is just

$$I_{spec}(f) = I(f) = I_{geom}(f). \tag{1.5}$$

It is worth to mention that the geometric side of this trace formula has already been proved in the previous paper [Wan15]. So in this paper, we will focus on the spectral side. The proof will be given in Section 7. Similarly, we will also prove the trace formula for the pair (G_D, H_D).

After proving the trace formula, we can prove a multiplicity formula for the tempered representations:

$$m(\pi) = m_{geom}(\pi),\ m(\pi_D) = m_{geom}(\pi_D). \tag{1.6}$$

Here $m_{geom}(\pi)$ (resp. $m_{geom}(\pi_D)$) is defined in the same way as $I_{geom}(f)$ except that we replace the distribution θ_f by the distribution character

θ_π (resp. θ_{π_D}) associated to the representation π (resp. π_D). For the complete definition of the multiplicity formula, see Section 8. Once this multiplicity formula has been proved, we can use the relations between the distribution characters θ_π and θ_{π_D} under the local Jacquet-Langlands correspondence to cancel out all the terms in the expression of $m_{geom}(\pi)+m_{geom}(\pi_D)$ except the term $c_{\theta_\pi,\mathcal{O}_{reg}}(1)$, which is the germ at the identity element. Then the work of Rodier ([Rod81]) shows that $c_{\theta_\pi,\mathcal{O}_{reg}}(1)=0$ if π is non-generic, and $c_{\theta_\pi,\mathcal{O}_{reg}}(1)=1$ if π is generic. Since all tempered representations of $GL_n(F)$ are generic, we get the following identity

$$m_{geom}(\pi)+m_{geom}(\pi_D)=1. \tag{1.7}$$

And this proves Theorem 1.2 for the p-adic case.

In the archimedean case, although we can use the same method as in the p-adic case (like Beuzart-Plessis did in [B15] for the GGP case), it is actually much easier. All we need to do is to show that the multiplicity is invariant under the parabolic induction, this will be done in Section 5 for both the p-adic and the archimedean case. Then since only $\mathrm{GL}_1(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$ have discrete series, we can reduce the problem to the trilinear GL_2 model case which has already been considered by Prasad ([P90]) and Loke ([L01]).

Another result of this paper is the epsilon dichotomy conjecture. To be specific, we prove a relation between the multiplicity and the exterior cube epsilon factor. The conjecture can be formulated as follows.

Conjecture 1.3. *With the same assumptions as in Conjecture 1.1, we have*

$$m(\pi)=1 \iff \epsilon(1/2,(\wedge^3\pi)\otimes\chi^{-1})=1,$$
$$m(\pi)=0 \iff \epsilon(1/2,(\wedge^3\pi)\otimes\chi^{-1})=-1.$$

Here $\epsilon(s,(\wedge^3\pi)\otimes\chi^{-1})$ is the epsilon factor of $(\wedge^3\phi_\pi)\otimes\chi^{-1}$ (NOT $\wedge^3(\phi_\pi\otimes\chi^{-1})$) where ϕ_π is the Langlands parameter of π.

In this paper, we always fix a Haar measure dx on F and an additive character ψ such that the Haar measure is selfdual for Fourier transform with respect to ψ. We use such dx and ψ in the definition of the ϵ factor. For simplicity, we will write the epsilon factor as $\epsilon(s,\pi)$ instead of $\epsilon(s,\pi,dx,\psi)$. Our result for Conjecture 1.3 is the following Theorem.

Theorem 1.4. *Let π be an irreducible tempered representation of $\mathrm{GL}_6(F)$ with central character χ^2.*

(1) *If $F=\mathbb{R}$, Conjecture 1.3 holds.*

(2) *If F is p-adic, and if π is not a discrete series or the parabolic induction of a discrete series of* $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$*, Conjecture 1.3 holds.*

Theorem 1.4 will be proved in Section 6 for the archimedean case, and in Section 8 for the p-adic case. Our method is to study the behavior of the multiplicity and the epsilon factor under the parabolic induction, and this is why in the p-adic case we have more restrictions.

Finally, our method can also be applied to all the reduced models of the Ginzburg-Rallis model coming from the parabolic induction. For some models, such results are well know (like the trilinear GL_2 model); but for the other models, as far as we know, no such results appeared in the literature. For details, see Appendix A.

1.2. **Organization of the paper and remarks on the proofs.** In Section 2, we introduce basic notations and conventions of this paper. We will also discuss the definitions and some basic facts of the intertwining operator and the Harish-Chandra-Schwartz space. In Section 3, we will study quasi-characters and strongly cuspidal functions. For Sections 2 and 3, we follow [B15] closely and provide details for the current case as needed.

In Section 4, we study the analytic and geometric properties of the Ginzburg-Rallis model. In particular, we show that it is a wavefront spherical variety and has polynomial growth as a homogeneous space. This gives us the weak Cartan decomposition. Then by applying those results, we prove some estimates for various integrals which will be used in later sections. The proofs of some estimations are similar to the GGP case in [B15], so we will skip it here and we refer the readers to the PhD thesis of the author ([Wan17]) for details of the proof.

In Section 5, we study an explicit element $\mathcal{L}_\pi$ in the Hom space coming from the (normalized) integration of the matrix coefficient. The goal is to prove that the Hom space is nonzero if and only if $\mathcal{L}_\pi$ is nonzero. The standard approach to prove such a result is by using the Plancherel formula together with the fact that the nonvanishing property of $\mathcal{L}_\pi$ is invariant under parabolic induction and unramified twist. However, there are two main difficulties in the proof of such a result for the Ginzburg-Rallis models. First, unlike the Gan-Gross-Prasad case, we do have nontrivial center for the Ginzburg-Rallis model. As a result, for many parabolic subgroups of $\mathrm{GL}_6(F)$ (those which do not have an analogue in the quaternion case, i.e. the one not of type (6), $(4,2)$ or $(2,2,2)$, we will call theses models "Type II models"), it is not clear why the nonvanishing property of $\mathcal{L}_\pi$ is invariant under the unramified twist. Instead, we show that for such parabolic subgroups, $\mathcal{L}_\pi$ will always be nonzero. This is Theorem 5.11, which will be proved in Section 6 for the archimedean case, and in Section 9 for the p-adic case. Another difficulty is that unlike the Gan-Gross-Prasad case, when we do parabolic induction, we don't always have the strongly tempered model (in the GGP case, one can always go up to the codimension one case which is strongly tempered, then run the parabolic induction process), we refer the readers to Section 6.2 of [SV12] for the definition of strongly tempered spherical variety. As a result, in order to prove the nonvanishing property of $\mathcal{L}_{\pi}$ is invariant under parabolic induction, it is not enough to just change the order of the integral. This is because if the model is not strongly tempered, $\mathcal{L}_{\pi}$ is defined via the normalized integral, not the original integral. We will find a way to deal with this issue in Section 5.3 and 5.4, but we have to treat the p-adic case and the archimedean case separately.

In Section 6, we prove our main Theorems for the archimedean case by reducing it to the trilinear GL_2 model, and then applying the results of Prasad ([P90]) and Loke ([L01]).

Starting from Section 7, we only need to consider the p-adic case. In Section 7, we state our trace formula. The geometric expansion of the trace formula has already been proved in the previous paper [Wan15]. By applying the results in Section 3, 4 and 5, we will prove the spectral side of the trace formula.

In Section 8, we prove the multiplicity formula by applying our trace formula. Using that, we prove our main Theorems for the p-adic case.

In Section 9, we prove the argument we left in Section 5 (i.e. Theorem 5.11) for the p-adic case. In other words, we show that for all Type II models, $\mathcal{L}_{\pi}$ is always nonzero. By some similar arguments as in Section 5, we only need to show that the Hom space for all Type II models is nonzero. Our method is to prove the local trace formula for these models which will give us a multiplicity formula. **The most important feature for those models is that all semisimple elements in the small group are split, i.e. there is no elliptic torus. As a result, the geometric side of the trace formula for those models only contains the germ at 1 (as in Appendix B of** [Wan15]**).** But by the work of Rodier [Rod81], the germ is always 1 for generic representations. Hence we know that the Hom space is always nonzero, and this proves Theorem 5.11.

In Appendix A, we will state some similar results for the reduced models of the Ginzburg-Rallis models coming from parabolic induction. Since the proof of those results are similar to the Ginzburg-Rallis model case, we will skip it here.

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2. Preliminaries

2.1. **Notation and conventions.** Let F be a p-adic field or $\mathbb{R}$, and let $|\cdot| = |\cdot|_F$ be the absolute value on F. For every connected reductive algebraic group G defined over F, let A_G be the maximal split center of G and let Z_G be the center of G. We denote by $X(G)$ the group of F-rational characters of G. Define $\mathfrak{a}_G$ =Hom$(X(G),\mathbb{R})$, and let $\mathfrak{a}_G^\ast = X(G)\otimes_{\mathbb{Z}}\mathbb{R}$ be the dual of $\mathfrak{a}_G$. We define a homomorphism $H_G : G(F) \to \mathfrak{a}_G$ by $H_G(g)(\chi) = \log(|\chi(g)|_F)$ for every $g \in G(F)$ and $\chi \in X(G)$. Let $\mathfrak{a}_{G,F}$ (resp. $\tilde{\mathfrak{a}}_{G,F}$) be the image of $G(F)$ (resp. $A_G(F)$) under H_G. In the real case, $\mathfrak{a}_G = \mathfrak{a}_{G,F} = \tilde{\mathfrak{a}}_{G,F}$; in the p-adic case, $\mathfrak{a}_{G,F}$ and $\tilde{\mathfrak{a}}_{G,F}$ are lattices in $\mathfrak{a}_G$. Let $\mathfrak{a}_{G,F}^\vee = \mathrm{Hom}(\mathfrak{a}_{G,F}, 2\pi\mathbb{Z})$ and let $\tilde{\mathfrak{a}}_{G,F}^\vee = \mathrm{Hom}(\tilde{\mathfrak{a}}_{G,F}, 2\pi\mathbb{Z})$. Set $\mathfrak{a}_{G,F}^\ast = \mathfrak{a}_G^\ast/\mathfrak{a}_{G,F}^\vee$. We can identify $i\mathfrak{a}_{G,F}^\ast$ with the group of unitary unramified characters of $G(F)$ by letting $\lambda(g) = e^{<\lambda, H_G(g)>}$ for $\lambda \in i\mathfrak{a}_{G,F}^\ast$ and $g \in G(F)$. For a Levi subgroup M of G, let $\mathfrak{a}_{M,0}^\ast$ be the subset of elements in $\mathfrak{a}_{M,F}^\ast$ whose restriction to $\tilde{\mathfrak{a}}_{G,F}$ is zero. Then we can identify $i\mathfrak{a}_{M,0}^\ast$ with the group of unitary unramified characters of $M(F)$ which is trivial on $Z_G(F)$.

Let $\mathfrak{g}$ be the Lie algebra of G. For a Levi subgroup M of G, let $\mathcal{P}(M)$ be the set of parabolic subgroups of G whose Levi part is M, $\mathcal{L}(M)$ be the set of Levi subgroups of G containing M, and let $\mathcal{F}(M)$ be the set of parabolic subgroups of G containing M. We have a natural decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$. Denote by $proj_M^G$ and $proj_G$ the projections of $\mathfrak{a}_M$ to each factors. The subspace $\mathfrak{a}_M^G$ has a set of coroots $\check{\Sigma}_M$, and for each $P \in \mathcal{P}(M)$, we can associate a positive chamber $\mathfrak{a}_P^+ \subset \mathfrak{a}_M$ and a subset of simple coroots $\check{\Delta}_P \subset \check{\Sigma}_M$. For such $P = MU$, we can also define a function $H_P : G(F) \to \mathfrak{a}_M$ by $H_P(g) = H_M(m_g)$ where $g = m_g u_g k_g$ is the Iwasawa decomposition of g.

According to Harish-Chandra, we can define the height function $\|\cdot\|$ on $G(F)$, taking values in $\mathbb{R}_{\geq 1}$. Then we define a log-norm σ on $G(F)$ by $\sigma(g) = \sup(1, \log(\|g\|))$. We also define $\sigma_0(g) = \inf_{z\in Z_G(F)}\{\sigma(zg)\}$. Similarly, we can define the log-norm function on $\mathfrak{g}(F)$ as follows: fix a basis $\{X_i\}$ of $\mathfrak{g}(F)$ over F. For $X \in \mathfrak{g}(F)$, let $\sigma(X) = \sup(1, \sup\{\log(|a_i|)\})$, where a_i is the X_i-coordinate of X.

Let M_{min} be a minimal Levi subgroup of G. For each $P_{min} \in \mathcal{P}(M_{min})$, let $\Psi(A_{min}, P_{min})$ be the set of positive roots associated to P_{min}, and let $\Delta(A_{min}, P_{min}) \subset \Psi(A_{min}, P_{min})$ be the subset of simple roots.

For $x \in G$ (resp. $X \in \mathfrak{g}$), let $Z_G(x)$ (resp. $Z_G(X)$) be the centralizer of x (resp. X) in G, and let G_x (resp. G_X) be the neutral component of $Z_G(x)$ (resp. $Z_G(X)$). Accordingly, let $\mathfrak{g}_x$ (resp. $\mathfrak{g}_X$) be the Lie algebra of G_x (resp. G_X). For a function f on $G(F)$ (resp. $\mathfrak{g}(F)$), and $g \in G(F)$, let ${}^g f$ be the g-conjugation of f, i.e. ${}^g f(x) = f(g^{-1}xg)$ for $x \in G(F)$ (resp. ${}^g f(X) = f(g^{-1}Xg)$ for $X \in \mathfrak{g}(F)$).

Denote by $G_{ss}(F)$ the set of semisimple elements in $G(F)$, and by $G_{reg}(F)$ the set of regular semisimple elements in $G(F)$. The Lie algebra versions are denoted by $\mathfrak{g}_{ss}(F)$ and $\mathfrak{g}_{reg}(F)$, respectively. For $x \in G_{ss}(F)$ (resp. $X \in \mathfrak{g}_{ss}(F)$), let $D^G(x)$ (resp. $D^G(X)$) be the Weyl determinant.

For two complex valued functions f and g on a set X with g taking values in the positive real numbers, we write $f(x) \ll g(x)$, and say that f is *essentially bounded* by g, if there exists a constant $c > 0$ such that for all $x \in X$, we have $|f(x)| \leq cg(x)$. We say f and g are *equivalent*, which is denoted by $f(x) \sim g(x)$, if f is essentially bounded by g and g is essentially bounded by f.

2.2. **Measures.** Through this paper, we fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$. If G is a connected reductive group, we may fix a non-degenerate symmetric bilinear form $< \cdot, \cdot >$ on $\mathfrak{g}(F)$ that is invariant under $G(F)$-conjugation. For any smooth compactly supported complex valued function $f \in C_c^\infty(\mathfrak{g}(F))$, we can define its Fourier transform $f \to \hat{f}$ to be

$$\hat{f}(X) = \int_{\mathfrak{g}(F)} f(Y)\psi(< X, Y >)dY \tag{2.1}$$

where dY is the selfdual Haar measure on $\mathfrak{g}(F)$ such that $\hat{\hat{f}}(X) = f(-X)$. Then we get a Haar measure on $G(F)$ such that the Jacobian of the exponential map equals 1. If H is a subgroup of G such that the restriction of the bilinear form to $\mathfrak{h}(F)$ is also non-degenerate, then we can define the measures on $\mathfrak{h}(F)$ and $H(F)$ by the same method.

If T is a subtorus of G such that the bilinear form is non-degenerate on $\mathfrak{t}(F)$, we can provide a measure on T by the method above, denoted by dt. On the other hand, we can define another measure $d_c t$ on $T(F)$ as follows: If T is split, we require the volume of the maximal compact subgroup of $T(F)$ is 1 under d_ct. In general, d_ct is compatible with the measure d_ct' defined on $A_T(F)$ and with the measure on $T(F)/A_T(F)$ of total volume 1. Then we have a constant number $\nu(T)$ such that $d_ct = \nu(T)dt$. In this paper, we will only use the measure dt, but in many cases we have to include the factor $\nu(T)$. Finally, if M is a Levi subgroup of G, we can define the Haar measure on $\mathfrak{a}_M^G$ such that the quotient

$$\mathfrak{a}_M^G/proj_M^G(H_M(A_M(F)))$$

is of volume 1.

2.3. **Induced representation and intertwining operator.** Given a parabolic subgroup $P = MU$ of G and an admissible representation (τ, V_τ) of $M(F)$, let $(I_P^G(\tau), I_P^G(V_\tau))$ be the normalized parabolic induced representation: $I_P^G(V_\tau)$ is the space of smooth functions $e : G(F) \to V_\tau$ such that

$$e(mug) = \delta_P(m)^{1/2}\tau(m)e(g),\ m \in M(F),\ u \in U(F),\ g \in G(F).$$

And the $G(F)$ action is just the right translation. Let K be a maximal compact subgroup of $G(F)$ that is in good position with respect to M.

For $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$, let τ_λ be the unramified twist of τ, i.e. $\tau_\lambda(m) = \exp(\lambda(H_M(m)))\tau(m)$ and let $I_P^G(\tau_\lambda)$ be the induced representation. By the Iwasawa decomposition, every function $e \in I_P^G(\tau_\lambda)$ is determined by its restriction on K, and that space is invariant under the unramified twist. i.e. for any λ, we can realize the representation $I_P^G(\tau_\lambda)$ on the space $I_{K\cap P}^K(\tau_K)$ which consists of functions $e_K : K \to V_\tau$ such that

$$e(mug) = \delta_P(m)^{1/2}\tau(m)e(g),\ m \in M(F) \cap K,\ u \in U(F) \cap K,\ g \in K.$$

Here τ_K is the restriction of τ to the group $K \cap M(F)$.

If τ is unitary, so is $I_P^G(\tau)$. The inner product on $I_P^G(V_\tau)$ can be realized as

$$(e, e') = \int_K (e'(k), e(k))_\tau dk.$$

Now we define the intertwining operator. Let M be a Levi subgroup of G. For $P = MU,\ P' = MU' \in \mathcal{P}(M)$, and $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$, define the intertwining operator $J_{P'|P}(\tau_\lambda) : I_P^G(V_\tau) \to I_{P'}^G(V_\tau)$ to be

$$J_{P'|P}(\tau_\lambda)(e)(g) = \int_{(U(F)\cap U'(F))\backslash U'(F)} e(ug)du.$$

In general, the integral above is not absolutely convergent. But it is absolutely convergent for $Re(\lambda)$ sufficiently large, and it is $G(F)$-equivariant. By restricting to K, we can view $J_{P'|P}(\tau_\lambda)$ as a homomorphism from $I_{K\cap P}^K(V_{\tau_K})$ to $I_{K\cap P'}^K(V_{\tau_K})$. In general, $J_{P'|P}(\tau_\lambda)$ can be meromorphically continued to a function on $\mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}/i\mathfrak{a}_{M,F}^\vee$.

If τ is irreducible, by Schur's lemma, the operator $J_{P|\bar{P}}(\tau_\lambda)J_{\bar{P}|P}(\tau_\lambda)$ is a scalar for generic λ. Here $\bar{P} = M\bar{U}$ is the opposite parabolic subgroup of $P = MU$ with respect to M. Let $j(\tau_\lambda)$ be the scalar, which is independent of the choice of P.

2.4. **Harish-Chandra-Schwartz space.** Let $P_{min} = M_{min}U_{min}$ be a minimal parabolic subgroup of G and let K be a maximal open compact subgroup in good position with respect to M_{min}. Consider the normalized induced representation

$$I_{P_{min}}^G(1) := \{e \in C^\infty(G(F)) \mid e(pg) = \delta_{P_{min}}(p)^{1/2}e(g) \ for\ all\ p \in P_{min}(F), g \in G(F)\}.$$

We equip the representation with the inner product

$$(e, e') = \int_K e(k)\bar{e'}(k)dk.$$

Let $e_K \in I_{P_{min}}^G(1)$ be the unique function such that $e_K(k) = 1$ for all $k \in K$.

Definition 2.1. *The Harish-Chandra function Ξ^G is defined to be*

$$\Xi^G(g) = (I_{P_{min}}^G(1)(g)e_K, e_K).$$

Remark 2.2. *The function Ξ^G depends on the various choices we made, but this doesn't matter since different choices give us equivalent functions and the function Ξ^G will only be used in estimations.*

We refer the readers to Proposition 1.5.1 of [B15] (or Proposition 2.8.3 of [Wan17]) for the basic properties of the function Ξ^G.

For $f \in C^\infty(G(F))$ and $d \in \mathbb{R}$, let

$$p_d(f) = \sup_{g \in G(F)} \{|f(g)|\Xi^G(g)^{-1}\sigma(g)^d\}.$$

If F is p-adic, we define the Harish-Chandra-Schwartz space to be

$$\mathcal{C}(G(F)) = \{f \in C^\infty(G(F)) | p_d(f) < \infty, \forall d > 0\}.$$

If $F = \mathbb{R}$, for $u, v \in \mathcal{U}(\mathfrak{g})$ and $d \in \mathbb{R}$, let

$$p_{u,v,d}(f) = p_d(R(u)L(v)f)$$

where "R" stands for the right translation, "L" stands for the left translation and $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra. We define the Harish-Chandra-Schwartz space to be

$$\mathcal{C}(G(F)) = \{f \in C^\infty(G(F)) | p_{u,v,d}(f) < \infty, \forall d > 0,\ u, v \in \mathcal{U}(\mathfrak{g})\}.$$

We also need the weak Harish-Chandra-Schwartz space $\mathcal{C}^w(G(F))$. For $d > 0$, let

$$\mathcal{C}_d^w(G(F)) = \{f \in C^\infty(G(F)) | p_{-d}(f) < \infty\}$$

if F is p-adic. And let

$$\mathcal{C}_d^w(G(F))=\{f\in C^\infty(G(F))|p_{u,v,-d}(f)<\infty,\forall u,v\in\mathcal{U}(\mathfrak{g})\}$$

if $F=\mathbb{R}$. We define

$$\mathcal{C}^w(G(F))=\cup_{d>0}\mathcal{C}_d^w(G(F)).$$

Also we can define the Harish-Chandra-Schwartz space (resp. weak Harish-Chandra-Schwartz space) with given unitary central character χ: let $\mathcal{C}(Z_G(F)\backslash G(F),\chi)$ (resp. $\mathcal{C}^w(Z_G(F)\backslash G(F),\chi)$) be the Mellin transform of the space $\mathcal{C}(G(F))$ (resp. $\mathcal{C}^w(G(F))$) with respect to χ.

2.5. **The Harish-Chandra-Plancherel formula.** Since the Ginzburg-Rallis model has nontrivial center, we only introduce the Plancherel formula with given central character. We fix an unitary character χ of $Z_G(F)$. For every $M\in\mathcal{L}(M_{min})$, fix an element $P\in\mathcal{P}(M)$. Let $\Pi_2(M,\chi)$ be the set of discrete series of $M(F)$ whose central character agrees with χ on $Z_G(F)$. Then $i\mathfrak{a}_{M,0}^*$ acts on $\Pi_2(M,\chi)$ by the unramified twist. Let $\{\Pi_2(M,\chi)\}$ be the set of orbits under this action. For every orbit $\mathcal{O}$, and for a fixed $\tau\in\mathcal{O}$, let $i\mathfrak{a}_{\mathcal{O}}^\vee$ be the set of $\lambda\in i\mathfrak{a}_{M,0}^*$ such that the representation τ and τ_λ are equivalent, which is a finite set. For $\lambda\in i\mathfrak{a}_{M,0}^*$, define the Plancherel measure to be

$$\mu(\tau_\lambda)=j(\tau_\lambda)^{-1}d(\tau)$$

where $d(\tau)$ is the formal degree of τ, which is invariant under the unramified twist, and $j(\tau_\lambda)$ is defined in Section 2.3. Then for $f\in\mathcal{C}(Z_G(F)\backslash G(F),\chi^{-1})$, the Harish-Chandra-Plancherel formula ([HC76], [W03]) is

$$\begin{aligned} f(g) &= \Sigma_{M\in\mathcal{L}(M_{min})}|W^M||W^G|^{-1}\Sigma_{\mathcal{O}\in\{\Pi_2(M,\chi)\}}|i\mathfrak{a}_{\mathcal{O}}^\vee|^{-1}\\ &\quad \int_{i\mathfrak{a}_{M,0}^*}\mu(\tau_\lambda)\mathrm{tr}(I_P^G(\tau_\lambda)(g^{-1})I_P^G(\tau_\lambda)(f))d\lambda. \end{aligned}$$

To simplify our notation, let $\Pi_{temp}(G,\chi)$ be the union of $I_P^G(\tau)$ for $P=MN$, $M\in\mathcal{L}(M_{min})$, $\tau\in\mathcal{O}$ and $\mathcal{O}\in\{\Pi_2(M,\chi)\}$. We define a Borel measure $d\pi$ on $\Pi_{temp}(G,\chi)$ such that

$$\int_{\Pi_{temp}(G,\chi)}\varphi(\pi)d\pi=\Sigma_{M\in\mathcal{L}(M_{min})}|W^M||W^G|^{-1}\Sigma_{\mathcal{O}\in\{\Pi_2(M,\chi)\}}|i\mathfrak{a}_{\mathcal{O}}^\vee|^{-1}\int_{i\mathfrak{a}_{M,0}^*}\varphi(I_P^G(\tau_\lambda))d\lambda$$

for every compactly supported function φ on $\Pi_{temp}(G,\chi)$. Here by saying a function φ is compactly supported on $\Pi_{temp}(G,\chi)$ we mean that it is supported on finitely many orbit $\mathcal{O}$ and for every such orbit $\mathcal{O}$, it is compactly supported. Note that the second condition is automatic if F is p-adic. Then the Harish-Chandra-Plancherel formula above becomes

$$f(g)=\int_{\Pi_{temp}(G,\chi)} \mathrm{tr}(\pi(g^{-1})\pi(f))\mu(\pi)d\pi.$$

We also need the matrical Paley-Wiener Theorem. Let $C^{\infty}(\Pi_{temp}(G,\chi))$ be the space of functions $\pi\in \Pi_{temp}(G,\chi)\rightarrow T_{\pi}\in \mathrm{End}(\pi)^{\infty}$ such that it is smooth on every orbits $\mathcal{O}$ as functions from $\mathcal{O}$ to $\mathrm{End}(\pi)^{\infty}\simeq \mathrm{End}(\pi_K)^{\infty}$. We define $\mathcal{C}(\Pi_{temp}(G,\chi))$ to be a subspace of $C^{\infty}(\Pi_{temp}(G,\chi))$ consisting of those $T:\pi\rightarrow T_{\pi}$ such that

(1) If F is p-adic, T is nonzero on finitely many orbits $\mathcal{O}$.
(2) If $F=\mathbb{R}$, for all parabolic subgroup $P=MU$ and for all differential operator with constant coefficients D on $i\mathfrak{a}_M^{\ast}$, the function $DT:\sigma\in \Pi_2(M,\chi)\rightarrow D(\lambda\rightarrow T_{I_P^G(\sigma_{\lambda})})$ satisfies $p_{D,u,v,k}(T)=\sup_{\sigma\in \Pi_2(M,\chi)} ||DT(\sigma)||_{u,v}N(\sigma)^k<\infty$ for all $u,v\in \mathcal{U}(\mathfrak{k})$ and $k\in \mathbb{N}$. Here $||DT(\sigma)||_{u,v}$ is the norm of the operator $\sigma(u)DT(\sigma)\sigma(v)$ and $N(\sigma)$ is the norm on the set of all tempered representations (See Section 2.2 of [B15]).

Then the matrical Paley-Wiener Theorem ([HC76], [W03]) states that we have an isomorphism between $\mathcal{C}(Z_G(F)\backslash G(F),\chi^{-1})$ and $\mathcal{C}(\Pi_{temp}(G,\chi))$ given by

$$f\in \mathcal{C}(Z_G(F)\backslash G(F),\chi^{-1})\rightarrow (\pi\in \Pi_{temp}(G,\chi)\rightarrow \pi(f)\in \mathrm{End}(\pi)^{\infty}),$$

$$T\in \mathcal{C}(\Pi_{temp}(G,\chi))\rightarrow f_T(g)=\int_{\Pi_{temp}(G,\chi)} \mathrm{tr}(\pi(g^{-1})T_{\pi})\mu(\pi)d\pi.$$

3. Strongly Cuspidal Functions and Quasi Characters

Throughout this section, assume that F is p-adic.

3.1. **Quasi-characters of $G(F)$.** If θ is a smooth function defined on $G_{reg}(F)$, invariant under $G(F)-$conjugation. We say it is a quasi-character on $G(F)$ if for every $x\in G_{ss}(F)$, there is a good neighborhood ω_x of 0 in $\mathfrak{g}_x(F)$, and for every $\mathcal{O}\in Nil(\mathfrak{g}_x)$, there exists $c_{\theta,\mathcal{O}}(x)\in \mathbb{C}$ such that

$$\theta(x\exp(X))=\Sigma_{\mathcal{O}\in Nil(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)\hat{j}(\mathcal{O},X) \tag{3.1}$$

for every $X\in \omega_{x,reg}$. We refer the readers to Section 3 of [W10] for the definition of good neighborhood. Here $\hat{j}(\mathcal{O},X)$ is the function on $\mathfrak{g}_{reg}(F)$ represent the Fourier transform of the nilpotent orbital integral, $Nil(\mathfrak{g}_x)$ is the set of nilpotent orbits of $\mathfrak{g}_x(F)$. The coefficients $c_{\theta,\mathcal{O}}(x)$ are called the germs of θ at x.

3.2. **Strongly cuspidal functions.** We say a function $f \in \mathcal{C}(Z_G(F)\backslash G(F),\chi)$ is strongly cuspidal if for every proper parabolic subgroup $P = MU$ of G, and for every $x \in M(F)$, we have

$$\int_{U(F)} f(xu)du = 0. \tag{3.2}$$

Some basic examples of strongly cuspidal functions are the matrix coefficients of supercuspidal representations, or the pseudo coefficients of discrete series (see Section 3.4). We will denote by $\mathcal{C}_{scusp}(Z_G(F)\backslash G(F),\chi)$ the subspace of strongly cuspidal functions in $\mathcal{C}(Z_G(F)\backslash G(F),\chi)$.

3.3. **Some properties of strongly cuspidal functions.** Same as in Section 4.1 of [Wan15], given a strongly cuspidal function $f \in \mathcal{C}(Z_G(F)\backslash G(F),\chi)$, we can define a function θ_f on $G_{reg}(F)$ with central character χ to be

$$\theta_f(x) = (-1)^{a_{M(x)}-a_G}\nu(G_x)^{-1}D^G(x)^{-1/2}J_{M(x)}(x,f),\ x \in G_{reg}(F). \tag{3.3}$$

Here $J_{M(x)}(x,f)$ is the weighted orbital integral defined in Section 2.4 of [Wan15], $M(x)$ is the centralizer of A_{G_x} in G, a_G is the dimension of A_G, and $a_{M(x)}$ is the dimension of $A_{M(x)}$. By Proposition 4.3 of [Wan15], θ_f is a quasi-character.

For the rest part of this section, we assume that G is $GL_n(D)$ for some division algebra D/F and $n \geq 1$. In particular, all irreducible tempered representation π of $G(F)$ is of the form $\pi = I_M^G(\tau)$ for some $\tau \in \Pi_2(M)$. For such π, let χ be the central character of π. For $f \in \mathcal{C}(Z_G(F)\backslash G(F),\chi^{-1})$ strongly cuspidal, define

$$\theta_f(\pi) = (-1)^{a_G-a_M}J_M^G(\tau,f). \tag{3.4}$$

Here $J_M^G(\tau,f)$ is the weighted character defined in Section 2.5 of [B15].

Proposition 3.1. *For every $f \in \mathcal{C}(Z_G(F)\backslash G(F),\chi^{-1})$ strongly cuspidal, we have*

$$\theta_f = \int_{\Pi_{temp}(G,\chi)} \theta_f(\pi)\bar{\theta}_\pi d\pi.$$

Proof. This is just Proposition 5.6.1 of [B15]. The only thing worthwhile to mention is that the function $D(\pi)$ in the loc. cit. is identically 1 in our case since we assume that $G = GL_n(D)$. □

To end this section, we need a local trace formula for strongly cuspidal functions. It will be used in Section 7 for the proof of the spectral side of our trace formula. For $f \in \mathcal{C}(Z_G(F)\backslash G(F),\chi^{-1})$, $f' \in$

$\mathcal{C}(Z_G(F)\backslash G(F),\chi)$ and $g_1,g_2\in G(F)$, set

$$K_{f,f'}^A(g_1,g_2)=\int_{Z_G(F)\backslash G(F)} f(g_1^{-1}gg_2)f'(g)dg.$$

By Proposition 1.5.1(v) of [B15], the integral above is absolutely convergent.

Theorem 3.2. *Assume that f is strongly cuspidal, define*

$$J^A(f,f')=\int_{Z_G(F)\backslash G(F)} K_{f,f'}^A(g,g)dg.$$

Then the integral above is absolutely convergent, and we have

$$J^A(f,f')=\int_{\Pi_{temp}(G,\chi)} \theta_f(\pi)\theta_{\bar{\pi}}(f')d\pi.$$

Proof. This is just Theorem 5.5.1 of [B15]. □

3.4. **Pseudo coefficients.** Recall that we assume $G=\mathrm{GL}_n(D)$ for some division algebra D/F. Let π be a discrete series of $G(F)$ with central character χ. For $f\in C_c^\infty(Z_G(F)\backslash G(F),\chi^{-1})$, we say f is a pseudo coefficient of π if the following conditions hold.

- $\mathrm{tr}(\pi(f))=1$.
- For all $\sigma\in\Pi_{temp}(G,\chi)$ with $\sigma\neq\pi$, we have $\mathrm{tr}(\sigma(f))=0$.

Lemma 3.3. *For all discrete series π of $G(F)$ with central character χ, the pseudo coefficients of π exist. Moreover, all pseudo coefficients are strongly cuspidal.*

Proof. The existence of the pseudo coefficient is proved in [BDK]. Let f be a pseudo coefficient, we want to show that f is strongly cuspidal. By the definition of f, we know that for all proper parabolic subgroup $P=MU$ of G, and for all tempered representations τ of $M(F)$, we have $\mathrm{tr}(\pi'(f))=0$ where $\pi'=I_P^G(\tau)$. Then by Section 5.3 of [B15], we know that f is strongly cuspidal. This proves the lemma. □

4. The Ginzburg-Rallis model

In this section, we study the analytic and geometric properties of the Ginzburg-Rallis model. Geometrically, we show it is a wavefront spherical variety. This gives us the weak Cartan decomposition. Analytically, we show it has polynomial growth as a homogeneous space. Then we prove some estimates for several integrals which will be used in Section 5 and Section 7. This is a technical section, readers may assume the results in this section at the beginning and come back for the proof later.

4.1. **Definition of the Ginzburg-Rallis model.** Let (G, H) be the pair (G, H) or (G_D, H_D) as in Section 1, and let $G_0 = M$. Then (G_0, H_0) is just the trilinear model of $GL_2(F)$ or $GL_1(D)$. We define a homomorphism $\lambda : U(F) \to F$ to be

$$\lambda(u(X, Y, Z)) = \mathrm{tr}(X) + \mathrm{tr}(Y).$$

Then the character ξ we defined in Section 1 can be written as $\xi(u) = \psi(\lambda(u))$ for $u \in U(F)$. Similarly, we can define λ on the Lie algebra of U. We also extend λ to $H(F)$ by making it trivial on $H_0(F)$.

Lemma 4.1. *The map $G \to H\backslash G$ has the norm descent property. For the definition of the norm descent property, see Section 18 of* [K05]*, or Section 1.2 of* [B15].

Proof. Since the map is obviously G-equivariant, by Proposition 18.2 of [K05], we only need to show that it admits a section over a nonempty Zariski-open subset. Let $\bar{P} = M\bar{U}$ be the opposite parabolic subgroup of $P = MU$ with respect to M, and let P' be the subgroup of $\bar{P}$ that consists of elements in $\bar{P}$ whose M-part is of the form $(1, h_1, h_2)$ where $h_1, h_2 \in GL_2(F)$ or $GL_1(D)$. By the Bruhat decompostion, the map $\phi : P' \to H\backslash G$ is injective and the image is a Zariski open subset of $H\backslash G$. Then the composition of ϕ^{-1} and the inclusion $P' \hookrightarrow G$ is a section on $Im(\phi)$. This proves the lemma. □

4.2. **The spherical pair** (G, H)**.** We say a parabolic subgroup $\bar{Q}$ of G is good if $H\bar{Q}$ is a Zariski open subset of G. This is equivalent to say that $H(F)\bar{Q}(F)$ is open in $G(F)$ under the analytic topology.

Proposition 4.2.
1. *There exist minimal parabolic subgroups of G that are good and they are all conjugated to each other by some elements in $H(F)$. If $\bar{P}_{min} = M_{min}\bar{U}_{min}$ is a good minimal parabolic subgroup, we have $H \cap \bar{U}_{min} = \{1\}$ and the complement of $H(F)\bar{P}_{min}(F)$ in $G(F)$ has zero measure.*
2. *A parabolic subgroup $\bar{Q}$ of G is good if and only if it contains a good minimal parabolic subgroup.*
3. *Let $\bar{P}_{min} = M_{min}\bar{U}_{min}$ be a good minimal parabolic subgroup and let $A_{min} = A_{M_{min}}$ be the split center of M_{min}. Set*

$$A_{min}^{+} = \{a \in A_{min}(F) \mid \mid \alpha(a) \mid \geq 1 \text{ for all } \alpha \in \Psi(A_{min}, \bar{P}_{min})\}.$$

   Then we have
   (a) $\sigma_0(h) + \sigma_0(a) \ll \sigma_0(ha)$ *for all* $a \in A_{min}^{+}$, $h \in H(F)$.
   (b) $\sigma(h) \ll \sigma(a^{-1}ha)$ *and* $\sigma_0(h) \ll \sigma_0(a^{-1}ha)$ *for all* $a \in A_{min}^{+}$, $h \in H(F)$.
4. $(1), (2)$ *and* (3) *also holds for the pair* (G_0, H_0).

Proof. (1) We first show the existence of a good minimal parabolic subgroup. In the quaternion case, we can just choose the lower triangle matrices, which form a good minimal parabolic subgroup by the Bruhat decomposition. (Note that in this case the minimal parabolic subgroup is not a Borel subgroup since G is not split). In the split case, we first show that it is enough to find a good minimal parabolic subgroup for the pair (G_0, H_0). Let B_0 be a good minimal parabolic subgroup for the pair (G_0, H_0). Since we are in the split case, B_0 is a Borel subgroup of G_0. Let $B = \bar{U}B_0$. It is a Borel subgroup of G. By the Bruhat decomposition, $\bar{U}P$ is open in G. Together with the fact that B_0 is a good Borel subgroup of (G_0, H_0), we know BH is open in G, which makes B a good minimal parabolic subgroup.

For the pair (G_0, H_0), let $B_0 = (B^+, B^-, B')$ where B^+ is the upper triangular Borel subgroup of GL_2, B^- is the lower triangular Borel subgroup of GL_2 and $B' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} B^- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is easy to see that $B^+ \cap B^- \cap B' = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$, which implies $B_0 \cap H_0 = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$. Then by comparing the dimensions, we know that B_0 is a good minimal parabolic subgroup.

Now we need to show that two good minimal parabolic subgroups are conjugated to each other by some elements in $H(F)$. Let $\bar{P}_{min}$ be the good minimal parabolic subgroup defined above, and let $\bar{P}'_{min}$ be another good minimal parabolic subgroup. We can always find $g \in G(F)$ such that $g\bar{P}_{min}g^{-1} = \bar{P}'_{min}$. Let $\mathcal{U} = H\bar{P}_{min}$ and let $\mathcal{Z} = G - \mathcal{U}$. If $g \in \mathcal{Z}$, then

$$H\bar{P}'_{min} = Hg\bar{P}_{min}g^{-1} \subset \mathcal{Z}g^{-1},$$

which is impossible since $H\bar{P}'_{min}$ is Zariski open and $\mathcal{Z}$ is Zariski closed. Hence $g \in \mathcal{U} \cap G(F) = \mathcal{U}(F)$. If $g \in H(F)\bar{P}_{min}(F)$, then we are done. So it is enough to show that

$$\mathcal{U}(F) = H(F)\bar{P}_{min}(F).$$

We have the following two exact sequence:

$$0 \to H^0(F, \bar{P}_{min}) \to H^0(F, H\bar{P}_{min}) \to H^0(F, H/H \cap \bar{P}_{min}),$$

$$0 \to H^0(F, H\cap\bar{P}_{min}) \to H^0(F, H) \to H^0(F, H/H\cap\bar{P}_{min}) \to H^1(F, H\cap\bar{P}_{min}) \to H^1(F, H).$$

Therefore it is enough to show that the map

$$H^1(F, H \cap \bar{P}_{min}) \to H^1(F, H) \tag{4.1}$$

is injective.

If G is split, by our construction, $H\cap\bar{P}_{min}=\mathrm{GL}_1$. Since $H^1(F,\mathrm{GL}_n)=\{1\}$ for any $n\in\mathbb{N}$, the map (4.1) is injective. If G is not split, by our construction, $H\cap\bar{P}_{min}=H_0$ and $H/H\cap\bar{P}_{min}=U$. Then the map (4.1) lies inside the exact sequence

$$0\to H^0(F,H_0)\to H^0(F,H)\to H^0(F,U)\to H^1(F,H_0)\to H^1(F,H)$$

It is easy to see that the map $H^0(F,H)\to H^0(F,U)$ is surjective, therefore (4.1) is injective. This finishes the proof.

For the rest part of (1), since we have already proved that two good minimal parabolic subgroups can be conjugated to each other by some elements in $H(F)$, it is enough to prove the rest part for a specific good minimal parabolic $\bar{P}_{min}$ we defined above, which is obvious from the construction of $\bar{P}_{min}$. This proves (1). The proof for the pair (G_0,H_0) is similar.

(2) Let $\bar{Q}$ be a good parabolic subgroup and let $P_{min}\subset\bar{Q}$ be a minimal parabolic subgroup. Set

$$\mathcal{G}=\{g\in G\mid g^{-1}P_{min}g\ is\ good\}.$$

This is a Zariski open subset of G since it is the inverse image of the Zariski open subset $\{\mathcal{V}\in Gr_n(\mathfrak{g})\mid \mathcal{V}+\mathfrak{h}=\mathfrak{g}\}$ of the Grassmannian variety $Gr_n(\mathfrak{g})$ under the morphism $g\in G\to g^{-1}\mathfrak{p}_{min}g\in Gr_n(\mathfrak{g})$, here $n=dim(P_{min})$. By (1), there exists good minimal parabolic subgroup, hence $\mathcal{G}$ is non-empty. Since $\bar{Q}$ is good, $\bar{Q}H$ is a Zariski open subset, hence $\bar{Q}H\cap\mathcal{G}\neq\emptyset$. So we can find $\bar{q}_0\in\bar{Q}$ such that $\bar{q}_0^{-1}P_{min}\bar{q}_0$ is a good parabolic subgroup. Let

$$\mathcal{Q}=\{\bar{q}\in\bar{Q}\mid \bar{q}^{-1}P_{min}\bar{q}\ is\ good\}.$$

Then we know $\mathcal{Q}$ is a non-empty Zariski open subset. Since $\bar{Q}(F)$ is dense in $\bar{Q}$, $\mathcal{Q}(F)$ is non-empty. Let $\bar{q}$ be an element of $\mathcal{Q}(F)$. Then the minimal parabolic subgroup $\bar{q}^{-1}P_{min}\bar{q}$ is good and is defined over F. This proves (2). The proof for the pair (G_0,H_0) is similar.

(3) By the first part of the proposition, two good minimal parabolic subgroups are conjugated to each other by some elements in $H(F)$. This implies that (a) and (b) do not depend on the choice of the minimal parabolic subgroups. Hence we may use the minimal parabolic subgroup $\bar{P}_{min}$ defined in (1). Next we show that (a) and (b) do not depend on the choice of M_{min}. Let M_{min},M_{min}' be two choices of Levi subgroup. Then there exists $\bar{u}\in\bar{U}_{min}(F)$ such that $M_{min}'=\bar{u}M_{min}\bar{u}^{-1}$ and $A_{min}'^{+}=\bar{u}A_{min}^{+}\bar{u}^{-1}$. Since $a^{-1}\bar{u}a$ is a contraction for $a\in A_{min}^{+}$, the sets $\{a^{-1}\bar{u}a\bar{u}^{-1}\mid a\in A_{min}^{+}\}$ and $\{a^{-1}\bar{u}^{-1}a\bar{u}\mid a\in A_{min}^{+}\}$ are bounded.

This implies that

$$\begin{aligned}\sigma_0(h\bar{u}a\bar{u}^{-1}) &\sim \sigma_0(ha),\\ \sigma(\bar{u}a\bar{u}^{-1}h\bar{u}a\bar{u}^{-1}) &\sim \sigma(a^{-1}ha),\\ \sigma_0(\bar{u}a\bar{u}^{-1}h\bar{u}a\bar{u}^{-1}) &\sim \sigma_0(a^{-1}ha)\end{aligned}$$

for all $a \in A_{min}^{+}$ and $h \in H(F)$. Therefore (a) and (b) do not depend on the choice of M_{min}. We may choose

$$M_{min} = \{\mathrm{diag}(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix}, \begin{pmatrix} a_5 & a_5 - a_6 \\ 0 & a_6 \end{pmatrix}) \mid a_i \in F^{\times}\}$$

in the split case, and choose

$$M_{min} = \{\mathrm{diag}(b_1, b_2, b_3) \mid b_j \in D^{\times}\}$$

in the non-split case.

For part (a), let $h = uh_0$ for $u \in U(F)$ and $h_0 \in H_0(F)$. Then we know $\sigma_0(h) \ll \sigma_0(h_0) + \sigma_0(u)$ and $\sigma_0(ha) = \sigma_0(uh_0a) \gg \sigma_0(u) + \sigma_0(h_0a)$. As a result, we may assume that $h = h_0 \in H_0(F)$. If we are in the non-split case, $Z_{H_0}(F)\backslash H_0(F)$ is compact, and the argument is trivial. In the split case, since the norm is K-invariant, by the Iwasawa decomposition, we may assume that h_0 is upper triangle. Then by using the same argument as above, we can get rid of the unipotent part. Hence we may assume that $h_0 = diag(h_1, h_2)$ with $h_1, h_2 \in F^{\times}$. By our choice of M_{min},

(4.2)

$$a = diag(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix}, \begin{pmatrix} a_5 & a_5 - a_6 \\ 0 & a_6 \end{pmatrix}) = diag(A_1, A_2, A_3)$$

with $\mid a_2 \mid\leq\mid a_1 \mid\leq\mid a_3 \mid\leq\mid a_4 \mid\leq\mid a_5 \mid\leq\mid a_6 \mid$. Since we only consider σ_0, we may assume that $\Pi a_i = 1$ and $h_1h_2 = 1$. (In general, after modulo the center, we can not make determinant equals 1, there should be some square class left. But since we are talking about majorization, the square class will not effect our estimation.) In order to make the argument also holds for the pair (G_0, H_0), here we only assume that $\mid a_2 \mid\leq\mid a_1 \mid, \mid a_3 \mid\leq\mid a_4 \mid, \mid a_5 \mid\leq\mid a_6 \mid$. It is enough to show that

$$\sigma(h_0) + \sigma(a) \ll \sigma(h_0a). \tag{4.3}$$

By our assumptions, $\sigma(h_0) \sim \log(max\{\mid h_1 \mid, \mid h_2 \mid\})$ and $\sigma(a) \sim \log(max\{\mid a_6 \mid, \mid a_4 \mid, \mid a_1 \mid\}) \sim \log(max\{\mid a_2^{-1} \mid, \mid a_3^{-1} \mid, \mid a_5^{-1} \mid\})$.

- If $|h_2| \geq 1$, we have $\sigma(h_0) \sim \log(\mid h_2 \mid), \parallel h_0A_3 \parallel\geq\mid a_6h_2 \mid$, and $\parallel h_0A_2 \parallel\geq\mid a_4h_2 \mid$. So if $max\{\mid a_6 \mid, \mid a_4 \mid, \mid a_1 \mid\} =\mid a_6 \mid$ or $\mid a_4 \mid$, (4.3) holds. By the same argument, if $max\{\mid a_2^{-1} \mid\mid a_3^{-1} \mid\mid a_5^{-1} \mid\} =\mid a_3^{-1} \mid$ or $\mid a_5^{-1} \mid$, (4.3) also holds. Now the only case left is when $max\{\mid a_6 \mid, \mid a_4 \mid, \mid a_1 \mid\} = \mid a_1 \mid$ and $max\{\mid a_2^{-1} \mid\mid a_3^{-1} \mid\mid a_5^{-1} \mid\} = \mid a_2^{-1} \mid$.

– If $\mid a_6 \mid \geq 1$, then $\parallel h_0 A_3 \parallel \geq \mid a_6 h_2 \mid$ and $\parallel h_0 A_1 \parallel \geq \mid a_2^{-1} h_2^{-1} \mid$. Hence $\parallel h_0 A_1 \parallel\parallel h_0 A_3 \parallel^2 \geq \mid a_2^{-1} a_6^2 h_2 \mid \geq \mid a_2^{-1} h_2 \mid$. In particular, (4.3) holds.
– If $\mid a_6 \mid < 1$, then $\mid a_5 \mid < 1$. In this case, $\parallel h_0 A_3 \parallel \geq \mid a_5^{-1} h_2 \mid$ and $\parallel h_0 A_1 \parallel \geq \mid a_1 h_2^{-1} \mid$. Hence $\parallel h_0 A_1 \parallel\parallel h_0 A_3 \parallel^2 \geq \mid a_5^{-2} a_1 h_2 \mid \geq \mid a_1 h_2 \mid$. In particular, (4.3) holds.

- If $|h_1| \geq 1$, the proof is similar, and we will skip it here.

This finishes the proof of (a) for both the (G, H) and the (G_0, H_0) pairs.

For part (b), the argument for σ_0 is an easy consequence of the argument for σ, so we will only prove the first one. We still let $h = uh_0$. By the definition of A_{min}^+, $a^{-1}ua$ is an extension of u (i.e. $\sigma(a^{-1}ua) \geq \sigma(u)$), so we can still reduce to the case when $h = h_0 \in H_0(F)$. For the non-split case, the argument is trivial since $a^{-1}h_0 a = h_0$. For the split case, we still let $a = diag(A_1, A_2, A_3)$ as above. It is enough to show that for all $h \in GL_2(F)$, we have

$$\parallel h \parallel \leq max\{\parallel A_i^{-1} h A_i \parallel, i = 1, 2, 3\}. \tag{4.4}$$

Let $h = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. We may assume that $|\det(h)| \geq 1$. Then $\parallel h \parallel = max\{|x_{ij}|\}$. If $\parallel h \parallel = |x_{11}|, |x_{21}|$ or $|x_{22}|$, it is easy to see that $\parallel h \parallel \leq \parallel A_1^{-1} h A_1 \parallel$. If $\parallel h \parallel = |x_{12}|$, then $\parallel h \parallel \leq \parallel A_2^{-1} h A_2 \parallel$. Therefore (4.4) holds, and this finishes the proof of (b).

(4) is already covered in the proof of (1), (2) and (3). ☐

The above proposition tells us $X = H\backslash G$ is a spherical variety of G and $X_0 = H_0\backslash G_0$ is a spherical variety of G_0. In the next proposition, we are going to show that X_0 is a wavefront spherical variety of G_0 (we refer the readers to Section 2.1 of [SV12] for the definition of wavefront spherical variety). We need to use this result for the weak Cartan decomposition of (G, H) and (G_0, H_0).

Proposition 4.3. *X_0 is a wavefront spherical variety of G_0.*

Proof. It's is enough to show that the little Weyl group W_{X_0} of X_0 is equal to the Weyl group of G_0, which is $(\mathbb{Z}/2\mathbb{Z})^3$. Here we use the method introduced by Knop in [Knop95] to calculate the little Weyl group. To be specific, let $B = B_1 \times B_2 \times B_3$ be a Borel subgroup of G_0. Without loss of generality, we may assume that B_i are the standard upper triangular Borel subgroup of GL_2. Let $\mathfrak{B}(X_0)$ be the set of all non-empty, closed, irreducible, B-stable subsets of X_0. It is easy to see that there is a bijection between $\mathfrak{B}(X_0)$ and the set of all non-empty, closed, irreducible, H_0-stable subsets of $G_0/B \simeq (\mathbb{P}^1)^3$. We can easily write down the orbits for the latter case: $(\mathbb{P}^1)^3$, X_{12}, X_{13}, X_{23} and Y where $X_{ij} = \{(a_1,a_2,a_3) \in (\mathbb{P}^1)^3 | a_i = a_j\}$ and $Y = \{(a_1,a_2,a_3) \in (\mathbb{P}^1)^3 | a_1 = a_2 = a_3\}$. Therefore $\mathfrak{B}(X_0)$ contain five elements

$$\mathfrak{B}(X_0) = \{X_0, Y_1, Y_2, Y_3, Z\} \tag{4.5}$$

where Z is the orbit of the identity element under the action of B, which is an irreducible subset of codimension 2. And all Y_i's are closed, irreducible, B-stable subsets of codimension 1, with $Y_1 = \{H_0\backslash(g, g'b, g') \mid b \in B_2, g, g' \in \mathrm{GL}_2\}$, $Y_2 = \{H_0\backslash(gb, g', g) \mid b \in B_1, g, g' \in \mathrm{GL}_2\}$, and $Y_3 = \{H_0\backslash(g, gb, g') \mid b \in B_2, g, g' \in \mathrm{GL}_2\}$. Now we study the action of the Weyl group $W = W_{G_0}$ of G_0 on the set $\mathfrak{B}(X_0)$.

Let $\Delta(G_0) = \{\alpha_1, \alpha_2, \alpha_3\}$ be the set of simple roots of G_0 with respect to the Borel subgroup B. Here α_i is the simple root of the i-th GL_2 with respect to B_i. For $i = 1, 2, 3$, let $w_i \in W$ be the simple reflection associated to α_i, and let P_i be the corresponding parabolic subgroup of G_0 (i.e. P_i has B_j on the j-th component for $i \neq j$, and has GL_2 on the i-th component). Since W is generated by the w_i's, it is enough to study the action of w_i on $\mathfrak{B}(X_0)$.

We first consider the action of w_1. It is easy to see that there are two non-empty, closed, irreducible, P_1-stable subsets of X_0: one is Y_1, the other one is X_0. Let

$$\mathfrak{B}(Y_1, P) = \{A \in \mathfrak{B}(X) \mid P_1 A = Y_1\},$$
$$\mathfrak{B}(X_0, P) = \{A \in \mathfrak{B}(X) \mid P_1 A = X_0\}.$$

We have $\mathfrak{B}(Y_1, P) = \{Y_1, Z\}$ and $\mathfrak{B}(X_0, P) = \{Y_2, Y_3, X_0\}$. By Theorem 4.2 of [Knop95], the action of w_1 on $\mathfrak{B}(X_0)$ is given by

$$w_1 \cdot X_0 = X_0, w_1 \cdot Y_1 = Y_1, w_1 \cdot Y_2 = Y_3, w_1 \cdot Y_3 = Y_2, w_1 \cdot Z = Z.$$

Similarly we can get the action of w_2 and w_3:

$$w_2 \cdot X_0 = X_0, w_2 \cdot Y_1 = Y_3, w_2 \cdot Y_2 = Y_2, w_2 \cdot Y_3 = Y_1, w_2 \cdot Z = Z;$$
$$w_3 \cdot X_0 = X_0, w_3 \cdot Y_1 = Y_2, w_3 \cdot Y_2 = Y_1, w_3 \cdot Y_3 = Y_3, w_3 \cdot Z = Z.$$

Hence the isotropy group of X_0 is W. By Theorem 6.2 of [Knop95], the little Weyl group W_{X_0} is just W. Therefore X_0 is a wavefront spherical variety of G_0. □

We need the weak Cartan decomposition for X_0 and X. Let $\bar{P}_0 = M_0\bar{U}_0$ be a good minimal parabolic subgroup of G_0, and let $A_0 = A_{M_0}$ be the maximal split center of M_0. Let

$$A_0^+ = \{a \in A_0(F) | \ |\alpha(a)| \geq 1, \ \forall \alpha \in \Psi(A_0, \bar{P}_0)\}.$$

Proposition 4.4. (1) *There exists a compact subset $\mathcal{K}_0 \subset G_0(F)$ such that*

$$G_0(F) = H_0(F)A_0^+\mathcal{K}_0. \tag{4.6}$$

(2) *There exists a compact subset $\mathcal{K} \subset G(F)$ such that*

$$G(F) = H(F)A_0^+\mathcal{K}. \tag{4.7}$$

Proof. We first prove that (1) implies (2). By the Iwasawa decomposition, there is a compact subgroup K of $G(F)$ such that $G(F) = P(F)K = U(F)M(F)K$. Now by part (1), there exists a compact subset $\mathcal{K}_0$ of $G_0(F) = M(F)$ such that $G_0(F) = H_0(F)A_0^+\mathcal{K}_0$. Let $\mathcal{K} = \mathcal{K}_0K$, then $H(F)A_0^+\mathcal{K} = U(F)H_0(F)A_0^+\mathcal{K}_0K = U(F)M(F)K = G(F)$. This proves (2).

Now we prove (1): in the non-split case, $A_0^+ = Z_{G_0}(F)$ and $Z_{G_0}(F)\backslash G_0(F)$ is compact, hence (1) is trivial. In the split case, if $F = \mathbb{R}$, since (G_0, H_0) is a wavefront spherical variety, (2) follows from Theorem 5.13 of [KKSS]. If F is p-adic, we refer the readers to Appendix A of my thesis [Wan17] for the explicit construction.

□

To end this section, we will show that the homogeneous space $X = H\backslash G$ has polynomial growth. We first recall the definition of polynomial growth in [Ber88].

Definition 4.5. *We say a homogeneous space $X = H\backslash G$ of G has polynomial growth if it satisfies the following condition:*

For a fixed compact neighborhood K of the identity element in G, there exist constants $d, C > 0$ such that for every $R > 0$, the ball $B(R) = \{x \in X \mid r(x) \leq R\}$ can be covered by less than $C(1+R)^d$ many $K-$ balls of the form $Kx, x \in X$. Here r is a function on X defined by $r(x) = \inf\{\sigma(g) \mid x = x_0g\}$ where $x_0 \in X$ is a fixed point.

Remark 4.6. *In our case, if we set $x_0 = 1$, then $r(x) = inf_{h\in H(F)}\sigma(hx)$. By Lemma 4.1, $r(x) = \sigma_{H\backslash G}(x)$.*

We first need a lemma.

Lemma 4.7. (1) *Let $\mathcal{K} \subset G(F)$ be a compact subset. We have $\sigma_{H\backslash G}(xk) \sim \sigma_{H\backslash G}(x)$ for all $x \in H(F)\backslash G(F), k \in \mathcal{K}$.*

(2) *For all $a \in A_0^+$, we have*

$$\sigma_{H\backslash G}(a) \sim \sigma_{Z_G\backslash G}(a) = \sigma_0(a). \tag{4.8}$$

Here the last equation is just the definition of σ_0.

Proof. (1) is trivial. For (2), since the map $G \to H\backslash G$ has the norm descent property (Lemma 4.1), we may assume that

$$\sigma_{H\backslash G}(x) = \inf_{h\in H(F)} \sigma_G(hx). \tag{4.9}$$

Then we obviously have the inequality $\sigma_{H\backslash G}(g) \ll \sigma_0(g)$ for all $g \in G(F)$. So we only need to show that $\sigma_0(a) \ll \sigma_{H\backslash G}(a)$ for all $a \in A_0^+$. By (4.9), it is enough to show that for all $a \in A_0^+$ and $h \in H(F)$, we have

$$\sigma_0(a) \ll \sigma_0(ha). \tag{4.10}$$

We can write $h = uh_0$ for $u \in U(F), h_0 \in H_0(F)$. For all $u \in U(F), g_0 \in G_0(F)$, we have $\sigma_0(ug_0) \gg \sigma_0(g_0)$. This implies $\sigma_0(ha) \gg \sigma_0(h_0a)$. So it is enough to show that for all $a \in A_0^+$ and $h_0 \in H_0(F)$, we have $\sigma_0(a) \ll \sigma_0(h_0a)$. This just follows from Proposition 4.2(3). □

Proposition 4.8. *$H(F)\backslash G(F)$ has polynomial growth as a $G(F)$-homogeneous space.*

Proof. By Proposition 4.4, there exists a compact subset $\mathcal{K} \subset G(F)$ such that $G(F) = H(F)A_0^+\mathcal{K}$. Since $H(F) \cap A_0^+ = Z_G(F)$, together with the lemma above, there exists a constant $c_0 > 0$ such that

$$B(R) \subset H(F)\{a \mid a \in A_0^+/Z_G(F), \sigma_0(a) \leq c_0R\}\mathcal{K}$$

for all $R \geq 1$. Hence we only need to show that there exists a positive integer $N > 0$ such that for all $R \geq 1$, the subset $\{a \in A_0^+/Z_G(F) \mid \sigma_0(a) < R\}$ can be covered by less than $(1+R)^N$ subsets of the form C_0a with $a \in A_0^+$ and $C_0 \subset A_0^+$ be a compact subset with nonempty interior. This is trivial. □

4.3. **Some estimates.** In this section, we are going to prove several estimates for various integrals which will be used in later sections. The main ingredients of the proofs are the results we proved in the previous two sections (i.e. Section 4.1 and 4.2). The proofs are similar to the GGP case considered in [B15], hence we will skip it here. We refer the readers to Section 4.3 and 4.4 of my thesis [Wan17] for details of the proofs.

Lemma 4.9. (1) *There exists $\epsilon > 0$ such that the integral*

$$\int_{Z_{H_0}(F)\backslash H_0(F)} \Xi^{G_0}(h_0)e^{\epsilon\sigma_0(h_0)}dh_0 \tag{4.11}$$

is absolutely convergent.

(2) *There exists $d>0$ such that the integral*

$$\int_{Z_H(F)\backslash H(F)} \Xi^G(h)\sigma_0(h)^{-d}dh \tag{4.12}$$

is absolutely convergent.

(3) *For all $\delta>0$, there exists $\epsilon>0$ such that the integral*

$$\int_{Z_G(F)\backslash H(F)} \Xi^G(h)e^{\epsilon\sigma_0(h)}(1+\mid\lambda(h)\mid)^{-\delta}dh \tag{4.13}$$

is absolutely convergent.

Proof. The proof is similar to the GGP case (Lemma 6.5.1(i), (ii), (iii) of [B15]), we will skip it here. We refer the readers to Lemma 4.3.1 of [Wan17] for details of the proof. □

Let $C\subset G(F)$ be a compact subset with nonempty interior. Define the function $\Xi_C^{H\backslash G}(x)=vol_{H\backslash G}(xC)^{-1/2}$ for $x\in H(F)\backslash G(F)$. If C' is another compact subset with nonempty interior, then $\Xi_C^{H\backslash G}(x)\sim\Xi_{C'}^{H\backslash G}(x)$ for all $x\in H(F)\backslash G(F)$. We will only use the function $\Xi_C^{H\backslash G}$ for majorization. From now on, we will fix a particular C, and set $\Xi^{H\backslash G}=\Xi_C^{H\backslash G}$. The next proposition gives some basic properties for the function $\Xi^{H\backslash G}$.

Proposition 4.10. (1) *Let $\mathcal{K}\subset G(F)$ be a compact subset. We have $\Xi^{H\backslash G}(xk)\sim\Xi^{H\backslash G}(x)$ for all $x\in H(F)\backslash G(F)$ and $k\in\mathcal{K}$.*

(2) *Let $\bar{P}_0=M_0\bar{U}_0$ be a good minimal parabolic subgroup of G_0 and let $A_0=A_{M_0}$ be the split center of M_0. Set*

$$A_0^+=\{a_0\in A_0(F)\mid\mid\alpha(a)\mid\geq 1\ for\ all\ \alpha\in\Psi(A_0,\bar{P}_0)\}.$$

Then there exists $d>0$ such that

$$\Xi^{G_0}(a)\delta_P(a)^{1/2}\sigma_{Z_{G_0}\backslash G_0}(a)^{-d}\ll\Xi^{H\backslash G}(a)\ll\Xi^{G_0}(a)\delta_P(a)^{1/2} \tag{4.14}$$

for all $a\in A_0^+$.

(3) *There exists $d>0$ such that the integral*

$$\int_{H(F)\backslash G(F)}\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{-d}dx$$

is absolutely convergent.

(4) *For all $d>0$, there exists $d'>0$ such that*

$$\int_{Z_H(F)\backslash H(F)}\Xi^G(hx)\sigma_0(hx)^{-d'}dh\ll\Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{-d}$$

for all $x\in H(F)\backslash G(F)$.

Proof. The proof is similar to the GGP case (Proposition 6.7.1 of [B15]), we will skip it here. We refer the readers to Proposition 4.4.1 of [Wan17] for details of the proof. □

Proposition 4.11. *Let $\bar{Q} = M_Q\bar{U}_Q$ be a good parabolic subgroup of G. Let $H_{\bar{Q}} = H \cap \bar{Q}$, and let $G_{\bar{Q}} = \bar{Q}/U_{\bar{Q}}$ be the reductive quotient of $\bar{Q}$. Then we have*

1. *$H_{\bar{Q}} \cap U_{\bar{Q}} = \{1\}$, hence we can view $H_{\bar{Q}}$ as a subgroup of $G_{\bar{Q}}$. We also have $\delta_{\bar{Q}}(h_{\bar{Q}}) = \delta_{H_{\bar{Q}}}(h_{\bar{Q}})$ for all $h_{\bar{Q}} \in H_{\bar{Q}}(F)$.*
2. *There exists $d > 0$ such that the integral*
$$\int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(h_{\bar{Q}})\sigma_0(h_{\bar{Q}})^{-d}\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}$$
is absolutely convergent. Moreover, if we are in the (G_0, H_0)-case (this means that we replace the pair (G, H) in the statement by the pair (G_0, H_0)), for all $d > 0$, the integral
$$\int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \Xi^{G_{\bar{Q}}}(h_{\bar{Q}})\sigma_0(h_{\bar{Q}})^{d}\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}dh_{\bar{Q}}$$
is absolutely convergent.

Proof. The proof is similar to the GGP case (Proposition 6.8.1 of [B15]), we will skip it here. We refer the readers to Lemma 4.4.2 of [Wan17] for details of the proof. □

5. Explicit intertwining operator

In this section, we study an explicit element $\mathcal{L}_\pi$ in the Hom space given by the (normalized) integral of the matrix coefficients. The main result of this section is to show that the Hom space is nonzero if and only if $\mathcal{L}_\pi \neq 0$ (i.e. Theorem 5.3). In Sections 5.1 and 5.2, we define $\mathcal{L}_\pi$ and prove some basic properties of it. In Sections 5.3 and 5.4, we study the behavior of $\mathcal{L}_\pi$ under parabolic induction. Since we can not always reduce to the strongly tempered case, we have to treat the p-adic case and the archimedean case separately. In Section 5.5, we prove Theorem 5.3. Then in Section 5.6, we discuss some applications of Theorem 5.3, which are Corollary 5.13 and Corollary 5.15. These two results will play essential roles in our proofs of the main results of this paper.

5.1. **A normalized integral.** Let χ be an unitary characters of $F^\times$, and let $\eta = \chi^2$. In Section 1, we define the character $\omega \otimes \xi$ on $H(F)$. By Lemma 4.9(2), for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, the integral
$$\int_{Z_H(F)\backslash H(F)} f(h)\omega \otimes \xi(h)dh$$

is absolutely convergent and defines a continuous linear form on the space $\mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$.

Proposition 5.1. *The linear form*

$$f\in \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})\to \int_{Z_H(F)\backslash H(F)} f(h)\omega\otimes\xi(h)dh$$

can be extended continuously to $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$.

Proof. Let $a:\mathbb{G}_m(F)\to Z_{G_0}(F)$ be a homomorphism defined by $a(t)=diag(t,t,1,1,t^{-1},t^{-1})$ in the split case, and $a(t)=diag(t,1,t^{-1})$ in the non-split case. Then we know $\lambda(a(t)ha(t)^{-1})=t\lambda(h)$ for all $h\in H(F)$ and $t\in\mathbb{G}_m(F)$.

If F is p-adic, fix an open compact subgroup $K\subset G(F)$ (not necessarily maximal), it is enough to prove that the linear form

$$f\in \mathcal{C}_K(Z_G(F)\backslash G(F),\eta^{-1})\to \int_{Z_H(F)\backslash H(F)} f(h)\omega\otimes\xi(h)dh$$

can be extended continuously to $\mathcal{C}_K^w(Z_G(F)\backslash G(F),\eta^{-1})$ for all K. Here $\mathcal{C}_K(Z_G(F)\backslash G(F),\eta^{-1})$ (resp. $\mathcal{C}_K^w(Z_G(F)\backslash G(F),\eta^{-1})$) is the space of bi-$K$-invariant elements in $\mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$ (resp. $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$). Let $K_a=a^{-1}(K\cap Z_{G_0}(F))$. It is an open compact subset of $F^\times$. Then for $f\in\mathcal{C}_K(Z_G(F)\backslash G(F),\eta^{-1})$, we have

$$\begin{aligned}
&\int_{Z_H(F)\backslash H(F)} f(h)\omega\otimes\xi(h)dh\\
=\ & meas(K_a)^{-1}\int_{K_a}\int_{Z_H(F)\backslash H(F)} f(a(t)^{-1}ha(t))\xi(h)\omega(a(t)^{-1}ha(t))dhd^\times t\\
=\ & meas(K_a)^{-1}\int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\int_{K_a}\xi(a(t)ha(t)^{-1})d^\times tdh\\
=\ & meas(K_a)^{-1}\int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\int_F 1_{K_a}(t)\psi(t\lambda(h))\mid t\mid^{-1}dtdh.
\end{aligned}$$

Here dt is the Haar measure on F, $d^\times t=\frac{dt}{|t|}$ is the Haar measure on $F^\times$ which also induces a Haar measure $d^\times t$ on K_a, and $meas(K_a)$ is the volume of K_a under the measure $d^\times t$. In the last equality above, we use the fact that for all $\phi\in C_c^\infty(F^\times)\subset C_c^\infty(F)$, we have $\int_{F^\times}\phi(t)d^\times t=\int_F\phi(t)\frac{dt}{|t|}$. Now the function $x\in F\mapsto\int_{K_a}\psi(tx)\mid t\mid^{-1}dt$ is the Fourier transform of the function $\mid\cdot\mid^{-1}1_{K_a}\in C_c^\infty(F)$, so it also belongs to $C_c^\infty(F)$. Hence the last integral above is essentially bounded by

$$\int_{Z_H(F)\backslash H(F)}|f(h)|(1+\mid\lambda(h)\mid)^{-\delta}dh$$

for all $\delta > 0$. By Lemma 4.9(3), we know that the integral above is also absolutely convergent for all $f \in \mathcal{C}_K^w(Z_G(F)\backslash G(F), \eta^{-1})$. Thus the linear form can be extended continuously to $\mathcal{C}_K^w(Z_G(F)\backslash G(F), \eta^{-1})$.

If F=$\mathbb{R}$, recall that for $g \in G(F)$ and $f \in C^\infty(G(F))$, we have defined ${}^g f(x) = f(g^{-1}xg)$. Let Ad_a be a smooth representation of $F^\times$ on $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$ given by $Ad_a(t)(f) = {}^{a(t)}f$. This induces an action of $\mathcal{U}(\mathfrak{gl}_1(F))$ on $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$, which is still denoted by Ad_a. Let $\Delta = 1-(t\frac{d}{dt})^2 \in \mathcal{U}(\mathfrak{gl}_1(F))$. By elliptic regularity (see Lemma 3.7 of [BK14]), for all integer $m \geq 1$, there exist $\varphi_1 \in C_c^{2m-2}(F^\times)$ and $\varphi_2 \in C_c^\infty(F^\times)$ such that $\varphi_1 * \Delta^m + \varphi_2 = \delta_1$. This implies

$$Ad_a(\varphi_1)Ad_a(\Delta^m) + Ad_a(\varphi_2) = Id.$$

Therefore for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, we have

$$\begin{aligned}
& \int_{Z_H(F)\backslash H(F)} f(h)\omega \otimes \xi(h)dh \\
= & \int_{Z_H(F)\backslash H(F)} (Ad_a(\varphi_1)Ad_a(\Delta^m)f)(h)\xi(h)\omega(h)dh \\
& + \int_{Z_H(F)\backslash H(F)} (Ad_a(\varphi_2)f)(h)\xi(h)\omega(h)dh \\
= & \int_{Z_H(F)\backslash H(F)} (Ad_a(\Delta^m)f)(h)\omega(h)\int_{F^\times} \varphi_1(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
& + \int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\int_{F^\times} \varphi_2(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times t dh \\
= & \int_{Z_H(F)\backslash H(F)} (Ad_a(\Delta^m)f)(h)\omega(h)\int_{F} \varphi_1(t)\psi(t\lambda(h))\delta_P(a(t)) \mid t \mid^{-1} dtdh \\
& + \int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\int_{F} \varphi_2(t)\psi(t\lambda(h))\delta_P(a(t)) \mid t \mid^{-1} dtdh.
\end{aligned}$$

Here the second equation is to take the transform $h \mapsto a(t)^{-1}ha(t)$ in both integrals and the extra $\delta_P(a(t))$ is the Jacobian. Also as in the p-adic case, dt is the Haar measure on F, $d^\times t = \frac{dt}{|t|}$ is the Haar measure on $F^\times$, and for all $\phi \in C_c^m(F^\times) \subset C_c^m(F)$, we have $\int_{F^\times} \phi(t)d^\times t = \int_F \phi(t)\frac{dt}{|t|}$. For $i = 1, 2$, the functions $f_i : x \in F \to \int_F \varphi_i(t)\delta_P(a(t))|t|^{-1}\psi(tx)dt$ are the Fourier transforms of the functions $t \to \varphi_i(t)\delta_P(a(t))|t|^{-1} \in C_c^{2m-2}(F)$. Hence f_1 and f_2 are essentially bounded by $(1+|x|)^{-2m+2}$. By applying Lemma 4.9(3) again, we know that for $m \geq 2$, the last two integrals above are absolutely convergent for all $f \in \mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$. Therefore the linear form can be extended continuously to $\mathcal{C}^w(Z_G(F)\backslash G(F), \eta^{-1})$. $\square$

Denote by $\mathcal{P}_{H,\xi}$ the continuous linear form on $\mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$ defined above. In other words, it is defined by

$$\mathcal{P}_{H,\xi}:\ f\in \mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})\rightarrow \int_{Z_H(F)\backslash H(F)}^{*} f(h)\omega\otimes\xi(h)dh.$$

Lemma 5.2. (1) *For all $f\in \mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$, and $h_0,h_1\in H(F)$, we have*

$$\mathcal{P}_{H,\xi}(L(h_0)R(h_1)f)=\omega\otimes\xi(h_0)\omega\otimes\xi(h_1)^{-1}\mathcal{P}_{H,\xi}(f)$$

where R (resp. L) is the right (resp. left) translation.

(2) *Let $\varphi\in C_c^\infty(F^\times)$, and let $\varphi'(t)=\mid t\mid^{-1}\delta_P(a(t))\varphi(t)$. We can view both φ and φ' as elements in $C_c^\infty(F)$. Let $\hat{\varphi}'$ be the Fourier transform of φ' with respect to ψ. Then we have*

$$\mathcal{P}_{H,\xi}(Ad_a(\varphi)f)=\int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\hat{\varphi}'(\lambda(h))dh$$

for all $f\in \mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$. Note that the last integral is absolutely convergent by Lemma 4.9(3).

Proof. Since both sides of the equality are continuous in $\mathcal{C}^w(Z_G(F)\backslash G(F))$, it is enough to check (1) and (2) for $f\in \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$. In this case, $\mathcal{P}_{H,\xi}(f)=\int_{Z_H(F)\backslash H(F)} f(h)\omega\otimes\xi(h)dh$. Then (1) follows from change variables in the integral. For (2), we have

$$\begin{aligned}
\mathcal{P}_{H,\xi}(Ad_a(\varphi)f) &= \int_{Z_H(F)\backslash H(F)} Ad_a(\varphi)(f)\omega\otimes\xi(h)dh\\
&= \int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\int_{F^\times}\varphi(t)\xi(a(t)ha(t)^{-1})\delta_P(a(t))d^\times tdh\\
&= \int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\int_{F}\varphi(t)\psi(t\lambda(h))\delta_P(a(t))|t|^{-1}dtdh\\
&= \int_{Z_H(F)\backslash H(F)} f(h)\omega(h)\hat{\varphi}'(\lambda(h))dh.
\end{aligned}$$

This finishes the proof of the Lemma. □

5.2. **Definition and properties of $\mathcal{L}_\pi$.** Let π be a tempered representation of $G(F)$ with central character η. For all $T\in \mathrm{End}(\pi)^\infty$, define

$$\mathcal{L}_\pi(T)=\mathcal{P}_{H,\xi}(\mathrm{tr}(\pi(g^{-1})T))=\int_{Z_H(F)\backslash H(F)}^{*}\mathrm{tr}(\pi(h^{-1})T)\omega\otimes\xi(h)dh.$$

By Proposition 5.1, together with the fact that the map $T\in \mathrm{End}(\pi)^\infty\rightarrow (g\rightarrow \mathrm{tr}(\pi(g^{-1})T)\in \mathcal{C}^w(Z_G(F)\backslash G(F),\eta^{-1})$ is continuous, we know

$\mathcal{L}_\pi : \mathrm{End}(\pi)^\infty \to \mathbb{C}$ is a continuous linear form in $\mathrm{End}(\pi)^{-\infty}$. Here $\mathrm{End}(\pi)^{-\infty}$ is the topological dual of $\mathrm{End}(\pi)^\infty$ endowed with the strong topology. By Lemma 5.2, for any $h, h' \in H(F)$, we have

$$\mathcal{L}_\pi(\pi(h)T\pi(h')) = \omega \otimes \xi(hh')\mathcal{L}_\pi(T). \tag{5.1}$$

For $e, e' \in \pi$, define $T_{e,e'} \in \mathrm{End}(\pi)^\infty$ to be $e_0 \in \pi \mapsto (e_0, e')e$. Set $\mathcal{L}_\pi(e, e') = \mathcal{L}_\pi(T_{e,e'})$. Then we have

$$\mathcal{L}_\pi(e, e') = \int_{Z_H(F)\backslash H(F)}^{*} (e, \pi(h)e')\omega \otimes \xi(h)dh.$$

If we fix e', by (5.1), the map $e \in \pi \to \mathcal{L}_\pi(e, e')$ belongs to $Hom_H(\pi, \omega \otimes \xi)$. Since $Span\{T_{e,e'} \mid e, e' \in \pi\}$ is dense in $\mathrm{End}(\pi)^\infty$ (in p-adic case, they are equal), we have $\mathcal{L}_\pi \neq 0 \Rightarrow m(\pi) \neq 0$. The purpose of this section is to prove the other direction.

Theorem 5.3. *For all $\pi \in \Pi_{temp}(G, \eta)$, we have*

$$\mathcal{L}_\pi \neq 0 \iff m(\pi) \neq 0.$$

Our proof for this result is based on the method developed by Waldspurger ([W12, Proposition 5.7]) and by Beuzart-Plessis ([B15, Theorem 8.2.1]) for the GGP models. See also [SV12, Theorem 6.2.1]. The key ingredient in the proof is the Plancherel formula together with the fact that the nonvanishing property of $\mathcal{L}_\pi$ is invariant under the parabolic induction and the unramified twist. For the rest part of this section, we discuss some basic properties of $\mathcal{L}_\pi$.

Lemma 5.4. *With the notation above, the followings hold.*

(1) *The map $\pi \in \Pi_{temp}(G, \eta) \to \mathcal{L}_\pi \in \mathrm{End}(\pi)^{-\infty}$ is smooth in the following sense: For all parabolic subgroup $Q = LU_Q$ of G, $\sigma \in \Pi_2(L)$, and for all maximal compact subgroup K of $G(F)$, the map $\lambda \in i\mathfrak{a}_{L,0}^* \to \mathcal{L}_{\pi_\lambda} \in \mathrm{End}(\pi_\lambda)^{-\infty} \simeq \mathrm{End}(\pi_K)^{-\infty}$ is smooth. Here $\pi_\lambda = I_Q^G(\sigma_\lambda)$ and $\pi_K = I_{Q\cap K}^K(\sigma_K)$.*

(2) *For $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, assume that its Fourier transform $\pi \in \Pi_{temp}(G, \eta) \to \pi(f)$ is compactly supported (this is always true in p-adic case). Then we have*

$$\int_{Z_H(F)\backslash H(F)} f(h)\omega \otimes \xi(h)dh = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(f))\mu(\pi)d\pi$$

with both integrals being absolutely convergent.

(3) *For $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ and $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$, assume that the Fourier transform of f is compactly supported.*

Then we have

$$\int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(f))\overline{\mathcal{L}_\pi(\pi(\bar{f}'))}\mu(\pi)d\pi$$
$$= \int_{Z_H(F)\backslash H(F)} \int_{Z_H(F)\backslash H(F)} \int_{Z_G(F)\backslash G(F)} f(hgh')f'(g)dg\omega\otimes\xi(h')dh'\omega\otimes\xi(h)dh$$

where the left hand side is absolutely convergent and the right hand side is convergent in that order but is not necessarily absolutely convergent.

Proof. The proof is similar to the GGP case (Lemma 8.2.1 of [B15]), we will skip it here. We refer the readers to Lemma 6.2.2 of [Wan17] for details of the proof. □

The next lemma is about the asymptotic properties for elements in $Hom_H(\pi,\omega\otimes\xi)$.

Lemma 5.5. (1) *Let π be a tempered representation of $G(F)$ with central character η and let $l\in Hom_{H(F)}(\pi,\omega\otimes\xi)$ be a continuous $(H(F),\omega\otimes\xi)$-equivariant linear form. Then there exist $d>0$ and a continuous semi-norm ν_d on π such that*

$$|l(\pi(x)e)|\leq \nu_d(e)\Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^d$$

for all $e\in\pi$ and $x\in H(F)\backslash G(F)$.

(2) *For all $d>0$, there exist $d'>0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$ such that*

$$|\mathcal{P}_{H,\xi}(R(x)L(y)\varphi|\leq \nu_{d,d'}(\varphi)\Xi^{H\backslash G}(x)\Xi^{H\backslash G}(y)\sigma_{H\backslash G}(x)^d\sigma_{H\backslash G}(y)^{d'}$$

for all $\varphi\in\mathcal{C}_d^w(Z_G(F)\backslash G(F),\eta^{-1})$ and $x,y\in H(F)\backslash G(F)$.

Proof. The proof is similar to the GGP case (Lemma 8.3.1 of [B15]), we will skip it here. We refer the readers to Lemma 6.2.3 of [Wan17] for details of the proof. □

5.3. **Parabolic induction for the p-adic case.** Assume that F is p-adic in this section. Let π be a tempered representation of $G(F)$ with central character η. There exists a parabolic subgroup $\bar{Q}=LU_{\bar{Q}}$ of G, together with a discrete series $\tau\in\Pi_2(L)$ such that $\pi=I_{\bar{Q}}^G(\tau)$. By Proposition 4.2, we may assume that $\bar{Q}$ is a good parabolic subgroup. We can further assume that the inner product on π is given by

$$(e,e')=\int_{Q(F)\backslash G(F)}(e(g),e'(g))_\tau dg,\ \forall e,e'\in\pi=I_Q^G(\tau). \tag{5.2}$$

Let $H_{\bar{Q}}=H\cap \bar{Q}$. For $T\in \mathrm{End}(\tau)^{\infty}$, define

$$\mathcal{L}_{\tau}(T)=\int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \mathrm{tr}(\tau(h_{\bar{Q}}^{-1})T)\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}\omega\otimes\xi(h_{\bar{Q}})dh_{\bar{Q}}.$$

The integral above is absolutely convergent by Proposition 4.11(2) together with the assumption that τ is a discrete series.

Proposition 5.6. *With the notation above, we have*

$$\mathcal{L}_{\pi}\neq 0 \iff \mathcal{L}_{\tau}\neq 0.$$

Proof. For $e,e'\in \pi^{\infty}$, by (5.2), we have

$$\mathcal{L}_{\pi}(e,e')=\int_{Z_H(F)\backslash H(F)}^{*}\int_{\bar{Q}(F)\backslash G(F)} (e(g),e'(gh))_{\tau}dg\omega\otimes\xi(h)dh.$$

Same as in the previous sections, let $a:\mathbb{G}_m(F)\rightarrow Z_{G_0}(F)$ be a homomorphism defined by $a(t)=diag(t,t,1,1,t^{-1},t^{-1})$ in the split case, and $a(t)=diag(t,1,t^{-1})$ in the non-split case. Since $e,e'\in\pi^{\infty}$, there exists an open compact subgroup K_0 of $G(F)$ such that the functions $e,e':G(F)\rightarrow \tau$ are bi-K_0-invariant. Let $K_a=a^{-1}(K_0\cap Z_{G_0}(F))\subset F^{\times}$, which is an open compact subset. By Proposition 5.1, we have

$$\begin{aligned}\mathcal{L}_{\pi}(e,e') &= \int_{Z_H(F)\backslash H(F)}^{*}\int_{\bar{Q}(F)\backslash G(F)} (e(g),e'(gh))_{\tau}dg\omega\otimes\xi(h)dh\\
&= meas(K_a)^{-1}\int_{Z_H(F)\backslash H(F)}\int_{\bar{Q}(F)\backslash G(F)} (e(g),e'(gh))_{\tau}dg \qquad (5.3)\\
&\quad\times\int_F 1_{K_a}(t)\psi(t\lambda(h))\mid t\mid^{-1}dt\omega(h)dh.\end{aligned}$$

By the same proposition, the last two integrals $\int_{Z_H(F)\backslash H(F)}\int_{\bar{Q}(F)\backslash G(F)}$ above is absolutely convergent. Since $\bar{Q}$ is a good parabolic subgroup, by Proposition 4.2, we can choose the Haar measures compatibly so that for all $\varphi\in L_1(\bar{Q}(F)\backslash G(F),\delta_{\bar{Q}})$, we have

$$\int_{\bar{Q}(F)\backslash G(F)}\varphi(g)dg=\int_{H_{\bar{Q}}(F)\backslash H(F)}\varphi(h)dh.$$

Then (5.3) becomes

$$\begin{aligned}\mathcal{L}_{\pi}(e,e') &= meas(K_a)^{-1}\int_{Z_H(F)\backslash H(F)}\int_{H_{\bar{Q}}(F)\backslash H(F)} (e(h'),e'(h'h))_{\tau}dh'\\
&\quad\times\int_F 1_{K_a}(t)\psi(t\lambda(h))\mid t\mid^{-1}dt\omega(h)dh.\end{aligned}$$

The integral $\int_{Z_H(F)\backslash H(F)}\int_{H_{\bar{Q}}(F)\backslash H(F)}$ above is absolutely convergent because the integral $\int_{Z_H(F)\backslash H(F)}\int_{\bar{Q}(F)\backslash G(F)}$ in (5.3) is absolutely convergent. By switching the two integrals, making the transform $h\rightarrow h'h$ and decomposing $\int_{Z_H(F)\backslash H(F)}$ as $\int_{H_{\bar{Q}}(F)\backslash H(F)}\int_{Z_H(F)\backslash H_{\bar{Q}}(F)}$, we have

$$\mathcal{L}_{\pi}(e,e')=meas(K_a)^{-1}\int_{(H_{\bar{Q}}(F)\backslash H(F))^2} f(h,h')dhdh'$$

where

$$\begin{aligned} f(h,h') &= \int_{Z_H(F)\backslash H_{\bar{Q}}(F)} (e(h),e'(h_{\bar{Q}}h'))_{\tau}\omega(h_{\bar{Q}})\omega(h^{-1}h') \\ &\quad\times\int_F 1_{K_a}(t)\psi(t\lambda(h'))\psi(t\lambda(h_{\bar{Q}}))\psi(-t\lambda(h))\mid t\mid^{-1}dtdh_{\bar{Q}} \\ &= \int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h),\tau(h_{\bar{Q}})e'(h'))_{\tau}\omega(h_{\bar{Q}})\omega(h^{-1}h') \\ &\quad\times\int_F 1_{K_a}(t)\psi(t\lambda(h'))\psi(t\lambda(h_{\bar{Q}}))\psi(-t\lambda(h))\mid t\mid^{-1}dtdh_{\bar{Q}}. \end{aligned} \tag{5.4}$$

Here we use the equation $\delta_{H_{\bar{Q}}}(h_{\bar{Q}})=\delta_{\bar{Q}}(h_{\bar{Q}})$ in the second equality. We first show that the integral (5.4) is absolutely convergent for any $h,h'\in H_{\bar{Q}}(F)\backslash H(F)$. In fact, since K_a is compact, it is enough to show that for any $h,h'\in H_{\bar{Q}}(F)\backslash H(F)$, the integral

$$\int_{Z_H(F)\backslash H_{\bar{Q}}(F)}\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h),\tau(h_{\bar{Q}})e'(h'))_{\tau}dh_{\bar{Q}}$$

is absolutely convergent. This just follows from Proposition 4.11(2) together with the assumption that τ is a discrete series. Then by switching the two integrals in (5.4), we have

$$\begin{aligned} f(h,h') &= \int_F 1_{K_a}(t)\int_{Z_H(F)\backslash H_{\bar{Q}}(F)}\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h),\tau(h_{\bar{Q}})e'(h'))_{\tau}\omega(h_{\bar{Q}}) \\ &\quad\times\psi(t\lambda(h_{\bar{Q}}))dh_{\bar{Q}}\omega(h^{-1}h')\psi(t\lambda(h'))\psi(-t\lambda(h))\mid t\mid^{-1}dt. \end{aligned}$$

For $t\in K_a$, by changing the variable $h_{\bar{Q}}\rightarrow a(t)h_{\bar{Q}}a(t)^{-1}$ in the inner integral (note that the Jacobian of such transform is 1 since $a(t)\in K_0$), we have

$$\begin{aligned} & \int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h),\tau(h_{\bar{Q}})e'(h'))_{\tau}\omega(h_{\bar{Q}})\psi(t\lambda(h_{\bar{Q}}))dh_{\bar{Q}} \\ = & \int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h),\tau(a(t)^{-1}h_{\bar{Q}}a(t))e'(h'))_{\tau}\omega(h_{\bar{Q}})\psi(\lambda(h_{\bar{Q}}))dh_{\bar{Q}} \\ = & \int_{Z_H(F)\backslash H_{\bar{Q}}(F)} \delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}(e(h),\tau(h_{\bar{Q}})e'(h'))_{\tau}\omega(h_{\bar{Q}})\psi(\lambda(h_{\bar{Q}}))dh_{\bar{Q}} \\ = & \mathcal{L}_{\tau}(e(h),e'(h')). \end{aligned}$$

Here we use the fact that e' is bi-K_0-invariant. Then we have

$$f(h,h') = \int_F 1_{K_a}(t)\mathcal{L}_{\tau}(e(h),e(h'))\omega(h^{-1}h')\psi(t\lambda(h'))\psi(-t\lambda(h)) \mid t\mid^{-1} dt.$$

If $\mathcal{L}_{\pi}(e,e')\neq 0$, there exist $h,h'\in H_{\bar{Q}}(F)\backslash H(F)$ such that $f(h,h')\neq 0$, and hence $\mathcal{L}_{\tau}(e(h),e(h'))\neq 0$. This proves that $\mathcal{L}_{\pi}\neq 0\Rightarrow\mathcal{L}_{\tau}\neq 0$.

For the other direction, if $\mathcal{L}_{\tau}\neq 0$, we can find $v_0,v_0'\in\tau^{\infty}$ such that $\mathcal{L}_{\tau}(v_0,v_0')\neq 0$. We choose a small open subset $\mathcal{U}\subset H_{\bar{Q}}(F)\backslash H(F)$ and let $s:\mathcal{U}\to H(F)$ be an analytic section of the map $H(F)\to H_{\bar{Q}}(F)\backslash H(F)$. For $f,f'\in C_c^{\infty}(\mathcal{U})$, define $\varphi,\varphi'\in C_c^{\infty}(\mathcal{U},\tau^{\infty})$ to be $\varphi(h)=f(h)v_0,\varphi'(h)=f'(h)v_0'$. Set

$$e_{\varphi}(g) = \begin{cases} \delta_{\bar{Q}}^{1/2}(l)\tau(l)\varphi(h) & \text{if} \quad g=lus(h) \text{ with } l\in L(F),u\in U_{\bar{Q}}(F),h\in\mathcal{U}; \\ 0 & \text{else.} \end{cases}$$

This is an element of π^{∞}. Similarly we can define $e_{\varphi'}$. By the above discussion, we have

$$\mathcal{L}_{\pi}(e_{\varphi},e_{\varphi'}) = meas(K_a)^{-1}\int_{(H_{\bar{Q}}(F)\backslash H(F))^2} f(h,h')dhdh'$$

where

$$f(h,h') = \int_F 1_{K_a}(t)\mathcal{L}_{\tau}(e_{\varphi}(h),e_{\varphi'}(h'))\omega(h^{-1}h')\psi(t\lambda(h'))\psi(-t\lambda(h)) \mid t\mid^{-1} dt.$$

Combining with the definition of e_{φ} and $e_{\varphi'}$, we have

$$\begin{aligned} \mathcal{L}_{\pi}(e_{\varphi},e_{\varphi'}) = & meas(K_a)^{-1}\mathcal{L}_{\tau}(v_0,v_0')\times\int_{\mathcal{U}^2}\int_F 1_{K_a}(t)f(h)\overline{f'(h')} \\ & \omega(s(h)^{-1}s(h'))\psi(t\lambda(s(h')))\psi(-t\lambda(s(h))) \mid t\mid^{-1} dtdhdh'. \end{aligned}$$

Now if we take $\mathcal{U}$ small enough, we can choose a suitable section $s:\mathcal{U}\to H(F)$ such that for all $t\in K_a$ and $h\in s(\mathcal{U})$, we have $\psi(t\lambda(h))=$

$\omega(h) = 1$. Then the integral above becomes

$$\begin{aligned}\mathcal{L}_\pi(e_\varphi, e_{\varphi'}) &= meas(K_a)^{-1}\mathcal{L}_\tau(v_0, v_0') \int_{\mathcal{U}^2}\int_F 1_{K_a}(t) f(h)\overline{f'(h')}|t|^{-1} dt dh dh' \\ &= \mathcal{L}_\tau(v_0, v_0') \int_{\mathcal{U}^2} f(h)\overline{f'(h')} dh dh'.\end{aligned}$$

Thus we can easily choose f and f' so that $\mathcal{L}_\pi(e_\varphi, e_{\varphi'}) \neq 0$. This proves $\mathcal{L}_\tau \neq 0 \Rightarrow \mathcal{L}_\pi \neq 0$. □

5.4. **Parabolic induction for $F = \mathbb{R}$.** Assume that $F = \mathbb{R}$ in this section. It is very hard to directly study any arbitrary parabolic induction because of the way that we normalize the integral. Instead, we first study the parabolic induction for $\bar{P}$, then we study all other parabolic subgroups contained in $\bar{P}$. This is allowable since in the archimedean case, the discrete series only appear on $\mathrm{GL}_1(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_1(D)$. Let π be a tempered representation of G with central character η. There exists a tempered representation π_0 of G_0 such that $\pi = I_{\bar{P}}^G(\pi_0)$. We assume that the inner product on π is given by

$$(e, e') = \int_{\bar{P}(F)\backslash G(F)} (e(g), e'(g))_{\pi_0} dg, \ e, e' \in \pi = I_{\bar{P}}^G(\pi_0). \tag{5.5}$$

For $T \in \mathrm{End}(\pi_0)^\infty$, define

$$\mathcal{L}_{\pi_0}(T) = \int_{Z_H(F)\backslash H_0(F)} \mathrm{tr}(\pi_0(h_0^{-1})T)\omega(h_0) dh_0.$$

The integral above is absolutely convergent by Lemma 4.9(1) together with the fact that π_0 is tempered.

Proposition 5.7. *With the notation above, we have*

$$\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_{\pi_0} \neq 0.$$

Proof. For $e, e' \in \pi^\infty$, we have

$$\mathcal{L}_\pi(e, e') = \int^*_{Z_H(F)\backslash H(F)} \int_{\bar{P}(F)\backslash G(F)} (e(g), e'(gh)) dg \omega \otimes \xi(h) dh.$$

Same as in Proposition 5.1, we can find $\varphi_1 \in C_c^{2m-2}(F^\times)$ and $\varphi_2 \in C_c^\infty(F^\times)$ such that $\varphi_1 * \Delta^m + \varphi_2 = \delta_1$, and we have

$$\begin{aligned} \mathcal{L}_\pi(e,e') &= \int_{Z_H(F)\backslash H(F)} Ad_a(\Delta^m)(\int_{\bar{P}(F)\backslash G(F)} (e(g),e'(gh))dg) \\ &\times \int_F \varphi_1(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdh \\ &+ \int_{Z_H(F)\backslash H(F)} \int_{\bar{P}(F)\backslash G(F)} (e(g),e'(gh)) \\ &\times \int_F \varphi_2(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdgdh. \end{aligned} \tag{5.6}$$

Here $Ad_a(\Delta^m)$ acts on the function $\int_{\bar{P}(F)\backslash G(F)}(e(g),e'(gh))dg$ for the variable h. It is clear that this action commutes with the integral $\int_{\bar{P}(F)\backslash G(F)}$. Also since $\bar{P}$ is a good parabolic subgroup, by Proposition 4.2, we can choose Haar measure compatibly so that for all $\varphi \in L_1(\bar{P}(F)\backslash G(F), \delta_{\bar{P}})$, we have

$$\int_{\bar{P}(F)\backslash G(F)} \varphi(g)dg = \int_{U(F)} \varphi(h)dh.$$

Therefore (5.6) becomes

$$\begin{aligned} \mathcal{L}_\pi(e,e') &= \int_{Z_H(F)\backslash H(F)} \int_{U(F)} Ad_a(\Delta^m)((e(u),e'(uh)))du \\ &\times \int_F \varphi_1(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdh \\ &+ \int_{Z_H(F)\backslash H(F)} \int_{U(F)} (e(u),e'(uh)) \\ &\times \int_F \varphi_2(t)\delta_P(a(t)) \mid t \mid^{-1} \psi(t\lambda(h))\omega(h)dtdudh. \end{aligned}$$

Here $Ad_a(\Delta^m)$ acts on the function $(e(u),e'(uh))$ for the variable h. By changing the order of integration $\int_{Z_H(F)\backslash H(F)} \int_{U(F)}$ and decomposing the integral $\int_{Z_H(F)\backslash H(F)}$ by $\int_{U(F)} \int_{Z_H(F)\backslash H_0(F)}$ (this is allowable since the outer two integrals are absolutely convergent by Proposition 5.1), together with the fact that Ad_a is the identity map on H_0, we have

$$\begin{aligned} \mathcal{L}_\pi(e,e') &= \int_{U(F)} \int_{U(F)} Ad_a(\Delta^m)(\mathcal{L}_{\pi_0}(e(u),e'(uu')))\varphi_1'(\lambda(u'))du'du \\ &+ \int_{U(F)} \int_{U(F)} \mathcal{L}_{\pi_0}(e(u),e'(uu'))\varphi_2'(\lambda(u'))du'du \end{aligned}$$

where $\varphi_i'(s)=\int_F \varphi_i(t)\delta_P(a(t)) \mid t\mid^{-1} \psi(ts)dt$ is the Fourier transforms of the function $\varphi_i(t)\delta_P(a(t)) \mid t\mid^{-1}$ for $i=1,2$. Here $Ad_a(\Delta^m)$ acts on the function $\mathcal{L}_{\pi_0}(e(u),e'(uu'))$ for the variable u'. In particular, this implies $\mathcal{L}_\pi\neq 0\Rightarrow \mathcal{L}_{\pi_0}\neq 0$.

For the other direction, if $\mathcal{L}_{\pi_0}\neq 0$, we can choose $v_1,v_2\in\pi_0^\infty$ such that $\mathcal{L}_{\pi_0}(v_1,v_2)\neq 0$. Choose $f_1,f_2\in C_c^\infty(U(F))$. For $i=1,2$, same as in the p-adic case, define

$$e_{f_i}(g)=\begin{cases}\delta_{\bar{P}}(l)\pi_0(l)f_i(u)v_i & \text{if} \quad g=l\bar{u}u \text{ with } l\in G_0(F),u\in U(F),\bar{u}\in\bar{U}(F);\\ 0 & \text{else.}\end{cases}$$

These are elements in π^∞, and we have

$$\begin{aligned}\mathcal{L}_\pi(e_{f_1},e_{f_2}) &= \int_{U(F)}\int_{U(F)}\mathcal{L}_{\pi_0}(v_1,v_2)f_1(u)Ad_a(\Delta^m)(f_2(uu')))\varphi_1'(\lambda(u'))du'du\\ &+\int_{U(F)}\int_{U(F)}\mathcal{L}_{\pi_0}(v_1,v_2)f_1(u)f_2(uu')\varphi_2'(\lambda(u'))du'du.\end{aligned}\tag{5.7}$$

Here $Ad_a(\Delta^m)$ acts on the function $f_2(uu')$ for the variable u'. Then we can easily find f_1,f_2 such that (5.7) is non-zero. This proves $\mathcal{L}_{\pi_0}\neq 0\Rightarrow\mathcal{L}_\pi\neq 0$, and finishes the proof of the proposition. □

Now for a tempered representation π_0 of $G_0(F)$ whose central character equals η when restricting on $Z_G(F)$, we can find a good parabolic subgroup $\bar{Q}_0=L_0U_0$ of $G_0(F)$ and a discrete series τ of L_0 such that $\pi_0=I_{\bar{Q}_0}^{G_0}(\tau)$. We still assume that the inner product on π_0 is given by

$$(e,e')=\int_{\bar{Q}_0(F)\backslash H_0(F)}(e(g),e'(g))_\tau dg,\ e,e'\in\pi_0=I_{\bar{Q}_0}^{G_0}(\tau).\tag{5.8}$$

Let $H_{\bar{Q}}=H_0\cap\bar{Q}_0$. For $T\in\mathrm{End}(\tau)^\infty$, define

$$\mathcal{L}_\tau(T)=\int_{Z_H(F)\backslash H_{\bar{Q}}(F)}\mathrm{tr}(\tau(h_{\bar{Q}}^{-1})T)\delta_{H_{\bar{Q}}}(h_{\bar{Q}})^{1/2}\omega(h_{\bar{Q}})dh_{\bar{Q}}.$$

The integral above is absolutely convergent by Proposition 4.11(2) together with the assumption that τ is a discrete series.

Proposition 5.8. *With the notation above, we have*

$$\mathcal{L}_{\pi_0}\neq 0\iff\mathcal{L}_\tau\neq 0.$$

Proof. Since we are in (G_0,H_0) case, the integral defining $\mathcal{L}_{\pi_0}$ is absolutely convergent. Together with (5.8), we have

$$\mathcal{L}_{\pi_0}(e,e')=\int_{Z_H(F)\backslash H_0(F)}\int_{\bar{Q}_0(F)\backslash G_0(F)}(e(g),e'(gh))_\tau\omega(h)dgdh.$$

The integral above is absolutely convergent by Lemma 4.9. Same as in the previous Propositions, the integral $\bar{Q}_0(F)\backslash G_0(F)$ can be replaced by $H_{\bar{Q}}(F)\backslash H_0(F)$. This implies

$$\mathcal{L}_{\pi_0}(e,e') = \int_{Z_H(F)\backslash H_0(F)} \int_{H_{\bar{Q}}(F)\backslash H_0(F)} (e(h'), e'(h'h))_\tau \omega(h) dh' dh.$$

By switching the two integrals, changing the variable $h \to h'h$ and decomposing the integral $\int_{Z_H(F)\backslash H_0(F)}$ by $\int_{H_{\bar{Q}}(F)\backslash H_0(F)} \int_{Z_H(F)\backslash H_{\bar{Q}}(F)}$, we have

$$\mathcal{L}_\pi(e,e') = \int_{H_{\bar{Q}}(F)\backslash H_0(F)} \int_{H_{\bar{Q}}(F)\backslash H_0(F)} \mathcal{L}_\tau(e(h), e'(h'))\omega(h)^{-1}\omega(h') dh dh'.$$

This proves $\mathcal{L}_{\pi_0} \neq 0 \Rightarrow \mathcal{L}_\tau \neq 0$.

For the other direction, if $\mathcal{L}_\tau \neq 0$, there exist $v_1, v_2 \in \tau^\infty$ such that $\mathcal{L}_\tau(v_1,v_2) \neq 0$. Let $s : \mathcal{U} \to H_0(F)$ be an analytic section over an open subset $\mathcal{U}$ of $H_{\bar{Q}}(F)\backslash H_0(F)$ of the map $H_0(F) \to H_{\bar{Q}}(F)\backslash H_0(F)$. Choose $f_1, f_2 \in C_c^\infty(\mathcal{U})$. For $i = 1,2$, define

$$e_{f_i}(g) = \begin{cases} \delta_{\bar{Q}}(l)\tau(l)f_i(h)v_i & \text{if} \quad g = lus(h) \text{ with } l \in L_0(F), u \in U_0(F), h \in \mathcal{U} \\ 0 & \text{else.} \end{cases}$$

These are elements in π_0^∞, and we have

$$\mathcal{L}_{\pi_0}(e_{f_1}, e_{f_2}) = \int_{\mathcal{U}} \int_{\mathcal{U}} f_1(h)\overline{f_2}(h')\omega(s(h))^{-1}\omega(s(h'))\mathcal{L}_\tau(v_1,v_2) dh dh'.$$

Then we can easily choose f_1, f_2 such that $\mathcal{L}_{\pi_0}(e_{f_1}, e_{f_2}) \neq 0$. This proves the other direction, and finishes the proof of the Proposition. □

Now let π be a tempered representation of $G(F)$. Then we can find a good parabolic subgroup $L_0U_0 = \bar{Q} \subset \bar{P}(F)$ and a discrete series τ of L_0 such that $\pi = I_{\bar{Q}}^G(\tau)$ (note that we are in archimedean case, only $GL_1(F), GL_2(F)$ and $GL_1(D)$ have discrete series). Combining Proposition 5.7 and Proposition 5.8, we have the following Proposition.

Proposition 5.9. *With the notation above, we have*

$$\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_\tau \neq 0.$$

5.5. **Proof of Theorem 5.3.** Let π be a tempered representation of $G(F)$ with central character η. We already know $\mathcal{L}_\pi \neq 0 \Rightarrow m(\pi) \neq 0$. We are going to prove the other direction. If $m(\pi) \neq 0$, let $0 \neq l \in Hom_{H(F)}(\pi^\infty, \omega \otimes \xi)$. We first prove

(1): For all $e \in \pi^\infty$ and $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, the integral

$$\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg \tag{5.9}$$

is absolutely convergent.

In fact, this is equivalent to the convergence of

$$\int_{H(F)\backslash G(F)} \mid l(\pi(x)e) \mid \int_{Z_H(F)\backslash H(F)} \mid f(hx) \mid dhdx.$$

By Proposition 4.10(4), for all $d > 0$ and $x \in H(F)\backslash G(F)$, we have

$$\int_{Z_H(F)\backslash H(F)} \mid f(hx) \mid dh \ll \Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{-d}. \tag{5.10}$$

On the other hand, by Lemma 5.5(1), there exists $d' > 0$ such that for all $x \in H(F)\backslash G(F)$, we have

$$\mid l(\pi(x)e) \mid \ll \Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{d'}. \tag{5.11}$$

Then (1) follows from (5.10), (5.11), and Proposition 4.10(3).

Now we can compute (5.9) in two different ways. First, it is easy to see that

$$\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg = l(\pi(f)e). \tag{5.12}$$

On the other hand,

$$\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg = \int_{H(F)\backslash G(F)} l(\pi(x)e) \int_{Z_H(F)\backslash H(F)} f(hx)\omega\otimes\xi(h)dhdx.$$

By Lemma 5.4(2), if the map $\Pi \in \Pi_{temp}(G,\eta) \to \Pi(f)$ is compactly supported, we have

$$\begin{aligned} &\int_{Z_G(F)\backslash G(F)} l(\pi(g)e)f(g)dg \\ = &\int_{H(F)\backslash G(F)} l(\pi(x)e) \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\Pi(\Pi(f)\Pi(x^{-1}))\mu(\Pi)d\Pi dx. \end{aligned} \tag{5.13}$$

For $T \in C_c^\infty(\Pi_{temp}(G,\eta))$, by applying (5.12) and (5.13) to the function $f = f_T$ corresponding to T by the matrical Paley-Wiener Theorem in Section 2.5, we have

$$l(T_\pi e) = \int_{H(F)\backslash G(F)} l(\pi(x)e) \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\Pi(T_\Pi \Pi(x^{-1}))\mu(\Pi)d\Pi dx \tag{5.14}$$

for all $e \in \pi^\infty$. Now assume that $\pi = I_Q^G(\sigma)$ for some good parabolic subgroup $Q = LU_Q$ of G, and $\sigma \in \Pi_2(L)$. Let

$$\mathcal{O} = \{Ind_Q^G(\sigma_\lambda) \mid \lambda \in i\mathfrak{a}_{L,0}^*\} \subset \Pi_{temp}(G,\eta) \tag{5.15}$$

be the connected component containing π. Choose $e_0 \in \pi^\infty$ such that $l(e_0) \neq 0$, and let $T_0 \in \mathrm{End}(\pi)^\infty$ with $T_0(e_0) = e_0$. We can easily find an element $T^0 \in C_c^\infty(\Pi_{temp}(G,\eta))$ such that

$$T_\pi^0 = T_0,\ Supp(T^0) \subset \mathcal{O}.$$

By applying (5.14) to the case $e = e_0$ and $T = T^0$, we know there exists $\lambda \in i\mathfrak{a}_{L,0}^*$ such that $\mathcal{L}_{\pi_\lambda} \neq 0$ where $\pi_\lambda = Ind_Q^G(\sigma_\lambda)$. By Proposition 5.6 and Proposition 5.9, this implies $\mathcal{L}_{\sigma_\lambda} \neq 0$. We need a Lemma:

Lemma 5.10. *For all $\lambda \in i\mathfrak{a}_{L,0}^*$, we have*

$$\mathcal{L}_\sigma \neq 0 \iff \mathcal{L}_{\sigma_\lambda} \neq 0.$$

Proof. In Appendix B of [Wan15], we divided the reduced models $(L, H_{\bar{Q}})$ into two types. Type I contain all reduced models in the $\mathrm{GL}_3(D)$ case together with its analogue in the $\mathrm{GL}_6(F)$ case. To be specific, in the $\mathrm{GL}_6(F)$ case, it contain the reduced models associated to the parabolic subgroups of type (6), $(4,2)$ and $(2,2,2)$. Type II contain all the rest reduced models in the $\mathrm{GL}_6(F)$ case.

If we are in Type I case, there are three models: the Ginzburg-Rallis model which correspondent to the parabolic subgroups of type (6) (resp. type (3)) in the $\mathrm{GL}_6(F)$ case (resp. $\mathrm{GL}_3(D)$ case); the "middle" model which correspondent to the parabolic subgroups of type $(4,2)$ (resp. type $(2,1)$) in the $\mathrm{GL}_6(F)$ case (resp. $\mathrm{GL}_3(D)$ case) and the trilinear GL_2 model which correspondent to the parabolic subgroups of type $(2,2,2)$ (resp. type $(1,1,1)$) in the $\mathrm{GL}_6(F)$ case (resp. $\mathrm{GL}_3(D)$ case). For those models, it is easy to see from the definition that the nonvanishing property of $\mathcal{L}_\sigma$ is invariant under the unramified twist.

For Type II models, it is not clear from the definition that the unramified twist will preserve the nonvanishing property. Instead, we are going to prove a much stronger statement. We claim that for all Type II models, $\mathcal{L}_\sigma$ is always nonzero. This will definitely implies our Lemma. Hence in order to prove the Lemma, we only need to prove the following theorem. The proof will be given in later sections. □

Theorem 5.11. *If $(L, H_{\bar{Q}})$ is of Type II, then $\mathcal{L}_\sigma \neq 0$ for all $\sigma \in \Pi_2(L)$.*

Theorem 5.11 will be proved in Section 6 for the archimedean case, and in Section 9 for the p-adic case.

Remark 5.12. *By the same argument as in this section, we can have a similar formula as* (5.14) *for $\mathcal{L}_\sigma$. Since σ is a discrete series, the connected component containing it (as defined in* (5.15)*) does not contains other element (i.e. $\mathcal{O} = \{\sigma\}$). By applying the same argument as* *above, we know that $m(\sigma) \neq 0 \Rightarrow \mathcal{L}_\sigma \neq 0$ (**The upshot is that since σ is a discrete series, we don't need to worry about the unramified twist issue**). Here $m(\sigma)$ is the multiplicity of the reduced model. Therefore in order to prove Theorem 5.11, we only need to show that for all Type II models, the multiplicity $m(\sigma)$ is always nonzero.*

Now back to the proof of Theorem 5.3. By applying Lemma 5.10, we know $\mathcal{L}_\sigma \neq 0$. Applying Proposition 5.6 and Proposition 5.9 again, we have $\mathcal{L}_\pi \neq 0$. This proves the other direction, and finishes the proof of Theorem 5.3.

5.6. **Some consequences of Theorem 5.3.** **If** $F = \mathbb{R}$, let π be a tempered representation of $G(F)$ with central character η. Since we are in the archimedean case, there exists a tempered representation π_0 of G_0 such that $\pi = I_{\bar{P}}^G(\pi_0)$. We have the following result.

Corollary 5.13. *$m(\pi) = m(\pi_0)$.*

Proof. Similar to Theorem 5.3, we can show that $m(\pi_0) \neq 0 \iff \mathcal{L}_{\pi_0} \neq 0$. Then by applying Proposition 5.7, we have

$$m(\pi) \neq 0 \iff \mathcal{L}_\pi \neq 0 \iff \mathcal{L}_{\pi_0} \neq 0 \iff m(\pi_0) \neq 0.$$

By the results in [JSZ11] and [P90], we have $m(\pi),\ m(\pi_0) \leq 1$. Hence the above equivalence implies that $m(\pi) = m(\pi_0)$. □

If F **is p-adic,** let π be a tempered representation of $\mathrm{GL}_6(F)$ with central character η. We can find a good parabolic subgroup $\bar{Q} = LU_Q$ and a discrete series σ of $L(F)$ such that $\pi = I_{\bar{Q}}^G(\sigma)$. By the construction of the local Jacquet-Langlands correspondence, we know that $\pi_D \neq 0$ if and only if $\bar{Q}$ is of Type I. In fact, the local Jacquet-Langlands correspondence established in [DKV84] gives a bijection between the discrete series. Then the map can be extended naively to all the tempered representations via the parabolic induction (note that all tempered representations of GL_n are the full induction of some discrete series of Levi subgroups). Therefore, in order to make $\pi_D \neq 0$, the Levi subgroup L should have an analogue in $\mathrm{GL}_3(D)$, which is equivalent to say that $\bar{Q}$ is of Type I.

Corollary 5.14. *If $\bar{Q}$ is of Type II, Theorem 1.2 holds.*

Proof. By the discussion above, we have $\pi_D = 0$. So we only need to show that $m(\pi) = 1$. Since $m(\pi) \leq 1$, it is enough to prove that $m(\pi) \neq 0$. By Theorem 5.11, we know $\mathcal{L}_\sigma \neq 0$. Together with Proposition 5.6, we have $\mathcal{L}_\pi \neq 0$. Combining with Theorem 5.3, we have $m(\pi) \neq 0$. This proves the Corollary. □

Now let π be a tempered representation of $G(F)$ with central character η (note that $G(F)$ can be both $\mathrm{GL}_6(F)$ and $\mathrm{GL}_3(D)$). We can find a good parabolic subgroup $\bar{Q} = LU_Q$ and a discrete series σ of $L(F)$ such that $\pi = I_{\bar{Q}}^G(\sigma)$. **We assume that $\bar{Q}$ is of Type I.**

Corollary 5.15. (1) $m(\pi) = m(\sigma)$.

(2) *Let $\mathcal{K} \subset \Pi_{temp}(G, \eta)$ be a compact subset. Then there exists an element $T \in \mathcal{C}(\Pi_{temp}(G, \eta))$ such that $\mathcal{L}_\pi(T_\pi) = m(\pi)$ for all $\pi \in \mathcal{K}$.*

Proof. (1) follows from the same proof as in Corollary 5.13. For (2), it is enough to show that for all $\pi' \in \Pi_{temp}(G, \eta)$, there exists $T \in \mathcal{C}(\Pi_{temp}(G, \eta))$ such that $\mathcal{L}_\pi(T_\pi) = m(\pi)$ for all π in some neighborhood of π' in $\Pi_{temp}(G, \eta)$. Since $m(\sigma)$ is invariant under the unramified twist for Type I models, combining with part (1) and Corollary 5.14, we know that the map $\pi \to m(\pi)$ is locally constant (In fact, we even know that the map is constant on each connected components of $\Pi_{temp}(G, \eta)$). If $m(\pi') = 0$, we just take $T = 0$, and there is nothing to prove.

If $m(\pi') \neq 0$, then we know $m(\pi) = 1$ for all π in the connected component containing π'. By Theorem 5.3, we can find $T' \in \mathrm{End}(\pi')^\infty$ such that $\mathcal{L}_{\pi'}(T') \neq 0$. Let $T^0 \in \mathcal{C}(\Pi_{temp}(G, \eta))$ be an element with $T^0_{\pi'} = T'$. By Lemma 5.4(1), the function $\pi \to \mathcal{L}_\pi(T^0_\pi)$ is a smooth function. The value at π' is just $\mathcal{L}_{\pi'}(T') \neq 0$. Hence we can find a smooth compactly supported function φ on $\Pi_{temp}(G, \eta)$ such that $\varphi(\pi)\mathcal{L}_\pi(T^0_\pi) = 1$ for all π belonging to a small neighborhood of π'. Then we just need to take $T = \varphi T^0$ and this proves the Corollary. □

6. The archimedean case

In this section, we prove our main theorems (Theorem 1.2 and Theorem 1.4) for the case when $F = \mathbb{R}$. The main ingredient of the proof is Corollary 5.13, which allows us to reduce the problem to the trilinear GL_2 model case. Then by applying the results of Prasad ([P90]) and Loke ([L01]), we can prove the two main theorems. The proof of Theorem 5.11 for the archimedean case will be given at the end of Section 6.2.

6.1. The trilinear GL_2 model.

In this subsection, we recall the results on the trilinear GL_2 model. Let $G_0(F) = GL_2(F) \times GL_2(F) \times GL_2(F)$, and let $H_0(F) = GL_2(F)$ diagonally embed into $G_0(F)$. For a given irreducible representation $\pi_0 = \pi_1 \otimes \pi_2 \otimes \pi_3$ of $G_0(F)$, assume that the central character of π_0 equals χ^2 on $Z_{H_0}(F)$ for some unitary character χ of $F^\times$. χ will induce a one-dimensional representation ω_0

of $H_0(F)$. Let

$$m(\pi_0)=dim(Hom_{H_0(F)}(\pi_0,\omega_0)). \tag{6.1}$$

Similarly, we have the quaternion algebra version: let $G_{0,D}(F)=GL_1(D)\times GL_1(D)\times GL_1(D)$, and let $H_{0,D}(F)=GL_1(D)$. We can still define the multiplicity $m(\pi_{0,D})$. The following theorem has been proved by Prasad in his thesis [P90] under the assumption that at least one π_i is a discrete series (i=1,2,3), and by Loke in [L01] for the case when π_0 is a principal series.

Theorem 6.1. *With the notation above, if π_0 is an irreducible generic representation of $G_0(F)$, let $\pi_{0,D}$ be the Jacquet-Langlands correspondence of π_0 to $G_{0,D}(F)$ if it exists; otherwise let $\pi_{0,D}=0$. Then we have*

(1) $m(\pi_0)+m(\pi_{0,D})=1$.
(2) $m(\pi_0)=1 \iff \epsilon(1/2,\pi_1\times\pi_2\times\pi_3\times\chi^{-1})=1$,
$m(\pi_0)=0 \iff \epsilon(1/2,\pi_1\times\pi_2\times\pi_3\times\chi^{-1})=-1$.

Remark 6.2. *Both Prasad's result and Loke's result are based on the assumption that the product of the central characters of π_i $(i=1,2,3)$ is trivial. In our case, we assume that the product of the central characters is χ^2. But we can always reduce our case to their cases by replacing π_1 by $\pi_1\otimes(\chi^{-1}\circ\det)$.*

6.2. **Proof of Theorem 1.2 and Theorem 1.4.** Let π be an irreducible tempered representation of $GL_6(F)$, with $F=\mathbb{R}$. There exists a tempered representation $\pi_0=\pi_1\otimes\pi_2\otimes\pi_3$ of $G_0(F)$ such that $\pi=Ind_{\bar{P}}^{G}(\pi_0)$. Let π_D be the Jacquet-Langlands correspondence of π to $GL_3(D)$. Similarly we can find a tempered representation $\pi_{0,D}$ of $G_{0,D}(F)$ such that $\pi=Ind_{\bar{P}_D}^{G_D}(\pi_{0,D})$. It is easy to see that $\pi_{0,D}$ is the Jacquet-Langlands correspondence of π_0 to $G_{0,D}(F)$. Note that π_D and $\pi_{0,D}$ may be zero. In fact, they are nonzero if and only if π_0 is a discrete series. By Corollary 5.13, $m(\pi)=m(\pi_0)$ and $m(\pi_D)=m(\pi_{0,D})$. Then by applying Theorem 6.1, we have $m(\pi)+m(\pi_D)=m(\pi_0)+m(\pi_{0,D})=1$. This proves Theorem 1.2.

For Theorem 1.4, by Theorem 6.1, it is enough to show that

$$\epsilon(1/2,(\wedge^3\pi)\otimes\chi^{-1})=\epsilon(1/2,\pi_1\times\pi_2\times\pi_3\times\chi^{-1}).$$

For $i=1,2,3$, let ϕ_i be the Langlands parameter of π_i. Then the Langlands parameter of π is $\phi_\pi=\phi_1\oplus\phi_2\oplus\phi_3$. This implies

$$\begin{aligned}&\wedge^3(\phi_\pi)=\wedge^3(\phi_1\oplus\phi_2\oplus\phi_3)\\ =\;&(\phi_1\otimes\phi_2\otimes\phi_3)\oplus(\det(\phi_2)\otimes\phi_1)\oplus(\det(\phi_3)\otimes\phi_1)\\ &\oplus(\det(\phi_1)\otimes\phi_2)\oplus(\det(\phi_3)\otimes\phi_2)\oplus(\det(\phi_1)\otimes\phi_3)\oplus(\det(\phi_2)\otimes\phi_3).\end{aligned}$$

By our assumption on the central character, we have $\det(\phi_\pi)=\det(\phi_1)\otimes \det(\phi_2)\otimes\det(\phi_3)=\chi^2$. Therefore $(\det(\phi_2)\otimes\phi_1\otimes\chi^{-1})^\vee=\det(\phi_1)^{-1}\otimes \det(\phi_2)^{-1}\otimes\chi\otimes\phi_1=\det(\phi_3)\otimes\phi_1\otimes\chi^{-1}$. This implies

$$\begin{aligned}&\epsilon(1/2,\det(\phi_2)\otimes\phi_1\otimes\chi^{-1})\epsilon(1/2,\det(\phi_3)\otimes\phi_1\otimes\chi^{-1})\\ =\ &\det(\det(\phi_2)\otimes\phi_1\otimes\chi^{-1})(-1)=\det(\phi_1)\otimes\det(\phi_2)^2\otimes\chi^{-2}(-1)=\det(\phi_1)(-1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\epsilon(1/2,\det(\phi_1)\otimes\phi_2\otimes\chi^{-1})\epsilon(1/2,\det(\phi_3)\otimes\phi_2\otimes\chi^{-1}) &= \det(\phi_2)(-1),\\ \epsilon(1/2,\det(\phi_1)\otimes\phi_3\otimes\chi^{-1})\epsilon(1/2,\det(\phi_2)\otimes\phi_3\otimes\chi^{-1}) &= \det(\phi_3)(-1).\end{aligned}$$

Combining the three equations above, we have

$$\begin{aligned}\epsilon(1/2,(\wedge^3\pi)\otimes\chi^{-1}) &= \det(\phi_1)\otimes\det(\phi_2)\otimes\det(\phi_3)(-1)\epsilon(1/2,\phi_1\otimes\phi_2\otimes\phi_3\otimes\chi^{-1})\\ &= \chi^2(-1)\epsilon(1/2,\phi_1\otimes\phi_2\otimes\phi_3\otimes\chi^{-1})=\epsilon(1/2,\pi_1\times\pi_2\times\pi_3\times\chi^{-1}).\end{aligned}$$

This proves Theorem 1.4.

Now the only thing left is to prove Theorem 5.11 for the archimedean case. By Remark 5.12, we only need to prove the multiplicity is nonzero for all Type II models. Since we are in the archimedean case, there are only three type II models: Type $(2,2,1,1)$, $(2,1,1,1,1)$ and $(1,1,1,1,1,1)$. Type $(1,1,1,1,1,1)$ case is trivial since both L and $H_{\bar{Q}}$ are abelian in this case. For Type $(2,1,1,1,1)$, by canceling the GL_1 part (which is abelian), we are considering the model $(\mathrm{GL}_2(F),T)$ where $T=\{\begin{pmatrix}a&0\\0&b\end{pmatrix}|a,b\in F^\times\}$ is the maximal torus. The multiplicity is always nonzero by the archimedean Rankin-Selberg theory of Jacquet and Shalika ([JS90]).

For Type $(2,2,1,1)$, by canceling the GL_1 part, we are considering the following model: $M(F)=\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$, and

$$M_0(F)=\{m(a,b,c)=\begin{pmatrix}a&0\\c&b\end{pmatrix}\times\begin{pmatrix}a&0\\c&b\end{pmatrix}|a,b\in F^\times,\ c\in F\}.$$

The character on $M_0(F)$ is given by $\omega(m(a,b,c))=\chi(ab)$. Let $B(F)$ be the lower Borel subgroup of $\mathrm{GL}_2(F)$. It is isomorphic to $M_0(F)$, hence we can also view ω as a character on $B(F)$. Let $\pi_3=I_B^G(\omega)$, it is a principal series of $\mathrm{GL}_2(F)$. For any irreducible tempered representation $\pi_1\otimes\pi_2$ of $M(F)$, by the Frobenius reciprocity, we have

$$Hom_{M_0(F)}(\pi_1\otimes\pi_2,\omega)=Hom_{\mathrm{GL}_2(F)}(\pi_1\otimes\pi_2,\pi_3).$$

Here $\mathrm{GL}_2(F)$ maps diagonally into $M(F)$. Therefore the Hom space is isomorphic to the Hom space of the trilinear GL_2 model for the representation $\pi_0=\pi_1\otimes\pi_2\otimes\pi_3$. Since π_3 is a principal series, $\pi_{0,D}=0$. By Theorem 6.1, $m(\pi_0)=1\neq 0$. Hence the Hom space is nonzero and this proves Theorem 5.11. **Now the proofs of our main theorems (Theorem 1.2 and Theorem 1.4) are complete for the archimedean case.**

7. The trace formula

For the rest part of this paper, we assume that F is p-adic. In this section, we will prove our trace formula. We will define the distribution on Section 7.1. In Section 7.2, we write down the geometric expansion which has already been proved in [Wan15]. In Section 7.3, we prove the spectral expansion.

7.1. The distribution.

Let χ be an unitary character of $F^\times$, and let $\eta = \chi^2$. For $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$, define the function $I(f, \cdot)$ on $H(F)\backslash G(F)$ to be

$$I(f,x) = \int_{Z_H(F)\backslash H(F)} f(x^{-1}hx)\omega \otimes \xi(h)dh.$$

By Lemma 4.9(2), the above integral is absolutely convergent. The following Proposition together with Proposition 4.10(3) tell us that the integral

$$I(f) := \int_{H(F)\backslash G(F)} I(f,x)dx$$

is also absolutely convergent for all $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$, and it defines a continuous linear form

$$\mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1}) \to \mathbb{C} : \ f \to I(f).$$

Proposition 7.1. (1) *There exist $d > 0$ and a continuous semi-norm ν on $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ such that*

$$|I(f,x)| \leq \nu(f)\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^d$$

for all $f \in \mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ and $x \in H(F)\backslash G(F)$.

(2) *For all $d > 0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$ such that*

$$|I(f,x)| \leq \nu_d(f)\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{-d}$$

for all $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$ and $x \in H(F)\backslash G(F)$.

Proof. The proof is similar to the GGP case (Proposition 7.1.1 of [B15]), we will skip it here. We refer the readers to Appendix B of [Wan17] for details of the proof. □

7.2. **The geometric expansion.** Let's recall the results in the previous paper [Wan15] for the geometric expansion of $I(f)$. Let $\mathcal{T}$ be a subset of subtori of H_0 defined as follows:

- If $H_0(F)=\mathrm{GL}_2(F)$, then $\mathcal{T}$ contain the trivial torus $\{1\}$ and all the non-split torus T_v for $v\in F^{\times}/(F^{\times})^2, v\neq 1$. Here $T_v(F)=\{\begin{pmatrix} a & bv \\ b & a\end{pmatrix}\in H_0(F)\mid a,b\in F,\ (a,b)\neq(0,0)\}$.
- If $H_0(F)=\mathrm{GL}_1(D)$, then $\mathcal{T}$ contain the subtorus T_v for $v\in F^{\times}/(F^{\times})^2$ with $v\neq 1$. Here $T_v(F)\subset \mathrm{GL}_1(D)$ is isomorphic to the quadratic extension $F(\sqrt{v})$ of F.

Let θ be a quasi-character on $G(F)$ with central character η^{-1}, and let $T\in\mathcal{T}$. If $T=\{1\}$, we are in the split case. In this case, we have the unique regular nilpotent orbit $\mathcal{O}_{reg}$ in $\mathfrak{g}(F)$ and we define $c_\theta(1)=c_{\theta,\mathcal{O}_{reg}}(1)$. If $T=T_v$ for some $v\in F^{\times}/(F^{\times})^2$ with $v\neq 1$, we take $t\in T_v(F)$ to be a regular element. It is easy to see that in both cases $G_t(F)$ is F-isomorphic to $\mathrm{GL}_3(F_v)$ with $F_v=F(\sqrt{v})$. Let $\mathcal{O}_v$ be the unique regular nilpotent orbit in $\mathfrak{gl}_3(F_v)$, and let $c_\theta(t)=c_{\theta,\mathcal{O}_v}(t)$.

In Section 3.3, for any $f\in C_{c,scusp}^{\infty}(Z_G(F)\backslash G(F),\eta^{-1})$, we have defined a quasi-character θ_f on $G(F)$. Let $c_f(t)=c_{\theta_f}(t)$. Also we define a function Δ on $H_{ss}(F)$ to be

$$\Delta(x)=\mid \det((1-ad(x)^{-1})_{|U(F)/U_x(F)})\mid_F .$$

Definition 7.2. *For $f\in C_{c,scusp}^{\infty}(Z_G(F)\backslash G(F),\eta^{-1})$, define the geometric side of the trace formula to be*

$$I_{geom}(f)=\sum_{T\in\mathcal{T}}|W(H_0,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)} c_f(t)D^H(t)\Delta(t)\chi(\det(t))dt.$$

The integral above is absolutely convergent by Proposition 5.2 of [Wan15].

The following theorem has been proved in Theorem 5.4 of [Wan15]. It gives the geometric expansion of our trace formula.

Theorem 7.3. *For every $f\in C_{c,scusp}^{\infty}(Z_G(F)\backslash G(F),\eta^{-1})$, we have $I(f)=I_{geom}(f)$.*

7.3. **The spectral expansion.**

Theorem 7.4. *For all $f\in\mathcal{C}_{scusp}(Z_G(F)\backslash G(F),\eta^{-1})$, define*

$$I_{spec}(f)=\int_{\Pi_{temp}(G,\eta)}\theta_f(\pi)m(\bar{\pi})d\pi.$$

Here $\theta_f(\pi)$ is defined in (3.4). *Then we have $I(f)=I_{spec}(f)$.*

The purpose of the rest part of this section is to prove this Theorem. We follow the method developed by Beuzart-Plessis in [B15] for the GGP case. We fix $f \in \mathcal{C}_{scusp}(Z_G(F)\backslash G(F), \eta^{-1})$. For all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$, define

$$\begin{aligned}
K^A_{f,f'}(g_1,g_2) &= \int_{Z_G(F)\backslash G(F)} f(g_1^{-1}gg_2)f'(g)dg,\ g_1,g_2 \in G(F),\\
K^1_{f,f'}(g,x) &= \int_{Z_H(F)\backslash H(F)} K^A_{f,f'}(g,hx)\omega\otimes\xi(h)dh,\ g,x \in G(F),\\
K^2_{f,f'}(x,y) &= \int_{Z_H(F)\backslash H(F)} K^1_{f,f'}(h^{-1}x,y)\omega\otimes\xi(h)dh,\ x,y \in G(F),\\
J_{aux}(f,f') &= \int_{H(F)\backslash G(F)} K^2_{f,f'}(x,x)dx.
\end{aligned}$$

Proposition 7.5. (1) *The integral defining $K^A_{f,f'}(g_1,g_2)$ is absolutely convergent. For all $g_1 \in G(F)$, the map*

$$g_2 \in G(F) \to K^A_{f,f'}(g_1,g_2)$$

belongs to $\mathcal{C}(Z_G(F)\backslash G(F), \eta^{-1})$. For all $d > 0$, there exists $d' > 0$ such that for all continuous semi-norm ν on $\mathcal{C}^w_{d'}(Z_G(F)\backslash G(F), \eta^{-1})$, there exists a continuous semi-norm μ on $\mathcal{C}(Z_G(F)\backslash G(F), \eta)$ such that

$$\nu(K^A_{f,f'}(g,\cdot)) \leq \mu(f')\Xi^G(g)\sigma_0(g)^{-d}$$

for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $g \in G(F)$.

(2) *The integral defining $K^1_{f,f'}(g,x)$ is absolutely convergent. For all $d > 0$, there exist $d' > 0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}(Z_G(F)\backslash G(F), \eta)$ such that*

$$|K^1_{f,f'}(g,x)| \leq \nu_{d,d'}(f')\Xi^G(g)\sigma_0(g)^{-d}\Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{d'}$$

for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $g,x \in G(F)$.

(3) *The integral defining $K^2_{f,f'}(x,y)$ is absolutely convergent. We have*

$$K^2_{f,f'}(x,y) = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(y^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi \tag{7.1}$$

for all $f' \in \mathcal{C}(Z_G(F)\backslash G(F), \eta)$ and $x,y \in G(F)$.

(4) *The integral defining $J_{aux}(f,f')$ is absolutely convergent. For all $d > 0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F), \eta)$ such that $|K^2_{f,f'}(x,x)| \leq \nu_d(f')\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{-d}$ for all $f' \in$*

$\mathcal{C}(Z_G(F)\backslash G(F),\eta)$ and $x\in H(F)\backslash G(F)$. Moreover, the linear map

$$f'\in \mathcal{C}(Z_G(F)\backslash G(F),\eta)\rightarrow J_{aux}(f,f') \tag{7.2}$$

is continuous.

Proof. (1) follows from Theorem 5.5.1(1) of [B15]. (2) follows from part (1) together with Lemma 4.9(2) and Lemma 5.5(2). For (3), the absolutely convergence follows from part (2) and Lemma 4.9(2). The equation (7.1) follows from Lemma 5.4(3).

For (4), by Lemma 5.4(1), the section

$$T(f'):\pi\in \Pi_{temp}(G,\eta)\mapsto \overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}\pi(f)\in \mathrm{End}(\pi)^{\infty}$$

is smooth. It is also compactly supported since we are in the p-adic case. Then by the matrical Paley-Wiener Theorem, there exists a unique element $\varphi_{f'}\in \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1})$ such that $\pi(\varphi_{f'})=\overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}\pi(f)$ for all $\pi\in \Pi_{temp}(G,\eta)$. Since f is strongly cuspidal, by Lemma 5.3.1(1) of [B15], $\varphi_{f'}$ is also strongly cuspidal. Then by (7.1), we have

$$\begin{aligned} K_{f,f'}^{2}(x,x) &= \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_{\pi}(\pi(x)\pi(f)\pi(x^{-1}))\overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}\mu(\pi)d\pi \\ &= \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_{\pi}(\pi(x)\pi(\varphi_{f'})\pi(x^{-1}))\mu(\pi)d\pi \\ &= \int_{Z_H(F)\backslash H(F)} \varphi_{f'}(x^{-1}hx)\omega\otimes\xi(h)dh = I(\varphi_{f'},x). \end{aligned}$$

Here the third equation follows from Lemma 5.4(2). Then by Proposition 7.1, for all $d>0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(Z_G(F)\backslash G(F),\eta)$ such that $|K_{f,f'}^{2}(x,x)|\leq \nu_d(\varphi_{f'})\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{-d}$ for all $f'\in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$ and $x\in H(F)\backslash G(F)$. Combining with Proposition 4.10(3), we know the integral defining $J_{aux}(f,f')$ is absolutely convergent. Finally, in order to prove the rest part of (4), it is enough to show that the map

$$\mathcal{C}(Z_G(F)\backslash G(F),\eta)\rightarrow \mathcal{C}(Z_G(F)\backslash G(F),\eta^{-1}):f'\mapsto \varphi_{f'}$$

is continuous. By the matrical Paley-Wiener Theorem, it is enough to show that the map

$$f'\in \mathcal{C}(Z_G(F)\backslash G(F),\eta)\mapsto (\pi\in \Pi_{temp}(G,\eta)\rightarrow \pi(\varphi_{f'})=\overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}\pi(f))\in \mathcal{C}(\Pi_{temp}(G,\eta))$$

is continuous. This just follows from Lemma 5.4(1). □

Proposition 7.6. *For all* $f'\in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$*, we have*

$$J_{aux}(f,f')=\int_{\Pi_{temp}(G,\chi)}\theta_f(\pi)\overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}d\pi. \tag{7.3}$$

Proof. Let's first prove the equation formally, i.e. without considering the convergence issue. Consider the triple integral defining $J_{aux}(f,f')$:
(7.4)

$$\int_{H(F)\backslash G(F)}\int_{Z_H(F)\backslash H(F)}\int_{Z_H(F)\backslash H(F)} K_{f,f'}^A(h_1^{-1}x,h_2x)\omega\otimes\xi(h_1h_2)dh_1dh_2dx.$$

By changing the variables $x\to h_1^{-1}x,\ h_2\to h_2h_1$, and then combining dh_1 and dx, we have

$$\begin{aligned} J_{aux}(f,f') &= \int_{H(F)\backslash G(F)}\int_{Z_H(F)\backslash H(F)}\int_{Z_H(F)\backslash H(F)} K_{f,f'}^A(h_1^{-1}x,h_2x)\omega\otimes\xi(h_1h_2)dh_1dh_2dx \\ &= \int_{Z_G(F)\backslash G(F)}\int_{Z_H(F)\backslash H(F)} K_{f,f'}^A(g,hg)\omega\otimes\xi(h)dhdg. \end{aligned}$$

By switching of order of integration, we have

$$J_{aux}(f,f')=\int_{Z_H(F)\backslash H(F)}\int_{Z_G(F)\backslash G(F)} K_{f,f'}^A(g,hg)dg\omega\otimes\xi(h)dh. \tag{7.5}$$

Combining with Theorem 3.2, we have

$$\begin{aligned} J_{aux}(f,f') &= \int_{Z_H(F)\backslash H(F)}\int_{\Pi_{temp}(G,\eta)}\theta_f(\pi)\mathrm{tr}(\bar{\pi}(R(h^{-1})f'))d\pi\omega\otimes\xi(h)dh \\ &= \int_{\Pi_{temp}(G,\eta)}\theta_f(\pi)\int_{Z_H(F)\backslash H(F)}\mathrm{tr}(\bar{\pi}(R(h^{-1})f'))\omega\otimes\xi(h)dhd\pi \qquad (7.6) \\ &= \int_{\Pi_{temp}(G,\eta)}\theta_f(\pi)\overline{\mathcal{L}_{\pi}(\pi(\overline{f'}))}d\pi. \end{aligned}$$

This proves the proposition. However, in general, the triple integral (7.4) is not absolutely convergent. As a result, we need to introduce some truncated functions before changing the variables and switching the order of integration. The argument is similar to the GGP case (Proposition 9.2.2 of [B15]), we will skip it here. We refer the readers to Proposition 8.2.3 of [Wan17] for details of the proof. □

Now we are ready to prove Theorem 7.4. Recall that $I(f)=\int_{H(F)\backslash G(F)}I(f,x)dx$ where $I(f,x)=\int_{Z_H(F)\backslash H(F)}f(x^{-1}hx)\omega\otimes\xi(h)dh$. By applying Lemma 5.4, we have

$$I(f,x)=\int_{\Pi_{temp}(G,\eta)}\mathcal{L}_{\pi}(\pi(x)\pi(f)\pi(x)^{-1})\mu(\pi)d\pi. \tag{7.7}$$

By Corollary 5.15, there exists a function $f' \in \mathcal{C}(Z_G(F)\backslash G(F),\eta)$ such that

$$\mathcal{L}_\pi(\pi(\overline{f'})) = m(\pi)$$

for all $\pi \in \Pi_{temp}(G,\eta)$ with $\pi(f) \neq 0$. By Theorem 5.3, for all $\pi \in \Pi_{temp}(G,\eta)$, $\mathcal{L}_\pi \neq 0$ if and only if $m(\pi)=1$. Then (7.7) becomes

$$I(f,x) = \int_{\Pi_{temp}(G,\eta)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x)^{-1})\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi.$$

Combining with Proposition 7.5(3), we have $I(f,x) = K^2_{f,f'}(x,x)$. Therefore $I(f) = J_{aux}(f,f')$. By the previous Proposition, together with the fact that $\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} = m(\pi) = m(\bar{\pi})$, we have

$$I(f) = J_{aux}(f,f') = \int_{\Pi_{temp}(G,\eta)} \theta_f(\pi) m(\bar{\pi}) d\pi = I_{spec}(f).$$

This finishes the proof of Theorem 7.4.

8. Proof of Theorem 1.2 and Theorem 1.4: the p-adic case

In this section, we prove our main theorems (Theorem 1.2 and Theorem 1.4) for the p-adic case. In Section 8.1, we introduce the multiplicity formulas for both the Ginzburg-Rallis model and all the reduced models. We show that Theorem 1.2 follows from the multiplicity formula. Also we prove that the multiplicity formula is compatible with the parabolic induction. In Section 8.2, by applying the trace formula in Section 7, we prove the multiplicity formula, which implies Theorem 1.2. In Section 8.3, we prove Theorem 1.4.

8.1. **The multiplicity formula.** Let π be an irreducible tempered representation of $G(F)$ with central character $\eta = \chi^2$. Similar to Section 7.2, we define the geometric multiplicity to be

$$m_{geom}(\pi) = \sum_{T\in\mathcal{T}} |W(H_0,T)|^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)} c_\pi(t) D^{H_0}(t)\Delta(t)\chi(\det(t))^{-1}dt.$$

Here $c_\pi(t) = c_{\theta_\pi}(t)$ is the germ associated to the distribution character θ_π. The multiplicity formula is just

$$m(\pi) = m_{geom}(\pi). \tag{8.1}$$

Let $\pi = I_{\bar{Q}}^G(\tau)$ for some good parabolic subgroup $\bar{Q} = LU_Q$ and some discrete series τ of $L(F)$. We also need the geometric multiplicity for the reduced model $(L, H_{\bar{Q}})$ where $H_{\bar{Q}} = H \cap \bar{Q}$. If $\bar{Q}$ is of Type II, we are in the $\mathrm{GL}_6(F)$ case, define

$$m_{geom}(\tau) = c_{\theta_{tau},\mathcal{O}_{reg}}(1).$$

By the work of Rodier [Rod81], we know $m_{geom}(\tau)$ is always equal to 1 in this case. If $\bar{Q}$ is of Type I, the definition of $m_{geom}(\tau)$ is very similar to $m_{geom}(\pi)$: it is still the summation of the integrals of the germs on $Z_H(F)\backslash T(F)$ for $T \in \mathcal{T}$. For details, see Appendix A. The following lemma tells us the relations between $m_{geom}(\pi)$ and $m_{geom}(\tau)$.

Lemma 8.1. *With the notation above, we have*

$$m_{geom}(\pi) = m_{geom}(\tau).$$

Proof. This is a direct consequence of Lemma 2.3 of [W12]. In fact, if $\bar{Q}$ is of Type I, by applying the lemma, we know the germs associated to π and τ are the same:

$$\Delta(t)^{1/2} c_{\pi}(t) = \Delta^L(t)^{1/2} c_{\tau}(t),\ \forall t \in T_{reg}(F), T \in \mathcal{T}.$$

Hence $m_{geom}(\pi) = m_{geom}(\tau)$. Here Δ^L is the analogue of the function Δ for the reduced model $(L, H_{\bar{Q}})$.

If $\bar{Q}$ is of Type II, by applying the lemma, we know the germ $c_{\pi}(t)$ is zero for all $t \in T_{reg}(F), T \in \mathcal{T}$ with $T \neq \{1\}$. Therefore we have $m_{geom}(\pi) = c_{\pi}(1) = 1 = c_{\tau}(1) = m_{geom}(\tau)$. This proves the lemma. □

Proposition 8.2. *Theorem 1.2 follows from the multiplicity formula* (8.1).

Proof. The proof has already appeared in Section 6.2 of the previous paper [Wan15], we will skip it here. □

By the proposition above, we only need to prove (8.1). We will only prove it for the GL_6 case since the quaternion case follows from the same argument.

8.2. **The proof of** (8.1)**.** In this section, the group $G(F)$ will be $\mathrm{GL}_6(F)$. Recall that π is of the form $\pi = I_{\bar{Q}}^{G}(\tau)$ for some good parabolic subgroup $\bar{Q} = LU_Q$ and some discrete series τ of $L(F)$.

Lemma 8.3. *If $\bar{Q}$ is of Type II, the multiplicity formula* (8.1) *holds.*

Proof. In Lemma 8.1, we have proved that $m_{geom}(\pi) = 1$. By Corollary 5.14, we also know $m(\pi) = 1$. Hence the multiplicity formula (8.1) holds. □

If $\bar{Q}$ is of Type I, there are only three possibilities: type (6), type (4, 2) and type (2, 2, 2). Since both $m(\pi)$ and $m_{geom}(\pi)$ are compatible with the parabolic induction, we only need to prove $m(\tau) = m_{geom}(\tau)$. **We first assume that** (8.1) **holds for the** (2, 2, 2) **and** (4, 2) **case, we are going to prove** (8.1) **for the** (6) **case, i.e. the case when** π **is a discrete series.**

Proposition 8.4. *Under the hypothesis above, if π is a discrete series, the multiplicity formula* (8.1) *holds.*

Proof. By Theorem 7.3 and Theorem 7.4, for every strongly cuspidal function $f \in C_c^{\infty}(Z_G(F)\backslash G(F), \eta)$, we have

$$I_{geom}(f) = I(f) = \int_{\Pi_{temp}(G,\eta^{-1})} \theta_f(\Pi) m(\bar{\Pi}) d\Pi. \tag{8.2}$$

By Proposition 3.1, we have

$$I_{geom}(f) = \int_{\Pi_{temp}(G,\eta^{-1})} \theta_f(\Pi) m_{geom}(\bar{\Pi}) d\Pi. \tag{8.3}$$

Let $\Pi_2(G,\eta^{-1}) \subset \Pi_{temp}(G,\eta^{-1})$ be the subset consisting of discrete series, and let $\Pi'_{temp}(G,\eta^{-1}) = \Pi_{temp}(G,\eta^{-1}) - \Pi_2(G,\eta^{-1})$. Then by (8.2) and (8.3), we have

$$\begin{aligned} &\int_{\Pi'_{temp}(G,\eta^{-1})} \theta_f(\Pi) m(\bar{\Pi}) d\Pi + \int_{\Pi_2(G,\eta^{-1})} \theta_f(\Pi) m(\bar{\Pi}) d\Pi \\ = &\int_{\Pi'_{temp}(G,\eta^{-1})} \theta_f(\Pi) m_{geom}(\bar{\Pi}) d\Pi + \int_{\Pi_2(G,\eta^{-1})} \theta_f(\Pi) m_{geom}(\bar{\Pi}) d\Pi. \end{aligned} \tag{8.4}$$

For $\pi' \in \Pi'_{temp}(G,\eta^{-1})$, by our hypothesis and Lemma 8.3, we have $m(\pi') = m_{geom}(\pi')$. Therefore (8.4) becomes

$$\int_{\Pi_2(G,\eta^{-1})} \theta_f(\Pi) m(\bar{\Pi}) d\Pi = \int_{\Pi_2(G,\eta^{-1})} \theta_f(\Pi) m_{geom}(\bar{\Pi}) d\Pi. \tag{8.5}$$

Now let $f \in C_c^{\infty}(Z_G(F)\backslash G(F), \eta)$ be a pseudo coefficient of $\bar{\pi}$. The existence of such f was proved in Lemma 3.3, the lemma also shows that f is strongly cuspidal. For such f and for any $\Pi \in \Pi_2(G,\eta)$, we have $\theta_f(\Pi) = \mathrm{tr}(\Pi(f))$. Hence $\theta_f(\Pi) \neq 0$ if and only if $\Pi = \bar{\pi}$. Therefore (8.5) becomes

$$\theta_f(\bar{\pi}) m(\pi) = \theta_f(\bar{\pi}) m_{geom}(\pi).$$

This implies $m_{geom}(\pi) = m(\pi)$, and finishes the proof of the Proposition. □

By the proposition above, we only need to prove (8.1) for the $(2,2,2)$ and $(4,2)$ cases. For the $(4,2)$ case, we only need to prove

$$m(\tau) = m_{geom}(\tau) \tag{8.6}$$

where τ is a discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$. By the same argument as in the previous proposition, we are reducing to prove (8.6) for all tempered representations τ which are not discrete series. Then by applying Lemma 8.3, we are reducing to the $(2,2,2)$ case.

For the $(2,2,2)$ case, we need to prove

$$m(\tau) = m_{geom}(\tau) \tag{8.7}$$

where τ is a discrete series of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. By Lemma 8.3, we know (8.7) holds for all tempered representations τ which are not discrete series (note that there is no Type I reduced model for the trilinear GL_2 model). Then (8.7) follows from the same argument as in the previous proposition. This finishes the proof of the multiplicity formula (8.1), and hence the proof of Theorem 1.2.

8.3. **The proof of Theorem 1.4: the p-adic case.** Let π be an irreducible tempered representation of $\mathrm{GL}_6(F)$ with central character χ^2. Let $\pi = I_{\bar{Q}}^G(\tau)$ for some good parabolic subgroup $\bar{Q} = LU_Q$ and some discrete series τ of $L(F)$. By our assumptions in Theorem 1.4, $\bar{Q}$ can not be of type (6) or type $(4,2)$. Then there are two possibilities: $\bar{Q}$ is of type $(2,2,2)$ or $\bar{Q}$ is of Type II.

If $\bar{Q}$ is of type $(2,2,2)$. By a similar argument as in the archimedean case, Theorem 1.4 will follow from Prasad's results for the trilinear GL_2 model ([P90]).

If $\bar{Q}$ is of Type II, by Corollary 5.14, $m(\pi) = 1$. Hence it is enough to prove the following proposition.

Proposition 8.5. *If $\bar{Q}$ is of Type II, we have $\epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = 1$.*

Proof. Since $\bar{Q}$ is of Type II, it is contained in some Type II maximal parabolic subgroups. There are only two Type II maximal parabolic subgroups: type $(5,1)$ and type $(3,3)$.

If $\bar{Q}$ is contained in the parabolic subgroup $Q_{5,1}$ of type $(5,1)$, then there exists a tempered representation $\sigma = \sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_5(F) \times \mathrm{GL}_1(F)$ such that $\pi = I_{Q_{5,1}}^G(\sigma)$. Let ϕ_i be the Langlands parameter of σ_i for $i = 1, 2$. Then $\phi = \phi_1 \oplus \phi_2$ is the Langlands parameter for π. This implies

$$\wedge^3(\phi) \;=\; \wedge^3(\phi_1 \oplus \phi_2) = \wedge^3(\phi_1) \oplus (\wedge^2(\phi_1) \otimes \phi_2).$$

Since the central character of π is χ^2, $\det(\phi) = \det(\phi_1) \otimes \det(\phi_2) = \chi^2$. Therefore $(\wedge^3(\phi_1) \otimes \chi^{-1})^\vee = \wedge^2(\phi_1) \otimes \det(\phi_1)^{-1} \otimes \chi = \wedge^2(\phi_1) \otimes \det(\phi_2) \otimes \chi^{-1} = \wedge^2(\phi_1) \otimes \phi_2 \otimes \chi^{-1}$. This implies

$$\epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = \det(\wedge^3(\phi_1) \otimes \chi^{-1})(-1) = \det(\phi_1)^6(-1)\chi^{-10}(-1) = 1.$$

If $\bar{Q}$ is contained in the parabolic subgroup $Q_{3,3}$ of type $(3,3)$, then there exists a tempered representation $\sigma = \sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$ such that $\pi = I_{Q_{3,3}}^G(\sigma)$. Let ϕ_i be the Langlands parameter of σ_i for

$i=1,2$. Then $\phi=\phi_1\oplus\phi_2$ is the Langlands parameter for π. This implies

$$\begin{aligned}\wedge^3(\phi) &= \wedge^3(\phi_1\oplus\phi_2)\\ &= (\wedge^2(\phi_1)\otimes\phi_2)\oplus(\phi_1\otimes\wedge^2(\phi_2))\oplus\det(\phi_1)\oplus\det(\phi_2).\end{aligned}$$

Since the central character of π is χ^2, $\det(\phi)=\det(\phi_1)\otimes\det(\phi_2)=\chi^2$. Therefore $(\wedge^2(\phi_1)\otimes\phi_2\otimes\chi^{-1})^\vee=(\phi_1\otimes\det(\phi_1)^{-1})\otimes(\wedge^2(\phi_2)\otimes\det(\phi_2)^{-1})\otimes\chi=\phi_1\otimes\wedge^2(\phi_2)\otimes\chi^{-1}$ and $(\det(\phi_1)\otimes\chi^{-1})^\vee=\det(\phi_2)\otimes\chi^{-1}$. This implies

$$\begin{aligned}\epsilon(1/2,(\wedge^3\pi)\otimes\chi^{-1}) &= \det(\wedge^2(\phi_1)\otimes\phi_2\otimes\chi^{-1})(-1)\times(\det(\phi_1)\otimes\chi^{-1})(-1)\\ &= \det(\wedge^2(\phi_1))^3(-1)\times\det(\phi_2)^3(-1)\times\chi^{-10}(-1)\times\det(\phi_1)(-1)\\ &= (\det(\phi_1)^2(-1))^3\times(\det(\phi_2)(-1))^3\times\det(\phi_1)(-1)\\ &= \det(\phi_1)(-1)\times\det(\phi_2)(-1)=\chi^2(-1)=1.\end{aligned}$$

This finishes the proof of the proposition and hence the proof of Theorem 1.4. □

Now the proofs of Theorem 1.2 and Theorem 1.4 are complete. The only thing left is to prove Theorem 5.11 for the p-adic case. This will be done in the next section.

9. The proof of Theorem 5.11

In this section, we will prove Theorem 5.11 for the p-adic case, the archimedean case has already been proved in Section 6. By Remark 5.12, we only need to prove that the multiplicity is nonzero for all Type II reduced models $(L,H_{\bar{Q}})$. Our method is similar to the Ginzburg-Rallis model case we considered in previous sections. In other word, we need a local trace formula for such models, which gives us a multiplicity formula for such models in terms of the germs associated to the distribution characters of the representations. **The key feature for Type II models is that all semisimple elements in $H_{\bar{Q}}$ is split. As a result, in the geometric side of our trace formula, we only have the germ at the identity element. Hence the multiplicity formula is always 1 by the work of Rodier** [Rod81]**.**

Let $(L,H_{\bar{Q}})$ be a Type II reduced model. For simplicity, we use $\omega\otimes\xi$ to denote the character $\omega\otimes\xi|_{H_{\bar{Q}}}$ on $H_{\bar{Q}}(F)$. Given an irreducible tempered representation τ of $L(F)$ whose central character α equals η on $Z_G(F)$, we want to show that

$$m(\tau):=dim(Hom_{H_{\bar{Q}}(F)}(\tau,(\omega\otimes\xi)\otimes\delta_{H_{\bar{Q}}}^{1/2}))\neq 0. \tag{9.1}$$

When $\bar{Q}$ is a Borel subgroup, (9.1) is trivial since both L and $H_{\bar{Q}}$ are abelian in this case. From now on, we assume that $\bar{Q}$ is not a Borel subgroup. Moreover, by induction, we may assume that (9.1) holds for all good parabolic subgroups $\bar{Q}'$ with $\bar{Q}' \subsetneq \bar{Q}$ (i.e. (9.1) holds for all reduced models of the model $(L, H_{\bar{Q}})$). Let α be a character of $Z_L(F)$ such that $\eta = \alpha|_{Z_G(F)}$. As in the Ginzburg-Rallis model case, for $f \in C_{c,scusp}^{\infty}(Z_L(F)\backslash L(F), \alpha^{-1})$, we can define the distribution $I_Q(f)$. The next proposition gives the trace formula for such model.

Proposition 9.1. *For all $f \in C_{c,scusp}^{\infty}(Z_L(F)\backslash L(F), \alpha^{-1})$, we have*

$$I_{Q,geom}(f) = I_Q(f) = I_{Q,spec}(f)$$

where $I_{Q,geom}(f) = c_{\theta_f,\mathcal{O}_{reg}}(1)$, and

$$I_{Q,spec}(f) = \int_{\Pi_{temp}(L,\alpha)} \theta_f(\tau) m(\bar{\tau}) d\tau.$$

Proof. The geometric expansion $I_Q(f) = I_{Q,geom}(f)$ is proved in Appendix B of the previous paper [Wan15]. The proof for the spectral expansion is similar to the Ginzburg-Rallis model case we proved in Theorem 7.4, we will skip it here. The only thing we want to point out is that in the proof of Theorem 7.4, we need to use Theorem 5.3 whose proof relies on Theorem 5.11. Now by our inductional hypothesis, we know that Theorem 5.11 holds for all the reduced models of $(L, H_{\bar{Q}})$, this would imply Theorem 5.3 for the model $(L, H_{\bar{Q}})$. □

By applying the proposition above, together with the same arguments as in Section 8.2, we can prove a multiplicity formula:

$$m(\tau) = m_{geom}(\tau) := c_{\theta_\tau,\mathcal{O}_{reg}}(1)$$

for all tempered representations τ of $L(F)$. Then by the work of Rodier, we have $m(\tau) = 1 \neq 0$. This finishes the proof of Theorem 5.11.

Appendix A. The Reduced Models

In this appendix, we will state some similar results for the reduced models of the Ginzburg-Rallis model, which appear naturally under the parabolic induction. To be specific, we have analogue results of Theorem 1.2 and Theorem 1.4 for those models. Since the proof is the same as the Ginzburg-Rallis model case, we will skip it here. We still use $(L, H_{\bar{Q}})$ to denote the reduced model, and use $m(\tau)$ to denote the multiplicity.

A.1. **Type II models.** If $(L, H_{\bar{Q}})$ is of Type II, the following result has already been proved in Section 9.

Theorem A.1. *With the assumptions above, we have*

$$m(\tau) = c_{\theta_\tau, \mathcal{O}_{reg}}(1) = 1$$

for all tempered representation τ of $L(F)$.

A.2. **The trilinear GL_2 model.** Let Q be the parabolic subgroup of $\mathrm{GL}_6(F)$ (resp. $\mathrm{GL}_3(D)$) of $(2,2,2)$ type (resp. $(1,1,1)$ type) which contains the lower minimal parabolic subgroup. Then it is easy to see that the model (L, H_Q) is just the trilinear GL_2 model. To make our notation simple, in this section we will temporarily let $G(F) = GL_2(F) \times GL_2(F) \times GL_2(F)$, and let $H(F) = GL_2(F)$ diagonally embeded into G. For a given irreducible representation π of $G(F)$, assume that the central character ω_π equals χ^2 on $Z_H(F)$ for some character χ of $F^\times$. χ will induce a one-dimensional representation ω of $H(F)$. Let $m(\pi) = \dim(\mathrm{Hom}_{H(F)}(\pi, \omega))$.

Similarly, we have the quaternion algebra version with the pair $G_D(F) = GL_1(D) \times GL_1(D) \times GL_1(D)$ and $H_D(F) = GL_1(D)$. We can still define the multiplicity $m(\pi_D)$. The following theorem has been proved by Prasad ([P90]) and Loke ([L01]) for general generic representations using different method. This can also be deduced essentially from Waldspurger's result for the model $(SO(4) \times SO(3), SO(3))$. By using our method in this paper, we can prove the tempered case.

Theorem A.2. *If π is a tempered representation of $G(F)$, and let π_D be the Jacquet-Langlands correspondence of π to $G_D(F)$. Then*

$$m(\pi) + m(\pi_D) = 1.$$

We can also prove a multiplicity formula for this model in the p-adic case. Let (G, H) be either (G, H) or (G_D, H_D) defined as above, and let F be a p-adic field. Let $\mathcal{T}$ be the subset of subtori of H_0 defined in Section 7.2. Let θ be a quasi-character on $Z_G(F)\backslash G(F)$ with central character $\eta = \chi^2$, and let $T \in \mathcal{T}$. If $T = \{1\}$, then we are in the split case. There is a unique regular nilpotent orbit in $\mathfrak{g}(F)$, and we define $c_\theta(1) = c_{\theta, \mathcal{O}_{reg}}(1)$. If $T = T_v$ for some $v \in F^\times/(F^\times)^2, v \neq 1$. Let $t \in T_v(F)$ be a regular element. Then G_t is abelian, and the germ of the quasi-character at t is just itself. So we define $c_\theta(t) = \theta(t)$. Then for all tempered representation π of $G(F)$, we have the multiplicity formula

$$m(\pi) = m_{geom}(\pi) := \sum_{T \in \mathcal{T}} \mid W(H,T) \mid^{-1} \nu(T) \int_{Z_G(F)\backslash T(F)} c_\pi(t) D^H(t) \chi(\det(t))^{-1} dt$$

where $c_\pi(t) = c_{\theta_\pi}(t)$.

A.3. **The middle model.** Choose Q be the parabolic subgroup of $\mathrm{GL}_6(F)$ (resp. $\mathrm{GL}_3(D)$) of $(4,2)$ type (resp. $(2,1)$ type) which contains the lower minimal parabolic subgroup. Then the reduced model (L, H_Q) we get is the following (once again to make our notation simple, we will use $(G,\ H)$ instead of (L, H_Q)): Let $G = GL_4(F) \times GL_2(F)$ and let $P = MU$ be the parabolic subgroup of $G(F)$ with the Levi part $M(F)$ isomorphic to $GL_2(F) \times GL_2(F) \times GL_2(F)$ (i.e. P is the product of the second $GL_2(F)$ and the parabolic subgroup $P_{2,2}$ of the first $GL_4(F)$). The unipotent radical $U(F)$ consists of elements of the form

$$u = u(X) := \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ X \in M_2(F). \tag{A.1}$$

The character ξ on $U(F)$ is defined to be $\xi(u(X)) = \psi(\mathrm{tr}(X))$. Let $H_0(F) = \mathrm{GL}_2(F)$ diagonally embeded into $M(F)$. For a given irreducible representation π of $G(F)$, assume the central character ω_π equals χ^2 on $Z_H(F)$ for some character χ of $F^\times$. χ will induce a one-dimensional representation ω of $H_0(F)$. Combining ξ and ω, we get a one-dimensional representation $\omega \otimes \xi$ of $H(F) := H_0(F) \ltimes U(F)$. Let

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \omega \otimes \xi). \tag{A.2}$$

This model can be thought as the "middle model" between the Ginzburg-Rallis model and the trilinear GL_2 model.

Similarly, for the quaternion algebra case, we can define the multiplicity $m(\pi_D)$. The following theorem is an analogue of Theorem 1.2 and Theorem 1.4 for this model.

Theorem A.3. *For any tempered representation π of $G(F)$, let π_D be the Jacquet-Langlands correspondence of π to $G_D(F)$. Then the followings hold.*

(1) $m(\pi) + m(\pi_D) = 1$.
(2) *If $F = \mathbb{R}$, we have*

$$m(\pi) = 1 \iff \epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = 1,$$
$$m(\pi) = 0 \iff \epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = -1.$$

(3) *If F is p-adic and if π is not a discrete series, we have*

$$m(\pi) = 1 \iff \epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = 1,$$
$$m(\pi) = 0 \iff \epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = -1.$$

We can also prove a multiplicity formula for this model in the p-adic case. Let (G,H) be either (G,H) or (G_D,H_D) defined as above, and let F be a p-adic field. Let $\mathcal{T}$ be the subset of subtori of H_0 defined in Section 7.2. Let θ be a quasi-character on $Z_G(F)\backslash G(F)$ with central character $\eta=\chi^2$, and let $T\in\mathcal{T}$. If $T=\{1\}$, we still define $c_\theta(1)=c_{\theta,\mathcal{O}_{reg}}(1)$. If $T=T_v$ for some $v\in F^\times/(F^\times)^2, v\neq 1$, and $t\in T_v$ is a regular element, $G_t=GL_2(F_v)\times GL_1(F_v)$. Let $\mathcal{O}_t=\mathcal{O}_1\times\mathcal{O}_2$ where $\mathcal{O}_1$ is the unique regular nilpotent orbit in $\mathfrak{gl}_2(F_v)$ and $\mathcal{O}_2=\{0\}$ is the unique nilpotent orbit in $\mathfrak{gl}_1(F_v)$. We define $c_\theta(t)=c_{\theta,\mathcal{O}_t}(t)$. Then for all irreducible tempered representation π of $G(F)$, we have the multiplicity formula

$$m(\pi)=m_{geom}(\pi):=\sum_{T\in\mathcal{T}}\mid W(H,T)\mid^{-1}\nu(T)\int_{Z_G(F)\backslash T(F)}c_\pi(t)D^H(t)\chi(\det(t))^{-1}\Delta(t)dt$$

where $c_\pi(t)=c_{\theta_\pi}(t)$, and $\Delta^L(t)=|\det((1-ad(x)^{-1})|_{U(F)/U_x(F)})|_F$.

Remark A.4. *When the central character of π is trivial, the middle model can be viewed as the Gan-Gross-Prasad model for* $(\mathrm{SO}_6,\mathrm{SO}_3)$.

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